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OSU ST 6540 Homework #5

§4.4 #1

$$\mathbb{P} = \begin{array}{c} \\ \end{array}\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}$$

$$\mathbb{P} = \begin{bmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

This is a success runs M.C. which restarts when there are 4 successes in a row.

$p+q=1$ First we note all states communicate and since $\exists\, i \ni P_{ii} > 0$ we know the M.C. is regular; therefore the limiting distribution exists and is given by $\pi^+ = \pi^+ P$

$\pi^+ = \pi^+ P$ $\qquad \sum \pi_i = 1$

$\pi_0 = \pi_0 q + \pi_1 q + \pi_2 q + \pi_3 q + \pi_4$ ← won't use this

$\pi_1 = p\pi_0$

$\pi_2 = p\pi_1 = p^2\pi_0$ $\qquad \Rightarrow$

$\pi_3 = p\pi_2 = p^2\pi_1 = p^3\pi_0$

$\pi_4 = p\pi_3 = p^2\pi_2 = p^3\pi_1 = p^4\pi_0$

Using $\sum \pi_i = 1$

$\pi_0 + \pi_0 p + \pi_0 p^2 + \pi_0 p^3 + \pi_0 p^4 = 1$

$$\pi_0 = \frac{1}{1+p+p^2+p^3+p^4}$$

$$\pi_i = \frac{p^i}{1+p+p^2+p^3+p^4} \qquad i = 1,2,3,4$$

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§ 4.4 E 2

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & P_1 & & \\ & P_2 & & \\ & P_3 & & \end{bmatrix} \qquad T = \min_{n \geq 0} X_n = 0$$

$\mu_0 = 1$

$$u_i = P(\; X_T = 0 \mid X_0 = i\;) \qquad i = 1, 2, 3$$

$\cancel{\mu_0 = 0 \text{ vs } \mu_0 = 1 \text{ in}}$ absorbing case

$$v_i = E[\,T \mid X_0 = i\,] \qquad i = 1, 2, 3$$

We have to consider 3 starting states

$$v_1 = 1 + P_{11} v_1 + P_{12} v_2 + P_{13} v_3$$

$$v_2 = 1 + P_{21} v_1 \;\; P_{21} v_2 + P_{23} v_3$$

$$v_3 = 1 + P_{31} v_1 + P_{32} v_2 + P_{23} v_3$$

we solve these for v_i

$$\begin{bmatrix} (P_{11}-1) & P_{12} & P_{13} \\ P_{21} & (P_{22}-1) & P_{23} \\ P_{13} & P_{23} & (P_{33}-1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

we solve this in Matlab. See attached code + output.

$$v_1 = 950/161 = 5.906$$

(Note the book has $v_1 = m_{10}$)

$$m_0 = 1 + v_1$$

$$1/_{1+v_1} = \frac{1}{1+5.906} = .1449 = \pi_0 \quad \text{- as expected.}$$

OSU ST 5640 HW5 4.4.E2

```
x1= [ .4-1 .2 .3 ];
x2= [ .2 .5-1 .1 ];
x3= [ .3 .4 -1 ];
P=[x1;x2;x3];
b=[-1 -1 -1];
u = linsolve( P , b' );
rats(u)
```

```
ans = 3×14 char array
    '    950/161   '
    '    860/161   '
    '    790/161   '
```

```
950/161
```

```
ans = 5.9006
```

```
1/(1+950/161)
```

```
ans = 0.1449
```

___/___/___

4.4 P1

$$P = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \end{array} \qquad 0 < \alpha \quad \beta < 1$$

$f_{ii}^{(n)}$ is the probability of first return to $0 \mid X_0 = 0$. The chain must look like this

$X_0 = 0, \; X_1 = 1, \; X_2 = 1, \; \dots X_{n-1} = 1, \; X_n = 0 \qquad X_k = 1 \; \forall \; 1 \le k \le n-1$

Now $P(X_0 = 0, X_1 = 1, \dots X_{n-1} = 1, X_n = 0) = P_{01} \, P_{11}^{(n-2)} \, P_{10}$

$$= \alpha \, (1-\beta)^{n-2} \beta$$

$f_{00}^{(1)}$ is probability of first return in one step

ie P_{00} so $f_{00}^{(1)} = 1 - \alpha$

m_0 = mean return time $= E[R_i \mid X_0 = i] = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$

$$m_0 = f_{00}^{(1)} + \sum_{n=2}^{\infty} n f_{ii}^{(n)} = (1-\alpha) + \sum_{n=2}^{\infty} n \alpha\beta \, (1-\beta)^{n-2}$$

$$= (1-\alpha) + \frac{\alpha\beta}{1-\beta} \sum_{n=2}^{\infty} (1-\beta)^{n-1}$$

$$= (1-\alpha) + \frac{\alpha\beta}{1-\beta} \Big[\sum_{n=1}^{\infty} n(1-\beta)^{n-1} - \underbrace{(1-\beta)^0}_{= 1} \Big] \qquad \text{Eq 1}$$

Let $\gamma = 1 - \beta$ and notice that $\gamma < 1$ so

$$\sum_{n=1}^{\infty} n \gamma^{n-1} = \frac{d}{d\gamma} \sum_{n=0}^{\infty} \gamma^n = \frac{d}{d\gamma} \frac{1}{1-\gamma} = \frac{+1}{(1-\gamma)^2}$$

4.4 p'

(P2)

putting this into our eq'

$$m_0 = (1-\alpha) + \frac{\alpha\beta}{1-\beta}\left[\frac{1}{(1-(1-\beta))^2} - 1\right]$$

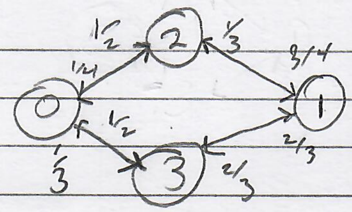
$$= (1-\alpha) + \frac{\alpha\beta}{1-\beta}\left[\frac{1}{\beta^2} - \frac{\beta^2}{\beta^2}\right]$$

$$= (1-\alpha) + \frac{\alpha}{1-\beta}\left[\frac{(1-\beta)(1+\beta)}{\beta}\right]$$

$$= (1-\alpha) + \frac{\alpha(1+\beta)}{\beta} = \frac{\beta(1-\alpha) + \alpha(1+\beta)}{\beta}$$

$$= \frac{\beta + \alpha}{\beta}$$

___/___/___ §4.4 p2



This is not a regular MC.

$\neg \exists\, i \ni P_{ii} > 0$

Thm 4.4 may apply - we know there is one recurrent class $C = \{0,1,2,3\}$ but need $\lim_{n\to\infty} P_{ii}^{(n)} > 0$. Looking at the block structure of P - we might not get this. $P^{(n)}$ has 0 along diagonal for all odd n

What if we just solve equations? $\sum \pi_i = 1$ $\pi_j = \sum \pi_i P_{ij}$

$\hookrightarrow$ Using Thm 4.3

$$P = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$

$$P = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \\ 1/4 & 3/4 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \end{bmatrix}\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

$\pi_0 = \pi_2 P_{20} + \pi_3 P_{30}$
$\pi_1 = \pi_2 P_{21} + \pi_3 P_{31}$
$\pi_2 = \pi_0 P_{02} + \pi_1 P_{01}$
$\pi_3 = \pi_0 P_{03} + \pi_1 P_{13}$
$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

$$\Rightarrow \begin{bmatrix} -1 & 0 & P_{20} & P_{30} \\ 0 & -1 & P_{21} & P_{31} \\ P_{03} & P_{13} & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

lets see what matlab says
(see next page)

$\pi = (.1507, .3493, .1918, .3082)$

$= (11/73, 51/146, 14/73, 45/146)$

OSU ST 6540 4.4 p2

```
P20=1/4;
P30 =1/3;
P21=3/4;
P31=2/3;
P03=1/2;
P13=2/3;
x1= [-1 0 P20 P30  ];
x2= [0 -1 P21 P31  ];
x3= [P03 P13 0 -1  ];
x4 = [1 1 1 1];
P=[x1;x2;x3;x4]
```

```
P = 4×4
   -1.0000         0    0.2500    0.3333
         0   -1.0000    0.7500    0.6667
    0.5000    0.6667         0   -1.0000
    1.0000    1.0000    1.0000    1.0000
```

```
b=[0 0 0 1]
```

```
b = 1×4
     0     0     0     1
```

```
u = linsolve( P , b' );
rats(u)
```

```
ans = 4×14 char array
    '     11/73    '
    '     51/146   '
    '     14/73    '
    '     45/146   '
```

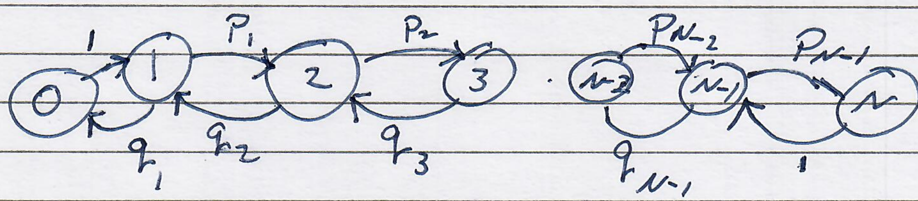
```
u
```

```
u = 4×1
    0.1507
    0.3493
    0.1918
    0.3082
```

§ 4.4 pblm 3

$$P = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & N-1 & N \end{array} \\ \left[\begin{array}{cccccc} 0 & 1 & 0 & & 0 & 0 \\ q_1 & 0 & p_1 & & & \\ 0 & q_2 & 0 & p_2 & & \\ & q_3 & 0 & p_3 & & \\ & & \ddots & q_{N-1} & 0 & p_{N-1} \\ & & & & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array}$$

0, N reflect



All states are communicating, but $\neg \exists\, i \ni P_{ii} > 0$

The chain is irreducible. Also - we don't know that P is not regular from the data given.

The period of the chain is not 1 so we can't ~~[illegible]~~ ~~[illegible]~~ conclude it's regular from the fact that it's finite state and irreducible.

There may be a stationary distribution which we can calculate using ~~[illegible]~~ $\sum \pi = 1$ $\pi_j = \sum_i \pi_i P_{ij}$

It makes sense that $\lim_{n\to\infty} P_{ij}^{(n)}$ might not converge for periodic chains - but we (I) can't say for sure that a limiting distribution does or does not exist yet.

using

$$\sum_{i=0}^{n} \pi_i = 1 \qquad \pi_j = \sum_{i=0}^{n} \pi_i P_{ij}$$

We have ① $\pi_0 = q_1 \pi_1$ ② $\pi_1 = \pi_0 + q_2 \pi_2$

③ $\pi_2 = p_1 \pi_1 + q_3 \pi_3$

$\vdots$

$\pi_i = p_{i-1} \pi_{i-1} + q_{i+1} \pi_{i+1}$

and

$\sum \pi_i = 1$

$\pi_1 = \pi_0 / q_1$

② $\Rightarrow \quad \pi_2 = \frac{1}{q_2} \pi_1 - \pi_0 = \frac{\frac{\pi_0}{q_1} - \pi_0}{q_2} = \frac{(1-q_1)\pi_0}{q_1 q_2} = \frac{p_1 \pi_0}{q_1 q_2}$

③ $\Rightarrow \quad \pi_3 = \frac{1}{q_3}\left(\pi_2 - p_1 \pi_1\right) = \frac{p_1 \pi_0 / q_1 q_2 - p_1/q_1 \, \pi_0}{q_3}$

$= \pi_0 \, p_1 \frac{\left(\frac{1}{q_1 q_2} - \frac{1}{q_1}\right)}{q_3} = \frac{\pi_0 p_1 (1-q_2)}{q_1 q_2 q_3}$

$= \pi_0 \frac{p_1 p_2}{q_1 q_2 q_3}$

$\vdots$

$\pi_N = \frac{p_1 p_2 \cdots p_{N-1}}{q_1 q_2 \cdots q_N} \pi_0$ Putting these

into $\sum \pi_i = 1$ we have

$$\pi_0 \left(1 + \frac{1}{q_1} + \frac{p_1}{q_1 q_2} + \frac{p_1 p_2}{q_1 q_2 q_3} + \cdots + \frac{p_1 \cdots p_{N-1}}{q_1 \cdots q_N}\right) = 1$$

$$\pi_0 = \frac{1}{\left[1 + \frac{1}{q_1} + \frac{p_1}{q_1 q_2} + \frac{p_1 p_2}{q_1 q_2 q_3} + \cdots + \frac{p_1 \cdots p_{N-1}}{q_1 \cdots q_N}\right]}$$ and finally

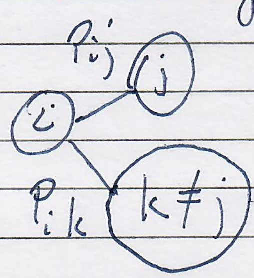
$$\pi_k = \frac{p_1 p_2 \cdots p_{k-1}}{q_1 \cdots q_k} \Big/ \left[1 + \frac{1}{q_1} + \frac{p_1}{q_1 q_2} + \cdots + \frac{p_1 \cdots p_{N-1}}{q_1 \cdots q_N}\right]$$

§ 4.4 p5

P transition matrix finite state MC. which is regular. $M = \| m_{ij} \|$

First - the book only defines m_{ii}. Let $T_j = \min\{ n \geq 1 ; X_n = j \}$ and consider $E[T_j \mid X_0 = i] = m_{ij}$. Then m_{ii} is as defined in the book, m_{ij} $i \neq j$ is not really a return time - its the first hitting time.

A ~~B~~ transition diagram helps with the first step setup



done

once at k we have expected wait m_{kj} ie m_{kj} steps are expected to get from k to j.

Taking into account that if $i \neq j$ at least one step is required, and all of the possible $k \neq j$ we have

$$m_{ij} = 1 + \sum_{k \neq j} P_{ik} \, m_{kj}$$

§ 4.4 P3

Taking $m_{ij} = 1 + \sum_{k \neq j} P_{ik} m_{kj}$

~~and~~ multiplying by π_i, and then suming over i

$$\sum_i \pi_i m_{ij} = \sum \pi_i \sum_{k \neq j} \sum_i \pi_i P_{ik} m_{kj}$$

now remembering that $\sum \pi_i = 1$ and

$\pi^T = \pi P_{ik} \Rightarrow \pi_i = \sum_i \pi_i P_{ik}$ we have

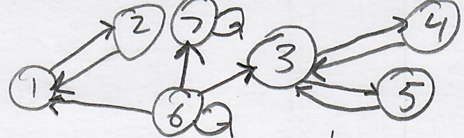
$\sum_i \pi_i m_i = 1 + \sum_{k \neq j} \pi_k m_{kj}$ subtracting

$$\sum_i \pi_i m_{ij} - \sum_{k \neq j} \pi_k m_{kj} = 1 \Rightarrow$$

$$\pi_j m_{jj} = 1$$

$\_\_\_|\_\_\_|\_\_\_$

ε^2



§4.5 $\quad S = \{$ $\{1,2\}$ recurrent aperiodic
$\{3,4,5\}$ recurrent period 3
$\{6\}$ transient
$\{7\}$ absorbing $\}$

For $i\,j \in \{1,2\}$ $\quad P_{ij}^{(n)}$ has a limit

$j \in \{3,4,5\} \Rightarrow P_{ij}^{(n)}$ does not have a limit, but time averages do converge.

$i = 6$, $j \in \{1,2\}$ we'll have to rely on first step analysis

$$\mathbb{P} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{cc|ccc|cc} 1/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 1/6 & 1/6 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

First we solve
$$\pi_1 = \pi_1 P_{11} + \pi_2 P_{21}$$
$$\pi_2 = \pi_1 P_{12} + \pi_2 P_{22}$$
$$\pi_1 + \pi_2 = 1$$
to get
$$\pi_1 = \tfrac{1}{3}\pi_1 + \tfrac{1}{4}\pi_2$$
$$\pi_1 = \tfrac{3}{8}\pi_2$$
$$\tfrac{3}{8}\pi_2 + \pi_2 = 1$$

$$\pi_1 = \frac{3}{11} \qquad \pi_2 = \frac{7}{11}$$

From which we conclude

$$\lim_{n\to\infty} P_{11}^{(n)} = \frac{3}{11} \quad \text{and} \quad \lim_{n\to\infty} P_{21}^{(n)} = \frac{3}{11}$$

(1) is not accessible from (3) and from this

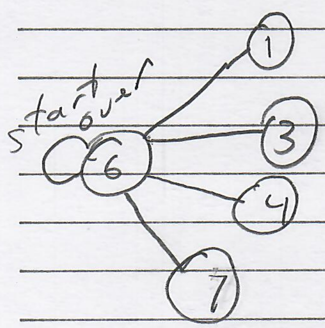
$$\lim_{n\to\infty} P_{31}^{(n)} = 0$$

§ 4.5 E2

Since 3 belongs to the periodic class

$\{3, 4, 5\}$ $\lim_{n \to \infty} P_{33}^{(n)}$ does not exist.

Now we do a first step analysis on 6



$P_{61}\ \pi_1$

out of time

might finish

before class

___/___/___

Problem From Class

$$p = \begin{array}{cc|c} C_1 & C_2 & \\ \left[\begin{array}{c|c} \mathbb{P}_1 & 0 \\ \hline 0 & \mathbb{P}_2 \end{array}\right] & & \begin{array}{c} C_1 \\ C_2 \end{array} \end{array}$$

$S = \{ \{C_1\}, \{C_2\} \}$

$|C_1| = n_1 \quad |C_2| = n_2$

all elemens of C_i communicate hence each C_i is positive recurrent.

$\pi^{(1)} \quad \pi^{(2)} \quad !$ stationary distribution for $\mathbb{P}_1 \quad \mathbb{P}_2$

a) $0 < \alpha < 1 \qquad \pi(\alpha) = (\alpha\pi^{(1)}, (1-\alpha)\pi^{(2)})$ is a stationary distribution for P. To show this we only need to demonstrate that $\pi(\alpha)$ satisfies $\sum \pi(\alpha)_i = 1$ and that $\pi(\alpha)^{\dagger} = \pi(\alpha)^{\dagger} P$

Well start with verifying $\sum \pi(\alpha) = 1$

$$\sum_i \pi(\alpha)_i = \sum_{C_1} \alpha \pi^{(1)} + \sum_{C_2} (1-\alpha)\pi^{(2)}$$

$$= \alpha\left(\sum_{C_1} \pi^{(1)}\right) + (1-\alpha)\sum_{C_2} \pi^{(2)} = \alpha + 1 - \alpha = 1$$

since both $\pi^{(1)}$ and $\pi^{(2)}$ are distributions on C_1, C_2. Next we'll write out $\pi(\alpha)^{\dagger} P$ in block form

$$[\underset{C_1}{\alpha\pi^{(1)\dagger}}, \underset{C_2}{(1-\alpha)\pi^{(2)\dagger}}] \begin{array}{cc} C_1 & C_2 \\ \left[\begin{array}{cc} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_2 \end{array}\right] \end{array}$$

$$= [\alpha\, \pi^{(1)\dagger} \mathbb{P}_1, (1-\alpha)\, \pi^{(2)\dagger} \mathbb{P}_2]$$

$$= [\alpha\pi^{(1)\dagger}, 1-\alpha\, \pi^{(2)\dagger}] = \pi^{\dagger}(\alpha)$$

the last step is ~~because~~ $\pi^{(1)}$ $\pi^{(2)}$ are stationary distributions

b) Suppose π is a stationary distribution for P, then

$\pi^+ = \pi^+ P$, and writing this out in block form over C_1 C_2

$$[\pi_{C_1}^+, \pi_{C_2}^+] = [\pi_{C_1}^+, \pi_{C_2}^+] \begin{bmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_2 \end{bmatrix} \qquad \text{eq } 1$$

$$= [\pi_{C_1}^+ \mathbb{P}_1, \pi_{C_2}^+ \mathbb{P}_2]$$

We now invoke the uniqeness of the stationary distributions for $\mathbb{P}_1$ $\mathbb{P}_2$ since they are positive recurrent to see that

$[\pi_{C_1}^+, \pi_{C_2}^+] = [\pi^{(1)}, \pi^{(2)}]$ we can insert a scale factor in eq 1 without affecting the argument i.e

$$[\alpha \pi_{C_1}^+, (1-\alpha) \pi_{C_2}^+] = [\alpha \pi^{(1)}, (1-\alpha) \pi^{(2)}]$$

i.e. any stationary distribution for P will have this form.