de Rham Cohomology

and a crash course on differential geometry

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Credits

- Introduction to Smooth Manifolds, John M. Lee
- "de Rham Theorem", Marco Perez (http://www1.mat.uniroma1.it/people/piazza/deRham-thm.pdf)

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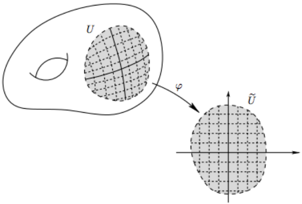
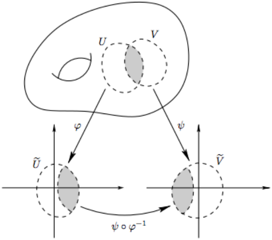
- A smooth manifold is a manifold who's transition maps, $\phi_a \circ \phi_b^{-1}$, are smooth maps (as defined for maps between R^n)

Smooth Manifolds

Visualization





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- Define $C^\infty(M, N)$ to be the set of smooth maps from M to N. $C^\infty(M) := C^\infty(M, \mathbb{R})$

Tangent Space
Definition

- let $p \in M$ A derivation at p is a linear map $X : C^\infty(M) \to R$ such that $X(f \cdot g) = X(f) \cdot g(p) + f(p) \cdot X(g)$

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- All derivations of R^n are of this form. if $v = v^i e_i$ then $D_v f(p) = v^i \frac{\partial f}{\partial x^i}(a)$, so the derivations $\frac{\partial}{\partial x^i}$ form a basis for T_pR^n

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- Likewise if $\left\{x^i\right\}$ are local coordinates for a point $p \in M$ then the derivations $\frac{\partial}{\partial x^i}$ form a basis for T_pM
- Thus one can think of the tangent space as the directional derivatives on a smooth manifold.

Tangent Space

Push Forward

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- If one looks at a coordinate neighborhood of p, one will find that the pushforward is none other than the jacobian of the map $\phi F \phi^{-1}$.

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- We can pushforward sections on M to N by applying F_*, however, these maps aren't always sections on N

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- Analogous to the tangent bundle and the space of sections, define cotangent bundle and it's sections $T^* M = \coprod_{p \in M} T_p^* M$ and $\mathfrak{T}^*(M)$
- For local coordinates $\left\{x_p^i\right\}$, to the basis $\frac{\partial}{\partial x^i}$ of $T_p M$, there is a dual basis $\left\{\epsilon^i\right\}$ for $T_p^* M$.

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Pullback

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- Unlike the pushforward, the pullback of a section of N will be a section of M
- note that $(F \circ G)^* = G^* \circ F^*$ and that $Id^* = Id_{T^*_p M}$ thus $T^* : Man^\infty \to Vect_R$ is a contravariant functor

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Differential Map

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- A generalization of this differential map to tensor spaces is the main object of study for de Rham Cohomology.

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- note that $T^1(V) = V^*$

- if $T \in T^k(V), S \in T^{k'}(V)$ define $T \otimes S \in T^{k+k'}(V)$ by $T \otimes S(X_1, ..., X_{k+k'}) = T(X_1, ..., X_k) \cdot S(X_{k+1}, ..., X_{k+k'})$

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Tensor Product

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- If $\{e_i\}$ is a basis for V and $\{\epsilon_j\}$ is the dual basis for V^*, then $\epsilon_{j_1} \otimes ... \otimes \epsilon_{j_k}\}$ is a basis for $T^k(V)$. Thus $T^k(V) \cong \underbrace{V^* \otimes ... \otimes V^*}_{k-times}$

Tensors

Tensor space and bundle

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- For the basis $\{\epsilon^i\}$ of V^*, and an ordered k-tuple I, define the elemetary alternating k-tensor ϵ^I, by

$$\epsilon^I(X_1, ..., X_k) = \det \begin{pmatrix} \epsilon^{i_1}(X_1) & \cdots & \epsilon^{i_k}(X_1) \\ \vdots & & \vdots \\ \epsilon^{i_1}(X_k) & \cdots & \epsilon^{i_k}(X_k) \end{pmatrix}$$

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- then the set $\{\ \epsilon^I |$ I is increasing$\}$ is a basis for $\Lambda^k(V)$
- some simple consequences of this is that $\Lambda^k(V) = 0$ if $k > dim(V)$ and that $\Lambda^1(V) = T^1(V)$

- define $Alt : T^k(V) \to \Lambda^k(V)$ by $AltT(X_1, ..., X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (sgn\sigma) T(X_{\sigma 1}, ..., X_{\sigma k})$

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- The key properties of the wedge product are billinearity, associativity, anticommutativity ($\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$) and the two following formula:
$\epsilon^{i_1} \wedge ... \wedge \epsilon^{i_k} = \epsilon^I$
$\omega^1 \wedge ... \wedge \omega^k(X_1, ..., X_k) = det\omega^i(X_j)$

Differential Forms

- Define the space of alternating k-tensors at p, $\Lambda_p^k(M) = \Lambda^k(T_pM)$ and the bundle $\Lambda^k(M) = \coprod_{p \in M} \Lambda_p^k(M)$.

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- in a neighborhood around a point p, $\{dx^i\}$ is a basis for T^*_pM so a section $\omega \in \Omega^k(M)$ can be written locally as
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- in a neighborhood around a point p, $\{dx^i\}$ is a basis for T_p^*M so a section $\omega \in \Omega^k(M)$ can be written locally as $\omega = \omega_I \cdot dx^{i_1} \wedge ... \wedge dx^{i_k}$ (ω_I are smooth functions)
- Note:$\Omega^0(M) = C^\infty(M)$, $\Omega^1(M) = \mathfrak{T}^*(M)$ and $\Omega^k(M) = 0$ if $k > dim(M)$

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  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
  3)$d(d(\omega)) = 0$.

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- It is non-trivial to show that this map exists and is unique, the proof can be found on page 215 of Lee.
- if $\{x^i\}$ are local coordinates around p, there is an explicit formulation for the map d.
  $d(\sum_I' \omega_I dx^{i_1} \wedge ... \wedge dx^{i_k})=\sum_I' \left[\sum_i \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}\right]$

- if $d\omega = 0$, we call ω closed, and if there is an η such that $d\eta = \omega$ we say it is exact. If ω is exact, then it is closed, but the converse is not true. Measuring the extent of this failure is the point of de Rham Cohomology.

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- Why is this important? As the notation $d\omega$ suggests, we are able to integrate these forms. Evaluating the integrals of these forms is implicitly linked to the differential of said forms by means of stokes theorem
  $\int_M d\omega = \int_{\partial M} \omega$

- if $d\omega = 0$, we call ω closed, and if there is an η such that $d\eta = \omega$ we say it is exact. If ω is exact, then it is closed, but the converse is not true. Measuring the extent of this failure is the point of de Rham Cohomology.
- Why is this important? As the notation $d\omega$ suggests, we are able to integrate these forms. Evaluating the integrals of these forms is implicitly linked to the differential of said forms by means of stokes theorem
  $\int_M d\omega = \int_{\partial M} \omega$
- For example, if $\gamma : I \to M$ is a smooth curve, and ω a 1-form, then we can evaluate $\int_\gamma \omega$ by means of the fundamental theorem of calculus when ω is exact.

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- But this is easy, since $G^*d(fdx^{i_1} \wedge ... \wedge d^{i_k}) = G^*(df \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}) = d(f \circ G) \wedge d(x^{i_1} \circ G) \wedge ... \wedge d(x^{i_k} \circ G) = d((f \circ G)d(x^{i_1} \circ G) \wedge ... \wedge d(x^{i_k} \circ G)) = dG^*(fdx^{i_1} \wedge ... \wedge dx^{i_k})$

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- What's more is that they are actually invarient under homotopy equivalence, yes the *continous* homotopy equivalence. This means that de Rham cohomology is actually a topological invarient
- This is the first clue toward the central theorem of de Rham, that the singular cohomology of a smooth manifold is isomorphic to it's de Rham cohomology.

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- We will then find that $\tilde{h}$ is the chain homotopy we are looking for, that is: $\tilde{h}(d\omega) + d(\tilde{h}\omega) = F^*\omega - G^*\omega$

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- I'll do the second case, as it is more enlightening, the first case is rather simple however.

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- For $k \geq 1$, $H^k_{dR}(R^n) = 0$

de Rham Cohomology

Mayer-Vietoris Sequence

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\begin{tikzcd}
 & \mathcal{U} \arrow[dr, "k"] & \\
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consider the pullbacks i^*, j^*, k^*, l^* the following sequence is exact:

$$0 \to \Omega^k(X) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \bigoplus \Omega^k(V) \xrightarrow{i^* - j_*} \Omega(U \cap V) \to 0$$

de Rham Cohomology
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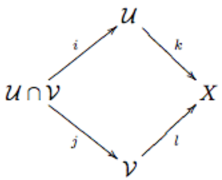

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Which it turn yields the long exact sequence

$$\ldots \xrightarrow{\Delta} H^k_{dR}(X) \xrightarrow{k^* \oplus l^*} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{i^* - j^*} H^k_{dR}(U \cap V) \xrightarrow{\Delta}$$

$$\xrightarrow{\Delta} H^{k+1}_{dR}(X) \to \ldots$$

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- It can be shown that the normal singular cohomology with coefficients in R is equivalent to the smooth singular cohomology on smooth manifolds.

- fix a k-form ω on M and a simplex σ. Pullback ω by σ and we have a k-form on Δ^K, define:
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de Rham's Theorem

$H(\Psi) : H_{dR}(M) \to H_\infty(M)$ is an isomorphism for all smooth manifolds