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ST 6540
HW 2

§3.1 E1

$\{X_n\}$ m.c. on $S = \{0,1,2\}$ $\mathbb{P} = \begin{bmatrix} .1 & .2 & .7 \\ .9 & .1 & 0 \\ .1 & .8 & .1 \end{bmatrix}$

$(P_0, P_1, P_2) = (.3, .4, .3)$

$P_i = P(\{X_0 = i\})$

Find $P(X_0 = 0, X_1 = 1, X_2 = 2)$

$P(X_0 = 0, X_1 = 1, X_2 = 2) = P_0 \, \mathbb{P}_{01} \, \mathbb{P}_{12}$

$P_0 = .3 \quad \mathbb{P}_{01} = .2 \quad \mathbb{P}_{12} = 0 \quad \therefore \; P(X_0 = 0, X_1 = 1, X_2 = 2) = 0$

§3.1 E2

$\mathbb{P} = \begin{bmatrix} .7 & .2 & .1 \\ 0 & .6 & .4 \\ .5 & 0 & .5 \end{bmatrix}$ $S = \{0,1,2\}$

Find $P(\{X_2 = 1, X_3 = 1\} / X_1 = 0)$ and

$P(X_1 = 1, X_2 = 1 \mid X_0 = 0)$

$P(X_2 = 1, X_3 = 1 \mid X_1 = 0) = \mathbb{P}_{01} \, \mathbb{P}_{11} = .2 \cdot .6 = .12$

$P(X_1 = 1, X_2 = 1 \mid X_0 = 0) = \mathbb{P}_{01} \, \mathbb{P}_{11} = .12$

§ 3.1 E5 $\{X_i\}$ MC. $S = \{0, 1, 2\}$

$$\mathbb{P} = \begin{bmatrix} .3 & .2 & .5 \\ .5 & .1 & .4 \\ .5 & .2 & .3 \end{bmatrix} \qquad P_0 = .5 \quad P_1 = .5$$

Find $P(X_0 = 1, X_1 = 1, X_2 = 0)$ and $P(X_1 = 1, X_2 = 1, X_3 = 0)$

Note $P_2 = 0$ since $\sum_S P_i = 1$

$P(X_0 = 1, X_1 = 1, X_2 = 0) = P_1 \mathbb{P}_{11} \mathbb{P}_{10} = .5 \; .1 \; .5$

First think about $P(X_1 = 1)$

$P(X_1 = 1) = P(X_1 = 1 \mid X_0 = 1) + P(X_1 = 1 \mid X_0 = 0)$

(don't have to consider $X_0 = 2$ since $P_2 = 0$)

$= P_1 \mathbb{P}_{11} + P_0 \mathbb{P}_{01}$

Now $P(X_1 = 1, X_2 = 1, X_3 = 0) = P(X_1 = 1, X_2 = 1) \times$

$P(X_3 = 0 \mid X_1 = 1, X_2 = 1) = P(X_1 = 1, X_2 = 1)\, P(X_3 = 0 \mid X_2 = 1)$

$= P(X_1 = 1)\, P(X_2 = 1 \mid X_1 = 1)\, P(X_3 = 0 \mid X_2 = 1)$

$= (P_1 \mathbb{P}_{11} + P_0 + \mathbb{P}_{01})\, \mathbb{P}_{11} \mathbb{P}_{10}$

§3.1 P2

$\{X_i\}$ M.C. $S = \{0, 1\}$ $\quad \mathbb{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{bmatrix}$

$\alpha \in (0, 1)$

Find $P(X_0 = 0, X_1 = 0, X_2 = 0)$

$P(X_0 = 0, X_1 = 0, X_2 = 0) = \mathbb{P}_{00} \mathbb{P}_{00} = (1-\alpha)^2$

Find probability no error is recived at $n = 2$.

We need to consider the cases that no errors occured, and that an even number $e \leq n$ of errors occur.

Let A = event no error occurs, then

$P(A) = P(X_0 = 0, X_1 = 1, X_2 = 0) + P(X_0 = 0, X_1 = 0, X_2 = 0)$

$= \mathbb{P}_{01} \mathbb{P}_{10} + \mathbb{P}_{00} \mathbb{P}_{00}$

$= \alpha^2 + (1-\alpha)^2 = 2\alpha^2 + 2\alpha + 1$

§ 3.1 P3

$\{X_i\}$ M.C. $S = \{G, D\}$

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} G & D \end{array} \\ \begin{array}{c} G \\ D \end{array} & \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} \end{array}$$

Let $X_0 = G$

Find $P(X_0 = G, X_1 = G, \dots X_4 = G, X_5 = D)$

The probability the first defective item is at 5.

$$P(X_0 = G, X_1 = G, \dots X_5 = D) = P_{GG} P_{GG} P_{GG} P_{GG} P_{GD}$$

$$= \alpha^4 (1-\beta)$$

§ 3.2 E1

$\{X_n\}$ M.C. $S = \{0,1,2\}$ $\mathbb{P} = \begin{bmatrix} .1 & .2 & .7 \\ .2 & .2 & .6 \\ .6 & .1 & .3 \end{bmatrix}$

Find $\mathbb{P}^{(2)} = \mathbb{P}^2$ the 2-step transition matrix

a) $\mathbb{P}^{(2)} = \mathbb{P} \times \mathbb{P} = \begin{bmatrix} .47 & .13 & .33 \\ .42 & .14 & .38 \\ .2 & .16 & .52 \end{bmatrix}$

b) $P(X_3 = 1 \mid X_1 = 0) = .13$

c) $P(X_3 = 1 \mid X_0 = 0) =$

$P(X_3 = 1 \mid X_1 = 0, X_0 = 0) + P(X_3 = 1 \mid X_1 = 1, X_0 = 0)$

$+ P(X_3 = 1 \mid X_1 = 2, X_0 = 0)$

$= \mathbb{P}^{(2)}_{01} + \mathbb{P}^{(2)}_{11} + \mathbb{P}^{(2)}_{21}$

$= .13 + .14 + .16$

§ 3.2 E2

$\{X_n\}$ M.C. $\quad S = \{0, 1, 2\} \quad \mathbb{P} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix}$

Find $P(X_n = 0 \mid X_0 = 0)$ for $n = 0, 1, 2, 3, 4$

$P(X_0 = 0 \mid X_0 = 0) = 1$

$P(X_1 = 0 \mid X_0 = 0) = \mathbb{P}_{00}$

$P(X_2 = 0 \mid X_0 = 0) = \mathbb{P}_{00}^{(2)}$

$\mathbb{P}_{00}^{(3)} = P(X_3 = 0 \mid X_0 = 0) = \sum_{k=0}^{2} \mathbb{P}_{0k} \mathbb{P}_{k0}^{(2)}$

$=$

$P_{00}^{(4)} = P(X_4 = 0 \mid X_0 = 0) = \mathbb{P}_{00}^{4} = \sum_{k=0}^{2} \mathbb{P}_{0k} \mathbb{P}_{k0}^{(3)}$

§ 3.2. P1

$$\mathbb{P} = \begin{bmatrix} .4 & .3 & .2 & .1 \\ .1 & .4 & .3 & .2 \\ .3 & .2 & .1 & .4 \\ .2 & .1 & .4 & .3 \end{bmatrix} \qquad S = \{0,1,2,3\}$$

$p_i = 1/4 \quad i = 0,1,2,3$ Show $P(X_n = 1) = 1/4$

$$p_k^{(n)} = \sum_{j=0}^{3} p_j \, P_{jk}^{(n)} \quad \forall n$$

We notice the rows of $\mathbb{P}$ are permutations of each other.

$P(X_0 = 1) = p_1 = 1/4 \qquad P(X_1 = 1) = \sum_{j=0}^{3} p_j \mathbb{P}_{j1}^{(1)} = \frac{1}{4} \sum_{j=0}^{3} \mathbb{P}_{j1}$

we notice that we can use the stochastic property of $\mathbb{P}$; $\sum_{j=0}^{n} \mathbb{P}_{jk} = 1$ to see that

$P(X_1 = 1) = 1/4$.

Now we know that $\mathbb{P}^{(n)} = \mathbb{P}^n$, and that $\mathbb{P}^{(n)}$ is the n-step transition probabilities so $\mathbb{P}^n$ is stochastic. From this we can conclude

$$P(X_n = 1) = \sum_{j=0}^{n} p_j \mathbb{P}_{j1}^{(n)} = 1/4 \sum_{j=0}^{n} \mathbb{P}^n = 1/4.$$

Anytime $p_i \sim \text{Uniform}(\alpha)$ we will have the general result that $P(X_n = k) = 1/\alpha$

Quick proof $\mathbb{P}^n$ is stochastic. $P^n = P^{n-1}P =: QP$

let Q_i be row i P^j be col j. Then

let P be stochastic, and assume $Q = P^{n-1}$ is.

$$\sum_j^n P_{ij} = \sum_j QP = \sum_j Q_i \cdot P^j = \sum_j \sum_k Q_{ik} P_{kj}$$

$$= \sum_k Q_{ik} \sum_j P_{kj} = \sum Q_{ik} \cdot 1 = 1 \cdot 1 \quad \therefore P^n \text{ stochastic}$$

§3.2 P2

$\{X_n\}$ MC, $\mathbb{P} = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{bmatrix} \end{array}$, Find $P(X_5=0 \mid X_0=0)$

$$\mathbb{P}_{ij}^{(n)} = P(X_n=j \mid X_0=i) = \sum_{k=0}^{1} \mathbb{P}_{ik}\mathbb{P}_{kj}^{n-1}$$

$$P(X_5=0 \mid X_0=0) = \sum_{k=0}^{1} \mathbb{P}_{0k}\mathbb{P}_{k0}^{4} = \mathbb{P}_{00}\mathbb{P}_{00}^{4} + \mathbb{P}_{01}\mathbb{P}_{10}^{4}$$

$$\mathbb{P}_{00}^{4} = \mathbb{P}_{00}\mathbb{P}_{00}^{3} + \mathbb{P}_{01}\mathbb{P}_{10}^{3} = (1-\alpha)(3\alpha^2(1-\alpha)+(1-\alpha)^3) + \alpha(3\alpha(1-\alpha)^2+\alpha^3)$$

$$\mathbb{P}_{10}^{4} = \mathbb{P}_{10}\mathbb{P}_{00}^{3} + \mathbb{P}_{11}\mathbb{P}_{10}^{3} = \alpha(3\alpha^2(1-\alpha)+(1-\alpha)^3) + (1-\alpha)(3\alpha(1-\alpha)^2+\alpha^3)$$

$$\mathbb{P}_{00}^{3} = \mathbb{P}_{00}\mathbb{P}_{00}^{2} + \mathbb{P}_{01}\mathbb{P}_{10}^{2} = (1-\alpha)((1-\alpha)^2+\alpha^2) + \alpha(2\alpha(1-\alpha)$$

$$= 3\alpha^2(1-\alpha) + (1-\alpha)^3$$

$$\mathbb{P}_{10}^{3} = \mathbb{P}_{10}\mathbb{P}_{00}^{2} + \mathbb{P}_{11}\mathbb{P}_{10}^{2} = \alpha(1-2\alpha+2\alpha^2) + (1-\alpha)2\alpha(1-\alpha)$$

$$= 3\alpha(1-\alpha)^2 + \alpha 3$$

$$\mathbb{P}_{00}^{2} = \mathbb{P}_{00}\mathbb{P}_{00} + \mathbb{P}_{01}\mathbb{P}_{10} = (1-\alpha)^2 + \alpha^2$$

$$\mathbb{P}_{10}^{2} = \mathbb{P}_{10}\mathbb{P}_{00} + \mathbb{P}_{11}\mathbb{P}_{10} = \alpha\cdot(1-\alpha) + (1-\alpha)\alpha = 2\alpha(1-\alpha)$$

Putting our expesions for $\mathbb{P}_{00}^{4}$ and $\mathbb{P}_{00}^{5}$ into

$P(X_5=0 \mid X_0=0) =$

$$(1-\alpha)\left[6\alpha^2(1-\alpha)^2 + (1-\alpha)^4 + \alpha^4\right] + \alpha\left[4\alpha^3(1-\alpha) + \alpha(1-\alpha)^3 + 3\alpha(1-\alpha)^3\right]$$

$$= 6\alpha^2(1-\alpha)^3 + (1-\alpha)^5 + \alpha^4(1-\alpha) + 4\alpha^4(1-\alpha) + \alpha^2(1-\alpha)^3 + 3\alpha^2(1-\alpha)^3$$

$$= 10\alpha^2(1-\alpha)^3 + (1-\alpha)^5 + 5\alpha^4(1-\alpha)$$

§3.2 p5

$$\mathbb{P} = \begin{bmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{X_n\} \text{ M.C.} \quad S = \{0,1,2\}$$

$T = \min\{n \geq 0 : X_n = 2\}$ Find $\mathbb{P}\{X_3 = 0 \mid X_0 = 0, T > 3\}$

This problem was hard for me!

$P(X_3 = 0 \mid X_0 = 0, T > 3) = P(X_3 = 0, T > 3 \mid X_0 = 0) / P(T > 3 \mid X_0 = 0)$

Now consider events; $\{X_3 = 0\} \subset \{T > 3\} \Rightarrow$

$P(X_3 = 0, T > 3 \mid X_0 = 0) = P(X_3 = 0 \mid X_0 = 0) = \mathbb{P}^3_{00}$

If we use hint $\{T > 3\} = \{X_3 \neq 2\}$ then

$P(T > 3 \mid X_0 = 0) = P(X_3 \neq 2 \mid X_0 = 0)$

$= P(X_3 = 1 \mid X_0 = 0) + P(X_3 = 0 \mid X_0 = 0)$

$= \mathbb{P}^3_{01} + \mathbb{P}^3_{00}$ so

$P(X_3 = 0 \mid X_0 = 0, T > 3) = \mathbb{P}^3_{00} / \mathbb{P}^3_{01} + \mathbb{P}^3_{00}$

Why was it hard? $T > 3 \Rightarrow \{X_3 \neq 2\}$

$T > 3 \Rightarrow \{T > 3\} \cap \{T > 2\} \cap \{T > 1\} \Rightarrow$

$X_3 \neq 3 \quad X_2 \neq 2 \quad \cap \quad X_1 \neq 2$

§ 3.3 E2

$X_n = \#$ balls in A

$X_n \to X_{n}+1$ when ball comes from B and A is selected

$X_n \to X_n - 1$ when ball comes from A and B is selected

$X_n \to X_n$ when ball comes from A and A is selected or when ball comes from B and B is selected

Consider events;

A - ball comes from A
$$P(A) = X_n / N$$

B - ball comes from B
$$P(B) = \frac{N - X_n}{N}$$

C - urn A is chosen
$$P(C \mid A) = p \cdot X_n / N \qquad P(C \mid B) = p \cdot (N - X_n)/N$$
by $\perp$

D - urn B is chosen
$$P(D \mid A) = (1-p)\, X_n / N \qquad P(D \mid B) = (1-p)\,(N - X_n)/N$$
by $\perp$

Now we have information to form $\mathbb{P}$

$$\mathbb{P}_{ij} = \begin{cases} P(X_{n+1} = i+1 \mid X_n = i) & j = i+1 \\ P(X_{n+1} = i-1 \mid X_n = i) & j = i-1 \\ P(X_{n+1} = i \mid X_n = i) & j = i \\ 0 & \text{otherwise} \end{cases}$$

and we have edge cases

$X_n = N \quad X_n = 0$

$P(X_{n+1} = N+1 \mid X_n = N) = 0$
$P(X_{n+1} = -1 \mid X_n = 0) = 0$

don't need these $-1, N+1$ not in S

$$\mathbb{P}_{ij} = \begin{cases} P(C \mid B) = p \cdot (N - i)/N & j = i+1 \\ P(D \mid A) = (1-p)\, i/N & j = i-1 \\ P(C \mid A) + P(D \mid B) = p \cdot \frac{i}{N} + (1-p)\frac{N-i}{N} & j = 0 \\ 0 & \text{otherwise} \end{cases}$$

§3.3 E5

$X_n = \#$ red balls $\qquad X_0 = 1 \qquad S = \{0, 1, 2\}$

$$X_n \begin{cases} X_n + 1 & \text{when a green ball is selected} \\ X_n - 1 & \text{when a red ball is selected} \end{cases}$$

$\mathbb{P}_{00} = 0 \quad \mathbb{P}_{01} = 1 \quad \mathbb{P}_{02} = 0$

$\mathbb{P}_{10} = 1/2 \quad \mathbb{P}_{11} = 0 \quad \mathbb{P}_{12} = 1/2$

$\mathbb{P}_{20} = 0 \quad \mathbb{P}_{21} = 1 \quad \mathbb{P}_{22} = 0$

§ 3.2 P1

$N = 6$

$X_n = \#\text{red} = N - (\#\text{green} + \#\text{blue})$ $\qquad X_0 = 3$

$$S = \{3, 2, 1, 0\}$$

$$X_n \begin{cases} \nearrow X_n - 1 & \text{when one red and one green are selected} \\ \searrow X_n & \text{otherwise} \end{cases}$$

So $\mathbb{P}_{ij} = \begin{cases} \mathbb{P}(X_{n+1} = i-1 \mid X_n = i) & j = i-1 \\ \mathbb{P}(X_{n+1} = i \mid X_n = i) & j = i \\ 0 & \text{otherwise} \end{cases}$

Now Consider events

$$P((\text{Red}, \text{Blue}) \mid X_0 = 3) = \frac{1}{2} \cdot \frac{N-3}{N-1} = P((\text{Blue}, \text{Red}) \mid X_0 = 3)$$

and more generally

$$P((\text{Red}, \text{Blue}) \mid X_0 = k) = \frac{k}{N} \cdot \frac{N-k}{N-1} \qquad k = 3, 2, 1$$

And

$$P(\neg (\text{Red}, \text{Blue}) \mid X_0 = k) = 1 - \left(\frac{k}{N} \frac{N-k}{N-1}\right)$$

Finally,

$$\mathbb{P} = \begin{array}{c} \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 2 & 1 & 0 \end{array} \\ \begin{bmatrix} 1 - \frac{3}{6} \cdot \frac{3}{5} & \frac{3}{6} \cdot \frac{3}{5} & 0 & 0 \\ 0 & 1 - \frac{2}{6} \cdot \frac{4}{5} & \frac{2}{6} \cdot \frac{4}{5} & 0 \\ 0 & 0 & 1 - \frac{1}{6} \cdot \frac{5}{5} & \frac{1}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 7/10 & 3/10 & 0 & 0 \\ 0 & 11/15 & 4/15 & 0 \\ 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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§ 3.3 p 5

$B_i = \text{Bernoulli}(p)$ ← they didn't say fair coin

$S = \{0, H, HH, HHT\}$

Allowed Transitions

$0 \to H$ when H observed
$0 \to 0$ when T observed

$H \to HH$ when H observed
$H \to 0$ when T observed

$HH \to 0$ when H observed
$HH \to HHT$ when T observed

Probabilities

$P(X_{n+1} = H \mid X_n = 0) = p$ $\quad P(X_{n+1} = 0 \mid X_n = 0) = 1-p$

$P(X_{n+1} = HH \mid X_n = H) = p^2$ $\quad P(X_{n+1} = 0 \mid X_n = H) = p(1-p)$

$P(X_{n+1} = 0 \mid X_n = HH) = p \cdot p^2$ $\quad P(X_{n+1} = HHT \mid X_n = HH) = (1-p)p^2$

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ H \\ HH \\ HHT \end{array} \begin{array}{c} \begin{array}{cccc} 0 & H & HH & HHT \end{array} \\ \begin{bmatrix} 1-p & p & 0 & 0 \\ p(1-p) & 0 & p^2 & 0 \\ p^3 & 0 & (1-p)p^2 & \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

§3.3 p 9

$\{X_n\}$ M.C. $X_n = \#$ balls in A

N = total # balls $\qquad N - X_n = \#$ balls in B

$S = \{0, \dots N\}$

Transitions

$X_n \to X_{n+1}$ B is selected, Ball retured to A

$X_n \to X_{n-1}$ A is selected, ball returned to B

$X_n \to X_n$ B selected, ball returned to B
(or)
A selected, ball returned to A

Probabilities

$P(X_{n+1} = i+1 \mid X_n = i)$

$P(\text{B selected} \mid \text{ball returned to A}) = \left(\frac{N - X_n}{N}\right)\left(\frac{X_n}{N}\right)$ by ⊥

$P(X_{n+1} = i-1 \mid X_n = i)$

$P(\text{A selected} \mid \text{ball returned to B}) = \frac{X_n}{N}\left(\frac{N - X_n}{N}\right)$ by ⊥

$$P(X_{n+1} = i \mid X_n = i) = \left(\frac{N - X_n}{N}\right)\left(\frac{N - X_n}{N}\right) + \frac{X_n}{N} \cdot \frac{X_n}{N}$$

$$P_{ij} = \begin{cases} P(X_{n+1} = i+1 \mid X_n = i) & j = i+1 \\ P(X_{n+1} = i-1 \mid X_n = i) & j = i-1 \\ P(X_n = i \mid X_n = i) & j = i \\ 0 & \text{otherwise} \end{cases}$$