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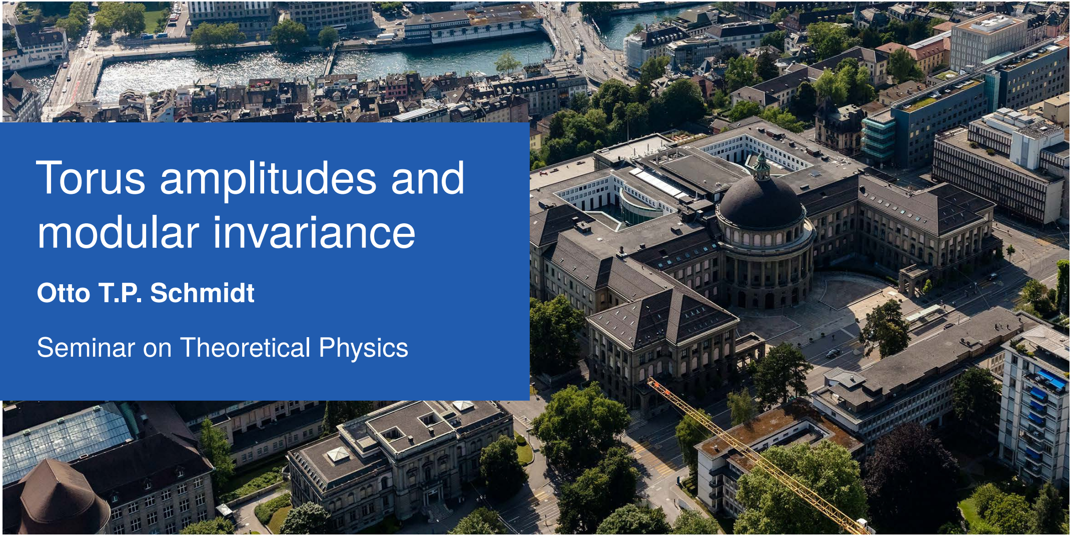

Torus amplitudes and modular invariance

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Seminar on Theoretical Physics

Outline

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Interactions and observables

In the study of string interactions, the ultimate goal will be the assignment of a probability for a certain process and the prediction of a physical cross section.

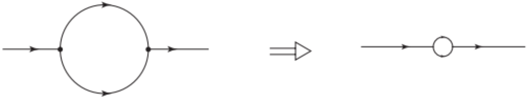
As outlined in Section 22, the computation of an observable cross section involves a series of steps:

1. Canonical representation of string diagram through moduli space
2. Compute scattering amplitude by means of conformal field theory
3. Convert scattering amplitude into a cross section

Loop amplitudes in string theory

In order to obtain accurate scattering amplitudes of processes, one needs to include contributions from loops in string diagrams.

These loops can be seen as contributions from the next higher order pertubation. Graphically we consider the following processes:



Ultraviolet divergence

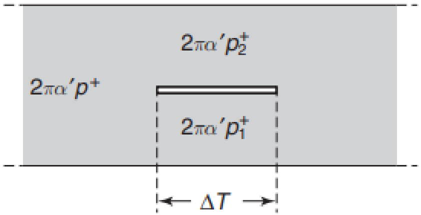
Amplitudes from virtual processes as depicted before can lead to ultraviolet (UV) divergences in quantum field theory (QFT).

Whereas QFT must employ complex renormalizations to deal with these UV divergences, we do not encounter these problems in string theory.

Outline

One-loop open strings

Before approaching the moduli space of tori, lets consider a one-loop open string with light-cone momentum p^+. This will serve as an intuitive analogon.
The light-cone diagram is:

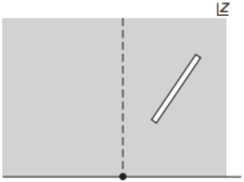


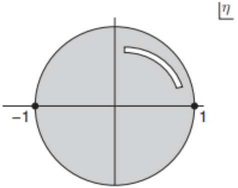
For fixed external momentum p^+ we find the two parameters: $\Delta T \in (0, \infty)$ and $p_1^+ \in (0, p^+)$.
$\rightarrow$ The class of Riemann surfaces of this process has two moduli.

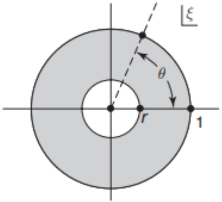
Canonical annulus

Use $w = \tau + i\sigma$ and apply conformal transformations:

1. Exponential map: $z = exp[\frac{w}{2\alpha' p^+}]$
2. Linear fractional transformation: $\eta = \frac{1+iz}{1-iz}$
3. Canonical annulus: *A region in $\mathbb{C}$ that is topologically an annulus can be mapped conformally to a canonical annulus*

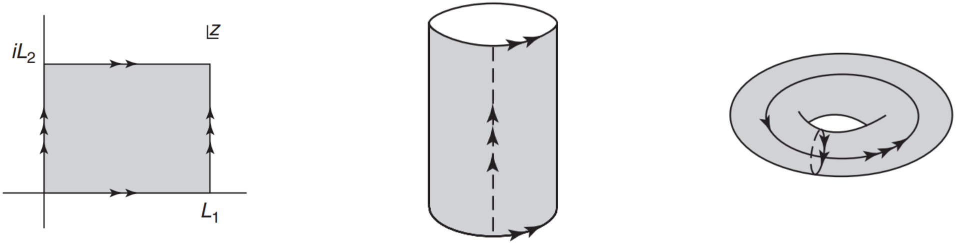






Rectangular tori

In order to apply the concept of moduli spaces to a torus, we need to assure that a torus is indeed a Riemann surface.
Consider a rectangular region of $\mathbb{C}$. By applying the analytic identifications $z \sim z + L_1$ and $z \sim z + iL_2$ we obtain a torus. This shows that the region remains a Riemann surface.
Graphically:



Parametrisation

We have:

Rectangular torus

$z \sim z + L_1$ and $z \sim z + iL_2$

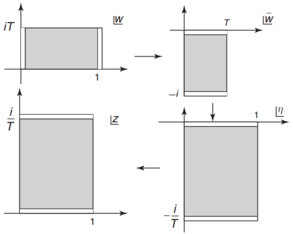
By applying $z^{'} = \frac{z}{L_1}$ the identifications become:

Torus parameter T

$z^{'} \sim z^{'} + 1$ and $z^{'} \sim z^{'} + iT$ with $T = \frac{L_2}{L_1}$

Ultraviolet divergence

T is a parameter of the torus but does not yet define the moduli space, i.e. tori with different T can be conformally equivalent.
Consider the following series of conformal maps to a rectangular torus with $T < 1$:



Rectangular torus

Tori with T and $\frac{1}{T}$ are conformally equivalent
$\rightarrow$ The moduli space can be chosen to be $T \in (0, 1]$ or $T \in [1, \infty)$

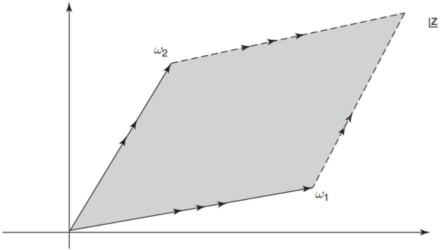
General tori

Rectangular tori represent only a subset off all conformally inequivalent tori. Let's construct a more general class of tori:

General construction of a torus Riemann surface

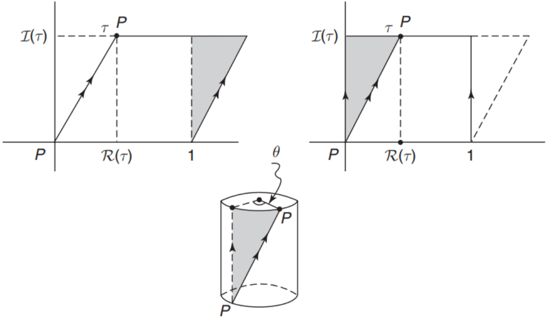
Choose $\omega_1, \omega_2 \in \mathbb{C}$ with $\mathrm{Im}(\frac{\omega_1}{\omega_2})$.
A torus is obtained by the indentifications $z \sim z + \omega_1$ and $z \sim z + \omega_2$.
By scaling we obtain $z \sim z + 1$ and $z \sim z + \tau$ with $\tau = \frac{\omega_2}{\omega_1}$, $\mathrm{Im}(\tau) > 0$.

$\rightarrow$ Note that for $\tau = iT$ ($\Leftrightarrow \mathrm{Re}(\tau) = 0$) we consider the rectangular torus.



Twisting the torus

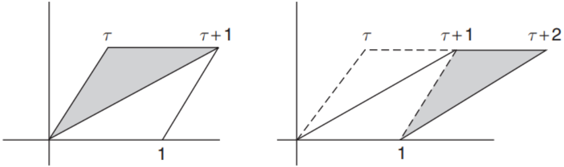
Intuitively, if a cylinder is twisted and the end surfaces are connected, we expect a different torus.
Formally: Consider $\mathrm{Re}(\tau) \neq 0$ and a point $P = 0 = \tau$. We can reconstruct the rectangular *fundamental domain* by using the identification $z \sim z + 1$.
Graphically:



Twisting parameter

The point P is no longer identified with a point on the perpendicular. Indeed the degree of twisting is parametrised by $\theta = 2\pi\,\mathrm{Re}(\tau)$. How does the twisting angle θ affect the torus parameter τ?

Consider the map $\tau \to \tau + 1$:



With the identification $z \sim z + 1$ we can conclude $\tau \sim \tau + 1$ and hence $\theta \sim \theta + 2\pi$.

Note:

The "twisting" does not correspond to actual torsion. It is the mere identification of the points P.

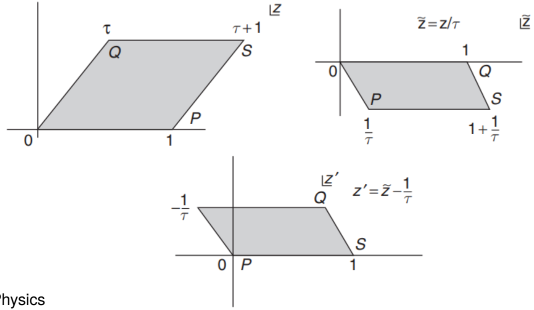
Moduli space I

So far we have established that $\mathrm{Im}(\tau) > 0 \Leftrightarrow \tau \in \mathbb{H}$. However, the identification $\tau \sim \tau + 1$ implies that the space of inequivalent tori is smaller. Indeed:

Strip $\mathcal{S}_0$

$\mathcal{S}_0 = \{-\frac{1}{2} < \mathrm{Re}(\tau) < \frac{1}{2}, \mathrm{Im}(\tau) > 0\}$

Is $\mathcal{S}_0$ the moduli space of tori?
For rectangular tori we found that $T, \frac{1}{T}$ yield equivalent tori. Since $\tau = iT$ we have $\tau' = \frac{i}{T} = -\frac{1}{\tau}$.



Moduli space II

We have found two identifications with generators:

T-module transform

$T\tau = \tau + 1$

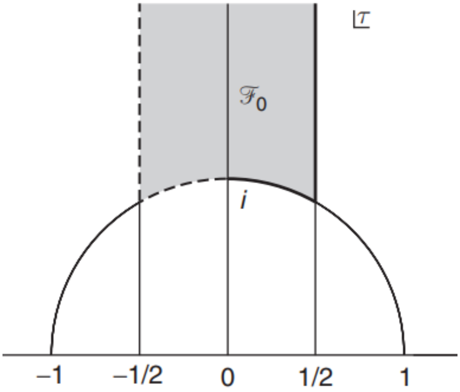
S-module transform

$S\tau = -\frac{1}{\tau}$

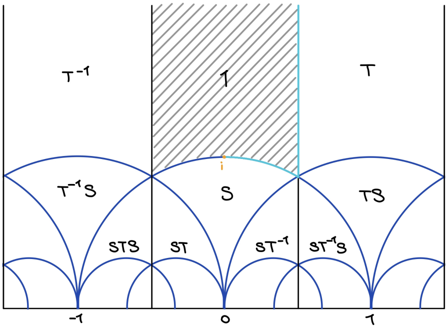
The corresponding fundamental domain should be a subset of $\mathcal{S}_0$. Indeed, the S-module transform identifies points in $|\tau| < 1$ with points in $|\tau| > 1$.
Therefore we can postulate:

Fundamental domain $\mathcal{F}_0$

$\mathcal{F}_0 = \{-\frac{1}{2} < \mathrm{Re}(\tau) < \frac{1}{2}, \mathrm{Im}(\tau) > 0, |\tau| \geq 1 \text{ and } \mathrm{Re}(\tau) \geq 0 \text{ if } |\tau| = 1\}$



τ
$\mathscr{F}_0$
i
–1
–1/2
0
1/2
1



T^{-1}
1
T
$T^{-1}S$
S
TS
i
STS
ST
ST^{-1}
$ST^{-1}S$
-1
0
1

Modular group $PSL(2,\mathbb{Z})$

Consider a general linear fractional transformation $g \in G$:

$$g\tau = \frac{a\tau + b}{c\tau + d}, \ \ \mathrm{Im}(g\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \tag{1}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.
Equivalently we can use a matrix representation:

$$[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det[g] = 1 \tag{2}$$

$\rightarrow g$ satisfies the group homomorphism $\phi : G \rightarrow G$, $[g_1 g_2] \mapsto [g_1][g_2]$.
We call G the modular group $PSL(2,\mathbb{Z})$.

In matrix notation we see that $[T] = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $[S] = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

Proving $\mathcal{F}_0$

Claim

For all $\tau \in \mathbb{H}$ exists $g \in G'$ such that $g\tau \in \mathcal{F}_0$
$\rightarrow \mathcal{F}_0$ contains exactly one copy of each inequivalent torus.

Step 1: For each τ there is $g \in G'$ such that $\mathrm{Im}(g\tau)$ is largest.
Step 2: Show that $\tau' = T^n g\tau \in \mathcal{S}_0$ really is in $\bar{\mathcal{F}}_0$.
Step 3: Send $\tau \in \bar{\mathcal{F}}_0$ to $\tau \in \mathcal{F}_0$ via T- or S-transform.

Outline

1. Motivation

2. The moduli space of tori
   - One-loop open strings
   - Rectangular tori
   - General tori
   - Fundamental domain

3. Torus partition function
   - Compactified and Minkowsi closed strings
   - T-duality
   - Partition function via state trace

4. Modular invariance of the amplitude

Compactified bosonic string

Consider one compactified coordinate $X \sim X + 2\pi r$. In a free field theory the action can be written as $S = \frac{1}{2\pi} \int \partial X \bar{\partial} X$

For such coordinates the boundary conditions read:

BC

$X_0(z + \tau, \bar{z} + \bar{\tau}) = X_0(z, \bar{z}) + 2\pi r n'$

and

$X_0(z + 1, \bar{z} + 1) = X_0(z, \bar{z}) + 2\pi r n$

The solution to the classical equations of motion reads:

$$X_0^{(n,n')}(z, \bar{z}) = \frac{2\pi r}{2i\tau_2}\left(n'(z - \bar{z}) + n(\tau \bar{z} - \bar{\tau} z)\right) \tag{3}$$

$\rightarrow$ one dimension is compactified

Partition function

Obtaining the space-time form:

$$p_L = \frac{m}{2r} + nr \text{ and } p_R = \frac{m}{2r} - nr \tag{4}$$

For $r \to \infty$ we get continuous momenta $p_L = p_R = \frac{m}{2r}$.
For a compactified closed string:

$$Z(r) = \int e^{-S} = \frac{1}{|\eta|^2} \sum_{m,n} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \tag{5}$$

Return to Minkowski space

For $r \to \infty$ with $q = e^{2\pi i \tau}$:

$$
\begin{aligned}
Z(r) &= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk q^{\frac{1}{2}k^2} \bar{q}^{\frac{1}{2}} \\
&= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk e^{2\pi i \frac{1}{2}k^2 \tau} e^{-2\pi i \frac{1}{2}k^2 \bar{\tau}} \\
&= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk e^{2\pi i \frac{1}{2}k^2 2i \operatorname{Im}\{\tau\}} \\
&= \frac{1}{|\eta|^2} \int_{-\infty}^{\infty} dk e^{-2\pi k^2 \operatorname{Im}\{\tau\}} \\
&\propto \frac{1}{|\eta|^2} \frac{1}{\operatorname{Im}(\tau)^{\frac{1}{2}}}
\end{aligned}
$$

T-duality

Consider again the momentum and winding for the compactified string:

$$p = \frac{m}{r} \text{ and } w = nr \tag{6}$$

We see that if we interchange $m \to n$ and $\frac{1}{r} \to r$, we interchange the meaning of winding and momentum.
This change however leaves the partition function invariant and we conclude T-duality.

Partition function via state trace

In quantum statistics:

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) \propto \sum_{\Phi} \langle\Phi| H |\Phi\rangle \tag{7}$$

The trace should run over all states of the torus. This means we need to find a propagator which propagates through all configurations in τ and σ. Lets postulate:

$$Z(\tau) \propto \mathrm{Tr}\left(q^{L_0}\bar{q}^{\bar{L_0}}\right) \tag{8}$$

with $q = e^{2\pi i\tau}$ and $L_0 = \sum \alpha_{-m}\alpha_m + \frac{1}{2}p_L^2$.

Is this a reasonable approach? Two hints:

I:

$$\mathrm{Tr}\left(q^{L_0}\right) \propto \mathrm{Tr}\left(q^{\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n}\right) = \prod_{n=1}^{\infty}(1 + q^n + q^{2n} + \ldots) = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{9}$$

This is the partition function of a boson from the Bose-Einstein distribution.

II:

$$\begin{aligned} q^{L_0}\bar{q}^{\bar{L_0}} &= e^{2\pi i(\tau_1 + i\tau_2)L_0} e^{-2\pi i(\tau_1 - i\tau_2)\bar{L_0}} \\ &= e^{2\pi i\tau_1(L_0 - \bar{L_0})} e^{-2\pi\tau_2(L_0 + \bar{L_0})} \end{aligned}$$

We can interpret $L_0 - \bar{L_0}$ as the momentum P which generates translation in σ. Similarly $L_0 + \bar{L_0}$ can be seen as Hamiltonian H which generates τ translation.
The propagator has then the interpretation of a τ and σ sweep.

The *Dedekind eta function* is defined as:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{10}$$

Therefore we get:

$$(q\bar{q})^{-\frac{1}{24}} \operatorname{Tr}\left(q^{L_0} \bar{q}^{\bar{L_0}}\right) = \frac{1}{|\eta|^2} \tag{11}$$

Outline

The amplitude

The correctly normalized one-loop vacuum amplitude reads:

$$A_0^{g=1} = \frac{V_{26}}{l_s^{26}} \int_{\mathcal{F}_0} \frac{d^2\tau}{4(\mathrm{Im}(\tau))^2} Z(\tau, \bar{\tau}) \tag{12}$$

This amplitude is carries the correct physical intuition: each possible form of a one-loop interaction is parametrised by τ. To get a probability measure for a process we therefore need to weight every $\tau \in \mathcal{F}_0$ with the partition function $Z(\tau, \bar{\tau})$

The partition function for the 24-dimensional transverse coordinate space is:

$$Z(\tau, \bar{\tau}) = (\mathrm{Im}(\tau))^{-12} |\eta(\tau)|^{-48} \tag{13}$$

Modular invariance