Comprehensive Compendium of 3D Ultrasound Reconstruction Approaches

Notation and Definitions

- $I(t) : [0, T] \to \mathbb{R}^{H \times W}$: 2D grayscale ultrasound image function
- $P(t) : [0, T] \to SE(3)$: probe pose function
- $\{(I_i, P_i)\}_{i=1}^N$: discrete set of image-pose pairs
- $\Omega \subset \mathbb{R}^3$: 3D reconstruction space
- $V : \Omega \to \mathbb{R}$: reconstructed 3D volume
- $\mathcal{P} = \{(x, y, z, i) | (x, y, z) \in \mathbb{R}^3, i \in \mathbb{R}\}$: point cloud space

Approach 1: Neural Operator for 3D Volume Reconstruction

Formulation

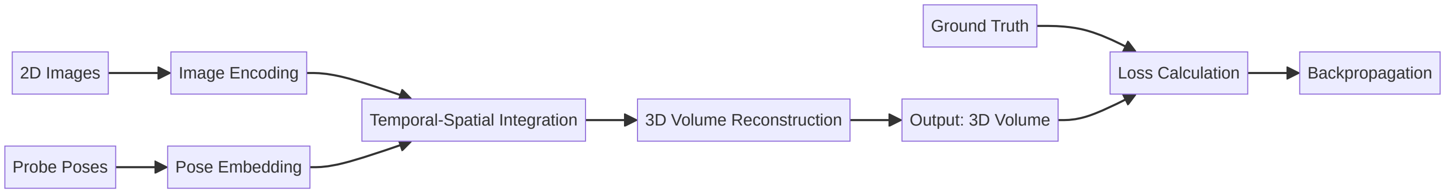
$$\mathcal{G} : (L^2([0, T]; \mathbb{R}^{H \times W}) \times C([0, T]; SE(3))) \to L^2(\Omega; \mathbb{R})$$

such that $\mathcal{G}(I, P) = V$, where $V : \Omega \to \mathbb{R}$ is the reconstructed 3D volume.

Algorithm

1. Define neural networks:
   - $\mathcal{E} : \mathbb{R}^{H \times W} \to \mathbb{R}^d$ (image encoder)
   - $\mathcal{Q} : SE(3) \to \mathbb{R}^d$ (pose embedder)
   - $\mathcal{F} : (\mathbb{R}^d \times \mathbb{R}^d)^N \to L^2(\Omega; \mathbb{R})$ (integrator)
2. For each (I_i, P_i) pair:
   a. Compute $E_i = \mathcal{E}(I_i)$
   b. Compute $Q_i = \mathcal{Q}(P_i)$
3. Integrate: $V = \mathcal{F}(\{E_i, Q_i\}_{i=1}^N)$
4. Define loss: $\mathcal{L} = \alpha \|V - V_{gt}\|_2^2 + \beta(1 - \text{SSIM}(V, V_{gt}))$
5. Train using gradient descent:
   $\theta \leftarrow \theta - \eta \nabla_\theta \mathcal{L}$

Mermaid Chart



Approach 2: Neural Operator for Point Cloud Reconstruction

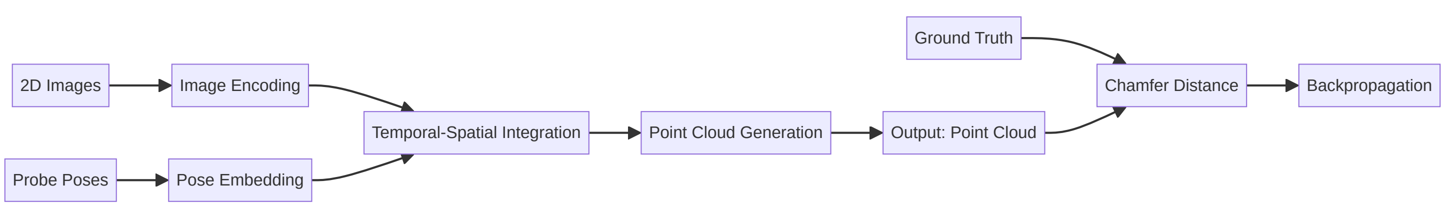
Formulation

$$\mathcal{G} : (L^2([0,T];\mathbb{R}^{H\times W}) \times C([0,T];SE(3))) \to \mathcal{P}$$

Algorithm

1. Define neural networks:
   - $\mathcal{E} : \mathbb{R}^{H\times W} \to \mathbb{R}^d$ (image encoder)
   - $\mathcal{Q} : SE(3) \to \mathbb{R}^d$ (pose embedder)
   - $\mathcal{F} : (\mathbb{R}^d \times \mathbb{R}^d)^N \to \mathcal{P}$ (point cloud generator)
2. For each (I_i, P_i) pair:
   a. Compute $E_i = \mathcal{E}(I_i)$
   b. Compute $Q_i = \mathcal{Q}(P_i)$
3. Generate point cloud: $\{(x_k, y_k, z_k, i_k)\}_{k=1}^K = \mathcal{F}(\{E_i, Q_i\}_{i=1}^N)$
4. Define loss: $\mathcal{L} = d_{Chamfer}(\{(x_k, y_k, z_k, i_k)\}, \{(x_k^{gt}, y_k^{gt}, z_k^{gt}, i_k^{gt})\})$
5. Train using gradient descent:
   $\theta \leftarrow \theta - \eta \nabla_\theta \mathcal{L}$

Mermaid Chart



Approach 3: Optical Flow Minimization for Frame Alignment

Formulation

Find $\{T_i\}_{i=1}^N$, where $T_i \in SE(3)$, such that:

$$\arg\min_{\{T_i\}} \sum_{i=1}^{N-1} \|\text{OpticalFlow}(I_i \circ T_i, I_{i+1} \circ T_{i+1})\|_2^2$$

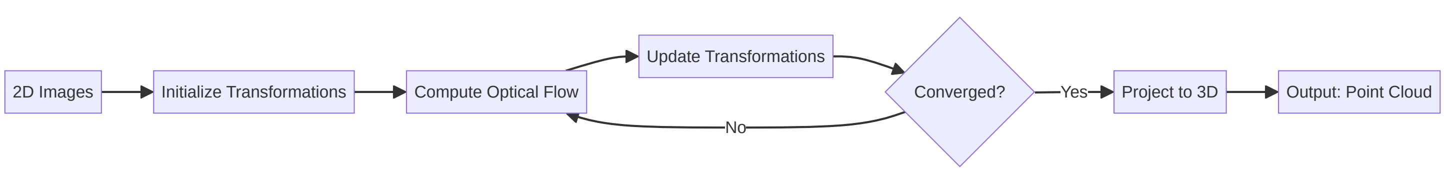
subject to $T_1 = I$ (identity transformation)

Algorithm

1. Initialize $\{T_i\}_{i=1}^N$ as identity transformations
2. For each iteration (max*iter):
   a. For $i = 1$ to $N - 1$:
   i. Compute optical flow: $F_i = \text{OpticalFlow}(I_i \circ T_i, I * i + 1 \circ T * i + 1)$
   ii. Estimate ΔT_i from F_i using least squares
   iii. Update $T_i \leftarrow T_i \circ \Delta T_i$
   b. Compute loss: $\mathcal{L} = \sum * i = 1^{N-1} \|F_i\|\_2^2$
   c. If $\mathcal{L} < \epsilon$ or iteration > max_iter, break
3. Reconstruct 3D points:
   For each pixel (u, v) in each image I_i:
   a. Back-project to 3D: $(x, y, z) = T_i^{-1} K^{-1} (u, v, 1)$
   b. Add $(x, y, z, I_i(u, v))$ to point cloud

Where K is the camera intrinsic matrix.

Mermaid Chart



Approach 4: Skewer-to-Curve Transformation

Formulation

Define $S(s) = (0, 0, sL)$, $s \in [0, 1]$, L is skewer length.
Find $C(s) : [0, 1] \to \mathbb{R}^3$ such that $C(s) = (x(s), y(s), z(s))$

Constraints

1. $C(0) = (0, 0, 0)$ and $C(1) = (x_f, y_f, L)$
2. $\|C'(s)\| \approx L$ for all s
3. Minimize $\sum_{i=1}^{N-1} \|\text{OpticalFlow}(I_i \circ T(s_i), I_{i+1} \circ T(s_{i+1}))\|_2^2$

Algorithm

1. Initialize $C(s) = S(s)$
2. For each iteration (max*iter):
   a. Compute transformations: $T(s_i) = C(s_i) \circ S(s_i)^{-1}$, $s_i = i/(N-1)$
   b. Calculate optical flow: $F_i = \text{OpticalFlow}(I_i \circ T(s_i), I * i + 1 \circ T(s * i + 1))$
   c. Update $C(s)$:
   i. Represent $C(s) = \sum * k = 1^K \alpha_k \phi_k(s)$, where ϕ_k are basis functions
   ii. Solve for $\{\alpha_k\}$ to minimize $\sum \|F_i\|^2$ subject to constraints
   d. If change in $C(s)$ < ϵ or iteration > max_iter, break
3. Final reconstruction:
   For each pixel (u, v) in each image I_i:
   a. Compute $s_i = i/(N-1)$
   b. Back-project to 3D: $(x, y, z) = T(s_i)^{-1} K^{-1}(u, v, 1)$
   c. Add $(x, y, z, I_i(u, v))$ to point cloud

Mermaid Chart

Approach 5: Time as a Spatial Dimension

Formulation

Let $V_0 : \Omega_4 \to \mathbb{R}$ be the initial 4D volume.
Find $T : \Omega_4 \to \Omega_3$ such that $V(T(x, y, z, t)) = V_0(x, y, z, t)$

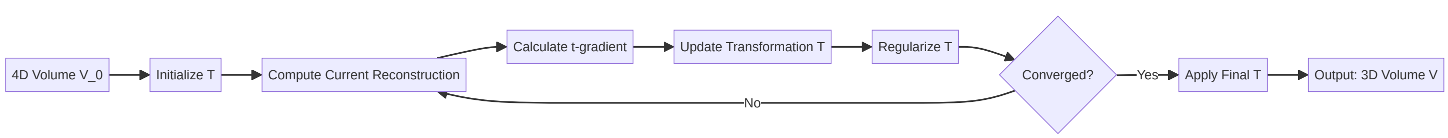
Constraints

1. T is a diffeomorphism
2. Minimize variation of V along original t direction
3. Well-conditioned Jacobian of T

Algorithm

1. Initialize $T_0(x, y, z, t) = (x, y, z)$
2. For each iteration (max*iter):
   a. Compute current reconstruction: $V_k(x, y, z) = V_0(T_k^{-1}(x, y, z))$
   b. Calculate t-gradient: $G_k = \nabla_t V_k \circ T_k^{-1}$
   c. Update T: $T * k + 1 = T * k - \lambda G_k$
   d. Regularize $T * k + 1$:
   i. Smooth: $T_{k+1} = T_{k+1} * \text{GaussianKernel}$
   ii. Enforce invertibility: $\det(\nabla T_{k+1}) > \epsilon$
   e. If $||T_{k+1} - T_k|| < \delta$ or iteration > max_iter, break
3. Final reconstruction: $V(x, y, z) = V_0(T^{-1}(x, y, z))$

Mermaid Chart



Approach 6: Dense Matching for Ultrasound Frames

Formulation

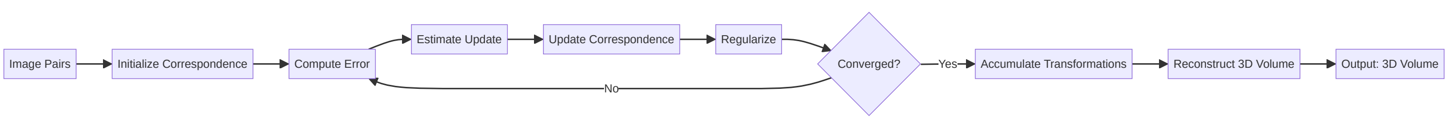
For each pair (I_i, I_{i+1}), find $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that:
$I_i(x, y) \approx I_{i+1}(F_i(x, y))$ for all (x, y)

Algorithm

1. For each pair (I_i, I_{i+1}):
   a. Initialize $F_i(x, y) = (x, y)$
   b. For iteration = 1 to max*iter:
   i. Compute error: $E_i(x, y) = I_i(x, y) - I * i + 1(F * i(x, y))$
   ii. Estimate update: $\Delta F_i = \mathrm{NCC}(E_i, I * i + 1)$
   iii. Update: $F_i(x, y) \leftarrow F_i(x, y) + \lambda \Delta F_i(x, y)$
   iv. Regularize: $F_i = \mathrm{BilateralFilter}(F_i)$
   v. If $\|\Delta F_i\| < \epsilon$, break
2. Accumulate: $T_i = F_1 \circ F_2 \circ ... \circ F_{i-1}$
3. Reconstruct:
   For each voxel (x, y, z) in V:
   a. Find i such that $z \in [i\Delta z, (i+1)\Delta z]$
   b. Compute $(x', y') = T_i^{-1}(x, y)$
   c. Set $V(x, y, z) = I_i(x', y')$

Where NCC is Normalized Cross-Correlation.

Mermaid Chart



Approach 7: Visual SLAM Inspired Reconstruction

Formulation

Simultaneously estimate:

1. Map $M : \Omega \to \mathbb{R}$ (3D tissue structure)
2. Trajectory $\tau : [0, T] \to SE(3)$ (probe poses)

Optimize $\arg\min_{M,\tau} \sum_{i=1}^{N} \|I_i - \mathrm{Proj}(M, \tau(t_i))\|^2 + \lambda R(M, \tau)$

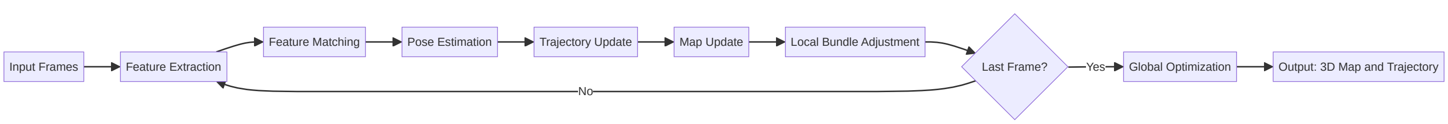
Where Proj is the projection function and R is a regularization term.

Algorithm

1. Initialize M_0 as empty map, τ_0 as identity trajectory
2. For each frame I_i, $i = 1$ to N:
   a. Feature extraction: $F_i = \mathrm{FAST}(I_i)$
   b. Feature matching: $M_i = \mathrm{Match}(F_i, F_{i-1})$
   c. Pose estimation: $\Delta\tau_i = \mathrm{PnP}(M_i, M_{i-1})$
   d. Update trajectory: $\tau_i = \tau_{i-1} \circ \Delta\tau_i$
   e. Update map:
   For each matched feature f in M_i:
   i. If f is new, triangulate and add to M
   ii. Else, update existing point in M via Kalman filter
   f. Local bundle adjustment:
   Optimize $\arg\min_{M',\tau'} \sum_{j=i-k}^{i} \|I_j - \mathrm{Proj}(M', \tau'(t_j))\|^2$
   Update M and τ with optimized M' and τ'
3. Global optimization:
   Optimize $\arg\min_{M,\tau} \sum_{i=1}^{N} \|I_i - \mathrm{Proj}(M, \tau(t_i))\|^2 + \lambda R(M, \tau)$

Where FAST is Features from Accelerated Segment Test, and PnP is Perspective-n-Point algorithm.

Mermaid Chart



Approach 8: Large Point Cloud Construction

Formulation

Given a sequence of frames $\{I_i\}_{i=1}^{N}$ and initial pose estimates $\{P_i\}_{i=1}^{N}$, construct a dense point cloud $\mathcal{C} = \{(x_j, y_j, z_j, i_j)\}_{j=1}^{M}$

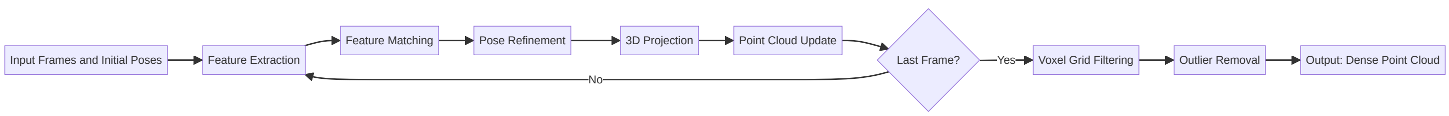
Algorithm

1. Initialize empty point cloud $\mathcal{C} = \{\}$
2. For each frame I_i, $i = 1$ to N:
   a. Extract features: $F_i = \mathrm{SIFT}(I_i)$
   b. If $i > 1$:

i. Match features: $M_i = \text{FlannMatcher}(F_i, F_{i-1})$
ii. Refine pose: $P_i = \text{ICP}(P_i, P_{i-1}, M_i)$
c. Project frame to 3D:
For each pixel (u, v) in I_i:
i. Depth estimation: $d = \text{EstimateDepth}(I_i, u, v)$
ii. Back-project: $(x, y, z) = P_i^{-1} K^{-1}(u, v, d)$
iii. Add $(x, y, z, I_i(u, v))$ to $\mathcal{C}$
3. Voxel grid filter: $\mathcal{C} = \text{VoxelGrid}(\mathcal{C}, \text{voxel\_size})$
4. Statistical outlier removal: $\mathcal{C} = \text{SOR}(\mathcal{C}, k, \alpha)$

Where SIFT is Scale-Invariant Feature Transform, ICP is Iterative Closest Point, SOR is Statistical Outlier Removal.

Mermaid Chart



Approach 9: Parametric Curve with Inflection Points

Formulation

Find a parametric curve $C(s) : [0, 1] \to \mathbb{R}^3$ with n inflection points, minimizing:

$$\arg\min_C \sum_{i=1}^{N-1} \|\text{OpticalFlow}(I_i \circ T(s_i), I_{i+1} \circ T(s_{i+1}))\|^2$$

subject to:

1. $C(0) = (0, 0, 0)$, $C(1) = (x_f, y_f, z_f)$
2. $\|C'(s)\| \approx L$ for all s
3. $C''(s_k) = 0$ for $k = 1, ..., n$ (inflection points)

Algorithm

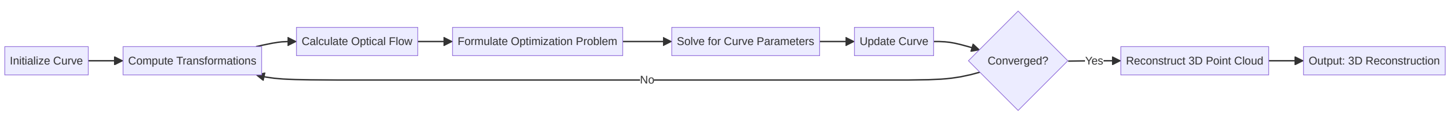
1. Initialize $C(s)$ as a straight line
2. Define basis: $C(s) = \sum_{k=0}^{K} \alpha_k B_k(s)$, where B_k are cubic B-splines
3. For iteration = 1 to max*iter:
   a. Compute transformations: $T(s_i) = C(s_i) \circ S(s_i)^{-1}$, $s_i = i/(N-1)$
   b. Calculate optical flow: $F_i = \text{OpticalFlow}(I_i \circ T(s_i), I * i + 1 \circ T(s * i + 1))$

c. Formulate optimization problem:
$\min *\alpha * k \sum *i = 1^{N-1} \|F_i\|^2$
subject to:
i. $C(0) = (0, 0, 0)$, $C(1) = (x_f, y_f, z_f)$
ii. $\|C'(s)\| \approx L$ for sampled s
iii. $C''(s_k) = 0$ for $k = 1, ..., n$
d. Solve for $\{\alpha_k\}$ using constrained optimization (e.g., SQP)
e. Update $C(s)$ with new $\{\alpha_k\}$
f. If change in $C(s)$ < ϵ, break

4. Final reconstruction:
For each pixel (u, v) in each image I_i:
a. Compute $s_i = i/(N-1)$
b. Back-project to 3D: $(x, y, z) = T(s_i)^{-1} K^{-1} (u, v, 1)$
c. Add $(x, y, z, I_i(u, v))$ to point cloud

Where SQP is Sequential Quadratic Programming.

Mermaid Chart



Approach 10: Multi-Modal Data Fusion

Formulation

Given ultrasound images $\{I_i\}_{i=1}^N$, IMU data $\{A_i\}_{i=1}^N$, and pressure data $\{P_i\}_{i=1}^N$, estimate 3D volume V and probe trajectory τ.

Optimize:
$\arg\min_{V,\tau} \sum_{i=1}^N (\|I_i - \text{Proj}(V, \tau(t_i))\|^2 + \lambda_1 \|\ddot{\tau}(t_i) - A_i\|^2 + \lambda_2 \|P_i - f(\tau(t_i))\|^2)$

Where f estimates pressure based on probe position.

Algorithm

1. Initialize V_0 as empty volume, τ_0 as constant velocity trajectory
2. For each time step $i = 1$ to N:
   a. Image processing:

i. Feature extraction: $F_i = \text{SURF}(I_i)$

ii. If $i > 1$: Match features $M_i = \text{Match}(F_i, F_{i-1})$

b. IMU integration:

i. Predict pose: $\hat{\tau}_i = \text{IntegrateIMU}(\tau_{i-1}, A_i)$

c. Pressure-based correction:

i. Estimate depth: $d_i = \text{EstimateDepth}(P_i)$

ii. Correct z-component: $\hat{\tau}_i(z) = d_i$

d. Pose optimization:

Optimize $\tau_i = \arg\min_\tau(\|M_i - \text{Project}(V_{i-1}, \tau)\|^2 + \|\tau - \hat{\tau}_i\|^2)$

e. Volume update:

i. Back-project I_i to 3D using τ_i

ii. Fuse with V_{i-1} using TSDF (Truncated Signed Distance Function)

3. Global optimization:

   Optimize full cost function over all frames for V and τ

Where SURF is Speeded Up Robust Features.

Mermaid Chart