

## Section 7.4: Integrating rational functions via partial fractions.

Rational functions:

$$f(x) = \frac{P(x)}{Q(x)} \quad P, Q - \text{polynomials}$$

$$\left\{ \text{rational function} \right\} \longrightarrow \left\{ \text{sum of partial fractions} \right\}$$

$$\underbrace{\frac{x^3 - x^2 + 8x}{(x^2 + 1)(x - 2)^2}}_{\text{rational function}} = \underbrace{\frac{x - 1}{x^2 + 1}}_{\text{partial fractions}} + \underbrace{\frac{4}{(x - 2)^2}}_{\text{partial fractions}}$$

$$f(x) = \frac{P(x)}{Q(x)}$$

$f$  is proper, if  $\deg P < \deg Q$   
 $f$  is improper, if  $\deg P \geq \deg Q$ .

Partial fraction decomposition applies only to proper rational functions!

For an improper  $f$ : apply the long division:

$$f(x) = \frac{P(x)}{Q(x)} \xrightarrow{\text{long division}} S(x) + \frac{R(x)}{Q(x)} \left\{ \begin{array}{l} \text{proper} \\ \text{rational} \\ \text{function} \end{array} \right.$$

# Outline of the PFD:

Take  $f = \frac{P(x)}{Q(x)}$  - proper

1. Factor  $Q(x)$ ,  $\rightarrow$  2 types of irreducible factors:  $(ax+b)$  OR  $(ax^2+bx+c)$  w/  $b^2-4ac < 0$

2. Make a sum of partial fractions: for each factor of the form in the left column, add terms from the right column:

Terms of $Q(x)$	Terms in PFD
$(ax+b)$	$\frac{A}{ax+b}$
$(ax+b)^r$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_r}{(ax+b)^r}$
$(ax^2+bx+c)$ w/ $b^2-4ac < 0$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^r$ w/ $b^2-4ac < 0$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$

3. Equate

$$f(x) = \text{PFD},$$

then determine the undefined coefficients

## Evaluate integrals:

Ex. 1  $\int \frac{x^3 + x}{x-1} dx = \int \left( x^2 + x + 2 + \frac{2}{x-1} \right) dx \quad (\Rightarrow)$

improper rational f-n

Perform the long division

$$\begin{array}{r} x^2 + x + 2 \\ x-1 \overline{) x^3 + x} \\ \underline{x^3 - x^2} \phantom{+ 2} \\ 2x^2 + x \phantom{+ 2} \\ \underline{2x^2 - 2x} \phantom{+ 2} \\ 3x \phantom{+ 2} \\ \underline{3x - 3} \\ 3 \end{array}$$

$$(\Rightarrow) \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln |x-1| + C$$

Ex. 2  $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$  } Case I:  
unique linear factors

proper rational function

$$\begin{aligned} 2x^3 + 3x^2 - 2x &= x(2x^2 + 3x - 2) \\ &= x \cdot 2(x - (-2))(x - \frac{1}{2}) \\ &= \underline{x} \cdot (\underline{x+2})(\underline{2x-1}) \end{aligned}$$

$$2x^2 + 3x - 2 = 0$$

$$x_{1,2} = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm 5}{4}$$

$$x_1 = -2$$

$$x_2 = \frac{1}{2}$$

$$\frac{x^2 + 2x - 1}{x(x+2)(2x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}$$

Multiply both sides by  $x(x+2)(2x-1)$ :

$$x^2 + 2x - 1 = A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2)$$

For unique linear factors: plug in roots of the denominator

$$x = -2:$$

$$-1 = B(-2)(-5) \Rightarrow B = -\frac{1}{10}$$

$$x = 0:$$

$$-1 = A \cdot 2(-1) \Rightarrow A = \frac{1}{2}$$

$$x = \frac{1}{2}:$$

$$\frac{1}{4} = C \cdot \frac{1}{2} \cdot \frac{5}{2} \Rightarrow C = \frac{1}{5}$$

$$\int \left( \frac{1}{2x} - \frac{1}{10(x+2)} + \frac{1}{5(2x-1)} \right) dx$$

$$= \frac{1}{2} \ln|x| - \frac{1}{10} \ln|x+2| + \frac{1}{5} \cdot \frac{1}{2} \ln|2x-1| + K$$

Ex. 3  $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

improper rational function

} Case II;  
repeated  
linear factors

Long division:

$$\begin{array}{r} x+1 \\ x^3 - x^2 - x + 1 \overline{) x^4 - 2x^2 + 4x + 1} \\ \underline{x^4 - x^3 - x^2 + x} \phantom{+ 1} \\ x^3 - x^2 + 3x + 1 \\ \underline{x^3 - x^2 - x + 1} \\ 4x \end{array}$$

$$\int \left( x+1 + \frac{4x}{x^3 - x^2 - x + 1} \right) dx$$

proper

Factor the denominator:

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x^3 - x^2) - (x - 1) \\ &= x^2(x - 1) - (x - 1) \\ &= (x - 1)(x^2 - 1) = (x - 1)(x - 1)(x + 1) \\ &= (x - 1)^2(x + 1) \end{aligned}$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiply by the common denominator:

$$4x = A \underbrace{(x-1)^2}_{x^2 - 2x + 1} + B \underbrace{(x-1)(x+1)}_{x^2 - 1} + C(x+1)$$

Equate the coefficients at corresponding powers of  $x$ :

$$x^2:$$

$$0 = A + B$$

$$\Rightarrow \underline{A = -B}$$

$$x:$$

$$\underline{4 = -2A + C}$$

$$x^0:$$

$$\underline{0 = A - B + C}$$

$$4 = -2A + C$$

$$0 = A - (-A) + C$$

$$\begin{cases} 4 = -2A + C \\ 0 = 2A + C \end{cases}$$

$$\Rightarrow 4 = 2C$$

$$C = 2$$

$$A = -1$$

$$B = 1$$

$$\int \left( x+1 + \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx$$

$$= \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + K$$

Ex. 4 Compute

$$\int \underbrace{\frac{2x^2 - x + 4}{x^3 + 4x}}_{\text{proper rational function}} dx$$

Case III:  
unique (unrepeated)  
irreducible quadratic  
factors

Factor the denominator:

$$x^3 + 4x = x(x^2 + 4)$$

$$D = 0^2 - 4 \cdot 4 = -16 < 0$$

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiply by the common denominator:

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

Equating the coefficients:

$x^2$ :

$$2 = A + B \Rightarrow B = 1$$

$x^1$ :

$$-1 = C \Rightarrow C = -1$$

$x^0$ :

$$4 = 4A \Rightarrow A = 1$$

$$\int \left( \frac{1}{x} + \frac{x-1}{x^2+4} \right) dx = \int \left( \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4} \right) dx$$

$$= \ln|x| + \frac{1}{2} \ln|x^2+4| - \frac{1}{2} \arctan\left(\frac{x}{2}\right) + K$$

$$\int \frac{x}{x^2+4} dx = \left| \begin{array}{l} u = x^2+4 \\ du = 2x dx \\ \frac{1}{2} du = x dx \end{array} \right| = \int \frac{\frac{1}{2} du}{u} =$$

$$= \frac{1}{2} \ln|u| + K_1 = \frac{1}{2} \ln|x^2+4| + K_1$$

$$\int \frac{1}{x^2+a^2} dx = \int \frac{1}{a^2 \left( \left( \frac{x}{a} \right)^2 + 1 \right)} dx$$

$$= \frac{1}{a^2} \int \frac{1}{\left( \frac{x}{a} \right)^2 + 1} dx = \left| \begin{array}{l} u = \frac{x}{a} \\ du = dx/a \\ a du = dx \end{array} \right| =$$

$$= \frac{1}{a^2} \int \frac{1}{u^2+1} a du = \frac{1}{a} \int \frac{1}{1+u^2} du$$

$$= \frac{1}{a} \arctan(u) + K_2 = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + K_2$$

Ex 5.

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}$$

proper rational function

Case IV:  
repeated irreducible quadratic factors

$$\frac{-x^3+2x^2-x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply by the common denominator:

$$-x^3+2x^2-x+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$x=0:$$

$$1 = A \Rightarrow A = 1$$



Equate coefficients at similar powers of  $x$ :

$$x^4:$$

$$0 = A + B \Rightarrow B = -1$$

$$x^3:$$

$$-1 = C \Rightarrow C = -1$$

$$x^2:$$

$$2 = 2A + B + D \Rightarrow D = 1$$

$$x^1:$$

$$-1 = C + E = E = 0$$

$$x^0:$$

$$1 = A \Rightarrow A = 1$$

$$\int \left( \frac{1}{x} + \frac{-1 \cdot x - 1}{x^2 + 1} + \frac{1 \cdot x}{(x^2 + 1)^2} \right) dx$$

$$= \int \left( \frac{1}{x} - \frac{x}{x^2 + 1} - \frac{1}{x^2 + 1} + \frac{x}{(x^2 + 1)^2} \right) dx$$

$$= \ln|x| - \frac{1}{2} \ln|x^2 + 1| - \arctan x - \frac{1}{2(x^2 + 1)} + k$$

$$\int \frac{x}{(x^2 + 1)^2} dx = \left| \begin{array}{l} u = x^2 + 1 \\ du = 2x dx \\ \frac{1}{2} du = x dx \end{array} \right|$$

$$= \int \frac{\frac{1}{2} du}{u^2} = \frac{1}{2} \int u^{-2} du = \frac{1}{2} \frac{u^{-1}}{-1} + k_0$$

$$= -\frac{1}{2} \frac{1}{x^2 + 1} + k_0$$

Ex. 5'  $\int \frac{1}{(x^2+1)^2} dx = \left| \begin{array}{l} x = \tan \theta \\ dx = \sec^2 \theta d\theta \end{array} \right|$

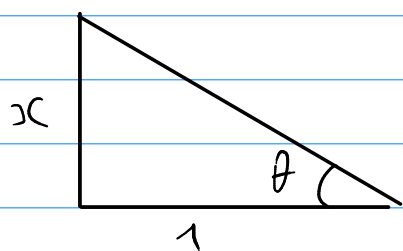
$$= \int \frac{1}{(\underbrace{\tan^2 \theta + 1}_{\sec^2 \theta})^2} \cdot \sec^2 \theta d\theta = \int \frac{\sec^2 \theta d\theta}{(\sec^2 \theta)^2}$$

$$= \int \frac{d\theta}{\sec^2 \theta} = \int \cos^2 \theta d\theta = \left| \begin{array}{l} \text{half-angle} \\ \text{formula} \end{array} \right|$$

$$= \frac{1}{2} \int (1 + \cos 2\theta) d\theta = \frac{1}{2} \left( \theta + \frac{\sin 2\theta}{2} \right) + K$$

$$= \frac{\theta}{2} + \frac{1}{2} \sin \theta \cos \theta + K$$

$$\sin 2A = 2 \sin A \cos A$$



$$\theta = \arctan x$$

$$x = 1 \cdot \tan \theta$$

$$\text{opp} = \text{adj} \cdot \tan \theta$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+1}}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{x^2+1}}$$

$$\equiv \frac{1}{2} \left( \arctan x + \frac{x}{x^2+1} \right) + K.$$

Ex. 6

$$\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} dx = \int \frac{(4x^2 - 4x + 3) + x - 1}{4x^2 - 4x + 3} dx$$

improper

$$= \int \left( 1 + \frac{x-1}{4x^2 - 4x + 3} \right) dx$$

proper

(=)

$$\Delta = (-4)^2 - 4 \cdot 4 \cdot 3 = -2 \cdot 16 = -32 < 0$$

$$\frac{x-1}{4x^2-4x+3} = \frac{Ax+B}{4x^2-4x+3}$$

$$\int \frac{x-1}{4x^2-4x+3} dx \stackrel{\text{complete the square}}{=} \int \frac{x-1}{(2x)^2 - 2 \cdot 2x + 1^2 + 2} dx$$

$$= \int \frac{x-1}{(2x-1)^2 + 2} dx = \left| \begin{array}{l} u = 2x-1 \\ du = 2 dx \\ \frac{1}{2} du = dx \\ x = \frac{u+1}{2} \end{array} \right|$$

$$= \int \frac{\frac{u+1}{2} - 1}{u^2 + 2} \cdot \frac{1}{2} du = \frac{1}{4} \int \frac{u-1}{u^2 + 2} du$$

$$= \frac{1}{4} \int \frac{u}{u^2 + 2} du - \frac{1}{4} \int \frac{du}{u^2 + 2}$$

$$= \frac{1}{8} \ln |u^2 + 2| - \frac{1}{4} \frac{1}{\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) + K$$

$$= \frac{1}{8} \ln |(2x-1)^2 + 2| - \frac{1}{4\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) + K.$$

(=)

$$x + \frac{1}{8} \ln |(2x-1)^2 + 2| - \frac{1}{4\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) + K.$$

# Rationalizing substitutions

$\sqrt[n]{g(x)} \mapsto$  apply  $u = \sqrt[n]{g(x)}$   
(guaranteed to work for  $g(x) = ax + b$ )

Ex. 7  $\int \frac{\sqrt{x+3}}{x} dx = \left| \begin{array}{l} u = \sqrt{x+3} \\ u^2 = x+3 \\ 2u du = dx \\ x = u^2 - 3 \end{array} \right| = \int \frac{u}{u^2 - 3} 2u du$

$$= 2 \int \frac{u^2}{u^2 - 3} du = 2 \int \frac{(u^2 - 3) + 3}{u^2 - 3} du = 2 \int \left( 1 + \underbrace{\frac{3}{u^2 - 3}}_{\text{proper}} \right) du$$

$$\frac{3}{(u - \sqrt{3})(u + \sqrt{3})} = \frac{A}{u - \sqrt{3}} + \frac{B}{u + \sqrt{3}}$$

Multiply by the common denominator:

$$3 = A(u + \sqrt{3}) + B(u - \sqrt{3})$$

$$u = \sqrt{3}:$$

$$3 = 2A\sqrt{3} \Rightarrow A = \frac{\sqrt{3}}{2}$$

$$u = -\sqrt{3}:$$

$$3 = -2\sqrt{3}B \Rightarrow B = -\frac{\sqrt{3}}{2}$$

$$2 \int \left( 1 + \frac{\sqrt{3}}{2} \left( \frac{1}{u - \sqrt{3}} - \frac{1}{u + \sqrt{3}} \right) \right) du$$

$$= 2u + \sqrt{3} (\ln |u - \sqrt{3}| - \ln |u + \sqrt{3}|) + K$$

$$= 2u + \sqrt{3} \ln \left| \frac{u - \sqrt{3}}{u + \sqrt{3}} \right| + K$$

$$= 2\sqrt{x+3} + \sqrt{3} \ln \left| \frac{\sqrt{x+3} - \sqrt{3}}{\sqrt{x+3} + \sqrt{3}} \right| + K.$$