Econometrica, Vol. 48, No. 2 (March, 1980)

A GENERAL EQUILIBRIUM APPROACH TO MARXIAN ECONOMICS

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In the first part of the paper, a model is proposed which places the Marxian and Sraffian conceptions of a capitalist economy in a general equilibrium framework. A central concern of these writers is that the economy be *reproducible*; this is incorporated formally into the equilibrium definition. Capitalists maximize profits subject to a capital constraint and workers are paid a subsistence wage. Equilibrium existence theorems are proved.

In the second part, the welfare properties of the equilibria are examined—which, in the Marxian tradition, involve the notion of *exploitation*. It is shown that the possibility of exploitation is necessary and sufficient for all equilibria to sustain positive profits, if a certain technological condition holds. Finally, the notion of a subsistence bundle is dispensed with, and a Marxian determination of workers' consumption is proposed.

In addition to placing the formal Marxian model into a general equilibrium context, the specification of production here is more general than the usual Leontief or von Neumann technologies: production sets are assumed to be only convex.

INTRODUCTION

MARXIAN AND SRAFFIAN ECONOMIC MODELS differ from neoclassical general equilibrium models in a number of ways which make them appear to be at least less general and perhaps even incommensurable, in a logical sense, with the latter. They differ in at least these ways:

1. Formal Marxian and Sraffian models employ linear production sets, while neoclassical models are studied in a more general convex environment.
2. In Marxian models, competition among capitalists somehow leads to an "equilibrium" characterized by *equal profit rates* in all sectors, while in neoclassical models firms *maximize profits*.
3. Marxian and Sraffian analyses avoid the concept of *neoclassical general equilibrium* and consider instead a different solution concept which shall be called here *reproducibility*.
4. In neoclassical models, workers consume bundles of goods which they choose, while in Marxian models there is assumed to be a *subsistence bundle* of goods for workers.

The purpose of this paper is to show the commensurability of the two types of models in the sense that there exists a reasonable economic environment in which the Marxian solutions can be viewed as a type of general equilibrium. The Marxian analysis adds, among other things, a useful characterization of non-trivial equilibria by use of the exploitation concept. The program shall be, first, to justify the adoption of the Marxian equilibrium criterion of reproducibility; second, to show that the equilibrium concept here defined leads to the Marxian–Sraffian

[1] A discussion with Leonid Hurwicz gave rise to the investigation reported in this paper. I thank also J. P. Benassy and A. Mas-Colell for several helpful discussions and the referees of this journal for their useful comments. Research was partially supported by a grant from the Social Science Research Council (US); a version of this paper was first read at the 1978 European Econometric Society meetings in Geneva.

equal-profit-rate prices in a linear economy; third, to replace the linear production model with general convex production sets and to show that Marxian concepts are still definable and that Marxian solutions still exist; fourth, to define exploitation in the convex model and to prove a generalization of what Morishima calls the Fundamental Marxian Theorem for these economies; and fifth, to discard the usual assumption of Marxian analysis of an exogenously given subsistence bundle in favor of a mechanism which provides for a *social determination* of workers' necessary consumption.

In short, the paper attempts to place the Marxian model in a general equilibrium framework and to derive an important theorem of the Marxiam theory of value and exploitation.

1. THE MARXIAN LINEAR MODEL: AN EXAMPLE

Let us take the simplest case. Let A be an $n \times n$ physical input-output matrix, assumed to be indecomposable and productive. The row vector of direct labor inputs into production is L. There are no joint products, no alternative processes to produce the same output, and all processes take unit time. There is no fixed capital. Each worker has a subsistence bundle b, a column vector of commodities. We normalize prices taking the wage as 1. Then the Marxian equilibrium price vector is a vector p such that there exists a profit rate π such that:

$$\begin{aligned} (1.1) \qquad & pb = 1, \\ & p = (1+\pi)(pA+L). \end{aligned}$$

Labor values are defined as the row vector $\Lambda = \Lambda A + L$, and the rate of exploitation is $e = (1/\Lambda b) - 1$. It can be shown that a pair (π, p) satisfying (1.1) exists if and only if $e \geqslant 0$, and the pair (π, p) is unique. (For a discussion of these facts, see Morishima [**6**, Chapters 5, 6].)

How can the Marxian solution (π, p) be viewed as an equilibrium under the behavioral postulate of profit maximization? Let us suppose there are N capitalists, the νth one possessing capital in the form of an endowment vector of commodities. Capital must be advanced to operate production out of the value of his endowment. Thus, if capitalist ν faces prices p, he can choose to operate processes $1, \dots, n$ at levels $x^\nu = \langle x_1^\nu, \dots, x_n^\nu \rangle$ subject to his capital constraint:

$$(1.2) \qquad (pA+L)x^\nu \leqslant p\omega^\nu.$$

Notice there is no financial capital market: capitalists are limited in the extent of their production by the level of internal finance.[2] Time is essential in production, in the sense that capitalists must pay today for inputs before revenues are received tomorrow.

[2] In a sequel to this paper the author has added a capital market to this model, allowing capitalists to borrow from each other at a rate of interest, with the equilibrium condition that total net borrowings are zero (Roemer [**11**]). This does not change the results reported here. For simplicity of exposition, we do not present this generalization here.

The capitalist's profit maximization problem is: given p, choose $x^\nu \geqslant 0$ to maximize $(px^\nu - (pA+L)x^\nu)$ subject to (1.2). Let $A^\nu(p)$ be the set of activity levels x^ν which solve this program and $A(p) = \Sigma_\nu A^\nu(p)$. $A(p)$ is the set of aggregate activity levels which can occur at prices p as a consequence of individual profit maximization subject to capital constraints.

What equilibrium or solution concept is adopted in Marxian–Sraffian analysis? The concern is with whether the economic system can reproduce itself: whether it can produce enough output to replenish the inputs used, and to reproduce the workers for another period of work. The concept of a worker's subsistence bundle is a logical one to adopt here, for one is not concerned with *satisfying* all economic agents (who may wish to maximize their utilities subject to budget constraints) but rather with investigating whether the economic system is capable of *reproducing* itself. Marx's investigation of the laws of motion of capitalist society attempts to uncover how capitalist society reproduces itself, a process which is multi-faceted, not simply economic. (Within neoclassical theory, the Marxian notion of reproducibility is most evident in growth models.)

In the linear model under consideration, we shall define the vector $(p, x^1, \ldots, x^n)$ to be a *reproducible solution* if and only if (a) $x \geqq Ax + (Lx)b$, where $x = \Sigma x^\nu$ (reproducibility); (b) $x^\nu \in A^\nu(p)$ (profit maximization); (c) $pb = 1$ (subsistence wage); (d) $Ax + (Lx)b \leqq \omega = \Sigma \omega^\nu$ (feasible production).

Condition (a) states that at the profit-maximizing aggregate vector of outputs x, intermediate inputs must be at least replenished: this is the reproducibility concept; condition (d) states that production at levels x must be possible from given physical endowments ω. The price vector p of a reproducible solution shall be called *viable*.[3]

PROPOSITION 1.1: *Let the linear model $\{A, L, b\}$ be as specified, with A productive and indecomposable, and such that the rate of exploitation is $e > 0$. Then there exists at most one viable price vector, and that is the vector p^* which generates equal profit rates in all sectors. There exists a reproducible solution if and only if $\omega \in C^*$, where C^* is a convex cone in the nonnegative orthant to be specified below.*

PROOF: Let $M = A + bL$; M is the matrix of "augmented" input coefficients. M is indecomposable (since A is); and the assumption $e > 0$ implies there exists a positive column vector x^* such that $x^* > Mx^*$.

If x is the aggregate activity vector associated with a reproducible solution (RS), it follows that $x \geqq Mx$, and since $(I-M)^{-1} > 0$ by the indecomposability of M, $x > 0$. Hence at a RS, all activities must be operated. Capitalists facing prices p, in maximizing profits, will only operate those processes generating the maximal profit *rate*, and they will operate those processes in (any) convex combination to the limit of their capital constraints (by linearity). Hence, for all processes to

[3] The criterion for economic reproducibility, part (a) of the above definition, is strong. One might prefer a weaker criterion: for example, that the economy should not embark upon a path which would lead to the depletion of the stock of some necessary good. This, however, would complicate the analysis greatly.

operate, it is necessary that the price vector p generate the *same* profit rate in all sectors. By the Frobenius–Perron Theorem, there is a unique such vector up to scale, and its scale is determined by part (c) of the definition of RS. Thus there is at most one viable price vector, p^*.

Let C^* be the convex cone defined by:

$$C^* = \{\omega \in \mathbb{R}^n_+ | (\exists \times \geqslant 0)(Mx = \omega) \text{ and } x \geqslant Mx\}.$$

C^* is a convex cone, and it is nonempty since there exists a balanced growth path x^* such that

$$x^* = (1+\pi)Mx^*, \quad x^* > 0,$$

where π is the profit rate associated with p^*. ($e > 0$ guarantees $\pi > 0$, from Morishima's Fundamental Marxian Theorem.) Hence $Mx^* \in C^*$, with appropriate scaling of x^*. It follows from previous remarks that C^* is a closed non-empty convex cone in the nonnegative orthant R^n_+. (Note the only point of C^* not in the strictly positive orthant is the origin.)

We proceed to show the existence of a RS if $\omega \in C^*$. Let $\bar{x}$ be a vector, which exists if $\omega \in C^*$, such that $M\bar{x} = \omega$ and $\bar{x} \geqslant M\bar{x}$. Let p^* rule, as those are the only possible viable prices. Notice in this case that any activity vector which uses up capitalist ν's capital is in fact a profit-maximizing one, since all processes generate equal profit rates at p^*. It follows that:

$$A(p^*) = \{x \geqslant 0 | p^*Mx = p^*\omega\}.$$

(Notice any x in this set can be decomposed as $x = \Sigma x^\nu$, where $px^\nu = p\omega^\nu$, and therefore $x^\nu \in A^\nu(p^*)$.) In particular, it follows that $\bar{x} \in A(p^*)$. All conditions of the definition of RS are satisfied.

Conversely, let $\{p^*, x^1, \ldots, x^N\}$ be a RS. By linearity each capitalist must spend the entire value of his endowment if he is maximizing profits; thus

$$p^*Mx^\nu = p^*\omega^\nu \forall \nu,$$

and so

$$p^*Mx = p^*\omega \tag{1.4}$$

where $x = \Sigma x^\nu$. From part (d) of the definition of RS, $Mx \leqslant \omega$; since $p^* > 0$, a well-known fact from the Frobenius–Perron Theorem, it follows that $Mx = \omega$ from (1.4). From part (a) of RS, $x \geqslant Mx$, and consequently $\omega \in C^*$.

Hence C^* is precisely the cone of initial aggregate endowments for which reproducible solutions exist. *Q.E.D.*

It is worth noting that the establishment of equal-profit-rate (EPR) equilibrium prices has a different genesis in this model from traditional Marxian heuristic discussions. The usual story is something like this: if prices p prevailed which gave rise to profit rate differentials between sectors, then capitalists would shift production from low to high profit rate sectors, cutting prices in the latter in a competitive drive. This process would continue until prices were established where profit rates were equalized. A *particular mechanism* whereby EPR prices are achieved is described; in the present model it is shown there is only one price vector capable of reproducing the system, independent of the mechanism by which it is achieved. This distinction is perhaps an important one, as it appears there is no guarantee that a dynamic process of price adjustment will converge to an equal-profit-rate price vector (see Nikaido [**8**]).[4]

Marxian equilibrium, at least in the linear case considered here, can therefore be seen to be a type of general equilibrium. Production and production decisions are specified with the same degree of specificity as in neoclassical analysis, but consumption is only specified insofar as it must be to discuss the reproduction of the system. This does not necessarily mean that neoclassical *general* equilibrium is a philosophical or technical advance over the Marxian *reproducible* solution concept. Rather, the difference arises because Marx was concerned with the capability of a social formation to reproduce itself independent of the (subjective) wills of the participants. Furthermore, the main drama in the capitalist mode of production is accumulation, or the consequences of profit maximization by capitalists. It would be incorrect to elevate consumption decisions based on subjective preferences to the same hierarchy of specificity as production decisions based on profit maximization in describing a capitalist economy. Secondly, in Marx's conception workers had little latitude in making consumption decisions. Their consumption patterns were socially determined, as shall be discussed below. Hence, specifying the consumption problem as an optimization problem is not an appropriate generalization of the Marxian conception, for it is antithetical to the Marxian class-based behavioral determinations. Capitalists optimize, but workers are forced to take what they can get; they live in a world where any optimizing they may do obfuscates the narrow boundaries of their behavior. (For a further discussion of this, see Roemer [**10**].)

Finally, it should be noticed that reproducible solutions only exist if the initial social endowment vector ω lies in a certain cone, that is, suitably close to the balanced growth ray. This leads to natural questions of stability. If $\omega \notin C^*$, is there a dynamic equilibrium concept which pushes the economy toward an endowment which does lie in C^*? If $\omega \in C^*$, under a dynamic specification of the economy, will the initial endowment remain in C^*? The answer: there is no guarantee that the economy tends towards endowments at which reproducible solutions exist. Whether or not such occurs depends upon the technology M. As this question is not central to the concerns of the investigation here, its pursuit is relegated to Appendix I.

2. MARXIAN EQUILIBRIUM IN A CONVEX ECONOMY

Formal Marxian economic analysis to date has been in the context of linear models. Morishima's book [**6**] represents the main presentation of this theory in

[4] There are two issues concerning equalization of profit rates: the equalization of profit rates among capitalists, and among sectors. If one introduces a capital market, then it can be shown competition among capitals equalizes the profit rate among capitalists, and the requirement of reproducibility equalizes the profit rate across sectors—in the linear model.

which the model of production is a linear one, similar to the one presented in the previous section. In a later paper, Morishima [**7**] generalized his model to a von Neumannesque linear economy taking account of joint production and alternative processes. Other writers who have examined linear Marxian economics include Samuelson [**12**], Wolfstetter [**15**], Okishio [**9**], von Weizsäcker [**14**], Maarek [**5**], and Brody [**1**]. The task here is to generalize the model presented above to a more general, not necessarily linear, but convex economy. First, production shall be defined and it shall be shown that the rate of exploitation is a well-defined concept. Then it shall be shown that under appropriate assumptions, reproducible solutions exist. A generalization of Morishima's Fundamental Marxian Theorem is finally proved.

2A. *Production*

The νth capitalist faces a production possibilities set P^ν. There are n commodities which can be produced, and labor. Vectors $\alpha^\nu \in P^\nu$ shall be written as $(2n+1)$ vectors, as follows:

$$\alpha^\nu = (-\alpha_0^\nu, -\underset{\sim}{\alpha}^\nu, \bar{\alpha}^\nu),$$

where α_0^ν is the direct labor input, $\underset{\sim}{\alpha}^\nu$ is the nonnegative n-vector of commodity inputs, and $\bar{\alpha}^\nu$ is the nonnegative n-vector of commodity outputs. For notational convenience, write $\hat{\alpha}^\nu$ for the n-vector of net outputs, $\alpha^\nu = \bar{\alpha}^\nu - \underset{\sim}{\alpha}^\nu$. It is assumed that:

Assumption A1: $(\forall\nu)(0 \in P^\nu)$.

Assumption A2: $(\forall\nu)(P^\nu$ is convex).

Assumption A3: $(\forall\nu)(P^\nu$ is closed).

Assumption A4: $(\forall \alpha P^\nu)(\alpha_0 \geqslant 0$ and $\underset{\sim}{\alpha} \geqq 0)$. $(\hat{\alpha} \geqslant 0 \Rightarrow \alpha_0 > 0$, where $\alpha = (-\alpha_0, -\underset{\sim}{\alpha}, \bar{\alpha})$, $\hat{\alpha} = \bar{\alpha} - \underset{\sim}{\alpha}$.)

In addition, let $P = \Sigma\, P^\nu$ be the aggregate production set.

Assumption A5: ($\forall$ commodity n-vectors c) $(\exists \alpha' \in P)$ $(\hat{\alpha}' \geqq c)$.

It follows that A1–A5 are also true for P.

Assumptions A1–A3 need no comment. A4 is a strong form of no free lunch: labor is not a producible commodity, and is necessary for the production of net output. A5 says that all (nonlabor) commodities are producible, and that any vector of nonnegative net outputs can be produced.

An explanation is due for why inputs are separated from outputs in the definition of production sets. Time is essential in this sense: capitalists must pay for inputs prior to production, and must use their capital $p\omega^\nu$ to do this. They cannot borrow against future revenues to finance today's production. Hence the differentiation between inputs and outputs must be made. For example, in the linear model of the previous section, a typical activity vector would be written, under this convention, as:

$$\langle -l_i, -a_{1i}, -a_{2i}, \ldots, -a_{ni}, 0, 0, \ldots, 1_i, 0, \ldots 0\rangle.$$

Under the convention which ignores time this activity vector would be written:

$$\langle -l_i, -a_{1i}, \ldots, -a_{i-1,i}, 1-a_{ii}, -a_{i+1,i}, \ldots, -a_{ni}\rangle.$$

In the latter case it appears as if there is no cost to the capitalist operating this activity from use of input i. This, however, is incorrect. In the model as described, the capitalist must lay out cost $p_i a_{ii}$, among other costs; this is captured only by viewing production as $(2n+1)$-vectors.

2B. *Exploitation*

Exploitation can now be defined.

DEFINITION: The *labor value of a nonnegative commodity bundle B*, l.v.(B), is defined as

$$\min_{\phi(B)} \alpha_0, \quad \text{where } \phi(B) = \{\alpha \in P \mid \hat{\alpha} \geqslant B\}.$$

DEFINITION: The *rate of exploitation at a point $\alpha \in P$* is

$$e(\alpha) = \frac{\alpha_0}{\text{l.v.}(\alpha_0 b)} - 1.$$

If $\alpha_0 \geqslant 0$, define $e(\alpha) = 0$.

PROPOSITION 2.1: *$e(\alpha)$ is well-defined $\forall \alpha \in P$.*

PROOF: It is only necessary to show that l.v.(B) is well-defined for all commodity vectors $B \geqslant 0$. The set $\phi(B)$ is nonempty by A5; g.l.b. α_0 exists since $\{\alpha_0 \mid (\alpha_0, -\underline{\alpha}, \bar{\alpha}) \in \phi(B)\}$ is bounded below by 0, by A4. It is then only necessary to show that

$$\alpha_0^* = \min_{\phi(B)} \alpha_0$$

is achieved at a point $\alpha^* = (-\alpha_0^*, -\underline{\alpha}^*, \bar{\alpha}^*) \in P$.

Choose a sequence $\alpha^i = (-\alpha_0^i, -\underline{\alpha}^i, \bar{\alpha}^i)$ in $\phi(B)$ such that $\alpha_0^i \to \alpha_0^*$. It is claimed that $\{\alpha^i\}$ is bounded. For let

$$\beta^i = \frac{1}{\|\alpha^i\|}\alpha^i, \quad \|\alpha^i\| = \max_j |\alpha_j^i|.$$

Since $0 \in P$ and P is convex, $\beta^i \in P \cdot \{\beta^i\}$ is bounded. Since P is closed, $\beta^i \to \beta^* = (-\beta_0^*, \beta^*, \bar{\beta}^*) \in P$. Suppose $\{\alpha^i\}$ is unbounded; then $\beta_0^i \to 0$ since $\alpha_0^i \to \alpha_0^*$. Hence $\beta_0^* = 0$. But $\hat{\beta}^* \geqslant 0$ which contradicts A4 applied to the set P.

Therefore $\{\alpha^i\}$ is bounded, so by the closedness of P, a convergent subsequence of $\{\alpha^i\}$ can be chosen converging to the required vector α^*. *Q.E.D.*

REMARK: Note that l.v.$(\alpha_0 b)$ is the labor time "socially necessary" to produce subsistence for workers laboring at social production point $\alpha = (-\alpha_0, -\underset{\sim}{\alpha}, \bar{\alpha})$. $e(\alpha)$ then gives the ratio of surplus labor time $(\alpha_0 - \text{l.v.}(\alpha_0 b))$ to necessary labor time $(\text{l.v.}(\alpha_0 b))$, the Marxian definition. In particular, it is entirely possible that $e(\alpha) < 0$. It can be easily shown that $e(\alpha)$ reduces to the ordinary rate of exploitation e in the linear case of Section 1.

2C. *Capitalist Behavior*

The behavior of capitalists is profit maximizing. Capitalist ν possesses capital ω_t^ν at time t. We take price vectors as vectors of commodity prices only and take the wage as numeraire.

We view the model in a temporary equilibrium framework. Capitalists face prices today, and their objective is to maximize the expected value of tomorrow's endowment ω_{t+1}^ν. Tomorrow's endowment comes from two sources: the output from production carried on today, and commodities carried over from today to tomorrow; commodity speculation may take place. Formally, if capitalist ω expects prices p_{t+1}^ν to rule tomorrow, his program is:

$$\text{choose } \alpha^\nu, \delta^\nu \in R_+^n \text{ to max } (p_{t+1}^\nu \bar{\alpha}^\nu + p_{t+1}^\nu \delta^\nu)$$

$$\text{subject to } \alpha_0^\nu + p_t \underset{\sim}{\alpha}^\nu \leqslant p_t \omega_t^\nu,$$

$$p_t \delta^\nu = p_t \omega_t^\nu - (p_t \underset{\sim}{\alpha}^\nu + \alpha_0^\nu).$$

(That is, capitalists seek to maximize the expected value of tomorrow's endowment $\omega_{t+1}^\nu = \delta^\nu + \bar{\alpha}^\nu$ subject to the budget constraint on production costs and commodity speculation that expenditures cannot exceed present capital $p_t \omega_t^\nu$.)

For the body of the paper, for simplicity, we shall assume expectations are stationary: $p_{t+1}^\nu = p_t = p$, $\forall \nu$. (Or, the analysis can be conceived of as applying in a general expectations framework, at a stationary state, if one exists.) The details of the more general analysis, where price expectations are not stationary, are presented in Appendix II.

To this end, we observe that with stationary expectations, the capitalist's objective function takes this simple form:

PROPOSITION 2.2: *When $p_{t+1} = p_t = p$, the capitalist's program is:*

$$\textit{choose } \alpha^\nu \in p^\nu \textit{ to maximize } p\bar{\alpha}^\nu - (p\underset{\sim}{\alpha}^\nu + \alpha_0^\nu)$$

$$\textit{subject to } \alpha_0^\nu + p\underset{\sim}{\alpha}^\nu \leqslant p\omega^\nu.$$

PROOF: The program is:

choose α^ν, δ^ν to max $p\bar{\alpha}^\nu + p\delta^\nu$

subject to (i) $\alpha_0^\nu + p\underline{\alpha}^\nu \leqslant p\omega^\nu$,

(ii) $p\delta^\nu = p\omega^\nu - (p\underline{\alpha}^\nu + \alpha_0^\nu)$.

Substituting from (ii) into the objective function, and recognizing that $p\omega^\nu$ is a constant, yields the result. *Q.E.D.*

NOTE: This Proposition justifies the specification of the capitalist's program of Section 1.

NOTE: From now on, all time subscripts are dropped as it is assumed $p_{t+1}^\nu = p_t$ for all ν. See Appendix II for general discussion of the temporary equilibrium model.

DEFINITION: The *feasible production set* for capitalist ν at prices p is:

$$B^\nu(p) = \{\alpha^\nu \in P^\nu \mid \alpha_0^\nu + p\underline{\alpha} \leqslant p\omega^\nu\}.$$

(The feasible set consists of those production points at which the capitalist can afford to produce with his capital.)

LEMMA 2.3: $\forall \nu, p \geqslant 0$, *$B^\nu(p)$ is nonempty, convex, and compact.*

PROOF: $B^\nu(p)$ is nonempty since it contains 0. Convexity and closedness are obvious. Boundedness: notice $\{\alpha_0^\nu \mid \alpha^\nu = (-\alpha_0^\nu, -\underline{\alpha}^\nu, \bar{\alpha}^\nu) \in B^\nu)(p)\}$ is bounded by $p\omega^\nu$. By the argument given in the proof of Proposition 2.1, $B^\nu(p)$ is bounded. *Q.E.D.*

DEFINITION: The *profit maximizing set* for capitalist ν at prices p is:

$$A^\nu(p) = \{\alpha^\nu \in B^\nu(p) \mid p\hat{\alpha}^\nu - \alpha_0^\nu \text{ is maximized}\}.$$

PROPOSITION 2.4: *$A^\nu(p)$ is a nonempty, compact, convex for all ν, $p \geqslant 0$.*

PROOF: Nonemptiness: notice that profits $p(\bar{\alpha}^\nu - \underline{\alpha}^\nu) - \alpha_0^\nu$ are bounded for $\alpha^\nu \in B^\nu(p)$ since $B^\nu(p)$ is bounded (Lemma 2.3). Hence, there is a sequence $\{\alpha^{\nu,i}\} \in B^\nu(p)$ whose profits converge to the maximum value, since $B^\nu(p)$ is compact, there is a limit vector $\alpha^{\nu*} = \lim_i \{\alpha^{\nu,i}\}$ which achieves those maximum profits in $B^\nu(p)$.

Closedness of $A^\nu(p)$ is obvious; boundedness follows from boundedness of B and convexity of $A^\nu(p)$ follows directly from convexity of $B^\nu(p)$. *Q.E.D.*

2C. *Equilibrium*

DEFINITION: A *reproducible solution* for the economy specified is as a pair (p, α) $p \geqslant 0$, $\alpha \in P$ such that: (a) $\alpha = (-\alpha_0, \underset{\sim}{\alpha}, \bar{\alpha})$ and $\hat{\alpha} \geqslant \alpha_0 b$; (b) $\alpha \in A(p) \equiv \Sigma_\nu A^\nu(p)$; (c) $pb = 1$; (d) $\underset{\sim}{\alpha} + \alpha_0 b \leqslant \omega$.

This definition follows exactly the definition of RS in the previous section.

To prove the existence of reproducible solutions in the general case, it is useful to introduce another concept.

DEFINITION: Let $W^1, \ldots, W^N$ be positive numbers. (W is to be thought of as the money wealth of capitalist ν.) Let

$$\bar{B}^\nu(p) = \{\alpha^\nu \in P^\nu | \alpha_0^\nu + p\underset{\sim}{\alpha}^\nu \leqslant W^\nu\}$$

and

$$\bar{A}^\nu(p) = \{\alpha^\nu \in B^\nu(p) | p\hat{\alpha}^\nu - \alpha_0^\nu \text{ is maximized}\}.$$

A pair (p, α) is said to be a *quasi-reproducible solution*: (QRS) if (a) $\alpha = (-\alpha_0, \underset{\sim}{\alpha}, \bar{\alpha})$ and $\hat{\alpha} \geqslant \alpha_0 b$; (b) $\alpha \in \bar{A}(p) \equiv \Sigma \bar{A}^\nu(p)$; (c) $pb = 1$.

That is, a QRS takes no account of the vector of physical endowments, and no account of the feasibility of the production plan, in the sense of part (d) of definition RS.

NOTE: Lemma 2.3 and Proposition 2.4 hold for the sets $\bar{B}^\nu(p)$ and $\bar{A}^\nu(p)$.

The method for showing the existence of a RS shall be, first, to demonstrate existence of a QRS.

THEOREM 2.5. *Let $b > 0$. Under Assumptions* A1–A4, *and stationary expectations, a quasi-reproducible solution exists, for any nonnegative values $W^1, \ldots, W^N$.*

REMARK: For technical reasons, a different proof is required if b has components which are 0. This will be provided subsequently.

The proof shall rely on the following lemma.

LEMMA (Gale, Nikaido): *Let the correspondence $z(p): S \to T$ be upper-hemi-continuous* (*uhc*) *from the simplex S to the compact set T. Let $z(p)$ be closed and convex for all p, and $pz(p) \leqslant 0$. Then $\exists \bar{p}$ and $\bar{z} \in z(\bar{p})$ such that $\bar{z} \leqslant 0$.*

LEMMA 2.6: *$\bar{B}^\nu(p)$ is lower-hemi-continuous.*

PROOF: Let $\alpha' \in \bar{B}^\nu(p)$, $p^\mu \to p$. We wish to produce a sequence $\alpha'^\mu \in \bar{B}(p^\mu)$

such that $\alpha'^{\mu} \to \alpha'$. Let

$$\alpha'^{\mu} = \begin{cases} \alpha & \text{if } \alpha' \in \bar{B}^{\nu}(p^{\mu}), \\ \lambda^{\mu}\alpha & \text{if } \alpha' \notin \bar{B}^{\nu}(P^{\mu}), \end{cases} \quad \text{where}$$

$$\lambda^{\mu} = \max\left\{\lambda \,|\, \lambda\left[\alpha_0' + \sum_1^n p_j^{\mu} \underline{\alpha}_j\right] \leqslant W^{\nu}\right\}.$$

λ^{μ} is well-defined. By definition, $\alpha'^{\mu} \in \bar{B}^{\nu}(p^{\mu})$. Furthermore, $p^{\mu} \to p$ and $\alpha' \in \bar{B}^{\nu}(p)$ imply $\lambda^{\mu} \to 1$. Hence $\alpha'^{\mu} \to \alpha'$. *Q.E.D.*

LEMMA 2.7: *$\bar{A}(p)$ is upper-hemi-continuous.*

PROOF: We show that $\bar{A}^{\nu}(p)$ is uhc for any ν. It follows that $\bar{A}(p)$ is uhc.

Let $p^{\mu} \to p$, $\alpha^{\mu} \in \bar{A}^{\nu}(p^{\nu})$, $\alpha^{\mu} \to \alpha$. We show $\alpha \in \bar{A}^{\nu}(p)$. Since $p^{\mu} \to p$ and $\alpha^{\mu} \to \alpha$ it is easily seen that $\alpha \in \bar{B}^{\nu}(p)$. It must be shown α is a profit maximizer in $\bar{B}^{\nu}(p)$. Suppose not. Then $\exists \alpha' \in \bar{B}^{\nu}(p)$, such that

$$p\hat{\alpha}' - \alpha_0' = M' > M = p\alpha' - \alpha_0.$$

Notice $p^{\mu}\hat{\alpha}^{\mu} - \alpha_0^{\mu} = M^{\mu} \to M$.

By Lemma 2.6, there is a sequence $\alpha'^{\mu} \in \bar{B}^{\nu}(p^{\mu})$ such that $\alpha'^{\mu} \to \alpha'$. Hence $p^{\mu}\hat{\alpha}' - \alpha_0' \to M'$. Thus for large μ, profits made by capitalist ν at prices p at point α'^{μ} are larger than profits made at α^{μ} at those prices. This contradicts the fact that $\alpha^{\mu} \in \bar{A}^{\nu}(p^{\mu})$. It follows that α must have been a profit maximizer at prices p for ν. *Q.E.D.*

PROOF OF THEOREM 2.5: Since $b > 0$, $S = \{p \geqslant 0 | pb = 1\}$ is a simplex. Define for $p \in S$:

$$z(p) = \left\{\left(\sum_{\nu} \alpha_0^{\nu}\right) b - \sum_{\nu} \hat{\alpha}^{\nu} | \forall \nu(-\alpha_0^{\nu}, -\underline{\alpha}^{\nu}, \bar{\alpha}^{\nu}) \in \bar{A}^{\nu}(p)\right\}.$$

(a) $pz(p) \leqslant 0$. Notice $-pz(p) = \Sigma_{\nu}\ (p\hat{\alpha}^{\nu} - \alpha_0^{\nu})$ which is simply the sum of the profits made by the capitalists at the chosen points α^{ν}. Hence $-pz(p) \geqslant 0$ since at worst capitalists operate at zero profits by producing at $0 \in P^{\nu}$.

(b) T, the set of images of z, is compact. The set $\bar{A} = \bigcup_{p \in S} \bar{A}(p)$ is bounded. For if $\alpha \in \bar{A}$, then $\alpha_0 \leqslant W$. It has been shown previously that $\{\alpha \in P | \alpha_0 \leqslant W\}$ is bounded.

Furthermore the set $\bar{A}$ is closed: let $\{\alpha^{\mu}\}$ be a bounded subsequence of $\bar{A}$. We write $\alpha^{\mu} = \alpha^{\mu}(p^{\mu})$ to identify the price vector with which α^{μ} is associated. There must exist a convergent subsequence of the $\{p^{\mu}\}$, so without loss of generality, assume $p^{\mu} \to p$. That α^{μ} converges to a vector $\alpha \in \bar{A}(p)$ is a direct consequence of the upper-hemi-continuity of $\bar{A}(p)$, by the Lemma 2.7.

Hence the set $\bar{A}$ is compact. Now $T = \bigcup_{p \in S} z(p)$ is the image of the compact set $\bar{A}$ under a continuous function ($\alpha \in \bar{A}$; $F(\alpha) = \alpha_0 b - \hat{\alpha}$), and is therefore itself compact.

(c) $z(p)$ is convex. Since $\bar{A}^{\nu}(p)$ is convex, this follows immediately.

(d) $z(p)$ is closed. This follows from closedness of $\bar{A}^{\nu}(p)$.

(e) $z(p)$ is uhc. This follows from the upper-hemi-continuity of $\bar{A}(p)$ (Lemma 2.7). For let $p^{\mu} \to p$, $z^{\mu} \in z(p^{\mu})$, $z^{\mu} \to z$. Write $z^{\mu} = \alpha_0^{\mu} b - \hat{\alpha}^{\mu}$ where $\alpha^{\mu} = (-\alpha_0^{\mu}, -\underline{\alpha}^{\mu}, \bar{\alpha}^{\mu}) \in \bar{A}(p^{\mu})$. Since $\{\alpha_0^{\mu}\}$ are bounded by W it follows that $\{\alpha^{\mu}\}$ are bounded and hence possess a convergent subsequence. Let the subsequence converge to $\alpha = (-\alpha_0, -\underline{\alpha}, \bar{\alpha})$. It follows that $z = \alpha_0 b - \hat{\alpha}$. By the uhc of $\bar{A}(p)$, $\alpha \in \bar{A}(p)$ and hence $z \in z(p)$.

(f) The conditions of the Gale–Nikaido Lemma are all satisfied. There exist therefore, $\bar{p} \in S$ and $\bar{z} \in z(\bar{p})$, $\bar{z} = \bar{\alpha}_0 b - \hat{\alpha}$, such that $\bar{z} \leqslant 0$. But this is precisely condition (b) of the definition that $(\bar{p}, \alpha)$ be a quasi-reproducible solution, while condition (a) of that definition holds since $\bar{\alpha} \in \bar{A}(\bar{p})$. *Q.E.D. Theorem* 2.5.

COROLLARY 2.8: *Let* $(W^1, \ldots, W^N)$ *be any vector of wealths. Then there exists a set of endowments*

$$\bar{\omega} = (\omega^1, \ldots, \omega^N)$$

such that a reproducible solution (p, α) *with respect to* $\bar{\omega}$ *exists under stationary expectations, where* $p\omega^{\nu} = W^{\nu}$ *for all* ν.

PROOF: By Theorem 2.5 a QRS (p, α) exists with respect to wealths $(W^1, \ldots, W^N)$. Say $\alpha = (-\alpha_0, \underline{\alpha}, \bar{\alpha})$. Let ω be any vector such that

$$\omega \geqslant \alpha_0 b + \underline{\alpha} \quad \text{and} \quad p\omega = W = \sum W^{\nu}.$$

(Such ω exists, since by definition of QRS, we have $p(\alpha_0 b + \underline{\alpha}) \leqslant W$.) Since $p\omega = W$, ω may be decomposed (in perhaps many ways) as $\omega = \Sigma\, \omega^{\nu}$ such that $p\omega^{\nu} = W^{\nu}$. It follows by checking the definition that (p, α) is a RS with respect to endowments $(\omega^1, \ldots, \omega^N)$. *Q.E.D.*

The question might arise: why not adopt the notion of quasi-reproducibility as the equilibrium notion, instead of the stronger concept of reproducibility? There are several reasons. First, although the magnitudes W have been called "wealths" to give an intuitive rendition to the proof, it is misleading to think of them as such. In particular, the W^{ν} are given before prices are obtained. If W^{ν} were money in the bank, therefore, it would not have much meaning with prices undetermined. We cannot assign a valuation to capital before prices are set (shades of the Cambridge controversy). Second, the QRS notion takes no account of feasibility of production, and hence of the notion of time in production. It is true that a QRS replenishes those inputs it uses up (definition, part (a)); but where does it get those inputs in the first place? There is no market for initial inputs: somehow each capitalist can automatically cash in his capital W for the necessary physical inputs into production. Clearly this notion is not a substitute for the RS notion: it is, however, a mathematical convenience in proving the existence of the latter.

Let us comment on the domain of possible initial endowments. Notice in this case we do not have the characterization of the initial endowment domain as a cone, as was true in the linear case. However, Corollary 2.8 shows that endowments held by different capitalists can replicate ex post (i.e., after prices are determined) any desired distribution of capital values among capitalists.

We proceed to the existence of reproducible solutions in the general case that $b \geqslant 0$. (The reason the proof of Theorem 2.5 does not apply to the case $b \geqslant 0$ is that $S = \{p | pb = 1\}$ is no longer a simplex if b possesses zero components.) We make the following assumption.

ASSUMPTION A6 (Indecomposability): $(\alpha \in P, \alpha = \Sigma\, \alpha^{\nu}, \hat{\alpha} \geqslant \alpha_0 b)\ (\forall_j \ni : b_j = 0)$ $(\exists \nu)\ (\underline{\alpha}_j^{\nu} > 0)$.

This is an indecomposability assumption, because it says that if a particular commodity is not a subsistence commodity ($b_j = 0$) then at any reproducible point in the production set, that good must be an input into some capitalist's production process. Put another way, Assumption A6 says there are two kinds of goods: workers' consumption goods, and "intermediate" goods which must enter the production process if the economy is capable of reproduction.

Viewed from the Marxian vantage point of evaluating the reproducibility of economies, this assumption is not difficult to justify. If a good is not consumed by workers and is not needed as an intermediate input for reproducible states then it is an economic appendage in a sense. It may be an intermediate good which *could* be used to reach reproducible states at higher profits that can be achieved without its use. In either case, however, the good in question and all production processes in which it appears could be eliminated and a reproducible solution would still exist.

To be more precise, suppose Assumption A6 does not hold. Let $J = \{j | b_j = 0$ and $\exists \alpha \in P,\ \hat{\alpha} \geqslant \alpha_0 b,\ \alpha = \Sigma\, \alpha^{\nu}$ and $\underline{\alpha}_j^{\nu} = 0 \forall \nu\}$. Then define new production sets

$$\bar{P}^{\nu} = \{\alpha \in P^{\nu} | \forall_j \in J, \underline{\alpha}_j^{\nu} = 0\}.$$

It is easily verified that Assumptions A1–A6 hold for the restricted production set $\bar{P}^{\nu}$ and $\bar{P} = \Sigma\, \bar{P}^{\nu}$. We may now consider the new economy with production sets $\bar{P}^{\nu}$ to consist only of the original commodities minus those in the set J. On the new economy specified, Assumption A6 therefore holds. Then, as is shown below, a reproducible solution exists in the restricted economy.

THEOREM 2.9: *Let* $b \geqslant 0$. *Under Assumptions* A1–A4, A6 *and stationary expectations a quasi-reproducible solution exists, for any nonnegative values* $W^1, \ldots, W^N$.

PROOF: Choose a sequence $b^{\mu} \to b$ where $b^{\mu} > 0$. By Theorem 2.5, there is a quasi-reproducible solution (p^{μ}, α^{μ}) corresponding to b^{μ}. The proof shall construct a quasi-reproducible solution at b.

(a) $\{\alpha^{\mu}\}$ possess a subsequence converging to a vector $\alpha \in P$. This follows since $\alpha_0^{\mu} \leqslant W\ \forall \mu$, and so by the standard convexity argument, $\{\alpha^{\mu}\}$ is bounded.

(b) $\{p^\mu\}$ is bounded. Suppose not. Then for some j, $p_j^\mu \underset{\mu}{\to} \infty$. Since $p^\mu b^\mu = 1$, $b_j^\mu \underset{\mu}{\to} 0$ and so $b_j = 0$. Since $\alpha^\mu \to \alpha$ and $b^\mu \to b$ and $\hat{\alpha}^\mu \geqslant \alpha_0^\mu b^\mu$ it follows that $\hat{\alpha} \geqslant \alpha_0 b$—that is, α is reproducible at b. Decomposing α^μ into the individual capitalists' profit -maximizing points gives:

$$\alpha^\mu = \sum_\nu (\alpha^\nu)^\mu.$$

Since $\{\alpha^\mu\}$ is bounded, we can write

$$\alpha = \sum \alpha^\nu \quad \text{where} \quad \alpha^\nu = \lim_\mu (\alpha^\nu)^\mu.$$

Now by Assumption A6, for some ν, $\underset{\sim}{\alpha}_j^\nu > 0$ since $b_j = 0$ and $\hat{\alpha} \geqslant \alpha_0 b$. Hence for large μ, $(\underset{\sim}{\alpha}_j^\nu)^\mu$ are bounded away from zero and positive since $(\underset{\sim}{\alpha}_j^\nu)^\mu \underset{\mu}{\to} \underset{\sim}{\alpha}_j^\nu > 0$.

This, however, contradicts the assumption that capitalist ν is always within his capital constraint. For:

$$(\alpha_0^\nu)^\mu + \sum_1^n p_j^\mu (\underset{\sim}{\alpha}_j^\nu)^\mu \leqslant W^\nu$$

by definition; however $p_j^\mu \underset{\mu}{\to} \infty$ and $(\underset{\sim}{\alpha}_j^\nu)^\mu > \varepsilon$ for all large μ and so the inequality fails for large μ.

Hence, by contradiction, $\{p^\mu\}$ is bounded.

(c) Consequently $\{p^\mu\}$ possesses a subsequence converging to a price vector p. By the upper hemi-continuity of the correspondence $\bar{A}(p)$ it follows that $\alpha \in \bar{A}(p)$ and hence (p, α) is a QRS at b. *Q.E.D.*

COROLLARY 2.10: *Let* $b \geqslant 0$. *Let* $(W^1, \ldots, W^N)$ *be any vector of wealths. Under Assumptions* A1–A4, A6, *there exists a set of endowments*

$$\bar{\omega} = (\omega^1, \ldots, \omega^N)$$

such that a reproducible solution (p, α) *with respect to* $\bar{\omega}$ *exists under stationary expectations, where* $p\omega^\nu = W^\nu$ *for all* ν.

PROOF: Same as Corollary 2.8.

Notice that the reproducible solution shown to exist by these theorems is possibly one of complete inaction, $\alpha^\nu = 0\ \forall \nu$. If the total profits $(p\hat{\alpha} - \alpha_0)$ are zero at the solution, then the solution of complete inaction is reproducible. This motivates the question to be investigated next: what condition will guarantee that the reproducible solutions do not include the trivial one of complete inaction? What guarantees the existence of reproducible solutions with positive profits?

3. THE FUNDAMENTAL MARXIAN THEOREM

Morishima has shown that the necessary and sufficient condition for the profit rate in a linear economy to be positive is that the rate of exploitation be positive

(Morishima [**6**, Chapter 6]). He has generalized this theorem to the linear von Neumann economy under certain assumptions (Morishima [**7**]). He considers this theorem "fundamental" because it gives a characterization of when the profit rate is positive: from the point of view here, this certainly is a fundamental necessity for reproducibility under stationary expectations, since it is easily seen that a point generating negative social profits cannot be reproducible. (Proof: $p\hat{\alpha} < \alpha_0 \Rightarrow p\hat{\alpha} < p\alpha_0 b \Rightarrow \hat{\alpha} \ngeqslant \alpha_0 b$.) The generalization of this theorem to the present model provides a characterization of economies where the reproducible solutions, known to exist, are nontrivial.

To prove the generalization of this fundamental theorem we require the following assumption.

ASSUMPTION A7 (Independence of Production): $(-\alpha_0, -\underline{\alpha}, \bar{\alpha}) \in P$, $\hat{\alpha} \geqslant 0$, and $0 \leqslant c \leqslant \hat{\alpha}$; then $\exists(-\alpha_0', -\underline{\alpha}', \bar{\alpha}') \in P$ such that $\bar{\alpha}' - \underline{\alpha}' \geqslant c$ and $\alpha_0' < \alpha_0$.

Assumption A7 deserves a comment. It says that if a bundle of net outputs $\hat{\alpha}$ can be produced with α_0 labor, then any smaller bundle can be produced with less labor. This is called independence of production because it fails when there is a good which can only be produced as a joint product in fixed proportions with some other good. Assumption A7 does not rule out joint products: it does rule out products which are only joint in the most severe sense. An example where Assumption A7 fails, and the consequences thereof, is provided later.

THEOREM 3.1 (Fundamental Marxian Theorem): *The following statements are equivalent under Assumptions* A1–A7 (*and stationary expectations*): (*A*) *There exists a point* $\alpha \in P$ *such that* $e(\alpha) > 0$. (*B*) *There exists a reproducible solution yielding total profits.* (*C*) *All reproducible solutions yield positive total profits.* (*D*) *All reproducible solutions yield positive rates of exploitation.*

The heart of Theorem 3.1 is the provision of a necessary and sufficient condition for reproducible solutions to be nontrivial: namely, the *possibility* of positive exploitation, in the sense of statement (A). The possibility of positive exploitation is the counterpart of the premise that an economy be technically productive, as is captured in a Hawkins–Simons condition on an input-output matrix; positive exploitation is here shown to be the *sine qua non* of capitalist production. Notice that the vector α of statement (A) need not be actually producible—it may violate the capital constraint of some capitalists. Furthermore, Theorem 3.1 states that reproducible solutions either always yield positive profits or always yield zero profits, even as the viable price vector changes. This simple social-technological characteristic, of the possibility of exploitation, therefore provides a characterization of economies into those which are unambiguously nontrivial and those which are completely trivial.

The proof of Theorem 3.1 depends on the following lemma.

LEMMA 3.2: *If* $e(\alpha^*) = 0$ *at reproducible solution* (p^*, α^*), *then* $p^*\hat{\alpha}^* - \alpha_0^* = 0$.

PROOF: Since α^* is reproducible, $\hat{\alpha}^* \geqq \alpha_0^* b$. $e(\alpha^*) = 0$ means:

$$\min_{\alpha \in \phi(\alpha_0^* b)} \alpha_0 = \alpha_0^*, \quad \text{where} \quad \phi(\alpha_0^* b) = \{\alpha \in P | \alpha = (-\alpha_0, -\underline{\alpha}, \bar{\alpha}), \hat{\alpha} \geqq \alpha_0^* b\}.$$

Suppose $\hat{\alpha} \geq \alpha_0^* b$ (i.e., $\hat{\alpha}^* \neq \alpha_0^* b$). Then by Assumption A7, $\exists \alpha^{**} \in P$, $\alpha^{**} = (-\alpha_0^{**}, -\underline{\alpha}^{**}, \bar{\alpha}^{**})$, $\hat{\alpha}^{**} \geqq \alpha_0^* b$ and $\alpha_0^{**} < \alpha_0^*$. But $\alpha^{**} \in \phi(\alpha_0^* b)$, thus contradicting that $e(\alpha^*) = 0$ (since α_0^* is evidently not $\min_\phi \alpha_0$). Hence $\hat{\alpha}^* = \alpha_0^* b$. Therefore $p^* \hat{\alpha}^* = p \alpha_0^* b = \alpha_0^*$. *Q.E.D.*

PROOF OF THEOREM 3.1: Method of proof: (B)$\Rightarrow$(A)$\Rightarrow$(C)$\Rightarrow$(B), (D)$\Rightarrow$(A), (C)$\Rightarrow$(D).

(i) Notice (B)$\Rightarrow$(A) follows from Lemma 3.2 since for any reproducible solution α, $e(\alpha) \geq 0$. ($e(\alpha) \geq 0$ since $\hat{\alpha} \geqq \alpha_0 b$ at a reproducible solution.)

(ii) (C)$\Rightarrow$(B) and (D)$\Rightarrow$(A) are trivial since reproducible solutions exist.

(iii) (C)$\Rightarrow$(D) by Lemma 3.2.

(iv) (A)$\Rightarrow$(C): Let $\alpha = (-\alpha_0, -\underline{\alpha}, \bar{\alpha}) \in P$, $e(\alpha) > 0$. By definition, there is a point $\alpha' = (-\alpha_0', -\underline{\alpha}', \bar{\alpha}') \in P$ such that $\hat{\alpha}' \geqq \alpha_0 b$ and $\alpha_0' < \alpha_0$. Assume contention (C) is false: then there is a reproducible solution (p, β) which produces zero total profits. Each capitalist ν makes zero profits at β^ν where $\beta = \Sigma \beta^\nu$. Since $\hat{\alpha}' \geqq \alpha_0 b$ and $\alpha_0 > \alpha_0'$, $p\hat{\alpha}' > \alpha_0'$. Decompose α' into $\alpha' = \Sigma \alpha'^\nu$. Some capitalist ν must make positive profits at α'^ν, since total profits are positive at α'. A small positive multiple $\lambda \alpha'^\nu$ of α'^ν lies in $B^\nu(p)$, and capitalist ν makes positive profits at $\lambda \alpha'^\nu$; thus, $\beta^\nu \notin A^\nu(p)$, a contradiction. *Q.E.D.*

Since Theorem 3.1 was proved with the aid of the "independence" Assumption A7, it is worthwhile to provide an example showing that Assumption A7 or something similar to it is necessary.[5]

EXAMPLE 1 (where Assumption A7 fails, profits>0, and $e = 0$): An example will be constructed of a quasi-reproducible solution, with respect to a certain wealth vector. It will follow that a reproducible solution with respect to an endowment vector can be constructed with the desired property, according to Corollary 2.8.

There are two goods and labor. There is one linear production process, specified by $(-1; 0, 0; 1, 2)$, and one capitalist. The production set is the set of points equal or inferior to a multiple of $(-1; 0, 0; 1, 2)$. The capitalist's capital is 10 units. Notice Assumptions A1–A6, are all satisfied but Assumption A7 fails. (For example, as much labor is required to produce $\hat{\alpha} = (1, 1)$ as to produce $\hat{\alpha} = (1, 2)$.) Let $b = (1, 1)$. It can easily be verified that for any price vector p in the simplex $pb = 1$ the profit maximizing set is:

$$A^1(p) = A(p) = (-10; 0, 0; 10, 20).$$

[5] Since this writing, the author has shown that Assumption A7 is in fact necessary for the equivalence of positive profits and positive exploitation. If A7 is violated for P, an economy can always be constructed with P as production set, in which reproducible solutions with positive profits and zero exploitation occur. (Available in "A Necessary and Sufficient Condition for the Equivalence of Positive Profits and Positive Exploitation," from the author.)

The point $\alpha = (-10; 0, 0; 10, 20)$ is a reproducible solution for any such p, since $\alpha_0 b = (10, 10) \leqslant (10, 20)$.

Choose $p = (1/2, 1/2)$. Then profits at $(-10; 0, 0; 10, 20)$ are positive. The rate of exploitation, however, is zero at α, since the socially necessary labor required to produce the bundle (10, 10) is 10; that is, one must produce (10, 20) to produce (10, 10).

This example provides more insight into the "positive profits-negative surplus value" debate introduced by Steedman. (For a summary of the discussion, see Steedman [**13**, Chapter 11].) Steedman pointed out that if one defines labor values additively in a joint production model, positive profits can coexist with negative "surplus value." Morishima's answer to the problem (Morishima [**6** and **7**]) was to define embodied labor in the von Neumann model similar to the way it has been defined in this paper. Under various assumptions, Morishima [**7**] proved the equivalence of positive profits and positive exploitation. The above example of a simple von Neumann model shows that if independence of production activities (Assumption A7) is not assumed, even the more general definition of embodied labor will not guarantee the equivalence of positive profits and positive exploitation, although, in any case, the perversity of Steedman's negative exploitation cannot occur.

4. THE SOCIAL DETERMINATION OF THE VALUE OF LABOR POWER

In the model presented, there remains one aspect which is not sufficiently general. The subsistence bundle, b, is taken as given. Determining the "value of labor power" (which is tantamount to determining the bundle b) is a problem with a long history. Marx was clear, at least, that the subsistence bundle was not intended as a biologically minimum consumption, but as a level of consumption which was determined by factors including an "historical and moral element." Yet the formal models of Marxian economics have made no attempt to include this feature. (Morishima, for example, simply posits an exogenous consumption bundle for workers, as has been done here.) In Sraffa's model, the question is also left unresolved. It is simply observed that the profit and wage rates are inversely related, so that once one is set, so is the other. No argumentation is provided, however, to determine either rate within the system.

This hiatus is a particularly important one in Marxian value theory. For by assuming workers' consumption to be fixed before the drama of exchange and production—that is, the circuit of capital—occurs, we are assuming workers to be somehow above the social process of production and exchange—and this is a violation of the most fundamental Marxian tenet. Workers' necessary consumption should be socially determined, and not taken as an exogenous datum of the system. To do the latter is to examine a robot economy, where the "factor" labor power might just as well enter production as part of the input-output matrix in the form of commodities which workers consume, eclipsing entirely the concious element of human work which is at the heart of the Marxian vision. What distinguishes labor power from other inputs into production, at the abstract level of modelling engaged in here, must be that its value (that is, the consumption b) does not enter as a technological datum, but is determined by social interactions.

We desire, then, to relax the assumption of an exogenously given bundle b, but to maintain the Marxian notion of workers' *necessary* consumption. Indeed, the notion of necessity is important to preserve the rationale behind the investigation of *reproducible* solutions, as opposed to some more subjectively specified concept of equilibrium.

There is a straightforward indication in Marx as to how we should proceed to determine the value of labor power. Workers' necessary consumption is determined from two directions: by the consumption necessities which a particular mode of production requires for purely technological reasons, and by the form of consciousness created among workers by a particular mode of production. An example of the first of these causations is that modern technologies require workers to possess certain skills, the education for which thereby becomes a part of the workers' necessary consumption. The second determination is contained in Marx's materialist philosophy: that consciousness is determined fundamentally by one's relation to the means of production.

The value of labor power, then, possesses a complex determination, whose origins are in the method of production. We therefore postulate:

POSTULATE B1: There is a continuous function $b(\alpha^1, \alpha^2, \ldots, \alpha^N) = b$ defined on $P^1 \times P^2 \times \ldots \times P^N$ into $\mathbb{R}^n_+$. Furthermore, there exist $b_* > 0$ and b^* such that $b_* \leqslant b(\alpha^1, \ldots, \alpha^N) \leqslant b^*$ for any point $(\alpha^1, \ldots, \alpha^N)$.

Necessary consumption thus arises as a consequence of the particular technology adopted. Postulating the existence of such a primitive function b is obviously very general, and in this sense cannot be considered to provide a theory of the determination of workers' consumption. (Similarly, it can be argued that postulating primitive preference orderings in neoclassical theory is so general as to not provide a theory of consumer behavior.)

If the value of labor power is socially determined in the above sense, do we have a meaningful economic model—do reproducible solutions exist?

DEFINITION: *A reproducible solution with the value of labor power socially determined*, with respect to endowments $(\omega^1, \ldots, \omega^N)$, is a set $(p, \alpha^1, \ldots, \alpha^N, b)$ such that: (a) $\alpha^\nu \in A^\nu(p)$, $b = b(\alpha^1, \ldots, \alpha^N)$; (b) $\Sigma\, \hat{\alpha}^\nu \geqslant \Sigma\, \alpha_0^\nu b$; (c) $pb = 1$; (d) $\Sigma\, (\alpha_0^\nu b + \underline{\alpha}^\nu) \leqslant \Sigma\, \omega^\nu$; where $A^\nu(p)$ is defined in terms of the capital constraint $p(\alpha_0^\nu b + \underline{\alpha}^\nu) \leqslant p\omega^\nu$, as before.

THEOREM 4.1: *Let $(W^1, \ldots, W^N)$ be any vector of wealths. There exists a set of endowments*

$$\bar{\omega} = (\omega^1, \ldots, \omega^N)$$

with respect to which a reproducible solution, with value of labor power socially determined, exists; and $p\omega^\nu = W^\nu$, for all ν, where p is the viable price vector.

The proof of this theorem is a technical application of Debreu's social equilibrium existence theorem (Debreu [**2**]), and is not provided here due to space limitations.

Similarly, the rate of exploitation can be defined in this context:

DEFINITION: The *rate of exploitation with the value of labor power socially determined* is defined on points $a \in P^1 \times \ldots \times P^N$. Let a be given, $a = (\alpha^1, \ldots, \alpha^N)$, $\alpha = \Sigma\, \alpha^\nu$. Then

$$e(a) = \frac{\alpha_0}{\text{l.v.}(\alpha_0 b(a))} - 1$$

where $\text{l.v.}(\alpha_0(b(\alpha))$ is defined as before.

A version of the Fundamental Marxian Theorem continues to hold:

THEOREM 4.2: *Under Assumptions* A1–A5, A7: *the rate of exploitation with the value of labor power socially determined is positive at a reproducible solution* $(p, \alpha^1, \ldots, \alpha^N, b)$ *if and only if total profits are positive there. Furthermore,* $e(\alpha^1, \ldots, \alpha^N) > 0$ *if and only if there exists a point* $\alpha \in P$ *such that* $e_b(\alpha) > 0$, *where* $e_b(\alpha)$ *is the rate of exploitation at* α *evaluated for workers consumption fixed at b.*

NOTE: It is not necessary for this theorem to assume the function $b(\alpha^1, \ldots, \alpha^N)$ is continuous.

Theorem 4.2 is weaker than Theorem 3.1. It is nevertheless a "fundamental" Marxian theorem as it shows that the *sine qua non* for reproducible capitalist production at positive profits is positive exploitation. It can be shown that Theorem 4.2 is necessarily weaker than Theorem 3.1. There are economies which fulfill all the postulates including B1, yet (a) positive and zero profits can both occur at reproducible solutions, and (b) $e(\alpha^1, \ldots, \alpha^N) > 0$ occurs but all reproducible solutions yield zero profit. This being the case, Theorem 4.2 is evidently the strongest version of Theorem 3.1 which is true when the value of labor power is socially determined. Examples verifying claims (a) and (b) are available from the author.

5. SUMMARY

It has been the aim of this paper to place the Marxian-Sraffian economic model within the jurisdiction of general equilibrium analysis, and to generalize the usual Marxian analysis to economic environments other than the linear one. In Section 1 it was shown that by introducing capital constraints, the linear Marxian model can be viewed as one of the general equilibrium genre. The equal-profit-rate price equilibria of Sraffa-Morishima analysis are shown to be the only price vectors associated with reproducible solutions under suitable assumptions.

In Section 2, the general notion of Marxian equilibrium (reproducible solution) was discussed, and the production environment was generalized to one of convex production sets. Exploitation was defined. The existence of Marxian general equilibrium in the case of a given subsistence bundle was proved. In Section 3 it was shown that exploitation is the key concept in these economies, as it provides a necessary and sufficient condition for the existence of non-trivial equilibria.

An exogenously determined subsistence bundle for workers indicates, however, an incomplete aspect of the theory. In Section 4 it was posited that a Marxian determination of workers' consumption makes the consumption bundle socially determined. The nonutilitarian basis of this premise should be noted. It can be shown that a reproducible solution, with the value of labor power socially determined, exists.

The reader may have noticed that the concept of labor value plays no role in describing exchange in this model; it enters only in the definition of exploitation. Although this is unusual in Marxian analysis, it is quite appropriate, as the author believes that the only role labor values should play in a Marxian model is in defining exploitation. Efforts to use labor values in both the theories of exchange and exploitation have led to much obfuscation in Marxian economics.

Finally, a caveat should be added on the lineage of the model. It is perhaps an injustice to Marx to characterize models such as this one as Marxian. Certain aspects of capitalist production are captured here, but perhaps not those aspects most important to Marx. Two important omissions must be mentioned: first, exploitation as here defined, would exist also under socialism. For Marx, exploitation consists not simply in the existence of a surplus, of a "productive" economy, but in the property relations which determine the appropriation of the surplus. Second, the capitalist labor process is eclipsed in the present model: there is no theory of what determines technology, the real wage, and the organization of production. The intended scope of the present paper has been, more narrowly, a discussion of those aspects of Marxian economics which have the clearest counterparts in neoclassical general equilibrium theory.

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Manuscript received October, 1977; revision received January, 1979.

APPENDIX I

ON THE STABILITY OF REPRODUCIBLE SOLUTIONS[5]

In this Appendix, we attempt to dynamize the model of Section 1 and ask: if ω, the initial endowment vector does not lie in the cone C^*, what happens?

The entire discussion is carried out for the case of a linear productive indecomposable matrix M.

We first define a weaker equilibrium notion than reproducible solution. Individual capitalists are not concerned with whether a given equilibrium is *reproducible*; they simply maximize profits. This motivates the following definition.

[5] This appendix has benefited from a discussion with A. Mas-Colell.

DEFINITION: A *competitive equilibrium* (CE) with respect to initial endowments $(\omega^1, \ldots, \omega^N)$ is a pair (p, x) such that: (a) $x \in A(p)$, (b) $pb = 1$, (c) $Ax + b(Lx) \leqslant \omega (Mx \leqslant \omega)$. That is, a CE which is also reproducible is a RS. Even if $\omega \notin C^*$, competitive equilibria exist:

THEOREM A.1: *Given* $(\omega^1, \ldots, \omega^N)$, *a competitive equilibrium exists.*

PROOF: Let $S = \{p | pb = 1\}$, the simplex. (Assume $b > 0$.) Construct the correspondence

$$z(p) = \left\{ Mx - \omega | x = \sum_{x^\nu} x^\nu \in A^\nu(p) \right\}.$$

It is easily verified that $z(p)$ satisfies the requirements of the Gale-Nikaido Lemma; hence a vector $\bar{p}$ and $\bar{x}$ exist such that $M\bar{x} - \omega \leqslant 0$ and $\bar{x} \in A(\bar{p})$. *Q.E.D.*

NOTE: The "fundamental Marxian theorem" does not, in general, hold for *competitive equilibria* (as opposed to *reproducible solutions*). For example, let $A + bL \equiv M$ be a productive positive matrix (and so $e > 0$), but suppose the aggregate endowment of some good is zero. Then the only competitive equilibrium will be the one of inaction (and hence zero profits), since all processes are blocked from operation by the lack of one good.

If $\omega \notin C^*$, what happens? We shall say a competitive equilibrium is obtained, (p, x). The initial endowments for the *next* period shall be

$$\omega(t+1) = \omega(t) - Mx + x;$$

that is, the old endowment plus net outputs $x - Mx$.

DEFINITION: A good is in *excess supply* at a $CE(p, x)$ if $\omega_i > (Mx)_i$.

LEMMA A.2: *If good i is in excess supply at CE* (p, x) *then it is not produced*:

$$\omega_i > (Mx)_i \Rightarrow x_i = 0.$$

Also,

$$\omega_i > (Mx)_i \Rightarrow p_i = 0.$$

LEMMA A.3: *Let* (p, x) *be a CE for technology* $\{A, L, b\}$ *for which it is assumed that* $e > 0$. *If* $p\omega > 0$, *then* (p, x) *generates positive total profits.*

PROOF OF LEMMA A.3: Suppose there were zero total profits at (p, x). Then each capitalist makes zero profits. Since $p\omega > 0$, for at least one capitalist, $p\omega^\nu > 0$. This capitalist would operate a positive profit rate activity if there were one: so all activities must have nonpositive profit rates, hence:

$$p \leqslant pM.$$

Let x^* be the column eigenvector of M: we know $Mx^* < x^*$, by Frobenius-Perron. We know $p = pM$ is impossible by Frobenius, since M has a unique row eigenvector, and it is associated with eigenvalue $1/(1+\pi) < 1$. Hence

$$p \leq pM.$$

Post-multiplying by x^* gives:

$$px^* < pMx^*.$$

But $x^* = (1+\pi)Mx^*$ which implies

$$px^* > pMx^*.$$

This contradicts the original assumption. *Q.E.D.*

PROOF OF LEMMA A.2: Consider capitalist with endowment ω^ν. Let (p, x) be a CE. Say $p\omega > 0$. By Lemma A.3, there are positive total profits; hence there must be a positive profit rate process, and so capitalist ν can certainly operate this process and make positive profits. By linearity capitalist ν will therefore operate to the limit of his capital constraint. That is he will choose a vector of activity levels x^ν such that:

(A.1) $\quad pMx^\nu = p\omega^\nu.$

On the other hand, if $p\omega^\nu = 0$, then certainly (A.1) holds also. Hence, adding gives

$$pMx = p\omega$$

or

(A.2) $\quad p(Mx-\omega) = 0.$

By (A.2) it follows that any good which is in excess supply at CE (p, x) has zero price. Consequently, the profit rate for operating the activity which produces that good is nonpositive. However, by Lemma A.2, we know there are positive profit rate processes: hence the nonpositive profit rate processes will not be operated. *Q.E.D.*

Using Lemma A.3, it is possible to construct a complete taxonomy of the CE for the case of an indecomposable matrix M. We shall not do this here but instead will indicate how this taxonomy is constructed for a 2×2 matrix M.

Let M be 2×2, and let d_1 and d_2 be the (positive) input requirement vectors for operating the first and second processes, respectively, at unit levels:

$$d_1 = M(1, 0),$$

$$d_2 = M(0, 1).$$

Let us suppose d_i lies as pictured in Figure 1, which is the positive orthant in input-requirements space. Then any vector of outputs x will generate input requirements lying in the cone A_1OA_2. Suppose the initial endowment vector lies to the right of the cone, as pictured. Then, by feasibility, any competitive equilibrium (p, x) must have its input requirements Mx lying in the rectangle $C\omega DO$ to the southwest of ω; since $Mx \in A_1OA_2$ also, it follows that good 1 must be in excess supply at competitive equilibrium. It follows by Lemma A.2 that $p_1 = 0 = x_1$. Thus only process 2 is operated, and process 2 is operated to the limit of the capital constraints.

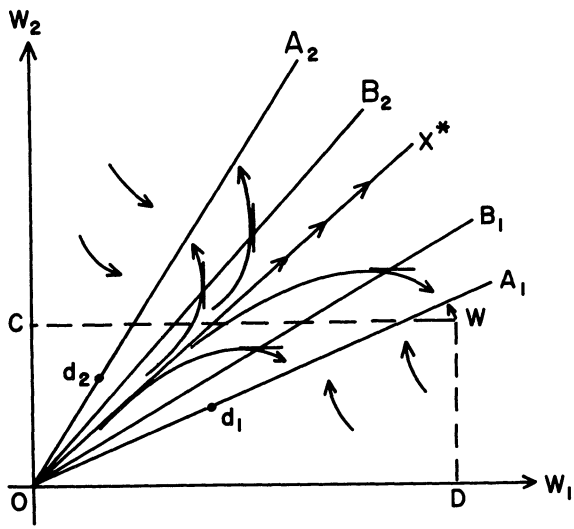

FIGURE 1

To get a dynamic story we must replace the discrete model here with a continuous model. We shall not do this rigorously here. The idea is as follows: we think of ω not as a stock of inputs to be used up, but as a stock of capital generating capital services over a large number of periods. At each period, the initial endowment of capital, ω, is adjusted by the net product $x - Mx$, in the sense discussed:

$$\omega(t+1) = \omega(t) - M(x(t)) + x(t).$$

We have observed that if $\omega(t)$ lies to the right of the cone A_1OA_2 then we get a solution $x(t)$ with the property:

$$x_1(t) = 0,$$

$$x_2(t) > 0.$$

Furthermore, $M(x(t)) > 0$ and we therefore have

$$\omega_1(t+1) < \omega_1(t),$$

$$\omega_2(t+1) > \omega_2(t).$$

(Observe, also, $x_2(t) > (M(x(t)))_2$, by productivity of M.) Hence, the net endowment vector moves towards the cone, as indicated by the arrow in Figure 1.

We can carry out this analysis for other positions of the initial endowment vector, and the phase diagram of Figure 1 is generated.

In Figure 1, x^* is the balanced growth path. B_1OB_2 is the cone C^*, in which reproducible solutions exist. A_1OA_2 is the cone of feasible input requirements. If ω lies precisely on the balanced growth path, then there is a competitive equilibrium which keeps it there. However if ω lies elsewhere in the cone A_1OA_2, the endowment vector, over time, moves away from the balanced path x^* towards the boundary of the cone. (On the cone's boundaries, behavior is not continuous.) If ω lies initially outside the cone, it moves towards the cone. From the figure, it can be seen that the difference between $\omega \in B_1OB_2$ and $\omega \notin B_1OB_2$ is this: if and only if $\omega \in B_1OB_2$, the endowments of both goods increase during that period.

Hence, the dynamics are unstable. ω does not converge to a value inside B_1OB_2.

There is a second case, namely when the factor intensities are reversed, as in Figure 2. In the first case each process is relatively intensive in its own input. In the factor reversal case two, the phase diagram of Figure 2 holds. Thus, the case of factor intensity reversal does generate stability and a convergence of endowments towards the domain B_1OB_2 where reproducible solutions exist.

This discussion shows that in general there is no convergence towards reproducible solutions. It is possible, though perhaps not fruitful, to interpret the paths which lead away from the cone of reproducible endowments as the paths leading to the "disproportionality crises" of Marx.

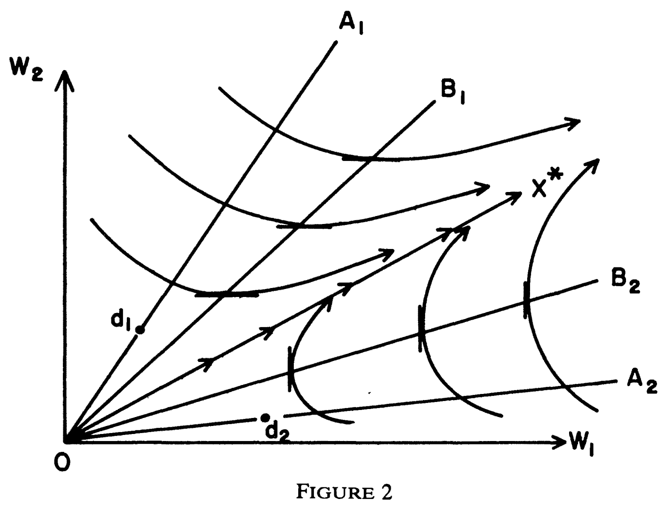

FIGURE 2

APPENDIX II

ON THE TEMPORARY EQUILIBRIUM NATURE OF THE MODEL

We here formulate the problem in a temporary equilibrium context.

ASSUMPTION A: Let $S=\{p|pb=1\}$. (We assume $b>0$ so S is a simplex.) There is for each capitalist ν a continuous expectation function $\psi^\nu: S^{T+1} \Rightarrow \mathbb{R}^n_+$; $\psi^\nu(p_t;\ p_{t-1},\ p_{t-2}, \ldots,\ p_{t-T})=p^\nu_{t+1}$ which describes the expected prices for capitalist ν.

REMARK: Note it is not necessary to assume that $p^\nu_{t+1} \in S$. Of course, p_{t+1} can only be equilibrium prices if $p_{t+1} \in S$. But the existence of a temporary equilibrium does not depend on that expectation.

With expectations, as has been shown, the capitalist's program at price p_t, becomes:

$$\begin{aligned}&\text{choose } \alpha^\nu \in P^\nu,\ \delta^\nu \in \mathbb{R}^n_+\\ &\text{to maximize } p^\nu_{t+1}\bar{\alpha}^\nu + p^\nu_{t+1}\delta^\nu\\ &\text{subject to}\quad \text{(i)}\ \alpha^\nu_0 + p_t\underset{\sim}{\alpha}^\nu \leqslant p_t\omega^\nu_t,\\ &\qquad\qquad\quad \text{(ii)}\ p_t\delta^\nu \leqslant p_t\omega_t - (p_t\underset{\sim}{\alpha}^\nu + \alpha^\nu_0)\end{aligned}$$

where $p^\nu_{t+1}=\psi^\nu(p_t, p_{t-1}, \ldots, p_{t-T})$. Analogous to the definitions of the feasible and profit maximizing sets defined in the body of paper, let:

$$B^\nu(p_t)=\{(\alpha^\nu, \delta^\nu)|\text{(i) and (ii) are satisfied}\},$$
$$A^\nu(p_t)=\{(\alpha^\nu, \delta^\nu)\in B^\nu(p_t)|p^\nu_{t+1}\bar{\alpha}^\nu + p^\nu_{t+1}\delta^\nu \text{ is maximized}\}.$$

DEFINITION: A *temporary equilibrium* is a set $\{p_t, \alpha^\nu, \delta^\nu;\ \nu=1, N\}$ such that:

(A.3) $\quad \forall\nu(\alpha^\nu, \delta^\nu)\in A^\nu(p_t),$

(A.4) $\quad \alpha_0 b + \underset{\sim}{\alpha} + \delta \leqslant \omega.$

(Condition (A.4) states that the profit maximizing plans which are chosen by capitalists are feasible for the economy as a whole.) Notice a temporary equilibrium is not (necessarily) a reproducible solution. It is only a competitive equilibrium, in the sense of Appendix I.

THEOREM: *Under the Assumption* A1–A4 *and Assumption* A *on the expectation functions* ψ^ν, *a temporary equilibrium exists.*

SKETCH OF PROOF:
1. $B^\nu(p_t)$ is lower-hemi-continuous (proved as in Lemma 2.6).
2. Define $z(p_t)=\{\omega-(\alpha_0 b+\underset{\sim}{\alpha}+\delta)|(\alpha, \delta)\in A(p_t)\}$. Notice $p_t z(p_t)\leqslant 0\ \forall p_t$ by the definition of the feasible sets $B^\nu(p_t)$.
3. $z(p_t)$ is upper-hemi-continuous: this follows, as in the proof of Theorem 2.5, from the lhc of $B^\nu(pt)$ and the continuity of the functions ψ^ν.
4. $z(p_t)$ is convex and closed and its image set is compact.
5. It follows by the Gale-Nikaido lemma that $\exists p_t,\ p_t z(p_t)\leqslant 0$ which proves the Theorem.

In the spirit of the prior definitions, we define reproducible solutions in the temporary equilibrium context.

DEFINITION: A *temporary reproducible solution* is a set $\{p_t, \alpha^\nu, \delta^\nu\}$ such that: (i) $\{p_t, \alpha^\nu, \delta^\nu\}$ is a temporary equilibrium; (ii) $\bar{\alpha}+\delta \geqslant \omega_t$. (This definition reduces to the old definition when $p^\nu_{t+1}=p_t, \forall\nu$. $\bar{\alpha}+\delta$ constitute the new endowments, ω_{t+1}.)

NOTE: A temporary reproducible solution does not assume price expectations are realized. The question posed is: do there exist prices p_t which, given capitalists' expectations, will reproduce the system next period?

We cannot, however, prove that temporary reproducible solutions exist, even by restricting the initial endowment vectors as was done in Corollary 2.8. An example shows this. Suppose the expectation functions are the same for all capitalists and have this property: $(\forall p_t)(p^1_{t+1}=0)$, where p^1_{t+1} is the expected price of good 1. Suppose we are in a simple 2 good, 2 process Leontief technology. Then no capitalist will operate process 1 at a temporary equilibrium. If process 2 uses some good 1, then the stocks of good 1 will surely be run down if the economy operates at all. Hence, the only possible temporary reproducible solution is with no production. Therefore, the only activity that can take place, for any p_t, is speculation in commodities. However, since $p^2_{t+1}>0$ and $p^1_{t+1}=0$ for all capitalists, all capitalists will desire to hold only good 2. Hence, there is no feasible solution, assuming some endowment of good 1 exists in the economy.

Hence, in general, the notion of temporary reproducibility is not an interesting one with general expectation functions. One can, however, ask whether *stationary states* exist for a given expectation structure $\{\psi^\nu\}$. We assume, which is a standard assumption of the temporary equilibrium literature:

ASSUMPTION: $(\forall p, \forall^\nu)\psi^\nu(p; p, \ldots, p)=p$. A stationary state is a sequence of temporary equilibria $\{p^t, \alpha^\nu_t, \delta^\nu_t\}$ such that $p_t=p$ for all sufficiently large t, and $\{p_t, \alpha^\nu_t, \delta^\nu_t\}$ are reproducible.

THEOREM: *Let the production sets $P^\nu=P$ be a cone. Then, for suitable initial endowments ω, a stationary state exists.*

PROOF: According to Karlin's [4] generalization of the von Neumann model, a von Neumann equilibrium exists if the production sets are cones satisfying our Assumptions A1–A4 (and, in addition, free disposability). We employ the balanced growth path and price vector of the von Neumann equilibrium to generate a stationary state in our model.

First, modify the production set to eliminate direct labor, and express commodity outputs as a function solely of commodity inputs:

$$\tilde{P}=\{(\alpha_0 b+\underset{\sim}{\alpha}, \bar{\alpha})|(-\alpha_0, -\underset{\sim}{\alpha}, \bar{\alpha})\in P\}.$$

$\tilde{P}$ is a cone with the required properties. By Karlin's theorem, there is a ray of maximal expansion in $\tilde{P}$:

$$(\exists(\alpha^*_0 b+\underset{\sim}{\alpha}^*, \bar{\alpha}^*)\in\tilde{P})(\exists g\in\mathbb{R}, g>-1)((1+g)(\alpha^*_0 b+\underset{\sim}{\alpha}^*)=\bar{\alpha}^*), \tag{A.5}$$

$$(\exists p\geqslant 0)(\forall(\alpha_0 b+\underset{\sim}{\alpha}, \bar{\alpha})\in\tilde{P})(p\bar{\alpha}\leqslant(1+g)p(\alpha_0 b+\underset{\sim}{\alpha})). \tag{A.6}$$

We proceed to examine whether p can support a stationary state. That is, suppose p has existed in the past for T periods. So $p^\nu_{t+1}=p$, for all ν, by the expectation assumption. Will p continue to support a reproducible solution in the future? That is, are expectations realizable?

Since $p_{t+1}=p_t=p$, the capitalists' objective function becomes, simply:

$$\max p\bar{\alpha}-(p\underset{\sim}{\alpha}+\alpha_0),$$

$$(-\alpha_0, -\underset{\sim}{\alpha}, \bar{\alpha})\in B^\nu(p).$$

(As is shown in Proposition 2.2, there is no speculation in commodities under stationary expectation.) By property (A.5) of the von Neumann-Karlin equilibrium, the rate of profit at the process $(\alpha^*_0, \underset{\sim}{\alpha}^*, \bar{\alpha}^*)$ is $\rho=1+g$, while the rate of profit at all other production points is not more than ρ, by property (A.6). Since P is a cone, capitalists can, therefore, maximize profits by investing all their capital $p\omega$ in process $(\alpha^*_0, \underset{\sim}{\alpha}^*, \bar{\alpha}^*)$. If endowments ω lie along the balanced growth path, then this action constitutes a reproducible solution at p. Furthermore, by property (A.5), next period's endowments will continue to lie on the balanced growth path, and p will generate a stationary state.

Q.E.D.

Finally, one can remark that one cannot, in general, weaken the assumption that the endowment vector must lie on the von Neumann ray to guarantee the existence of a stationary state. If we have the cone technology of Figure 1, Appendix I, and if ω lies off the von Neumann ray, but in the cone B_1OB_2, the equal-profit-rate price vector, which is the von Neumann price ray, will generate reproducible solutions for a while, but then will not, as endowments pass across the rays $\overline{OB_1}$ and $\overline{OB_2}$.

In summary, the temporary equilibrium treatment of the model shows: (a) temporary (competitive) equilibria always exist, for any continuous expectation function and initial endowments; (b) temporary reproducible solutions do not, in general, exist; (c) if the production sets are a cone, then stationary states of temporary reproducible solutions exist, if the initial endowment vector lies along the balanced growth path; (d) if the endowment vector does not lie on the balanced growth path, then, in general, stationary states do not exist even with the cone technology; (e) in stationary states, the "fundamental Marxian theorem" holds. (This is an immediate consequence of Theorem 3.1.)

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