Visualizing the Continuum Hypothesis through Discrete and Continuous Distributions and Partial Moments

Fred Viole

Abstract

This paper presents a pedagogical visualization of Cantor's Continuum Hypothesis (CH) using analogies from probability theory and Nonlinear Nonparametric Statistics (NNS). Rather than attempting a proof, we illustrate the conceptual difference between countable and uncountable infinities by comparing discrete and continuous uniform distributions and the boundary behavior of partial moments. The *degree* parameter in the partial moment formalism provides an analytic transition between discrete (cardinal) and continuous (measure) regimes. This analogy is designed to illuminate, not resolve, the CH, which remains independent of standard set theory (ZFC).

1 Introduction

The *Continuum Hypothesis* (CH), first posed by Georg Cantor, states:

$$\nexists S \text{ such that } |\mathbb{Z}| < |S| < |\mathbb{R}|.$$

It asks whether there exists a set of cardinality strictly between that of the integers and the real numbers. Cantor proved $\mathbb{R}$ is uncountable, and that $|\mathbb{R}| > |\mathbb{Z}|$. However, Gödel (1940) and Cohen (1963) established that CH can neither be proved nor disproved from Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC).

Here we build a pedagogical model comparing discrete and continuous distributions and connecting this dichotomy to the analytic structure of partial moments. The goal is to make visible why only two regimes—discrete and continuous—naturally arise.

2 Discrete and Continuous Uniform Distributions

In probability theory, a discrete uniform distribution assigns equal mass to a countable number of points, while a continuous uniform assigns equal density across an interval. These correspond conceptually to countable and uncountable sets.

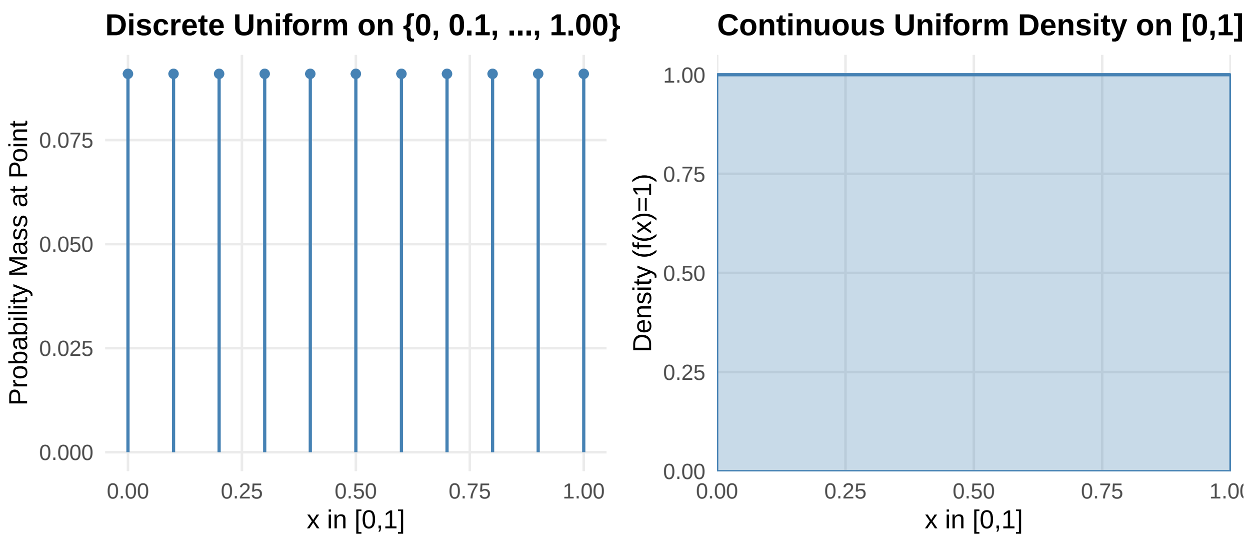

Figure 1: **Figure 1.** Discrete vs. continuous uniform on $[0, 1]$ (steelblue). Left: a discrete uniform with 11 rational bins. Right: a continuous uniform density. The visual metaphor shows how continuous models "fill" the gaps between discrete support points. This illustrates measure-based continuity, not set-theoretic cardinality.

3 Partial Moments and Measure-Theoretic Sets

Partial moments quantify cumulative deviations of random variables from a threshold q:

$$\mathrm{LPM}(n, q) = \int_{\{X \leq q\}} (q - X)^n \, dP, \quad \mathrm{UPM}(n, q) = \int_{\{X > q\}} (X - q)^n \, dP.$$

Each integrates over measurable subsets of $(\Omega, \mathcal{F}, P)$, where $\{X \leq q\}$ and $\{X > q\}$ belong to the σ-algebra $\mathcal{F}$. These subsets are sets of *measure*, not of distinct *size* in the Cantorian sense.

Degree Zero as a Cardinal Boundary. When degree $= 0$, the partial moment reduces to

$$\mathrm{LPM}(0, q) = P(X \leq q),$$

which counts how many observations fall below q. This behaves as a set of *size*—a discrete aggregation of atoms—though expressed as a Lebesgue integral. For degree > 0, the integrand $(q - X)^n$ introduces weighting, producing a continuous smoothing. Thus, degree $= 0$ marks the analytic boundary between the discrete (count-based) and continuous (measure-based) regimes.

4 Binary Nature of Smoothness and Cardinality

The transition from degree $= 0$ to degree > 0 in the `LPM.ratio` framework creates an analytic binary: once smoothness is invoked, discreteness disappears. This mirrors the Continuum

Hypothesis, which asks whether a third category can exist between countable and uncountable sets. To posit such a state would require a new mathematical ontology—something neither atomic nor dense.

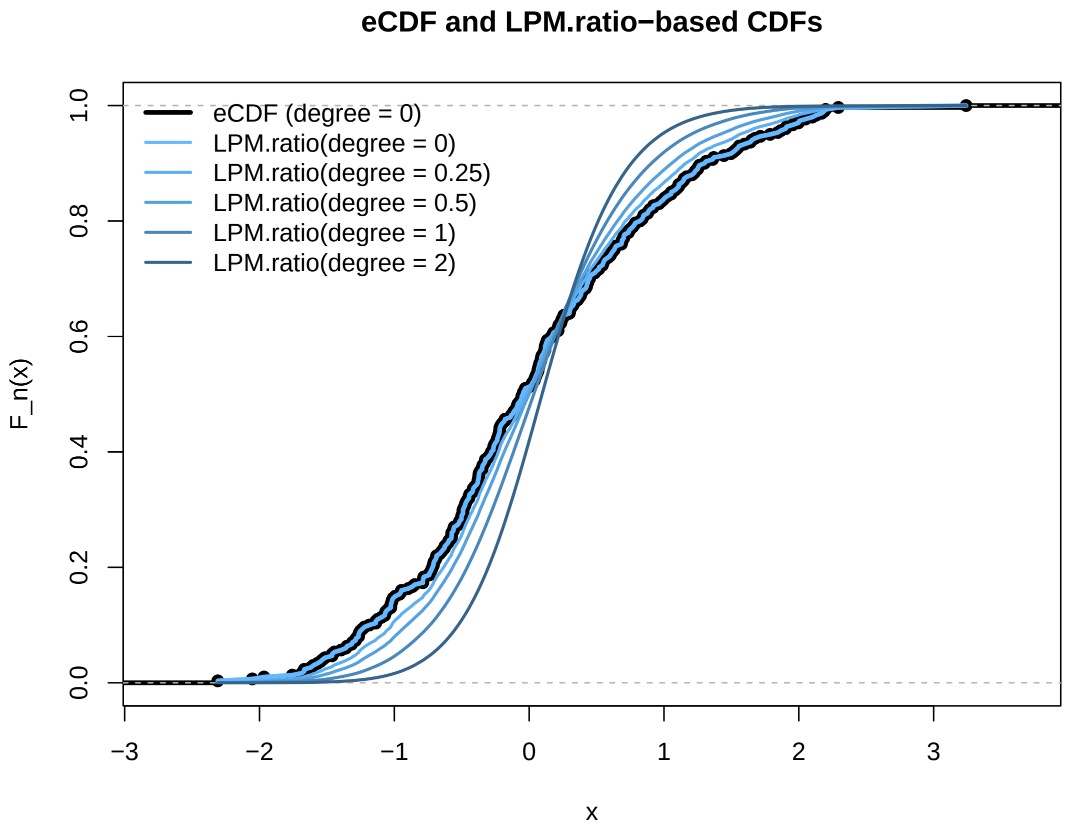

Figure 2: Empirical CDF (black) versus `LPM.ratio`-based CDFs (steelblue). At degree = 0, the empirical step function counts atoms; for degree > 0, smooth weighting produces continuous measure. This analytic shift parallels the discrete–continuous divide underlying the Continuum Hypothesis.

5 Definition: Cardinality

Two sets A and B have the same *cardinality*, written $|A| = |B|$, if there exists a bijection $f : A \to B$ that is one-to-one and onto. The *cardinality* of a set A, denoted $|A|$, is the equivalence class of all sets bijective with A. For finite sets, $|A|$ equals the number of elements. For infinite sets:

$$|\mathbb{N}| = \aleph_0, \quad |\mathbb{R}| = 2^{\aleph_0}.$$

The Continuum Hypothesis asks whether there exists a κ such that

$$\aleph_0 < \kappa < 2^{\aleph_0}.$$

6 Measure and Cardinality: Two Different Languages

Measure theory quantifies "how much," while set theory quantifies "how many." A set can have zero measure but infinite (countable) size (e.g., $\mathbb{Q}$) or have zero measure yet the same cardinality as $\mathbb{R}$ (e.g., the Cantor set). Thus, having more area does not imply greater cardinality.

7 Pedagogical Analogy and CH

The binary transition between degree $= 0$ and degree > 0 reproduces the discrete–continuous dichotomy formalized by CH. Within the analytic framework, there exists no intermediate regime that is both discrete and continuous; likewise, CH posits no cardinality between the countable and the continuum. The LPM analogy does not prove CH but makes visible why the continuum seems singular.

8 Conclusion

At degree $= 0$, partial moments operate as sets of size; for degree > 0, they become sets of measure. This binary transition parallels the logical structure of CH: asserting an intermediate state would require a new mathematical object beyond discreteness and continuity. The analogy remains pedagogical but provides intuition for Cantor's insight that between counting and measuring, no natural middle ground exists.

References

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