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OSU ST 6540 HW#8

3.8.1 $X(0)=1$ let ξ be r.v.

$P(\xi=0)=1/2$ $P(\xi=2)=1/2$ Let

$X(n)=\xi_1^{(n)}+\cdots\xi_{X(n-1)}^{(n)}$. Now we

have a branching process ($\xi_i^{(n)}$).

let $\mu=E[\xi]=0\cdot\frac{1}{2}+2\cdot\frac{1}{2}=1$, as in

the book, using $E[X(n)]=E[\xi]E[X(n-1)]$

and iterating $E[X(n)]=\mu^n=1$

$Var(\xi)=E[\xi^2]-E[\xi]^2=(0^2\cdot\frac{1}{2}+2^2\frac{1}{2})-1$

$=1$, and now a direct appication

of 3.99 $Var(X(n))=Var(\xi)E[\xi]^{n-1}\cdot\begin{cases} n & \mu=1 \\ \frac{1-\mu^n}{1-\mu} & \mu\neq 1\end{cases}$

gives us

$Var(X(n))=1\cdot 1^{n-1}\cdot n=n$

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3.8.2

$P(\xi=0)=a \quad P(\xi=1)=b \quad P(\xi=2)=c$

$a+b+c=1$

$\mu = E[\xi] = 0\cdot a + 1\cdot b + 2\cdot c = b+2c$

$E[X(n)] = \mu^n = (b+2c)^n$

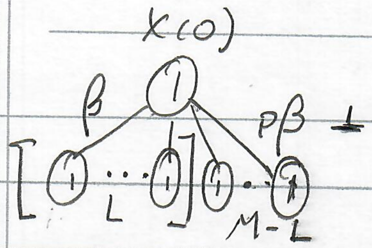
$Var(\xi) = (0^2\cdot a + 1^2\cdot b + 2^2\cdot c) - (b+2c)^2$

$= b+4c-b^2-2bc-4c^2$

$Var(X(n)) = [b+4c-b^2-2bc-4c^2]\cdot(b+2c)^{n-1}$

$\times \dfrac{1-(b+2c)^n}{1-(b+2c)}$

3.8: P1



(bad picture)

$$\xi = \nu_1 + \dots \nu_{M-L} + L\theta$$

$P(\theta = 1) = \beta \quad P(\theta = 0) = 1 - \beta$

$P(\nu = 1) = p\beta \quad P(\nu = 0) = 1 - p\beta$

The mean number of offspring $E[X(1)]$

where $X(1) = \xi$

$E[X(1)] = E[\xi]\, E[X(0)] = E[\xi]$ since

$X(0) = 1$

$$E[\xi] = \sum_{i=1}^{M-L} E[\nu] + L\, E[\theta]$$

$= (M-L)p\beta + L\beta$. Setting $\alpha = p\beta$, $N = M-L$

$E[X(1)] = \alpha N + \beta(M-N)$

Since $\theta \perp \nu_i$ $\quad \nu_i$ iid

$P(\xi = 0) = P(\nu_1 = 0, \nu_2 = 0, \dots, \nu_N = 0, \theta = 0)$

$= P(\nu = 0)^N P(\theta = 0) = (1-\alpha)^N (1-\beta)$

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3-8.3

First assume $P(\text{Boy}) = P(\text{Girl}) = 1/2$

Let ξ be a r.v. denoting # childeren

$P(\xi = 1) = 0$

$P(\xi = 2) = P(\text{First Child} = \text{Boy}, \text{Second Child} = \text{Girl})$
$= 1/2 \cdot 1/2$

Let C_i = Child i B = Boy G = Girl

$P(\xi = 3) = P(C_1 = B, C_2 = B, C_3 = G) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$\vdots$

$P(\xi = k) = (\frac{1}{2})^{k-1} \cdot 1/2 = (\frac{1}{2})^k$

Let η be a r.v. denoting # boys

$P(\eta = 1) = 0$

$P(\eta = 2) = P(C_1 = B, C_2 = B, C_3 = G) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$P(\eta = 3) = P(C_1 = B, C_2 = B, C_3 = B, C_4 = G) = (1/2)^4$

$\vdots$

$P(\eta = k) = (1/2)^k \, \frac{1}{2} = (\frac{1}{2})^{k+1}$

3.9.E7

$\xi \sim Pois(\lambda) \qquad \lambda = 1.1$

$u_n = P(X(n)=0 \mid X(0)=1) \qquad u_n = \phi(u_{n-1})$

$u(0)=0$. Now we need the g.f.

of $Pois(\lambda)$ $\quad \phi(s) = E[s^{\xi}] = \sum_{k=0}^{\infty} p_k s^k$

$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} s^k e^{-\lambda} = \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} e^{-\lambda}$

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{-\lambda(1-s)}$

$u(0)=0 \Rightarrow u_1 = e^{-\lambda(1-0)} = e^{-\lambda}$

$u_2 = e^{-\lambda(1-e^{-\lambda})}$ We'll use matlab

to calculate the exact values for $u_1, \ldots u_5$

For u_∞ we need to solve for

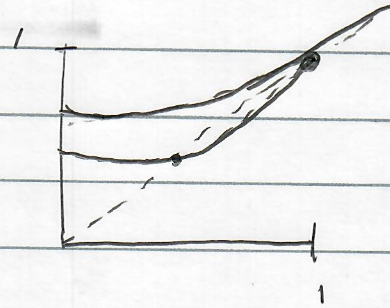
the smallest solution to $u = \phi(u)$

We know 1 is a solution, $\phi'(s) =$

$\frac{d}{ds} e^{-\lambda(1-s)}\Big|_1 = e^{-\lambda} \lambda e^{\lambda s}\Big|_1 = \lambda > 1$

3.9 ε1

case 1
case 2



Case 1 $\phi'(s)\big|_1 < 1$

Case 2 $\phi'(s)\big|_1 > 1$ we're in case 2

so the solution is given by

$u = e^{-\lambda(1-u)}$. Taking derivative of this

$$\frac{d}{du} u = \frac{d}{du} e^{-\lambda} e^{-\lambda u} \Rightarrow 1 = e^{-\lambda} \lambda e^{-\lambda u}$$

$$e^{-\lambda u} = \frac{e^{\lambda}}{\lambda} \Rightarrow \log e^{-\lambda u} = \lambda - \log \frac{1}{\lambda}$$

$$\Rightarrow u = \log \tfrac{1}{\lambda} - \lambda \big/ \lambda = \frac{\log 1/\lambda}{\lambda} - 1$$

3.9.E2

$P(\xi=0)=p_0 \quad P(\xi=2)=p_2 \quad p_0+p_2=1$

$\phi(s) = E[s^{\xi}] = s^0 p_0 + s^2 p_2$

$= 1 \cdot p_2 + s^2 p_2$

$X(0)=1 \quad X(1)=\xi \quad X(n)=\xi_1^{(n)}+\cdots \xi_{X(n-1)}^{(n)}$

So $\phi_n(s)=E[s^{X(n)}]$ $\quad$ So $\phi_0=E[s^1]=1$

$\phi_1(s)=E[s^{X(1)}]=E[s^{\xi}]$ $\quad$ in general

$\phi_{n+1}(s)=\phi_n[\phi(s)]$

The g.f. of $X(1)$ is $\phi_1(s)=E[s^{\xi}]$

$= 1-p_2+s^2 p_2$

§ 3.9. P1

ξ r.v. describing offspring. Assume

$P(\text{Boy}) = P(\text{Girl}) = 1/2$. $X(0) = 1$

$X(1) = \xi \qquad X(n+1) = \sum_{i=1}^{X(n)} \xi_i^{(n)}$

What is P(a given couple has k males)

$k = 0, 1, 2, 3$ in this case

$P_0 = \frac{1}{4} + \frac{3}{4} \cdot \left(\frac{1}{2}\right)^3$ For the other

case we need binomial probabilities

let $\eta \sim \text{Binom}(3, 1/2)$

$P_1 = \frac{3}{4} P(\eta = 1) = \frac{3}{4}\left[\frac{3!}{2!1!}\left(\frac{1}{2}\right)^1\left(1-\frac{1}{2}\right)^{3-1}\right] = 9/32$

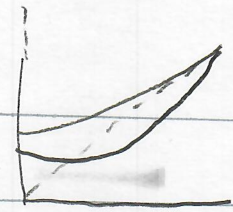
$P_2 = \frac{3}{4} P(\eta = 2) = \frac{3}{4}\left[\frac{3!}{1!2!}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{3-2}\right] = 9/32$

$P_3 = \frac{3}{4} P(\eta = 3) = \frac{3}{4}\left[\frac{3!}{0!3!}\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{3-3}\right] = 3/32$

Now let u_k be the probability of

extinction of males at generation k, and

§3.9.P1



Then $u_0 = 0$ $\quad u_1 = p_0 = \phi(u_0)$ ...

Calculating $\phi'(1) = \left.\dfrac{d\phi(s)}{ds}\right|_{s=1} = p_1 + 2sp_2 + 3s^2 p_3$

See our Matlab script; $\phi'(1) = \frac{9}{8} > 1$

so we seek a solution $u \in (0,1)$ to $u = \phi(u)$. Again see our Matlab script where we calculate the three roots $u_1 \cong -4.7689$ $\quad u_2 = 1.0$ $\quad u_3 \cong .7689$

Choosing the root in $(0,1)$

$u_\infty = P(\text{extinction of males}) = \dfrac{499}{649} \approx .7689$

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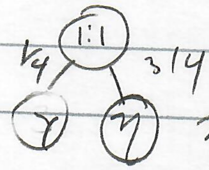
§3.9.P2



$\eta \sim \text{Geom}(\tfrac{1}{2})$ $\qquad \gamma \sim \text{Binom}(3, \tfrac{1}{2})$

$\xi = \gamma + \eta$ $\qquad$ let $X(0) = 0 \qquad X(1) = \xi$

$$X(n+1) = \sum_{i=1}^{X(n)} \xi_i^{(n)}$$

Assume $P(\text{Boy}) = P(\text{Girl}) = \tfrac{1}{2}$. What is $P(\text{Given couple has } k \text{ males})$? η will contribute 1 and $\gamma \sim \text{Binom}(3, \tfrac{1}{2})$ will determine the number from the portion of parents having 3 children.

$$P_0 = \frac{1}{4} P(\gamma = 0) = \frac{1}{4}\left(\frac{3!}{0!\,3!}\left(\frac{1}{2}\right)^0 \left(1 - \frac{1}{2}\right)^3\right) = 1/32$$

$$P_1 = \frac{3}{4} + \frac{1}{4} P(\gamma = 1) = \frac{3}{4} + \frac{1}{4}\left(\frac{3!}{1!\,2!}\left(\frac{1}{2}\right)^1 \left(1 - \frac{1}{2}\right)^{3-1}\right) = \frac{27}{32}$$

$$P_2 = \cancel{\frac{3}{4}} + \frac{1}{4} P(\gamma = 2) = 1/4\left(\frac{3!}{2!\,1!}\left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^{3-2}\right) = 3/32$$

$$P_3 = \frac{1}{4} P(\gamma = 3) = 1/4\left(\frac{3!}{3!\,0!}\left(\frac{1}{2}\right)^0 \left(1 - \frac{1}{2}\right)^3\right) = 1/32$$

Like before $\phi(s) = E[s^{\xi}] = s^0 P_0 + s P_1 + s^2 P_2 + s^3 P_3$

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§3.9.P2 ②

First check $\phi'(s)\Big|_{s=1}$

$\phi'(s) = p_1 + 2sp_2 + 3s^2 p_3 = 9/8 > 1$ so we

seek solution of $u = \phi(u) \in (0,1)$

See our Matlab script, there are

three roots $u_1 = -4.236$ $u_2 = .2361$ $u_3 = 1.0$

$u_\infty = P(\text{extinction of males}) = \frac{72}{305} \approx .2361$

3.9.P1

```
p0=1/4+3/4*(1/2)^3;
p1=9/32;
p2=9/32;
p3=3/32;

derivate_1=p1+2*p2+3*p3;
rats(derivate_1)
```

```
ans = 
'      9/8     '
```

```
p11=p1-1;
p=[p3 p2 p11 p0];
r = roots(p)
```

```
r = 3×1
   -4.7689
    1.0000
    0.7689
```

```
rats(r)
```

```
ans = 3×14 char array
    '  -3095/649   '
    '       1      '
    '    499/649   '
```

```

```

3.9.P2

```
p0=1/32;
p1=27/32;
p2=3/32;
p3=1/32;

derivate_1=p1+2*p2+3*p3;
rats(derivate_1)
```

```
ans = 
'      9/8     '
```

```
p11=p1-1;
p=[p3 p2 p11 p0];
r = roots(p)
```

```
r = 3×1
   -4.2361
    1.0000
```

```
    0.2361
```

```
rats(r)
```

```
ans = 3×14 char array
    '  -1292/305   '
    '       1      '
    '     72/305   '
```