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OSU ST 6540 HW#11

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DATE

6.4.E2

$X(t)$ BDP $\lambda_n = \alpha(N-n)$ $n = 0, 1, \dots N$

$\mu_n = \beta n$ We find the stationary distribution using 6.35 6.36 + 6.37

Setting $\theta_0 = 1$ $\theta_j = \dfrac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j}$ $j \geq 1$ $j \leq N$

Using $\pi_{j+1} = \theta_{j+1} \pi_0$

$\pi_1 = \theta_1 \pi_0 = \dfrac{\lambda_0}{\mu_1} = \dfrac{\alpha N}{\beta}$, $\pi_2 = \theta_2 \pi_0 = \dfrac{\alpha N \, \alpha(N-1)}{\beta \cdot 2\beta} \pi_0$

and $\pi_{n+1} = \dfrac{\alpha N \, \alpha(N-1) \cdots \alpha(N-n)}{\beta \cdot 2\beta \cdots (n+1)\beta} \pi_0$

$= \dfrac{\alpha^n \, N \cdot (N-1) \cdots (N-n)}{\beta^n \, (n+1)!} \pi_0$

$\pi_N = \dfrac{\alpha^N \, N!}{\beta^N (N+1)!} \pi_0$. We recognize

$\pi_{n+1} = \dfrac{\alpha^n}{\beta^n} \dfrac{N \cdot N-1 \cdots (N-n)}{(n+1)!} \cdot \dfrac{(N-(n+1)) \cdots (N-(N-1))}{(N-(n+1)) \cdots (N-(N-1))}$

$= \dfrac{\alpha^n}{\beta^n} \dfrac{N!}{(n+1)!(N-(n+1))!} \pi_0 = \dfrac{\alpha^n}{\beta^n} \binom{N}{n+1} \pi_0$

6.4.22

②

lol this derived for us in the book as a special case of the repairman model. following that same example for calculating π_0. Summing $\pi_0 = \theta_0 \pi_0, \ldots \pi_N = \theta_N \pi_0$ to get $\sum_{i=0}^{N} \pi_i = \pi_0 \sum_{i=0}^{N} \theta_i$ LHS is $\underline{1}$ due to $\underset{\sim}{\pi}$ being a probability distribution, so $\pi_0 = 1 / \sum_{i=0}^{N} \theta_i$. Now

$$\sum_{i=0}^{N} \theta_i = \sum_{i=0}^{N} \binom{N}{i} \left(\frac{\alpha}{\beta}\right)^i = \left(1 + \frac{\alpha}{\beta}\right)^N$$

from the binomial formula. So $\pi_0 = 1 / \left(1+\frac{\alpha}{\beta}\right)^N = \left(\frac{\beta}{\alpha+\beta}\right)^N$

and we see that

$$\pi_i = \frac{\alpha^i}{\beta^i} \binom{N}{i} \left(\frac{\beta}{\alpha+\beta}\right)^N = \binom{N}{i} \left(\frac{\alpha}{\beta}\right)^i \left(\frac{\beta}{\alpha+\beta}\right)^i \left(\frac{\beta}{\alpha+\beta}\right)^{N-i}$$

$$= \binom{N}{i} \left(\frac{\alpha}{\alpha+\beta}\right)^i \left(\frac{\beta}{\alpha+\beta}\right)^{N-i}$$

$$\underset{\sim}{\pi} \sim \text{Binom}\left(\frac{\alpha}{\alpha+\beta}, N\right)$$

☺ fun problem

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6.4.E5

$X(t)$ BDP ; $\lambda_n = \theta < 1$ $\mu_n = \frac{n}{n+1}$, we find the stationary distribution using 635 6.36 6.37

I <u>have</u> to set $\lambda_n = \alpha = \theta$ so we're not confused with notation.

$$\theta_0 = 1 \qquad \theta_j = \frac{\overbrace{\alpha \cdot \cdots \alpha}^{j\text{-times}}}{\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{j}{j+1}} = \frac{\alpha^j}{1/j+1} = (j+1)\,\alpha^j$$

$\pi_j = \alpha^j (j+1)\,\pi_0$. Like before

$$\pi_0 = 1/\sum_{i=0}^{\infty} \theta_i \qquad \sum_{i=0}^{\infty} \theta_i = \sum_{i=0}^{\infty} (i+1)\,\alpha^i$$

Now we look for a better way to express sum.

Hoping the ∞ binomial formula helps;

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \binom{2+k-1}{k} x^k = \sum_{k=0}^{\infty} \binom{k+1}{k} x^k$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)!}{k!\,(k+1-k)!}\, x! = \sum_{k=0}^{\infty} (k+1)\, x^k$$ is

what we need.

6.4.E5

(2)

Using our result; $\pi_0 = \frac{1}{\sum_{i=1}^{\infty} \theta_i} = \frac{1}{\frac{1}{(1-\alpha)^2}} = (1-\alpha)^2$

$\pi_j = \alpha^j (j+1)(1-\alpha)^2$

We don't recognise this as the kernel of any discrete distribution encountered so far.

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8.1.E1

$B(t) : t \geq 0$ S.B.M.

$P(B(4) \leq 3 \mid B(0) = 1) =$

$P(B(4) - B(0) \leq 3 - 1)$

We know $B(s+t) - B(s) \sim N(0, t\sigma^2)$

and $\sigma^2 = 1$ for S.B.M. letting $s = 0$

$t = 4$ $P(B(4) - B(0) \leq 2)$

$= \Phi_Z\left(\frac{2}{\sqrt{4}}\right)$ where Φ_Z is

CDF of $N(0,1)$ r.v.

$\Phi_Z(1) \cong .8413$

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8.1.21

2

$$P(\, B(9) - B(0) > c - 1\,) =$$

$$P(\, B(9) > c \mid B(0) = 1\,) = 0.10$$

$$\Rightarrow P(\, B(9) - B(0) \le c - 1\,) = 1 - .10 = .90$$

We seek $c \ni$

$$\Phi_Z\left(\frac{c-1}{\sqrt{9}} \right) = .90 \Rightarrow$$

$$\frac{c-1}{3} \approx 1.282 \qquad c \approx 4.846$$

(tried to use table got close to answer in back)

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8.1.P4

$\alpha_i \in \mathbb{R}$ $i=1,\dots n$. $B(t)$ S.B.M.

- lets consider increments

$B(t_i) - B(0)$, they are $\perp$ and

$N(0, t)$ distributed. Now we

write $E[\cancel{\alpha} \sum_{i=1}^{n} \alpha_i (B(t_i) - B(0))]$

$= \sum \alpha_i \; E[B(t_i) - B(0)] = 0$ since

$\cancel{E[\alpha}$ $B(t_i) - B(0) \sim N(0, t_i)$ $\forall i$

but since $B(0)=0$ we also have

$E[\sum_{i=1}^{n} \alpha_i B(t_i)] = 0$. Writing out the

$Var(\sum_{i=1}^{n} \alpha_i B(t_i)) = E[(\sum_{i=1}^{n} \alpha_i B(t_i))^2] - E[\sum \alpha_i B(t_i)]^2$

$= E[(\sum \alpha_i B(t_i))^2]$ from our result above.

Continuing using derivation of 8.10

as inspiration

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8.1. P4

2

$Var(\sum \alpha_i B(t_i)) =$

$E\left((\alpha_1 B(t_1) + \alpha_2 B(t_2) + \cdots \alpha_n B(t_n)) (\alpha_1 B(t_1) + \cdots + \alpha_n B(t_n)) \right)$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \alpha_j \; E[B(t_i) B(t_j)]$$

considering $\forall$ i,j the cases

$t_i < t_j$ $\quad t_j < t_i$ we have

$$* \quad Var(\sum \alpha_i B(t_i)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \min\{t_i, t_j\}.$$

Writing $E[B(t_i) B(t_j)]$ for the case

$t_i < t_j$ $\quad E[B(t_i) B(t_j)] = E[B(t_i)(B(t_j) - B(t_i) + B(t_i))]$

$= E[B(t_i)^2] + E[B(t_i)(B(t_j) - B(t_i))]$

$= E[B(t_i)^2] + E[B(t_i)] \, E[B(t_j) - B(t_i)]$

by $\perp$ increments but $E[B(t_i)] = 0$

since $B(t_i) = B(t_i) - B(0) \sim N(0, t_i)$

$E[B(t_i)B(t_j)] = t_i$ $\quad$ if $t_j < t_i$ we'll have

$E[B(t_i)B(t_j)] = t_j$ $\quad$ hence $*$