Two-dimensional Tunnel

You live in the anime world which is, of course, two-dimensional. You are trying to transport a polygon via a tunnel. Given all the points of the polygon, can you calculate the minimum height of the tunnel?

First Thoughts

Minimizing the height of the tunnel.
How? Rotating the polygon and find the minimum required height.
Polygon: $((x_1, y_1), (x_2, y_2), \ldots)$
Rotated: $((x'_1, y'_1), (x'_2, y'_2), \ldots)$

First Thoughts

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Polygon: $((x_1, y_1), (x_2, y_2), \ldots)$

Rotated: $((x'_1, y'_1), (x'_2, y'_2), \ldots)$

Minimal height = $\max(y') - \min(y')$

Heuristic Searching

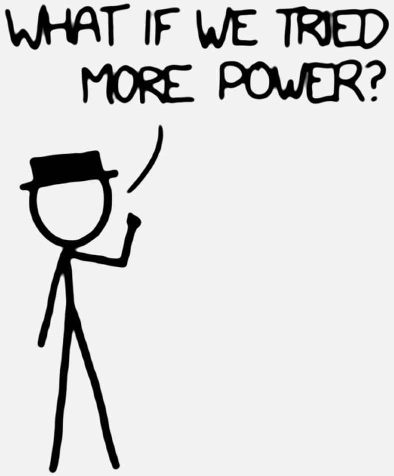
Polygon: $((x_1, y_1), (x_2, y_2), ...)$

Rotated: $((x'_1, y'_1), (x'_2, y'_2), ...)$

Minimal height = $\max(y') - \min(y')$

Simple heuristic search:

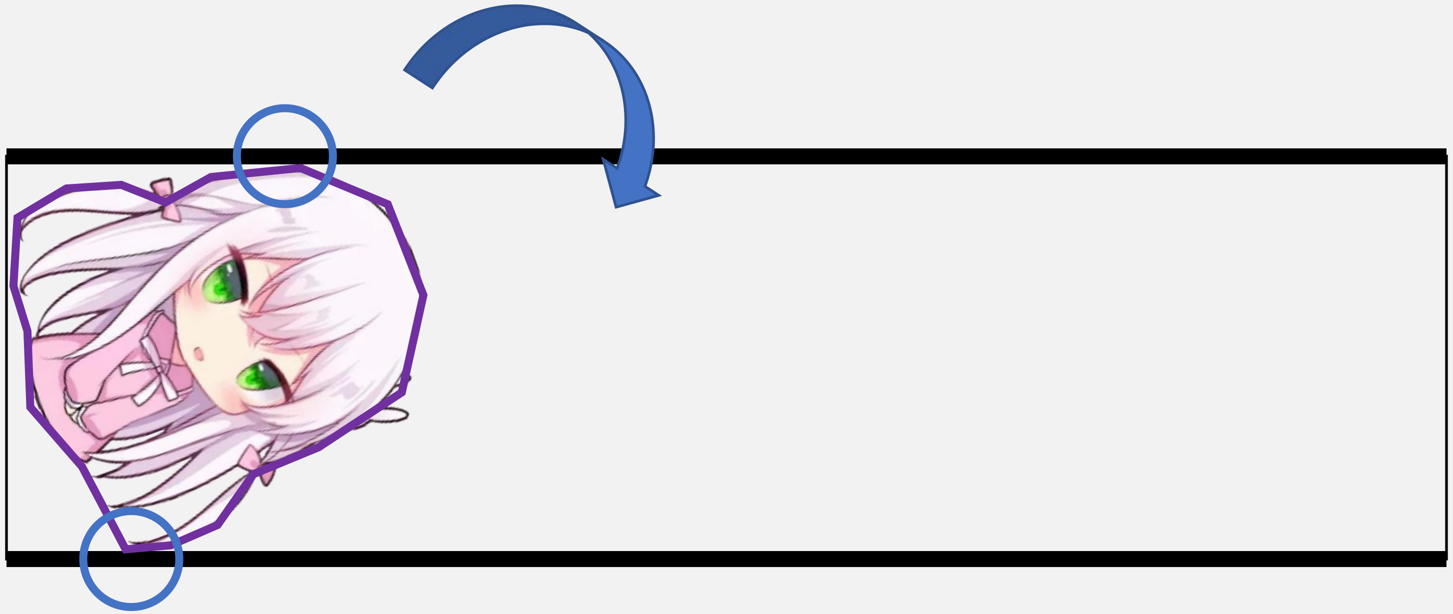
- Divide $[0, \pi)$ into n sections
- Select k best sections
- Divide them recursively


WHAT IF WE TRIED MORE POWER?

Accurate Solution?


OK

There should be 3 contact points between the polygon and the tunnel.



If there are 2 contact points, we can rotate the polygon.

Solution

For each pair of points (x_i, y_i), (x_j, y_j):

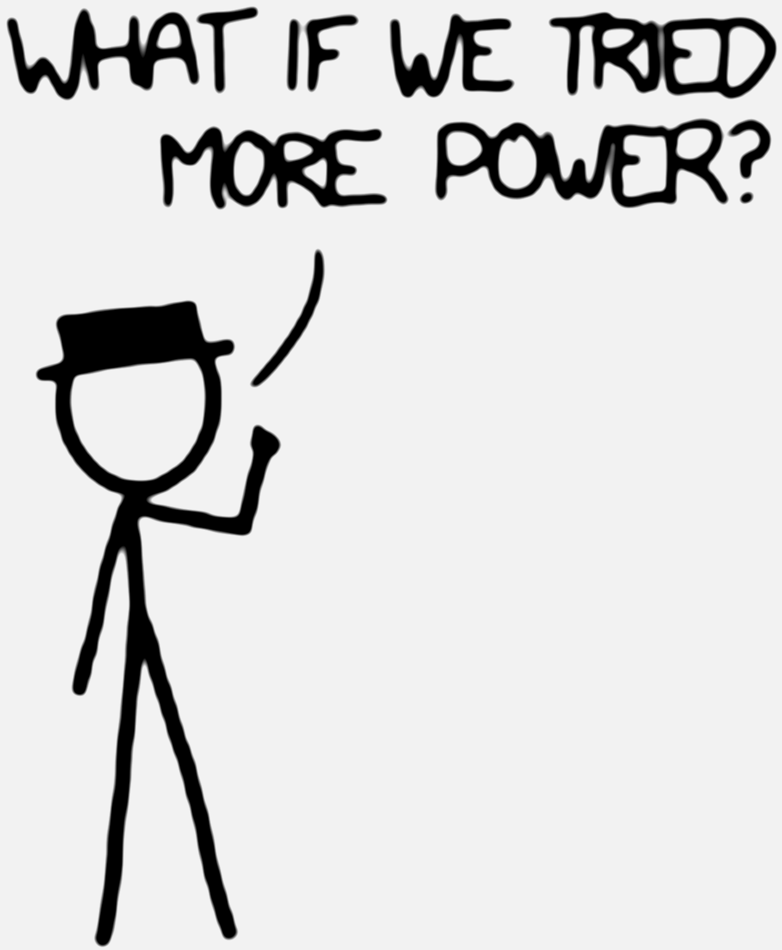
- Rotate the polygon so that $y'_i == y'_j$
    - $V = (x_j - x_i, y_j - y_i)$
    - $y' = ((x - x_i, y - y_i) \text{ cross } V) / |V|$
- Height = max(y') - min(y')
- If the new height is smaller
    - Update the minimal height

Analysis

If there are n points, it will loop n * (n - 1) / 2 times.

Then, n points will be rotated per each loop iteration.

Total time complexity: $O(n^3)$


WHAT IF WE TRIED MORE POWER?

Finding the Convex Hull

Solution

For each segment $(x_i, y_i), (x_{i+1}, y_{i+1})$:

- Rotate the polygon so that $y'_i == y'_{i+1}$
    - $V = (x_{i+1} - x_i, y_{i+1} - y_i)$
    - $y' = ((x - x_i, y - y_i) \text{ cross } V) / |V|$
- Height = max(y') - min(y')
- If the new height is smaller
    - Update the minimal height

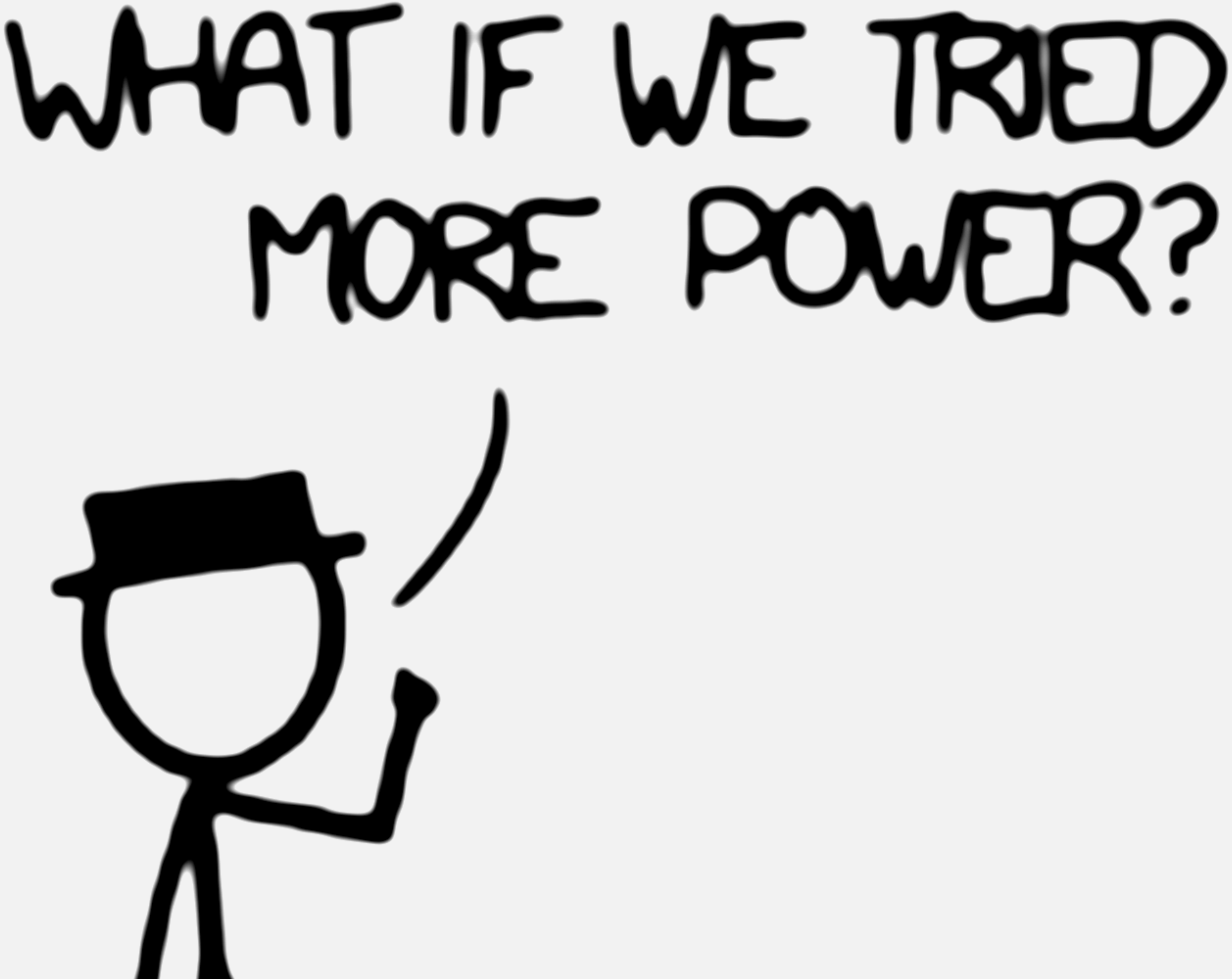
Analysis

If there are n points, it will take $O(n \log(n))$ to find the Convex Hull.

There are at most n points on it.

Then, n points will be rotated per each loop iteration.

Combined time complexity: $O(n^2)$


WHAT IF WE TRIED MORE POWER?

Solution

For each segment $(x_i, y_i), (x_{i+1}, y_{i+1})$:

- Move the pointer
    - Until we found the farthest point
    - Let it be (x_k, y_k)
- Height = distance between the segment and the point

Analysis

If there are n points, it will take O(n log(n)) to find the Convex Hull.
The pointer will move at most 2n times.
So, the loop is in linear time.
Combined time complexity: O(n log(n))

2D Convex Hull

Algorithm: Graham Scan
Time complexity: O(n log(n))

https://commons.wikimedia.org/wiki/File:GrahamScanDemo.gif

Graham Scan

- Find the point P_0 with smallest y_0
- Sort each point P based on the direction of (P_0, P) (forming a loop)
- Make a stack with P
- For each point, pop points not on the convex hull and then push itself
- Loop until it goes back to P_0