In[1]:=
```
Remove["Global`*"]
```

Tensor Derivative for Modified Hyperbolic Drucker Prager.

Part I. Define and Verify the Vector and Operator System.

(Develped by Yuan Feng 11/20/2016 at UC Davis)

(1) Define the vector system

In[2]:=
```
SetAttributes[δ, Orderless];
SetAttributes[σ, Orderless];
SetAttributes[s, Orderless];
SetAttributes[α, Orderless];
δ[i_Integer, j_Integer] := If[i == j, 1, 0];
δ[i_, i_] := 3;
s[i_, i_] := 0;
α[i_, i_] := 0;
σ[i_, i_] := -3 p;
```

(2) Define Operator Plus

In[11]:=
```
(*s[i_,j_]:= σ[i,j]+p δ[i,j];*)
(*σ[i_,j_]:=s[i,j]-p δ[i,j];*)
(*Refine[σ[i,j]+p δ[i,j],σ[i_,j_]+p δ[i_,j_]:=s[i,j]]*)
Unprotect[Plus];
σ[i_, j_] + p δ[i_, j_] := s[i, j]
Protect[Plus];
```

(3) Define Operator Multiplication.

In[14]:=
```
Unprotect[Times];
Times[δ[i_, j_], δ[i_, j_]] := δ[i, i] ;
Times[δ[i_, j_], δ[i_, k_]] := δ[k, j] ;
Times[σ[i_, j_], δ[i_, j_]] := σ[j, j];
Times[σ[i_, j_], δ[i_, k_]] := σ[j, k];
Times[s[i_, j_], δ[i_, j_]] := s[i, i];
Times[s[i_, j_], δ[i_, k_]] := s[j, k];
Times[α[i_, j_], δ[i_, j_]] := α[i, i];
Times[α[i_, j_], δ[i_, k_]] := α[j, k];
Times[a_, δ[i_, j_] + b_] := a δ[i, j] + a b;
Times[a_, σ[i_, j_] + b_] := a σ[i, j] + a b;
Times[a_, s[i_, j_] + b_] := a s[i, j] + a b;
Times[a_, α[i_, j_] + b_] := a α[i, j] + a b;
Times[s[i_, j_] , α[i_, j_]] := sα;
Protect[Times];
```

(4) Define Operator Power.

In[29]:=
```
Unprotect[Power];
Power[(s[i_, j_] - p α[i_, j_]), 2] := r^2;
Power[(s[i_, j_]), 2] := s^2;
Power[α[i_, j_], 2] := α^2;
Protect[Power];
```

(5) Define Operator D .

(5.1) For derivative over σ

In[34]:=
```
Unprotect[D];
D[p, σ[i_, j_]] := -1/3 δ[i, j];
D[s[i_, j_], σ[k_, l_]] := δ[i, k] δ[j, l] - 1/3 δ[i, j] δ[k, l];
D[s^2, σ[i_, j_]] := 2 s[i, j] ;
D[r, σ[i_, j_]] := 1/(3 r) (3 s[i, j] - 3 p α[i, j] + sα δ[i, j] - p α^2 δ[i, j]);
D[sα, σ[i_, j_]] := α[i, j];
D[Times[a_, b_], σ[i_, j_]] := D[a, σ[i, j]] b + a D[b, σ[i, j]];
D[Plus[a_, b_], σ[i_, j_]] := D[a, σ[i, j]] + D[b, σ[i, j]];
D[Power[a_, n_], σ[i_, j_]] := n Power[a, (n - 1)] D[a, σ[i, j]] ;
Protect[D];
```

(5.2) For derivative over α

```
In[44]:= Unprotect[D];
D[α[i_, j_], α[k_, l_]] := δ[i, k] δ[j, l];
D[r, σ[i_, j_]] := (-2 p s[i, j] + 2 p^2 α[i, j]) / (2 √(p^2 α^2 + s^2 - 2 p sα));
D[sα, α[i_, j_]] := s[i, j];
D[α^2, α[i_, j_]] := 2 α[i, j];
D[Times[a_, b_], α[i_, j_]] := (D[a, α[i, j]] ) (b) + (a) ( D[b, α[i, j] ]);
D[Plus[a_, b_], α[i_, j_]] := D[a, α[i, j]] + D[b, α[i, j] ];
D[Power[a_, n_], α[i_, j_]] := n Power[a, (n - 1)] D[a, α[i, j]] ;
Protect[D];
```

(6) Test Cases

(6.1) Simple Test Case

```
In[53]:= Simplify[D[ √((s[o, t] - p α[o, t] ) (s[o, t] - p α[o, t] ) ), σ[i, j]]]

Out[53]= (3 s[i, j] - 3 p α[i, j] + (sα - p α^2) δ[i, j]) / (3 √(s^2 + p (-2 sα + p α^2)))

In[54]:= Simplify[
  D[(3 s[i, j] - 3 p α[i, j] + (sα - p α^2) δ[i, j]) / (3 √(s^2 + p (-2 sα + p α^2))), σ[a, b]]]

Out[54]= ((3 s[a, b] - 3 p α[a, b] + (sα - p α^2) δ[a, b])
    (-3 s[i, j] + 3 p α[i, j] + (-sα + p α^2) δ[i, j]) + (s^2 + p (-2 sα + p α^2))
    (3 α[i, j] δ[a, b] + 9 δ[a, i] δ[b, j] + (3 α[a, b] + (-3 + α^2) δ[a, b]) δ[i, j])) /
  (9 (s^2 + p (-2 sα + p α^2))^(3/2))
```

(6.2) Advanced Test Case (Verified to existing tensor derivatives to Lecture notes)

```
In[55]:= testF = (( s[i, j] - p α[i, j]) (s[i, j] - p α[i, j]))^0.5 - √(2/3) η p ;
```

(6.2.1) Normal to yield surface. Correct!

```
In[56]:= testN = D[testF, σ[k, l]];

In[57]:= Simplify[testN]

Out[57]= (1. s[k, l] - 1. p α[k, l] +
     (0.333333 sα - 0.333333 p α^2 + 0.272166 (s^2 - 2 p sα + p^2 α^2)^0.5 η) δ[k, l]) /
  (s^2 - 2 p sα + p^2 α^2)^0.5
```

(6.2.2) Normal to yield surface for α. Correct!

```
In[58]:= testN4α = D[testF, α[k, l]];

In[59]:= Simplify[testN4α]

Out[59]= (p (-1. s[k, l] + 1. p α[k, l])) / (s^2 - 2 p sα + p^2 α^2)^0.5
```

(6.2.3) Normal to yield surface for η. Correct!

In[60]:= `testN4η = D[testF, η]`

Out[60]= $-\sqrt{\frac{2}{3}}\ p$

(6.2.4) Associated plastic flow and its derivative to σ. Correct!

In[61]:= `testM4σ = D[testN, σ[i, j]]`

Out[61]= $$0.5\left(-\left(\left(0.5\left(2\,s[i,j]-\frac{2}{3}\,p\,\alpha^2\,\delta[i,j]-2\left(p\,\alpha[i,j]-\frac{1}{3}\,s\alpha\,\delta[i,j]\right)\right)\left(2\,s[k,l]-\frac{2}{3}\,p\,\alpha^2\,\delta[k,l]-2\left(p\,\alpha[k,l]-\frac{1}{3}\,s\alpha\,\delta[k,l]\right)\right)\right)\Big/\left(s^2-2\,p\,s\alpha+p^2\,\alpha^2\right)^{1.5}\right)+\left(\frac{2}{9}\,\alpha^2\,\delta[i,j]\,\delta[k,l]-2\left(-\frac{1}{3}\,\alpha[k,l]\,\delta[i,j]-\frac{1}{3}\,\alpha[i,j]\,\delta[k,l]\right)+2\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\,\delta[i,j]\,\delta[k,l]\right)\right)\Big/\left(s^2-2\,p\,s\alpha+p^2\,\alpha^2\right)^{0.5}\right)$$

In[62]:= `Simplify[testM4σ];`

(6.2.5) Associated plastic flow and its derivative to α. Correct!

In[63]:= `testM4α = D[testN, α[i, j]]`

Out[63]= $$0.5\left(-\left(\left(0.5\left(-2\,p\,s[i,j]+2\,p^2\,\alpha[i,j]\right)\left(2\,s[k,l]-\frac{2}{3}\,p\,\alpha^2\,\delta[k,l]-2\left(p\,\alpha[k,l]-\frac{1}{3}\,s\alpha\,\delta[k,l]\right)\right)\right)\Big/\left(s^2-2\,p\,s\alpha+p^2\,\alpha^2\right)^{1.5}\right)+\left(-\frac{4}{3}\,p\,\alpha[i,j]\,\delta[k,l]-2\left(p\,\delta[i,k]\,\delta[j,l]-\frac{1}{3}\,s[i,j]\,\delta[k,l]\right)\right)\Big/\left(s^2-2\,p\,s\alpha+p^2\,\alpha^2\right)^{0.5}\right)$$

(6.2.6) Associated plastic flow and its derivative to η. Correct!

In[64]:= `testM4η = D[testN, η]`

Out[64]= $\frac{1}{3}\sqrt{\frac{2}{3}}\ \delta[k,l]$

In[65]:=

In[66]:=

In[67]:=

In[68]:=

Part 2. Tensor Derivative for the Model

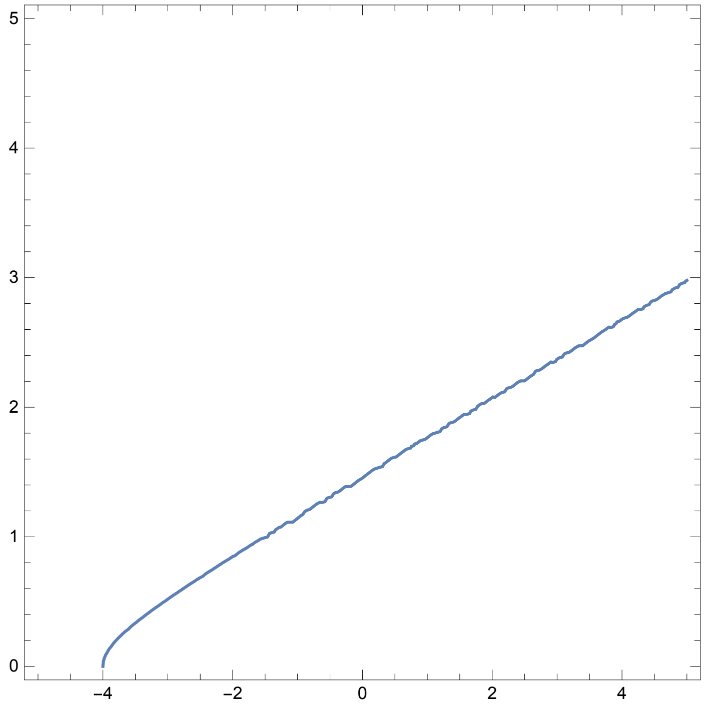
(7) Real Model: Modified Hyperbolic Yield Surface

(7.1) Yield Surface

In[69]:=
$$\Phi = 1 - \left(\left(p + \frac{\xi}{\eta}\right)^2 + \frac{1}{2}\,\frac{1}{\eta^2}\,(s[i,j] - p\,\alpha[i,j])\,(s[i,j] - p\,\alpha[i,j])\right)^{0.5};$$

In[70]:=
```
centerP = 5; range = 5; Slope = 0.3;
fig1 = ContourPlot[1 - ((p + centerP) ^2 - (q/Slope) ^2)^0.5 == 0,
    {p, -centerP, range}, {q, 0, range}];
Show[
 fig1]
```

Out[72]=



(7.1.1) Normal to the yield surface for σ. Namely, $\frac{\partial \Phi}{\partial \sigma_{ij}}$

In[73]:=
```
n = D[Φ, σ[k, l]];
```

In[74]:= `FullSimplify[n, ExcludedForms → Power[_]]`

Out[74]=
$$-\left(\left(0.5\left(-\frac{2}{3}\left(p+\frac{\xi}{\eta}\right)\delta[k,l]+\frac{1}{2}\left(\frac{1}{3}\alpha[i,j]\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)\delta[k,l]+\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)+s[i,j]\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\eta^2\right)+\frac{1}{\eta^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right)-p\,\alpha[i,j]\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\eta^2\right)+\frac{1}{\eta^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right)\right)\right)\right)\Big/\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)\right)\right)^{0.5}\right)$$

In[75]:= `FullSimplify[n];`

(7.1.2) Normal to the yield surface for α. Namely, $\frac{\partial\Phi}{\partial\alpha_{ij}}$

In[76]:= `n4α = D[Φ, α[k, l]]`

Out[76]=
$$-\left(\left(0.25\left(-\frac{p\,s[k,l]}{\eta^2}-p\left(-\frac{p\,\alpha[k,l]}{\eta^2}+\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)\delta[i,k]\,\delta[j,l]\right)\right)\right)\Big/\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)\right)\right)^{0.5}\right)$$

In[77]:= `FullSimplify[n4α];`

(7.1.3) Normal to the yield surface for η. Namely, $\frac{\partial\Phi}{\partial\eta}$

In[78]:= `n4η = D[Φ, η]`

Out[78]=
$$-\left(\left(0.5\left(-\frac{2\,\xi\left(p+\frac{\xi}{\eta}\right)}{\eta^2}+\frac{1}{2}\left(s[i,j]\left(-\frac{2\,s[i,j]}{\eta^3}+\frac{2\,p\,\alpha[i,j]}{\eta^3}\right)-p\,\alpha[i,j]\left(-\frac{2\,s[i,j]}{\eta^3}+\frac{2\,p\,\alpha[i,j]}{\eta^3}\right)\right)\right)\right)\Big/\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\eta^2}-\frac{p\,\alpha[i,j]}{\eta^2}\right)\right)\right)^{0.5}\right)$$

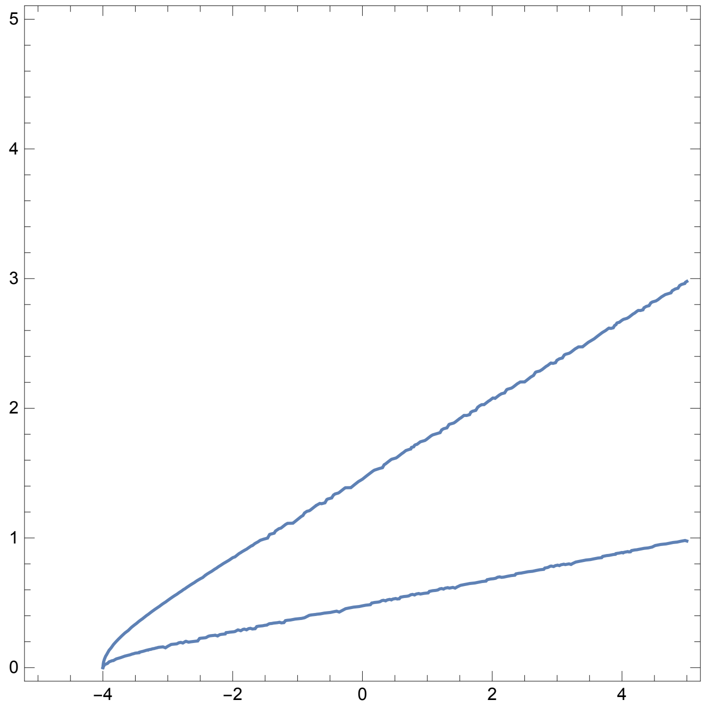
In[79]:= `FullSimplify[n4η];`

(7.2) Plastic Flow Potential

In[80]:=
$$\psi = 1-\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\,\frac{1}{\eta^2}(s[i,j]-p\,\alpha[i,j])\,(s[i,j]-p\,\alpha[i,j])\right)^{0.5};$$

In[81]:=
```
centerP2 = 5; range2 = 5; Slop2 = 0.1;
fig2 = ContourPlot[1 - ((p + centerP)^2 - (q/Slop2)^2)^0.5 == 0,
   {p, -centerP2, range2}, {q, 0, range2}];
Show[
 fig1,
 fig2]
```

Out[83]=



(7.2.1) Plastic flow direction, normal to the plastic potential. Namely, $\frac{\partial\psi}{\partial\sigma_{ij}}$.

In[84]:=
```
m = D[ψ, σ[k, l]]
```

Out[84]=

$$-\left(\left(0.5\left(-\frac{2}{3}\left(p+\frac{\xi}{\eta}\right)\delta[k,l]+\frac{1}{2}\left(\frac{1}{3}\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\delta[k,l]+\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)+s[i,j]\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\bar{\eta}^2\right)+\frac{1}{\bar{\eta}^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right)-p\,\alpha[i,j]\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\bar{\eta}^2\right)+\frac{1}{\bar{\eta}^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right)\right)\right)\right)\Bigg/\left(\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\right)\right)^{0.5}\right)$$

In[85]:=
```
Simplify[m];
```

(7.2.2) Useful derivative with plastic flow m_{ij} . For the stress, $\frac{\partial m_{kl}}{\partial \sigma_{ij}}$.

In[86]:=
```
m4σ = D[m, σ[i, j]]
```

Out[86]=

$$
\begin{aligned}
&-0.5\Bigg(\Bigg(\frac{2}{9}\,\delta[i,j]\,\delta[k,l]+\\
&\quad\frac{1}{2}\Bigg((8\,\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\bar{\eta}^2\right)+\frac{1}{\bar{\eta}^2}8\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\,\delta[i,j]\,\delta[k,l]\right)+\\
&\qquad 8\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\bar{\eta}^2\right)+\frac{1}{\bar{\eta}^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\,\delta[i,j]\,\delta[k,l]\right)\right)\Bigg)\Bigg)\Bigg/\\
&\quad\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\right)\right)^{0.5}-\\
&\quad\Bigg(0.5\left(\frac{1}{2}\left(\frac{8\,s[i,j]}{\bar{\eta}^2}-\frac{8\,p\,\alpha[i,j]}{\bar{\eta}^2}+8\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\right)-\frac{2}{3}\left(p+\frac{\xi}{\eta}\right)\delta[i,j]\right)\\
&\qquad\Bigg(-\frac{2}{3}\left(p+\frac{\xi}{\eta}\right)\delta[k,l]+\\
&\qquad\quad\frac{1}{2}\Bigg(\frac{1}{3}\,\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\delta[k,l]+\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\\
&\qquad\qquad\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\,\delta[i,j]\,\delta[k,l]\right)+s[i,j]\Bigg((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\bar{\eta}^2\right)+\\
&\qquad\qquad\frac{1}{\bar{\eta}^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\,\delta[i,j]\,\delta[k,l]\right)\Bigg)-p\,\alpha[i,j]\\
&\qquad\qquad\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\bar{\eta}^2\right)+\frac{1}{\bar{\eta}^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\,\delta[i,j]\,\delta[k,l]\right)\right)\Bigg)\Bigg)\Bigg)\Bigg/\\
&\quad\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2}-\frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\right)\right)^{1.5}\Bigg)
\end{aligned}
$$

In[87]:=
```
Simplify[m4σ];
```

(7.2.3) Useful derivative with plastic flow m_{ij} . For the backstress, $\frac{\partial m_{kl}}{\partial \alpha_{ij}}$.

In[88]:= **m4α = D[m, α[i, j]]**

Out[88]=

$$
\begin{aligned}
-0.5\, & \Bigg(\Bigg(\frac{1}{\bar{\eta}^2}3\, s[i,j]\, \delta[k,l] + \frac{1}{3}\left(-\frac{1}{\bar{\eta}^2}9\, p\, \alpha[i,j]\, \delta[k,l] + 9\left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\delta[k,l]\right) - \\
& \frac{1}{\bar{\eta}^2}9\, p\left(\delta[i,k]\,\delta[j,l] - \frac{1}{3}\delta[i,j]\,\delta[k,l]\right) - p\Bigg(\frac{1}{\bar{\eta}^2}3\,\alpha[i,j]\,\delta[k,l] + \\
& 9\left((\alpha[i,j]\,\delta[k,l]) \big/ \left(3\,\bar{\eta}^2\right) + \frac{1}{\bar{\eta}^2}\left(\delta[i,k]\,\delta[j,l] - \frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right)\Bigg)\Bigg) \Bigg/ \\
& \Bigg(2\left(\left(p + \frac{\xi}{\eta}\right)^2 + \frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right) - \right.\right. \\
& \left.\left. p\,\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\right)\right)^{0.5}\Bigg) - \\
& \Bigg(0.25\left(-\frac{9\,p\,s[i,j]}{\bar{\eta}^2} - p\left(-\frac{9\,p\,\alpha[i,j]}{\bar{\eta}^2} + 9\left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\right)\right) \\
& \Bigg(-\frac{2}{3}\left(p + \frac{\xi}{\eta}\right)\delta[k,l] + \frac{1}{2}\Bigg(\frac{1}{3}\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\delta[k,l] + \\
& \left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\left(\delta[i,k]\,\delta[j,l] - \frac{1}{3}\delta[i,j]\,\delta[k,l]\right) + s[i,j] \\
& \left((\alpha[i,j]\,\delta[k,l]) \big/ \left(3\,\bar{\eta}^2\right) + \frac{1}{\bar{\eta}^2}\left(\delta[i,k]\,\delta[j,l] - \frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right) - \\
& p\,\alpha[i,j]\Bigg((\alpha[i,j]\,\delta[k,l]) \big/ \left(3\,\bar{\eta}^2\right) + \frac{1}{\bar{\eta}^2} \\
& \left(\delta[i,k]\,\delta[j,l] - \frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\Bigg)\Bigg)\Bigg)\Bigg) \Bigg/ \\
& \left(\left(p + \frac{\xi}{\eta}\right)^2 + \frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right) - p\,\alpha[i,j]\left(\frac{s[i,j]}{\bar{\eta}^2} - \frac{p\,\alpha[i,j]}{\bar{\eta}^2}\right)\right)\right)^{1.5}\Bigg)
\end{aligned}
$$

In[89]:= **Simplify[m4α];**

(7.2.4) Useful derivative with plastic flow m_{ij} . For the slope, $\frac{\partial m_{kl}}{\partial \eta}$.

In[90]:= `m4η = D[m, η]`

Out[90]=

$$\left(0.25\left(-\frac{2}{3}\left(p+\frac{\xi}{\eta}\right)\delta[k,l]+\frac{1}{2}\left(\frac{1}{3}\alpha[i,j]\left(\frac{s[i,j]}{\overline{\eta}^2}-\frac{p\,\alpha[i,j]}{\overline{\eta}^2}\right)\delta[k,l]+\left(\frac{s[i,j]}{\overline{\eta}^2}-\frac{p\,\alpha[i,j]}{\overline{\eta}^2}\right)\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)+s[i,j]\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\overline{\eta}^2\right)+\frac{1}{\overline{\eta}^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right)-p\,\alpha[i,j]\left((\alpha[i,j]\,\delta[k,l])\Big/\left(3\,\overline{\eta}^2\right)+\frac{1}{\overline{\eta}^2}\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\right)\right)\right)\left(-\frac{2\,\xi\left(p+\frac{\xi}{\eta}\right)}{\eta^2}+\frac{1}{2}\left(s[i,j]\left(-\frac{1}{\overline{\eta}^3}2\,s[i,j]\,\text{OverBar}'[\eta]+\frac{1}{\overline{\eta}^3}2\,p\,\alpha[i,j]\,\text{OverBar}'[\eta]\right)-p\,\alpha[i,j]\left(-\frac{1}{\overline{\eta}^3}2\,s[i,j]\,\text{OverBar}'[\eta]+\frac{1}{\overline{\eta}^3}2\,p\,\alpha[i,j]\,\text{OverBar}'[\eta]\right)\right)\right)\right)\Big/$$

$$\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\overline{\eta}^2}-\frac{p\,\alpha[i,j]}{\overline{\eta}^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\overline{\eta}^2}-\frac{p\,\alpha[i,j]}{\overline{\eta}^2}\right)\right)\right)^{1.5}-$$

$$\left(0.5\left(\frac{2\,\xi\,\delta[k,l]}{3\,\eta^2}+\frac{1}{2}\left(\frac{1}{3}\alpha[i,j]\,\delta[k,l]\left(-\frac{1}{\overline{\eta}^3}2\,s[i,j]\,\text{OverBar}'[\eta]+\frac{1}{\overline{\eta}^3}2\,p\,\alpha[i,j]\,\text{OverBar}'[\eta]\right)+\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\left(-\frac{1}{\overline{\eta}^3}2\,s[i,j]\,\text{OverBar}'[\eta]+\frac{1}{\overline{\eta}^3}2\,p\,\alpha[i,j]\,\text{OverBar}'[\eta]\right)+s[i,j]\left(-\left((2\,\alpha[i,j]\,\delta[k,l]\,\text{OverBar}'[\eta])\Big/\left(3\,\overline{\eta}^3\right)\right)-\frac{1}{\overline{\eta}^3}2\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\text{OverBar}'[\eta]\right)-p\,\alpha[i,j]\left(-\left((2\,\alpha[i,j]\,\delta[k,l]\,\text{OverBar}'[\eta])\Big/\left(3\,\overline{\eta}^3\right)\right)-\frac{1}{\overline{\eta}^3}2\left(\delta[i,k]\,\delta[j,l]-\frac{1}{3}\delta[i,j]\,\delta[k,l]\right)\text{OverBar}'[\eta]\right)\right)\right)\right)\Big/$$

$$\left(\left(p+\frac{\xi}{\eta}\right)^2+\frac{1}{2}\left(s[i,j]\left(\frac{s[i,j]}{\overline{\eta}^2}-\frac{p\,\alpha[i,j]}{\overline{\eta}^2}\right)-p\,\alpha[i,j]\left(\frac{s[i,j]}{\overline{\eta}^2}-\frac{p\,\alpha[i,j]}{\overline{\eta}^2}\right)\right)\right)^{0.5}$$

In[91]:= `Simplify[m4η];`