Derivation of phase - field thermo - mechanical model

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Needs["VariationalMethods`"]
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Needs["VectorAnalysis`"]
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Papers used :

1. Wang et.al MSEA 481 - 482, 2008, 190 - 193
2. Thermomechanical waves in sma pathces udner small mechanical loadings
3. Applications of fully ocnservative schemes in nonlinear thermoelasticity : modeling Shape Memory Materials.
4. Use Defer and Hold command in mathematica for holding the equation, and HELP tutorial/NonStandardEvaluation

```
e1 = (D[u[x, y, t], x] + D[v[x, y, t], y])/Sqrt[2]
```

$$\frac{v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]}{\sqrt{2}}$$

```
e2 = (D[u[x, y, t], x] - D[v[x, y, t], y])/Sqrt[2]
```

$$\frac{-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]}{\sqrt{2}}$$

```
e3 = (D[u[x, y, t], y] + D[v[x, y, t], x])/2
```

$$\frac{1}{2}\left(u^{(0,1,0)}[x, y, t] + v^{(1,0,0)}[x, y, t]\right)$$

```
KE = ρ/2 * (D[u[x, y, t], t]^2 + D[v[x, y, t], t]^2)
```

$$\frac{1}{2}\rho\left(u^{(0,0,1)}[x, y, t]^2 + v^{(0,0,1)}[x, y, t]^2\right)$$

```
PE = a1/2 * e1^2 + a3/2 * e3^2 + a2/2 * (θ[x, y, t] - θ0) / θ0 * e2^2 -
   a4/4 * e2^4 + a6/6 * e2^6 + kg/2 * (D[e2, x]^2 + D[e2, y]^2)
```

$$\frac{a2\,(-\theta 0+\theta[x,y,t])\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta 0}-\frac{1}{16}a4\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^4+\frac{1}{48}a6\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^6+\frac{1}{4}a1\left(v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2+\frac{1}{8}a3\left(u^{(0,1,0)}[x,y,t]+v^{(1,0,0)}[x,y,t]\right)^2+\frac{1}{2}kg\left(\frac{1}{2}\left(-v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)^2+\frac{1}{2}\left(-v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)^2\right)$$

```
DE = η / 2 * ( (D[u[x, y, t], x, t])^2 +
     (D[u[x, y, t], y, t])^2 + (D[v[x, y, t], x, t])^2 + (D[v[x, y, t], y, t])^2 )
```

$$\frac{1}{2}\eta\left(u^{(0,1,1)}[x,y,t]^2+v^{(0,1,1)}[x,y,t]^2+u^{(1,0,1)}[x,y,t]^2+v^{(1,0,1)}[x,y,t]^2\right)$$

```
L = KE - PE
```

$$\frac{1}{2}\rho\left(u^{(0,0,1)}[x,y,t]^2+v^{(0,0,1)}[x,y,t]^2\right)-\frac{a2\,(-\theta 0+\theta[x,y,t])\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta 0}+\frac{1}{16}a4\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^4-\frac{1}{48}a6\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^6-\frac{1}{4}a1\left(v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2-\frac{1}{8}a3\left(u^{(0,1,0)}[x,y,t]+v^{(1,0,0)}[x,y,t]\right)^2-\frac{1}{2}kg\left(\frac{1}{2}\left(-v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)^2+\frac{1}{2}\left(-v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)^2\right)$$

```
H = VariationalD[L, {u[x, y, t], v[x, y, t]}, {x, y, t}]
```

$$\left\{\frac{1}{8}\left(-8\,\rho\,u^{(0,0,2)}[x,y,t]+\frac{4\,a2\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\theta^{(1,0,0)}[x,y,t]}{\theta 0}+\right.\right.$$
$$2\,a3\left(u^{(0,2,0)}[x,y,t]+v^{(1,1,0)}[x,y,t]\right)+$$
$$\frac{4\,a2\,(\theta 0-\theta[x,y,t])\left(v^{(1,1,0)}[x,y,t]-u^{(2,0,0)}[x,y,t]\right)}{\theta 0}+$$
$$6\,a4\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^2\left(v^{(1,1,0)}[x,y,t]-u^{(2,0,0)}[x,y,t]\right)+$$
$$5\,a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^4\left(-v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)+$$
$$4\,a1\left(v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)+4\,kg\left(v^{(1,3,0)}[x,y,t]-u^{(2,2,0)}[x,y,t]\right)+$$
$$\left.4\,kg\left(v^{(3,1,0)}[x,y,t]-u^{(4,0,0)}[x,y,t]\right)\right),$$
$$\frac{1}{8}\left(-8\,\rho\,v^{(0,0,2)}[x,y,t]+\frac{4\,a2\,\theta^{(0,1,0)}[x,y,t]\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)}{\theta 0}-\right.$$
$$\frac{4\,a2\,(\theta 0-\theta[x,y,t])\left(v^{(0,2,0)}[x,y,t]-u^{(1,1,0)}[x,y,t]\right)}{\theta 0}+$$
$$5\,a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^4\left(v^{(0,2,0)}[x,y,t]-u^{(1,1,0)}[x,y,t]\right)+$$
$$6\,a4\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^2\left(-v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)+$$
$$4\,a1\left(v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)-4\,kg\left(v^{(0,4,0)}[x,y,t]-u^{(1,3,0)}[x,y,t]\right)+$$
$$\left.\left.2\,a3\left(u^{(1,1,0)}[x,y,t]+v^{(2,0,0)}[x,y,t]\right)-4\,kg\left(v^{(2,2,0)}[x,y,t]-u^{(3,1,0)}[x,y,t]\right)\right)\right\}$$

```
EOM = Collect[H, {kg}]
```

$$\left\{\frac{1}{8}\left(-8\,\rho\,u^{(0,0,2)}[x,y,t]+\frac{4\,a2\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\theta^{(1,0,0)}[x,y,t]}{\theta 0}+\right.\right.$$
$$2\,a3\left(u^{(0,2,0)}[x,y,t]+v^{(1,1,0)}[x,y,t]\right)+$$
$$\frac{4\,a2\,(\theta 0-\theta[x,y,t])\left(v^{(1,1,0)}[x,y,t]-u^{(2,0,0)}[x,y,t]\right)}{\theta 0}+$$
$$6\,a4\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^2\left(v^{(1,1,0)}[x,y,t]-u^{(2,0,0)}[x,y,t]\right)+$$
$$5\,a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^4\left(-v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)+$$
$$\left.4\,a1\left(v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)\right)+$$
$$\frac{1}{8}\,kg\left(4\left(v^{(1,3,0)}[x,y,t]-u^{(2,2,0)}[x,y,t]\right)+4\left(v^{(3,1,0)}[x,y,t]-u^{(4,0,0)}[x,y,t]\right)\right),$$
$$\frac{1}{8}\left(-8\,\rho\,v^{(0,0,2)}[x,y,t]+\frac{4\,a2\,\theta^{(0,1,0)}[x,y,t]\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)}{\theta 0}-\right.$$
$$\frac{4\,a2\,(\theta 0-\theta[x,y,t])\left(v^{(0,2,0)}[x,y,t]-u^{(1,1,0)}[x,y,t]\right)}{\theta 0}+$$
$$5\,a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^4\left(v^{(0,2,0)}[x,y,t]-u^{(1,1,0)}[x,y,t]\right)+$$
$$6\,a4\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^2\left(-v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)+$$
$$\left.4\,a1\left(v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)+2\,a3\left(u^{(1,1,0)}[x,y,t]+v^{(2,0,0)}[x,y,t]\right)\right)+$$
$$\left.\frac{1}{8}\,kg\left(-4\left(v^{(0,4,0)}[x,y,t]-u^{(1,3,0)}[x,y,t]\right)-4\left(v^{(2,2,0)}[x,y,t]-u^{(3,1,0)}[x,y,t]\right)\right)\right\}$$

Simplify the higher order terms

```
FE2dHT = -a4/4 * e2^4 + a6/6 * e2^6
```

$$-\frac{1}{16}\,a4\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^4+\frac{1}{48}\,a6\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^6$$

```
H1 = VariationalD[FE2dHT, {D[u[x, y, t], x], D[v[x, y, t], y]}, {x, y, t}]
```

$$\left\{\frac{1}{8}\left(-2\,a4+a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^2\right)\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^3,\right.$$
$$\left.\frac{1}{8}\left(a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^5+2\,a4\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^3\right)\right\}$$

```
EOMx = Take[EOM, {1, 1}]
```

$$\left\{\frac{1}{8}\left(-8\,\rho\,u^{(0,0,2)}[x,y,t]+\frac{4\,a2\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\theta^{(1,0,0)}[x,y,t]}{\theta 0}+\right.\right.$$
$$2\,a3\left(u^{(0,2,0)}[x,y,t]+v^{(1,1,0)}[x,y,t]\right)+$$
$$\frac{4\,a2\,(\theta 0-\theta[x,y,t])\left(v^{(1,1,0)}[x,y,t]-u^{(2,0,0)}[x,y,t]\right)}{\theta 0}+$$
$$6\,a4\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^2\left(v^{(1,1,0)}[x,y,t]-u^{(2,0,0)}[x,y,t]\right)+$$
$$5\,a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^4\left(-v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)+$$
$$\left.4\,a1\left(v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)\right)+$$
$$\left.\frac{1}{8}\,kg\left(4\left(v^{(1,3,0)}[x,y,t]-u^{(2,2,0)}[x,y,t]\right)+4\left(v^{(3,1,0)}[x,y,t]-u^{(4,0,0)}[x,y,t]\right)\right)\right\}$$

```
EOMy = Take[EOM, {2, 2}]
```

$$\left\{\frac{1}{8}\left(-8\,\rho\,v^{(0,0,2)}[x,y,t]+\frac{4\,a2\,\theta^{(0,1,0)}[x,y,t]\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)}{\theta 0}-\right.\right.$$
$$\frac{4\,a2\,(\theta 0-\theta[x,y,t])\left(v^{(0,2,0)}[x,y,t]-u^{(1,1,0)}[x,y,t]\right)}{\theta 0}+$$
$$5\,a6\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^4\left(v^{(0,2,0)}[x,y,t]-u^{(1,1,0)}[x,y,t]\right)+$$
$$6\,a4\left(v^{(0,1,0)}[x,y,t]-u^{(1,0,0)}[x,y,t]\right)^2\left(-v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)+$$
$$\left.4\,a1\left(v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)+2\,a3\left(u^{(1,1,0)}[x,y,t]+v^{(2,0,0)}[x,y,t]\right)\right)+$$
$$\left.\frac{1}{8}\,kg\left(-4\left(v^{(0,4,0)}[x,y,t]-u^{(1,3,0)}[x,y,t]\right)-4\left(v^{(2,2,0)}[x,y,t]-u^{(3,1,0)}[x,y,t]\right)\right)\right\}$$

Stress tensor components

```
S11T = D[PE, D[u[x, y, t], x]] /.
  {(D[u[x, y, t], x] - D[v[x, y, t], y]) → √2 * E2, (D[u[x, y, t], x] + D[v[x, y, t], y]) →
    √2 * E1, (D[u[x, y, t], y] + D[v[x, y, t], x]) → 2 * E3}
```

$$\frac{a1\,E1}{\sqrt{2}}-\frac{a4\,E2^3}{\sqrt{2}}+\frac{a6\,E2^5}{\sqrt{2}}+\frac{a2\,E2\,(-\theta 0+\theta[x,y,t])}{\sqrt{2}\,\theta 0}$$

```
S12T = D[PE, D[u[x, y, t], y]] /.
  {(D[u[x, y, t], x] - D[v[x, y, t], y]) → √2 * E2, (D[u[x, y, t], x] + D[v[x, y, t], y]) →
    √2 * E1, (D[u[x, y, t], y] + D[v[x, y, t], x]) → 2 * E3}
```

$$\frac{a3\,E3}{2}$$

```
S21T = D[PE, D[v[x, y, t], x]] /.
  {(D[u[x, y, t], x] - D[v[x, y, t], y]) → √2 * E2, (D[u[x, y, t], x] + D[v[x, y, t], y]) →
    √2 * E1, (D[u[x, y, t], y] + D[v[x, y, t], x]) → 2 * E3}
```

$$\frac{a3\,E3}{2}$$

```
S22T = D[PE, D[v[x, y, t], y]] /.
  {(D[u[x, y, t], x] - D[v[x, y, t], y]) → √2 * E2, (D[u[x, y, t], x] + D[v[x, y, t], y]) →
    √2 * E1, (D[u[x, y, t], y] + D[v[x, y, t], x]) → 2 * E3}
```

$$\frac{a1\,E1}{\sqrt{2}} + \frac{a4\,E2^3}{\sqrt{2}} - \frac{a6\,E2^5}{\sqrt{2}} - \frac{a2\,E2\,(-\theta 0 + \theta[x, y, t])}{\sqrt{2}\,\theta 0}$$

Dissipational Stress components in Eq of motion

```
DEx = Simplify[EOMx /. {a1 → 0,  a2 → 0, a3 → 0, a4 → 0, a6 → 0, kg → 0, ρ → 0}]
```

$$\{0\}$$

```
DEy = Simplify[EOMy /. {a1 → 0,  a2 → 0, a3 → 0, a4 → 0, a6 → 0, kg → 0, ρ → 0}]
```

$$\{0\}$$

Thermal Dynamics - 2 D

```
Psi = - Cv * θ[x, y, t] * Log[θ[x, y, t]] + PE
```

$$-\text{Cv}\,\text{Log}[\theta[x,y,t]]\,\theta[x,y,t]+\frac{\text{a2}\,(-\theta 0+\theta[x,y,t])\left(-\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta 0}-\frac{1}{16}\,\text{a4}\left(-\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^4+\frac{1}{48}\,\text{a6}\left(-\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^6+\frac{1}{4}\,\text{a1}\left(\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^2+\frac{1}{8}\,\text{a3}\left(\text{u}^{(0,1,0)}[x,y,t]+\text{v}^{(1,0,0)}[x,y,t]\right)^2+\frac{1}{2}\,\text{kg}\left(\frac{1}{2}\left(-\text{v}^{(0,2,0)}[x,y,t]+\text{u}^{(1,1,0)}[x,y,t]\right)^2+\frac{1}{2}\left(-\text{v}^{(1,1,0)}[x,y,t]+\text{u}^{(2,0,0)}[x,y,t]\right)^2\right)$$

```
e = Psi - θ[x, y, t] * D[Psi, θ[x, y, t]]
```

$$-\text{Cv}\,\text{Log}[\theta[x,y,t]]\,\theta[x,y,t]+\frac{\text{a2}\,(-\theta 0+\theta[x,y,t])\left(-\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta 0}-\frac{1}{16}\,\text{a4}\left(-\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^4+\frac{1}{48}\,\text{a6}\left(-\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^6+\frac{1}{4}\,\text{a1}\left(\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^2-\theta[x,y,t]\left(-\text{Cv}-\text{Cv}\,\text{Log}[\theta[x,y,t]]+\frac{\text{a2}\left(-\text{v}^{(0,1,0)}[x,y,t]+\text{u}^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta 0}\right)+\frac{1}{8}\,\text{a3}\left(\text{u}^{(0,1,0)}[x,y,t]+\text{v}^{(1,0,0)}[x,y,t]\right)^2+\frac{1}{2}\,\text{kg}\left(\frac{1}{2}\left(-\text{v}^{(0,2,0)}[x,y,t]+\text{u}^{(1,1,0)}[x,y,t]\right)^2+\frac{1}{2}\left(-\text{v}^{(1,1,0)}[x,y,t]+\text{u}^{(2,0,0)}[x,y,t]\right)^2\right)$$

```
q = -k * (D[θ[x, y, t], x] + D[θ[x, y, t], y])
```

$$-\text{k}\left(\theta^{(0,1,0)}[x,y,t]+\theta^{(1,0,0)}[x,y,t]\right)$$

```
T1 = ρ * D[e, t]
```

$$\rho \left(-\text{Cv}\,\theta^{(0,0,1)}[x,y,t] - \text{Cv}\,\text{Log}[\theta[x,y,t]]\,\theta^{(0,0,1)}[x,y,t] + \frac{\text{a2}\,\theta^{(0,0,1)}[x,y,t]\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta 0} - \theta^{(0,0,1)}[x,y,t]\left(-\text{Cv}-\text{Cv}\,\text{Log}[\theta[x,y,t]]+\frac{\text{a2}\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta 0}\right) + \frac{1}{2\,\theta 0}\text{a2}\,(-\theta 0+\theta[x,y,t])\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right) - \frac{1}{4}\text{a4}\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^3\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right) + \frac{1}{8}\text{a6}\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^5\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right) + \frac{1}{2}\text{a1}\left(v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\left(v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right) - \theta[x,y,t]\left(-\frac{\text{Cv}\,\theta^{(0,0,1)}[x,y,t]}{\theta[x,y,t]} + \frac{\text{a2}\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right)}{2\,\theta 0}\right) + \frac{1}{4}\text{a3}\left(u^{(0,1,0)}[x,y,t]+v^{(1,0,0)}[x,y,t]\right)\left(u^{(0,1,1)}[x,y,t]+v^{(1,0,1)}[x,y,t]\right) + \frac{1}{2}\text{kg}\left(\left(-v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)\left(-v^{(0,2,1)}[x,y,t]+u^{(1,1,1)}[x,y,t]\right) + \left(-v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)\left(-v^{(1,1,1)}[x,y,t]+u^{(2,0,1)}[x,y,t]\right)\right)\right)$$

```
T3 = -k * (D[θ[x, y, t], x, x] + D[θ[x, y, t], y, y])
```

$$-\text{k}\left(\theta^{(0,2,0)}[x,y,t]+\theta^{(2,0,0)}[x,y,t]\right)$$

```
v11 = D[D[u[x, y, t], t], x]
```

$$u^{(1,0,1)}[x,y,t]$$

```
v12 = D[D[u[x, y, t], t], y]
```

$$u^{(0,1,1)}[x,y,t]$$

```
v21 = D[D[v[x, y, t], t], x]
```

$$v^{(1,0,1)}[x, y, t]$$

```
v22 = D[D[v[x, y, t], t], y]
```

$$v^{(0,1,1)}[x, y, t]$$

```
S11 = D[PE, D[u[x, y, t], x]]
```

$$\frac{a2\ (-\Theta 0 + \Theta[x, y, t])\ \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)}{2\,\Theta 0} -$$
$$\frac{1}{4} a4 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^3 +$$
$$\frac{1}{8} a6 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^5 + \frac{1}{2} a1 \left(v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)$$

```
S12 = D[PE, D[u[x, y, t], y]]
```

$$\frac{1}{4} a3 \left(u^{(0,1,0)}[x, y, t] + v^{(1,0,0)}[x, y, t]\right)$$

```
S21 = D[PE, D[v[x, y, t], x]]
```

$$\frac{1}{4} a3 \left(u^{(0,1,0)}[x, y, t] + v^{(1,0,0)}[x, y, t]\right)$$

```
S22 = D[PE, D[v[x, y, t], y]]
```

$$-\frac{a2\ (-\Theta 0 + \Theta[x, y, t])\ \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)}{2\,\Theta 0} +$$
$$\frac{1}{4} a4 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^3 -$$
$$\frac{1}{8} a6 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^5 + \frac{1}{2} a1 \left(v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)$$

```
T2 = ρ * (S11 * v11 + S12 * v12 + S21 * v21 + S22 * v22)
```

$$\rho \left(v^{(0,1,1)}[x, y, t] \left(-\frac{a2\ (-\theta 0 + \theta[x, y, t])\ \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)}{2\,\theta 0} + \frac{1}{4} a4 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^3 - \frac{1}{8} a6 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^5 + \frac{1}{2} a1 \left(v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right) \right) + \frac{1}{4} a3\, u^{(0,1,1)}[x, y, t] \left(u^{(0,1,0)}[x, y, t] + v^{(1,0,0)}[x, y, t]\right) + \left(\frac{a2\ (-\theta 0 + \theta[x, y, t])\ \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)}{2\,\theta 0} - \frac{1}{4} a4 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^3 + \frac{1}{8} a6 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^5 + \frac{1}{2} a1 \left(v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right) \right) u^{(1,0,1)}[x, y, t] + \frac{1}{4} a3 \left(u^{(0,1,0)}[x, y, t] + v^{(1,0,0)}[x, y, t]\right) v^{(1,0,1)}[x, y, t] \right)$$

```
T = T1 - T2 + T3
```

$$-\rho \left(v^{(0,1,1)}[x, y, t] \left(-\frac{a2\ (-\theta 0 + \theta[x, y, t])\ \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)}{2\,\theta 0} + \frac{1}{4} a4 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^3 - \frac{1}{8} a6 \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)^5 + \frac{1}{2} a1 \left(v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right) \right) + \frac{1}{4} a3\, u^{(0,1,1)}[x, y, t] \left(u^{(0,1,0)}[x, y, t] + v^{(1,0,0)}[x, y, t]\right) + \left(\frac{a2\ (-\theta 0 + \theta[x, y, t])\ \left(-v^{(0,1,0)}[x, y, t] + u^{(1,0,0)}[x, y, t]\right)}{2\,\theta 0} -
```
```

$$\frac{1}{4}\,a4\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^3+\frac{1}{8}\,a6\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^5+\frac{1}{2}\,a1\left(v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\Bigg)$$

$$u^{(1,0,1)}[x,y,t]+\frac{1}{4}\,a3\left(u^{(0,1,0)}[x,y,t]+v^{(1,0,0)}[x,y,t]\right)v^{(1,0,1)}[x,y,t]\Bigg)-$$

$$k\left(\theta^{(0,2,0)}[x,y,t]+\theta^{(2,0,0)}[x,y,t]\right)+$$

$$\rho\Bigg(-Cv\,\theta^{(0,0,1)}[x,y,t]-Cv\,Log[\theta[x,y,t]]\,\theta^{(0,0,1)}[x,y,t]+$$

$$\frac{a2\,\theta^{(0,0,1)}[x,y,t]\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta0}-$$

$$\theta^{(0,0,1)}[x,y,t]\left(-Cv-Cv\,Log[\theta[x,y,t]]+\frac{a2\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^2}{4\,\theta0}\right)+$$

$$\frac{1}{2\,\theta0}a2\,(-\theta0+\theta[x,y,t])\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)$$
$$\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right)-$$

$$\frac{1}{4}\,a4\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^3\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right)+$$

$$\frac{1}{8}\,a6\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)^5\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right)+$$

$$\frac{1}{2}\,a1\left(v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\left(v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right)-$$

$$\theta[x,y,t]\left(-\frac{Cv\,\theta^{(0,0,1)}[x,y,t]}{\theta[x,y,t]}+\frac{a2\left(-v^{(0,1,0)}[x,y,t]+u^{(1,0,0)}[x,y,t]\right)\left(-v^{(0,1,1)}[x,y,t]+u^{(1,0,1)}[x,y,t]\right)}{2\,\theta0}\right)+$$

$$\frac{1}{4}\,a3\left(u^{(0,1,0)}[x,y,t]+v^{(1,0,0)}[x,y,t]\right)\left(u^{(0,1,1)}[x,y,t]+v^{(1,0,1)}[x,y,t]\right)+$$

$$\frac{1}{2}\,kg\left(\left(-v^{(0,2,0)}[x,y,t]+u^{(1,1,0)}[x,y,t]\right)\left(-v^{(0,2,1)}[x,y,t]+u^{(1,1,1)}[x,y,t]\right)+\left(-v^{(1,1,0)}[x,y,t]+u^{(2,0,0)}[x,y,t]\right)\left(-v^{(1,1,1)}[x,y,t]+u^{(2,0,1)}[x,y,t]\right)\right)\Bigg)$$

```
Expand[T]
```

$$\text{Cv}\,\rho\,\theta^{(0,0,1)}[x,y,t] - \frac{\text{a2}\,\rho\,\theta[x,y,t]\,v^{(0,1,0)}[x,y,t]\,v^{(0,1,1)}[x,y,t]}{2\,\theta 0} - k\,\theta^{(0,2,0)}[x,y,t] + \frac{1}{2}\,\text{kg}\,\rho\,v^{(0,2,0)}[x,y,t]\,v^{(0,2,1)}[x,y,t] + \frac{\text{a2}\,\rho\,\theta[x,y,t]\,v^{(0,1,1)}[x,y,t]\,u^{(1,0,0)}[x,y,t]}{2\,\theta 0} + \frac{\text{a2}\,\rho\,\theta[x,y,t]\,v^{(0,1,0)}[x,y,t]\,u^{(1,0,1)}[x,y,t]}{2\,\theta 0} - \frac{\text{a2}\,\rho\,\theta[x,y,t]\,u^{(1,0,0)}[x,y,t]\,u^{(1,0,1)}[x,y,t]}{2\,\theta 0} - \frac{1}{2}\,\text{kg}\,\rho\,v^{(0,2,1)}[x,y,t]\,u^{(1,1,0)}[x,y,t] - \frac{1}{2}\,\text{kg}\,\rho\,v^{(0,2,0)}[x,y,t]\,u^{(1,1,1)}[x,y,t] + \frac{1}{2}\,\text{kg}\,\rho\,u^{(1,1,0)}[x,y,t]\,u^{(1,1,1)}[x,y,t] + \frac{1}{2}\,\text{kg}\,\rho\,v^{(1,1,0)}[x,y,t]\,v^{(1,1,1)}[x,y,t] - \frac{1}{2}\,\text{kg}\,\rho\,v^{(1,1,1)}[x,y,t]\,u^{(2,0,0)}[x,y,t] - k\,\theta^{(2,0,0)}[x,y,t] - \frac{1}{2}\,\text{kg}\,\rho\,v^{(1,1,0)}[x,y,t]\,u^{(2,0,1)}[x,y,t] + \frac{1}{2}\,\text{kg}\,\rho\,u^{(2,0,0)}[x,y,t]\,u^{(2,0,1)}[x,y,t]$$

```
Thermal =
 Expand[T /. {kg → 0, Log[θ[x, t]] → 0, D[u[x, y, t], t, t] → 0, D[v[x, y, t], t, t] → 0}]
```

$$\text{Cv}\,\rho\,\theta^{(0,0,1)}[x,y,t] - \frac{\text{a2}\,\rho\,\theta[x,y,t]\,v^{(0,1,0)}[x,y,t]\,v^{(0,1,1)}[x,y,t]}{2\,\theta 0} - k\,\theta^{(0,2,0)}[x,y,t] + \frac{\text{a2}\,\rho\,\theta[x,y,t]\,v^{(0,1,1)}[x,y,t]\,u^{(1,0,0)}[x,y,t]}{2\,\theta 0} + \frac{\text{a2}\,\rho\,\theta[x,y,t]\,v^{(0,1,0)}[x,y,t]\,u^{(1,0,1)}[x,y,t]}{2\,\theta 0} - \frac{\text{a2}\,\rho\,\theta[x,y,t]\,u^{(1,0,0)}[x,y,t]\,u^{(1,0,1)}[x,y,t]}{2\,\theta 0} - k\,\theta^{(2,0,0)}[x,y,t]$$

```
T = Simplify[Thermal]
```

$$-\frac{1}{2\,\theta 0}\left(-2\,\text{Cv}\,\theta 0\,\rho\,\theta^{(0,0,1)}[x,y,t] + \text{a2}\,\rho\,\theta[x,y,t]\left(v^{(0,1,0)}[x,y,t] - u^{(1,0,0)}[x,y,t]\right)\left(v^{(0,1,1)}[x,y,t] - u^{(1,0,1)}[x,y,t]\right) + 2\,k\,\theta 0\left(\theta^{(0,2,0)}[x,y,t] + \theta^{(2,0,0)}[x,y,t]\right)\right)$$

```
Ther = T /. {(-D[u[x, y, t], x] + D[v[x, y, t], y]) → -√2 * E2,
    (-D[u[x, y, t], x, t] + D[v[x, y, t], y, t]) → -√2 * E2dot}
```

$$-\frac{2\,\text{a2}\,\text{E2}\,\text{E2dot}\,\rho\,\theta[x,y,t] - 2\,\text{Cv}\,\theta 0\,\rho\,\theta^{(0,0,1)}[x,y,t] + 2\,k\,\theta 0\left(\theta^{(0,2,0)}[x,y,t] + \theta^{(2,0,0)}[x,y,t]\right)}{2\,\theta 0}$$

TeXForm[Ther]

```
-\frac{2 \text{a2} \text{E2} \text{E2dot} \rho  \theta (x,y,t)-2 \text{Cv} \text{$\theta $0
```

Simplify[Ther]

$$-\frac{\text{a2 E2 E2dot}\,\rho\,\theta[\text{x, y, t}]}{\theta 0}+\text{Cv}\,\rho\,\theta^{(0,0,1)}[\text{x, y, t}]-\text{k}\left(\theta^{(0,2,0)}[\text{x, y, t}]+\theta^{(2,0,0)}[\text{x, y, t}]\right)$$