Computational Nonlinear Mechanics

Assignment 1: Truss structure

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March 6, 2019

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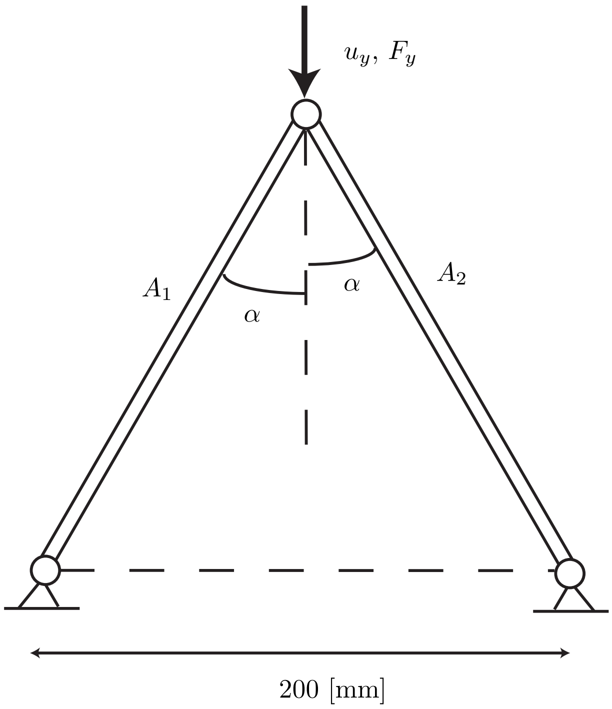
Computer assignment 1: Truss structure

Write your own code that includes load/time stepping, computation of geometrical nonlinear bar (element) force and stiffness, computation of hyperelastic and elasto-plastic stress and consistent tangent stiffness in 1D, solution of nonlinear system of equations by a Newton-Raphson scheme. Analyze the following truss structure:



- [a] Assume that the bars are equal, i.e., $A_1 = A_2 = A$, $\alpha = \pi/4$ reproduce the results given in Figure 3.8 a-d (p. 90) in Bonet & Wood (Remember that you deform 2 bars). Control the vertical displacement u_y of the initially upper joint. Also plot the constitutive behaviour in the bars for the purely elastic case.
- [b] Now assume an imperfection $A_2 = A_1 \cdot 1.05 = A \cdot 1.05$ that will cause an unsymmetric response. Otherwise assume the same conditions as in [a] and give the same type results as in [a]. Investigate the sensitivity of the imperfection by plotting how the initially upper joint will move during the loading.
- [c] Perform [b] once again but now for a smaller angle atan(α)= 1/3.

1 Theory

The following is a short summary of the used equations, which were adopted from chapter 3 of the course book [1]. Unless stated otherwise, the uppercase letters refer to the initial (reference) configuration of the problem, while the lowercase ones refer to the current configuration. The boldface letters denote vectors, matrices and tensors.

For a given load, solving the problem implies finding a configuration that simultaneously satisfies both the global (system) equilibrium equations and the constitutive equations. The equilibrium equations are expressed in terms of the residual (out-of-balance) vector $\boldsymbol{R}(\boldsymbol{x})$ as the balance between internal and external forces:

$$\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{F} = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N]^{\mathrm{T}}$ is the vector of current nodal positions; $\boldsymbol{T} = [\boldsymbol{T}_1, \boldsymbol{T}_2, \cdots, \boldsymbol{T}_N]^{\mathrm{T}}$ is the vector of internal nodal forces; $\boldsymbol{F} = [\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_N]^{\mathrm{T}}$ is the vector of external nodal forces, where it is assumed to be independent of the current nodal positions x (generally this is not true); and N is the number of nodes.

1.1 Hyperelasticity

In the case of hyperelastic material behaviour of a rod, i.e. material whose strain energy per unit volume V does not depend on the path taken by the rod as it moved from initial length L to the current length l, the internal truss forces $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$ can be computed as

$$\boldsymbol{T}_b = \frac{VE}{l} \ln\left(\frac{l}{L}\right) \boldsymbol{n} = \tau \frac{V}{l} \boldsymbol{n}, \quad \boldsymbol{T}_a = -\boldsymbol{T}_b. \tag{2}$$

Here, a Young's modulus like constant E has been used to relate Kirchhoff stress $\tau = E\varepsilon = \sigma v/V$ to logarithmic strain $\varepsilon = \ln(l/L)$.

Finding equilibrium position is carried out using Newton-Raphson method, which involves linearisation of the equilibrium equations. The linearisation yields the directional derivative $D\boldsymbol{T}^{(e)}(\boldsymbol{x}^{(e)})[\boldsymbol{u}^{(e)}]$, which gives the expression for the tangent stiffness matrix:

$$D\boldsymbol{T}^{(e)}(\boldsymbol{x}^{(e)})[\boldsymbol{u}^{(e)}] = \boldsymbol{K}^{(e)}\boldsymbol{u}^{(e)} = \begin{bmatrix} \boldsymbol{K}_{aa}^{(e)} & \boldsymbol{K}_{ab}^{(e)} \\ \boldsymbol{K}_{ba}^{(e)} & \boldsymbol{K}_{bb}^{(e)} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_a \\ \boldsymbol{u}_b \end{bmatrix} \tag{3}$$

with

$$\boldsymbol{K}_{aa}^{(e)} = \boldsymbol{K}_{bb}^{(e)} = \left(\frac{V}{v}\frac{d\tau}{d\varepsilon}\frac{a}{l} - \frac{2\sigma a}{l}\right) \boldsymbol{n} \otimes \boldsymbol{n} + \frac{\sigma a}{l}\boldsymbol{I} \tag{4}$$

$$\boldsymbol{K}_{ab}^{(e)} = \boldsymbol{K}_{ba}^{(e)} = -\boldsymbol{K}_{aa}^{(e)}, \tag{5}$$

where $d\tau/d\varepsilon = E$ is the elastic material tangent modulus.

1.2 Rate-independent plasticity

In the case of rate-independent finite strain plasticity with isotropic hardening, the stress is defined via elastic strain ε_e:

$$\tau = E\varepsilon_e = E\left(\varepsilon - \varepsilon_p\right) \tag{6}$$

$$\varepsilon_p = \int_0^t \dot{\varepsilon}_p dt \tag{7}$$

The onset of plastic deformation is governed by the yield condition, which for the problem at issue is

$$f(\tau, \bar{\varepsilon}_p) = |\tau| - \left(\tau_y^0 + H\bar{\varepsilon}_p\right) \leq 0, \quad \bar{\varepsilon}_p \geq 0, \tag{8}$$

where τ_y^0 is the initial yield stress, $\bar{\varepsilon}_p$ is the hardening parameter and H is a material property called plastic modulus. At its simplest, the hardening parameter is defined as the accumulated absolute plastic strain occurring over time:

$$\bar{\varepsilon}_p = \int_0^t \dot{\bar{\varepsilon}}_p dt, \quad \dot{\bar{\varepsilon}}_p = |\dot{\varepsilon}_p|, \quad \dot{\varepsilon}_p = \dot{\gamma}\frac{\partial f}{\partial \tau}, \tag{9}$$

where $\dot{\gamma}$ is plastic multiplier.

In a computational setting the time integration of (7) can only be performed approximately from a finite sequence of values determined at different time steps. In order to satisfy the yield condition exactly at each incremental time step, a return-mapping algorithm described in figure 1(a) is used, which employs incremental kinematics (see figure 1(b)).

If $f\left(\tau_{n+1}^{\text{trial}}, \bar{\varepsilon}_{p,n}\right) \leq 0$

$\Delta\gamma = 0$

Else

$\Delta\gamma = \dfrac{f_{n+1}^{\text{trial}}}{(E+H)}$

Endif

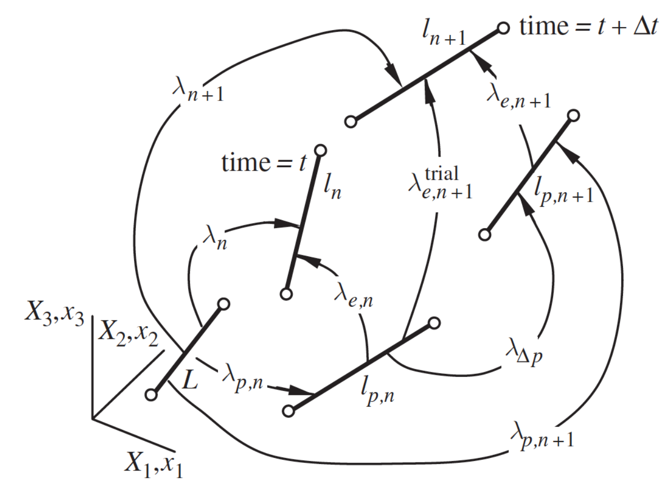
$\Delta\varepsilon_p = \Delta\gamma\,\text{sign}(\tau_{n+1}^{\text{trial}})$

$\tau_{n+1} = \tau_{n+1}^{\text{trial}} - E\Delta\varepsilon_p$

$\varepsilon_{p,n+1} = \varepsilon_{p,n} + \Delta\varepsilon_p$

$\bar{\varepsilon}_{p,n+1} = \bar{\varepsilon}_{p,n} + \Delta\gamma$

(a) Return-mapping algorithm.



(b) Incremental kinematics. $\lambda = l/L$.

Figure 1: Two figures from [1].

The last matter to be addressed in the presence of plasticity is the material tangent modulus $d\tau/d\varepsilon$. The tangent modulus derived from incremental considerations is generally not the same as the one obtained from the rate equations. The reason is that the incremental change in stress imposed by the chosen return-mapping algorithm is different from the continuous change in stress stemming from the rate equations. That is why the algorithmic tangent modulus is used:

$$\frac{d\tau_{n+1}}{d\varepsilon_{n+1}} = \frac{EH}{E+H} \tag{10}$$

When plasticity occurs, this replaces the elastic tangent stiffness $d\tau/d\varepsilon = E$.

2 Task A

In the first task, the plots from [1] were reproduced. In order to achieve that the presence of the second rod in the given problem needed to be accounted for. This was done by normalising the force by the total area of the two rods. At each displacement increment, the vertical force in the top node (DOF 4) was computed as a sum of the already acting force plus the force needed to move the top node by the increment of the displacement, i.e.

$$F_{n+1} = F_n + K^{\text{global}}_{4,j} \Delta u_j \tag{11}$$

The plots presented in figure 3.8 a-d of [1] were reproduced and are given in figure 2.

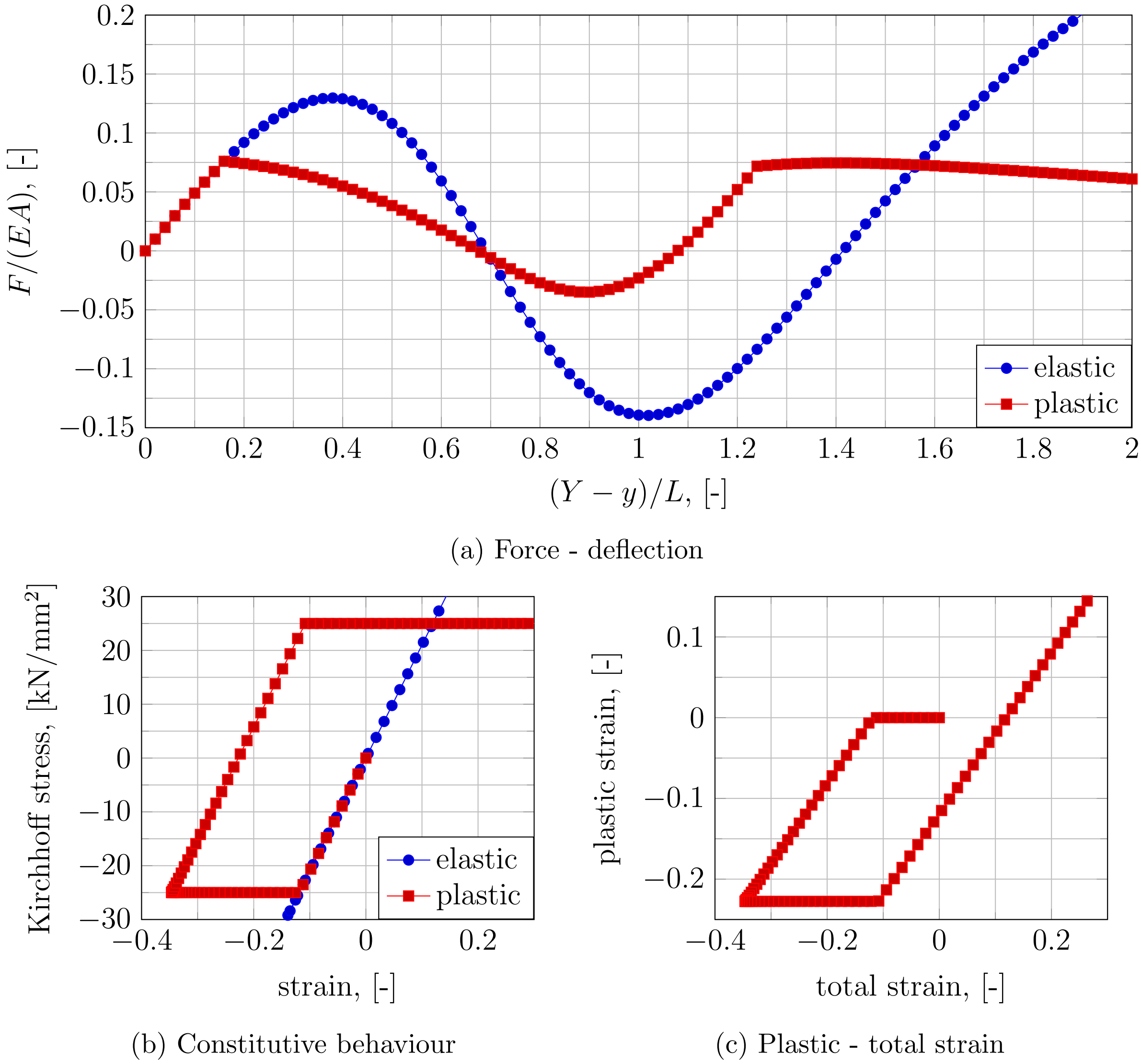


Figure 2: Plots of large deflection elasto-plastic behaviour from [1].

References

[1] J. Bonet and R. D. Wood. *Nonlinear continuum mechanics for finite element analysis*. Cambridge University Press, second edition, 2008.