Optimal measures for three-point energies and semidefinite programming

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Distances and energy

For a finite set of points on a compact subset of Eucldiean space, $\mathcal{C} \subset \Omega$, $\Omega \subset \mathbb{R}^d$ and a continuous function $f : \mathbb{R} \to (-\infty, \infty]$, the (discrete) f-potential energy of $\mathcal{C}$ is

$$E_f(\mathcal{C}) = \sum_{x \neq y \in \mathcal{C}} f(|x-y|^2).$$

When $\Omega = \mathbb{S}^{d-1}$, the surface of the sphere, $f(|x-y|^2)$ may take the alternative form
$g(\langle x, y\rangle) = f(2 - 2\langle x, y\rangle) = f(|x-y|^2)$.

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$g(\langle x, y\rangle) = f(2 - 2\langle x, y\rangle) = f(|x-y|^2)$.

(i) On $\mathbb{S}^2$, the electric potential according to Coulomb's law is $f(|x-y|^2) = \frac{1}{|x-y|}$.

(ii) $f(|x-y|^2) = 1/|x-y|^{12} - 1/|x-y|^6$, is a Lennard-Jones potential, approximating the interaction between a pair of neutral atoms.

Continuous energy on the sphere

For continuous f and a Borel measure μ on $\mathbb{S}^{d-1}$, we define the energy integral of μ with respect to f as

$$I_f(\mu) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} f(\langle x, y \rangle) d\mu(x) d\mu(y).$$

An attractive-repulsive potential function $f(\langle x, y \rangle)$ on the sphere would satisfy that f is decreasing away 1, but increasing towards 1.

Euclidean setting

$$I_f(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(|x-y|)d\mu(x)d\mu(y).$$

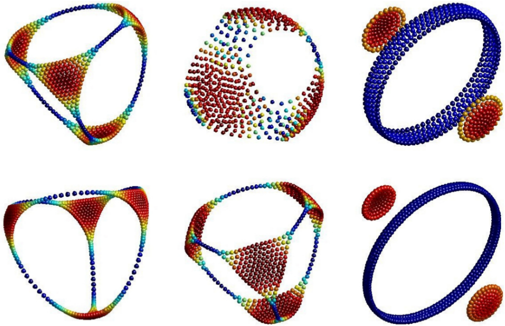


Figure: Numerical minimizers of f energies for $f'(t) = \frac{\tanh(a(1-\sqrt{2t}))+b}{\sqrt{2t}}$, for various a and b. [Source: J. H. von Brecht].

Euclidean setting

Let $\mathcal{P}(\mathbb{R}^d)$ be the space of Borel probability measures.

Theorem (Carillo, Figalli, Patacchini, '17)

For $f \in C^2$, **mildly repulsive**, *that is, satisfying $f(x) \sim -|x|^\alpha$ as $|x| \to 0$ for some $\alpha > 2$, solutions to*

$$\min_{\mu \in \mathcal{P}(\mathbb{R}^d)} I_f(\mu) = \min_{\mu \in \mathcal{P}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(|x-y|) d\mu(x) d\mu(y),$$

are finitely supported.

Note: An almost identical result for the sphere (with only the distance being replaced by the geodesic distance) was recently proved by O. Vlasiuk.

Discrete minimizers for polynomial potentials

Theorem (Bilyk, Glazyrin, Matzke, P., Vlasiuk, '19)

Assume that f is a polynomial in the distance squared, $|x-y|^2$, or equivalently in the inner product $\langle x, y\rangle$. Then there exists a discrete minimizer $\mu \in \mathcal{P}(\mathbb{S}^{d-1})$ of the energy I_f when optimizing over all Borel probability measures supported on the sphere.

Moreover, such a measure may have its support size bounded by a polynomial function in the degree of the polynomial f.

Functions of distances on spheres

$L^2(\mathbb{S}^{d-1})$ breaks down into a direct sum of unitary irreducible representations. V_l are eigenspaces corresponding to the l-th eigenvalue of the Laplacian on $\mathbb{S}^{d-1}$. The eigenfunctions are denoted $Y_{m,n}$. There exists polynomials $C_m : [-1, 1] \to \mathbb{C}$ for which

$$C_m(\langle x, y\rangle) = \sum_{n=1}^{\gamma_m} Y_{m,n}(x) Y_{m,n}(y),$$

where $\gamma_m = \dim V_m$.
The functions $f : [-1, 1] \to \mathbb{R}$ with positive coefficients in its Gegenbauer expansion

$$f(\langle x, y\rangle) = \sum_{m=0}^{\infty} \hat{f}_m C_m(\langle x, y\rangle),$$

$\hat{f}_m \geqslant 0$, are precisely the real valued *positive definite functions*, as

$$\sum_{i,j=1}^{N} c_i c_j f(\langle x_i, x_j\rangle)) = \sum_{m=0}^{k} \hat{f}_m \sum_{n=1}^{\gamma_m} \left(\sum_{i=1}^{N} c_i Y_{m,n}(x_i) \right)^2 \geqslant 0.$$

Full result on discrete minimizers

Denote $N_+ = \{n \geqslant 0 : \widehat{f}_n > 0\}$ and $N_- = \{n \geqslant 0 : \widehat{f}_n < 0\}$. Assume that $\#N_+ < \infty$, i.e. there are only finitely many terms with $\widehat{f}_n > 0$.

Theorem (Bilyk, Glazyrin, Matzke, P., Vlasiuk, '19)

Let the function $f \in C[-1,1]$ have Gegenbauer expansion $\sum_n \widehat{f}_n C_n(t)$ with

$$\#N_+ = \#\{n \geqslant 0 : \widehat{f}_n > 0\} < \infty,$$

i.e. with only finitely many positive definite terms. Then there exists a measure $\mu^* \in \mathcal{P}$ such that

$$\#\mathrm{supp}\, \mu^* \leqslant \sum_{n \in N_+ \cup \{0\}} \dim V_n, \tag{1}$$

and μ^* minimizes I_f, i.e.

$$I_f(\mu^*) = \inf_{\mu \in \mathcal{P}} I_f(\mu). \tag{2}$$

Theorem (Bilyk, Glazyrin, Matzke, P., Vlasiuk, '19)

If f is a real-analytic, nonnegative function which is not positive definite on $\mathbb{S}^{d-1}$, and μ is a minimizer of I_f. Then $(\mathrm{supp}(\mu))^\circ = \emptyset$.

Corollary

If f is a polynomial kernel with negative coefficients (f is not positive definite), then $(\mathrm{supp}(\mu))^\circ = \emptyset$.

p-Frame energies

For $\Omega = \{x \in \mathbb{F}^d \mid |x| = 1\}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the p-frame energy is the energy integral

$$I_{|t|^p}(\mu) = \int_\Omega \int_\Omega |\langle x, y \rangle|^p d\mu(x) d\mu(y)$$

computed for μ, probability Borel measures supported on Ω.

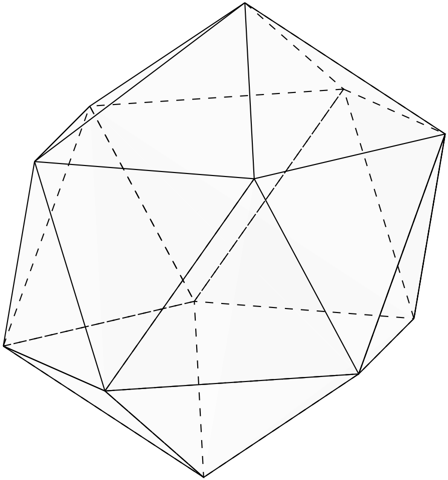


Figure: Icosahedron.

What are minimizers for p not even?

d	N	Energy	Range of p	Tight	Name
2	N	*	$[2N-4, 2N-2]$	t	regular $2N$-gon
d	d	$1/d$	$[0, 2]$	t	orthonormal basis
3	6	0.241202265916660	$[2, 4]$	t	icosahedron
3	11	0.142857142857143	$6-$		Reznick design
3	16	0.124867143799450	$[6, 8]$		icosahedron and dodecahedron
4	11	0.125000000000000	$4-$		small weighted design
4	24	0.096277507157493	$[4, 6]$		D_4 root vectors
4	60	0.047015486159502	$[8, 10]$		600-cell
5	16	0.118257675970387	$[2, 4]$		hemicube
5	41	0.061838820473855	$[4, 6]$		Stroud design
6	22	0.090559619406078	$[2, 4]$		cross-polytope and hemicube
6	63	0.042488105634495	$[4, 6]$		E_6 & E_6^* roots
7	28	0.071428571428571	$[2, 4]$	t	kissing E_8
7	91	0.030645893660944	$[4, 6]$		E_7 & E_7^* roots
8	36	0.059098639455782	3		mid-edges of regular simplex
8	120	0.022916666666667	$[4, 6]$	t	E_8 min. vec.

What are minimizers for p not even?

d	N	Energy	Range of p	Tight	Name
23	276	0.011594202898551	$[2, 4]$	t	equiangular lines
23	2300	0.002028985507246	$[4, 6]$	t	kissing Leech lattice
24	98280	0.000103419439357	$[8, 10]$	t	Leech lattice min. vec.

d	N	Energy	Range of p	Tight	Name
d	d	$1/d$	$[0, 2]$	t	orthonormal basis
3	9	0.222222222222222	$[2, 4]$	t	SIC-POVM
3	21	0.012610934678518	$[4, 6]$		union equiangular lines
4	16	0.146352549156242	$[2, 4]$	t	SIC-POVM
4	40	0.068301270189222	$[4, 6]$	t	Eisenstein structure on E_8
5	25	0.105319726474218	$[2, 4]$	t	SIC-POVM
5	85	0.041997097378053	$[4, 6]$		O_{10} & $(C_6 \times SU(4,2)) : C_2$ min. vec.
6	36	0.080272843473504	$[2, 4]$	t	SIC-POVM
6	126	0.027777777777778	$[4, 6]$	t	Eisenstein structure on K_{12}
d	d^2	$\frac{1+(d^2-1)(1/(d+1))^{3/2}}{d^2}$	$[2, 4]$	t	SIC-POVM (conjectured)

Convex geometry

$$\frac{n}{p}V_p(K,L) = \lim_{\epsilon\to 0^+}\frac{V(K+_p\epsilon L)-V(K)}{\epsilon}.$$

$K +_p \epsilon L$ denotes the convex body with support function of the form.

$$h(K+_p\epsilon L,u)^p = h(K,u)^p + \epsilon h(L,u)^p.$$

To each convex body K, there is a positive dS_K^p on $\mathbb{S}^{d-1}$ and

$$V_p(K,Q) = \frac{1}{n}\int_{\mathbb{S}^{d-1}} h(Q,u)^p dS_K^p(u),$$

for each convex body Q, where $h(Q,u)$ is the support function of Q.

$$h(\Pi_p K,u) = \left\{\int_{\mathbb{S}^{d-1}} |\langle u,v\rangle|^p dS_K^p(v)\right\}^{\frac{1}{p}}.$$

$$V_p(K,\Pi_p K) = \frac{1}{n}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} |\langle u,v\rangle|^p dS_K^p(v)dS_K^p(u).$$

Theorem (Bilyk, Glazyrin, Matzke, P., Vlasiuk, '19)

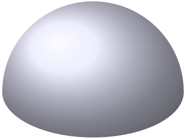
Suppose $p \notin 2\mathbb{N}$, and μ is a minimizer of the p-frame energy $I_{|t|^p}$ on $\mathbb{S}^{d-1}_{\mathbb{R}}$. Then the support of μ has empty interior.

Potentials and projective spaces

The projective spaces, $\mathbb{FP}^{d-1}$, for $\mathbb{F} = \mathbb{R}, \mathbb{C},$ or $\mathbb{H}$, reals, complex numbers, or quaternions, have a simple description as the space of lines in $\mathbb{F}^d$,

$$x\mathbb{F} = \{x\lambda \mid \lambda \in \mathbb{F}\backslash\{0\}\}.$$

Each line may be identified with a unit vector $x \in \mathbb{F}^d$, $|x| = 1$, so that the geodesic distance $\vartheta(x, y)$ on $\mathbb{FP}^{d-1}$ satisfies $\cos\vartheta(x, y) = 2|\langle x, y\rangle|^2 - 1$. Ω is used alternatively to space $\mathbb{FP}^{d-1}$ in what follows.



Analysis on projective spaces

Energies of distances for these spaces take form

$$\int_\Omega \int_\Omega f(\cos\vartheta(x,y))d\mu(x)d\mu(y).$$

To each projective space corresponds the decomposition

$$L^2(\Omega) = \bigoplus_{k\geqslant 0} V_k$$

into unitary irreducible representations. V_k are eigenspaces corresponding to the k-th eigenvalue of the Laplacian on Ω. The eigenfunctions are denoted $Y_{k,l}$. There exists complex polynomials $C_m : [-1,1] \to \mathbb{C}$ for which

$$C_k(\cos\vartheta(x,y)) = \sum_{l=1}^{\gamma_k} \overline{Y_{k,l}}(x) Y_{k,l}(y),$$

where $\gamma_k = \dim V_k$. C_k are a complete system of polynomials on $L^2[-1,1]$ and are given explicitly as a family of Jacobi polynomials.

Projective spaces and designs

There are complex polynomials C_k, which satisfy

$$C_k(\cos\vartheta(x,y)) = \sum_{j=1}^{\gamma_k} \overline{Y_{k,l}(x)}Y_{k,l}(y)$$

for all $x, y \in \Omega$. A weighted projective design can be defined in either of the two below ways.

(i) $\mathcal{C} \subset \Omega$, $W \subset \mathbb{R}_+^N$, $|\mathcal{C}| = N$ is a weighted t-design if

$$\sum_{i,j=1}^{N} C_k(\cos\vartheta(x_i,x_j))\omega_i\omega_j = 0 \text{ for } 1 \leqslant k \leqslant t.$$

(ii) If $\mathcal{P}_k$ is the space of polynomials over $\mathbb{F}^d$ of degree k for which $p(x\lambda) = |\lambda|^k p(x)$, $\lambda \in \mathbb{F}$, then $(\mathcal{C}, W)$ is a t-design if

$$\int_\Omega p(x)d\sigma(x) = \sum_{i=1}^{N} p(x_i)\omega_i$$

for all polynomials $p \in \bigoplus_{k=0}^{t} \mathcal{P}_k$.

Tight designs

A discrete set $\mathcal{C} \subset \Omega$ is called a *tight design* if it satisfies one of the following conditions.

(i) $\mathcal{C}$ is a design of degree $2m-1$ and there are m distances between its distinct elements, including at least one pair diameter apart;

(ii) $\mathcal{C}$ is a design of degree $2m$ and there are m distances between its distinct elements.

Projective tight designs

Table: A list of parameters for the known to exist real, complex, and quaternionic projective tight designs (besides trivial examples/SIC-POVMs). Here M denotes the strength of the design, and N is the size of the design.

d	N	M	Inner Products	$\mathbb{F}$	Name
3	6	2	$1/\sqrt{5}$	$\mathbb{R}$	icosahedron
7	28	2	$1/3$	$\mathbb{R}$	kissing configuration for E_8
8	120	3	$0, 1/2$	$\mathbb{R}$	roots of E_8 lattice
23	276	2	$1/5$	$\mathbb{R}$	tight simplex
23	2300	3	$0, 1/3$	$\mathbb{R}$	kissing configuration for Λ_{24}
24	98280	5	$0, 1/4, 1/2$	$\mathbb{R}$	minimal vectors of Λ_{24}
4	40	3	$0, 1/\sqrt{3}$	$\mathbb{C}$	Eisenstein structure on E_8
6	126	3	$0, 1/2$	$\mathbb{C}$	Eisenstein structure on K_{12}
3	15	2	$\sqrt{14}/7$	$\mathbb{H}$	tight simplex
5	165	3	$0, 1/2$	$\mathbb{H}$	quaternionic reflection group

Tight designs as minimizers

Definition

Let $f \in C^M(a,b)$. We say that f is absolutely monotonic of degree M if $f^{(k)}(t) \geqslant 0$ for $0 \leqslant k \leqslant M$ and $t \in (a,b)$.

Tight designs as minimizers

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Theorem (Bilyk, Glazyrin, Matzke, P., Vlasiuk, '19)

Let f be absolutely monotonic of degree $2m$, with $f^{(2m+1)}(t) \leqslant 0$ for $t \in (-1,1)$. Then

$$\mu_{\mathcal{C}} = \frac{1}{|\mathcal{C}|} \sum_{x \in \mathcal{C}} \delta_x$$

for a tight $2m$-design $\mathcal{C}$ is a minimizer of

$$I_f(\mu) = \int_\Omega \int_\Omega f(\cos\vartheta(x,y))\, d\mu(x) d\mu(y)$$

over $\mathcal{P}(\Omega)$, the set of probability measures on Ω.

Linear programming bounds

If $\mathcal{C}$ is a t-design, $\mathcal{A}(\mathcal{C}) = \{2|\langle x, y\rangle|^2 - 1 \mid x, y \in \mathcal{C}\}$, and h is a positive-definite polynomial, with $f(s) = h(s)$, $s \in \mathcal{A}(\mathcal{C})$, $h \leqslant f$, and $deg\ h \leqslant t$, then for $\mu \in \mathcal{P}(\Omega)$,

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$$I_f(\mu) \geqslant I_h(\mu) \qquad \text{by } f \geqslant h$$

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$$
\begin{aligned}
I_f(\mu) &\geqslant I_h(\mu) && \text{by } f \geqslant h \\
&\geqslant I_h(\sigma) && \text{by } h \text{ positive definite}
\end{aligned}
$$

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\begin{aligned}
I_f(\mu) &\geqslant I_h(\mu) && \text{by } f \geqslant h \\
&\geqslant I_h(\sigma) && \text{by } h \ \text{ positive definite} \\
&= I_h\left(\frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}}\delta_x\right) && \text{by } \mathcal{C} \ \text{ being a } \ t - \text{design}
\end{aligned}
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&= I_f\left(\frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}}\delta_x\right) && \text{by } \ f = h \ \text{ holding on } \mathcal{A}(\mathcal{C})
\end{aligned}$$

Linear programming bounds

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&\geqslant I_h(\sigma) && \text{by } h \text{ positive definite} \\
&= I_h\left(\frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}} \delta_x\right) && \text{by } \mathcal{C} \text{ being a } t-\text{design} \\
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\end{aligned}$$

When is our argument sharp?

Linear programming bounds

If $\mathcal{C}$ is a t-design, $\mathcal{A}(\mathcal{C}) = \{2|\langle x, y\rangle|^2 - 1 \mid x, y \in \mathcal{C}\}$, and h is a positive-definite polynomial, with $f(s) = h(s)$, $s \in \mathcal{A}(\mathcal{C})$, $h \leqslant f$, and $deg\ h \leqslant t$, then for $\mu \in \mathcal{P}(\Omega)$,

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I_f(\mu) &\geqslant I_h(\mu) && \text{by } f \geqslant h \\
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&= I_f\left(\frac{1}{|\mathcal{C}|}\sum_{x\in\mathcal{C}} \delta_x\right) && \text{by } f = h \text{ holding on } \mathcal{A}(\mathcal{C})
\end{aligned}$$

When is our argument sharp? Precisely when $\mathcal{C}$ is a tight t-design.

p-Frame energies

Corollary (Bilyk, Glazyrin, Matzke, P., Vlasiuk, '19)

If there exists a tight m-design on Ω and $p \in (2m-2, 2m)$, then minimizers of the p-frame energy, $I_{|t|^p}(\mu)$ are uniquely supported on tight m-designs.

Theorem (Bilyk, Glazyrin, Matzke, P., Vlasiuk, '19)

The 600-cell minimizes the p-frame energy for $p \in (8, 10)$ over probability measures on $\mathbb{S}^3$.

Three point energies

Let f be $O(d)$ invariant, so that $f(Qx, Qy, Qz) = f(x, y, z)$ for all $Q \in O(d)$ and let

$$I_f(\mu) = \iiint_{\mathbb{S}^{d-1}} f(\langle x, y\rangle, \langle y, z\rangle, \langle z, x\rangle) d\mu(x) d\mu(y) d\mu(z),$$

Question: For a given kernel f, what do minimizers of $I_f(\mu)$ look like?

Three point energies: Ex. conjecturally and proved optima

f	$\mathbb{S}^1$	$\mathbb{S}^2$	$\mathbb{S}^{d-1}$
$\lvert\langle x,y\rangle\langle y,z\rangle\langle z,x\rangle\rvert^p,\ p\in[0,1]$	ONB$^{(?)}$	ONB$^{(?)}$	ONB$^{(?)}$
$\lvert\langle x,y\rangle\langle y,z\rangle\langle z,x\rangle\rvert^{2k},\ k\in\mathbb{N}$	σ	σ	σ
$-\lvert(x-z)\times(y-z)\rvert$	σ	$\sigma^{(?)}$	$\sigma^{(?)}$
$-\lvert(x-z)\times(y-z)\rvert^2$	σ	σ	σ
$-\det^2\begin{pmatrix} \vert & \vert & \vert \\ x & y & z \\ \vert & \vert & \vert \end{pmatrix}$	σ	σ	σ
$-\left\lvert\det\begin{pmatrix} \vert & \vert & \vert \\ x & y & z \\ \vert & \vert & \vert \end{pmatrix}\right\rvert$	ONB$^{(?)}$	ONB$^{(?)}$	ONB$^{(?)}$

(E_8 minimal vectors for $\lvert\langle x,y\rangle\langle y,z\rangle\langle z,x\rangle\rvert^p$, $p\in[1,2]$, and $\mathbb{S}^7$?)

Positive definite three point potentials

A three-point potential function f is said to be *positive definite* on the sphere if, for all $z \in \mathbb{S}^{d-1}$, $f_z = f(\cdot, \cdot, z)$ is positive definite, i.e. for every signed Borel measure ν supported on $\mathbb{S}^{d-1}$,

$$I_{f_z}(\nu) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} f_z(x, y) d\nu(x) d\nu(y) \geqslant 0.$$

Potential function f is *conditionally positive definite* if for all $z \in \mathbb{S}^{d-1}$, and for every finite mean-zero signed Borel measure on $\mathbb{S}^{d-1}$, $I_{f_z}(\nu) \geqslant 0$.

Expansions for three point potentials

$$\mathrm{Harm}_k^d = \{f \in \mathbb{R}[x_1, \ldots, x_d] \mid f \text{ homogeneous}, \ \deg f = k, \ \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0\}$$

$$H = Stab(O(d), e)$$

Suppose there exist matrices $Z_k^e(x, y)$ for which

- $\forall x, y \in \mathbb{S}^{d-1}$ and $\forall g \in H$, $Z_k^e(g(x), g(y)) = Z_k^e(x, y)$,
- for all finite $\mathcal{C} \subset \mathbb{S}^{d-1}$, $\sum_{(c,c') \in \mathcal{C}^2} Z_k^e(c, c') \succeq 0$, *and*
- the entries of $Z_k^e(x, y)$ are polynomials.

Then for f $O(d)$-invariant, if

$$f(x, y, z) = \text{const.} + \sum_k \langle A_k, Z_k^z(x, y)\rangle, \ A_k \succeq 0,$$

$$I_f(\mu) = \iiint_{\mathbb{S}^{d-1}} f(\langle x, y\rangle, \langle y, z\rangle, \langle z, x\rangle) d\mu(x) d\mu(y) d\mu(z) \geqslant const.$$

Expansions for three point potentials

Theorem (Bachoc-Vallentin '07)

Such $Z_k^z(x,y)$ exist and may be taken of the form

$$Y_k^d(\langle z,x\rangle,\langle z,y\rangle,\langle x,y\rangle) = Y_k^d(u,v,t),\ \textit{where},\ \forall\ 0 \leqslant i,j \leqslant d-k,$$

$$(Y_k^d)(u,v,t) =$$

$$C_i^{d+2k}(u)C_j^{d+2k}(v)((1-u^2)(1-v^2))^{k/2}C_k^{d-1}\left(\frac{t-uv}{\sqrt{(1-u^2)(1-v^2)}}\right).$$

Determinant squared example

$$\det \begin{pmatrix} | & | & | \\ x & y & z \\ | & | & | \end{pmatrix} \cdot \det \begin{pmatrix} | & | & | \\ x & y & z \\ | & | & | \end{pmatrix} = \det \begin{pmatrix} 1 & \langle x, y \rangle & \langle x, z \rangle \\ \langle y, x \rangle & 1 & \langle y, z \rangle \\ \langle z, x \rangle & \langle z, y \rangle & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & t & u \\ t & 1 & v \\ u & v & 1 \end{pmatrix} = 1 - u^2 - v^2 - t^2 + 2tuv$$

where $u = \langle z, x \rangle$, $v = \langle z, y \rangle$, and $t = \langle x, y \rangle$ as before. The relevant submatrices of Y_0^3, Y_1^3, and Y_2^3 are then,

$$K_0 = \begin{pmatrix} uv & u\frac{3v^2-1}{2} \\ \frac{3u^2-1}{2}v & \frac{3u^2-1}{2}\frac{3v^2-1}{2} \end{pmatrix}, K_1 = \begin{pmatrix} t - uv & u(t-uv) \\ v(t-uv) & uv(t-uv) \end{pmatrix},$$

$$\text{and } K_2 = \left(2(t-uv)^2 - (1-u^2)(1-v^2) \right).$$

Determinant squared example

Symmetrizing and relabeling $K_0 \to K_0'$, $K_1 \to K_1'$, $K_2 \to K_2'$,

$$f(\langle x,y\rangle,\langle z,x\rangle,\langle z,y\rangle) = f(u,v,t) = \frac{2}{9} - \frac{1}{27}K_0' - \frac{2}{9}K_1' - \frac{5}{36}K_2',$$

$$\text{so that } I_f(\mu) = \frac{2}{9} - \frac{1}{27}I_{K_0'}(\mu) - \frac{2}{9}I_{K_1'}(\mu) - \frac{5}{36}I_{K_2'}(\mu) \leqslant \frac{2}{9}.$$

Finally, one may check that equality hold above precisely for isotropic measures on the sphere (and for σ in particular).

Surface measure minimizes cond. positive definite energies

Theorem (BFGMPV '20)

Suppose f is conditionally positive definite on $\mathbb{S}^{d-1}$. Then σ is a minimizer of I_f over $\mathcal{P}(\mathbb{S}^{d-1})$.

An example of a function which is conditionally positive definite but not positive definite is $f(x, y, z) = \langle x, y\rangle + \langle y, z\rangle + \langle z, x\rangle$.

Proposition

Let $k \in \mathbb{N}$ and $f(x, y, z) = (\langle x, y\rangle\langle y, z\rangle\langle z, x\rangle)^k$. Then f is conditionally positive definite.

Three point frame energies

Theorem (BFGMPV '20)

Surface measure minimizes I_f for $f(x,y,z) = (\langle x,y\rangle\langle y,z\rangle\langle z,x\rangle)^k$, and in particular equal weights on any spherical 2k-design gives a minimizer for I_f.