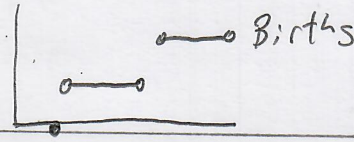
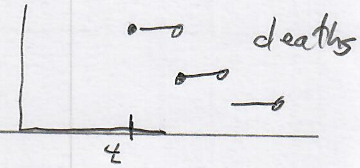
Brucke Campbell S_{Z_i} sojurns of Z
OSU ST6540 HW10 S_{Y_i} lifetimes





§6.3.E1

We'll borrow from §5.4 - using all the notation except the problem has $X(t)$ as # particles alive @ t. - denoted $M(t)$ in §5.4 Our lifetimes $Y_1, Y_2 \dots \perp \sim \exp(\mu)$ so the death rates $\mu_i = \mu \ \forall i$. Our birth rates are determined by the arrival P.P.(λ), and $\lambda_i = \lambda \ \forall i$.

Now we should show that the conditions of a BDP(μ_i, λ_i) are satisfied. Let $Z(\lambda)$ be the P.P.(λ) generating particles. We'll start by noting $P_{ij}(0) = P(X(0) = j \mid X(0) = i) = \delta_{ij}$ and $\mu_0 = 0$ because $Z(0) = 0$ and the fact that there can be no deaths until an arrival. We know the sojurn times of Z are exponential, and by definition the sojurns before a death are exponential as well.

We wold like to reason $X(t) = Z(t) - Y(t)$

Suppose $X(t) = N$ what can happen in an interval $t+h$? We can have a death $P_{i,i-1}(h) = \mu h + o(h)$ or a birth $P_{i,i+1}(h) = \lambda h + o(h)$ or no events

$P_{i,i}(h) = 1 - [P_{i,i+1}(h) + P_{i,i-1}(h)]$

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§ 6.3.E3 Let $\pi = \frac{\alpha}{\alpha+\beta}$ $\tau = \alpha+\beta$

$P_{00}(t) = (1-\pi) + \pi e^{-\tau t}$ $P_{01} = 1 - P_{00}(t) = \pi - \pi e^{-\tau t}$

$P_{10}(t) = (1-\pi) - (1-\pi) e^{-\tau t}$ $P_{11}(t) = 1 - P_{10}(t) = \pi + (1-\pi) e^{-\tau t}$

Using $P(V(t)=n) = \sum_{i=0}^{n} q_i P_{in}(t)$ (error in book?) version 4

where $q_i = P(V(0)=i)$ we see

$q_0 = 1-\pi$ $q_1 = \pi$

$P(V(t)=1) = q_0 P_{01}(t) + q_1 P_{11}(t)$

Inserting q's and P's from above

$P(V(t)=1) = (1-\pi)[\pi - \pi e^{-\tau t}] + \pi[\pi + (1-\pi) e^{-\tau t}]$

$= \pi - \cancel{\pi e^{-\tau t}} - \cancel{\pi^2} - \cancel{\pi^2 e^{-\tau t}} + \cancel{\pi^2} + \cancel{\pi e^{-\tau t}} - \cancel{\pi^2 e^{-\tau t}}$

$= \pi$ ✓ t

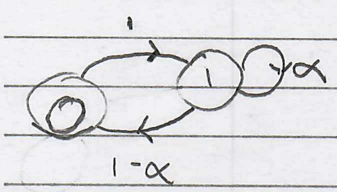
§ 6.3. P1 $\quad \xi_n \quad n = 0, 1, \dots \quad$ 2 state M.C. $\bar{c}$ $S = \{0, 1\}$

TPM $\quad P = \begin{bmatrix} 0 & 1 \\ 1-\alpha & \alpha \end{bmatrix} \quad$ let $N(t)$ $PP(\lambda)$

Show $X(t) = \xi_{N(t)}$ is 2 state B.D.P (λ_0, μ_1)

and find λ_0 μ_1 in terms of λ, α

First $\quad X(0) = \xi_{N(0)} = \xi_0 \quad$ ξ_0 could be 0 or 1



Starting at $N(0)$ suppose $\xi_0 = 0$, the sojurn of $N(t)$ to $N(t)+1$ is $\exp(\lambda)$ and we transition to state 1, then we sojurn in state 1 $\exp(\lambda)$ and either stay there with probability α or transition to 0 with probability $1-\alpha$. In terms of infinitesimal transition probabilities we have

$$P(X(t+h) = 1 \mid X(t) = 0) = \lambda h + o(h) \quad \text{and}$$

$$P(X(t+h) = 0 \mid X(t) = 1) = (1-\alpha)\lambda + o(h)$$

The internet helped me above for the second equation

I also reasoned through what happens when $\alpha = 0$

From the above $\lambda_0 = \lambda \quad \mu_1 = (1-\alpha)\lambda$

§6.3. P3 $\quad S = \{0, 1\}$

$P(X(0)=0) = 1-\pi \quad P(X(0)=1) = \pi \quad 0 < s < t$

$$E[X(s)X(t)] = \sum_{i=0}^{1} i \, P(X(s)X(t) = i)$$

since $X(s)x(t) = 1 \Rightarrow X(s) = 1, \; X(s) = 1$

we have

$$E[X(s)X(t)] = 0 \cdot P(X(s)X(t)=0) + 1 \, P(X(s)X(t)=1)$$

$= P(X(s)=1, X(t)=1)$ now condition

$$E[X(s)X(t)] = P(X(t)=1 \mid X(s)=1) \, P(X(s)=1)$$

$$= P(X(t-s)=1 \mid X(0)=1) \, P(X(s)=1)$$

Since we're starting in the stationary distribution $P(X(s)=1) = \pi$.

$$P(X(t-s)=1 \mid X(0)=1) = P_{11}(t-s) =$$

$$\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)(t-s)}$$ From 6.29c

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§6.3.P3

Also from 6.30d we have

$P_{11}(t-s) = \pi + (1-\pi)e^{-\tau(t-s)}$ $\tau = \alpha+\beta$

Since $P_{11}(t-s) = 1 - P_{10}(t-s)$ from T.P.M.

conditions on rows being probability distribution

* so $E[X(s)X(t)] = \pi - \pi P_{10}(t-s)$

Now $Cov(X(t), X(s)) = E[X(s)X(t)] - E[X(s)]E[X(t)]$

$E[X(s)] = \sum_{i=0}^{1} i \cdot P(X(s)=i) = P(X(s)=1) = \pi$

from our starting conditions, likewise

$E[X(t)] = \pi$, so

$Cov(X(s), X(t)) = \pi - \pi P_{10}(t-s) - \pi^2$ - Not sure

why * - if we use $P_{11}(t-s)$ - we have

$Cov(X(s), X(t)) = \pi P_{11}(t-s) - \pi^2 = \pi(\pi + 1 - \pi e^{-\tau(t-s)}) - \pi^2$

$= \pi(1-\pi)e^{-\tau(t-s)} = \pi(1-\pi)e^{-(\alpha+\beta)|t-s|}$ since $t>s>0$

§ 6.6.E2

$$X_1(t) \quad X_2(t) \quad \perp \quad S = \{0, 1\}$$

inf generating
$$A = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

$$Z(t) = X_1(t) + X_2(t)$$

Clearly $S_Z = \{0, 1, 2\}$ by considering possibilities for $X_1(t)$ $X_2(t)$, or using convolution formula

The Markov property for $X_1(t)$ $X_2(t)$

$$P^1_{ij}(t) = P(X_1(t+s) = j \mid X_1(s) = i) \ \forall \ s \geq 0$$

$$P^2_{ij}(t) = P(X_2(t+s) = j \mid X_2(s) = i) \ \forall \ s \geq 0$$

We need to show Z inherits this property

I googled this one because it wasn't obvious that Z is Markov. In fact I found lots of confusion and it's not true in general. We will have to use the fact that A is the same.

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§ 6.6.E2

Looking at the hint – we can just reason with the P_{ij}'s

$$P_{00}^{Z}(t) = P_{00}^{X_1}(t) \cdot P_{00}^{X_2}(t)$$

$$P_{01}^{Z}(t) = P_{00}^{X_1}(t) \cdot P_{01}^{X_2}(t) + P_{01}^{X_1}(t) \cdot P_{00}^{X_2}(t)$$

$$P_{02}^{Z} = P_{01}^{X_1}(t) \cdot P_{01}^{X_2}(t)$$

$$P_{10}^{Z} = P_{10}^{X_1}(t)\, P_{00}^{X_2}(t) + P_{00}^{X_1}(t)\, P_{10}^{X_2}(t)$$

$$P_{11}^{Z}(t) = P_{11}^{X_1}(t)\, P_{00}^{X_1}(t) + P_{00}^{X_1}(t)\, P_{11}^{X_2}(t) + P_{01}^{X_1}(t)\, P_{10}^{X_2}(t)$$

$$P_{12}^{Z}(t) = P_{11}^{X_1}(t)\, P_{01}^{X_2}(t) + P_{01}^{X_1}(t)\, P_{11}^{X_2}(t)$$

$$P_{20}^{Z}(t) = P_{10}^{X_1}(t)\, P_{10}^{X_2}(t)$$

$$P_{21}^{Z}(t) = P_{11}^{X_1}(t)\, P_{10}^{X_2}(t) + P_{10}^{X_1}(t)\, P_{11}^{X_2}(t)$$

$$P_{22}^{Z}(t) = P_{11}^{X_1}(t)\, P_{11}^{X_2}(t)$$

Now expressing in terms of

$\pi = \dfrac{\lambda}{\lambda+\mu}$ $\quad \tau = \lambda+\mu$ $\quad$ we have

Writing $\underset{\sim}{P}^z(t) = [P_0^z(t), P_1^z(t), P_2^z]$
the columns of $\underset{\sim}{P}^z(t)$ are

$$\underset{\sim}{P}_0^z(t) = \begin{bmatrix} [(1-\pi) + \pi e^{-\tau t}]^2 \\ [(1-\pi) + \pi e^{-\tau t}][\pi - \pi e^{-\tau t}] \cdot 2 \\ [\pi - \pi e^{-\tau t}]^2 \end{bmatrix}$$

$$\underset{\sim}{P}_1^z(t) = \begin{bmatrix} [(1-\pi) - (1-\pi) e^{-\tau t}][(1-\pi) + \pi e^{-\tau t}] \cdot 2 \\ [\pi + (1-\pi) e^{-\tau t}][(1-\pi) + \pi e^{-\tau t}] \cdot 2 + [\pi - \pi e^{-\tau t}][(1-\pi) - (1-\pi) e^{-\tau t}] \\ [\pi + (1-\pi) e^{-\tau t}][\pi - \pi e^{-\tau t}] \cdot 2 \end{bmatrix}$$

$$\underset{\sim}{P}_2^z(t) = \begin{bmatrix} [(1-\pi) - (1-\pi) e^{-\tau t}]^2 \\ [\pi + (1-\pi) e^{-\tau t}][(1-\pi) - (1-\pi) e^{-\tau t}] \cdot 2 \\ [\pi + (1-\pi) e^{-\tau t}]^2 \end{bmatrix}$$

6.6.P1

$Y_n \quad n = 0, 1, 2, \ldots$ a M.C. $\bar{c}$ T.P.M. $P = \|P_{ij}\|$

$N(t)$ + P.P. (λ) $\quad X(t) = Y_{N(t)} \quad t \geq 0$

$N(t) \sim Pois(t\lambda) \quad \forall\, t \in [0, \infty) \quad N(t) \in [0, 1, \ldots)$

so $\forall\, t \quad X(t) = Y_k$ for some k

and $S_X = S_Y$. Now $P(X(t) = j \mid X(0) = i)$

$= P(Y_{N(t)} = j \mid Y_0 = i) = P(Y_k = j \mid Y_0 = i) = P_{ij}^k$

We also know $P(Y_{n+1} = j \mid Y_0 = i_0, Y_1 = i_1, \ldots Y_n = i)$

$= P(Y_{n+1} = j \mid Y_n = i)$. Now let $t_n > t_{n-1} > \ldots t_0$

$P(X(t_n) = j \mid X(t_{n-1}) = i, X(t_{n-2}) = i_{n-2}, \ldots X(t_0) = i_0)$

$= P(Y_{k_n} = j \mid Y_{k_{n-1}} = i, \ldots Y_{k_0} = i_0) =$

$P(Y_{k_n} = j \mid Y_{k_{n-1}} = i)$ since

$k_n \geq k_{n-1} \geq \cdots \geq k_0$ from the properties of

$N(t)$ so $X(t)$ is M.C.

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6.6.P1 (2)

Like the 2-state situation we can investigate A through the infinitesimal transition probabilities

$$P(X(t+h)=j \mid X(t)=i) = (\lambda h) P_{ij} + o(h)$$

the prob of $N(t)$ having an event in $[t, t+h)$ times the probability of a transition from i to j. In our earlier notation

$$P(X(t+h)=j \mid X(t)=i) = P(Y_{N(t+h)}=j \mid Y_{N(t)}=i)$$

and when h is small $N(t+h)-N(t) = \begin{cases} 0 \\ 1 \end{cases}$

with prob $\begin{matrix} 1-(\lambda h)+o(h) \\ \lambda h + o(h) \end{matrix}$ from ch 5.