CSE 555 Spring 2009 Mid-Term Exam

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The exam is worth 100 points total and each question is marked with its portion. The exam is closed book/notes. You have 70 minutes to complete the exam. Use the provided white paper, write your name on the top of each sheet and number them. Write legibly.

Problem 1: "Recall" Questions (25pts)

Answer each in one or two sentences.

1. (5pts) What is the fundamental difference between Maximum Likelihood parameter estimation and Bayesian parameter estimation?
2. (5pts) What quantity is PCA maximizing during dimension reduction?
3. (5pts) Describe Receiver Operating Characteristics or ROC-Curves (illustrate if necessary).
4. (5pts) What is the Curse of Dimensionality?
5. (5pts) What is the key idea of the No Free Lunch Theorem?

Solution:
Answers are all directly in the notes.

Problem 2: Bayesian Reasoning (25pts)

Monty Hall Formulate and solve this classifical problem using the Bayes rule. Imagine you are on a gameshow and you're given the choice of three doors: behind one door is a car and behind the other two doors are goats. You have the opportunity to select a door (say No. 1). Then the host, who knows exactly what is behind each door and will not reveal the car, will open a different door (i.e., one that has a goat). The host then asks you if you want to switch your selection to the last remaining door.

1. (3pts) Formulate the problem using the Bayes rule, i.e., what are the random variables and the input data. What are the meaning of the prior and the posterior probabilities in this problem (one sentence each).

   Solution:
   Without the loss of generality, suppose that we chose door 1 at the beginning.
   Random Variables: C_i represents the state that the car is behind door $i, i \in [1, 2, 3]$, and H_j represents the state that the host opens door $j, j \in [2, 3]$.
   Input Data: The door that the host opens.
   Prior Probability: The probability of winning the car before the host opens the door.
   Posterior Probability: The probability of winning the car after the host opens the door.

2. (3pts) What are the probability values for the prior?

   Solution:
   $P(C_1) = \frac{1}{3}, P(C_2) = \frac{1}{3}, P(C_3) = \frac{1}{3}$.

3. (3pts) What are the probability values for the likelihood?

Solution:

$P(H_2|C_1) = \frac{1}{2}, P(H_3|C_1) = \frac{1}{2}$,
$P(H_2|C_2) = 0, P(H_3|C_2) = 1$,
$P(H_2|C_3) = 1, P(H_3|C_3) = 0$.

4. (3pts) Derive the posterior probability (include intermediate steps).

Solution:

(a) If we chose to stay after the host opened door 2, the probability of winning the car is:

$$P(C_1|H_2) = \frac{P(H_2|C_1)P(C_1)}{P(H_2)} = \frac{P(H_2|C_1)P(C_1)}{P(H_2|C_1)P(C_1) + P(H_2|C_2)P(C_2) + P(H_2|C_3)P(C_3)}$$
$$= \frac{1/2 \cdot 1/3}{1/2 \cdot 1/3 + 0 \cdot 1/3 + 1 \cdot 1/3} = \frac{1/6}{1/2} = \frac{1}{3}.$$

(b) If we chose to stay after the host opened door 3, the probability of winning the car is:

$$P(C_1|H_3) = \frac{P(H_3|C_1)P(C_1)}{P(H_3)} = \frac{P(H_3|C_1)P(C_1)}{P(H_3|C_1)P(C_1) + P(H_3|C_2)P(C_2) + P(H_3|C_3)P(C_3)}$$
$$= \frac{1/2 \cdot 1/3}{1/2 \cdot 1/3 + 1 \cdot 1/3 + 0 \cdot 1/3} = \frac{1/6}{1/2} = \frac{1}{3}.$$

$\Rightarrow$ Our posterior probability of winning the car if we chose not the switch is always $\frac{1}{3}$, no matter what door the host opened.

(c) If we chose to switch doors (from 1 to 3) after the host opened door 2, the probability of winning is:

$$P(C_3|H_2) = \frac{P(H_2|C_3)P(C_3)}{P(H_2)} = \frac{P(H_2|C_3)P(C_3)}{P(H_2|C_1)P(C_1) + P(H_2|C_2)P(C_2) + P(H_2|C_3)P(C_3)}$$
$$= \frac{1 \cdot 1/3}{1/2 \cdot 1/3 + 0 \cdot 1/3 + 1 \cdot 1/3} = \frac{1/3}{1/2} = \frac{2}{3}.$$

(d) If we chose to switch doors (from 1 to 2) after the host opened door 3, the probability of winning is:

$$P(C_2|H_3) = \frac{P(H_3|C_2)P(C_2)}{P(H_3)} = \frac{P(H_3|C_2)P(C_2)}{P(H_3|C_1)P(C_1) + P(H_3|C_2)P(C_2) + P(H_3|C_3)P(C_3)}$$
$$= \frac{1 \cdot 1/3}{1/2 \cdot 1/3 + 1 \cdot 1/3 + 0 \cdot 1/3} = \frac{1/3}{1/2} = \frac{2}{3}.$$

$\Rightarrow$ Our posterior probability of winning the car if we chose not the switch is always $\frac{2}{3}$, no matter what door the host opened.

5. (3pts) Is it in the contestant's advantage to switch his/her selection? Why?

Solution:

Yes, due to the posterior probabilities we calculated above.

6. (10pts) Now, consider the following twist. The host is having trouble remembering what is behind each of the doors. So, we cannot guarantee that he will not accidentally open the door for the car. We only know that he will not open the door the contestant has opened. Indeed, if he accidentally opens the door for the car, the contestant

wins. How does this change the situation? Is it now in the contestant's advantage to switch? Rederive your probabilities to justify your answer.

Solution:

(a) Again, let's assume that we chose door 1 without the loss of generality. The probability values for the likelihoods will change to:
$P(H_2|C_1) = \frac{1}{2}, P(H_3|C_1) = \frac{1}{2}$,
$P(H_2|C_2) = \frac{1}{2}, P(H_3|C_2) = \frac{1}{2}$,
$P(H_2|C_3) = \frac{1}{2}, P(H_3|C_3) = \frac{1}{2}$,
thus causing the posterior probabilities to change.

(b) It doesn't matter, because the posterior probabilities of switching or not switching is equivalent.

(c) i. If we chose to stay after the host opened door 2, the probability of winning the car is:

$$P(\text{winning}|H_2) = P(C_1|H_2) + P(C_2|H_2) = \frac{P(H_2|C_1)P(C_1)}{P(H_2)} + \frac{P(H_2|C_2)P(C_2)}{P(H_2)}$$
$$= \frac{1/2 \cdot 1/3}{1/2} + \frac{1/2 \cdot 1/3}{1/2} = \frac{2}{3}.$$

ii. If we chose to stay after the host opened door 3, the probability of winning the car is:

$$P(\text{winning}|H_3) = P(C_1|H_3) + P(C_3|H_3) = \frac{P(H_3|C_1)P(C_1)}{P(H_3)} + \frac{P(H_3|C_3)P(C_3)}{P(H_3)}$$
$$= \frac{1/2 \cdot 1/3}{1/2} + \frac{1/2 \cdot 1/3}{1/2} = \frac{2}{3}.$$

$\Rightarrow$ Our posterior probability of winning the car if we chose not the switch is always $\frac{2}{3}$, no matter what door the host opened.

iii. If we chose to switch doors (from 1 to 3) after the host opened door 2, the probability of winning is:

$$P(\text{winning}|H_2) = P(C_3|H_2) + P(C_2|H_2) = \frac{P(H_2|C_3)P(C_3)}{P(H_2)} + \frac{P(H_2|C_2)P(C_2)}{P(H_2)}$$
$$= \frac{1/2 \cdot 1/3}{1/2} + \frac{1/2 \cdot 1/3}{1/2} = \frac{2}{3}.$$

iv. If we chose to switch doors (from 1 to 2) after the host opened door 3, the probability of winning is:

$$P(\text{winning}|H_3) = P(C_2|H_3) + P(C_3|H_3) = \frac{P(H_3|C_2)P(C_2)}{P(H_3)} + \frac{P(H_3|C_3)P(C_3)}{P(H_2)}$$
$$= \frac{1/2 \cdot 1/3}{1/2} + \frac{1/2 \cdot 1/3}{1/2} = \frac{2}{3}.$$

$\Rightarrow$ Our posterior probability of winning the car if we chose not the switch is always $\frac{2}{3}$, no matter what door the host opened.

Problem 3: Discriminants (25pts)

Consider two univariate Gaussian distributions with means μ_1 and μ_2, respectively, and an arbitrary, but known and