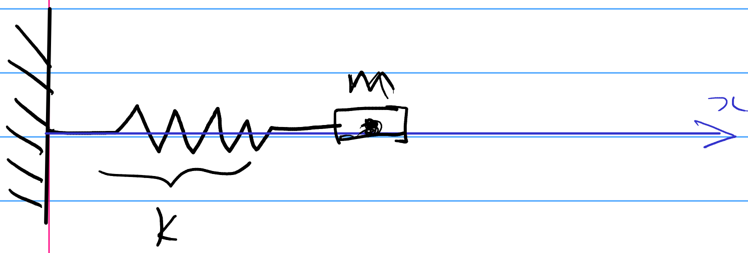
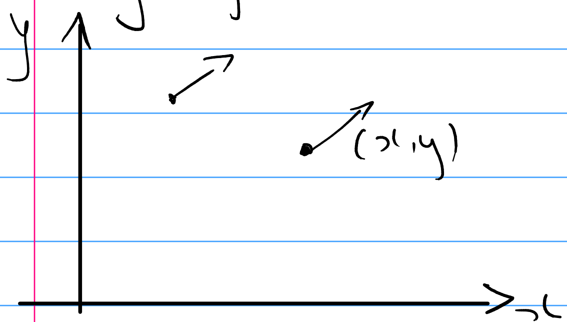
Section 9.3: Separable differential equations



$$\underbrace{\underbrace{F}_{ma}}_{mx''} = -k\underbrace{x}_{\text{deformation}}$$

$$mx'' = -kx \qquad x = x(t)$$

$$y = y(x)$$



$$\frac{dy}{dx} = F(x,y) \quad \Big\} \text{ general differential equation}$$

It is <u>separable</u>, if

$$F(x,y) = g(x)\cdot f(y),$$

$$\frac{dy}{dx} = g(x)\cdot f(y) = \frac{g(x)}{h(y)} \qquad h(y) = \frac{1}{f(y)}$$

We can separate the variables:

$$h(y)\,dy = g(x)\,dx$$

Now, integrate both sides:

$(*)$ $\int h(y)\,dy = \int g(x)\,dx$

↳ solve for $y = y(x)$

To see that $y(x)$ from $(*)$ solves the original equation, differentiate equation $(*)$:

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\int g(x)\,dx$$

(Chain rule:)

$$\frac{d}{dy}\left(\int h(y)\,dy\right)\cdot\frac{dy}{dx} = g(x)$$

$$h(y)\cdot\frac{dy}{dx} = g(x)$$

Solving a seperable differential equation

1. Separate variables
$h(y)\,dy = g(x)\,dx$

2. Integrate both sides
$\int h(y)\,dy = \int g(x)\,dx$

3. Solve for y $\qquad\qquad y = y(x)$

Ex. 1 Solve:

$$\frac{dy}{dx} = \frac{x^2}{y^2} \qquad \underbrace{y(0) = 2}_{\text{initial condition}}$$

Separate variables:

$$y^2\,dy = x^2\,dx$$

Integrate:

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

Solve for y:

$$y^3 = x^3 + 3C$$

$$y = \sqrt[3]{x^3 + 3C}$$

$$2 = \sqrt[3]{0^3 + 3c}$$
$$8 = 3c$$
$$c = 8/3$$

Ans: $y(x) = \sqrt[3]{x^3 + 8}$

Ex. 2 Solve
$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$

Separate variables:
$$(2y + \cos y)\,dy = 6x^2\,dx$$

Integrate:
$$\int (2y + \cos y)\,dy = \int 6x^2\,dx$$
$$y^2 + \sin y = 2x^3 + C \quad \} \text{ Ans}$$

Solve for y – ✗

Solve for x:
$$x^3 = \frac{1}{2}(y^2 + \sin y) \underbrace{- C/2}_{+C_1}$$
$$x = \sqrt[3]{\frac{1}{2}(y^2 + \sin y) + C_1}$$

Ex.3 Solve $y' = x^2 y$

$$\frac{dy}{dx} = x^2 y$$

$$\frac{dy}{y} = x^2\,dx$$

$$\int \frac{dy}{y} = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

$$|y| = e^{\ln|y|} = e^{\frac{x^3}{3}+C} = e^{\frac{x^3}{3}} \cdot \underbrace{e^{C}}_{K>0} = K \cdot e^{x^3/3}$$

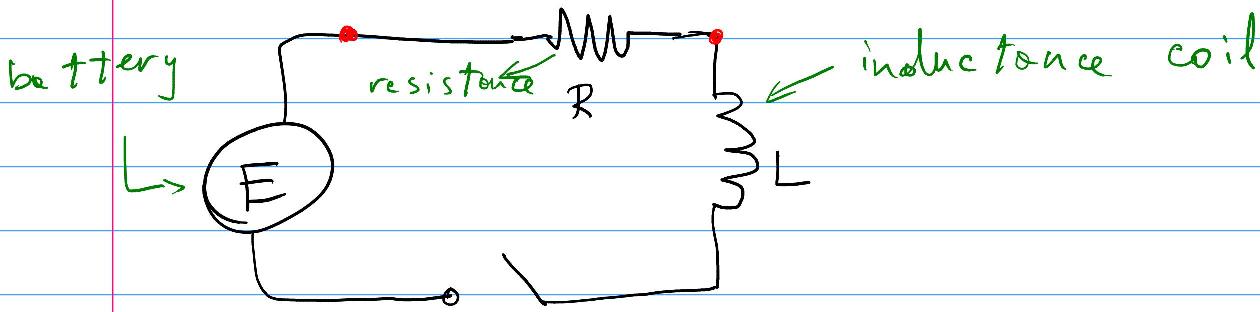
$$|y| = K \cdot e^{x^3/3} \qquad K>0$$

$$y = \pm K e^{x^3/3}$$

$$y = K e^{x^3/3} \qquad K \neq 0$$

Ans: $y = K \cdot e^{x^3/3}$ $\quad K$ in $(-\infty, \infty)$

Ex. 4 Current in an electric circuit



$L = 4\ H \qquad R = 12\,\Omega \qquad E(t) = 60\ V$

$I(0) = 0.$

$$R \cdot I = \underbrace{E(t) - L\frac{dI}{dt}}_{\text{Faraday's law}}$$

$$12 \cdot I = 60 - 4 \cdot \frac{dI}{dt} \qquad /4$$

$$3I = 15 - \frac{dI}{dt}$$

$$\frac{dI}{dt} = 15 - 3I = 3(5 - I)$$

$$\frac{dI}{5 - I} = 3\,dt$$

$$\int \frac{dI}{5 - I} = \int 3\,dt$$

$$-\ln|5 - I| = 3t + C_1$$

$$\ln|5 - I| = -3t + \underbrace{C}_{-C_1}$$

$$e^{\ln|5-I|} = e^{-3t+C} = \underbrace{e^{C}}_{K>0} \cdot e^{-3t}$$

$$|5-I| = Ke^{-3t} \quad (K>0)$$

$$5-I = \pm Ke^{-3t} \quad (K>0)$$

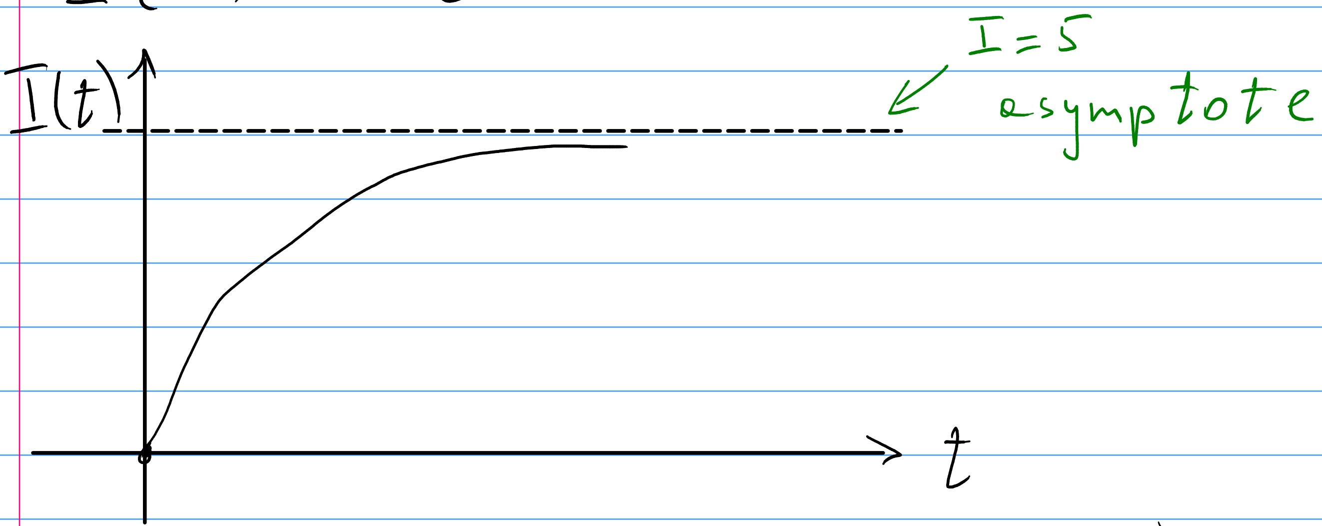
$$5-I = k\cdot e^{-3t} \quad k \neq 0$$

$$I = 5 - k\cdot e^{-3t} \quad k \text{ in } (-\infty, \infty)$$

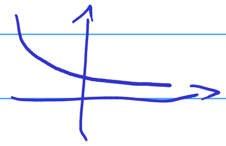
$$0 = I(0) = 5 - k\cdot e^{0} = 5-k$$

$$\Rightarrow k = 5$$

$$I(t) = 5 - 5e^{-3t}.$$



$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}\left(5 - \underset{\downarrow 0}{5e^{-3t}}\right) = 5$$

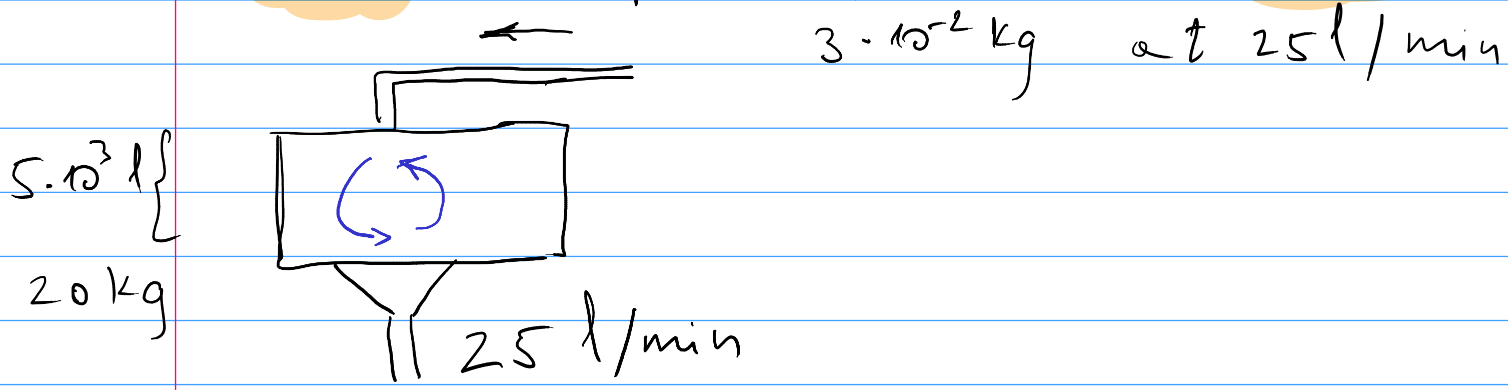


Mixing problems

Ex. 5

A tank contains 20 kg of salt dissolved in $5\cdot 10^3$ l of water. Brine w/ $3\cdot 10^{-2}$ kg of salt / 1 l enters the tank at 25 l/min. After mixing, the solution drains, also at 25 l/min.

Find: mass of salt in the tank in 30 min



$m(t)$ } mass of salt in the tank

$$\frac{dm}{dt} = 3\cdot 10^{-2}\cdot 25 - 25\cdot\frac{m(t)}{5\cdot 10^3}$$

$$\frac{dm}{dt} = \frac{3\cdot 25}{100} - \frac{m}{200} = \frac{150-m}{200}$$

$$\int\frac{dm}{150-m} = \int\frac{dt}{200}$$

$$-\ln|150-m| = \frac{t}{200} + C_1$$

$$e^{\ln|150-m|} = e^{-t/200 + C}$$

$$|150-m| = \underbrace{K}_{e^C>0} \cdot e^{-t/200}$$

$$150 - m = K e^{-t/200}$$

$$m(t) = 150 - K \cdot e^{-t/200}$$

Initial condition: $m(t) = 20$

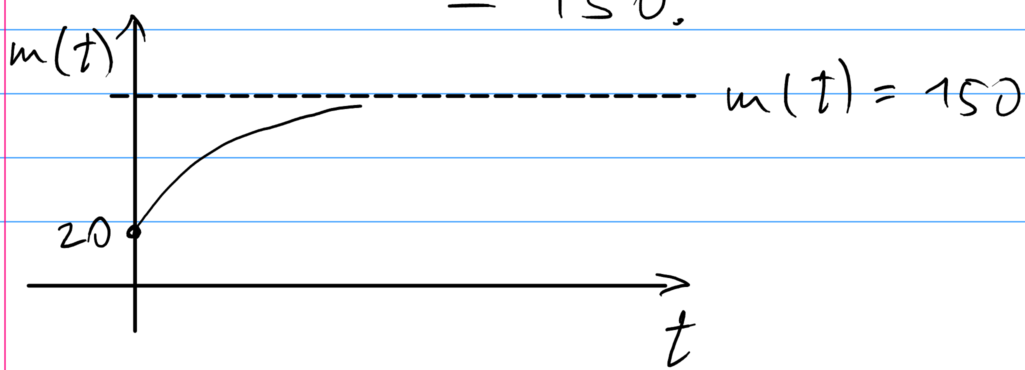
$$20 = m(0) = 150 - K \cdot \underbrace{e^{-0/200}}_{1} = 150 - K$$

$$\Rightarrow K = 130$$

Ans: $m(t) = 150 - 130 \cdot e^{-t/200}$

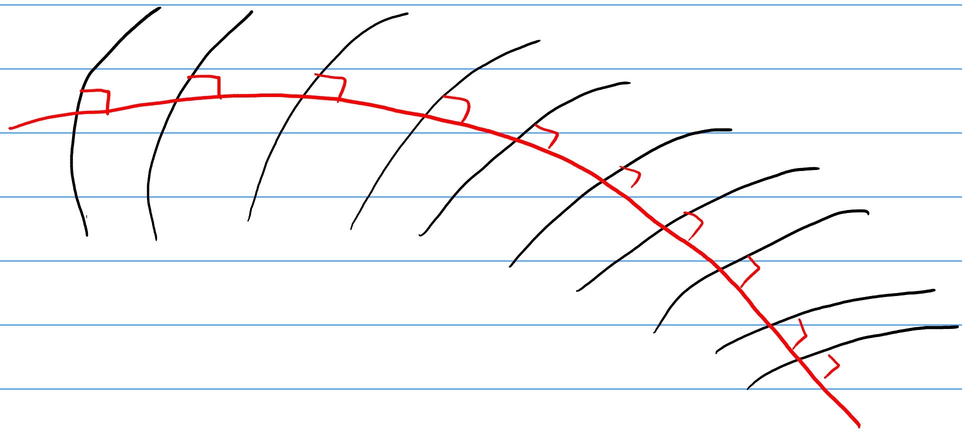
$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty}\left(150 - 130 \cdot \underbrace{e^{-t/200}}_{\downarrow 0}\right)$$

$$= 150.$$

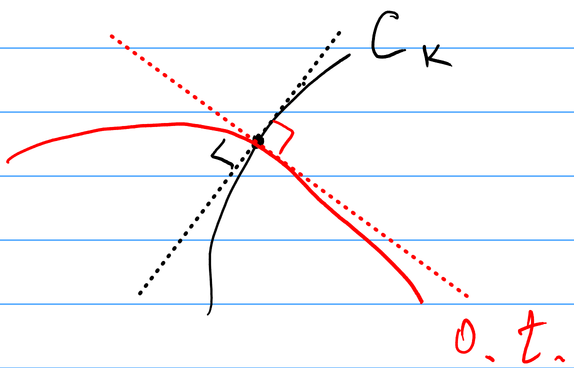


Orthogonal trajectories

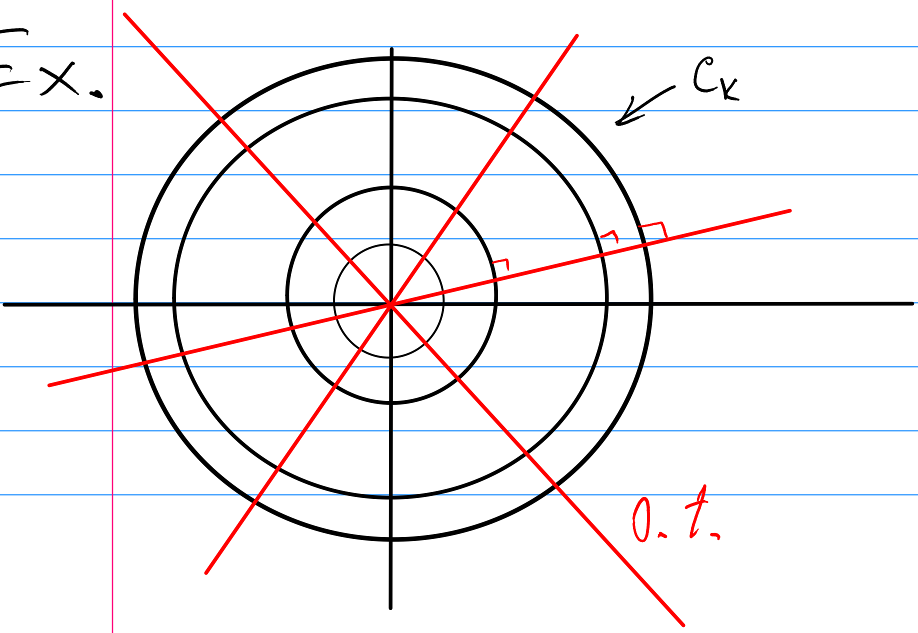
An o.t. of a family of curves $\{C_k\}_k$ is a curve C, intersecting each of C_k at the right angle (= orthogonally)



Angle between curves = angle between their tangents at the point of intersection



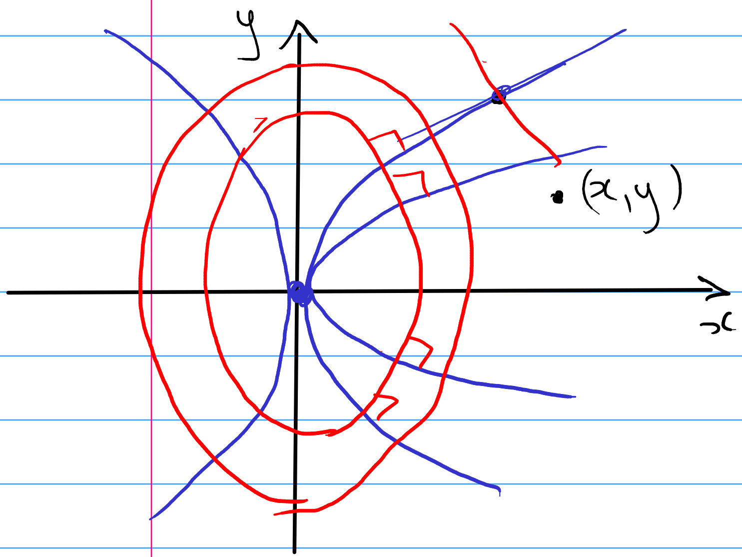
Ex.



$$\overbrace{x^2+y^2=k^2}^{\text{original curves}}$$

$$\underbrace{y=mx}_{\text{o.t.}}$$

Ex. 6 Find orthogonal trajectories to the family $x = ky^2$, k in $(-\infty, \infty)$



Idea: find slope of the tangent to the curve passing through (x,y); say m_1. Then, slope of the o.t through (x,y) m_2 is such that

$$m_1 \cdot m_2 = -1$$

1) Differentiate equation of the family:

$$\frac{d}{dx}\left(x = ky^2\right)$$

$$1 = k \cdot 2y y' = k \cdot 2y \cdot \frac{dy}{dx}$$

2). Eliminate k:

$$x = ky^2 \implies k = \frac{x}{y^2}, \quad \text{so}$$

$$1 = \frac{x}{y^2} \cdot 2y \cdot \frac{dy}{dx}$$

$$1 = \frac{2x}{y} \cdot \frac{dy}{dx}$$

So, slope of a blue curve through (x,y) is

$$\frac{dy}{dx} = \frac{y}{2x} = m_1$$

3). Take negative inverse to obtain the DE for orthogonal trajectory

eqn for o.t.

$$\frac{dy}{dx} = -\frac{2x}{y} = -\frac{1}{m_1} = m_2$$

4). Solve the DE from 3).;

$$\frac{dy}{dx} = -\frac{2x}{y}$$

$$y\,dy = -2x\,dx$$

$$\frac{y^2}{2} = -x^2 + C$$

Ans: $\underbrace{x^2 + \frac{y^2}{2} = C}$ } ellipses

Equation of the OT.

Say, the OT through $(1,2)$ is

obtained by using $x \mapsto 1$, $y \mapsto 2$:

$$1 + \frac{2^2}{2} = C \implies C = 3$$

Then this specific OT is

$$x^2 + \frac{y^2}{2} = 3.$$