__/__/__

$X: \Omega_X = \mathbb{N}$

$Y: \Omega_Y = [0,\infty)$

we need to invoke limiting argument on $P(X|Y)$

§ 5.1 Eg

$X \sim Pois(\lambda) \qquad \lambda \sim Exp(\theta) \qquad \theta > 0$

$$P(X=k) = \int_{\alpha=0}^{\alpha=\infty} P(X=k \mid \lambda=\alpha)\, P(\lambda=\alpha)\, d\alpha$$

via the law of total probability.

$$P(X=k) = \int_{\alpha=0}^{\alpha=\infty} \frac{(\alpha)^k e^{-\alpha}}{k!} \theta e^{-\theta\alpha}\, d\alpha$$

We recognize what may be a gamma kernel

$$\Gamma(a,b)(\alpha) = \frac{b}{\Gamma(a)} (b\alpha)^{a-1} e^{-b\alpha}$$

$$b=\theta \quad a=k+1 \Rightarrow \Gamma(k+1,\theta)(\alpha) = \frac{\theta}{k!} \theta^k \alpha^k e^{-\theta\alpha}$$

$\int_{\alpha=0}^{\alpha=\infty} \frac{\theta}{k!} (\alpha)^k e^{-\alpha(1+\theta)}\, d\alpha$. We recognize what is a Γ kernel. Lets see if we can evaluate integral using that. Let $Y \sim \Gamma(k+1, \theta+1)$, then

$f_Y(\alpha) = \frac{(1+\theta)}{k!} (\alpha(1+\theta))^k e^{-(1+\theta)\alpha}$ using

$\int_{\alpha=0}^{\alpha=\infty} f_Y(\alpha)\, d\alpha = 1$ we see $P(X=k) = \frac{\theta}{(1+\theta)^{k+1}}$ ☺ ok!

§ 5.1 E6

$X(t)$ P.P. $\lambda = 3$

a) $P(X(4) - X(0) = 0)$ is what were looking for. $X(4) - X(0) \sim Pois(4\lambda)$ $\therefore$

$P(X(4) - X(0) = 0) = e^{-4\lambda}$

b) The distribution of the first afternoon message is the waitting time to first event

$P(X(k) = 1) = \lambda e^{-\lambda} \sim exp(\lambda)$

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§ 5.1 P1 $X:\ \Omega = \mathbb{N}$

$\xi\ \ \Omega = [0,\infty)$

ξ_i ⊥ r.v. $\xi_i \sim \mathcal{E}xp(\lambda)$ $\sum_{j=1}^{k} \xi_k \sim \Gamma(k,\lambda)$

Let
$$X = \begin{cases} 0 & 1 < \xi_1 \\ 1 & \xi_1 < 1 < \xi_1 + \xi_2 \\ 2 & \xi_1 + \xi_2 \le 1 \le \xi_1 + \xi_2 + \xi_3 \\ \vdots & \\ k & \sum_{j=1}^{k-1} \xi_j \le 1 \le \sum_{j=1}^{k} \xi_j \end{cases}$$

Now lets look at $P(X=k)$

$$P(X=0) = P(\xi_1 > 1) = 1 - P(\xi_1 \le 1) = 1 - F_\xi(1)$$

recall $F_\xi(x) = 1 - e^{-\lambda x}$

$P(X=0) = e^{-\lambda x}$ which agrees $\bar{c}$ pois$(\lambda)_{k=0}$

Turned the page & saw the rest of the hint!

Set

$$P(X=k) = P\left(\sum_{j=1}^{k} \xi_j < 1 < \sum_{j=1}^{k+1} \xi_j\right) \qquad \text{let } A = \sum_{j=1}^{k} \xi_j$$

$$= P\left(1 < \xi_{k+1} + \alpha \ \middle|\ \sum_{j=1}^{k} \xi_j < 1\right)$$

$$= \int P(X=k \mid A=\alpha)\, P(A=\alpha)\, d\xi_1$$ what limits make sense for integral? $[0,1]$ since if $A > 1$ $P(X=k) = 0$ and $\Omega_{\xi_i} = [0,\infty)$ gives lower limit

$$P(X=k) = \int_0^1 P(X=k \mid A=\alpha)\, P(A=\alpha)\, d\alpha$$

§5.1 P1 remember $\alpha = \sum_{j=1}^{k} \xi_j$ (P2)

Now $P(X = k \mid A = \alpha,\ \alpha \in [0,1]) =$

$P(\ \alpha < 1 < \alpha + \xi_{k+1} \mid \alpha \in [0,1])$

$P(\ \xi_{k+1} > 1-\alpha\ ) = 1 - F_\xi[(1-\alpha)]$

Using $\alpha = \sum_{j=1}^{k} \xi_j \sim \text{Gamma}(k, \lambda)$ we have

$$P(X=k) = \int_0^1 P(X = k \mid A = \alpha)\, P(A=\alpha)\, d\alpha$$

$$= \int_0^1 P(X = k \mid A = \alpha, \alpha \in [0,1])\, P(A = \alpha)\, d\alpha$$

$$= \int_0^1 \left(1 - F_\xi[(1-\alpha)]\right) \Gamma(k, \lambda)(\alpha)\, d\alpha$$

Where $\Gamma(k,\lambda)(\alpha) = \frac{(\lambda x)^{\alpha-1} \lambda e^{-\lambda x}}{\Gamma(\alpha)}$ is pdf of

Gamma(k, λ), and

$F_\xi[x] = 1 - e^{-\lambda x}$ is CDF of $\mathcal{E}xp(\lambda)$ r.v.

___/___/___

§5.1 P5

$X(h) \sim Pois(\lambda h)$ $h > 0$

$P_k(h) = P(X(h) = k) = \frac{(\lambda h)^k}{k!} e^{-\lambda h}$

a) $P_0(h) = e^{-\lambda h}$ expandind the Taylor series of $e^{-\lambda h}$ about 0 $(\lim_{h\to 0})$ we have

$$e^{-\lambda h} = \sum_{n=0}^{\infty} \frac{(-\lambda h)^n}{n!} = 1 - \lambda h + o(h)$$

$$\lim_{h\to 0} \frac{1 - P_0(h)}{h} = \lim_{h\to 0} \frac{1 - [1 - \lambda h + o(h)]}{h} = \lim_{h\to 0} \lambda + \frac{o(h)}{h} = \lambda$$

b) $P_1(h) = \lambda h\, e^{-\lambda h} = \lambda h[1 - \lambda h + o(h)] \Rightarrow$

$$\lim_{h\to 0} \frac{P_1(h)}{h} = \lim_{h\to 0} \frac{\lambda h[1 - \lambda h + o(h)]}{h}$$

$$= \lim_{h\to 0} \lambda - \lambda^2 h + o(h) = \lambda$$

c) $P_2(h) = \frac{(\lambda h)^2}{2} e^{-\lambda h}$

$$\lim_{h\to 0} \frac{P_2(h)}{h} = \frac{(\lambda h)^2 [1 - \lambda h + o(h)]}{h} =$$

$$\lim_{h\to 0} \lambda^2 h - \lambda^3 h^2 + (\lambda h)^2 o(h) = 0$$

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5.2 E2

Let $p = \frac{240}{600}$ | prob of error

Consider P.P. ($N \cdot p$) approx to

$S_n = \varepsilon_1 + \cdots \varepsilon_n$ $\varepsilon_i \sim$ Bernoulli (p)

$Y \sim$ Binom $(3, p)$ $P(Y=0)$ is probability of 3 succesive error free pages.

$P(Y=0) \approx P($ X (P.P. $3 \cdot p$) has no arrivals in first period) by the Poisson approximation. let X be PP $3 \cdot p$

$P(X(1) = 0) = e^{-3p} = e^{-12}$

To me 240/600 is not small. Also how are erros distributed! What if they are all on the same page.

$p!$

§ 5.2 $\quad X(n,p) \sim \text{Binom}(n,p)$

$$\begin{matrix} n \to \infty \\ p \to 0 \end{matrix} \Rightarrow np = \lambda$$

a) Show $\lim\limits_{n\to\infty} P(X(n,p)=0) = e^{-\lambda}$

$$P(X(n,p)=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$\Rightarrow$$

$$\frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$\frac{n!}{(n-k)!} \xrightarrow[n\to\infty]{} 1$$

$$p = \frac{\lambda}{n} \Rightarrow (1-p)^{n-k} = \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \xrightarrow[n\to\infty]{} e^{-\lambda}$$

$$\lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \frac{\lambda^k}{n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\,n^k} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k} \leftarrow$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

$$\lim P(X(n,p)=0) = e^{-\lambda}$$

§ 5.2 #1 $\qquad X(n,p) \sim \text{Binom}(n,p)$

$$P(X(n,p) = k+1) = \frac{n!}{(k+1)!\,(n-(k+1))!}\, p^{(k+1)} (1-p)^{n-(k+1)}$$

$$P(X(n,p) = k) = \frac{n!}{k!\,(n-k)!}\, p^{k} (1-p)^{n-k}$$

$$\frac{P(X(n,p) = k+1)}{P(X(n,p) = k)} = \frac{k!\,(n-k)!}{(k+1)!\,(n-(k+1))!}\; \frac{p^{k+1}}{p^{k}}\; \frac{(1-p)^{n-(k+1)}}{(1-p)^{n-k}}$$

$$= \frac{(n-k)}{(k+1)}\, \frac{p}{1-p} = \frac{np - kp}{(k+1)(1-p)}$$

$$\lim_{n\to\infty} \frac{P(X(n,p) = k+1)}{P(X(n,p) = k)} = \lim_{n\to\infty} \frac{np}{(k+1)(1-p)} - \frac{kp}{(k+1)(1-p)}$$

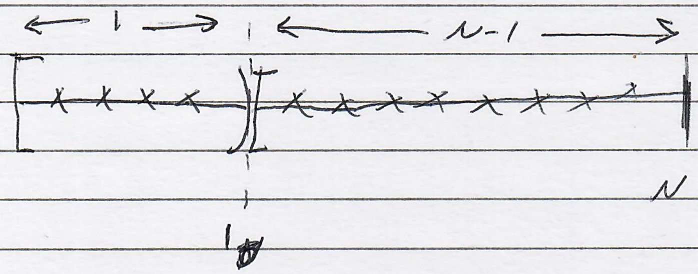
$\frac{\lambda}{n} = p$

$$= \lim_{n\to\infty} \left[\frac{\lambda}{(k+1)(1-\frac{\lambda}{n})} + \frac{k\,\frac{\lambda}{n}}{((k+1)(1-\frac{\lambda}{n})} \right]$$

$$= \frac{\lambda}{k+1}$$

second term $\to 0$

§ 5.2 p4



Let $X = $ # points in $[0,1)$

$P(X = k) = ?$ There are 2 bins

$[0,1)$ $[1,n)$ If a point falls in $[0,1)$ call it a success.

$P(\text{Point falls in } [0,1)) = 1/N$

$P(\text{Point falls in } [1,N]) = \frac{N-1}{N} = 1 - \frac{1}{N}$

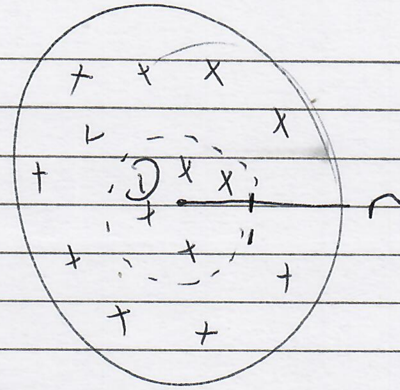
Let $p = 1/N$ since we have to account for the ways in which $X = k$, we see $X \sim \text{Binom}(N, p)$ and now we can take a limit

$$P(X = k) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\lim_{N\to\infty} P(X=k) = \lim_{N\to\infty} \frac{N!}{k!(N-k)!}\left(\frac{1}{N}\right)^k \left(1 - \frac{1}{N}\right)^{N-k}$$

$$= \lim_{N\to\infty} \frac{N!\, N^k}{k!(N-k)!} \frac{\left(1-\frac{1}{N}\right)^N}{\left(1-\frac{1}{N}\right)^k} = \frac{e^{-1}}{k!} \sim \cancel{\exp(1)}\ \text{Pois}(1)$$

5.2 P5



Let x = # points in D

Same setup as pblm 5.2 let $D = B(1)$ disk unit radius

$A = B(r) \setminus B(1)$ annulus

$|D| = \pi \qquad |A| = \pi(r^2-1)$

$P(\text{point falls in } D) = \frac{|D|}{\cancel{|A|}\ \pi r^2} \qquad P(\text{point falls in } A) = \frac{|A|}{\pi r^2}$

$= \frac{\pi(r^2-1)}{\pi(r^2)} = \left(1-\frac{1}{r^2}\right)$

Taking into account the ways in which points can be arranged in A, D,

$$P(X=k) = \binom{N}{k}\left(\frac{1}{r^2}\right)^k\left(1-\frac{1}{r^2}\right)^{N-k}$$

$$\lim_{\substack{N\to\infty \\ r\to\infty}} \quad \frac{N}{\pi r^2} = \lambda \quad \left\{ \frac{N!}{k!(N-k)!}\left(\frac{1}{r^2}\right)^k\left(1-\frac{1}{r^2}\right)^{N-k} \right.$$

$$\frac{1}{r^2} = \frac{\lambda\pi}{N}$$

$$\lim_{\substack{N\to\infty \\ r\to\infty}} \quad \frac{N}{\pi r^2} = \lambda \quad \frac{N!}{k!(N-k)!}\left(\frac{\lambda\pi}{N}\right)^k\left(1-\frac{\lambda\pi}{N}\right)^{N-k}$$

$$\frac{1}{k!}(\lambda\pi)^k \lim_{\substack{N\to\infty \\ r\to\infty}} \quad \frac{N}{\pi r^2} = \lambda \quad \frac{\left(1-\frac{\lambda\pi}{N}\right)^N}{\left(1-\frac{\lambda\pi}{N}\right)^k} = \frac{(\lambda\pi)^k e^{-\lambda\pi}}{k!} \sim \text{Pois}(\lambda\pi)$$

5.3. Σ9

W_n - waiting time to nth event

$W_n = t \iff X(t) = N[0,t) = n$

$W_r \le t$ means r events happened before t

This $\Rightarrow$ $X(t) \ge r$. Similarly if we've observed more than r events in $[0,t)$ we know the waiting ~~time~~ time is less than t. We can determine cdf of $\Gamma(n,\lambda)$ as sum of poisson

$$P(W_n \le t) = P(X(t) \ge n) = 1 - P(X(t) < n)$$

$$= 1 - \sum_{i=0}^{n-1} \frac{\lambda^i}{i!} e^{-\lambda}$$

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5.3 P1

$X(t)$ P.P. (λ) Show $\{W_1 > w_1, W_2 > w_2\}$

iff $\{X(w_1) = 0, \; X(w_2) - X(w_1) \in \{0, 1\}\}$

$W_1 > w_1$ means no arrivals by w_1 ie $X(w_1) = 0$

$W_2 > w_2$ means we have not observed 2

arrivals by time w_2 $\Leftrightarrow$ 0 arrivals

or 1 arrival between w_1 and w_2 $\boxtimes$

$$P(W_1 > w_1, \; W_2 > w_2) = P(X(w_1) = 0, \; X(w_2) - X(w_1) \in \{0, 1\})$$

Invoking $\perp$ increments for PP

$$P(W_1 > w_1 \; W_2 > w_2) = P(X(w_1) = 0) P(X(w_2) - X(w_1) \in \substack{0 \\ 1})$$

$X(w_1) \sim \text{Pois}(w_1 \lambda)$ $\quad X(w_2) - X(w_1) \sim \text{Pois}((w_2 - w_1)\lambda)$

$$P(X(w_1) = 0) = e^{-\lambda w_1}$$

$$P(X(w_2) - X(w_1) \in \{0, 1\}) = e^{-\lambda(w_2 - w_1)} +$$

$$\lambda(w_2 - w_1) e^{-\lambda(w_2 - w_1)}$$ union of $\perp$ events

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P2

5.3 P2

$$\frac{\partial^2}{\partial \omega_1 \partial \omega_2} P(X(\omega_1)=0)\, P(X(\omega_2)-X(\omega_1) \in \{0,1\})$$

$$\frac{\partial^2}{\partial \omega_1 \partial \omega_2} e^{-\lambda \omega_1} [1 + \lambda(\omega_2-\omega_1)]\, e^{-\lambda \omega_2} e^{\lambda \omega_1} =$$

$$\frac{\partial^2}{\partial \omega_1 \partial \omega_2} [1 + \lambda(\omega_2 - \omega_1)]\, e^{-\lambda \omega_2} =$$

$$\frac{\partial}{\partial \omega_2} -\lambda e^{-\lambda \omega_2} = \lambda^2 e^{-\lambda \omega_2}$$

5.3 p5

$X(t)$ P.P. (λ) $T \sim \mathcal{E}xp(\theta)$

Find pmf of $X(T)$

$$P(X(T)=k) = P(X(T)=k \mid T)$$

$$= \int_{t=0}^{\infty} P(X(t)=k \mid T=t)\, P(T=t)$$

use result from 5.1. E5

$$P(X(t)=k \mid T=t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$P(T=t) = \theta e^{-\theta t}$$

$$P(X(T)=k) = \int_{t=0}^{t=\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \theta e^{-\theta t}\, dt$$