设$A=\begin{pmatrix}7&2&-4\\0&4&5\\6&0&-5\end{pmatrix}$，$A^*$为A的伴随矩阵，

(1) 利用代数余子式，计算$|A|$，A^*，并验证$AA^*=|A|I$；

(2) 计算$r(A^*)$和$|A^*|$；

(3) 若A可逆，利用A^*计算A^{-1}，并求$|A^{-1}|$；若不可逆，说明理由.

按第2列展开.

$|A| = 2\cdot 30 + 4\cdot(-11)$

$= 16$

(1) $A_{11}=-20$ $A_{12}=30$ $A_{13}=-24$

$A_{21}=10$ $A_{22}=-11$ $A_{23}=12$

$A_{31}=26$ $A_{32}=-35$ $A_{33}=28$

$A^*=\begin{pmatrix}-20&10&26\\30&-11&-35\\-24&12&28\end{pmatrix}$

(2) $|A|\neq 0 \Rightarrow r(A)=3 \Rightarrow r(A^*)=3$.

$|A^*| = |A|^{3-1} = 16^2 = 256$

(3) A可逆. $A^{-1}=\frac{1}{16}A^* = \frac{1}{16}\begin{pmatrix}-20&10&26\\30&-11&-35\\-24&12&28\end{pmatrix}$

设A为3阶方阵，且$|A|=2$，求行列式$\left|(2A)^{-1}-\frac{1}{2}A^*\right|$.

$$\left|(2A)^{-1}-\frac{1}{2}A^*\right|$$

$$AA^*=|A|I \text{ 且 } |A|\neq 0$$

$$= \left|\frac{1}{2}A^{-1}-\frac{1}{2}A^*\right|$$

$$\Rightarrow A^*=|A|A^{-1}$$

$$=\left(\frac{1}{2}\right)^3\left|A^{-1}-A^*\right|$$

$$=\left(\frac{1}{2}\right)^3\left|A^{-1}-|A|A^{-1}\right|$$

$$=\left(\frac{1}{2}\right)^3\left|A^{-1}-2A^{-1}\right|$$

$$=\left(\frac{1}{2}\right)^3(-1)^3\left|A^{-1}\right|=\left(-\frac{1}{2}\right)^3|A|^{-1}=-\frac{1}{16}$$

设α,β,γ为3维列向量，3阶行列式$|\alpha-\beta,\beta-\gamma,\gamma-2\alpha|=4$，计算$|\alpha,\beta,\gamma|$.

$$|\alpha-\beta,\ \beta-\gamma,\ \gamma-2\alpha|$$

$$=|\alpha-\beta,\ \alpha-\gamma,\ \gamma-2\alpha|$$

$$=|\alpha-\beta,\ \alpha-\gamma,\ -\alpha|$$

$$=|-\beta,\ -\gamma,\ -\alpha|$$

$$=(-1)^{9}|\alpha,\ \beta,\ \gamma|$$

$$=4$$

$$|\alpha,\beta,\gamma|=-4$$

设A为n阶可逆方阵，证明：$(A^*)^T = (A^T)^*$.

$AA^* = |A|I$. $A^* = |A|A^{-1}$

$(A^*)^T = (|A|A^{-1})^T = |A|(A^{-1})^T = |A|(A^T)^{-1}$

$= |A^T|(A^T)^{-1}$

$= (A^T)^*$

设n阶方阵$A=\begin{pmatrix} a & 0 & \cdots & 0 & 1 \\ 1 & a & \ddots & \vdots & 1 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 1 & \vdots & \ddots & a & 1 \\ 1 & 0 & \cdots & 0 & a \end{pmatrix}$，其中$a$为任意常数，求$|A|$.

按第一行展开.

$$|A| = a\begin{vmatrix} a & & \vdots \\ & \ddots & \vdots \\ & & a \end{vmatrix} + (-1)^{n+1}\begin{vmatrix} 1 & a & & \\ \vdots & & \ddots & \\ \vdots & & & a \\ 1 & & & 0 \end{vmatrix}$$

第二个行列式按最后一行展开.

$$= a^n + (-1)^{n+1}(-1)^{n-1+1}\begin{vmatrix} a & & \\ & \ddots & \\ & & a \end{vmatrix}$$

$$= a^n - a^{n-2}$$

设$A=(a_{ij})_{3\times 3}$为非零方阵，且A的代数余子式A_{ij}满足$A_{ij}+a_{ij}=0$，$i,j=1,2,3$．求$|A|$.

$$A^*=\begin{pmatrix}A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{pmatrix}=\begin{pmatrix}-a_{11}&-a_{21}&-a_{31}\\-a_{12}&-a_{22}&-a_{32}\\-a_{13}&-a_{23}&-a_{33}\end{pmatrix}=-A^T$$

$$|A^*|=|A|^{3-1}=|-A^T|=(-1)^3|A^T|=(-1)^3|A|$$

$\therefore\ |A|^2=-|A|$ 有 $|A|=-1$ 或 0.

假设$|A|=0$.

$$|A|=0 \Longleftrightarrow r(A)<3 \Longleftrightarrow \begin{cases}r(A)=1 \Rightarrow r(A^*)=0\neq r(A^T)\\ r(A)=2 \Rightarrow r(A^*)=1\neq r(A^T)\end{cases}$$

与$A^*=-A^T$矛盾.

对$|A|=-1$举例若令 $A=\begin{pmatrix}-1&&\\&-1&\\&&-1\end{pmatrix}$，满足题设条件.

且 $\begin{vmatrix}-1&&\\&-1&\\&&-1\end{vmatrix}=-1$. 所以有$|A|=-1$

已知R^3的两组基为$B_1=\{\alpha_1,\alpha_2,\alpha_3\}$及$B_2=\{\beta_1,\beta_2,\beta_3\}$，其中

$$\alpha_1=(1,1,1)^T,\alpha_2=(0,1,1)^T,\alpha_3=(0,0,1)^T,$$
$$\beta_1=(1,0,1)^T,\beta_2=(0,1,-1)^T,\beta_3=(1,2,0)^T.$$

计算:

(1) 从基$\mathbf{B_1}$到基$\mathbf{B_2}$的过渡矩阵A;

(2) 已知γ在基$\mathbf{B_1}$下的坐标为$(1,-2,-1)^T$，求γ在基$\mathbf{B_2}$下的坐标;

(3) 已知σ在基$\mathbf{B_2}$下的坐标为$(5,7,-4)^T$，求σ在基$\mathbf{B_1}$下的坐标.

(1) $\begin{pmatrix}1&0&1\\0&1&2\\1&-1&0\end{pmatrix}=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}A$, $\left(\begin{array}{ccc|ccc}1&0&0&1&0&1\\1&1&0&0&1&2\\1&1&1&1&-1&0\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&0&1\\0&1&0&-1&1&+1\\0&0&1&1&-2&-2\end{array}\right)$

$0\ -1\ -1$ $\qquad 1\ -2\ -2$

$A=\begin{pmatrix}1&0&1\\-1&1&+1\\1&-2&-2\end{pmatrix}$

(2) 设 $\gamma_{B_2}=(y_1,y_2,y_3)^T$. $(\beta_1,\beta_2,\beta_3)\gamma_{B_2}=(\alpha_1,\alpha_2,\alpha_3)\gamma_{B_1}$

$A\gamma_{B_2}=\gamma_{B_1}=\begin{pmatrix}1\\-2\\-1\end{pmatrix}$, 解得 $\gamma_{B_2}=(5,7,-4)^T$

(3) $A\sigma_{B_2}=\sigma_{B_1}=(1,-2,-1)^T$

已知R^3的两组基为$\mathbf{B_1}=\{\alpha_1,\alpha_2,\alpha_3\}$及$\mathbf{B_2}=\{\beta_1,\beta_2,\beta_3\}$，其中

$$\alpha_1=(1,2,1)^T,\alpha_2=(2,3,3)^T,\alpha_3=(3,7,1)^T,$$

从基B_1到基B_2的过渡矩阵$A=\begin{pmatrix}-27 & -71 & -41\\ 9 & 20 & 9\\ 4 & 12 & 8\end{pmatrix}$，计算

(1) β_1,β_2,β_3在自然基下的坐标;

(2) 已知γ在基$\mathbf{B_1}$下的坐标为$(-2,1,1)^T$，求γ在自然基下的坐标和在基$\mathbf{B_2}$下的坐标.

(1) $(\beta_1,\beta_2,\beta_3)=(\alpha_1,\alpha_2,\alpha_3)A=\begin{pmatrix}1&2&3\\2&3&7\\1&3&1\end{pmatrix}\begin{pmatrix}-27&-71&-41\\9&20&9\\4&12&8\end{pmatrix}$

$=\begin{pmatrix}3&5&1\\1&2&1\\4&1&-6\end{pmatrix}$

$\beta_1=(3,1,4)^T$

$\beta_2=(5,2,1)^T$

$\beta_3=(1,1,-6)^T$

(2) $\gamma=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}1&2&3\\2&3&7\\1&3&1\end{pmatrix}\begin{pmatrix}-2\\1\\1\end{pmatrix}=(3,6,2)^T$

$A\gamma_{B_2}=\gamma_{B_1}=\begin{pmatrix}-2\\1\\1\end{pmatrix}$，解得$\gamma_{B_2}=\left(\frac{153}{4},-\frac{53}{2},\frac{83}{4}\right)^T$

已知 R^3 的两组基为 $B_1=\{\alpha_1,\alpha_2,\alpha_3\}$ 及 $B_2=\{\beta_1,\beta_2,\beta_3\}$， 其中 $\alpha_1=(1,1,0),\alpha_2=(1,0,1),\alpha_3=(0,2,1),\beta_1=(0,1,2),\beta_2=(0,1,0),\beta_3=(2,1,0)$. 求一个向量γ，使得γ在基$B_1$下的坐标与γ在基$B_2$下的坐标相等.

设 $\gamma_{B_1}=\gamma_{B_2}=X$

$$\gamma=\begin{pmatrix}1&1&0\\1&0&2\\0&1&1\end{pmatrix}X=\begin{pmatrix}0&0&2\\1&1&1\\2&0&0\end{pmatrix}X$$

$$\begin{pmatrix}1&1&-2\\0&-1&1\\-2&1&1\end{pmatrix}X=0$$

$$\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}X=0$$

$X=k(1,1,1)^T$, $\forall k$.

$\gamma=k(2,3,2)^T$.

设$\alpha_1,\alpha_2,\alpha_3$是$R^3$的一组基，求$\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3$到$\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$过渡矩阵.

$$\left(\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3\right)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&&\\&\frac{1}{2}&\\&&\frac{1}{3}\end{pmatrix}$$

$$(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}$$

$$(\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1)=\left(\alpha_1,\frac{1}{2}\alpha_2,\frac{1}{3}\alpha_3\right)A$$

$$(\alpha_1\alpha_2\alpha_3)\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&&\\&\frac{1}{2}&\\&&\frac{1}{3}\end{pmatrix}A$$

$$A=\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}=\begin{pmatrix}1&0&1\\2&2&0\\0&3&3\end{pmatrix}$$

设$\alpha_1=(1,2,1,3),\alpha_2=(-1,-1,0,-1),\alpha_3=(-1,-2,1,-1)$.

(1) 计算内积，验证

$$(\alpha_1+\alpha_2,\alpha_3)=(\alpha_1,\alpha_3)+(\alpha_2,\alpha_3)\text{且}(2\alpha_1,\alpha_3)=2(\alpha_1,\alpha_3);$$

(2) 计算$\alpha_1,\alpha_2,\alpha_3$的长度;

(3) 计算α_1,α_2间的夹角$<\alpha_1,\alpha_2>$及α_2,α_3间的夹角$<\alpha_2,\alpha_3>$的余弦值.

(1) $(\alpha_1+\alpha_2,\alpha_3)=((0,1,1,2),(-1,-2,1,-1))=-2+1-2=-3$

$(\alpha_1,\alpha_3)+(\alpha_2,\alpha_3)=-1-4+1-3+1+2+1=-3$ 同理.

(2). $|\alpha_1|=\sqrt{(\alpha_1,\alpha_1)}=\sqrt{1+4+1+9}=\sqrt{15}$,

$|\alpha_2|=\sqrt{(\alpha_2,\alpha_2)}=\sqrt{1+1+1}=\sqrt{3}$

(3) $\cos<\alpha_1,\alpha_2>=\dfrac{(\alpha_1,\alpha_2)}{|\alpha_1||\alpha_2|}=\dfrac{-1-2-3}{\sqrt{15}\sqrt{3}}=\dfrac{-6}{3\sqrt{5}}=-\dfrac{2}{\sqrt{5}}$