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OSU STAT 6540
Homework #3

§3.4 E1

$\{X_n\}$ M.C. $S = \{0,1,2,3\}$ $\mathbb{P} = \begin{bmatrix} .4 & .3 & .2 & .1 \\ 0 & .7 & .2 & .1 \\ 0 & 0 & .9 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find $E[T \mid X_0 = 0]$

$T = \min\{ n \geq 1 \mid X_n = 3\}$

$$E[T \mid X_0 = i] = v_i = 1 + \sum_{j=0}^{2} P_{ij} v_j$$

Since $\mathbb{P}$ is upper triangular we start with $i = 2$ and solve upwards.

$v_2 = 1 + P_{20} v_0 + P_{21} v_1 + P_{22} v_2$ since $P_{20} = P_{21} = 0$

$\Rightarrow$ $v_2 = 1/1 - P_{22}$

$v_1 = 1 + P_{10} v_0 + P_{11} v_1 + P_{12} v_2 = 1 + P_{11} v_1 + P_{12} v_2$ since $P_{10} = 0$ $\Rightarrow$

$$v_1 = \frac{1}{1-P_{11}}\left[1 + P_{12}(v_2)\right] = \frac{1}{(1-P_{11})}\left[1 + \frac{P_{12}}{1-P_{22}}\right]$$

Finally $v_0 = 1 + P_{00} v_0 + P_{01} v_1 + P_{02} v_2 \Rightarrow$

$v_0(1 - P_{00}) = 1 + P_{01} v_1 + P_{02} v_2$ $\Rightarrow$

$$v_0 = \frac{1}{1-P_{00}}\left[1 + \frac{P_{01}}{1-P_{11}}\left[1 + \frac{P_{12}}{1-P_{22}}\right] + \frac{P_{02}}{1-P_{22}}\right]$$

Putting values in

$$v_0 = \left(\frac{1}{1-.4}\right)\left[1 + \frac{.3}{1-.7}\left[1 + \frac{.2}{1-.9}\right] + \frac{.2}{1-.9}\right] = 10$$

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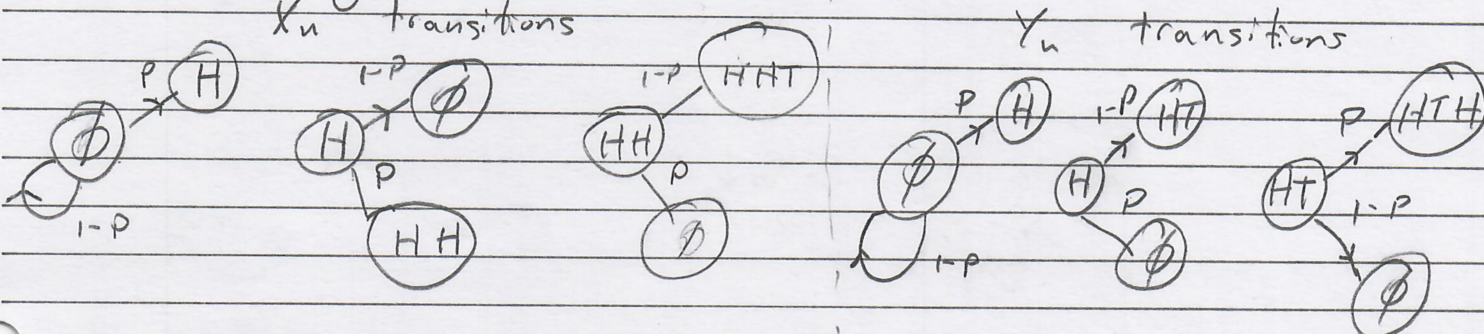
§ 3.4 p1

Let $\{B_n\}$ be sequence of Bernoulli(p) r.v. $\{X_n\}$ be the M.C. based on $\{B_n\}$ with state space $\{\emptyset, H, HH, HHT\}$ and $\{Y_n\}$ be the M.C. based on $\{B_n\}$ with state space $\{\emptyset, H, HT, HTH\}$. HTH, HHT are absorbing.



$$\mathbb{P}_X = \begin{bmatrix} 1-p & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ p & 0 & 0 & 1-p \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbb{P}_Y = \begin{bmatrix} 1-p & p & 0 & 0 \\ p & 0 & 1-p & 0 \\ 1-p & 0 & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We relabel $S_X, S_Y = \{0, 1, 2, 3\}$ and let

$T_X = \min\{n \geq 1, X_n = HHT\}$ $\quad T_Y = \min\{n \geq 1 \ \ Y_n = HTH\}$

Then we need to compare $E[T_X \mid X_0 = 0]$, $E[T_Y \mid Y_0 = 0\}$

Using $v_i = E[T \mid X_0 = i] = 1 + \sum_{j=0}^{2} \mathbb{P}_{ij} v_j$

§ 3.4 p1

p2

$\{X_n\}$ analysis

$$v_0 = 1 + P_{00} v_0 + P_{01} v_1 + \cancel{P_{02} v_2}$$

$$v_1 = 1 + P_{10} v_0 + \cancel{P_{11} v_1} + P_{12} v_2$$

$$v_2 = 1 + P_{20} v_0 + \cancel{P_{21} v_1} + \cancel{P_{22} v_2} = 1 + p v_0$$

$$\Rightarrow v_1 = 1 + (1-p) v_0 + p(1 + p v_0)$$

$$\Rightarrow v_0 = 1 + (1-p) v_0 + p(1 + (1-p) v_0 + p + p^2 v_0)$$

$$v_0 (1 - (1-p) - p(1-p) - p^3) = 1 + p + p^2$$

$$v_0 = \frac{1 + p + p^2}{p^2 - p^3} =: v_x$$

$\{Y_n\}$ analysis

$$v_0 = 1 + P_{00} v_0 + P_{01} v_1 + \cancel{P_{02} v_2}$$

$$v_1 = 1 + P_{10} v_0 + \cancel{P_{11} v_1} + P_{12} v_2$$

$$v_2 = 1 + P_{20} v_0 + \cancel{P_{21} v_1} + \cancel{P_{22} v_2} = 1 + (1-p) v_0$$

$$\Rightarrow v_1 = 1 + p v_0 + (1-p)(1 + (1-p) v_0)$$

$$\Rightarrow v_0 = 1 + (1-p) v_0 + p(1 + p v_0 + (1-p) + (1-p)^2 v_0)$$

$$v_0 (1 - (1-p) - p^2 - p(1-p)^2) = 1 + p + p(1-p)$$

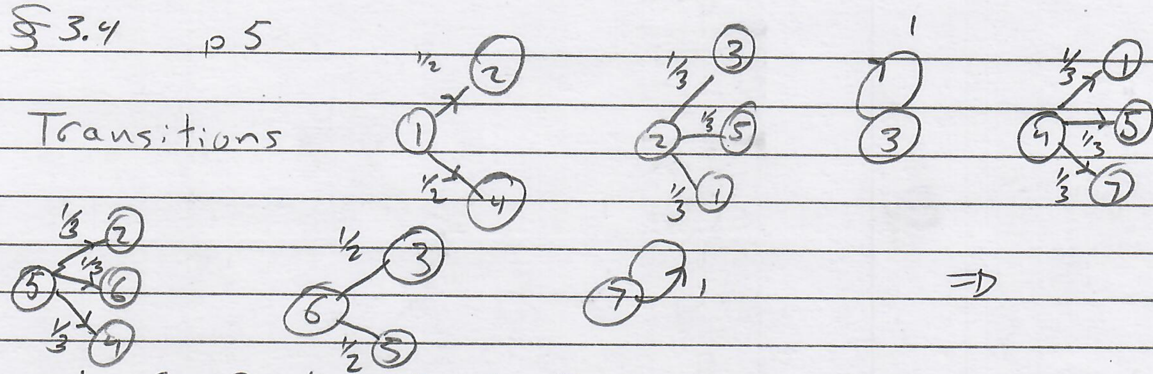
$$v_0 = \frac{1 + 2p - p^2}{p^2 - p^3} =: v_y$$

We see $v_x = v_y + \dfrac{p + 2p^2}{p^2 - p^3}$ since $p \in (0,1)$

$v_x > v_y$

§3.4 p5

Transitions



$$
\begin{array}{c|ccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\hline
 & 0 & \tfrac12 & 0 & \tfrac12 & 0 & 0 & 0 & 1\\
\mathbb{P} = & \tfrac13 & 0 & \tfrac13 & 0 & \tfrac13 & 0 & 0 & 2\\
 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3\\
 & \tfrac13 & 0 & 0 & 0 & \tfrac13 & 0 & \tfrac13 & 4\\
 & 0 & \tfrac13 & 0 & \tfrac13 & 0 & \tfrac13 & 0 & 5\\
 & 0 & 0 & \tfrac12 & 0 & \tfrac12 & 0 & 0 & 6\\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7
\end{array}
$$

let

$T = \min\{n \geq 1 \mid X_n = 3 \text{ or } X_n = 7\}$

$\mu_i = U_{i3} = P(X_T = 3 \mid X_0 = i)$

Now perform a first step analysis to get

$\mu_1 = P_{13} + P_{12}\mu_2 + P_{14}\mu_4 = \tfrac12 \mu_2 + \tfrac12 \mu_4$

$\mu_2 = P_{23} + P_{21}\mu_1 + P_{25}\mu_5 = \tfrac13 + \tfrac13 \mu_1 + \tfrac13 \mu_5$

* $\mu_4 = P_{43} + P_{41}\mu_1 + P_{45}\mu_5 = \tfrac13 \mu_1 + \tfrac13 \mu_5$

$\mu_5 = P_{52}\mu_2 + P_{54}\mu_4 + P_{56}\mu_6 = \tfrac13 \mu_2 + \tfrac13 \mu_4 + \tfrac13 \mu_6$

$\mu_6 = P_{63} + P_{65}\mu_5 = \tfrac12 + \tfrac12 \mu_5$

Rewriting

$-\mu_1 + \tfrac12 \mu_2 + \tfrac12 \mu_4 = 0$

$-\mu_2 + \tfrac13 \mu_1 + \tfrac13 \mu_5 = -\tfrac13$

$-\mu_4 + \tfrac13 \mu_1 + \tfrac13 \mu_5 = 0$

$-\mu_5 + \tfrac13 \mu_2 + \tfrac13 \mu_4 + \tfrac13 \mu_6 = 0$

$-\mu_6 + \tfrac12 \mu_5 = -\tfrac12$

$$
\Rightarrow
\begin{array}{ccccc}
1 & 2 & 4 & 5 & 6
\end{array}
$$

$$
\begin{bmatrix}
-1 & \tfrac12 & \tfrac12 & 0 & 0\\
\tfrac13 & -1 & 0 & \tfrac13 & 0\\
\tfrac13 & 0 & -1 & \tfrac13 & 0\\
0 & \tfrac13 & \tfrac13 & -1 & \tfrac13\\
0 & 0 & 0 & \tfrac12 & -1
\end{bmatrix}
\begin{bmatrix}\mu_1\\ \mu_2\\ \mu_4\\ \mu_5\\ \mu_6\end{bmatrix}
=
\begin{bmatrix}0\\ -\tfrac13\\ 0\\ 0\\ -\tfrac12\end{bmatrix}
$$

See next

I used Matlab

$\mu' = (\tfrac{7}{2}, \tfrac{3}{4}, \tfrac{5}{12}, \tfrac{2}{3}, \tfrac{5}{6})$

$(\mu_4 = 5/12)$

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§3.4 p6

$$\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \mathbb{P} = \begin{bmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$p+q=1$

$r=4$

Find $E[T \mid X_0=0]$ $T=\min\{n\geq 1: X_n=4$

Let $v_i = E[T \mid X_0=i]$, then

$v_0 = 1 + q v_0 + p v_1$

$v_1 = 1 + q v_0 + p v_2 \Rightarrow v_1 = 1 + q v_0 + p(1 + q v_0 + p(1+q v_0))$

$v_2 = 1 + q v_0 + p v_3 \Rightarrow v_2 = 1 + q v_0 + p(1+q v_0)$

$v_3 = 1 + q v_0$

Putting expession for v_1 into v_0 gives

$v_0 = 1 + q v_0 + p(1 + q v_0 + p(1+q v_0 + p(1+q v_0)))$

$v_0(1 - q - pq - p^2 q - p^3 q) = 1 + p + p^2 + p^3$

$v_0 = 1 + p + p^2 + p^3 / p^4$ after simplifying using $q = 1-p$

§ 3.4 p11

$$S = \{ RRGG,\ RGG,\ GRR,\ RG,\ GG,\ G,\ \phi \}$$

$X_0 = RRGG$

The probability that a single red ball is in the urn at the time ~~the~~ last green ball is selected is given by

$P(X_{n+1} = \phi \mid X_n = RG)$ This is $\mathbb{P}^{(n)}_{RG,\phi}$

$$\mathbb{P} = \begin{array}{c} \begin{array}{ccccccc} RRGG & RGG & GRR & RG & GG & G & \phi \end{array} \\ \left[\begin{array}{ccccccc} & 1/2 & 1/2 & & & & \\ & & & 2/3 & 1/3 & & \\ & & & 2/3 & & & 1/3 \\ & & & & & 1/2 & 1/2 \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & 1 \end{array}\right] \end{array} \begin{array}{c} \\ RRGG \\ RGG \\ GRR \\ RG \\ GG \\ G \\ \phi \end{array}$$

G is not absorbing - but it's <u>not</u> transient either.

We can read off the probability we're interested in from the transition matrix since by definition $P_{ij} = P(X_{n+1} = i \mid X_n = j)$

and we see

$$P(X_{n+1} = \phi \mid X_n = RG) = 1/3$$

§3.5 ε4

$\{B_n\}$ $\quad B_n \sim \text{Bernoulli}(p)$ $\quad$ let $\{X_n\}$ be based on B_n $\quad X_n =$ current run of successes.

The process stops when $X_n = 3$

Instead of modelling on $\{\emptyset, H, HH, HHH\}$ where $\emptyset$ has events T, HT, HHT and HHH is absorbing, let $S = \{0, 1, 2, 3\}$ then if $q = 1-p$

$$\mathbb{P} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} q & p & 0 & 0 \\ q & 0 & p & 0 \\ q & 0 & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

§3.5 ε5 $\quad$ let $\alpha = .8$ $\quad \beta = .95$

$X_0 =$

$$\mathbb{P} = \begin{array}{c} \begin{array}{cc} \text{Defective} & \text{Good} \end{array} \\ \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix} \end{array} \begin{array}{c} \text{Defective} \\ \text{Good} \end{array}$$

let $a = 1-\alpha$
$\quad b = 1-\beta$

Then putting into 3.31

$$\mathbb{P}^{(8)} = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^8}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

We will finish computations in matlab.

§ 3.5 E 7

$$\mathbb{P} = \begin{array}{c c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .3 & 0 & .7 & 0 \\ 0 & .3 & 0 & .7 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array} \qquad X_0 = 1$$

$$u_1 = P_{10} + P_{11} u_1 + P_{12} u_2 \qquad u_1 = .3 + .7 u_2$$

$$\Rightarrow$$

$$u_2 = P_{20} + P_{21} u_1 + P_{22} u_2 \qquad u_2 = .3 u_1$$

$$u_1 = .3/(1-.21) \qquad u_2 = .09/(1-.21)$$

$$= .3797 = 30/79 \qquad = .1139 = 9/79$$

Now $\mathbb{P}$ is a random walk with absorbing states $0, N=3$ and we can use the derived equations $u_i = \left((q/p)^i - (q/p)^N \right) / \left(1 - (q/p)^N\right)$

$i = 1, 2$, $q = .3$ $p = .7$

$$u_1 = \left(\frac{.3}{.7}\right) - \left(\frac{.3}{.7}\right)^3 \Big/ \left[1 - \left(\frac{.3}{.7}\right)^3\right] = .323$$

$$u_2 = \left(\frac{.3}{.7}\right)^2 - \left(\frac{.3}{.7}\right)^3 \Big/ \left[1 - \left(\frac{.3}{.7}\right)^3\right] = .097$$

Not sure why these don't match.

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§ 3.5 p 2

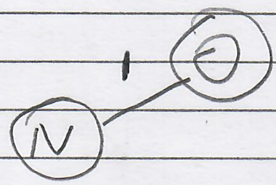
$\{X_n\}$ M.C. of ages components

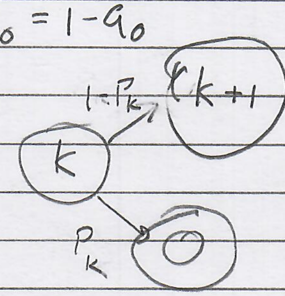
$S = \{0, 1, 2, \dots\}$ $\quad P(T=k) = a_k \quad k = 1, 2, \dots$

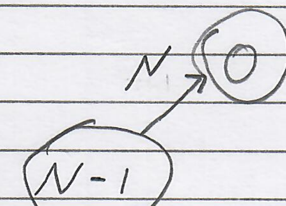
a) $p_i = a_{i+1} \Big/ \sum_{k \geq i+1}^{\infty} a_k \qquad q_i = 1 - p_i \qquad i = 1, \dots$

$p_0 = a_1 \qquad q_0 = 1 - a_0$

b) Consider transitions







$$\mathbb{P} = \begin{array}{c|cccccc|c} & 0 & 1 & 2 & 3 & N-1 & N & \\ \hline & p_0 & q_0 & 0 & \cdots & & & 0 \\ & p_1 & 0 & q_1 & & & & 1 \\ & p_2 & 0 & 0 & q_2 & & & 2 \\ & & & & \vdots & & & \\ & p_{N-2} & & & & & & \\ & 1 & & & & & & N-1 \end{array}$$

But here we don't have the same p_i's

we only have to consider the contribution of

a_k's for $k < N$

$$p_i = a_{i+1} \Big/ \sum_{k \geq i+1}^{N-1} a_k \qquad q_i = 1 - p_i \qquad p_0 = a_1$$

Note : above $p_0 = a_1$ because $p_0 = \dfrac{a_1}{\sum_{k=1}^{\infty} a_k} = a_1$

Here $p_0 = a_1$ because its literally

the probability $1 \to 0$ i.e. $\sum_{k=1}^{N-1} a_k \neq 1$

§3-6. P1

$$u_i = P[X_T = 0 \mid X_0 = i] \quad \text{satisfies}$$

$$u_i = q_i u_{i-1} + r_i u_i + p_i u_{i+1} \qquad i = 1, \dots, N-1$$

$$u_0 = 1 \qquad u_N = 0$$

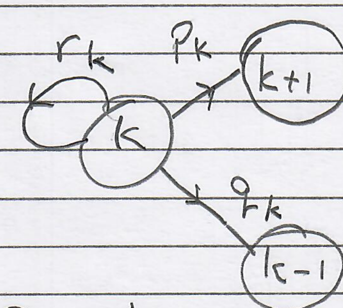
$$u_i = \rho_i + \cdots + \rho_{N-1} \Big/ \Big(1 + \sum_{j=1}^{N-1} \rho_j\Big) \quad ; \quad \rho_k = \frac{q_1 \cdots q_k}{p_1 \cdots p_k}$$

$k = 1, \dots, N-1$

We're asked to show the above for the general

RW

$$\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 & \cdots & & & \\ q_1 & r_1 & p_1 & 0 & \cdots & & \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots & \\ & & q_3 & r_3 & p_3 & & \\ & & & & q_{N-1} & r_{N-1} & p_{N-1} \\ 0 & 0 & \cdots & & & & \end{bmatrix}$$



We'll try to follow the form for the case $q_i = q \quad r = 0 \quad p_i = p$.

Let $x_k = u_k - u_{k-1} \qquad u_k = (p_k + q_k + r_k) u_k$

and put these into first step equations, using

$u_0 = 1 \; u_N = 0 \qquad u_i = q_i u_{i-1} + r_i u_i + p_i u_{i+1}$

$$(p_k + q_k + r_k) u_k = q_k u_{k-1} + r_k u_k + p_k u_{k+1}$$

$$\Rightarrow \quad 0 = p_k (u_{k+1} - u_k) - q_k (u_k - u_{k-1})$$

$$= p_k x_{k+1} - q_k x_k$$

It's comforting r_k drops out, because its not in our expression for the solution to u_i.

§ 3.6 p1 p.2

Our expressions for x_k are

$$x_2 = \frac{q_1}{p_1} x_1 \qquad x_3 = \frac{q_2}{p_2} x_2 = \frac{q_2}{p_2}\frac{q_1}{p_1} x_1 \quad \cdots \quad x_N = \frac{q_{N-1}}{p_{N-1}} x_{N-1} = \prod_{j=1}^{N-1} \frac{q_j}{p_j}$$

Using boundary cond to find μ_k;

$$x_1 = \mu_1 - \mu_0 = \mu_1 - 1$$

$$x_2 = \mu_2 - \mu_1 \quad \Rightarrow \quad x_1 + x_2 = \mu_2 - 1$$

$$\vdots$$

$$x_k = \mu_k - \mu_{k-1} \quad \Rightarrow \quad \sum_{i=1}^{k} x_k = \mu_k - 1$$

$$\vdots$$

$$x_N = \mu_N - \mu_{N-1} = -\mu_{N-1}$$

$$\mu_k = 1 + x_1 + \cdots + x_k \qquad \text{set } \rho_i = \prod_{j=1}^{i} \frac{q_j}{p_j}$$

$$\mu_k = 1 + x_1 + \rho_1 x_1 + \cdots + \rho_{k-1} x_1$$

$$\mu_N = 1 + x_1 \left(\sum_{j=1}^{N-1} \rho_j \right) = 0 \quad \Rightarrow \quad x_1 = \frac{-1}{1 + \sum_{j=1}^{N-1} \rho_j}$$

$$\Rightarrow \mu_k = 1 - [\rho_1 + \rho_2 + \cdots + \rho_k] / \left(1 + \sum_{j=1} \rho_j \right)$$

Simplifying

$$\mu_k = \left[1 + \sum_{k=1}^{N-1} \rho_k \right] - [\rho_k + \rho_2 + \cdots \rho_{k-1}] / 1 + \sum_{j=1}^{N-1} \rho_j$$

$$= (\rho_k + \cdots + \rho_{N-1}) / \left[1 + \sum_{j=1} \rho_j \right]$$