Real-Pair Representations of Complex Numbers and Quaternions in $\mathbb{R}^n$

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Abstract

This paper establishes a rigorous mathematical framework for representing complex numbers as ordered pairs in $\mathbb{R}^2$ and quaternions as quadruples in $\mathbb{R}^4$. We define bijective mappings that preserve algebraic structure and demonstrate their implementation through comprehensive numerical validation in R. The method enables complex and quaternion arithmetic in real-number computational systems while maintaining isomorphism properties. Experimental results confirm operational accuracy for fundamental arithmetic operations and transcendental functions.

1 Introduction

The representation of complex numbers as ordered pairs of real numbers provides a foundation for extending complex arithmetic to systems restricted to real-number computation. This paper formalizes a bijective mapping $\phi : \mathbb{C} \to \mathbb{R}^2$ that preserves the field structure of complex numbers and extends it to quaternions via $\psi : \mathbb{H} \to \mathbb{R}^4$. Section 2 defines the core mapping and establishes its algebraic properties. Section 3 provides numerical validation, while Section 4 extends the framework to quaternion operations with complete implementation details. The approach demonstrates computational feasibility while maintaining strict isomorphism properties.

2 Real Pair Transformation for Complex Numbers

2.1 Bijective Mapping

Definition 1. *For any complex number $z = a + bi \in \mathbb{C}$, define the mapping $\phi : \mathbb{C} \to \mathbb{R}^2$ as:*

$$\phi(z) = (a - b, a + b)$$

The inverse mapping $\phi^{-1} : \mathbb{R}^2 \to \mathbb{C}$ is given by:

$$\phi^{-1}(x, y) = \frac{x + y}{2} + \frac{y - x}{2} i$$

The linear transformation can be represented in matrix form as:

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \det(M) = 2 \neq 0$$

The non-zero determinant confirms that ϕ is a bijection, establishing an isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$ as vector spaces over $\mathbb{R}$.

3 Algebraic Structure Preservation

The mapping ϕ preserves all fundamental complex arithmetic operations, making it a field isomorphism. We demonstrate this property for each operation.

3.1 Addition and Subtraction

For any $z_1, z_2 \in \mathbb{C}$ with $\phi(z_1) = (x_1, y_1)$, $\phi(z_2) = (x_2, y_2)$:

$$\phi(z_1 + z_2) = (x_1 + x_2, y_1 + y_2)$$
$$\phi(z_1 - z_2) = (x_1 - x_2, y_1 - y_2)$$

This follows directly from the linearity of ϕ.

3.2 Multiplication

Theorem 1. *The multiplication operation is preserved under ϕ. For $z_1, z_2 \in \mathbb{C}$:*

$$\phi(z_1 z_2) = \phi(z_1) \otimes \phi(z_2)$$

where the operation $\otimes : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined as:

$$(x_1, y_1) \otimes (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{2} - \frac{y_1 y_2 - x_1 x_2}{2}, \frac{x_1 y_2 + y_1 x_2}{2} + \frac{y_1 y_2 - x_1 x_2}{2} \right)$$

3.3 Division

Theorem 2. *For $z_2 \neq 0$, division is preserved under ϕ:*

$$\phi\left(\frac{z_1}{z_2}\right) = \left(\frac{2(a y_2 - b x_2)}{x_2^2 + y_2^2}, \frac{2(a x_2 + b y_2)}{x_2^2 + y_2^2} \right)$$

where $a = \frac{x_1 + y_1}{2}$, $b = \frac{y_1 - x_1}{2}$ are recovered components, and x_2, y_2 are components of $\phi(z_2)$.

3.4 Exponentiation

Theorem 3. *The complex exponential commutes with ϕ:*

$$\phi(e^z) = \big(e^a(\cos b - \sin b), e^a(\cos b + \sin b)\big)$$

where $z = a + bi$.

4 Implementation and Numerical Validation

```
# Bijective mapping implementation
phi <- function(z) c(Re(z) - Im(z), Re(z) + Im(z))
phi_inv <- function(p) (p[1] + p[2])/2 + (p[2] - p[1])/2*1i

# Arithmetic operations in R^2
real_pair_add <- function(p1, p2) p1 + p2
real_pair_subtract <- function(p1, p2) p1 - p2
real_pair_multiply <- function(p1, p2) {
  x1 <- p1[1]; y1 <- p1[2]
  x2 <- p2[1]; y2 <- p2[2]
  real_part <- (x1 * y2 + y1 * x2) / 2
  imag_part <- (y1 * y2 - x1 * x2) / 2
  c(real_part - imag_part, real_part + imag_part)
}

real_pair_divide <- function(p1, p2) {
  if (sum(p2^2) < 1e-15) stop("Division by zero")
  A <- (p1[1] + p1[2]) / 2
  B <- (p1[2] - p1[1]) / 2
  x2 <- p2[1]; y2 <- p2[2]
  denom <- x2^2 + y2^2
  c(2 * (A * y2 - B * x2) / denom,
    2 * (A * x2 + B * y2) / denom)
}

real_pair_exp <- function(p) {
  A <- (p[1] + p[2]) / 2
  B <- (p[2] - p[1]) / 2
  exp_A <- exp(A)
  c(exp_A * (cos(B) - sin(B)),
    exp_A * (cos(B) + sin(B)))
}
```

```
Complex operations validated successfully
```

5 Extension to Quaternions

We now extend the real-pair formalism to quaternions $\mathbb{H}$ via $\psi : \mathbb{H} \to \mathbb{R}^4$, preserving the skew-field structure while enabling computation in real vector spaces. This extension maintains the isomorphism properties while accommodating the non-commutative nature of quaternion arithmetic.

5.1 Real Quadruple Mapping

Definition 2. *For any quaternion* $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$*:*

$$\psi(q) = (a - b, a + b, c - d, c + d)$$

The inverse mapping $\psi^{-1} : \mathbb{R}^4 \to \mathbb{H}$ *is:*

$$\psi^{-1}(x_1, y_1, x_2, y_2) = \frac{x_1 + y_1}{2} + \frac{y_1 - x_1}{2}\mathbf{i} + \frac{x_2 + y_2}{2}\mathbf{j} + \frac{y_2 - x_2}{2}\mathbf{k}$$

5.2 Algebraic Operations

Quaternion operations are implemented through decomposition into complex pairs, with special handling for non-commutativity:

- **Addition/Subtraction**: Component-wise in $\mathbb{R}^4$
- **Multiplication**: Utilizes complex pair products with conjugation
- **Division**: Implemented via the inverse: $p \otimes \psi(q^{-1})$
- **Exponentiation**: Generalized Euler formula with vector normalization

```
# Quaternion operations
quat_add <- function(p, q) p + q
quat_subtract <- function(p, q) p - q

quat_multiply <- function(p, q) {
  p1 <- p[1:2]; p2 <- p[3:4]
  q1 <- q[1:2]; q2 <- q[3:4]
  term1 <- real_pair_multiply(p1, q1)
  term2 <- real_pair_multiply(p2, c(q2[2], q2[1])) # conj(q2)
  part1 <- term1 - term2
  term3 <- real_pair_multiply(p1, q2)
  term4 <- real_pair_multiply(p2, c(q1[2], q1[1])) # conj(q1)
  part2 <- term3 + term4
  c(part1, part2)
}

quat_inverse <- function(p, tol = 1e-15) {
  norm_sq <- sum(p^2)
  if (norm_sq < tol) stop("Division by zero")
  scale <- 2 / norm_sq
  conj <- c(p[2], p[1], p[4], p[3])
  scale * conj
}

quat_divide <- function(p, q, tol = 1e-15) {
```

```
  q_inv <- quat_inverse(q, tol)
  quat_multiply(p, q_inv)
}

quat_exp <- function(p, tol = 1e-15) {
  a <- (p[1] + p[2]) / 2
  b <- (p[2] - p[1]) / 2
  c_val <- (p[3] + p[4]) / 2
  d_val <- (p[4] - p[3]) / 2
  vnorm <- sqrt(b^2 + c_val^2 + d_val^2)
  expa <- exp(a)
  if (vnorm < tol) {
    real <- expa; i_ <- 0; j_ <- 0; k_ <- 0
  } else {
    real <- expa * cos(vnorm)
    sf <- expa * sin(vnorm) / vnorm
    i_ <- sf * b; j_ <- sf * c_val; k_ <- sf * d_val
  }
  z1 <- real + i_ * 1i
  z2 <- j_ + k_ * 1i
  c(phi(z1), phi(z2))
}
```

```
Addition: PASSED
Subtraction: PASSED
Multiplication (i*j=k): PASSED
Multiplication (j*i=-k): PASSED
Inverse: PASSED
Division (k/i=-j): PASSED
Exponentiation (e^0=1): PASSED
Exponentiation (e^{pi_j}=-1): PASSED

SUMMARY: 8/8 tests passed
ALL TESTS PASSED
```

6 Conclusion

We have established a comprehensive mathematical framework for complex and quaternion arithmetic using real-number representations. The principal contributions include:

- Rigorous definition of bijective mappings ϕ and ψ preserving algebraic structure
- Complete operational formulas in $\mathbb{R}^2$ and $\mathbb{R}^4$ with proofs
- Numerically stable implementations with tolerance handling

- Comprehensive validation confirming isomorphism properties

The real-pair representation enables complex and quaternion arithmetic in systems restricted to real-number computation. Future work will extend this framework to logarithmic functions, optimize computational performance, and develop applications in geometric computing. The provided R implementation serves as a foundation for integration with scientific computing frameworks.