WORKING PAPER

i

By,

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* The author is grateful for the many discussions with Adam Bogart who initially presented the iterative graphs in search of an explanation. All errors are the sole responsibility of the author.

INTRODUCTION

i is typically defined as $i = \sqrt{-1}$ or $i^2 = -1$. We aren't given much in way of an explanation as to why this is; rather we are just supposed to accept this as fact. This is not denying the importance of ***i*** in complex numbers or its role. However, we aim to show an explanation as to why ***i*** is equivalent to its extension in the imaginary axis orthogonal to the real number axis.

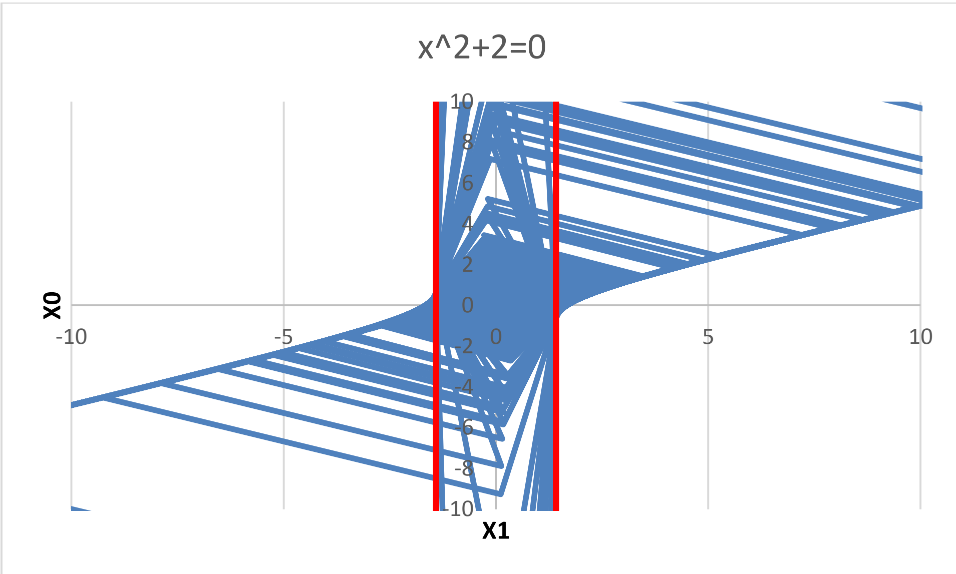
We discovered when using Newton-Raphson's method for finding roots on classically non converging equations, the roots are clearly identified and present themselves with a visualization of the classical method. The real number roots are the "attractors" for the iterative method when plotted against a lagged period of itself. These roots, present on the real number axis, happen to be the coefficients for ***i*** in the classical solutions. This is important to note, neither "attractor" is dominant, they are of equal strength. This is why "attractor" is in quotes, because of the repellent overall effect when the estimate is within those "attractors". A simple interpretation of this result implies *Both* when asking, "What is the root?" We are accustomed to *Either / Or* conditions using current notation of ± 1. However, in this instance, we refer to $\pm$ as the **Simultaneous** implementation of both numbers.

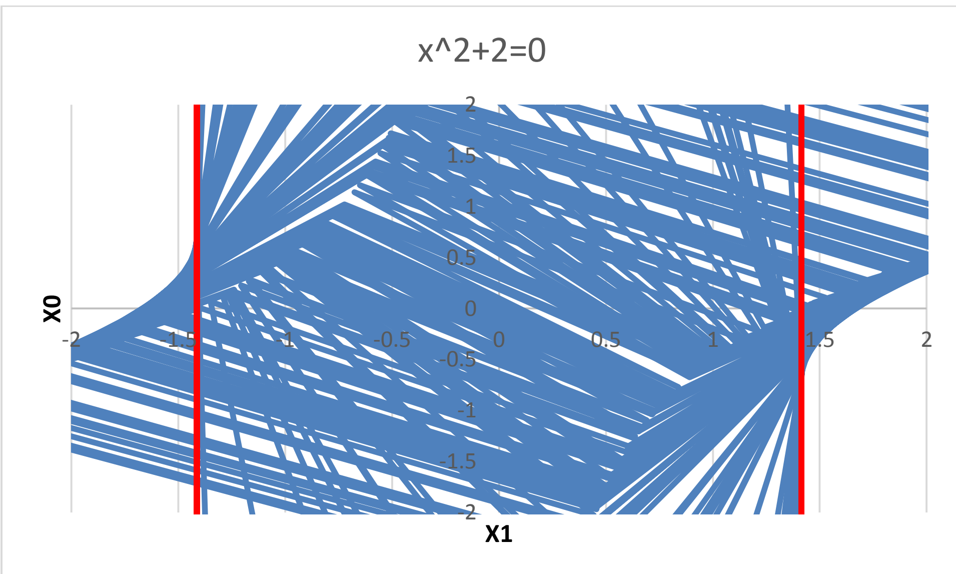
This is of course not possible under current knowledge and practice (we remind the reader that the term *imaginary* was penned by Descartes and used in a negative light). However, if we use ***i*** to represent the **Simultaneous** implementation of two real numbers, we find several consistencies with already accepted facts of ***i*** while reconciling the coincidental "attractors" present in the visualization.

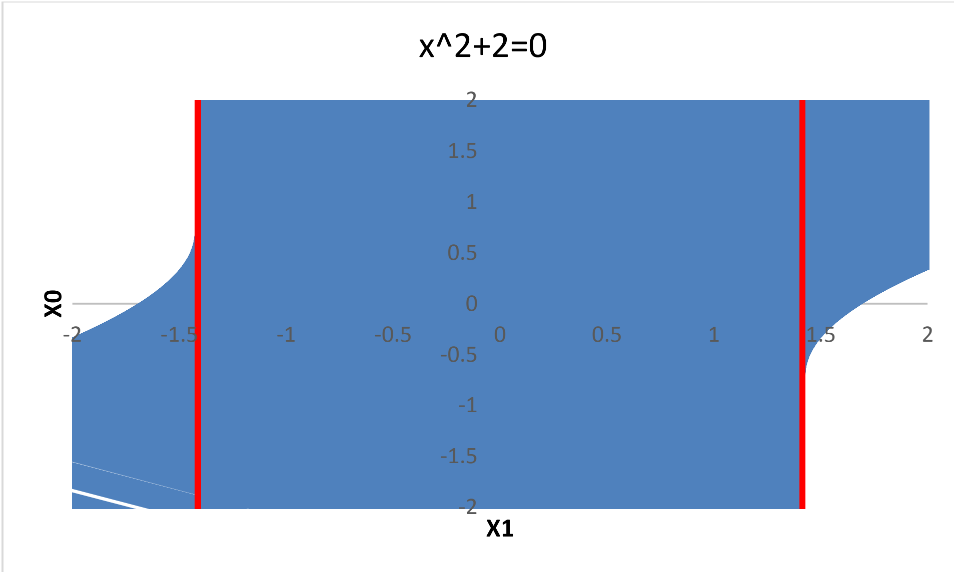
The **Simultaneous** condition is not foreign to current thought processes. For instance, qubits (quantum computing bits) are thought to be either a 0 or 1 simultaneously until measured. And light is believed to be both a particle and a wave through various experimentation. Thus, we feel this **Simultaneous** condition of the real numbers has ample support to warrant its investigation.

NEWTON-RAPHSON

We present the iterative method on the equation $x^2 + 2 = 0$. The "attractors" are clearly visualized at $\pm\sqrt{2}$ while the classical solution is $\sqrt{-2}$ *or* $\sqrt{2}i$. $\pm\sqrt{2}$ are the coefficients to i.







Figures 1,2 and 3. "Attractors" for the equation $x^2 + 2 = 0$, expanded range (top) and close up (bottom). Red lines equal $\pm\sqrt{2}$, coefficients for *i*. Increased iterations (5,000) and "attractors" hold.

We find Newton-Raphson to generate these results for all instances of $x^2 + C = 0$ for all positive C. *The "attractors" are* $\pm\sqrt{C}$ *which are the coefficients of* ***i*** *in the classical solution.*

The iterative routine (in R) to replicate these findings is available at:

https://github.com/OVVO-Financial/Number-Theory/blob/master/R/NEWTON%20ATTRACTOR.R

GRAPHING *i*, ADDING AND SUBTRACTING COMPLEX NUMBERS

We noted in the above section how the coefficients of ***i*** are the "attractors" on real axis, which are the solutions in the classical method. Figure 4 illustrates how ***i*** is a shared point from these real number representations. It is equidistant and has a 45° angle from each real. Thus it truly belongs to both, simultaneously.

SIMULTANEOUS REAL NUMBER REPRESENTATION

We can map each complex number to the corresponding real numbers with the following representation:

Proposition 1:

$$(\boldsymbol{A}+\boldsymbol{Bi}) \rightarrow \textbf{SIMULTANEOUS REAL NUMBERS } \boldsymbol{A} \pm \boldsymbol{B}$$

$$\boldsymbol{A} \pm \boldsymbol{B} = \{(\boldsymbol{A}-\boldsymbol{B}),(\boldsymbol{A}+\boldsymbol{B})\}\ \textit{where}\ \{(\boldsymbol{A}-\boldsymbol{B}),(\boldsymbol{A}+\boldsymbol{B})\} \in \mathbb{R}$$

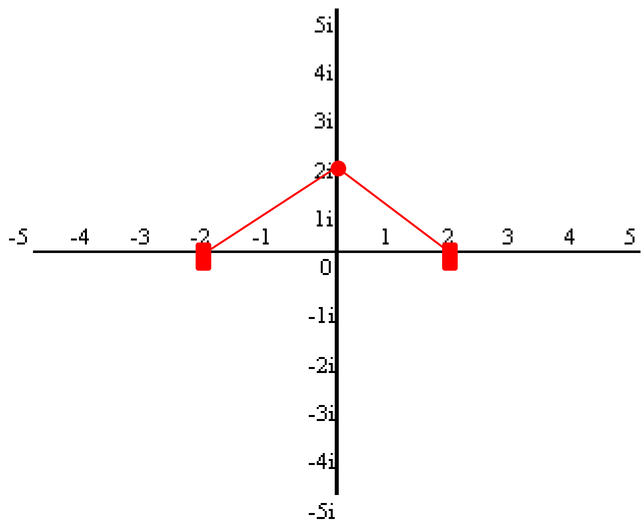


Figure 4. Representation of 2*i* simultaneous reals [-2,2].

ADDING AND SUBTRACTING COMPLEX NUMBERS USING SIMULTANEOUS REALS

ADDITION:

Using these real numbers' relationship to $\boldsymbol{i}$, we can verify the addition of complex numbers. For example, when adding the complex numbers:

$$\begin{array}{c} 3+2i \\ \underline{+(-1-3i)} \\ 2-i \end{array}$$

we can verify this with each complex number's simultaneous real number solution. Taking the one dimensional Minkowski sum of both simultaneous real number sets yields,

$$\begin{array}{c} [1,5] \\ \underline{+[2,-4]} \\ [3,1] \end{array}$$

$\{3,1\}$ are the simultaneous reals to $(2-i)$ following from **Proposition 1** $\{(\boldsymbol{A}-\boldsymbol{B}),(\boldsymbol{A}+\boldsymbol{B})\}$ where $\boldsymbol{A}=\mathbf{2}$ and $\boldsymbol{B}= \mathbf{-1}$ yields:

$$\{(\boldsymbol{A}-\boldsymbol{B}),(\boldsymbol{A}+\boldsymbol{B})\}=\{(2-(-1)),(2+(-1))\}$$

$$\{(\boldsymbol{A}-\boldsymbol{B}),(\boldsymbol{A}+\boldsymbol{B})\}=\{3,1\}$$

ADDITION EXAMPLE 2:

$$\begin{array}{c} 3+2i \\ \underline{+(-1+3i)} \\ 2+5i \end{array}$$

Where $\mathbf{A_1}=\mathbf{3},\mathbf{B_1}=\mathbf{2},\mathbf{A_2}=\mathbf{-1},\mathbf{B_2}=\mathbf{3}$

$$\begin{array}{rr} \{(\mathbf{A_1}-\mathbf{B_1}),(\mathbf{A_1}+\mathbf{B_1})\} & [1,5] \\ +\{(\mathbf{A_2}-\mathbf{B_2}),(\mathbf{A_2}+\mathbf{B_2})\} & [-4,2] \\ \hline & [-3,7] \end{array}$$

$\{-3,7\}$ are the simultaneous reals to $(2+5i)$.

SUBTRACTION:

When subtracting the complex numbers:

$$\begin{array}{c} 3+2i \\ \underline{-(-1+3i)} \\ 4-i \end{array}$$

The one dimensional Minkowski difference of simultaneous reals yields,

$$\begin{array}{c} [1,5] \\ \underline{-[-4,2]} \\ [5,3] \end{array}$$

$\{5,3\}$ are the simultaneous reals to $(4-i)$.

SUBTRACTION EXAMPLE 2:

$$\begin{array}{c} 2+2i \\ \underline{-(2-i)} \\ 3i \end{array}$$

Again, we can verify this with each equation's simultaneous real number solution.

$$\begin{array}{c} [0,4] \\ \underline{-[3,1]} \\ [-3,3] \end{array}$$

$\{-3,3\}$ are the simultaneous reals to $(3i)$.

MULTIPLYING COMPLEX NUMBERS

Starting with the canonical definition of $i^2 = -1$, we have simply $i \cdot i$. The simultaneous reals for ***i*** and procedure follows:

STEP 1: Multiply the columns

$$\begin{array}{r} [-1,1] \\ \times\,[-1,1] \\ \hline [1,1] \end{array}$$

STEP 2: Find the difference between the simultaneous reals in Step 1.

$[1,1]$ Difference = 0

STEP 3: Divide the difference by 2 **(remember the point is shared and simultaneous!)**. This is the coefficient to ***i*** in the complex number solution.

$0 = \frac{0}{2}$

STEP 4: Multiply the opposing terms

$$\begin{array}{r} [-1,1] \\ \times\,[-1,1] \\ \hline [-1,-1] \end{array}$$

STEP 5: Average the simultaneous reals from Step 4. This is the real part of the complex number. **(This is equivalent to half of the cross product)**.

$[-1,-1]$ Average = -1

So our solution using just the simultaneous reals is $(-1 + 0i)$ or $i^2 = -1$.

Using another example, $(2+5i)(6+7i)$ the simultaneous reals are $[-3,7],[-1,13]$.

STEP 1: Multiply the columns

$$\begin{array}{r} [-3,7] \\ \times\,[-1,13] \\ \hline [3,91] \end{array}$$

STEP 2: Find the difference between the simultaneous reals in Step 1.

$[3,91]$ Difference = 88

STEP 3: Divide the difference by 2. This is the coefficient to ***i*** in the complex number solution.

$44 = \frac{88}{2}$

STEP 4: Multiply the opposing terms

$$\begin{array}{r} [-3,7] \\ \times\,[-1,13] \\ \hline [-7,-39] \end{array}$$

STEP 5: Average the simultaneous reals from Step 4. This is the real part of the complex number.

$[-7,-39]$ Average = -23

So our solution using just the simultaneous reals is $(-23+44i)$. Here is the Wolfram Alpha link to the solution to verify

http://www.wolframalpha.com/input/?i=%282%2B5i%29*%286%2B7i%29

When the signs of $\boldsymbol{i}$ are opposing, the ordering of the simultaneous reals will differ following from **Proposition 1**. Using the different sign from the previous example, $(2+5i)(6-7i)$ the simultaneous reals are $[-3,7],[13,-1]$.

STEP 1: Multiply the columns

$$\begin{array}{r} [-3,7] \\ \times\ [13,-1] \\ \hline [-39,-7] \end{array}$$

STEP 2: Divide the difference by 2 from Step 1. This is the coefficient of $\boldsymbol{i}$ in the complex number.

$[-39,-7]$ Difference = 32 $\qquad 16=\frac{32}{2}$

STEP 3: Multiply the opposing terms

$$\begin{array}{r} [-3,7] \\ \times\ [13,-1] \\ \hline [91,3] \end{array}$$

STEP 4: Average the simultaneous reals from Step 3. This is the real part of the complex number.

$47=\frac{94}{2}$

So our division solution using just the simultaneous reals is $(47+16i)$. Here is the Wolfram Alpha link to the solution to verify.

http://www.wolframalpha.com/input/?i=%282%2B5i%29*%286-7i%29

DIVIDING COMPLEX NUMBERS

The procedure for dividing complex numbers involves multiplying by the conjugate to eliminate ***i*** from the denominator. The simultaneous reals for $\frac{i}{i}$ and procedure follows:

STEP 1: Square the conjugate of the denominator **(*i* in this case)** by multiplying columns

$$\begin{array}{r} [1,-1] \\ \times\ [1,-1] \\ \hline [1,\quad 1] \end{array}$$

STEP 2: Average the simultaneous reals from Step 1. This is the denominator in the complex number solution. **(This is equivalent to half of the dot product).**

[1,1] Average = 1

STEP 3: Multiply the columns

$$\begin{array}{rr} [-1,1] & i \\ \times\ [1,-1] & -i \\ \cline{1-1} [-1,-1] & \end{array}$$

STEP 4: Divide the difference by 2 from Step3. This is the coefficient of ***i*** in the complex number.

$[-1,-1]$ Difference = 0 $\qquad 0=\frac{0}{2}$

STEP 5: Multiply the opposing terms

$$\begin{array}{r} [-1,1] \\ \times\ [1,-1] \\ \hline [1,1] \end{array}$$

STEP 6: Average the simultaneous reals from Step 5. This is the real part of the complex number.

[1,1] Average = 1

So our solution using just the simultaneous reals is $\frac{1}{1}(1+0i)$ or $\frac{i}{i}=1$.

Using another example, $\frac{(2+5i)}{(6+7i)}$ the simultaneous reals are $[-3,7],[-1,13]$.

STEP 1: Square the conjugate of the denominator by multiplying columns

$$\begin{array}{r} [13,-1] \\ \times\,[13,-1] \\ \hline [169,1] \end{array}$$

STEP 2: Average the simultaneous reals from Step 1. This is the denominator in the complex number solution.

[169,1] Average = 85

STEP 3: Multiply the columns

$$\begin{array}{r} [-3,7] \\ \times\,[13,-1] \\ \hline [-39,-7] \end{array}$$

STEP 4: Divide the difference by 2 from Step 3. This is the coefficient of ***i*** in the complex number.

$[-39,-7]$ Difference = 32 $16=\frac{32}{2}$

STEP 5: Multiply the opposing terms

$$\begin{array}{r} [-3,7] \\ \times\,[13,-1] \\ \hline [91,3] \end{array}$$

STEP 6: Average the simultaneous reals from Step 5. This is the real part of the complex number.

$47=\frac{94}{2}$

So our division solution using just the simultaneous reals is $\frac{1}{85}(47+16i)$. Here is the Wolfram Alpha link to the solution to verify.

http://www.wolframalpha.com/input/?i=%282%2B5i%29%2F%286%2B7i%29

Using yet another example, $\frac{(2+5i)}{(6-7i)}$ the simultaneous reals are $[-3,7],[13,-1]$.

STEP 1: Square the conjugate of the denominator by multiplying columns

$$\begin{array}{r} [-1,13] \\ \times\,[-1,13] \\ \hline [1,169] \end{array}$$

STEP 2: Average the simultaneous reals from Step 1. This is the denominator in the complex number solution.

[1,169] Average = 85

STEP 3: Multiply the columns

$$\begin{array}{r} [-3,7] \\ \times\,[-1,13] \\ \hline [3,91] \end{array}$$

STEP 4: Find the difference between the simultaneous reals in Step 3.

[3,91] Difference = 88

STEP 5: Divide the difference by 2. This is the coefficient to ***i*** in the complex number solution.

$44 = \frac{88}{2}$

STEP 6: Multiply the opposing terms

$$\begin{array}{r} [-3,7] \\ \times\,[-1,13] \\ \hline [-7,-39] \end{array}$$

STEP 7: Average the simultaneous reals from Step 6. This is the real part of the complex number.

$[-7,-39]$ Average = -23

So our solution using just the simultaneous reals is $\frac{1}{85}(-23+44i)$. Here is the Wolfram Alpha link to the solution to verify.

http://www.wolframalpha.com/input/?i=%282%2B5i%29%2F%286-7i%29

FACTORIZATION

A quick note on factorization using complex numbers... Each complex number represents a specific situation involving the simultaneous reals. Squaring this complex number will reveal a real part equal to the N an individual is trying to factor per the multiplication routine presented above. For example, if trying to factor $N = 39$, there exists a specific complex number, when squared will have a real part equal to 39. This occurs for $(8 + 5i)$ whose simultaneous reals are [3,13]…the factors of 39!

Alternatively, there exists an $\boldsymbol{i}$ when added to the real number will yield its factors upon taking the square root of that resulting complex number. For example, when $N = 39$, taking the square root of the complex number $\sqrt{(39 + 80i)} = (8 + 5i)$.[1] This coefficient of $\boldsymbol{i}$ is also equal to $2(real \cdot imaginary)$ for the complex number factor, in this case $2(8 \cdot 5) = 80$.

This insight is inextricably related to Fermat's factorization method, $= A^2 - B^2$. Solving for $B^2 = (A^2 - N)$ tests all $\boldsymbol{i}$ for a specific real (vertically in the complex space). While solving for $A^2 = (B^2 + N)$ tests all reals for a specific $\boldsymbol{i}$ (horizontally in the complex space).

$B^2 = (A^2 - 39)$	$A^2 = (B^2 + 39)$
$B^2 = (8^2 - 39)$	$A^2 = (5^2 + 39)$
$B^2 = 25$	$A^2 = 64$
$B = 5$	$A = 8$

Unfortunately these insights have not yielded a direct computation of the factors of real numbers using their complex number representation. The author's GitHub page contains an expansive visualization and explanation of factorization in the complex space.[2]

[1] http://www.wolframalpha.com/input/?i=sqrt%2839%2B80i%29
[2] https://github.com/OVVO-Financial/Number-Theory/blob/master/Complex%20space.md

COMPLEX STEP DIFFERENTIATION

We will demonstrate how the simultaneous reals explanation is also consistent with the complex step differentiation method. The finite step method is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

but when very small h are used, there exists an issue with floating point numbers. The complex step differentiation avoids the floating point problem with its form

$$f'(x) = \frac{f(x+h(i))}{h}$$

For $f(x) = x^2$ when trying to find the derivative at $x = 1$ using $h = .0001$

$$f'(1) = \frac{(1+.0001i)^2}{.0001}$$

Execute the function in the numerator to find the coefficient of ***i***:

$$\begin{array}{r} [0.9999, 1.0001] \\ \times\ [0.9999, 1.0001] \\ \hline [0.99980001, 1.00020001] \end{array}$$

Average the difference of reals:

$$\frac{|[0.99980001 - 1.00020001]|}{2} = \frac{.0004}{2} = .0002$$

Divide by h:

$$f'(1) = \frac{.0002}{.0001} = 2$$

For $f(x) = x^2$ when trying to find the derivative at $x = 39$ using $h = .0001$

$$f'(39) = \frac{(39 + .0001i)^2}{.0001}$$

Execute the function in the numerator to find the coefficient of $\boldsymbol{i}$:

$$\begin{array}{r} [38.9999, 39.0001] \\ \times\, [38.9999, 39.0001] \\ \hline [1520.99220001, 1521.00780001] \end{array}$$

Average the difference of reals:

$$\frac{|[1520.99220001 - 1521.00780001]|}{2} = \frac{.0156}{2} = .0078$$

Divide by h:

$$f'(39) = \frac{.0078}{.0001} = 78$$

DISCUSSION

i isn't just $= \sqrt{-1}$ or $i^2 = -1$. It permits us to use a **Simultaneous** implementation of two real numbers. Newton-Raphson finds these real number roots, but it is torn between both of them. When plotting the iterative method over time one can feel and sense the frustration of the method as it wants to answer *Both* for the root solution.

We have demonstrated how to perform addition, subtraction, multiplication and division with complex numbers using only the real number roots initially discovered with Newton-Raphson. We verify these solutions with classical methods. We also illustrate the relationship of complex numbers to specific situations of real numbers, which can be used to determine factors of a number. Complex step differentiation is also fully compliant with the use of the simultaneous real numbers representing the complex number. Complex roots and exponentials are being actively researched.

The interpretation of these conditional real number roots or further insight into simultaneous manipulation of two real numbers would alleviate our reliance on ***i*** and further simplify equations with complex numbers.