Bruce Campbell

M ST 6540 HW#1

1.2 E6

$X \perp Y$ r.v. $\bar{c}$ distribution fn $F_X(x)$ $F_Y(y)$

a) Let $Z = \max\{X, Y\}$, then

$P(Z \le z) = F_Z(z) = P(\max\{X, Y\} \le z)$

$= P(X \le z \cap Y \le z)$ and since $X \perp Y$

$P(Z \le z) = P(X \le z)\, P(Y \le z) = F_X(z)\, F_Y(z)$

b) Let $W = \min\{X, Y\}$

$P(W \le w) = P(\min\{X, Y\} \le w) = 1 - P(\min\{X, Y\} > w)$.

Now $P(\min\{X, Y\} > w) = P(X > w)\, P(Y > w)$ since $X \perp Y$. Remembering that $P(X > w) = 1 - F_X(w)$ we see

$P(W \le w) = F_W(w) = 1 - (1 - F_X(w))(1 - F_Y(w))$

1.2. P13

Let π be a (random) permutation of $\{1, \dots 13\}$

Let $N = \{k : \pi(k) = k\}$ be the number of matches (fixed points). Let A_k be the event $\pi(k) = k$. $A_i \cap A_j = \emptyset \ \forall \ i \neq j$ ie $\{A_k\} \perp$

Now we can write $N = \mathbb{1}\{A_1\} + \dots \mathbb{1}\{A_{13}\}$

Use linearity of expectation and that $P(A_k) = 1/13 \ \forall \ k$ to get

$$E[N] = \sum_{k=1}^{13} E[\mathbb{1}\{A_k\}] = \sum_{k=1}^{13} \left[1 \cdot P(A_k) + 0 \cdot P(A_k^c)\right]$$

$$= \sum_{k=1}^{13} 1/13 = 1.$$

Notice $P(N = k) =$ ways to choose k fixed points $\times$ proportion permutations on $N-k$ with <u>no</u> fixed points

↑
derangements

1.2. P13

$X \perp Y \sim U[0,1]$ let $U = \max\{X, Y\}$ $V = \min\{X, Y\}$

Find joint pdf of U, V.

First $P(U \le u) = F_U(u) = P(\max\{X,Y\} \le u)$

$= P(X \le u \cap Y \le u) = P(X \le u) P(Y \le u)$

$= F_X(u) F_Y(u) = u^2$ since $F_X(x) = \int_0^x 1\, dv = x$ and

$F_Y(y) = \int_0^y 1\, dv$.

Similarly $F_V(v) = P(V \le v) = 1 - P(V > v)$

$= 1 - P(\min\{X,Y\} > v) = 1 - P(X > v) P(Y > v)$

$= 1 - (1 - F_X(v))(1 - F_Y(v)) = 1 - (1-v)^2$

U, V are not $\perp$

Try direct approach;

$P(V \le v, U \le u) = F_{V,U}(v,u) = P(U \le u) - P(V > v, U \le u)$ We partitioned $\Omega = \{V > v\} \cup \{V \le v\}$

and used law of total probability.

Now $P(U \le u) = P(X \le u, Y \le u) = F_X(u) F_Y(u)$ since

$X \perp Y$ $P(U \le u) = u^2$. $P(V > v, U \le u) =$

$P(\min\{X,Y\} > v, \max\{X,Y\} \le u) =$

$P(v < X \le u \cap v < Y \le u) = (F_X(u) - F_X(v))(F_Y(u) - F_Y(v))$

$= (u - v)^2$ from our calculations above

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1.2. p13 p2

$$F_{U,V}(u,v) = u^2 - (u-v)^2 = \int_{\alpha=0}^{\alpha=u} \int_{\beta=0}^{\beta=v} f_{U,V}(\alpha,\beta)\, d\alpha\, d\beta$$

differentiating both sides gives us $f_{U,V}(u,v)$

$$\frac{\partial^2}{\partial u \partial v} u^2 - (u-v)^2 = \frac{\partial^2}{\partial u \partial v} 2uv + v^2$$

$$= \frac{\partial}{\partial u} 2u + 2v = 2$$

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3rd edition of book has typo here!

1.3.8 $X \sim \text{Binom}(N,p) \quad Y \sim \text{Binom}(M,p)$

let $Z = X+Y$

Then since $X = \sum_{i=1}^{N} B_i \quad B_i \sim \text{Bernoulli}(p)$ and

$Y = \sum_{i=1}^{M} B_i$ we see that

$$Z = \sum_{i=1}^{M} B_i + \sum_{i=1}^{N} B_i = \sum_{j=1}^{M+N} B_i \sim \text{Binomial}(M+N, p)$$

Now $f_X(n) = \binom{N}{n} p^n (1-p)^{N-n} \quad f_Y(m) = \binom{M}{m} p^m (1-p)^{M-m}$

and $$f_Z(k) = \sum_{i=-\infty}^{\infty} f_X(k-i) f_Y(i) = \sum_{i=0}^{z} f_X(k-i) f_Y(i)$$

where we used the fact that $X, Y \geq 0$ in the limits. Now

$$f_Z(z) = \sum_{i=0}^{z} \binom{N}{i} p^i (1-p)^{N-i} \binom{M}{z-i} p^{z-i} (1-p)^{M-(z-i)}$$

$$= \sum_{i=0}^{z} \binom{N}{i}\binom{M}{z-i} p^z (1-p)^{M+N-z}$$

We can stop here because $p^z(1-p)^{M+N-z}$ is the kernel of a Binomial $(M+N, p)$ r.v.

Fun fact $\binom{N}{i}\binom{M}{z-i}$ is kernel of hypergeometric r.v. and we can conclude $\sum_{i=0}^{z} \binom{N}{i}\binom{M}{z-i} = \binom{N}{z}$

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1.3, p11

$X \perp Y \sim \text{geom}(\pi)$ $\quad U = \min\{X, Y\}$ $\quad V = \max\{X, Y\}$

$W = V - U$

$f_{U,W}(u, w) = P(U = u, W = w) = P(\min\{X, Y\} = u, W = w)$

Now look at events $A = \{X = u, Y > u, W = w\}$

$B = \{Y = u, X > u, W = w\}$ and our edge case

$C = \{X = u, Y = u, W = 0\}$

$A = \{X = u, Y > u, W = w\} = \{X = u, Y = w + u\}$

$B = \{Y = u, X > u, W = w\} = \{Y = u, X = w + u\}$

Since $X \perp Y$ and $A \cap B = \emptyset$ for $w \neq 0$

$f_{U,W}(u, w) = P(\min\{X, Y\} = u, W = w) =$

$P(A) + P(B) = P(\{X = u, Y = w + u\}) + P(\{Y = u, X = w + u\})$

$= P(X = u) P(Y = w + u) + P(Y = u) P(X = w + u)$

$= p^u (1 - p) p^{w+u} (1 - p) + p^u (1 - p) p^{w+u} (1 - p)$

$= 2 p^u (1 - p)^2 p^{w+u}$

$f_{U,W}(u, 0) = P(C) = P(X = u, Y = u, W = 0) = P(X = u, Y = u)$

$= P(X = u) P(Y = u) = p^u (1 - p) p^u (1 - p)$

Now we can get $f_U(u)$ by summing or direct calculation from order stats

1.3. p11 p2

$$f_U(u) = \sum_{w=0}^{\infty} f_{U,W}(u,w) = f_{U,W}(u,0) + \sum_{w=1}^{\infty} f_{U,W}(u,w)$$

$$= p^{2u}(1-p)^2 + 2p^{2u}(1-p)^2\left(\sum_{w=1}^{\infty} p^w - 1\right)$$

$$= p^{2u}(1-p)^2 \left[1 + \frac{2p}{1-p}\right] = p^{2u}(1-p)(1+p)$$

$$f_W(w) = \sum_{u=0}^{\infty} f_{U,W}(u,w)$$

$$f_W(0) = \sum_{u=0}^{\infty} f_{U,W}(u,0) = (1-p)^2 \sum_{u=0}^{\infty} p^{2u} = (1-p)^2/(1-p^2)$$

$$f_W(w) = \sum_{u=0}^{\infty} f_{U,W}(u,w) = 2(1-p)^2 p^w \sum_{u=1}^{\infty} p^{2u}$$

$$= 2(1-p)^2 p^w \frac{1}{1-p^2}$$

$$f_{U,W}(u,0) = f_U(u) f_W(0)$$

Revisit !

$$f_{U,W}(u,w) = f_U(u) f_W(w)$$

1.4.E6

$U \sim Unif[0,1]$

a) $Y = -\ln(1-U)$ $\quad F_Y(y) = P(Y \le y)$

$= P(-\ln(1-U) \le y) = P(1-U \ge e^{-y})$

$= P(U < 1-e^{-y}) = \int_{x=0}^{x=1-e^{-y}} 1\,dx = 1-e^{-y}$

$\Rightarrow$

$f_Y(y) = e^{-y}$ $\leftarrow$ we recognize kernel of exp r.v.

b) $W_n = U^n$ $\quad F_{W_n}(w) = P(W_n \le w)$

$= P(U^n \le y) = P(U \le w^{1/n}) = \int_{x=0}^{x=w^{1/n}} 1\,dx$

$= w^{1/n}$ $\quad \Rightarrow \quad f_W(w) = \frac{1}{n} w^{\frac{1-n}{n}}$

Note both fn's $g(x) = x^n$ $g(x) = -\ln(1-x)$ are monotonic inc on $[0,1]$ so change of variable formula can be applied directly.

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1.4.7

$S, T \perp \quad S, T \sim \exp(\lambda)$ find

$f_R(r) \quad R = S + T$

$$f_R(r) = \int_{\eta=0}^{\eta=r} f_S(r-\eta) f_T(\eta)\, d\eta \qquad r \geq 0$$

$$f_T(\eta) = \lambda e^{-\lambda\eta} \qquad f_S(r-\eta) = \lambda e^{-\lambda(r-\eta)}$$

$$f_R(r) = \int_0^r \lambda e^{-\lambda(r-\eta)} \lambda e^{-\lambda\eta}\, d\eta$$

$$= \lambda^2 e^{-\lambda r} \int_0^r e^{\lambda n} e^{-\lambda n}\, d\eta = \lambda^2 e^{-\lambda r} r$$

Which we recognize as Gamma$(\lambda, 2)$

1.5.E2

X = # draws to select given chip $\bar{c}$ out replacement

$X \in \{1, 2, 3, 4\}$ $\qquad E[X] = \sum_{i=1}^{4} P(X > i)$

$P(X \geq 1) = P(\{X > 1\} \cup \{X = 1\}) = (1 - 1/4) + 1/4 = 1$ ~~(makes sense)~~

$P(X \geq 2) = P(\{X > 2\} \cup \{X = 2\}) = P(\{X > 2\}) + P(X = 2)$

$= [1 - (1 - 1/4)(1 - 1/3)] + [(1 - 1/4) \cdot 1/3]$

$P(X = 1) = 1/4$

$P(X = 2) = (1 - 1/4) \cdot 1/3 = 1/4$

$P(X = 3) = (1 - 1/4)(1 - 1/3) \cdot 1/2 = 1/8$

$P(X = 4) = 1/8$ by normalization

$E[X] = 1 \cdot 1/4 + 2 \cdot 1/4 + 3 \cdot 1/8 + 4 \cdot \frac{1}{8}$

$= \frac{3}{4} + \frac{3}{8} + \frac{1}{2} = 2\ 1/8$

Note I tried to use $E[X] = \sum_{i=1}^{4} P(X \geq i)$

but didn't find that easier.

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1.5.2 $X_1, \dots X_n$ i.i.d. $\exp(\lambda)$ r.v.

Find $F_Z(z)$ where $Z = \min\{X_1, \dots, X_n\}$

$$P(Z \le z) = 1 - P(Z > z) = 1 - P(X_1 \le z,\ X_2 \le z, \dots X_n < z)$$

$$P(X_1 > z, \dots X_n > z) = \prod_{i=1}^{n} P(X_i > z) = \prod_{i=1}^{n} [1 - F_{X_i}(z)]$$

$$1 - F_{X_i}(z) = 1 - \int_0^z \lambda e^{-\lambda u} du = 1 - (1 - e^{-\lambda z}) = e^{-\lambda z}$$

$$\Rightarrow \prod_{i=1}^{n} [1 - F_x(z)] = e^{-n\lambda} \qquad \Rightarrow$$

$$F_Z(z) = 1 - e^{-n\lambda z}$$

1.5.7 $X_i \sim \exp(\lambda_i)$ $i = 1, \dots, n$ find d.f. of

$V = \min\{X_1, \dots X_n\}$

Just as above

$$P(V \le v) = 1 - P(V > v) = 1 - P(X_1 > v,\ X_2 > v, \dots, X_n > v)$$

$$= 1 - \prod_{i=1}^{n} [1 - F_{X_i}(v)]$$

$$= 1 - \prod_{i=1}^{n} e^{\lambda_i v}$$

$$= 1 - e^{-(\sum \lambda_i) v}$$

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2.1.1

$M \sim Binom(N, p)$ $\qquad X \mid M \sim Binom(M, \pi)$

$$P(X=x) = \sum_{m=0}^{N} P(X=x \mid M=m) P(M=m)$$

$$P(M=m) = \binom{N}{m} p^m (1-p)^{N-m}$$ and

$$P(X=x \mid M=m) = \binom{m}{x} \pi^x (1-\pi)^{m-x}$$

Now $P(X=x \mid M=m) = P(X=x, M=m) / P(M=m)$

but we can't have more succeses than trials

so $P(X=x, M=m) = 0$ when $x > m$

Finally

$$P(X=x) = \sum_{m=x}^{N} \binom{m}{x}\binom{N}{m} p^m (1-p)^{N-m} \pi^x (1-\pi)^{m-x}$$

$$= \frac{\pi^x}{(1-\pi)^x} (1-p)^N \sum_{m=x}^{N} \binom{m}{x}\binom{N}{m} \left(\frac{p}{1-p}\right)^m (1-\pi)^m$$

We can follow the steps in the book to conclude

$$P(X=x) = \binom{N}{k} (\pi p)^x (1-\pi p)^{N-x} \sim Binom(N, \pi p)$$

I could not follow the last step to evaluate

$$\sum_{m=x}^{N} \frac{p^m}{(m-x)!\,(N-m)!} \left(\frac{1-\pi}{1-p}\right)^m$$

2.6.4

$X \sim Binom(N, p)$ $\qquad$ $X | N \sim Binom(N, p)$

$N \sim Binom(M, q) \Rightarrow X \sim Binom(M, pq)$

$E[X] = Mpq$ from above.

So if $M = 20$ $\quad p = 1/2$ $\quad q = 1/4$

$E[X] = 5/2$

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2.2.1 $\{X_i\}$ i.i.d. r.v. d.f. F

$$X_N = \min\{k \geq 1 : X_k > A\}$$

$$\alpha = P(\{X_1 > A\}) \qquad M = E[X_N]$$

a) Consider

$$E[X_N \mid X_1 = x] = \begin{cases} x & x > A \\ E[X_N] & x \leq A \quad \text{since } X_1 \perp X_k \end{cases}$$

Then we can write

$$E[X_N] = \int_{u=0}^{u=\infty} E[X_N \mid X_1 = u]\, dF(u)$$

$$= \int_{u=0}^{u=A} E[X_N]\, dF(u) + \int_{u=A}^{u=\infty} u\, dF(u)$$

$$= E[X_N](1-\alpha) + \int_{u=A}^{u=\infty} u\, dF(u)$$

Since $P(X_1 \leq A) = 1-\alpha$ and $P(X_1 \leq A) = \int_{u=0}^{A} u\, dF(u)$

b) Rearranging the above

$$M = \frac{1}{\alpha}\int_{u=A}^{u=\infty} u\, dF(u)$$

c) $X_i \sim \exp(\lambda) \Rightarrow$

and

$$M = 1/\alpha \int_{u=A}^{u=\infty} u\, \lambda e^{-\lambda u}\, du$$

2.2.p1 p2

To solve the integral note

$$\int_{u=0}^{u=\infty} u\lambda e^{-\lambda u}\, du = \int_{u=0}^{u=A} u\lambda e^{-\lambda u}\, du + \int_{u=A}^{u=\infty}$$

$$\int x e^{ax} = \frac{x e^{ax}}{a} - \frac{e^{ax}}{a^2}$$ using integration by parts if I recall

$$\int_{u=A}^{u=\infty} (u\lambda) e^{-(u\lambda)}\, du$$ set $z = u\lambda$

$$\int_{z=A\lambda}^{z=\infty} z e^{-z} \frac{dz}{\lambda} = \frac{z e^{-z}}{-1} - e^{-z} \Big|_{z=A\lambda}^{z=\infty}$$

$$= A\lambda e^{-A\lambda} + e^{-A\lambda}$$

Putting this in $M = \frac{1}{\alpha}\int_{u=A}^{u=\infty} u\lambda e^{-\lambda u}\, du$

$$M = \frac{1}{\alpha} e^{-A\lambda} (A\lambda + 1)$$

hmm... revisit

2.3.E1

$N \sim \text{unif}(1, 2, \ldots, 6)$ $\quad Z \mid N \sim \text{Binomial}(N, 1/2)$

Find $E[Z]$ $Var(Z)$ by looking at Z as random sum

$$Z = \left\{ \sum_{i=1}^{N} X_i \right. \qquad X_i \sim \text{Bernoulli}(1/2)$$

$$E[Z] = \sum_{n=1}^{6} E[Z \mid N=n]\, P(n)$$

$$= \sum_{n=1}^{6} E[\sum_{i=1}^{n} X_i \mid N=n]\, P(n)$$

$$= \sum_{n=1}^{6} n\, E[X_i] \cdot P(n) \qquad E[X_i] = 1/2$$

$\Rightarrow$

$P(n)\ 1/6$

$$E[Z] = \frac{1}{12} \sum_{n=1}^{6} n = 21/12 = 7/4$$

For the variance we'll follow the argument in the book let $E[X_i] = \mu = 1/2$ $Var(X_i) = \sigma^2 =$ $1/2\,(1 - 1/2) = 1/4$
(abusing notation we recycle

$E[N] = \nu = \frac{21}{6}$ $\quad Var[N] = \tau^2 = 35/12$

Then $Var(Z) = \nu\sigma^2 + \mu^2\tau^2 = \frac{21}{6} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{35}{12}$

≈ 1.60416

To find the pmf we need the n-fold convolution of X_i

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2.3.21 p2

$$f_Z(z) = \sum_{n=1}^{6} f^{(n)}(z)\, P_N(n)$$

But we know $P_N(n) = 1/6$ and that $f^{(n)}(z)$ is the Binomial $(n, 1/2)$ $\therefore$

$$f_Z(z) = \sum_{n=1}^{6} \binom{n}{z} \left(\frac{1}{2}\right)^z \left(\frac{1}{2}\right)^{n-z} \cdot \frac{1}{6}$$

and we can put in $n = 1, 2, 3, 4, 5, 6$ to get the values.

x x out of time to compute!

out of time for last 2 problems

x x

2.3. p1

$N \sim \text{Poisson}(\lambda)$ $\qquad Z \mid N \sim \text{Binomial}(N, p)$

$Z = \sum_{i=1}^{N} X_i \qquad X_i \sim \text{Bernoulli}(p)$

$$E[Z] = \sum_{n=0}^{\infty} E[Z \mid N=n]\, P_N(n)$$

$$= \sum_{n=0}^{\infty} E\left[\sum_{i=1}^{n} X_i \mid N=n\right] P_N(n)$$

Now $P_N(n) = \frac{\lambda^n e^{-\lambda}}{n!}$ and

$E\left[\sum_{i=1}^{n} X_i \mid N=n\right] = np \qquad \Rightarrow$

$$E[Z] = \sum_{n=0}^{\infty} np \frac{\lambda^n e^{-\lambda}}{n!} = p \sum_{n=0}^{\infty} n \frac{\lambda^n e^{-\lambda}}{n!}$$

$$= p\, E[\exp(\lambda)] = p\lambda$$

We will use book for var

$$Var(Z) = E[N]\, Var(X_i) + E[X_i]^2\, Var(\lambda)$$

$$= \lambda p(1-p) + p^2 \lambda = \lambda p$$

bet $Z \sim \text{Poisson}(\lambda p)$

7.3.p2

$Z \sim \text{Binomial}(N, p)$ $\qquad N \sim \text{Binomia}(M, q)$

$$Z = \sum_{i=1}^{N} X_i$$

$$f_Z(z) = \begin{cases} P_N(0) & N = 0 \\ \sum_{n=1}^{M} f_{X_i}^{(n)}(z)\, P_N(n) & \end{cases}$$

We know the form of $f_{X_i}^{(n)}(z)$ as

Binomial ($\cancel{n}$, p) so like before

$$P(Z = z) = \sum_{n=z}^{M} \binom{n}{z} p^z (1-p)^{n-z} \binom{M}{n} q^n (1-q)$$

again no time to sort out

simplifying this. I promise to revisit

2.3.4

ξ_i iid $\quad E[\xi_i] = \mu \quad Var(\xi_i) = \sigma^2$

$$S_N = \sum_{n=1}^{N} \xi_i$$

$$E[S_N] = E[\xi_i]\,E[N]$$

$$Var(S_N) = E[N]\,Var(\xi_i) + E[\xi_i]^2\,Var(N$$

If $N \sim Pois(\lambda)$

$$E[S_N] = \mu\lambda, \quad Var[S_N] = \lambda\sigma^2 + \mu^2\lambda$$

If $N \sim Geom(p)$

$$E[S_N] = \mu\frac{1-p}{p} \qquad Var[S_N] = \frac{1-p}{p}\sigma^2 + \mu^2\frac{1-p}{p^2}$$

$$= \mu\lambda \qquad\qquad = \lambda\sigma^2 + \mu^2\frac{\lambda}{p}$$

Comaring when $\lambda \to \infty$

we see that the meas stay the same but the variance in the $N \sim Geom(p)$ case is larger