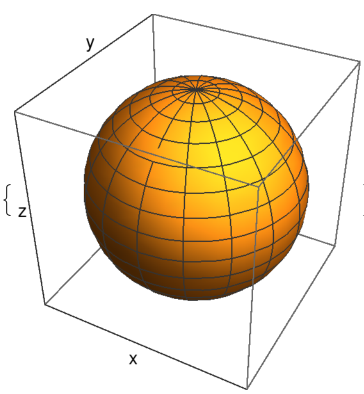
Global parameters

In[∘]:=
```
r = Sqrt[x^2 + y^2 + z^2] ;
rbm = {x, y, z};
tospherical = {x → r * Sin[θ] Cos[ϕ],
   y → r * Sin[θ] Sin[ϕ], z → r * Cos[θ], x^2 + y^2 + z^2 → r^2, Sqrt[x^2 + y^2 + z^2] → r};
tocartesian = {Sin[θ] → Sqrt[x^2 + y^2]/r, Cos[θ] → z/r,
   Sin[ϕ] → y/Sqrt[x^2 + y^2], Cos[ϕ] → x/Sqrt[x^2 + y^2]} ;
L[f_] := -I * {x, y, z} × Grad[f, {x, y, z}]
```

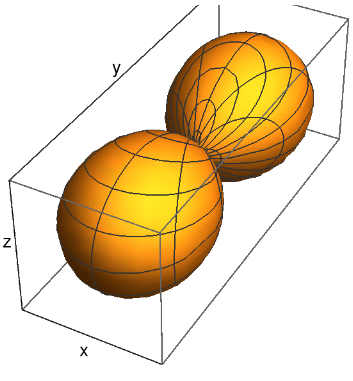
In[∘]:=
```
Y[l_, m_, θ_: θ, ϕ_: ϕ] :=
 If[m == 0, SphericalHarmonicY[l, m, θ, ϕ],
   If[m < 0, I * 1/Sqrt[2]
     (SphericalHarmonicY[l, -m, θ, ϕ] + (-1)^(Abs[m]+1) * SphericalHarmonicY[l, m, θ, ϕ]),
    1/Sqrt[2] (SphericalHarmonicY[l, -m, θ, ϕ] +
      (-1)^Abs[m] * SphericalHarmonicY[l, m, θ, ϕ])]] // FullSimplify
Yxyz[l_, m_, θ_: θ, ϕ_: ϕ] := ReplaceAll[
   TrigExpand[Y[l, m, θ, ϕ]], tocartesian
  ] // Simplify
Olm[l_, m_, θ_: θ, ϕ_: ϕ] := Sqrt[(4 π)/(2 l + 1)] * r^l * Yxyz[l, m, θ, ϕ]
```

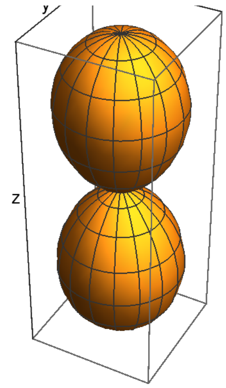
In[∘]:=
```
Table[SphericalPlot3D[Evaluate[Y[l, m, θ, ϕ]^2],
   {θ, 0, Pi}, {ϕ, 0, 2 π}, AxesLabel → {"x", "y", "z"}, Ticks → None,
   DisplayFunction → Identity, PlotRange → All], {l, 0, 2}, {m, -l, l}] // Short
```

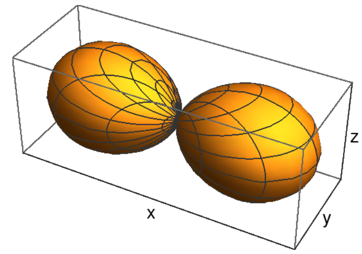
Out[∘]//Short= {{



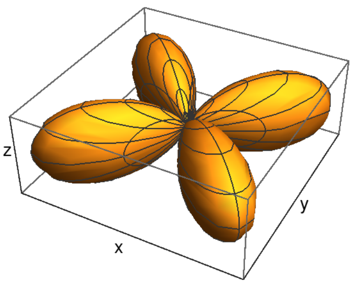
},

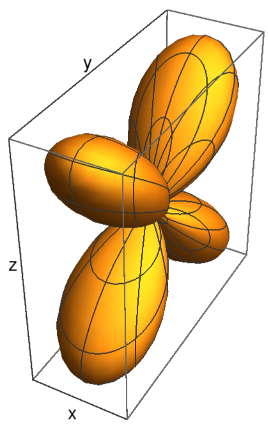
{ 

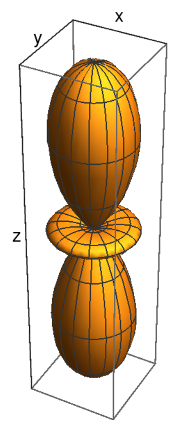
, 

, 

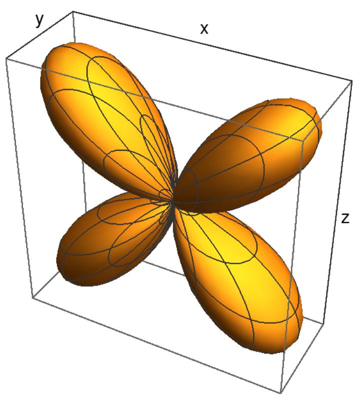
},

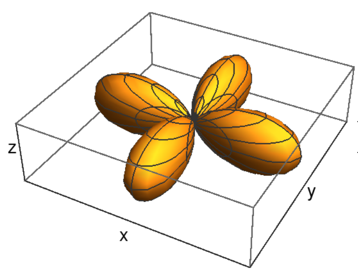
{ 

, 

, 

,



, 

}}

Basis

In[∘]:=
$$A = \sqrt{\frac{3}{4\pi}};$$
$$B = \sqrt{\frac{7}{4\pi}};$$
$$\phi x = A * \frac{x}{r};\quad \phi y = A * \frac{y}{r};\quad \phi z = A * \frac{z}{r};$$
$$\phi xyz = B * \sqrt{15} * \frac{xyz}{r^3};\quad \phi x\alpha = B * \frac{1}{2} * \frac{x\left(5x^2 - 3r^2\right)}{r^3};$$
$$\phi y\alpha = B * \frac{1}{2} * \frac{y\left(5y^2 - 3r^2\right)}{r^3};\quad \phi z\alpha = B * \frac{1}{2} * \frac{z\left(5z^2 - 3r^2\right)}{r^3};$$
$$\phi x\beta = B * \frac{\sqrt{15}}{2} * \frac{x\left(y^2 - z^2\right)}{r^3};\quad \phi y\beta = B * \frac{\sqrt{15}}{2} * \frac{y\left(z^2 - x^2\right)}{r^3};\quad \phi z\beta = B * \frac{\sqrt{15}}{2} * \frac{z\left(x^2 - y^2\right)}{r^3};$$

```
ϕp  = {ϕx, ϕy, ϕz};
ϕf = {ϕxyz, ϕxα, ϕyα, ϕzα, ϕxβ, ϕyβ, ϕzβ};
```

Electric Multipole

In[∘]:=
```
Q[l_, m_, basis_, basis2_, θ_: θ, ϕ_: ϕ] :=
  Module[{e = 1},
   electricpole = -e * Olm[l, m];
   Qmat = ReplaceAll[KroneckerProduct[basis, electricpole * basis2], tospherical];
   Qmat = Integrate[Qmat * Sin[θ], {θ, 0, π}, {ϕ, 0, 2 π}];
   Qmat
  ]

Q[0, 0, ϕp, ϕf] // MatrixForm
Q[2, 2, ϕp, ϕf] // MatrixForm
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[∘]//MatrixForm=
$$\begin{pmatrix} 0 & -\frac{9\left(x^2+y^2+z^2\right)}{10\sqrt{7}} & 0 & 0 & \frac{1}{2}\sqrt{\frac{3}{35}}\left(x^2+y^2+z^2\right) & 0 & 0 \\ 0 & 0 & \frac{9\left(x^2+y^2+z^2\right)}{10\sqrt{7}} & 0 & 0 & \frac{1}{2}\sqrt{\frac{3}{35}}\left(x^2+y^2+z^2\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{35}}\left(x^2+y^2+z^2\right) \end{pmatrix}$$

Magnetic Multipole

In[ ]:=
```
Clear[M, magneticpole]
M[l_, m_, basis_, basis2_, θ_: θ, ϕ_: ϕ] :=
 Module[{μB = 1},
  magneticpole[b_] := -μB * Grad[Olm[l, m], {x, y, z}].(2 * L[b])/(l + 1);
  ket := Map[magneticpole, basis2];
  Mx = ReplaceAll[KroneckerProduct[basis*, ket], tospherical] // FullSimplify;
  Mmat = Integrate[Mx * Sin[θ], {θ, 0, π}, {ϕ, 0, 2 π}];
  Mmat
 ]
M[1, 1, ϕp, ϕf] // MatrixForm
M[2, 2, ϕp, ϕf] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Toroidal Multipole

In[∘]:=
```
Clear[T]
Clear[troidalpole]

T[l_, m_, basis_, basis2_, θ_: θ, ϕ_: ϕ] :=
 Module[{μB = 1},
  t[b_] := {x, y, z}/(l + 1) × ((2 * L[b])/(l + 2));
  troidalpole[b_] := -μB * Grad[Olm[l, m], {x, y, z}].t[b];
  ket = Map[troidalpole, basis2];
  Tmat = ReplaceAll[KroneckerProduct[basis*, ket], tospherical] // FullSimplify;
  Tmat = Integrate[Tmat * Sin[θ], {θ, 0, π}, {ϕ, 0, 2 π}];
  Tmat
 ]
T[2, 0, ϕp, ϕf] * 4 √7 // MatrixForm // FullSimplify
T[2, 1, ϕp, ϕf] * 2 √21 // MatrixForm // FullSimplify
T[2, 2, ϕp, ϕf] * -4 √21 // MatrixForm // FullSimplify
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{8}{5} i \sqrt{3} \left(x^2+y^2+z^2\right) & 0 & 0 & \frac{8 i \left(x^2+y^2+z^2\right)}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{8}{5} i \sqrt{3} \left(x^2+y^2+z^2\right) & 0 & 0 & -\frac{8 i \left(x^2+y^2+z^2\right)}{\sqrt{5}} \\ 0 & 0 & 0 & -\frac{16}{5} i \sqrt{3} \left(x^2+y^2+z^2\right) & 0 & 0 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & \frac{8}{5} i \sqrt{3} \left(x^2+y^2+z^2\right) & 0 & 0 & -\frac{8 i \left(x^2+y^2+z^2\right)}{\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{8}{5} i \sqrt{3} \left(x^2+y^2+z^2\right) & 0 & 0 & \frac{8 i \left(x^2+y^2+z^2\right)}{\sqrt{5}} & 0 & 0 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{24}{5} i \sqrt{3} \left(x^2+y^2+z^2\right) & 0 & 0 & -\frac{8 i \left(x^2+y^2+z^2\right)}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & -\frac{24}{5} i \sqrt{3} \left(x^2+y^2+z^2\right) & 0 & 0 & -\frac{8 i \left(x^2+y^2+z^2\right)}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{16 i \left(x^2+y^2+z^2\right)}{\sqrt{5}} \end{pmatrix}$$

Electric Toroidal Multipole

In[ ]:=
```
Clear[G]
Clear[electrictroidal]

G[l_, m_, basis_, basis2_, θ_: θ, ϕ_: ϕ] :=
 Module[{e = 1},
  mα[b_] := (2 * L[b])/(l + 1);
  tβ[b_] := {x, y, z}/(l + 1) × ((2 * L[b])/(l + 2));
  g[b_] := Sum[-e * mα[tβ[b][[i]]][[j]] *
     D[Olm[l, m], rbm[[i]], rbm[[j]]], {i, 1, 3}, {j, 1, 3}];
  electrictroid = Map[g, basis2];
  Gmat = ReplaceAll[
     KroneckerProduct[basis, electrictroid], tospherical] // FullSimplify;
  Gmat = Integrate[Gmat * Sin[θ], {θ, 0, π}, {ϕ, 0, 2 π}];
  Gmat
 ]
G[3, 0, ϕp, ϕf] * (-8 * Sqrt[7])/3 // MatrixForm // FullSimplify
G[3, 1, ϕp, ϕf] * (-8 * Sqrt[7])/3 // MatrixForm // FullSimplify
G[3, 2, ϕp, ϕf] * (-8 * Sqrt[21])/3 // MatrixForm // FullSimplify
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -\frac{8}{5}\sqrt{3}\left(x^2+y^2+z^2\right) & 0 & 0 & \frac{8\left(x^2+y^2+z^2\right)}{\sqrt{5}} & 0 \\ 0 & \frac{8}{5}\sqrt{3}\left(x^2+y^2+z^2\right) & 0 & 0 & \frac{8\left(x^2+y^2+z^2\right)}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{16}{5}\sqrt{2}\left(x^2+y^2+z^2\right) & 0 & 0 & -8\sqrt{\frac{2}{15}}\left(x^2+y^2+z^2\right) \\ 0 & 0 & \frac{4}{5}\sqrt{2}\left(x^2+y^2+z^2\right) & 0 & 0 & -4\sqrt{\frac{2}{15}}\left(x^2+y^2+z^2\right) & 0 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -8\sqrt{\frac{3}{5}}\left(x^2+y^2+z^2\right) & 0 & 0 & \frac{8}{5}\left(x^2+y^2+z^2\right) & 0 \\ 0 & -8\sqrt{\frac{3}{5}}\left(x^2+y^2+z^2\right) & 0 & 0 & -\frac{8}{5}\left(x^2+y^2+z^2\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$