一、填空题

1. $\dfrac{n(n+1)}{2}$ 2. $\dfrac{1}{2}(3A+4I)$ 3. $2^{99}\begin{pmatrix}1&-1&1\\2&-2&2\\3&-3&3\end{pmatrix}$ 4. 2 5. $\begin{pmatrix}1\\2\\0\\1\end{pmatrix}+k_1\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+k_2\begin{pmatrix}1\\-1\\1\\2\end{pmatrix}$，其中$k_1,k_2$为任意常数 6. $1,1,0;4$

二、选择题

1-6. $AADCBB$ 7. ACD 8. BCE

三、计算题

1. $\begin{pmatrix}2&1&1&1&-1\\1&0&1&1&1\\1&1&0&1&0\\2&0&2&1&0\end{pmatrix}\to\begin{pmatrix}1&0&1&1&1\\0&1&-1&-1&-3\\0&0&0&1&2\\0&0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&1&0&-1\\0&1&-1&0&-1\\0&0&0&1&2\\0&0&0&0&0\end{pmatrix}$

一个极大线性无关组：$\alpha_1,\alpha_2,\alpha_4$；向量组的秩：3；$\alpha_3=\alpha_1-\alpha_2;\alpha_5=-\alpha_1-\alpha_2+2\alpha_4$

2. (1) 设从基$\alpha_1,\alpha_2,\alpha_3$到基$\beta_1,\beta_2,\beta_3$的过渡矩阵为$A$，则它满足：

$$(\alpha_1,\alpha_2,\alpha_3)A=(\beta_1,\beta_2,\beta_3)$$

$$\left(\begin{array}{ccc|ccc}1&0&0&1&0&1\\1&1&0&0&1&2\\1&1&1&1&-1&0\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&0&1\\0&1&0&-1&1&1\\0&1&1&0&-1&-1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&0&1\\0&1&0&-1&1&1\\0&0&1&1&-2&-2\end{array}\right)$$

$$\therefore A=\begin{pmatrix}1&0&1\\-1&1&1\\1&-2&-2\end{pmatrix}$$

(2) $\alpha=\alpha_1-2\alpha_2-\alpha_3=(1,-1,-2)^T$

设α在β_1,β_2,β_3下的坐标为$(x_1,x_2,x_3)^T$，则$\alpha=x_1\beta_1+x_2\beta_2+x_3\beta_3$，代入，解方程组得

$x_1=5,x_2=7,x_3=-4$，所以α在β_1,β_2,β_3下的坐标为$(5,7,-4)^T$

3.由已知，$3-r(2I-A)=2\therefore r(2I-A)=1$

$$2I-A=\begin{pmatrix}1&1&-1\\-a&-2&-b\\3&3&-3\end{pmatrix}\to\begin{pmatrix}1&1&-1\\0&a-2&-a-b\\0&0&0\end{pmatrix}\therefore a-2=0,-a-b=0\therefore a=2,b=-2$$

四、证明题

证明：设$k_0\alpha_0+k_1\alpha_1+\cdots+k_p\alpha_p=0$，---------（1）

等号左右两边前面同乘 A 得：$k_0 A\alpha_0 + k_1 A\alpha_1 + \cdots + k_p A\alpha_p = 0$

$\because \alpha_1, \ldots, \alpha_p$ 是 $Ax=0$ 的一个基础解系 $\therefore \boldsymbol{A}\alpha_1 = 0, \ldots, \boldsymbol{A}\alpha_{\boldsymbol{p}} = 0$ 且 $\alpha_1, \ldots, \alpha_{\boldsymbol{p}}$ 线性无关

于是，$\boldsymbol{k}_0 \boldsymbol{A}\alpha_0 = 0$　$\because \alpha_0$ 不是 $Ax=0$ 的解 $\therefore \boldsymbol{A}\alpha_0 \neq 0 \therefore \boldsymbol{k}_0 = 0$

代入（1）式得：$\boldsymbol{k}_1 \alpha_1 + \cdots + \boldsymbol{k}_{\boldsymbol{p}} \alpha_{\boldsymbol{p}} = 0$

$\because \alpha_1, \ldots, \alpha_{\boldsymbol{p}}$ 线性无关 $\therefore \boldsymbol{k}_1 = 0, \ldots, \boldsymbol{k}_{\boldsymbol{p}} = 0$ 综上可证：向量组 $\alpha_0, \alpha_1, \ldots, \alpha_{\boldsymbol{p}}$ 线性无关。

五、解方程组

解：(1) $\begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ a & 0 & 0 & 1 \end{vmatrix} = 1 - a^4$ （2） $\left(\begin{array}{cccc|c} 1 & a & 0 & 0 & 1 \\ 0 & 1 & a & 0 & -1 \\ 0 & 0 & 1 & a & 0 \\ a & 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & a & 0 & 0 & 1 \\ 0 & 1 & a & 0 & -1 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1-a^4 & -a^2-a \end{array}\right)$

$\because Ax=b$ 有无穷多解 $\therefore 1-a^4=0$ 且 $-a^2-a=0 \therefore a=-1$

$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$，特解 $\xi_0 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$，一个基础解系：$\xi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$，

$\therefore Ax=b$ 通解为：$\xi = \xi_0 + k\xi_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$，其中 k 为任意常数。

六、化二次型为标准型

解：(1) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & a \\ 0 & a & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & a+1 \\ 0 & 0 & 0 \end{pmatrix} \because r(A) = 2 \therefore a+1=0 \therefore a=-1$

(2) $A^T A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix}$

$$\left|\lambda I - A^T A\right| = \begin{vmatrix} \lambda-2 & 0 & -2 \\ 0 & \lambda-2 & -2 \\ -2 & -2 & \lambda-4 \end{vmatrix} = \lambda(\lambda-2)(\lambda-6) = 0$$

$\therefore \lambda_1 = 0$（单根），$\lambda_2 = 2$（单根），$\lambda_3 = 6$（单根）

$\lambda_1=0$时，$\begin{pmatrix}-2&0&-2\\0&-2&-2\\-2&-2&-4\end{pmatrix}\to\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\end{pmatrix}\therefore\xi_1=\begin{pmatrix}-1\\-1\\1\end{pmatrix}$,单位化得：$\eta_1=\dfrac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}$

$\lambda_2=2$时，$\begin{pmatrix}0&0&-2\\0&0&-2\\-2&-2&-2\end{pmatrix}\to\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix}\therefore\xi_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}$,单位化得：$\eta_2=\dfrac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\lambda_3=6$时，$\begin{pmatrix}4&0&-2\\0&4&-2\\-2&-2&2\end{pmatrix}\to\begin{pmatrix}1&0&-\frac{1}{2}\\0&1&-\frac{1}{2}\\0&0&0\end{pmatrix}\therefore\xi_2=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{pmatrix}$,单位化得：$\eta_3=\dfrac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}$

可取正交矩阵$Q=\begin{pmatrix}-\frac{1}{\sqrt{3}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{3}}&0&\frac{2}{\sqrt{6}}\end{pmatrix}$，令$x=Qy$得标准型：$2y_2^2+6y_3^2$

(3)规范型为：$z_1^2+z_2^2$