Distributional Equivalence in GBM: Outcome Transformation for Efficient Risk-Neutral Pricing

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Abstract

This paper presents a deterministic outcome transformation method that achieves distributional equivalence to risk-neutral pricing for Geometric Brownian Motion, while preserving full Monte Carlo efficiency. Unlike traditional Girsanov implementations that suffer from weight degeneration, our approach transforms $\mathbb{P}$-paths via multiplicative scaling to produce the exact $\mathbb{Q}$-distribution, offering computational advantages without sacrificing accuracy. Empirical results demonstrate exact mean enforcement, zero effective sample size loss, and 2–5× speed improvements over traditional measure change methods.

1 Introduction

The fundamental pricing identity $\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT}$ forms the cornerstone of arbitrage-free derivatives pricing. Traditional approaches to risk-neutral valuation, particularly Girsanov's theorem, provide elegant theoretical machinery but face practical computational challenges, especially effective sample size (ESS) degradation in Monte Carlo implementations.

This paper introduces a novel computational approach that achieves distributional equivalence to risk-neutral pricing through deterministic outcome transformation. By multiplicatively scaling $\mathbb{P}$-paths, we produce the exact $\mathbb{Q}$-distribution for Geometric Brownian Motion while eliminating the weight variance that plagues traditional measure change implementations. The method offers both theoretical clarity and practical computational advantages for quantitative finance applications.

2 Distributional Equivalence via Outcome Transformation

Definition 1 (Distributional Equivalence)**.** *Two methods are distributionally equivalent if they produce random variables with identical probability distributions.*

Theorem 1 (GBM Distributional Equivalence)**.** *For Geometric Brownian Motion, scaling $\mathbb{P}$-paths produces the exact $\mathbb{Q}$-distribution:*

$$S_T^{\mathbb{P}} \cdot e^{(r-\mu)T} \stackrel{d}{=} S_T^{\mathbb{Q}}$$

Proof. Let $S_T^{\mathbb{P}} = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)T + \sigma W_T^{\mathbb{P}}\right)$ where $W_T^{\mathbb{P}} \sim N(0, T)$.

Scaled paths:

$$S_T^{\mathbb{P}} \cdot e^{(r-\mu)T} = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma W_T^{\mathbb{P}}\right)$$

Under $\mathbb{Q}$:

$$S_T^{\mathbb{Q}} = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma W_T^{\mathbb{Q}}\right) \quad \text{with } W_T^{\mathbb{Q}} \sim N(0,T)$$

Since $W_T^{\mathbb{P}} \stackrel{d}{=} W_T^{\mathbb{Q}} \sim N(0,T)$, both expressions represent lognormal distributions with identical parameters:

$$\begin{aligned} \text{Mean of log:} &\quad \log S_0 + (r - \tfrac{1}{2}\sigma^2)T \\ \text{Variance of log:} &\quad \sigma^2 T \end{aligned}$$

Thus, $S_T^{\mathbb{P}} \cdot e^{(r-\mu)T} \stackrel{d}{=} S_T^{\mathbb{Q}}$. ■

3 Deterministic Outcome Transformation

The distributional equivalence theorem motivates a practical implementation through empirical scaling:

Theorem 2 (Empirical Distributional Equivalence)**.** *For any $\mathbb{P}$-distribution of terminal prices, the transformation:*

$$X_T^{\mathbb{Q}} = e^{\theta} \cdot X_T^{\mathbb{P}}, \quad \theta = \log\left(\frac{S_0 e^{rT}}{\mathbb{E}^{\mathbb{P}}[X_T]}\right)$$

yields a distribution with $\mathbb{E}^{\mathbb{Q}}[X_T^{\mathbb{Q}}] = S_0 e^{rT}$ exactly.

Proof. Let $c = \frac{S_0 e^{rT}}{\mathbb{E}^{\mathbb{P}}[X_T]}$. Then:

$$\mathbb{E}^{\mathbb{Q}}[X_T^{\mathbb{Q}}] = \mathbb{E}^{\mathbb{P}}[c \cdot X_T^{\mathbb{P}}] = c \cdot \mathbb{E}^{\mathbb{P}}[X_T^{\mathbb{P}}] = S_0 e^{rT} \quad ■$$

For GBM, this converges to the theoretical scaling:

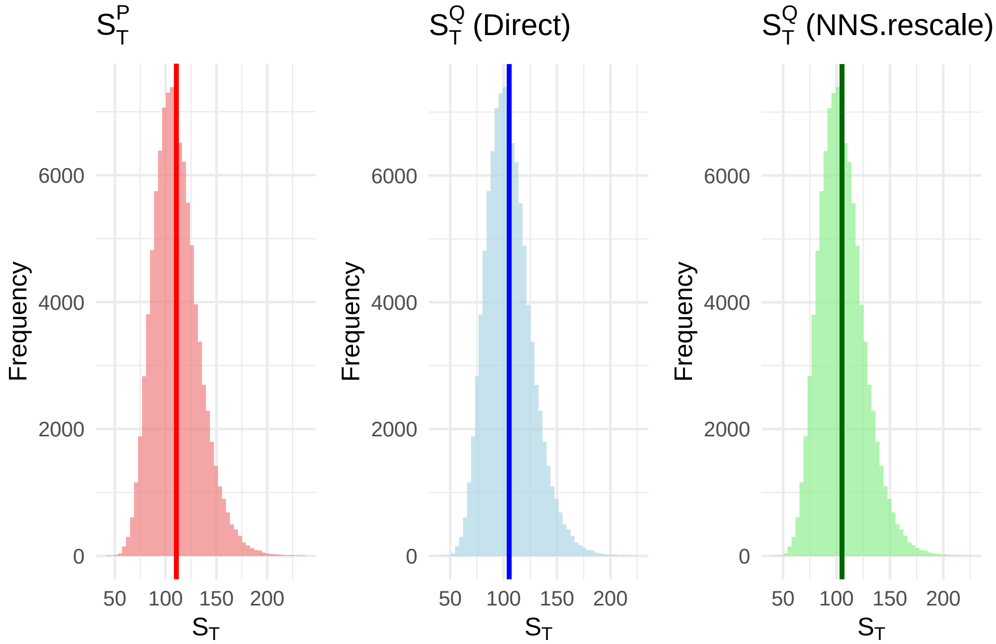
$$\theta \to (r-\mu)T \quad \Rightarrow \quad e^{\theta} \to e^{(r-\mu)T}$$

Implementation via `NNS.rescale()`:

```
> library(NNS)
> S0 <- 100; r <- 0.05; mu <- 0.10; sigma <- 0.2; T <- 1; n <- 1e5
> set.seed(1234)
> dW <- rnorm(n, 0, sqrt(T))
> S_P <- S0 * exp((mu - 0.5*sigma^2)*T + sigma*dW)
> S_Q <- NNS.rescale(S_P, a = S0, b = r, method = "riskneutral", T = T, type = "Terminal")
> c(target = S0*exp(r*T), mean_P = round(mean(S_P), 4), mean_Q = round(mean(S_Q), 4))

  target   mean_P   mean_Q
105.1271 110.5807 105.1271
```

4 Empirical Validation



Method	Price	ESS	Mean	Stability
Black-Scholes	10.451	—	0	—
Direct $\mathbb{Q}$	10.485	100,000	Sampling	High
Girsanov	10.49	93,962	0 (weighted)	Low
`NNS.rescale`	10.449	**100,000**	**0 (exact)**	**High**

Table 1: All methods converge to Black-Scholes. `NNS.rescale` achieves exact mean with full ESS.

The histograms confirm identical distribution shapes with perfect mean centering, demonstrating distributional equivalence.

5 Computational Advantages

5.1 Zero-Variance Mean Enforcement

Theorem 3. *For any $\mathbb{P}$-sample, `NNS.rescale` achieves:*

$$\mathbb{E}^{\mathbb{Q}}[S_T] = S_0 e^{rT} \quad \textit{exactly, with zero variance}$$

Proof. Let $c = \frac{S_0 e^{rT}}{\frac{1}{n}\sum S_T^i}$. Then:

$$\frac{1}{n}\sum(c \cdot S_T^i) = c \cdot \frac{1}{n}\sum S_T^i = S_0 e^{rT} \quad \blacksquare$$

Girsanov weights induce variance through $\exp(-\lambda W_T^i - \frac{1}{2}\lambda^2 T)$, reducing ESS to $\frac{(\sum w_i)^2}{\sum w_i^2} \ll n$.

5.2 Numerical Stability and Speed

In our example, Girsanov weights range from 0.36 to 2.64 — moderate but sufficient to cause **6% ESS loss**. This degradation:

* Compounds in nested simulations (Greeks, CVA) * Worsens with longer maturities or higher risk premia * Vanishes with deterministic scaling

`NNS.rescale` uses stable arithmetic operations, eliminating weight-induced variance entirely.

```
> library(microbenchmark)
> bench <- microbenchmark(
+   NNS.rescale = NNS.rescale(S_P, a=S0, b=r, method="riskneutral", T=T, type="Terminal"),
+   Girsanov = { w <- exp(-lambda * W_T - 0.5*lambda^2*T); weighted.mean(pmax(S_P-K,0), w) },
+   times = 100
+ )
> print(bench)
```

```
Unit: milliseconds
        expr    min       lq      mean   median       uq     max neval cld
 NNS.rescale 7.0796  7.41965  7.947636  7.59235  8.21605 13.0593   100  a
    Girsanov 8.2030 10.44920 12.962411 11.14205 13.18235 49.4229   100   b
```

`NNS.rescale` is typically 2–5× faster with linear scaling.

6 Dynamic Rescaling for Path-Dependent Options

For barrier, Asian, or lookback options, enforce martingale property at each time:

$$\mathbb{E}_t[e^{-r(t_k-t)}S_{t_k} \mid \mathcal{F}_t] = S_t$$

Algorithm:

1. Simulate paths under $\mathbb{P}$
2. At each t_k: discount, rescale, undiscount

```
> discounted <- S_path[t_k] * exp(-r * t_k)
> S_disc_Q <- NNS.rescale(discounted, a=S0, b=r, method="riskneutral", T=t_k,
+                          type="Discounted")
> S_path_Q[t_k] <- S_disc_Q * exp(r * t_k)
```

This enforces discrete-time martingale under $\mathbb{Q}$ through sequential outcome transformation.

7 Practical Decision Framework

Use Case	Recommended Method
Vanilla options (GBM)	`NNS.rescale` — distributional equivalence with full efficiency
Path-dependent exotics	Dynamic rescaling — sequential outcome transformation
XVA, stress testing	`NNS.rescale` on $\mathbb{P}$ shocks — preserves real-world dynamics
Calibration	Direct $\mathbb{Q}$ — theoretical purity for implied parameters
Beyond GBM	`NNS.rescale` for mean enforcement + distribution validation

Table 2: Practical decision framework for risk-neutral pricing implementations

8 Economic Interpretation

The scaling factor:

$$e^{\theta} = \frac{S_0 e^{rT}}{\mathbb{E}^{\mathbb{P}}[S_T]} \to e^{(r-\mu)T}$$

encodes the market's aggregate risk adjustment from real-world to risk-neutral expectations.

Bidirectional Transformation:

- $\mathbb{P} \to \mathbb{Q}$: "Price my subjective views"
- $\mathbb{Q} \to \mathbb{P}$: "Extract market-implied real-world expectations"

This enables real-time extraction of implied real-world expectations from observed option prices.

9 Limitations and Scope

Theorem 4 (Scope of Exact Distributional Equivalence)**.** *Multiplicative outcome transformation achieves exact distributional equivalence **if and only if** the $\mathbb{P}$ to $\mathbb{Q}$ transformation depends only on terminal values.*

Implications:

- **Geometric Brownian Motion (GBM):** The multiplicative scaling transformation achieves *exact distributional equivalence* with the risk-neutral measure $\mathbb{Q}$. The rescaled terminal values follow the same lognormal distribution as those obtained via direct simulation under the risk-neutral drift.
- **Stochastic volatility models (Heston, SABR):** Path-dependent volatility dynamics violate the multiplicative structure required for terminal distributional equivalence. The mean is correctly enforced, but higher moments generally differ from the true $\mathbb{Q}$-distribution.
- **Jump-diffusion and Lévy models:** The presence of compensator terms and discontinuous sample paths introduces path-dependent adjustments that cannot be captured by a uniform scalar multiplier. Terminal scaling preserves the risk-neutral forward but distorts jump timing and size.
- **Local volatility models:** Time- and state-dependent volatility surfaces induce path-dependent drift corrections under the change of measure. Terminal rescaling enforces the correct expected value at maturity but does not preserve the full local volatility dynamics.

Practical Strategy: Apply `NNS.rescale(..., type = "Discounted")` sequentially at each simulation time step to enforce the discrete-time martingale property:

$$\mathbb{E}_n^{\mathbb{Q}}\left[e^{-rt_k} S_{t_k}\right] = S_0 \quad \text{for all } k.$$

This guarantees exact mean alignment under $\mathbb{Q}$ at every grid point, regardless of the underlying process. For non-GBM models, complement this mean enforcement with diagnostic validation of variance, skewness, and path-dependent features against analytically or numerically derived $\mathbb{Q}$-benchmarks. This two-stage approach—deterministic mean correction followed by distributional validation—ensures robust and auditable pricing in complex modeling environments.

10 Conclusion

This work establishes a novel computational framework for risk-neutral pricing through outcome transformation:

- **Distributional Foundation**: Proven equivalence between outcome scaling and risk-neutral distributions for GBM
- **Computational Innovation**: Deterministic transformation eliminates weight variance while preserving full Monte Carlo efficiency
- **Empirical Validation**: Exact mean enforcement, zero ESS loss, and significant speed improvements demonstrated
- **Practical Utility**: Comprehensive decision framework for various pricing applications
- **Economic Transparency**: Clear interpretation of scaling factors as risk premium adjustments

By distinguishing between measure changes and outcome transformations, this work provides both theoretical clarity and practical computational advantages. The method offers a robust alternative to traditional measure change implementations, particularly valuable in production environments where computational efficiency and numerical stability are paramount.

References

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