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Bruce Campbell

§4.1 E8 OSU ST 6540 HW#4

Let $\{X_n\}$ be successive generations wealth

$$\mathbb{P}_{ij} = \begin{array}{c} \begin{array}{ccc} L & M & U \end{array} \\ \left[\begin{array}{ccc} .7 & .2 & .1 \\ .2 & .6 & .2 \\ .1 & .4 & .5 \end{array}\right] \end{array} \begin{array}{c} \\ L \\ M \\ U \end{array} \qquad S = \{L, M, U\}$$

The fraction of time spent in state $U$ is given by $\pi_U$.

We note that $\mathbb{P} > 0 \Rightarrow \mathbb{P}^n > 0$, also $\mathbb{P} > 0 \Rightarrow$ our 2 conditions for regularity;

$\forall\ i j\ \exists\ k_1, \dots k_r \ni P_{ik_1} \cdots P_{ik_r} > 0$ and

$\exists\ i \ni \mathbb{P}_{ii} > 0$. So we know $\pi$ exists and don't have to check $\mathbb{P}^{(N^2)}$

Since $\mathbb{P}$ is regular, thm 4.1 $\Rightarrow$ $\pi^t = \pi^t\,\mathbb{P}$

$$\Rightarrow \quad \pi_L P_{LL} + \pi_M P_{ML} + \pi_U P_{UL} = \pi_L$$

$$\pi_L P_{LM} + \pi_M P_{MM} + \pi_U P_{UM} = \pi_M$$

$$\pi_L P_{LU} + \pi_M P_{MU} + \pi_U P_{UU} = \pi_U$$

$$\begin{cases} \pi_L \frac{7}{10} + \pi_M \frac{2}{10} + \pi_U \frac{1}{10} = \pi_L \\ \pi_L \frac{2}{10} + \pi_M \frac{6}{10} + \pi_U \frac{4}{10} = \pi_M \\ \pi_L \frac{1}{10} + \pi_M \frac{2}{10} + \pi_U \frac{5}{10} = \pi_U \end{cases} \Rightarrow$$

$\pi_L + \pi_M + \pi_U = 1$

① $-3\pi_L + 2\pi_M + \pi_U = 0 \Rightarrow \pi_U = 3\pi_L - 2\pi_M$

② $2\pi_L - 4\pi_M + 4\pi_U = 0 \Rightarrow \pi_L = 6/7\,\pi_M$

③ $\pi_L + 2\pi_M - 5\pi_U = 0 \Rightarrow \frac{6}{7}\pi_M + 2\pi_M - 5\left(3 \cdot \frac{6}{7}\pi_M - 2\pi_M\right) = 0$

cant use eq ③ have to use constraint $\sum_S \pi_i = 1$

$\frac{6}{7}\pi_M + \pi_M + \left(3 \cdot \frac{6}{7}\pi_M - 2\pi_M\right) = 1$

$\frac{13}{7}\pi_M + \frac{4}{7}\pi_M = 1 \Rightarrow \pi_M = 7/17$

$\pi_L = \frac{6}{17}$

$\pi_U = 4/17$

Same as Matlab!

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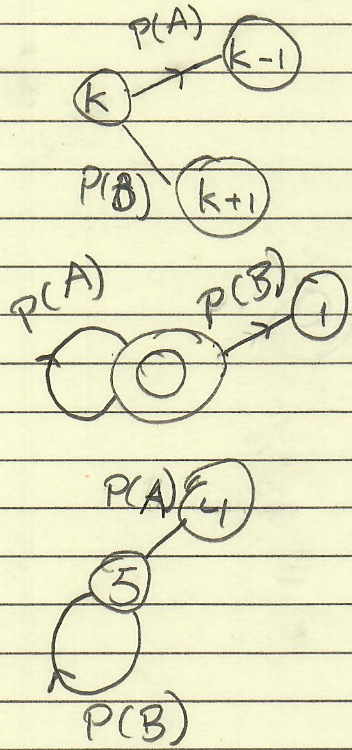
§ 4.1 p1

Let $\{X_n\}$ be the M.C. with $S = \{0, 1, \dots, 5\}$

$X_n$ = # balls in urn A.




$k \neq 0$

$P(A) = \text{Prob}(A \text{ is selected}) =: p$
$P(B) = 1 - P(A) =: q$

Then

$$\mathbb{P} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc|c} 0 & 1 & 2 & 3 & 4 & 5 & \\ \hline p & q & 0 & 0 & 0 & 0 & 0 \\ p & 0 & q & 0 & 0 & 0 & 1 \\ 0 & p & 0 & q & 0 & 0 & 2 \\ 0 & 0 & p & 0 & q & 0 & 3 \\ 0 & 0 & 0 & p & 0 & q & 4 \\ 0 & 0 & 0 & 0 & p & q & 5 \end{array}$$

We have to check $\mathbb{P}$ is regular

We see all states are communicating, and that $\exists\, i \ni P_{ii} > 0$. Writing out $\pi^t = \pi^t \mathbb{P}$

$\pi_0 p + \pi_1 p = \pi_0 \Rightarrow \pi_0 q = \pi_1 p$

$\pi_0 q + \pi_2 p = \pi_1 \Rightarrow \pi_1 p + \pi_2 p = \pi_1 \Rightarrow \pi_1 q = \pi_2 p$

$\pi_1 q + \pi_3 p = \pi_2 \Rightarrow \pi_2 p + \pi_3 p = \pi_2 \Rightarrow \pi_2 q = \pi_3 p$

$\pi_2 q + \pi_4 p = \pi_3 \Rightarrow \pi_3 p + \pi_4 p = \pi_3 \Rightarrow \pi_3 q = \pi_4 p$

$\pi_3 q + \pi_5 p = \pi_4 \Rightarrow \pi_4 p + \pi_5 p = \pi_4 \Rightarrow \pi_4 q = \pi_5 p$

$\pi_4 q + \pi_5 q = \pi_5 \Rightarrow \pi_5 p + \pi_5 q = \pi_5 \Leftarrow$ not useful

$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$ will be last eq.

§4.1 P1

Putting our expressions $\pi_{i+1} = \pi_i \frac{q}{p}$

into $\sum \pi_i = 1$

$$\pi_0 + \pi_0 \frac{q}{p} + \pi_1 \left(\frac{q}{p}\right)^2 + \cdots + \pi_1 \left(\frac{q}{p}\right)^5 = 1$$

$$\pi_0 = 1 \Big/ \sum_{i=0}^{5} \left(\frac{q}{p}\right)^i \quad , \quad \pi_i = \left(q/p\right)^i \Big/ \sum_{i=0}^{5} \left(\frac{q}{p}\right)^i$$

Finally - the fraction of time that urn A is empty in the long run is $\pi_0$

§ 4.1 p 4

$\{X_n\}$ M.C. $|S| < \infty$ has $\mathbb{P}$ and a limiting ditribution $\pi$. What fraction of transitions are from $k$ to $m$ in the long run?

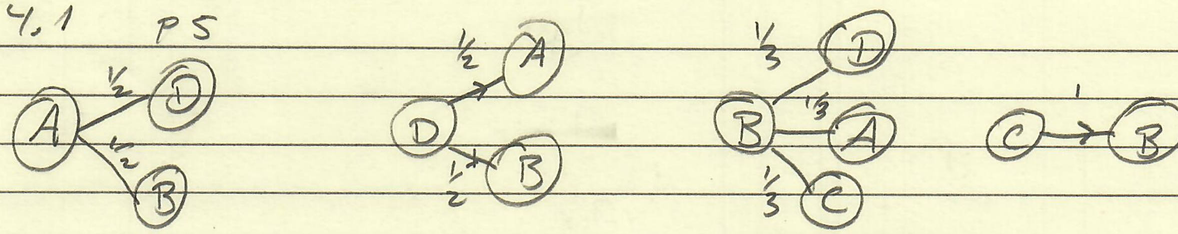
We know the proportion of time in states $k$ and $m$ are $\pi_k$, $\pi_m$

We expect the answer to be the proportion of time spent in $k$ times the proportion of transitions <u>from</u> $k$ to $m$, but this is just $P_{km}$

So

$\pi_k P_{km}$ is the long run fraction of transitions from $k \to m$.

§ 4.1 P5



$$\begin{array}{c} \\ \mathbb{P} = \end{array}
\begin{array}{c}
\begin{array}{cccc} A & B & C & D \end{array} \\
\left[\begin{array}{cccc} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{array}\right]
\end{array}
\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

We're asked to find $\pi_D$ - the long run probability of finding state D. All states are communicating but, notice we don't have our regularity condition $\exists\, i \ni P_{ii} > 0$. We'll finish calculations in Matlab. First checking $\mathbb{P}^{(5^2)} > 0$ and then solving

$$\pi^t = \pi^t \mathbb{P}$$

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§4.1 p10

$$\mathbb{P} = \begin{bmatrix} S^0 p \\ S^1 p \\ \vdots \\ S^{N-1} p \end{bmatrix}$$ where $S$ is cyclic permutation

$$S : (1, \dots n) \to (n, 1, \dots, n-1)$$

$$p = (p_0, \dots, p_N)$$

$$S^i p = (p_{N-(i+1)}, \dots, p_N, p_0, p_1, \dots p_{N-i})$$

We notice $\displaystyle\sum_{j=0}^{N} \mathbb{P}_{ji} = \sum_{j=0}^{N} (S^j p)_i$

$$= p_{N-(i+1)} + \dots + p_N + p_0 + \dots + p_{N-i}$$

$$= \sum p_i = 1$$ so $\mathbb{P}$ is doubly stochastic. Now, if $\mathbb{P}$ is regular, then we can conclude $\pi = (1/N, \dots 1/N) \sim Unif(N)$. We don't know $p$, but the assumption $0 < p_0 < 1$ tells us a few things. First, there must be at least one other non-zero $p_i$ s.t. $i \neq 0$ since $\sum p_i = 1$. There are also no absorbing states, since that would require $p_j = 1$ for some $j$.

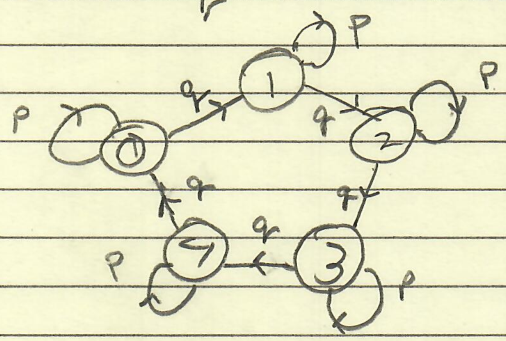
Let's consider the worst case $p_0, p_k \neq 0$, $p_i = 0 \ \forall i \neq 0, k$

$$\begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & \cdots & k & k+1 & \cdots & N \end{array} \\
\mathbb{P} = & \left[\begin{array}{ccccccc}
p & 0 & & q & 0 & & \\
0 & p & & 0 & q & & \\
0 & & \ddots & & & & \\
0 & & & p & & & q \\
q & & \cdots & & p & & \\
0 & & q & & 0 & & p
\end{array}\right]
\begin{array}{l} \\ \\ N-(k+1) \\ N-k \\ N-(k+1) \\ N \end{array}
\end{array}$$

Suppose $N=4$ $\quad p_0, p_1 \neq 0 \quad p_2 = p_3 = p_4 = 0 \Rightarrow p_0 = p \quad p_1 = 1-p = q$

$$\mathbb{P} = \begin{array}{cc} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} & \\ \left[\begin{array}{ccccc} p & q & 0 & 0 & 0 \\ 0 & p & q & 0 & 0 \\ 0 & 0 & p & q & 0 \\ 0 & 0 & 0 & p & q \\ q & 0 & 0 & 0 & p \end{array}\right] & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

and looking at our transitions we see




a similar argument for the general case should show all states are communicating -with some additional modulo k considerations

4.2 ≷6

§4.2 P

$P(T=1)=.1$ $\quad P(T=2)=.2$ $\quad P(T=3)=.3$ $\quad P(T=4)=.4$

$a_1$ $\quad a_2$ $\quad a_3$ $\quad a_4$

Let $X_n$ be age of component at $n$

$\{X_n\}$ is M.C. $\{X_n\}$ is a success run

M.C. with $S=\{0,1,2,3,4\}$. Here $X_0=0$

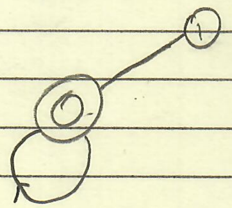
Let $T_k$ be lifetime of $k^{th}$ component. $\sum_{k=1}^{n} T_k$

gives us the time the $n^{th}$ bulb fails.

$\sum T_k + 1$ is when the $n+1^{th}$ bulb is put in service.

We're going to guess $p_k = a_{k+1} / (a_{k+1} + \dots + a_N)$

$k=0, N-1$ when $k=4$ we replace component



Better! Set up as replacement policy on

$S=\{0,1,2,3,4,5\}$

and we specify a new component upon reaching 5. This is consistent with since

$P(X_{n+1}=4 \mid X_n=3) = ?$ 1 or .4

Writing

$p_0$, $p_1$, $p_2$, $p_3$

$$\begin{array}{c} \\ \mathbb{P} = \end{array}
\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{cccccc}
a_1 & 1-a_1 & & & & \\
p_1 & & 1-p_1 & & & \\
p_2 & & & & & \\
p_3 & & & & & \\
p_4 & & & & & \\
1 & & & & &
\end{array}\right] \end{array}
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$p_0 = \dfrac{a_1}{a_1+a_2+a_3+a_4} = a_1$ $\qquad p_1 = \dfrac{a_2}{a_2+a_3+a_4} = P(X_{n+1}=2 \mid X_n=1)$

? check!

4.2 E6

$P(T=1) = a_1$

$P(T>1) = 1 - a_1 = \cancel{\ }$

$P(T>2) = a_3 + a_4$

$$P(X_{n+1} = k+1 \mid X_n = k) = \frac{P(X_{n+1} = k+1, X_n = k)}{P(X_n = k)}$$

$$P(X_1 = 1 \mid X_0 = 0) = P(X_1 = 1, X_0 = 0) / P(X_0 = 0)$$

$X_0 = 0 \Rightarrow P(X_0 = 0) = 1 \qquad P(X_1 = 1, X_0 = 0) = P(X_1 = 1) = a_1$

$P(X_2 = 2 \mid X_1 = 1) = P(T > 1) = P(T \geq 2 \mid T > 1)$

$P(X \leq 3) =$

$X \leq 1 \quad E_1$

$X \leq 2 \quad E_2 \qquad \cap E_i = E_1$

$X \leq 3 \quad E_3$

$4 \nearrow 0$

Let $p_k = \dfrac{q_{k+1}}{q_{k+1} \cdots q_{N-2}}$

$N = 4$ or $5$

sort!

$k = 0, 1, 2, \ \big| \ (4-2)$

$3 \ \big| \ (5-2)$

$N=5$ when $T=4$

$5-1$

$4 - 0 \Rightarrow$ replacement @ 5

THIS ONE!

$$\mathbb{P} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & p_0 & q_0 & 0 & 0 & 0 \\ 1 & p_1 & 0 & q_1 & 0 & 0 \\ 2 & p_2 & 0 & 0 & q_2 & 0 \\ 3 & p_3 & 0 & 0 & 0 & q_3 \\ 4 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Solve this system $\pi^t = \pi^t \mathbb{P}$

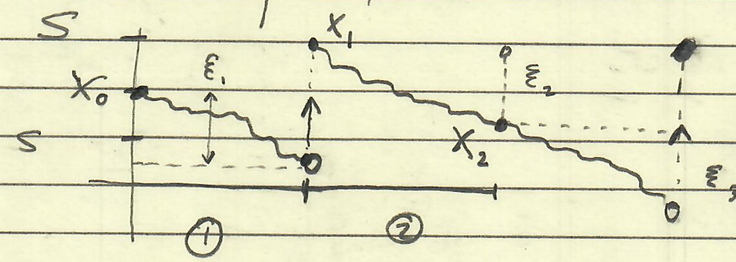
letting $A_j = a_j + \cdots a_3$

§ 4.2

$n = 0, 1, 2, \dots$ $\quad \xi_n$ demand $\quad \xi_i \perp \quad \xi_i \sim (a_0, a_1, \dots)$

$\sum a_k = 1$ $\quad X_n$ quanity on hand

$S, s$



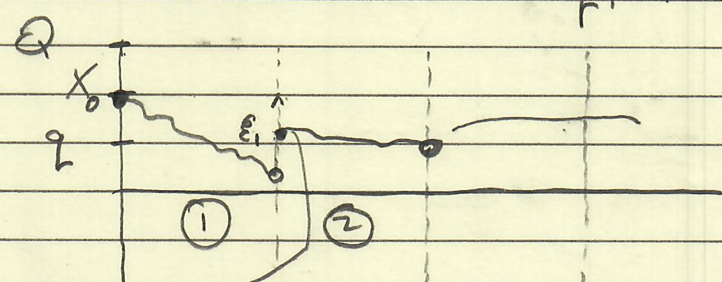
$$X_{n+1} = \begin{cases} X_n - \xi_{n+1} & s < X_n \le S \\ S - \xi_{n+1} & X_n \le s \end{cases}$$

$X_i \le S \ \forall i$ (an assumption since we could set $X_0 = S + k$ )

$\mathcal{S} = \{ S, S-1, \dots, 0, -1, \dots -S+1, \dots \}$

$\mathcal{S}$ is determined by $S$ and $a_k$'s

Lets draw the $q, Q$ policy



$X_2 = X_1 - \xi_2$ when $X_1 - \xi_2 > q$

$X_1 = X_0 - \xi_1 + Q$ when $X_0 - \xi_1 \le q$

$$X_{n+1} = \begin{cases} X_n - \xi_{n+1} + Q & X_n - \xi_{n+1} \le q \\ X_n - \xi_{n+1} & X_n - \xi_{n+1} > q \end{cases}$$

$$j = \begin{cases} i - \xi_{n+1} + q & i - \xi_{n+1} \le q \\ i - \xi_n & i - \xi_n > q \end{cases}$$

$$\xi_n = \begin{cases} i - (j + q) \\ i - j \end{cases}$$

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$j - i = \xi_n + Q$

$\xi_n = i - j - Q$

§ 4.2 P1

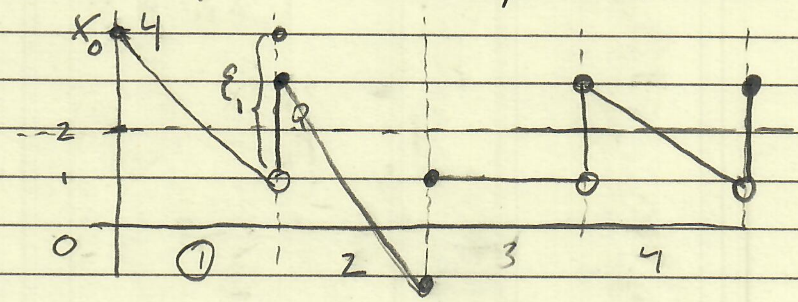
P2

a) Suppose $X_0 = 4$ $\xi_1 = 3$ $\xi_2 = 4$ $\xi_3 = 0$ $\xi_4 = 2$

Find $X_1, X_2, X_3, X_4$ for the $q, Q$ process

with $q = Q = 2$

Let's draw!




$X_1 = 3$ $X_2 = X_1 - \xi_2 + Q$
$= 1$

$X_3 = X_2 - \xi_3 + Q$
$= 3$

$X_4 = X_3 - \xi_4 + Q$
$= 3$

Now let $\xi_i$ i.i.d. $P(\xi = k) = (a_1, a_2, a_3, a_4)$

$$P_{ij} = P(X_{n+1} = j \mid X_n = i) = \begin{cases} P(\xi_{n+1} = i - j) & i > q \\ P(\xi_{n+1} = i - (j + Q)) & i \le q \end{cases}$$

What about assumption That demand that is not filled is lost (i.e. no backordering?) does this mean $X_n \geq 0$? If thats true we have 3 cases to worry about $X_n - \xi_{n+1} > q$, $X_n - \xi_{n+1} < q$

$X_n - \xi_{n+1} + Q < 0$ the last one $= 0$

$$X_{n+1} = \begin{cases} X_n - \xi_{n+1} + Q & X_n - \xi_{n+1} \le q, \\ Q & \text{when } X_n - \xi_{n+1} < 0 \\ X_n - \xi_{n+1} & X_n - \xi_{n+1} > q \end{cases}$$

§ 4.2 p 2

Let $(x, y) = (\#\text{ computers operating}, \begin{cases} 1 & \text{one day of labor on broken one} \\ 0 & \text{otherwise} \end{cases})$

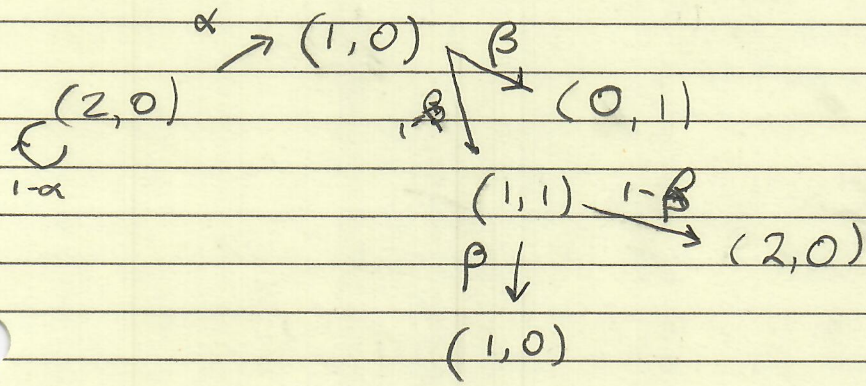
Then $S = \{ (2,0), (1,0), (1,1) \quad (0,1) \}$

= { both operational, one is working + second needs 2 days work

one is working + second needs one day work

one needs one day work + other needs 2 days }

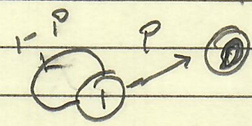
a)

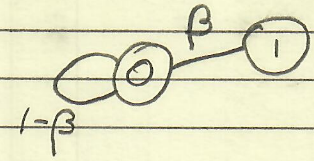



$$
\begin{array}{c|c}
 & \begin{array}{cccc} (2,0) & (1,0) & (1,1) & (0,1) \end{array} \\
\mathbb{P} = & \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ (1-\beta) & \beta & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} \\ (2,0) \\ (1,0) \\ (1,1) \\ (0,1) \end{array}
$$

§4.2 p6

let $X_n = \begin{cases} 1 & \text{computer operational} \\ 0 & \text{otherwise} \end{cases}$




We recognize $N \sim \text{Geom}(\beta)$ which we know to be memoryless - $P(N=1) = 1 - P(N \neq 1) = \beta$ i.e. success on first trial. If $0 \to 0$ the process starts over.

We can construct $\mathbb{P}$ from our transitions now

$$\begin{array}{c} \\ \mathbb{P} = \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-\beta & \beta \\ p & 1-p \end{bmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \end{array}$$

The long run proportion of time the computer is operational is given by $\pi_1$ wich can be obtained by solving $\pi^T = \pi^T \mathbb{P}$

$\pi_0 (1-\beta) + \pi_1 p = \pi_0 \quad \Rightarrow \quad \pi_0 = \frac{p}{\beta} \pi_1$

$\pi_0 \beta + \pi_1 (1-p) = \pi_1$

$\pi_0 + \pi_1 = 1 \quad \Rightarrow \quad \pi_1 = \frac{1}{1 + p/\beta}, \quad \pi_0 = \frac{\frac{p}{\beta}}{1 + \frac{p}{\beta}} = \frac{1}{1 + \frac{\beta}{p}}$

This makes qualitative sense; $\beta$ small $\Rightarrow N$ big $\Rightarrow \pi_1$ smaller as expected.

§4.3 §3

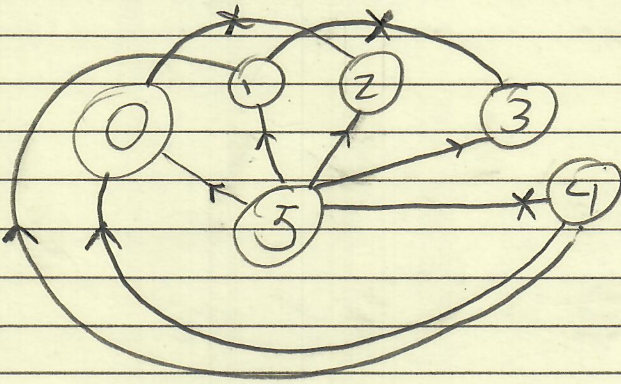
a) By inspection $0 \leftrightarrow 2$ $\quad 1 \leftrightarrow 3$ $\quad 5 \leftrightarrow 4$

$0 \not\to 1 \quad 0 \not\to 3 \quad 0 \not\to 4 \quad 0 \not\to 5$

$1 \not\to \{0, 2, 4, 5\} \quad 2 \not\to \{1, 3, 4, 5\} \quad 3 \not\to \{0, 2, 4, 5\}$

$4 \to \{0, 1, 5\}$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 4/5 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix} \end{array}$$



ignoring $P_{ii}$ transitions

$C_1 = \{0, 2\} \quad C_2 = \{1, 3\} \quad C_3 = \{5, 4\}$

Are the communicating classes.

Note $C_3 \to C_1$ and $C_3 \to C_2$ by $4 \to 0$, $4 \to 1$
and $C_3 \to C_1$, $C_3 \to C_2$ by $5 \to 0$ $5 \to 1$
All of the states are recurrent except for 4
and 5

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§4.3 E3

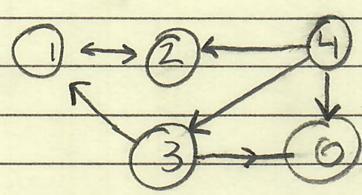
b)

$$\mathbb{P} = \begin{array}{c|cccccc|c} & 0 & 1 & 2 & 3 & 4 & 5 & \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 3/4 & 1/4 & 0 & 0 & 0 & 1 \\ & 0 & 1/8 & 7/8 & 0 & 0 & 0 & 2 \\ & 1/4 & 1/4 & 0 & 1/8 & 3/8 & 0 & 3 \\ & 1/3 & 0 & 1/6 & 1/4 & 1/4 & 0 & 4 \\ & 0 & 0 & 0 & 0 & 0 & 1 & 5 \end{array}$$

By inspection (our other choice is to calculate $\mathbb{P}^{(n)}$)

$C_0 = \{0\}$ $C_1 = \{5\}$ are absorbing - hence not transient.

we don't draw $\mathbb{P}_{ii}$ transitions



$C_2 = \{1, 2\}$ $C_3 = \{4\}$

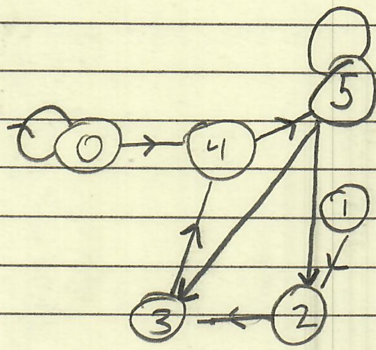
$1 \not\to \{2, 3, 4, 0\}$

$C_0 = \{0\}$ $C_1 = \{5\}$ $C_2 = \{1, 2\}$ $C_3 = \{4\}$ are the communicating classes

4, 3 are transient 0 1 2 5 are recurrent

§ 4.3 E 4

$$\mathbb{P} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$




$C_1 = \{0\}$ $\quad C_2 = \{1\}$

$C_3 = \{2, 3, 4, 5\}$

We only have to comput $d(0)$, $d(1)$ and $d(i)$ for one $i \in \{2, 3, 4, 5\}$

Since $P_{11} > 0$ $P_{55} > 0$ we have $d(0) = 1$ and $d(5) = 1$

Since periodicity is constant within a communicating class – $d(2) = d(3) = d(4) = d(5) = 1$

Finally for $d(1)$ notice the second column of $\mathbb{P}^{(n)}$ is always $0$ – i.e. $\mathbb{P}_{11}^{(n)} = 0 \ \forall n \geq 1$

$\therefore$ $d(1) = 0$