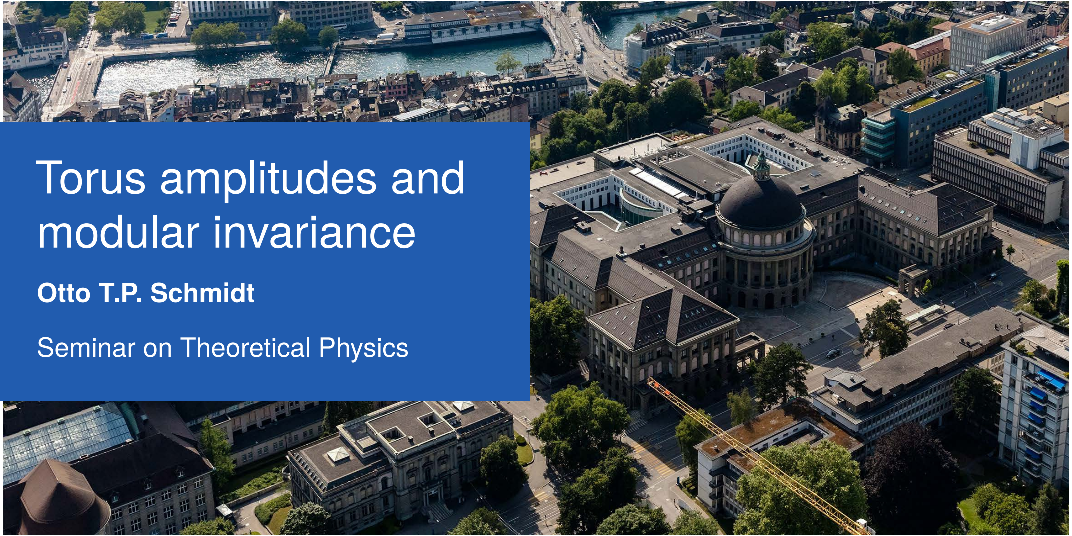
ETH zürich



Torus amplitudes and modular invariance

Otto T.P. Schmidt

Seminar on Theoretical Physics

Outline

1. Motivation
2. The moduli space of tori
   - One-loop open strings
   - Rectangular tori
   - General tori
   - Fundamental domain
3. Torus partition function
   - Single free boson in CFT
   - Modular invariance of partition function
   - Partition function via state trace
4. Modular invariance of the amplitude

Outline

Interactions and observables

In the study of string interactions, the ultimate goal will be the assignment of a probability for a certain process and the prediction of a physical cross section.

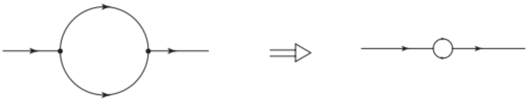
As outlined in Section 22, the computation of an observable cross section involves a series of steps:

1. Canonical representation of string diagram through moduli space
2. Compute scattering amplitude by means of conformal field theory
3. Convert scattering amplitude into a cross section

Loop amplitudes in string theory

In order to obtain accurate scattering amplitudes of processes, one needs to include contributions from loops in string diagrams.

These loops can be seen as contributions from the next higher order pertubation. Graphically we consider the following processes:



Ultraviolet divergence

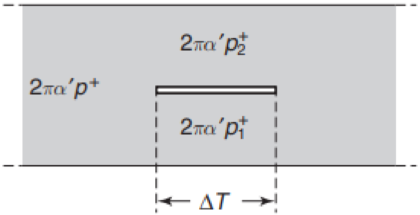
Amplitudes from virtual processes as depicted before can lead to ultraviolet (UV) divergences in quantum field theory (QFT).

Whereas QFT must employ complex renormalizations to deal with these UV divergences, we do not encounter these problems in string theory.

Outline

One-loop open strings

Before approaching the moduli space of tori, lets consider a one-loop open string with light-cone momentum p^+. This will serve as an intuitive analogon.
The light-cone diagram is:

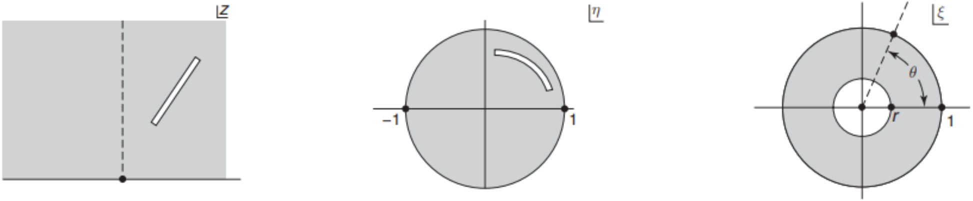


For fixed external momentum p^+ we find the two parameters: $\Delta T \in (0, \infty)$ and $p_1^+ \in (0, p^+)$.
$\rightarrow$ The class of Riemann surfaces of this process has two moduli.

Canonical annulus

Use $w = \tau + i\sigma$ and apply conformal transformations:

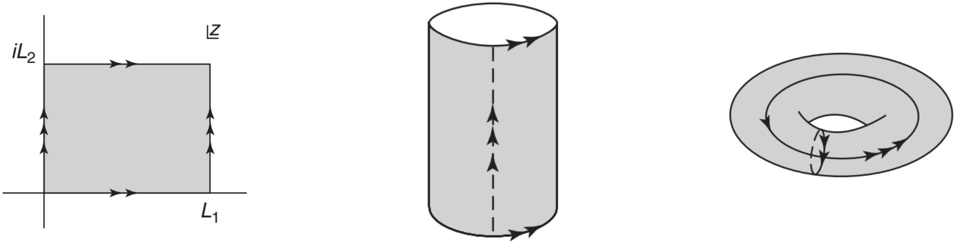
1. Exponential map: $z = exp[\frac{w}{2\alpha' p^+}]$
2. Linear fractional transformation: $\eta = \frac{1+iz}{1-iz}$
3. Canonical annulus: *A region in* $\mathbb{C}$ *that is topologically an annulus can be mapped conformally to a canonical annulus*



Rectangular tori

In order to apply the concept of moduli spaces to a torus, we need to assure that a torus is indeed a Riemann surface.

Consider a rectangular region of $\mathbb{C}$. By applying the analytic identifications $z \sim z + L_1$ and $z \sim z + iL_2$ we obtain a torus. This shows that the region remains a Riemann surface.
Graphically:



Parametrisation

We have:

Rectangular torus

$z \sim z + L_1$ and $z \sim z + iL_2$

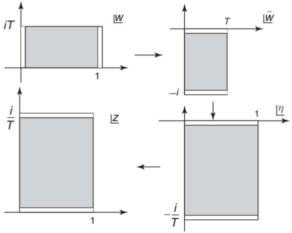
By applying $z' = \frac{z}{L_1}$ the identifications become:

Torus parameter T

$z' \sim z' + 1$ and $z' \sim z' + iT$ with $T = \frac{L_2}{L_1}$

Ultraviolet divergence

T is a parameter of the torus but does not yet define the moduli space, i.e. tori with different T can be conformally equivalent.
Consider the following series of conformal maps to a rectangular torus with $T < 1$:



Rectangular torus

Tori with T and $\frac{1}{T}$ are conformally equivalent
$\rightarrow$ The moduli space can be chosen to be $T \in (0, 1]$ or $T \in [1, \infty)$

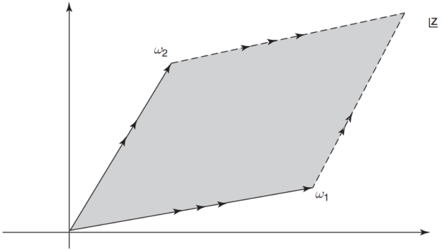
General tori

Rectangular tori represent only a subset of all conformally inequivalent tori. Let's construct a more general class of tori:

General construction of a torus as Riemann surface

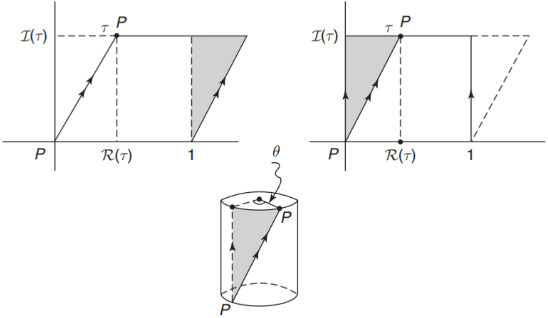
Choose $\omega_1, \omega_2 \in \mathbb{C}$ with $\mathrm{Im}(\frac{\omega_1}{\omega_2}) > 0$.
A torus is obtained by the indentifications $z \sim z + \omega_1$ and $z \sim z + \omega_2$.
By scaling we obtain $z \sim z + 1$ and $z \sim z + \tau$ with $\tau = \frac{\omega_2}{\omega_1}$, $\mathrm{Im}(\tau) > 0$.

$\rightarrow$ Note that for $\tau = iT$ ($\Leftrightarrow \mathrm{Re}(\tau) = 0$) we obtain the rectangular torus again.



Twisting the torus

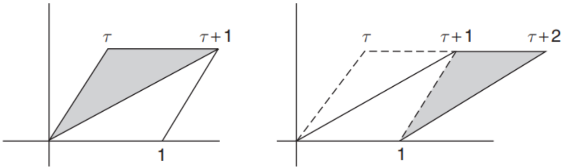
Intuitively, if a cylinder is twisted and the end surfaces are connected, we expect a different torus.
Formally: Consider $\mathrm{Re}(\tau) \neq 0$ and a point $P = 0 = \tau$. We can reconstruct the rectangular *fundamental domain* by using the identification $z \sim z + 1$.
Graphically:



Twisting parameter

The point P is no longer identified with a point on the perpendicular. Indeed the degree of twisting is parametrised by $\theta = 2\pi\,\mathrm{Re}(\tau)$. How does the twisting angle θ affect the torus parameter τ?

Consider the map $\tau \to \tau + 1$:



With the identification $z \sim z + 1$ we can conclude $\tau \sim \tau + 1$ and hence $\theta \sim \theta + 2\pi$.

Note:

The "twisting" does not correspond to actual torsion. It is the mere identification of the points P.

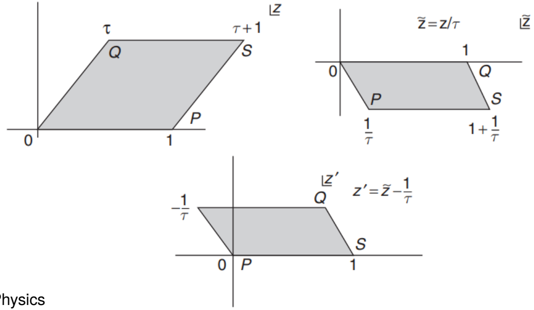
Moduli space I

So far we have established that $\mathrm{Im}(\tau) > 0 \Leftrightarrow \tau \in \mathbb{H}$. However, the identification $\tau \sim \tau + 1$ implies that the space of inequivalent tori is smaller. Indeed:

Strip $\mathcal{S}_0$

$\mathcal{S}_0 = \{-\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}, \mathrm{Im}(\tau) > 0\}$

Is $\mathcal{S}_0$ the moduli space of tori?
For rectangular tori we found that $T, \frac{1}{T}$ yield equivalent tori. Since $\tau = iT$ we have $\tau' = \frac{i}{T} = -\frac{1}{\tau}$.



Moduli space II

We have found two identifications with generators:

T-modular transform

$T\tau = \tau + 1$

S-modular transform

$S\tau = -\frac{1}{\tau}$

The corresponding fundamental domain should be a subset of $\mathcal{S}_0$. Indeed, the S-modular transform identifies points in $|\tau| < 1$ with points in $|\tau| > 1$.
Therefore we can postulate:

Fundamental domain $\mathcal{F}_0$

$\mathcal{F}_0 = \{-\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}, \mathrm{Im}(\tau) > 0, |\tau| \geq 1 \text{ and } \mathrm{Re}(\tau) \geq 0 \text{ if } |\tau| = 1\}$

Moduli space III

Why is $\mathcal{S}_0$ not the correct moduli space?

Consider the torus $\tau = i$. Via the T-modular transform we have:

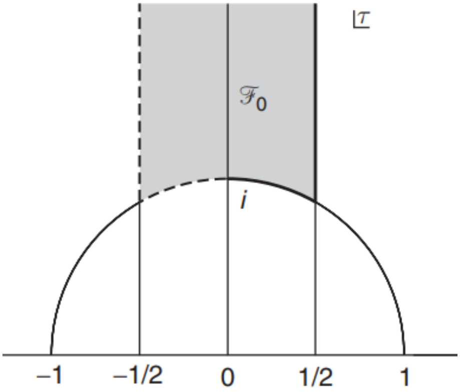
$$\tau_n = i + n \text{ for } n \geq 1 \tag{1}$$

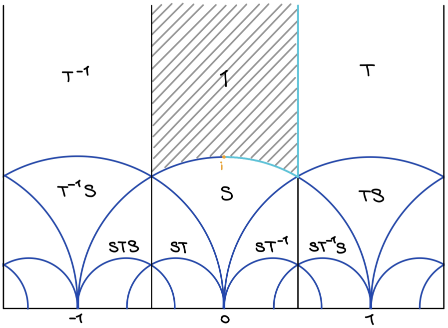
Using the S-modular transform we get:

$$-\frac{1}{\tau_n} = -\frac{n}{n^2+1} + \frac{i}{n^2+1} \tag{2}$$

These tori have $\mathrm{Re}(-\frac{1}{\tau_n}) \in [-\frac{1}{2}, 0]$ and $|-\frac{1}{\tau_n}| < 1$.
So they lie in $\mathcal{S}_0 - \mathcal{F}_0$.

$\rightarrow$ The complement of $\mathcal{F}_0$ in $\mathcal{S}_0$ therefore contains infinitely many copies of $\tau = i$. So we have to at least exclude this part of $\mathcal{S}_0$.


Im τ
$\mathcal{F}_0$
i
−1
−1/2
0
1/2
1


T⁻¹
1
T
T⁻¹S
S
TS
STS
ST
ST⁻¹
ST⁻¹S
i
-1
0
1

Modular group $PSL(2,\mathbb{Z})$

Consider a general linear fractional transformation $g \in G$:

$$g\tau = \frac{a\tau + b}{c\tau + d}, \;\; \mathrm{Im}(g\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \tag{3}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.
Equivalently we can use a matrix representation:

$$[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; \det[g] = 1 \tag{4}$$

$\rightarrow g$ satisfies the group homomorphism $\phi : G \rightarrow G$, $[g_1 g_2] \mapsto [g_1][g_2]$.
We call G the modular group $PSL(2,\mathbb{Z})$.

In matrix notation we see that $[T] = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $[S] = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

Proving $\mathcal{F}_0$

Let G' be the set of projective transformations generated by T- and S-modular transforms.

Claim

For all $\tau \in \mathbb{H}$ exists $g \in G'$ such that $g\tau \in \mathcal{F}_0$
$\rightarrow \mathcal{F}_0$ contains exactly one copy of each inequivalent torus.

Step 1: For each τ there is $g \in G'$ such that $\mathrm{Im}(g\tau)$ is largest.
Step 2: Show that $\tau' = T^n g\tau \in \mathcal{S}_0$ really is in $\bar{\mathcal{F}}_0$.
Step 3: Send $\tau \in \bar{\mathcal{F}}_0$ to $\tau \in \mathcal{F}_0$ via T- or S-transform.

Outline

Compactified bosonic string

Consider one compactified dimension $X \sim X + 2\pi r$. In a free field theory the action can be written as $S = \frac{1}{2\pi} \int \partial X \bar{\partial} X$.
Intuitively we expect the partition function to count all states of the single free boson.

For a compactified closed string a Euclidean path integral yields the following expression for the partition function (using $q = e^{2\pi i \tau}$):

$$Z_r(\tau, \bar{\tau}) = \int e^{-S} = \frac{1}{|\eta|^2} \sum_{m,n} q^{\frac{1}{2} p_L^2} \bar{q}^{\frac{1}{2} p_R^2} \tag{5}$$

where

$$p_L = \frac{m}{2r} + nr \text{ and } p_R = \frac{m}{2r} - nr \tag{6}$$

and $p = \frac{m}{r}, w = nr$.

In order to understand this result we need to remember that the Hilbert space of the single free boson on a compactified world-sheet is in fact a product state:

$$\mathcal{H} = \mathcal{H}_{osc.} \otimes \bigoplus_{p,w} \mathcal{H}_{p,w} \tag{7}$$

where $\mathcal{H}_{osc.}$ is the bosonic Fock space generated by the mode operators α_{-n} and $\bigoplus_{p,w} \mathcal{H}_{p,w}$ is the Hilbert space for different winding and momentum states.
Let's denote the vectors as

$$|N, p\rangle = \alpha_{-n_1} \dots \alpha_{-n_N} |p\rangle \text{ with } \sum n_i = N \tag{8}$$

We can therefore expect:

$$Z = Z_{osc.} \times \sum_{p,w} Z_{p,w} \tag{9}$$

So the partition function counts the oscillation modes of the string and the quantised momenta/winding.

Dedekind eta function

The *Dedekind eta function* is defined as:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) \tag{10}$$

It is a modular form with weight $\frac{1}{2}$ and obeys:

$$\eta(\tau + 1) = e^{i\frac{\pi}{12}}\eta(\tau) \text{ and } \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau) \tag{11}$$

T-modular transform

The partition function must remain invariant under a comformal transformation. Otherwise the inequivalent tori are not counted properly.

Under a T-modular transform the partition function we obtained via a Euclidean path integral behaves like:

$$\begin{aligned} Z_r(\tau+1,\bar{\tau}+1) &= \frac{1}{|\eta|^2}\sum_{m,n} q^{\frac{1}{2}(\frac{m}{2r}+nr)^2} e^{2\pi i \frac{1}{2}(\frac{m}{2r}+nr)^2} \bar{q}^{\frac{1}{2}(\frac{m}{2r}-nr)^2} e^{-2\pi i \frac{1}{2}(\frac{m}{2r}-nr)^2} \\ &= \frac{1}{|\eta|^2}\sum_{m,n} q^{\frac{1}{2}(\frac{m}{2r}+nr)^2} \bar{q}^{\frac{1}{2}(\frac{m}{2r}-nr)^2} e^{2\pi i mn} \\ &= Z_r(\tau,\bar{\tau}) \end{aligned}$$

S-modular transform

To show modular invariance under a S-modular transform we need to make use of the Poisson resummation:

Poisson resummation

Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function $f \in \mathcal{S}(\mathbb{R})$. Then:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \tag{12}$$

We can write the partition function more aptly:

$$Z_r(\tau,\bar{\tau}) = \frac{1}{|\eta|^2} \sum_{p_L\in\Sigma, p_R\in\bar{\Sigma}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} = \frac{1}{|\eta|^2} \sum_{(p,w)\in\mathbb{Z}^2} e^{-\pi \begin{bmatrix} p \\ w \end{bmatrix}^T A \begin{bmatrix} p \\ w \end{bmatrix}} \tag{13}$$

with $A = -\begin{bmatrix} \frac{1}{2r^2}\tau_2 & i\tau_1 \\ i\tau_1 & 2r^2\tau_2 \end{bmatrix}$

Using

$$\mathcal{F}(\exp\left\{-\pi \begin{bmatrix} p \\ w \end{bmatrix}^T A \begin{bmatrix} p \\ w \end{bmatrix}\right\}) = \det\{A\}^{-1} \exp\left\{-\pi \begin{bmatrix} p \\ w \end{bmatrix}^T A^{-1} \begin{bmatrix} p \\ w \end{bmatrix}\right\} \tag{14}$$

we see that under the S-modular transform the partition function becomes (assuming $\Sigma = \Sigma^*$):

$$\begin{aligned} Z_r(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) &= \frac{1}{|\tau||\eta|^2} \sum_{p_L \in \Sigma} e^{-\pi(\frac{1}{-i\tau})p_L^2} \sum_{p_R \in \bar{\Sigma}} e^{-\pi(\frac{1}{i\bar{\tau}})p_R^2} \\ &= \frac{1}{|\tau||\eta|^2} \sum_{p_L \in \Sigma^*} \sqrt{-i\tau} e^{i\pi\tau p_L^2} \sum_{p_R \in \bar{\Sigma}^*} \sqrt{i\bar{\tau}} e^{-i\pi\bar{\tau}p_R^2} \\ &= \frac{1}{|\tau||\eta|^2} |\tau| \sum_{p_L \in \Sigma, p_R \in \bar{\Sigma}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \\ &= Z_r(\tau, \bar{\tau}) \end{aligned}$$

T-duality

Consider again the momentum and winding for the compactified string:

$$p = \frac{m}{r} \text{ and } w = nr \tag{15}$$

Let's interchange $p \to w$.
So:

$$p'_L = \frac{w}{2} + p = \frac{m}{2r} + nr = p_L$$
$$p'_R = \frac{w}{2} + p = \frac{m}{2r} - nr = p_R$$

This change however leaves the partition function invariant. We can conclude T-duality for the compactified closed string partition function:

$$Z_r(\tau, \bar{\tau}) = Z_{\frac{1}{2r}}(\tau, \bar{\tau}) \tag{16}$$

Return to flat space

For $r \to \infty$ we get continuous momenta $p_L = p_R = \frac{m}{2r}$. The contribution from winding $w = nr$ leads to a fast oscillating factor.
With $q = e^{2\pi i \tau}$ we can transform back to flat space:

$$
\begin{aligned}
Z(r) &= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk q^{\frac{1}{2}k^2} \bar{q}^{\frac{1}{2}k^2} \\
&= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk e^{2\pi i \frac{1}{2}k^2 \tau} e^{-2\pi i \frac{1}{2}k^2 \bar{\tau}} \\
&= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk e^{2\pi i \frac{1}{2}k^2 2i \,\mathrm{Im}\{\tau\}} \\
&= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk e^{-2\pi k^2 \,\mathrm{Im}\{\tau\}} \\
&\propto \frac{1}{|\eta|^2} \frac{1}{\mathrm{Im}(\tau)^{\frac{1}{2}}}
\end{aligned}
$$

Partition function via state trace

A alternative approach to obtain the partition function is the state trace.

In quantum statistics:

$$Z = \mathrm{Tr}\Big(e^{-\beta H}\Big) \propto \sum_{\Phi} \langle\Phi|\, e^{-\beta H}\, |\Phi\rangle \tag{17}$$

The trace should run over all states of the torus. This means we need to find a propagator which propagates through all state configurations of the torus.

The partition function can be split up into a part corresponding to the oscillation modes of the world-sheet string and the momentum.

The momentum states are counted via:

$$\sum_{p_L \in \Sigma, p_R \in \bar{\Sigma}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \xrightarrow{r \to \infty} \int_{-\infty}^{\infty} dk q^{\frac{1}{2}k^2} \bar{q}^{\frac{1}{2}k^2} \propto \frac{1}{\mathrm{Im}(\tau)^{\frac{1}{2}}} \tag{18}$$

A possible way of counting the oscillation states is:

$$\sum_{\Phi} \langle \Phi | e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0} | \Phi \rangle = \mathrm{Tr}\Big(q^{L_0} \bar{q}^{L_0} \Big) \tag{19}$$

Is this a reasonable result? Two hints:

I:

$$\mathrm{Tr}\left(q^{L_0}\right) \propto \mathrm{Tr}\left(q^{\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n}\right) = \prod_{n=1}^{\infty}(1 + q^n + q^{2n} + \dots) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \tag{20}$$

This is the partition function of a boson from the Bose-Einstein distribution.
And:

$$\mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\right) = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \frac{1}{\eta(\tau)} \tag{21}$$

II:

$$\begin{aligned} q^{L_0} \bar{q}^{\bar{L_0}} &= e^{2\pi i(\tau_1 + i\tau_2)L_0} e^{-2\pi i(\tau_1 - i\tau_2)\bar{L_0}} \\ &= e^{2\pi i\tau_1(L_0 - \bar{L_0})} e^{-2\pi\tau_2(L_0 + \bar{L_0})} \end{aligned}$$

We can interpret $L_0 - \bar{L_0}$ as the momentum P which generates translation in σ. Similarly $L_0 + \bar{L_0}$ can be seen as Hamiltonian H which generates τ translation.
The propagator has then the interpretation of a τ and σ sweep.

Light-cone partition function

So far the partition function has been derived for a single free boson in one dimension:

$$Z(\tau,\bar{\tau})_{1d} = \frac{1}{|\eta(\tau)|^2}\frac{1}{(\mathrm{Im}\,\tau)^{\frac{1}{2}}} \tag{22}$$

The generalisation for 24 transverse dimensions in light-cone gauge is:

Light-cone partition function

$$Z(\tau,\bar{\tau})_{l.c.} = (\frac{1}{|\eta(\tau)|^2}\frac{1}{(\mathrm{Im}\,\tau)^{\frac{1}{2}}})^{24} = \frac{1}{|\eta(\tau)|^{48}}\frac{1}{(\mathrm{Im}\,\tau)^{12}} \tag{23}$$

Outline

The amplitude

The correct one-loop vacuum amplitude reads:

$$A_0^{g=1} \propto \int_{\mathcal{F}_0} \frac{d^2\tau}{4(\mathrm{Im}(\tau))^2} Z(\tau, \bar{\tau}) \tag{24}$$

This amplitude carries the correct physical intuition: each possible form of a one-loop interaction is parametrised by τ. To get a probability measure for a process we therefore need to weight every $\tau \in \mathcal{F}_0$ with the partition function $Z(\tau, \bar{\tau})$.

We can rewrite the amplitude using the light-cone Hamiltonian and momentum (with $\tau = \tau_1 + i\tau_2$):

$$A_0^{g=1} \propto \int_{\mathcal{F}_0} \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2} \int \frac{d^{24}p}{(2\pi)^{24}} \operatorname{Tr}\left(e^{-2\pi\tau_2 H_{l.c.}} e^{-2\pi i\tau_1 P_{l.c.}}\right) \tag{25}$$

Connection to moduli space and twisting angle

Consider $\tau_1 = 0$. In this case τ_2 plays the role of a Euclidean time. We know that $\tau_1 = 0$ corresponds to a rectangular torus.
The partition function counts the number of states propagating around the torus in τ_2 direction and weights them with $e^{-2\pi\tau_2 H_{l.c.}}$.

Now consider a cylinder with length τ_2 whose ends are identified. We can twist the ends by the twist angle $\theta = 2\pi\tau_1$. This twist is then induced by $P_{l.c.}$.

Modular invariance

The proposed amplitude is only valid if equivalent tori, i.e. equivalent interactions, result in the same amplitude.
We therefore need to have invariance under T- and S-modular transforms.
The integration domain $\mathcal{F}_0$ is by definition modular invariant.
Consider a general modular transformation g on the measure:

$$\begin{aligned} d(g\tau) &= [\frac{a(c\tau+d)-(a\tau+b)c}{(c\tau+d)^2}]d\tau \\ &= [\frac{ad-bc}{(c\tau+d)^2}]d\tau \end{aligned}$$

So $d^2\tau \to |c\tau+d|^{-4}d^2\tau$
Also:

$$\mathrm{Im}(g\tau) = |c\tau+d|^{-2}\,\mathrm{Im}(\tau) \tag{26}$$

Therefore the measure transforms like:

$$\frac{d^2\tau}{\mathrm{Im}(\tau)^2} \rightarrow \frac{|c\tau + d|^{-4} d^2\tau}{(|c\tau + d|^{-2}\,\mathrm{Im}(\tau))^2} = \frac{d^2\tau}{\mathrm{Im}(\tau)^2} \tag{27}$$

As previously shown, the partition function is modular invariant (T- and S-modular transform).

This makes the whole amplitude modular invariant.

Conclusion

We have set out to perform the steps:

1. Canonical representation of string diagram through moduli space
2. Compute scattering amplitude by means of conformal field theory

We have constructed the moduli space of the torus and with this knowledge defined a amplitude for one-loop processes.

With this knowledge we are able to compute contributions from second order pertubation terms to the scattering amplitude and hence make more precise predictions about the physical cross section of an interaction.

References

Zwiebach, B. (2004). A First Course in String Theory. Cambridge: Cambridge University Press. doi:10.1017/CBO9780511841682

Blumenhagen et. al. (2013). Basic Concepts of String Theory. Springer

Ginsparg, P. (1988). Les Houches, Session XLIX