FERMAT'S LAST THEOREM FROM THE GENERALIZED LAW OF COSINES

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ABSTRACT. We will show how Fermat's Last Theorem is a special case from the generalized Law of Cosines. Using the associated Pythagorean properties, we will demonstrate how the equivalence to known (at Fermat's time) geometric relationships fails for exponents higher than 2.

1. INTRODUCTION

Fermat's Last Theorem (FLT) stated that the more general Pythagorean equation $c^n = a^n + b^n$ had no solutions for $a, b,$ and c in positive integers, if n is an integer greater than 2. Fermat famously claimed to have a general proof of his conjecture, but left us with no indication.

One obvious immediate insight is Fermat's Pythagorean influence. In fact, Fermat was intimately involved with right triangles. An example of this influence is Fermat's Right Triangle Theorem, where he states that no square number can be a congruent number. A congruent number is a positive integer that is the area of a right triangle.

The Law of Cosines is quite amenable to the right triangle case due to the property of $\cos(90°) = 0$. We can also show how the Law of Cosines can be generalized and further how this generalization holds for known triangles such as equilateral, and isosceles right triangles.

2. Law of Cosines

The Law of Cosines in a triangle says: The square of any side is equal to the sum of the squares of the other two sides minus twice the product of those sides times the cosine of the angle between them. Let us first consider the right triangle.

Proposition 2.1: *Fermat's Last Theorem is a right triangle general equation from the Law of Cosines.*

Let *a*, *b*, and *c* be the legs of a triangle opposite angles A, B, and C. The Law of Cosines states:

$$c^2 = a^2 + b^2 - 2ab\cos(C) \tag{1}$$

More generally, this is equivalent to the form (when $n = 2$):

$$c^n = a^n + b^n - (ab^{n-1} + ba^{n-1})\cos(C) \tag{2}$$

Proof. When $C = 90°$ we have a right triangle and equation (2) collapses to Fermat's equation since $\cos(90°) = 0$.

$$c^n = a^n + b^n \tag{3}$$

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Lemma 2.2: *All right triangles follow the Pythagorean Theorem.*

$$c^2 = a^2 + b^2 \tag{4}$$

$$c = \sqrt{a^2 + b^2} \tag{5}$$

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Equation (5) represents the maximum descent for Pythagoras from Proposition 2.1. We can reconstruct, via infinite ascent, a general Pythagorean right triangle.

Proposition 2.3*: When we construct a general right triangle equation from Pythagoras we are not equivalent to Fermat.*

$$c^n \neq (a^n + b^n)$$

Proof.

$$\begin{aligned}
&c = \sqrt{a^2 + b^2},\\
&c^2 = a^2 + b^2,\\
&c^3 = a^2\sqrt{a^2 + b^2} + b^2\sqrt{a^2 + b^2},\\
&c^4 = a^4 + 2a^2b^2 + b^4,\\
&\vdots\\
&c^n = (a^2 + b^2)^{n/2} \neq (a^n + b^n) \qquad (6)
\end{aligned}$$

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We can also show that the generalized Law of Cosines holds for equilateral triangles, thus supporting FLT for the equilateral case.

Proposition 2.4*: The generalized Law of Cosines holds for equilateral triangles.*

Proof. When $a = b = c$ and $C = 60°$ such that $\cos(60°) = \frac{1}{2}$ we can restate equation (2) as:

$$c^n = c^n + c^n - (cc^{n-1} + cc^{n-1})\cos(C) \qquad (7)$$

Substituting and simplifying:

$$c^n = c^n + c^n - (c^n + c^n)\cos(60°)$$

$$c^n = c^n + c^n - \frac{(2c^n)}{2}$$

Yielding the equality below for any power n for the equilateral triangle:

$$c^n = c^n \qquad (8)$$

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Offering further instances of the generalized Law of Cosines, we can demonstrate how an isosceles right triangle must be of the Pythagorean form in Lemma 2.2. Isosceles right triangles (where $a = b$) inherently support FLT due to Pythagoras' constant $\left(\sqrt{2}\right)$ present in the isosceles hypotenuse definition $\left(a\sqrt{2}\right)$. This irrationality of Pythagoras' constant automatically eliminates any possibility of integer solutions to equation (3).

***Proposition 2.5**: The generalized Law of Cosines holds for isosceles right triangles and an isosceles right triangle cannot exist for $n \neq 2$.*

Proof. When $a = b$, $c = a\sqrt{2}$ and $C = 90°$ we can restate equation (2) as:

$$(a\sqrt{2})^n \;\; = a^n + a^n - (aa^{n-1} + aa^{n-1})\cos(C) \tag{9}$$

Substituting and simplifying when $\cos(90°) = 0$:

$$(a\sqrt{2})^n \;\; = a^n + a^n$$

Equivalent to:

$$2^{n/2}\, a^n \;\; = 2a^n \tag{10}$$

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Thus FLT is specific to the right triangle and trivial to the isosceles case. In this sense, FLT can be viewed simply as a reaffirmation of Pythagoras' Theorem. Given Fermat's intimate knowledge of right triangles, i.e., Fermat's Right Triangle Theorem, FLT's complete reliance on Pythagoras should not come as a surprise.

The generalized right triangle Fermat conjectured simply cannot be Pythagorean, thus a contradiction following from Proposition 2.3. Since a congruent number is a positive integer that is the area of a right triangle, no congruent number can exist in Fermat's generalization. If no congruent number can exist, then by definition no simultaneous rational sides to this conjectured right triangle can exist, of which integers satisfy.