let $N(t)$ be P.P. (λ) ξ_k = lifetime of kth arrival.

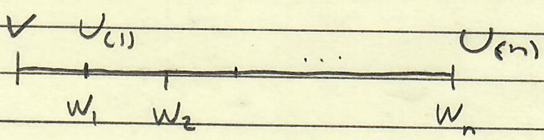
Let $G(t)$ be c.d.f. of ξ $\xi_i \perp$ i.i.d.

let $X(t)$ = # customers in store $Y(t)$ = # arrived and departed. Then $N(t) = X(t) + Y(t)$

Consider W_k - waiting time to kth arrival; then k is in store at t iff $W_k + \xi_k \geq t$. Now let $E_k = \{ W_k + \xi_k \geq t \}$, then $P(X(t) = k \mid N(t) = n) =$

$P(\sum_{k=1}^{n} E_k \mid N(t) = n)$.

Now $\{W_k\}$ jointly distributed as order stat of Unif$[0,t]$ - conditional on n. Any sum over all W_k can be cast as sum over the $U_k \sim$ Unif$[0,t]$ r.v.s.



So

$$P(X(t) = k \mid N(t) = n) = P(\sum 1(U_k + \xi_k \geq t) = k)$$

rhs is Binomial(n,p) with $p = P(U_k + \xi_k \geq t)$

We need the convolution thm to evaluate p

let $Z = U + \xi$ $U \sim$ Unif$[0,t]$ ξ $\bar{c}$ CDF G

$$F_Z(z) = \int_{u=-\infty}^{u=\infty} F_U(z-u) F_\xi(u) \quad \text{or} \quad f_Z(z) = \int_{-\infty}^{\infty} f_U(z-u) f_\xi(u)\, du$$

$$= \int_{u=-\infty}^{u=\infty} \frac{z-u}{t} G(u)$$

How to find joint distribution? Note

$Y(t) = j,\ X(t) = k \Rightarrow N(t) = j+k$. Now write

$P(X(t)=k, Y(t)=j) = P(X(t)=k, N(t)=j+k) =$

$P(X(t)=k \mid N(t)=j+k)\, P(N(t)=j+k)$

$$\frac{(j+k)!}{(k)!\,(j)!} p^k (1-p)^j \frac{(\lambda t)^{j+k}}{(j+k)!} e^{-\lambda t}$$

More about $P(U_k + \xi_k \geq t)$. First $P(U \leq z)$

$= F_U(z) = \int_{u=0}^{u=t} \frac{1}{t}\, du$, $P(U > z) = 1 - F_U(z)$

In our convolution formula there are two ways to convolve $F_{U+\xi}(z) = \int_{u=-\infty}^{u=\infty} F_U(z-u) F_\xi(u)\, du$

$= \int_{u=-\infty}^{u=\infty} F_U(u)\, F_\xi(z-u)\, du$. Using the second form

and noting $F_U(u) = 0$ $u < 0$ we have

$P(U + \xi > z) = \int_0^\infty \frac{1}{t} G(z-u)\, du$. Close ;^

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5.4.P1 $W_1, \dots W_n$ arrivals $PP(\lambda)$

$X(1) = n$ $k < n$ Find density of

$W_1, W_2, \dots W_{k-1}, W_{k+1}, \dots W_n \mid W_k = w, \quad X(1) = n$

We note our $U_{(i)} \sim U[0,1]$

The order stats of k^{th} O.S. $U_{(k)}$

$$f_{U_{(k)}}(u) = \frac{n!}{(k-1)!(n-k)!} u^{k-1} (1-u)^{n-k}$$

$U_{(k)} \sim \text{Beta}(k, n+1-k)$

We can make prob arg

$$f_{W_1, \dots W_n \mid X(1)=n}(w_1, \dots w_n) = n!$$

How can we have $k-1$ to left of $U_{(k)}$

$n-k+1$ to right of $U_{(k)}$ given that $U_{(k)} = w$?

We can also write $f_{W_1, \dots W_{k-1}, W_{k+1}, \dots W_n \mid X(1)=n, W_k = w}$

$$= f_{W_1, \dots W_n \mid X(1)=n} \Big/ f_{W_k} = \frac{n!}{\frac{n!}{(k-1)!(n-k)!} u^{k-1} (1-u)^{n-k}}$$

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5.4.P1

2

$$f_{W_1,\dots W_{k-1},W_{k+1},\dots W_n \mid X(1)=n,\, W_k=w} = \frac{n!}{\dfrac{n!}{(k-1)!\,(n-k)!}\, w^{k-1}(1-w)^{n-k}}$$

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5.4 P5

$N(t)$ P.P. (λ) find $\lim\limits_{\substack{n \to \infty \\ \frac{n}{t} = \beta}} P(W_1 = t \mid N(t) = n)$

$W_n \sim \Gamma(n, \lambda)$, $W_1 \sim \Gamma(1, \lambda) - \exp(\lambda)$

$f_{W_n}(t) = \frac{\lambda}{\Gamma(n)} (\lambda t)^{n-1} e^{-\lambda t}$

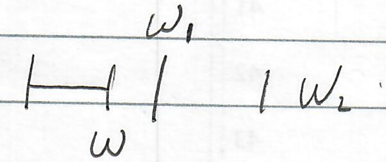
We know $f_{W_1, \dots W_n \mid N(t) = n}(t) = n!\, t^{-n}$

Do we marginalize over $W_n, W_{n-1}, \dots W_2$

- remembering $W_1 < W_2 \dots < W_n$

I needed a LOT of help here

$P(W_1 > w \mid N(t) = n)$



$\{W_1 > w\} \Rightarrow$ all waiting times to the right of w. - Same as prob of all n $U_{(i)} > w$ $U_i \perp \Rightarrow$

$P(W_1 > w \mid N(t) = n) = P(U_1 > w, U_2 > w, \dots, U_n > w)$

$= \prod_{i=1}^{n} P(U_i > w) = \left(1 - \frac{w}{t}\right)^n$

Now let $\frac{n}{t} = \beta$ and take limit

$\lim\limits_{n \to \infty} \left(1 - \frac{\beta w}{n}\right)^n = e^{-\beta w}$

5.4 P5

②

$$\lim_{n\to\infty} P(W_1 > w \mid N(t) = n) = 1 - F_{W_1|N}(w) = e^{-\beta w}$$

$$F_{W_1|N}(w) = 1 - e^{-\beta w}$$

$$\frac{d}{dw} F_{W_1|N}(w) = \beta e^{-\beta w} \Rightarrow$$

$W_1 | N \sim \exp(\beta)$. My notation is bad.

What really happens to N when we take limit? $W_1 | N$ becomes something that does not depend on N.

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Could not figure out part b)

5.6.E4

$X(t)$ P.P.(2) $Y(t)$ P.P.(4)

Treat Y as a marking for X (X, Y)?; Then the sojourn times of X determine Y, which violates conditions of a marked P.P.. What if we combine X, Y to make one P.P.(2+4) and then mark each arrival of $Z = X+Y$ accordingly? Let G be discrete r.v. $P(G = \text{male}) = 1/3$ $P(G = \text{female}) = 2/3$, then (Z, G) should provide the answers were looking for. Each customer enters at a rate of 6, and is independently male $\bar{c}$ prob $p = 1/3$ and female $\bar{c}$ prob $1 - 1/3$

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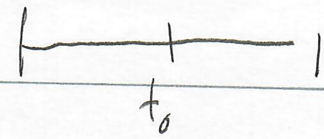

Let t_0 be an arbitrary time, and

T be the time 2 men arrive

$P(Y(T-t_0)=0,\ X(T-t_0)=2)$ Let $S=T-t_0$

then $P(Y(S)=0, X(S)=2)$ is what we're

looking for. Due to the independence of

Y, X - BUT WHAT IS S? It's a RV,

$P(Y(S)=0 \mid X(S)=2) = P(Y(S)=0, X(S)=2) / P(X(S)=2)$

b $P(X(S)\geq 2,\ Y(S)\leq 2)$

What is S? C

6 ways

MMFF

MFFM

MFMF

FFMM

FMFM FMMF

$Z = X+Y$ if $X=2$

LoL is it this easy?

x x

~~$P(\text{male, male}) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$~~ $\frac{6}{9}$

MMFF | F MFMF | F

MFMF | F

$6\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{24/3}{81/3} = \frac{8}{9\cdot 3} = \frac{8}{27}$

$8 + 1 = 8$ 9

5.6 P3

$$X(t) = \sum_{i=1}^{N(t)} Y_i \qquad Y_i \sim \text{Geom}(p)$$

Then following the book - if T is random failure time, we have

$\{T > t\}$ iff $\{X(t) < a\}$

Using 5.34 with $X(t)$ and a we have

$$P(X(t) < a) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} G^{(n)}(a)$$

$$= P(T > t) \Rightarrow$$

$$E[T] = \int_{t=0}^{t=\infty} P(T > t)\, dt = \int_{t=0}^{t=\infty} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} G^{(n)}(a)\, dt$$

$$= \sum_{n=0}^{\infty} \int_{t=0}^{t=\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t}\, dt \; G(a)\, dt$$

The integral can be evaluated noting we have Gamma$(n+1, \lambda)$ kernel.

$$E[T] = \frac{1}{\lambda} \sum_{n=0}^{\infty} G^{(n)}(a)$$

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5.6 P3

I Looked up the n-fold convolution of Geom(p) ; Y_i iid Geom(p) $\Rightarrow$ $\sum_{i=1}^{n} Y_i \sim$ Negative Binomial (n, p)

If I recall - there's a nice argument why this is the case. We should also be able to verify directly with convolution formula. - revisit both of these.

We still need to calculate $G^{(n)}(a)$ $\forall n$

$G^{(n)}(a)$ = cdf $NB(n,p)$ which I looked up. It's a regularized incomplete β-fn. This is looking more complex than might be necessary.

5.6 P3

Instead of using cdf we note

$$G^n(a) = P(N.B.(n,p) \le a)$$

$$= \sum_{i=1}^{a} \xi_i \quad \text{where} \quad \xi_i = \binom{i+n-1}{i}(1-p)^i p^n$$

are the N.B(n,p) probabilities.

Now $$E[T] = \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{i=0}^{a} \binom{i+n-1}{i}(1-p)^i p^n$$

Hopefully exchanging sums will simplify

$$E[T] = \frac{1}{\lambda} \sum_{i=0}^{a} \sum_{n=0}^{\infty} \frac{(i+n-1)!}{i!\,(n-1)!}(1-p)^i p^n$$

$$= \frac{1}{\lambda} \sum_{i=0}^{a} \frac{(1-p)^i}{i!} \sum_{n=0}^{\infty} \frac{(i+n-1)!}{(n-1)!} p^n$$

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Note: this is in Ross, but I tried not to revisit when solving

Extra HW

$X(t)$ P.P. (λ) each arrival type 1 $\bar{c}$ prob p

type 2 $\bar{c}$ prob $1-p$ $\perp$ ✓ arrivals.

$X_1(t) = \#$ type $1 \in [0, t)$ $X_2(t) = \#$ type $2 \in [0, t)$

Prove $X_1(t)$ P.P. $(p\lambda)$ $X_2(t)$ P P $((1-p)\lambda)$

and $X_1 \perp X_2$

What do we need to show? ① $X_1(0) = X_2(0) = 0$

② $\perp$ increments poisson distributed.

① is easy $X_1(t) + X_2(t) = X(t)$ + $X(0) = 0$

$X_1(t)$ $X_2(t) \geq 0 \Rightarrow X_1(0) = X_2(0) = 0.$

For ② we will use conditioning and L.T.P.

in the form of Thm 3.2

$X | N \sim \text{Binom}(N, p)$ $N \sim \text{Pois}(\lambda) \Rightarrow$

$X \sim \text{Pois}(p\lambda)$

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Extra H.W. 2

First establish $X_1 | N \sim \text{Binom}(N, p)$

$P(X_1(t) = m | X(t) = n)$ is the probability of m Bernoulli succeses in n trials.

For now on $N = X$ - so

$X_1 | X \sim \text{Binom}(X, p)$ and since $X(t) \sim \text{Pois}(\lambda t)$ we have our result $X_1(t) \sim \text{Pois}(p\lambda t)$. Applying similar argument to X_2 $X_2(t) \sim \text{Pois}((1-p)\lambda t)$

Now we need to establish $X_1(t) \perp X_2(t)$ this <u>should</u> follow trivially by the independence of our labeling as type 1, type 2.

I tried to show $P(X_1(t) = m, X_2(t) = n) = P(X_1(t) = m) P(X_2(t) = n)$

(revisit) $P(X_1(t) = m, X_2(t) = n) = P(X_1(t) = m | X_2(t) = n) P(X_2 = n)$

$P(X_1(t) = m | X_2(t) = n) = P(X_1(t) = m | X(t) = n+m) \sim \text{Binom}$ x^+ r