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OSU ST 6540 HW

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§ 6.1.E2

$PBP(\{\lambda_k\})$ $\quad k = 0, 1, 2, 3$

$W_3 = \sum_{k=1}^{3} S_k$ $\quad \lambda_0 = 1 \quad \lambda_1 = 3 \quad \lambda_2 = 2 \quad \lambda_3 = 5 \quad \lambda_i \neq \lambda_j \quad i \neq j$

so $E[W_3] = E[\sum_{k=0}^{2} S_k] = \sum_{k=0}^{2} E[S_k]$

$S_0 \sim exp(1) \quad S_1 \sim exp(3) \quad S_2 \sim exp(2)$

$E[W_3] = \frac{1}{1} + \frac{1}{3} + \frac{1}{2} = 11/6$

$E[W_1 + W_2 + W_3] = E[W_1] + E[W_2] + E[W_3]$

$E[W_1] = E[\sum_{k=0}^{0} S_k] = E[S_0] = 1$

$E[W_2] = E[\sum_{k=0}^{1} S_k] = E[S_0] + E[S_1] = 1 + \frac{1}{3} = 4/3$

$\Rightarrow E[W_1 + W_2 + W_3] = 1 + 4/3 + 11/6 = 25/6$

$Var(W_3) = Var(S_0) + Var(S_1) + Var(S_2)$

$= 1/1^2 + 1/3^2 + 1/2^3 = 1 + 1/9 + 1/4 = \frac{36+4+9}{36} = \frac{49}{36}$

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§6.1.P2

Yule process $\lambda_k = \alpha + k\beta$ $\quad X(0)=0$

$P_n(t)$ can be calculated from the equations for a P.B.B. $(\{\lambda_k\})$

$$P_0(t) = e^{-\alpha t} \qquad P_1(t) = \alpha\left(\frac{1}{\beta} e^{-\alpha t} + \frac{1}{\beta} e^{-(\alpha+\beta)t}\right)$$

$$P_n(t) = \prod_{k=0}^{n-1} (\alpha + k\beta)\left[B_{0,n} e^{-\alpha t} + \cdots B_{n,n} e^{-(\alpha+n\beta)t}\right]$$

$$B_{0,n} = \frac{1}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)\cdots(\lambda_n-\lambda_0)} = \frac{1}{(\beta)\cdot(2\beta)\cdots n\beta} = \frac{1}{n!\,\beta^n}$$

$$B_{k,n} = \frac{1}{(\lambda_0-\lambda_k)(\lambda_1-\lambda_k)\cdots(\lambda_{k-1}-\lambda_k)(\lambda_{k+1}-\lambda_k)\cdots(\lambda_n-\lambda_k)}$$

↑(no k-term obv.)

$$= \frac{1}{(-k\beta)((1-k)\beta)((2-k)\beta)\cdots((-1)\beta)(1\cdot\beta)(2\cdot\beta)\cdots((n-k)\beta)}$$

$$= \frac{1}{\beta^n \gamma_k} \qquad \text{where}$$

$$\gamma_k = (n-k)! \prod_{i=0}^{k-1} (i-k)$$

$$B_{n,n} = \frac{1}{(\lambda_0-\lambda_n)(\lambda_1-\lambda_n)\cdots(\lambda_{n-1}-\lambda_n)} = \frac{1}{(n\beta)\cdot((n-1)\beta)((n-2)\beta)\cdots\beta} = \frac{1}{n!\,\beta^n}$$

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§6.1.P2 ②

Let's try to simplify

$P_0(t) = e^{-\alpha t}$ $\quad P_1(t) = \frac{\alpha}{\beta} e^{-\alpha t}(1 + e^{-\beta t})$

$$P_n(t) = \prod_{k=0}^{n-1}(\alpha + k\beta)\left[\frac{1}{n!\,\beta^n} e^{-\alpha t} + \dots + \frac{e^{-\alpha t - k\beta t}}{\beta^n \gamma_k} + \dots \frac{e^{-\alpha t - n\beta t}}{n!\,\beta^n}\right]$$

we can pull out a $e^{-\alpha t}$ and look at γ_k

hoping it $n!$ No but

$$\gamma_k = \frac{1}{(-1)^k\; 1\cdot 2\cdot \cdot (k-2)(k-1)(k)\,(n-k)!} = \frac{1}{(-1)^k\, k!\,(n-k)!}$$

$$P_n(t) = \frac{\prod_{k=0}^{n-1} \alpha + k\beta}{\beta^n} e^{-\alpha t}\left[\frac{1}{n!} + \frac{e^{-2\beta}}{(n-2)!\,2!} + \frac{e^{-3\beta}}{(n-3)!\,3!(-1)^3} + \right.$$

$$\left. \dots \frac{e^{-(n-1)\beta t}}{(-1)^{n-1}(n-1)!\,1!} + \frac{e^{-n\beta t}}{(-1)^n\, n!}\right]$$

No idea how to get a set of $P_n(t)$

from this that I recognize as a

common distribution (x x / ~)

§ 6.1.P5

$\{W_1 > t\}$ same event as $\{X(t)=0\}$

$\{W_2 > t+h\}$ same event as $\{\{X(t+h)=0\} \cup \{X(t+h)=1\}\}$ Since $\{X(t+h)=0\}$ $\{X(t+h)=1\}$ are mutually exclusive we can write

$$P(W_1 > t,\ W_2 > t+h) = P(X(t)=0,\ X(t+h)=0 \cup X(t+h)=1)$$

$$= P(X(t)=0,\ X(t+h)=0) + P(X(t)=0,\ X(t+h)=1)$$

$$= P(X(t+h)=0 \mid X(t)=0)\, P(X(t)=0) +$$

$$P(X(t+h)=1 \mid X(t)=0)\, P(X(t)=0)$$

$$= P(X(t)=0)\left[P(X(h)=0 \mid X(0)=0) + P(X(h)=1 \mid X(0)=0)\right]$$

by Markov property so we get that

$$P(W_1 > t,\ W_2 > t+h) = P_0(t)\left[P_0(h) + P_1(h)\right]$$

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§6.1 P5

2

Our expression for $W_1 W_2$ is a tail probability

Let $t+h=\eta$, $P(W_1 > t, W_2 > \eta)$

$$= \int_{x_1=t}^{\infty} \int_{x_2=\eta}^{\infty} f_{W_1 W_2}(x_1, x_2)\, dx_1\, dx_2$$

From here we can use our expression and take derivatives.

$$P(W_1 > t, W_2 > \eta) = P_0(t)\left[P_0(\eta - t) + P_1(\eta - t)\right]$$

$$= \int_{x_1=t}^{\infty} \int_{x_2=\eta}^{\infty} f_{W_1, W_2}(x_1, x_2)\, dx_1\, dx_2$$

$$\frac{\partial^2}{\partial t \partial \eta} P_0(t)\left[P_0(\eta - t) + P_1(\eta - t)\right] = f_{W_1, W_2}(t, \eta)$$

Now we have tedious algebra

$$f_{W_1, W_2}(t, \eta) = \frac{\partial}{\partial \eta} \frac{\partial}{\partial t}\left[e^{-\lambda_0 t}\left[e^{-\lambda_0(\eta - t)} + \lambda_0 \left(\frac{1}{\lambda_1 - \lambda_0} e^{-\lambda_0(\eta - t)} + \frac{1}{\lambda_0 - \lambda_1} e^{-\lambda_1(\eta - t)} \right)\right]\right]$$

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§6.1, P5

③

LHS simplifies

$$e^{-\lambda_0 \eta} + \frac{\lambda_0}{\lambda_1-\lambda_0} e^{-\lambda_0 \eta} + \frac{\lambda_0}{\lambda_0-\lambda_1} e^{-\lambda_0 t+\lambda_1 t} e^{-\lambda_1 \eta}$$

Taking $\frac{\partial}{\partial t}$ First

$$f_{W_1,W_2}(t,\eta) = \frac{\partial}{\partial \eta} \frac{\lambda_0 e^{-\lambda_1 \eta}}{\lambda_0-\lambda_1} \cdot (\lambda_0-\lambda_1) e^{-t(\lambda_0-\lambda_1)}$$

$$= \lambda_0 \lambda_1 e^{-\lambda_1 \eta} e^{-t(\lambda_0-\lambda_1)}$$

We already know $f_{S_0,S_1}(s_0,s_1) = \lambda_0 e^{-\lambda_0 s_0} \lambda_1 e^{-\lambda_1 s_1}$

Since $S_0 \sim \exp(\lambda_0)$ $S_1 \sim \exp(\lambda_1)$ $S_0 \perp S_1$

But it looks like we have another tedious calculation to do using

f_{W_1,W_2} or note $\{W_1 > t\} \cap \{W_2 > t+s\}$

Same event as $\{W_1 > t\} \cap \{W_2 - W_1 > s\}$

Which is same as $\{S_0 > t\} \cap \{S_1 > s\}$

So we can say

$$f_{S_0 S_1}(t, s) = f_{W_1 W_2}(t, t-s) = \lambda_0 \lambda_1 e^{-\lambda_1 t-s} e^{-t(\lambda_0-\lambda_1)}$$

$$= \lambda_0 e^{-\lambda_0 t} \lambda_1 e^{-\lambda_1 s}$$

GO

Hard problem!

I wanted to see how the $P_1(t)$ came out

§ 6.1.7

$S_0 \sim \exp(\lambda_0)$ $S_1 \sim \exp(\lambda_1)$ $S_2 \sim \exp(\lambda_2)$ $\perp$

$$P(S_0 > t) = 1 - P(S_0 \le t) = 1 - (1 - e^{-\lambda_0 t}) = e^{-\lambda_0 t}$$

For $P(S_0 + S_1 > t)$ we first calculate

$p_{S_0+S_1}(t)$ using the convolution formula

$$p_{S_0+S_1}(t) = \int_{-\infty}^{\infty} p_{S_1}(t-\eta)\, p_{S_0}(\eta)\, d\eta$$

supp $p_{S_1}(t-\eta) = \{t > \eta\}$ supp $p_{S_0}(\eta) = [0,\infty)$

$$p_{S_0+S_1}(t) = \int_0^t \lambda_1 e^{-\lambda_1(t-\eta)} \lambda_0 e^{-\lambda_0 \eta}\, d\eta$$

$$= \lambda_1\lambda_0 e^{-\lambda_1 t} \int_0^t e^{-\eta(\lambda_0-\lambda_1)}\, d\eta = \frac{\lambda_1\lambda_0 e^{-\lambda_1 t}}{(\lambda_1-\lambda_0)} + e^{-\eta(\lambda_0-\lambda_1)}\Big|_0^t$$

$$= \frac{\lambda_1\lambda_0}{\lambda_1-\lambda_0}\left(e^{-t\lambda_0} - e^{-\lambda_1 t}\right)$$

$$P(S_0+S_1 < t) = \int_0^t \frac{\lambda_1\lambda_0}{\lambda_1-\lambda_0}\left(e^{-\eta\lambda_0} - e^{-\lambda_1\eta}\right) d\eta$$

$$= \frac{\lambda_1\lambda_0}{\lambda_1-\lambda_0}\left[-\frac{1}{\lambda_0} e^{-\eta\lambda_0}\Big|_0^t + \frac{1}{\lambda_1} e^{-\eta\lambda_1}\Big|_0^t \right]$$

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Jus

§ 6.1.7

(2)

$$= \frac{\lambda_1}{\lambda_0 - \lambda_1}\left[e^{-t\lambda_0} - 1\right] + \frac{\lambda_0}{\lambda_1 - \lambda_0}\left[e^{-t\lambda_1} - 1\right]$$

$$= \frac{\lambda_1}{\lambda_0 - \lambda_1} e^{-t\lambda_0} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-t\lambda_1} + \frac{\lambda_1}{\lambda_1 - \lambda_0} - \frac{\lambda_0}{\lambda_1 - \lambda_0}$$

$$= \frac{\lambda_1}{\lambda_0 - \lambda_1} e^{-t\lambda_0} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-t\lambda_1} + 1$$

finally

$$P(S_0 + S_1 > t) = \frac{\lambda_1}{\lambda_0 - \lambda_1} e^{-\lambda_0 t} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$

Now for $P(S_1 + S_2 + S_3 > t)$ we'll use same proceedure using $(S_1 + S_2)$ pdf

$$P_{S_0+S_1+S_2} = \int_0^t \lambda_2 e^{-\lambda_2(t-\eta)}\left[\frac{\lambda_1\lambda_0}{\lambda_1 - \lambda_0}\left(e^{-\lambda_0\eta} - e^{-\lambda_1\eta}\right)\right] d\eta$$

$$= \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1 - \lambda_0)} e^{-\lambda_2 t} \int_0^t e^{-\eta(\lambda_0 - \lambda_2)} - e^{-\eta(\lambda_1 - \lambda_2)}\, d\eta$$

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$$= \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)}\left[\frac{1}{(\lambda_2-\lambda_0)} e^{-\eta(\lambda_0-\lambda_2)}\Big|_0^t - \frac{1}{(\lambda_2-\lambda_1)} e^{-\eta(\lambda_1-\lambda_2)}\Big|_0^t \right]$$

$$= \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}\left[e^{-t(\lambda_0-\lambda_2)} - 1\right] - \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_1)}\left[e^{-t(\lambda_1-\lambda_2)} - 1\right]$$

$$= \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)} e^{-t(\lambda_0-\lambda_2)} - \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_1)} e^{-t(\lambda_1-\lambda_2)} + $$

$$\left[\frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_1)} - \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)} \right] = P_{S_0+S_1+S_2}(t)$$

Now $\cancel{P_{S_0+S_1+S_2}}$ $P(S_1+S_2+S_3 \le t) =$

$$= \Big\{ \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)} \int_0^t \left[e^{-\eta(\lambda_0-\lambda_2)} - 1\right] d\eta - \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_1)} \int_0^t \left[e^{-\eta(\lambda_1-\lambda_2)} - 1\right] d\eta$$

$$= \frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)(\lambda_2-\lambda_0)} \left[e^{-\eta(\lambda_0-\lambda_2)} - 1\right]\Big|_0^t -$$

$$\frac{\lambda_2\lambda_1\lambda_0}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_1)(\lambda_2-\lambda_1)} \left[e^{-\eta(\lambda_1-\lambda_2)} - 1\right]\Big|^t$$

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$P(S_1 + S_2 + S_3 \le t) =$

$$\frac{\lambda_2 \lambda_1 \lambda_0}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)^2} \left[e^{-t(\lambda_0 - \lambda_2)} - t \right] - \frac{\lambda_2 \lambda_1 \lambda_0}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_1)^2} \left[e^{-t(\lambda_1 - \lambda_2)} - t \right.$$

Something went wrong

will revisit

§ 6.1.P13

$\lambda_0 = \lambda_1 = \cdots = \lambda \quad \forall i$

$P_0(t) = e^{-\lambda t}$ From 6.5 we have

$$P_1(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda x} e^{-\lambda x}\, dx = (\lambda t) e^{-\lambda t}$$

$$P_2(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda x} (\lambda x) e^{-\lambda x}\, dx$$

$$= \lambda e^{-\lambda t} \lambda \int_0^t x\, dx = \frac{(\lambda t)^2}{2} e^{-\lambda t}$$

Continuing this process we see that

$P_{n-1}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}$ and that

$$P_n(t) = \lambda e^{-\lambda t} \int_0^t e^{\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x}\, dx$$

$$= \lambda e^{-\lambda t} \frac{\lambda^{n-1}}{(n-1)!} \int_0^t x^{n-1}\, dx$$

$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

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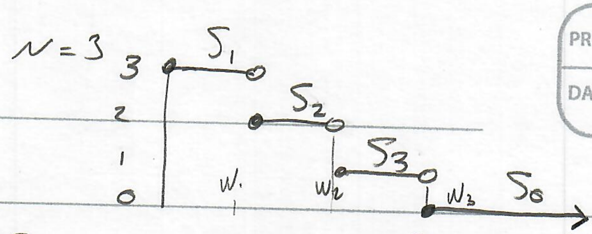

§6.2 E2

P.O.P. $\{ \mu_0 = 0, \ \mu_1 = 3, \ \mu_2 = 2, \ \mu_3 = 5 \}$ $\quad X(0) = 3$

$W_3 = \sum_{i=1}^{3} S_i \qquad S_i \sim exp(\mu_i) \qquad i = 1, 2, 3$

so $E[W_3] = E[S_1] + E[S_2] + E[S_3]$

$= \frac{1}{3} + \frac{1}{2} + \frac{1}{5} = \frac{5}{6} + \frac{1}{5} = \frac{31}{30}$

$E[W_1 + W_2 + W_3] = E[W_1] + E[W_2] + E[W_3]$

$E[W_1] = E[S_1] = 1/3$

$E[W_2] = E[\sum_{i=1}^{2} S_i] = 1/3 + 1/2 = 5/6$

What's the point of calculating this?

→ $E[W_1 + W_2 + W_3] = \frac{1}{3} + 5/6 + 31/30 = 66/30$

$Var(W_3) = E[W_3^2] - E[W_3]^2$ or better

$Var(W_3) = Var(S_1) + Var(S_2) + Var(S_3)$

$= 1/9 + 1/4 + 1/25 = \frac{1}{9} + \frac{29}{100} = \frac{261}{900}$

§ 6.2. E2

$X(t)$ L.D.P. $\{ \mu_k = k\alpha \}$ $X(0) = N = 5$

$\alpha = 2$

$$P_2(t) = P(X(t) = 2) = \frac{5!}{2!\,3!} e^{-2\alpha t} (1 - e^{-\alpha t})^3$$

P_2 for Binom$(5, e^{-\alpha t})$

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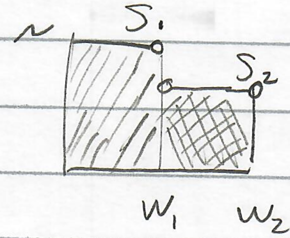
§6.2.P3



Ha! This is why one would calculate $E[W_1 + W_2 + \ ]$ (see my question about 6.2.E2)

The are of a sample path is given by ...

$N S_N + (N-1) S_{N-1} + \cdots 1 \cdot S_1$ and since

$W_N = S_N \quad W_{N-1} = S_N + S_{N-1} \cdots \quad W_1 = S_N + S_{N-1} \cdots S_1$

we see that mean area can be calculated

as $E[W_1 + W_2 + \cdots W_N] =$

$E[(S_N + S_{N-1} + \cdots S_1) + (S_N + S_{N-1} + \cdots S_2) + \cdots + S_N]$

$= E[N S_N + (N-1) S_{N-1} + \cdots + S_1]$ so

$E[W_1 + W_2 + \cdots W_N] = N\,E[S_N] + (N-1)\,E[S_{N-1}]$

$+ \cdots 1 \cdot E[S_1] = N \frac{1}{\mu_N} + (N-1) \frac{1}{\mu_{N-1}} + \cdots 1 \cdot \frac{1}{\mu_1}$

since $S_i \sim \exp(\mu_i)$

6.2. P2

$X(t)$ P.D.P ($\mu_k = \theta \quad \forall \quad k = 1, 2, \ldots N$)

$X(0) = N$

$P_N(t) = e^{-\theta t}$, but since $\mu_i = \mu_j$ we can't use 6.13. From problem 6.1.33 and looking at eq's 6.12 defining a P.D.P. we think $X(t) = N - Y(t)$ & $Y \sim Pois(\theta t)$

$P_n(t) = N - \tilde{P}_n(t) \qquad \tilde{P}_n(t) = \frac{(\theta t)^n}{n!} e^{-\theta t}$

Putting this into i) defining PDP ($\mu_k = \theta$);

$P(X(t+h) = k-1 \mid X(t) = k) =$

$P(N - Y(t+h) = k-1 \mid N - Y(t) = k) =$

$P(Y(t+h) = N-k+1 \mid Y(t) = N-k) =$

$P(Y(t+h) - Y(t) = 1) = \theta h + o(h)$

from the definition of a P.P. in Ch 5

There's more to this! What about conditions ii) iii) for PDP. And what is S for $N - Y(t)$? We have to restrict $Y(t) \leq N$. Where to put extra

$P_{N-1}(t) = P(X(t) = N-1) = P(Y(t)=1)$

$= \theta t \, e^{-\theta t}$

$P_{N-2}(t) = P(X(t) = N-2) = P(Y(t)=2) = \frac{(\theta t)^2}{2} e^{-\theta t}$

$\vdots$

$P_2(t) = P(X(t)=2) = P(Y(t) = N-2) = \frac{(\theta t)^{N-2}}{(N-2)!} e^{-\theta t}$

$P_1(t) = P(X(t)=1) = P(Y(t) = N-1) = \frac{(\theta t)^{N-1}}{(N-1)!} e^{-\theta t}$

and now we see where the extra mass goes - all of the time we spend in state 0! To get a normalized $P_i(t)$ (for every t the process has to be somewhere) we require $\sum P_i(t) = 1$ so we must have

$$P_0(t) = 1 - \sum_{i=1}^{N} P_i(t)$$