设4×3矩阵$A=\begin{pmatrix}1&2&3\\-2&-1&-1\\-1&1&a\\0&3&5\end{pmatrix}$，向量$b=\begin{pmatrix}4\\d-5\\-3\\1\end{pmatrix}$，求a, d的取值范围，使得方程组$AX=b$有唯一解.

$AX=b$ 有唯一解 $\Leftrightarrow r(A)=r(A|b)=n=3$

$$\left(\begin{array}{ccc:c}1&2&3&4\\-2&-1&-1&d-5\\-1&1&a&-3\\0&3&5&1\end{array}\right)\to\begin{pmatrix}1&2&3&4\\0&3&5&d+3\\0&3&a+3&1\\0&3&5&1\end{pmatrix}\to\begin{pmatrix}1&2&3&4\\0&3&5&1\\0&0&a-2&0\\0&0&0&d+2\end{pmatrix}$$

$d=-2$, $a\neq 2$

设$A=\begin{pmatrix}1&2&3\\-2&-1&-1\\0&3&5\end{pmatrix}$, $B=\begin{pmatrix}-2&1&-1\\-1&0&1\\0&2&3\end{pmatrix}$, 计算:

(1) $3A, 2B, 3A+2B$;

(2) $AB, BA, AB-BA$;

(3) 记$A^2=AA$, I为单位阵, 计算$A^2-3A+2I$;

(1) $3A=\begin{pmatrix}3&6&9\\-6&-3&-3\\0&9&15\end{pmatrix}$, $2B=\begin{pmatrix}-4&2&-2\\-2&0&2\\0&4&6\end{pmatrix}$

$3A+2B=\begin{pmatrix}-1&8&7\\-8&-3&-1\\0&13&21\end{pmatrix}$

(2) $AB=\begin{pmatrix}-4&7&10\\5&-4&-2\\-3&10&18\end{pmatrix}$, $BA=\begin{pmatrix}-4&-8&-12\\-1&1&2\\-4&7&13\end{pmatrix}$

$AB-BA=\begin{pmatrix}0&15&22\\6&-5&-4\\1&3&5\end{pmatrix}$

(3) $A^2=\begin{pmatrix}-3&9&16\\0&-6&-10\\-6&12&22\end{pmatrix}$, $3A=\begin{pmatrix}3&6&9\\-6&-3&-3\\0&9&15\end{pmatrix}$, $A^2-3A+2I=\begin{pmatrix}-4&3&7\\6&-1&-7\\-6&3&9\end{pmatrix}$

设$AB=C$，$A=\begin{pmatrix}1&0&0\\-9&7&8\\-2&2&3\\1&-3&0\end{pmatrix}$，$B=\begin{pmatrix}2&0&1&7\\1&0&2&5\\2&1&0&3\end{pmatrix}$，

(1) 求C;

(2) 将C的第2列用A的列向量组线性表示;

(3) 验证C的第3行等于B的行向量组以A的第3行的元素为系数的线性组合.

$$C=\begin{pmatrix}2&0&1&7\\5&8&5&-4\\4&3&2&5\\-1&0&-5&-8\end{pmatrix}$$

$$\begin{pmatrix}0\\8\\3\\0\end{pmatrix}=0\begin{pmatrix}1\\-9\\-2\\1\end{pmatrix}+0\begin{pmatrix}0\\7\\2\\-3\end{pmatrix}+1\begin{pmatrix}0\\8\\3\\0\end{pmatrix}$$

$$(4\ 3\ 2\ 5)=-2(2\ 0\ 1\ 7)+2(1\ 0\ 2\ 5)+3(2\ 1\ 0\ 3)$$

举例说明：存在矩阵A, B, C，使得AB = AC，但B ≠ C.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

设$\alpha=(1,2)^T$. 将α看作2×1矩阵，$\alpha^T=(1,2)$看作是1×2矩阵，令$A=\alpha\alpha^T$. 计算A和$(x_1,x_2)\ A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$A=\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$(x_1. x_2)A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=(x_1+2x_2, 2x_1+4x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$=x_1^2+4x_1x_2+4x_2^2$$

已知$AB=0$，即矩阵A与B的乘积为零矩阵，且$B=\begin{pmatrix}1&2&3\\-2&1&-1\\0&3&3\end{pmatrix}$，$r(A)=1$，求$AX=0$的通解.

· 由AB有意义知，A有3列，$n=3$.

设$AX=0$的基础解系有s个解向量.

· 由$r(A)=1$，$s=n-r(A)=3-1=2$.

也就是说任意两个无关的$AX=0$的解向量就构成$AX=0$的一个基础解系.

· 由$AB=0$知，B的列都为$AX=0$的解.

其中任两列都是无关的，取前两列，$AX=0$通解为

$k_1(1,-2,0)^T+k_2(2,1,3)^T \quad \forall k_1,k_2$.

设$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} B = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$，求$B$.

设 $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 有

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} B = \begin{pmatrix} 1a+2c & 1b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$$

$$\begin{cases} a+2c=5 \\ 3a+4c=11 \end{cases} \Rightarrow \begin{cases} a=1 \\ c=2 \end{cases} \qquad \begin{cases} b+2d=11 \\ 3b+4d=25 \end{cases} \Rightarrow \begin{cases} b=3 \\ d=4 \end{cases}$$

$\therefore$ $B=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

设$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}B = B\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$，求$B$.

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}a+2c & b+2d\\c & d\end{pmatrix} = \begin{pmatrix}a & 2a+b\\c & 2c+d\end{pmatrix}$$

$$a+2c=a \Rightarrow c=0$$

$$b+2d=2a+b \Rightarrow d=a.$$

$$\therefore B=\begin{pmatrix}a & b\\0 & a\end{pmatrix}.\ \forall a,b.$$

设$E_1=\begin{pmatrix}1&0&0\\k&1&0\\0&0&1\end{pmatrix}$, $E_2=\begin{pmatrix}1&0&0\\0&k&0\\0&0&1\end{pmatrix}$, $E_3=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$, 计算:

(1) E_1A, E_2A, E_3A;

(2) AE_1, AE_2, AE_3.

(1) $E_1A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\ka_{11}+a_{21}&ka_{12}+a_{22}&ka_{13}+a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$

$E_2A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\ka_{21}&ka_{22}&ka_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$

$E_3A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{31}&a_{32}&a_{33}\\a_{21}&a_{22}&a_{23}\end{pmatrix}$

$AE_3=\begin{pmatrix}a_{11}&a_{13}&a_{12}\\a_{21}&a_{23}&a_{22}\\a_{31}&a_{33}&a_{32}\end{pmatrix}$

(2) $AE_1=\begin{pmatrix}a_{11}+ka_{12}&a_{12}&a_{13}\\a_{21}+ka_{22}&a_{22}&a_{23}\\a_{31}+ka_{32}&a_{32}&a_{33}\end{pmatrix}$

$AE_2=\begin{pmatrix}a_{11}&ka_{12}&a_{13}\\a_{21}&ka_{22}&a_{23}\\a_{31}&ka_{32}&a_{33}\end{pmatrix}$