# FERMAT'S LAST THEOREM FROM THE GENERALIZED LAW OF COSINES

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Abstract. We show how Fermat's equation arises from a generalized Law of Cosines for right triangles and how this generalization holds for equilateral and isosceles right triangles. Using the associated Pythagorean properties, we demonstrate how the equivalence to known (at Fermat's time) geometric relationships fails for exponents higher than 2.

## 1. Introduction

Fermat's Last Theorem (FLT) stated that the more general Pythagorean equation $c^n = a^n + b^n$ had no solutions for $a, b$, and $c$ in positive integers, if $n$ is an integer greater than 2. Fermat famously claimed to have a general proof of his conjecture, but left us with no indication.

One obvious immediate insight is Fermat's Pythagorean influence. In fact, Fermat was intimately involved with right triangles. An example of this influence is Fermat's Right Triangle Theorem, where he states that no square number can be a congruent number [2]. A congruent number is a positive integer that is the area of a right triangle [1].

The Law of Cosines is quite amenable to the right triangle case due to the property of $\cos(90°) = 0$. The Law of Cosines can be generalized to higher exponents and this generalization holds for known triangles such as equilateral, and isosceles right triangles. If we accept without question that Fermat was indeed discussing a right triangle, the generalized law of cosines is not necessary to demonstrate that Fermat's equation arises from the right triangle instance.

## 2. Law of Cosines

The Law of Cosines in a triangle states: The square of any side is equal to the sum of the squares of the other two sides minus twice the product of those sides times the cosine of the angle between them [3].

Let us first consider the right triangle.

**Proposition 2.1.** *Fermat's equation is a right triangle general equation from the Law of Cosines.*

Let $a, b$, and $c$ be the legs of a triangle opposite angles A, B, and C. The Law of Cosines states:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

More generally, this is equivalent to the form (when $n = 2$):

$$c^n = a^n + b^n - (ab^{n-1} + ba^{n-1})\cos(C) \tag{1}$$

*Proof.* When $C = 90°$ we have a right triangle and the generalized Law of Cosines collapses to Fermat's equation since $\cos(90°) = 0$.

$$c^n = a^n + b^n \tag{2}$$

□

## 3. Pythagoras's Theorem

All right triangles follow the Pythagorean Theorem $c^2 = a^2 + b^2$ [4].

**Lemma 3.1.** *The maximum descent of Pythagoras's Theorem for* $c^{\mathbb{Z}} s.t. \mathbb{Z} > 0$ *is:*

$$c = \sqrt{a^2 + b^2}$$
$$c = (a^2 + b^2)^{\frac{1}{2}}$$

We can reconstruct, via infinite ascent, a general Pythagorean right triangle.

**Proposition 3.2.** *When we construct a general right triangle equation from Pythagoras we are no longer equivalent to Fermat.*

*Proof.*

$$\begin{aligned} c &= (a^2 + b^2)^{\frac{1}{2}} \equiv \sqrt{a^2 + b^2} \\ c^2 &= (a^2 + b^2)^{\frac{2}{2}} \equiv (a^2 + b^2) \\ c^3 &= (a^2 + b^2)^{\frac{3}{2}} \equiv a^2\sqrt{a^2 + b^2} + b^2\sqrt{a^2 + b^2} \\ c^4 &= (a^2 + b^2)^{\frac{4}{2}} \equiv a^4 + 2a^2b^2 + b^4 \\ &\vdots \\ c^n &= (a^2 + b^2)^{\frac{n}{2}} \neq (a^n + b^n) \end{aligned}$$

□

## 4. Generalized Law of Cosines

**Proposition 4.1.** *The generalized Law of Cosines is true for all equilateral triangles.*

*Proof.* When $a = b = c$ and $C = 60°$ such that $\cos(60°) = \frac{1}{2}$ we can restate 1 as:

$$c^n = c^n + c^n - (cc^{n-1} + cc^{n-1})\cos(C)$$

Substituting and simplifying:

$$c^n = c^n + c^n - (c^n + c^n)\cos(60°)$$
$$c^n = c^n + c^n - \frac{2c^n}{2}$$

Yielding the equality below for any power $n$ for the equilateral triangle:

$$c^n = c^n$$

□

Offering further instances of the generalized Law of Cosines, we prove how an isosceles right triangle cannot be of the Pythagorean form in Lemma 3.1. Isosceles right triangles (where $a = b$) inherently support FLT due to Pythagoras's constant ($\sqrt{2}$) present in the isosceles hypotenuse definition of ($a\sqrt{2}$) [5]. This irrationality of Pythagoras's constant automatically eliminates any possibility of integer solutions to 2.

**Proposition 4.2.** *The generalized Law of Cosines holds for isosceles right triangles and an isosceles right triangle cannot exist for $n \neq 2$.*

*Proof.* When $a = b, c = a\sqrt{2}$ and $C = 90°$ we can restate 1 as:

$$(a\sqrt{2})^n = a^n + a^n - (aa^{n-1} + aa^{n-1})\cos(C)$$

Substituting and simplifying when $\cos(90°) = 0$:

$$(a\sqrt{2})^n = a^n + a^n$$

Equivalent to:

$$2^{\frac{n}{2}} a^n = 2a^n$$

□

## 5. Conclusion

Fermat's Last Theorem is specific to the right triangle and trivial to the isosceles case. In this sense, FLT can be viewed simply as a reaffirmation of Pythagoras's Theorem. Given Fermat's intimate knowledge of right triangles, e.g., Fermat's Right Triangle Theorem, FLT's complete reliance on Pythagoras should not come as a surprise.

The generalized right triangle Fermat conjectured simply cannot be Pythagorean, thus a contradiction following from Proposition 3.2. Wiles' proof of FLT [6] shows that all elliptic curves are modular and solutions for $n > 2$ would be associated with a non-modular elliptical curve, hence no solutions exist. Restating the last sentence combined with associated terms from both methods will clearly illustrate the similarity of both arguments:

> **Wiles'** [The preceding] proof shows that all **elliptic curves** [right triangles] are **modular** [Pythagorean] and solutions for $n > 2$ would be associated with a **non-modular** [non-Pythagorean] **elliptical curve** [right triangle], hence no solutions exist.

## References


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