设A是3阶方阵，将A的第1列与第2列对换得B，再把B的第2列加到第3列得C．求可逆阵Q，使得AQ = C.

$$B = AE_{12}, \quad C = BE_{32}(1)$$

$$= AE_{12}E_{32}(1)$$

$$Q = E_{12}E_{32}(1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

设$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & a \\ 0 & a & -1 \end{pmatrix}$，且$r(A)=2$，求$a$的值.

$\because r(A)=2$，且 $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$，$\begin{pmatrix} 0 \\ 1 \\ 0 \\ a \end{pmatrix}$ 无关.

$\therefore \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ a \\ -1 \end{pmatrix}$

$\therefore a=-1$

n阶

设A和B为同阶方阵，证明：若AB可逆，则A可逆且B可逆.

AB可逆 $\Longleftrightarrow r(AB)=n$.

$\because r(AB) \le r(A) \le n$

$\therefore \quad r(A)=n$

$\therefore \quad A$可逆

同理 B可逆

设$A_{m\times n}$，$B_{n\times m}$，且$n < m$，证明：$(AB)X = 0$有非零解.

由A, B的型知，

$r(A) \le n$，$r(B) \le n$.

由$r(AB) \le \min\{r(A), r(B)\}$知，

$r(AB) \le n < m$.

由$r(AB) < m$知

$(AB)X = 0$有非零解.

设A为m × 3矩阵，B为3 × p矩阵．若$r(A) = 2$，$r(B) = 3$，证明$r(AB) = 2$.

$$r(AB) \le \min\{r(A), r(B)\}$$

$$\Rightarrow r(AB) \le 2$$

$$r(AB) + n \ge r(A) + r(B). \quad n = 3$$

$$\Rightarrow r(AB) \ge 2 + 3 - 3 = 2.$$

$$\therefore r(AB) = 2.$$

设$A_{m\times n}B_{n\times p}=C_{m\times p}$且$r(A)=n$. 证明: $r(C)=r(B)$

显然 $r(C)\leq r(B)$. 要证 $r(C)\geq r(B)$.

有 $r(C)+n\geq r(A)+r(B)$.

$\therefore r(C)\geq r(B)$.

设分块矩阵$A=\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}$，且$B$和$D$可逆.

(1) 证明A可逆，并求A^{-1}；

(2) 根据(1)计算$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{pmatrix}$的逆矩阵.

(1) 设 $\begin{pmatrix} E & F \\ G & H \end{pmatrix}\begin{pmatrix} B & 0 \\ C & D \end{pmatrix}=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

则 $EB+FC=I \qquad FD=0$

$GB+HC=0 \qquad HD=I$

则 $E=B^{-1} \qquad F=0$

$G=-D^{-1}CB^{-1} \qquad H=D^{-1}$

(2) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1}=\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$-\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1}=\begin{pmatrix} 4 & -9 \\ -2 & 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1}=\begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & -9 & 1 & -2 \\ -2 & 4 & 0 & 1 \end{pmatrix}$

计算以下方阵的行列式:

$$\begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix},\ \begin{pmatrix}0 & 2 & 0\\1 & 0 & 0\\0 & 0 & 3\end{pmatrix},\ \begin{pmatrix}0 & 2 & 0\\1 & 0 & 0\\0 & 0 & 6\end{pmatrix},\ \begin{pmatrix}0 & 2 & 3\\1 & 0 & 0\\0 & 4 & 6\end{pmatrix}$$

$= 6 \qquad = -6 \qquad = -12 \qquad = 0$

设α_1,β_1,β_2均为2维行向量，且方阵$A=\begin{pmatrix}\alpha_1\\\beta_1\end{pmatrix}$，方阵$B=\begin{pmatrix}\alpha_1,\\-\beta_2\end{pmatrix}$. 若$|A|=-4$，$|B|=1$，求$|A+B|$.

$$|A+B|=\begin{vmatrix}2\alpha_1\\\beta_1-\beta_2\end{vmatrix}=2\begin{vmatrix}\alpha_1\\\beta_1-\beta_2\end{vmatrix}$$

$$=2\left(\begin{vmatrix}\alpha_1\\\beta_1\end{vmatrix}+\begin{vmatrix}\alpha_1\\-\beta_2\end{vmatrix}\right)$$

$$=2(-4+1)=-6$$

设$\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2$均为4维列向量，且方阵$\mathrm{A}=(\alpha_1,\alpha_2,\alpha_3,\beta_1)$，方阵$\mathrm{B}=(\alpha_1,\alpha_3,-\alpha_2,\beta_2)$. 若$|A|=-4$，$|B|=1$，求$|A+B|$.

$$|A+B| = |2\alpha_1, \alpha_2+\alpha_3, \alpha_3-\alpha_2, \beta_1+\beta_2|$$

$$=2\left(|\alpha_1, \alpha_2, \alpha_3, \beta_1+\beta_2| + |\alpha_1, \alpha_3, -\alpha_2, \beta_1+\beta_2|\right)$$

$$=2\left(|\alpha_1,\alpha_2,\alpha_3,\beta_1| + |\alpha_1,\alpha_3,-\alpha_2,\beta_2| + |\alpha_1,\alpha_2,\alpha_3,\beta_1| + |\alpha_1,\alpha_3,-\alpha_2,\beta_2|\right)$$

$$=2(-4+1+(-4)+1) = -12$$

设$A=\begin{pmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{pmatrix}$，将$A$的第三行与第一行对换得到$B$，再将$B$的第二列的$(-2)$倍加到第3列得$C$，计算$|A|$，$|B|$，$|C|$.

$$|A|=\begin{vmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{vmatrix}=\begin{vmatrix}1&2&3&4\\0&-1&-2&-7\\0&-2&-8&-10\\0&-7&-10&-13\end{vmatrix}=\begin{vmatrix}1&2&3&4\\0&-1&-2&-7\\0&0&-4&4\\0&0&4&36\end{vmatrix}$$

$$=\begin{vmatrix}1&2&3&4\\0&-1&-2&-7\\0&0&-4&4\\0&0&0&40\end{vmatrix}=160$$

$$|B|=-160=|C|$$

设A,B为3阶方阵，且$|A|=2,|B|=-1$，计算行列式$\begin{vmatrix} 0 & 2A \\ -B & AB \end{vmatrix}$. (提示，$\begin{vmatrix} A & C \\ 0 & B \end{vmatrix}=|A||B|$)

$$\begin{vmatrix} 0 & 2A \\ -B & AB \end{vmatrix} = (-1)^{3\times 3}\begin{vmatrix} -B & AB \\ 0 & 2A \end{vmatrix}$$

$$= (-1)^{3\times 3}(-1)^{3}(2)^{3}\begin{vmatrix} B & -AB \\ 0 & A \end{vmatrix}$$

$$= 8|B||A| = -16$$

设A,B为3阶方阵，且$|A|=-3,|B|=2$，根据行列式的运算律，计算行列式

(1) $\left|\frac{1}{3}A^TB^{-1}\right|$;

(2) $\left|\left(\frac{1}{3}A^3B^{-2}\right)^{-2}\right|$.

(1) $\left|\frac{1}{3}A^TB^{-1}\right| = \left(\frac{1}{3}\right)^3 |A^T|\,|B^{-1}| = \frac{1}{27}|A|\,|B|^{-1} = -\frac{1}{18}$

(2) $\left(\frac{1}{3}A^3B^{-2}\right)^{-1} = 3\,(B^{-2})^{-1}(A^3)^{-1} = 3B^2A^{-3}$

$\left|\left(\frac{1}{3}A^3B^{-2}\right)^{-2}\right| = \left|(3B^2A^{-3})^2\right| = |3B^2A^{-3}|^2$.

$= \left(3^3|B^2A^{-3}|\right)^2 = \left(3^3|B|^2|A|^{-3}\right)^2 = 16$

根据行列式的计算公式，计算行列式$\begin{vmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{vmatrix}$.

$$= \begin{vmatrix} a & & & \\ & a & & \\ & & d & \\ & & & d \end{vmatrix} + \begin{vmatrix} a & & & \\ & & b & \\ & c & & \\ & & & d \end{vmatrix} + \begin{vmatrix} & & & b \\ & a & & \\ & & d & \\ c & & & \end{vmatrix} + \begin{vmatrix} & & & b \\ & & b & \\ & c & & \\ c & & & \end{vmatrix}$$

$$= a^2d^2 - abcd - abcd + b^2c^2$$

$$= (ad-bc)^2$$

设n阶方阵$A = \begin{pmatrix} a_1+\lambda_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2+\lambda_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & a_3+\lambda_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & a_n+\lambda_n \end{pmatrix}$，求$|A|$.

(提示，$\begin{vmatrix} a_1+\lambda_1 & a_1 \\ a_2 & a_2+\lambda_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_1 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ a_2 & a_2 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix}$)

$$|A| = \sum_{j=1}^{n}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{n} \lambda_i\right) a_j\right] + \prod_{j=1}^{n} \lambda_j .$$

设A为n阶方阵，$A^TA=I$，且$|A|<0$，求$|A+I|$.

$$|A+I|=|A+A^TA|=|I+A^T||A|$$

由 $(A+I)^T=A^T+I$ 知. $\underbrace{|A+I|=|A+I||A|}$

由 $|A^T|=|A|$ 且 $A^TA=I$ 知

$$|A^TA|=|A^T||A|=|A|^2=1,$$

由 $|A|<0$ 知. $|A|=-1$，由 知. $|A+I|=0$.