A Pedagogical Model for Fermat's Last Theorem Inspired by the Law of Cosines

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Abstract

We construct an algebraic form that mimics the structure of the Law of Cosines. This form reduces to the standard Law for $n = 2$ and to Fermat's equation for a right angle. We then use this model, applied to specific triangles, to illustrate geometrically the impossibility of Fermat's equation holding for $n > 2$. By examining the behavior of this constructed form for equilateral and isosceles right triangles, we demonstrate its internal coherence and show how it reproduces both the stability of Pythagorean structure at $n = 2$ and its collapse for higher powers. This serves as a visual and algebraic analogy to the theorem, complementing but not replacing sophisticated proofs like Wiles'.

1 Introduction

Fermat's Last Theorem states that the equation

$$a^n + b^n = c^n,$$

has no solutions in positive integers (a, b, c) for integer $n > 2$. While Wiles' proof rests on the modularity of elliptic curves, the origins of FLT are geometric: it extends the Pythagorean relationship $a^2 + b^2 = c^2$ to higher powers.

Here, we explore a constructed algebraic form inspired by the Law of Cosines. We demonstrate that this form maintains consistency for $n = 2$, collapses to Fermat's equation under $C = 90°$, and reveals the algebraic divergence that underlies the impossibility of integer solutions for $n > 2$. This model is presented as a pedagogical tool to build intuition about FLT, not as an alternative mathematical representation or proof.

2 A Constructed Form Inspired by the Law of Cosines

The classical Law of Cosines for any triangle with sides a, b, c and opposite angles A, B, C is:

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

We define a constructed form that parallels this structure:

$$c^n = a^n + b^n - (ab^{n-1} + ba^{n-1})\cos(C), \tag{1}$$

where $n \in \mathbb{R}^+$.

Important Note: This is not the Law of Cosines for $n \neq 2$, but rather a form designed to explore the consequences of extending the Pythagorean relationship.

When $n = 2$,

$$c^2 = a^2 + b^2 - (ab + ba)\cos(C) = a^2 + b^2 - 2ab\cos(C),$$

so equation (1) reproduces the standard Law of Cosines.

2.1 The Right Triangle Case

For $C = 90°$, we have $\cos(90°) = 0$. Equation (1) reduces to:

$$c^n = a^n + b^n,$$

which is precisely Fermat's equation. Thus, under orthogonality, the constructed form collapses naturally to the FLT form. This validates the model's structural compatibility at the quadratic case and its algebraic consistency for higher n.

3 The Pythagorean Baseline and Infinite Ascent

All right triangles satisfy:

$$c = (a^2 + b^2)^{1/2}.$$

By iterating powers, we obtain:

$$c^n = (a^2 + b^2)^{n/2}.$$

Comparing this with $c^n = a^n + b^n$ shows that:

$$(a^2 + b^2)^{n/2} \neq a^n + b^n \quad \text{for } n > 2,$$

and equality holds only at $n = 2$.

This *infinite ascent* framework visually illustrates the divergence of the Pythagorean structure as the exponent increases. The exponentiation process preserves form but not equality, echoing the essence of Fermat's impossibility statement.

4 Symmetric Demonstrations

4.1 Equilateral Triangle

Let $a = b = c$ and $C = 60°$ so that $\cos(60°) = \frac{1}{2}$. Substituting into (1):

$$c^n = c^n + c^n - (cc^{n-1} + cc^{n-1})\cos(60°),$$

$$c^n = 2c^n - 2c^n \cdot \frac{1}{2} = c^n.$$

The equality holds for all n, confirming internal consistency under complete geometric symmetry.

4.2 Isosceles Right Triangle

Let $a = b$ and $C = 90^\circ$. Then (1) becomes:

$$c^n = 2a^n.$$

But geometrically, $c = a\sqrt{2}$, so:

$$(a\sqrt{2})^n = 2a^n \Rightarrow 2^{n/2}a^n = 2a^n.$$

Equality holds only for $n = 2$. For $n > 2$, the left-hand side exceeds the right-hand side, illustrating exactly where the Pythagorean structure collapses. Thus, the constructed form correctly identifies the critical exponent of breakdown.

5 Philosophical Analogy to Wiles' Work

We can draw a philosophical analogy between this model and Wiles' proof based on the concept of structural incompatibility:

> Wiles proved that all elliptic curves over $\mathbb{Q}$ are modular, and that non-modular curves would correspond to impossible Fermat-type equations. In a very general philosophical sense, our constructed model shows that all right triangles are Pythagorean for $n = 2$, and any extension of the Pythagorean form (for $n > 2$) fails to maintain structural coherence with elementary geometry.

This analogy is strictly philosophical and conceptual, it is not a mathematical isomorphism between elliptic curves and triangles. Both frameworks, however, illustrate in their respective domains how higher-order consistency can be prohibited within a well-defined structural system.

6 Conclusion

The constructed form presented here provides an algebraically coherent diagnostic tool for understanding Fermat's Last Theorem geometrically. It:

- coincides with the classical Law of Cosines when $n = 2$,
- collapses to Fermat's equation when $C = 90^\circ$,
- preserves equality under full symmetry (equilateral case),
- reveals inconsistency under $n > 2$ for the isosceles right case.

As a pedagogical model, it successfully illustrates the fundamental theme of higher-order incompatibility within a consistent structural system. It does not replace Wiles' proof but provides an accessible, geometric perspective on the same ultimate conclusion: the Pythagorean relationship does not generalize to higher powers.

References

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3. Wiles, Andrew. "Modular Elliptic Curves and Fermat's Last Theorem." *Annals of Mathematics*, 141(3), 1995.