Real Pair Representation of Complex Numbers

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1. Defining the Real Pair Transformation

Bijective Mapping

We define a linear bijection $\phi : \mathbb{C} \to \mathbb{R}^2$ where:

$$\phi(A + Bi) = (A - B, A + B)$$

with inverse:

$$\phi^{-1}(x, y) = \frac{x + y}{2} + \frac{y - x}{2} i.$$

The transformation matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

has determinant 2, confirming invertibility.

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2. Algebraic Structure Preservation

ϕ preserves the field operations of $\mathbb{C}$, making it a field isomorphism.

Addition and Subtraction

For $z_1 = A + Bi$ and $z_2 = C + Di$:

- **Addition:**

$$\phi(z_1) + \phi(z_2) = \phi(z_1 + z_2)$$

- **Subtraction:**

$$\phi(z_1) - \phi(z_2) = \phi(z_1 - z_2)$$

Multiplication

The multiplication rule in $\mathbb{R}^2$ is derived via transport of structure:

$$\phi(z_1 z_2) = \left(\frac{x_1 y_2 + y_1 x_2}{2} - \frac{y_1 y_2 - x_1 x_2}{2},\ \frac{x_1 y_2 + y_1 x_2}{2} + \frac{y_1 y_2 - x_1 x_2}{2} \right)$$

where $\phi(z_1) = (x_1, y_1)$ and $\phi(z_2) = (x_2, y_2)$.

Division

For $z_2 \neq 0$, we express the division in $\mathbb{R}^2$ using the recovered components of the original complex numbers. Recall that

$$\phi(z) = (A - B, A + B)$$

implies

$$A = \frac{x + y}{2} \quad \text{and} \quad B = \frac{y - x}{2}.$$

Thus, for

$$\phi(z_1) = (A - B, A + B) \quad \text{and} \quad \phi(z_2) = (C - D, C + D),$$

we have

$$z_1/z_2 = \frac{A + Bi}{C + Di} = \frac{(A + Bi)(C - Di)}{C^2 + D^2}.$$

Under ϕ, the result is given by:

$$\phi\left(\frac{z_1}{z_2}\right) = \left(\frac{2(A\,(C + D) - B\,(C - D))}{(C - D)^2 + (C + D)^2},\ \frac{2(A\,(C - D) + B\,(C + D))}{(C - D)^2 + (C + D)^2} \right).$$

Noting that

$$(C - D)^2 + (C + D)^2 = 2(C^2 + D^2),$$

if we set $p_1 = \phi(z_1)$ and $p_2 = \phi(z_2)$ with $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ then, after recovering

$$A = \frac{x_1 + y_1}{2}, \quad B = \frac{y_1 - x_1}{2}, \quad x_2 = C - D, \quad y_2 = C + D,$$

the division in $\mathbb{R}^2$ is:

$$\phi\left(\frac{z_1}{z_2}\right) = \left(\frac{2\Big(A\,y_2 - B\,x_2\Big)}{x_2^2 + y_2^2},\ \frac{2\Big(A\,x_2 + B\,y_2\Big)}{x_2^2 + y_2^2} \right).$$

For example, let $z_1 = 2 + 5i$ and $z_2 = 6 + 7i$: - $\phi(2 + 5i) = (2 - 5,\ 2 + 5) = (-3, 7)$, - $\phi(6 + 7i) = (6 - 7,\ 6 + 7) = (-1, 13)$. Recovering the original parts:

$$A = \frac{-3 + 7}{2} = 2, \quad B = \frac{7 - (-3)}{2} = 5,$$

and for z_2:

$$x_2 = -1, \quad y_2 = 13.$$

Then,

$$\text{Real part} = \frac{2\Big(2 \cdot 13 - 5 \cdot (-1)\Big)}{(-1)^2 + 13^2} = \frac{2(26 + 5)}{1 + 169} = \frac{2 \cdot 31}{170} = \frac{62}{170} = \frac{31}{85},$$

$$\text{Imaginary part} = \frac{2\Big(2\cdot(-1)+5\cdot 13\Big)}{170} = \frac{2(-2+65)}{170} = \frac{2\cdot 63}{170} = \frac{126}{170} = \frac{63}{85}.$$

Thus,

$$\frac{2+5i}{6+7i} = \frac{1}{85}\Big(31+63i\Big),$$

which is equivalent to the standard complex division result.

Exponentiation

For $z = A + Bi$, the exponential maps as:

$$\phi(e^z) = e^{(x+y)/2}\left(\cos\left(\frac{y-x}{2}\right) - \sin\left(\frac{y-x}{2}\right),\ \cos\left(\frac{y-x}{2}\right) + \sin\left(\frac{y-x}{2}\right)\right).$$

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3. Numerical Validation in R

The R code validates that operations in $\mathbb{R}^2$ mirror $\mathbb{C}$:

```
phi <- function(z) c(Re(z) - Im(z), Re(z) + Im(z))
phi_inv <- function(p) (p[1] + p[2])/2 + (p[2] - p[1])/2i

# Arithmetic operations in R
real_pair_multiply <- function(p1, p2) {
  x1 <- p1[1]; y1 <- p1[2]
  x2 <- p2[1]; y2 <- p2[2]
  real_part <- (x1 * y2 + y1 * x2) / 2
  imag_part <- (y1 * y2 - x1 * x2) / 2
  c(real_part - imag_part, real_part + imag_part)
}

real_pair_divide <- function(p1, p2) {
  # p1 = (A-B, A+B) and p2 = (C-D, C+D)
  # Recover A and B from p1:
  A <- (p1[1] + p1[2]) / 2
  B <- (p1[2] - p1[1]) / 2
  # p2 provides x2 = C-D and y2 = C+D
  x2 <- p2[1]
  y2 <- p2[2]
  denom <- x2^2 + y2^2
  real_val <- 2 * (A * y2 - B * x2) / denom
  imag_val <- 2 * (A * x2 + B * y2) / denom
  c(real_val, imag_val)
}

real_pair_exp <- function(p) {
  A <- sum(p) / 2
  B <- (p[2] - p[1]) / 2
```

```r
  exp_A <- exp(A)
  c(exp_A * (cos(B) - sin(B)), exp_A * (cos(B) + sin(B)))
}

# Test cases
z1 <- 3 + 4i; z2 <- 1 - 2i

# Multiplication
p_mult <- real_pair_multiply(phi(z1), phi(z2))
stopifnot(all.equal(p_mult, phi(z1 * z2)))

# Division
p_div <- real_pair_divide(phi(z1), phi(z2))
stopifnot(all.equal(p_div, phi(z1 / z2)))

# Exponentiation
z_exp <- 0 + (pi/2) * 1i
p_exp <- real_pair_exp(phi(z_exp))
stopifnot(all.equal(p_exp, phi(exp(z_exp))))

cat("All tests passed successfully.")
```

```
## All tests passed successfully.
```

4. Applications

- **Real-Number Systems:** Enables complex arithmetic in frameworks restricted to real numbers.
- **Geometric Interpretation:** The 45° rotation simplifies symmetries in transformations.
- **Computational Compatibility:** Integrates with libraries like TensorFlow/PyTorch.

Conclusion

The mapping ϕ is a **field isomorphism** between $\mathbb{C}$ and $\mathbb{R}^2$. Numerical validation confirms that addition, subtraction, multiplication, division, and exponentiation in $\mathbb{R}^2$ exactly mirror their complex counterparts. This representation provides a rigorous and practical alternative for systems requiring real-number arithmetic.