SUMMARY OF CALC II

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1. Methods of integration

1.1. Integration by parts.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Definite integral:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Compact notation:

$$\int u\,dv = uv - \int v\,du.$$

1.2. Trigonometric integrals.

Depending on the type of factors in the product, use the following approaches.

Odd positive powers of sine or cosine present.
If an odd power of either function is present: use u-sub for the cofunction, keep one factor of the function, and use the identity $\sin^2 x + \cos^2 x = 1$ to get everything in terms of the cofunction *aka* u.

Only even positive powers of sine and cosine.
Use the half-angle identities to reduce the degree:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

A factor of an even power of secant, or an odd power of tangent.
If secant is even, combine $\sec^2 x\,dx$, let $u = \tan x$. If tangent is odd, combine $\sec x \tan x\,dx$, substitute $u = \sec x$. Then use the identity $\tan^2 x + 1 = \sec^2 x$ to get everything in terms of the respective u.

Power of secant is odd, power of tangent is even.
Use $\tan^2 x + 1 = \sec^2 x$ to rewrite in terms of secant, then apply the reduction formula:

$$\int \sec^n x\,dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, \qquad n \geq 2.$$

If different arguments of the trig functions present in the product.
Reduce to the same angle by using the sum formulas, or use product to sum identities:

$$\begin{aligned}
\sin A \cos B &= \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)\\
\sin A \sin B &= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)\\
\cos A \cos B &= \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right).
\end{aligned}$$

1.3. **Trigonometric substitution.**

Integrals involving the form $a^2 - x^2$.

Substitute $x = a\sin\theta$ $(-\pi/2 \leq \theta \leq \pi/2)$ and use the identity $\sin^2\theta + \cos^2\theta = 1$

Integrals involving the form $a^2 + x^2$.

Substitute $x = a\tan\theta$ $(-\pi/2 < \theta < \pi/2)$ and use the identity $\tan^2\theta + 1 = \sec^2\theta$

Integrals involving the form $x^2 - a^2$.

Substitute $x = a\sec\theta$ $(0 \leq \theta < \pi/2)$ and use the identity $\tan^2\theta + 1 = \sec^2\theta$

Completing the square.

$a(x^2 + bx) = a(x + (b/2))^2 - a(b/2)^2$

1.4. **Partial fractions.**

Decomposing rational functions.

Step 1. Use long division if the degree of the numerator is greater than or equal to the degree of the denominator, i.e. rewrite as follows

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}, \qquad deg(R(x)) < deg(g(x)).$$

Step 2. Completely factor the denominator into linear and irreducible (with negative discriminant) quadratic factors: $(ax+b)^n$ and/or $(ax^2+bx+c)^m$.

Step 3. For each distinct linear factor, $(ax+b)^n$, the partial fraction decomposition will include the sum

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Step 4. For each distinct irreducible quadratic factor, $(ax^2+bx+c)^m$ with $b^2 - 4ac < 0$, the partial fraction decomposition will include the sum

$$\frac{A_1x + B_1}{(ax^2+bx+c)} + \frac{A_2x + B_2}{(ax^2+bx+c)^2} + \frac{A_3x + B_3}{(ax^2+bx+c)^3} + \cdots + \frac{A_mx + B_m}{(ax^2+bx+c)^m}$$

Rationalizing substitution.

If the integrand involves a higher-order root $\sqrt[n]{f(x)}$, make a substitution of the form $u = \sqrt[n]{f(x)}$ to change the integrand into a rational function of u. This substitution is guaranteed to work for roots of the form $\sqrt[n]{ax+b}$, that is, when the function inside the root is linear.

1.5. **Integration tables.**

Generally one has to rewrite the integrand or make a substitution before a standard integral may be applied. On an exam you may be given a selection of standard integrals, and you will have to integrate a given function using the appropriate one.

1.6. **Approximate integration.**

Midpoint rule.
Use a Riemann sum where the x_i^* are chosen to be the midpoints of the respective subintervals:

$$\int_a^b f(x)\,dx \approx \Delta x[f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)].$$

Trapezoidal rule.
Rather than rectangles, the area is approximated using trapezoids.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)],$$
$$x_i = a + i\Delta x.$$

Simpson's rule.
For Simpson's Rule quadratic approximations for the curve in each subinterval are used to approximate the integral. Number n must be even for this method.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Midpoint rule error.

$$|error| \le \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \le K$ for all $a \le x \le b$.

Trapezoid rule error.

$$|error| \le \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \le K$ for all $a \le x \le b$.

Simpson's rule error.

$$|error| \le \frac{K(b-a)^5}{180n^4}$$

where $|f^{(4)}(x)| \le K$ for all $a \le x \le b$.

1.7. **Improper integrals.**

We define an improper integral using limits as shown below. It is said to be *convergent* if the limit exists (as a finite real number). Otherwise, the integral is called *divergent*.

Improper integrals of the Ist kind: infinite bounds.

(1) $\int_a^\infty f(x)\,dx = \lim_{n\to\infty} \int_a^n f(x)\,dx$
(2) $\int_{-\infty}^b f(x)\,dx = \lim_{n\to-\infty} \int_n^b f(x)\,dx$
(3) $\int_{-\infty}^\infty f(x)\,dx = \lim_{m\to-\infty} \int_m^a f(x)\,dx + \lim_{n\to\infty} \int_a^n f(x)\,dx$ (if both limits exist)

Improper integrals of the IInd kind: infinite function values.

(1) $f(x)$ has a discontinuity at $x = a$. Then $\int_a^b f(x)\,dx = \lim_{n\to a^+} \int_n^b f(x)\,dx$

(2) $f(x)$ has a discontinuity at $x = b$. Then $\int_a^b f(x)\,dx = \lim_{n\to b^-} \int_a^n f(x)\,dx$
(3) $f(x)$ has a discontinuity at $x = c$ and $a < c < b$. Then

$$\int_a^b f(x)\,dx = \lim_{m\to c^-} \int_a^m f(x)\,dx + \lim_{n\to c^+} \int_n^b f(x)\,dx$$

(if both limits exist)

Inverse powers $\frac{1}{x^p}$
. In what follows a is a positive finite number.

(1) If $p > 1$ then $\displaystyle\int_a^\infty \frac{dx}{x^p}$ is convergent.
(2) If $p \le 1$ then $\displaystyle\int_a^\infty \frac{dx}{x^p}$ is divergent.
(3) If $p \ge 1$ then $\displaystyle\int_0^a \frac{dx}{x^p}$ is divergent.
(4) If $p < 1$ then $\displaystyle\int_0^a \frac{dx}{x^p}$ is convergent.

Comparison theorem.

Suppose $0 \le f(x) \le g(x) \le h(x)$ for all x in the interval (a, b), where a and/or b may be infinite.
(1) If $\int_a^b f(x)\,dx$ is divergent, then $\int_a^b g(x)\,dx$ is divergent.
(2) If $\int_a^b h(x)\,dx$ is convergent, then $\int_a^b g(x)\,dx$ is convergent.

2. Geometric applications of integration

2.1. **Arc length.**

Graph of a function of x.
If $y = f(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$ on $a \le x \le b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Graph of a function of y.
If $x = g(y)$ is continuous on $[c, d]$, then the length of the curve $x = g(y)$ on $c \le y \le d$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Arc length function.
Let f be a smooth function on $[a, b]$. The *arc length function* s is defined by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

Arc length notation.
We can formally think about the differential of the arc length function, ds, by setting

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

2.2. **Surface area of a solid of revolution.**

The

$$A = \int_\alpha^\beta 2\pi r\, ds,$$

where r is the radius of the surface, ds is the element of arc length, see above. If the axis of rotation is the x or y axis then the radius is one of x, y, $f(x)$, or $f(y)$. See also section 5.1 below, for the formula of ds for parametric curves.

3. Physics applications

3.1. Hydrostatic force and pressure.

The force exerted by a fluid on a thin plate is $F = mg = \rho g d$, where ρ is the density of the fluid, g acceleration of the free fall, d depth at which the plate is submerged. The pressure P on the plate is defined to be the force per unit area: $P = F/A = \rho g d$. To find hydrostatic force when the depth is not constant express the approximate force on a thin slice to express the force as an integral of the variable pressure over the entire shape.

3.2. Center of mass.

Moments.
If we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points (x_1, y_1), (x_2, y_2), $(x_3, y_3), \ldots,$ (x_n, y_n) in the xy-plane, then the *moment of the system about the y-axis* is defined by

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the *moment of the system about the x-axis* is defined by

$$M_x = \sum_{i=1}^{n} m_i y_i.$$

Center of mass.
The *center of mass* is given by the coordinates

$$(\overline{x}, \overline{y}) = (M_y/m, M_x/m),$$

where $m = \sum m_i$.

Centroid.
The center of mass of a solid plate is called the *centroid* of the region it occupies. If $\mathcal{R}$ is the region in the xy-plane bounded above by $y = f(x)$ and below by $y = g(x)$ over the interval $[a, b]$, then the centroid of $\mathcal{R}$ is

$$(\overline{x}, \overline{y}) = \left(\frac{1}{A} \int_a^b x \left[f(x) - g(x) \right] dx, \;\; \frac{1}{2A} \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx \right),$$

where A is the area of $\mathcal{R}$.

3.3. Separable differential equations.

A *separable equation* is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as (a function of x) $\times$ (a function of y). In other words, it has the form

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

This allows to rewrite such an equation as follows:

$$\frac{dy}{g(y)} = f(x) dx,$$

assuming $g(y) \neq 0$. Then integration on both sides gives:

$$\int \frac{dy}{g(y)} = \int f(x) dx.$$

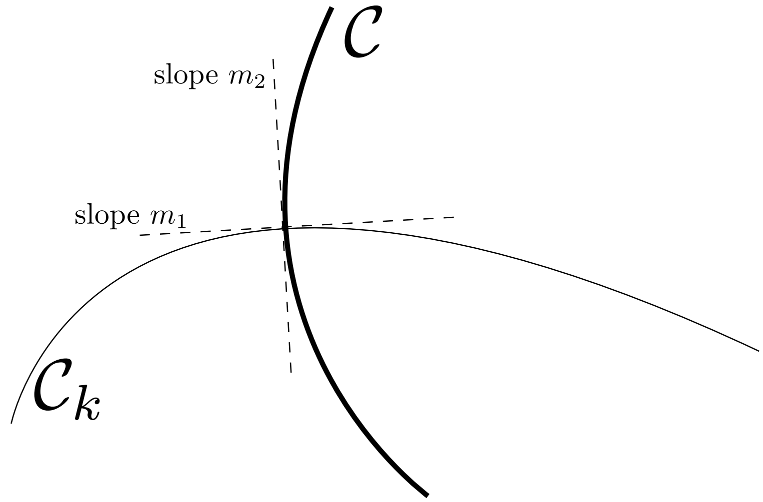

FIGURE 1. Orthogonal trajectory $\mathcal{C}$ intersecting a curve from the given family $\mathcal{C}_k$.

Solving for y the last equality gives it as a function $y(x)$. The constant of integration is found from the initial conditions, if available; otherwise the solution of the original equation is a family of functions.

Orthogonal Trajectory.
An *orthogonal trajectory* of a family of curves $\{\mathcal{C}_k\}$ indexed by k is a curve $\mathcal{C}$ that intersects each curve of the family orthogonally (at a right angle). This means, if m_1 is the slope of the tangent to a curve $\mathcal{C}_k$ from the family at a point $P(x, y)$, and m_2 is the slope of the tangent to the orthogonal trajectory $\mathcal{C}$ at the point P, then

$$m_1 \cdot m_2 = -1.$$

3.4. Exponential growth and decay.

There are many applications where a function is proportional to its first derivative. In other words, $\frac{dy}{dx} = ky$. This differential equation is called the *natural law of growth* $(k > 0)$ *or decay* $(k < 0)$.

4. Series

4.1. **Sequences.**

(1) A sequence is a function whose domain is the set of positive integers: $f(n)$ is defined on positive integer n.

A sequence can also be thought of as a list of numbers $\{a_n\}$.

(2) A *recursive* sequence uses previous terms to define later terms.
(3) If $\lim_{n\to\infty} a_n$ is a finite real number then we say the sequence is *convergent*. If the limit is not a real number then we say the sequence is *divergent*.
(4) A sequence of the form $\{ar^{n-1}\}$ is called a *geometric sequence*.
(5) A sequence that alternates signs is called an *alternating sequence*.
(6) A sequence a_n is *bounded above* if there is a number M such that $a_n \le M$ for all n.
(7) A sequence a_n is *bounded below* if there is a number M such that $a_n \ge M$ for all n.
(8) A sequence a_n is *bounded* if it is both bounded above and below.
(9) A sequence a_n is *increasing* if $a_n \le a_{n+1}$ for all n.
(10) A sequence a_n is *decreasing* if $a_n \ge a_{n+1}$ for all n.
(11) A sequence a_n is *monotonic* if it is either increasing or decreasing.

Theorem 4.1.1. *A geometric sequence $\{ar^{n-1}\}$ is convergent if and only if $-1 < r \le 1$.*

Theorem 4.1.2 (Squeeze theorem)**.** *If $a_n \le b_n \le c_n$ for all $n > K$, where K a fixed integer, and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L,$$

where L is a finite real number.

Corollary 4.1.2.1. *(1)* $\lim_{n\to\infty} -|a_n| \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty} |a_n|$

(2) Suppose a_n is an alternating sequence. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} |a_n| \ne 0$, then $\lim_{n\to\infty} a_n$ diverges.

Theorem 4.1.3. *A bounded monotonic function is convergent.*

4.2. **Number series.**

(1) If $\{a_k\}_{k=1}^{\infty}$ is a sequence, then $a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k = \sum a_k$ is an *(infinite) seres.*
(2) The *n-th partial sum* is $s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$
(3) The *sum* of the *series* is

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n.$$

(4) The series is called *convergent* if the limit defining the sum is a finite real number and this number is then the *sum of the series*. If the limit is not a finite number (does not exist or is infinite), the series is called *divergent*.
(5) The *n-th remainder* of the series $\sum_{k=1}^{\infty} a_k$ is

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots = \sum_{k=n+1}^{\infty} a_k.$$

(6) The two previous points give

$$\sum_{k=1}^{\infty} a_k = s_n + R_n$$

for any n.

(7) A series of the form $\sum_{k=1}^{\infty} ar^{k-1}$ is called a *geometric series*. The number r is called the *common ratio* of the series.

(8) If cancellation occurs in consecutive terms of a series, this series is called *telescoping*.

(9) A series $\sum a_k$ is called *absolutely convergent* if the series $\sum |a_k|$ converges.

(10) A series $\sum a_k$ is called *conditionally convergent* if the series $\sum a_k$ converges, but the series $\sum |a_k|$ diverges.

4.2.1. *Tests to determine convergence of series.*

Geometric series.
If $\sum_{k=1}^{\infty} ar^{k-1}$ is a geometric series, then

(1) $s_n = \frac{a(1-r^n)}{(1-r)}$, n-th partial sum
(2) $\sum_{k=1}^{\infty} ar^{k-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

Telescoping series.
Middle terms in the partial sums cancel so one can use the definition to find the value of the infinite sum.

Test for divergence.
If the series $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{k\to\infty} a_k = 0$. The contrapositive is (and is also true): If $\lim_{k\to\infty} a_k \neq 0$ then $\sum_{k=1}^{\infty} a_k$ diverges.

Harmonic series.
$\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

p-series.
$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Comparison test.
If $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$ are sequences with $0 \leq a_k \leq b_k \leq c_k$ then

(1) If $\sum_{k=1}^{\infty} a_k$ is divergent then $\sum_{k=1}^{\infty} b_k$ also is divergent.
(2) If $\sum_{k=1}^{\infty} c_k$ is convergent then $\sum_{k=1}^{\infty} b_k$ also is convergent.

The integral test.
Assume f is a continuous, positive, decreasing function on $[K, \infty)$ such that $a_k = f(k)$ for all integers $k \geq K$. Then $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\int_K^{\infty} f(x)\, dx$ is convergent

Limit comparison test.
Suppose $a_k \geq 0$, $b_k \geq 0$, and $\lim_{k\to\infty} \frac{a_k}{b_k} = c$ where c is a real number with $c > 0$. Then $\sum a_k$ converges if and only if $\sum b_k$ converges.

Alternating series test.
Suppose $\sum_{k=1}^{\infty} a_k$ is an alternating series. If

(1) $|a_{k+1}| \leq |a_k|$ (the sequence $\{|a_k|\}$ is decreasing), and
(2) $\lim_{k\to\infty} |a_k| = 0$.

then the series is convergent.

Absolute convergence.
If $\sum |a_k|$ converges, then $\sum a_k$ converges.

Ratio test.
Let $\sum a_k$ be a series with $\lim\limits_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = L$ (L is not necessarily a real number, can be $+\infty$)

(1) If $0 \le L < 1$, then $\sum a_k$ is absolutely convergent.
(2) If $L > 1$ or $L = \infty$, then $\sum a_k$ is divergent.
(3) If $L = 1$, then the test fails (is inconclusive).

Root test.
Let $\sum a_k$ be a series with $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = L$ (L is not necessarily a real number, can be $+\infty$)

(1) If $0 \le L < 1$, then $\sum a_k$ is absolutely convergent.
(2) If $L > 1$ or $L = \infty$, then $\sum a_k$ is divergent.
(3) If $L = 1$, then the test fails (is inconclusive).

Note that the ratio and root test always fail simultaneously, so after one fails applying the other is redundant.

Theorem 4.2.1. *If the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, then*

(1) $\displaystyle\sum_{k=1}^{\infty} [a_k + b_k] = \left(\sum_{k=1}^{\infty} a_k\right) + \left(\sum_{k=1}^{\infty} b_k\right)$
(2) $\displaystyle\sum_{k=1}^{\infty} [a_k - b_k] = \left(\sum_{k=1}^{\infty} a_k\right) - \left(\sum_{k=1}^{\infty} b_k\right)$
(3) $\displaystyle\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k$*, where c is a constant*

If either series is divergent, then the above equalities do not necessarily hold.

4.2.2. *Estimating the sum of a series with s_n.*

Accuracy of an estimate with s_n.
The error in using the partial sum s_n to estimate the infinite sum is the remainder, R_n:

$$R_n = s - s_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k.$$

This tacitly assumes that the series under consideration is convergent, so that s is defined.

Error based on integral test.
Assume f is a continuous, positive, decreasing function on $[n, \infty)$ such that $a_k = f(k)$ for all integers $k \ge n$. Then $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$

Error based on comparison test.
If we know $0 \le a_k \le b_k$, then $0 \le \sum_{k=n+1}^{\infty} a_k \le \sum_{k=n+1}^{\infty} b_k$

Error for an alternating series.
Suppose $\sum_{k=1}^{\infty} a_k$ is an alternating series. If

(1) $|a_{n+1}| \le |a_k|$ (the sequence $\{|a_k|\}$ is decreasing), and
(2) $\lim_{k\to\infty} |a_k| = 0$.

then

$$|R_n| \leq |a_{n+1}|.$$

4.3. **Power series.**

A series of the form

$$\sum_{k=0}^{\infty} c_k(x-a)^k$$

where c_k, and a are constants and x is a variable, is called a *power series in* $(x-a)$*, or centered at* a*, or about* a*.*

Theorem 4.3.1. *For the convergence of a given power series* $\sum c_k(x-a)^k$*, there are three possibilities.*

(1) The series converges for $x = a$ *only. In this case we say the radius of convergence of the series is* $R = 0$ *and the interval of convergence is* $\{a\}$*.*
(2) The series converges for all x*. In this case we say the radius of convergence of the series is* $R = \infty$ *and the interval of convergence is* $(-\infty, \infty)$*.*
(3) There is a positive real number R *where the series converges for all* x *with* $|x-a| < R$ *and diverges for all* x *with* $|x-a| > R$*. In this case we say the radius of convergence of the series is* R*. The interval of convergence is one of the intervals* $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, *or* $[a-R, a+R]$*.*

4.4. **Representing functions using power series.**

Formula for Geometric Series.

$$\sum_{k=1}^{\infty} x^{k-1} = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{when} \quad |x| < 1.$$

Theorem 4.4.1. *Let* $f(x) = \sum c_k(x-a)^k$ *be a power series with radius of convergence* $R > 0$*. Then*

(1) f *is continuous on the interval* $(a-R, a+R)$
(2) f *is differentiable on the interval* $(a-R, a+R)$ *and its derivative is*

$$f'(x) = \frac{d}{dx}\sum_k c_k(x-a)^k = \sum_k \frac{d}{dx} c_k(x-a)^k = \sum_k k c_k(x-a)^{k-1}.$$

(3) f *may be integrated on a closed interval contained in* $(a-R, a+R)$ *and*

$$\int f(x)\,dx = \int \sum_k c_k(x-a)^k\,dx = \sum_k \int c_k(x-a)^k\,dx = \sum_k \frac{c_k}{k+1}(x-a)^{k+1}.$$

4.5. Taylor and Maclaurin series.

Definition 4.5.1. *The Taylor Series of a function f centered at a is the power series expansion of $f(x)$ about a and is in the form*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Definition 4.5.2. *The Maclaurin Series of a function f is the power series expansion of $f(x)$ about 0 and is in the form*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k.$$

As usual, we assume that $0! = 1$.

Some important Maclaurin series:

(1) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$ $R = +\infty$

(2) $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \dots$ $R = +\infty$

(3) $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \dots$ $R = +\infty$

(4) $(1+x)^p = \sum_{k=0}^{\infty} \binom{k}{p} x^k = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 \dots$ $R = 1$

(5) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \dots$ $R = 1$

(6) $\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$ $R = 1$

(7) $\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots$ $R = 1$

Observe that the series for $\ln(1+x)$ and $\arctan x$ can be obtained from the series for $1/(1+x)$ and $1/(1+x^2)$ respectively, by integration. Binomial coefficients are given by the formula

$$\binom{p}{n} = \frac{p(p-1)(p-2)\dots(p-k+1)}{k!},$$

where k is any real number; k is a positive integer.

4.6. Approximating functions using polynomials.

Suppose f is a function with Taylor series about a given by $\sum a_k(x-a)^k$. We call the n-th partial sum of the series the *n-th degree Taylor polynomial*.

We denote the n-th degree Taylor polynomial by $T_n(x)$, so that

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem 4.6.1. *Let T_n denote the n-th degree Taylor polynomial of a function, f, about a and let $R_n = f - T_n$. If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of the Taylor series when $|x-a| < R$.*

Theorem 4.6.2 (Taylor's Inequality). *If $|f^{(n+1)}(x)| \leq M$ for all x with $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d.$$

5. Polar coordinates and conic sections

5.1. Parametric equations.

A set of equations that are defined using a single independent variable are called *parametric equations*. Typically, we use t, called the *parameter*, as the independent variable to define the functions $x(t)$ and $y(t)$ (and perhaps $z(t)$). These equations define a *parametric curve*, C, in the plane (or in 3-space if $z(t)$ is given) such that the points on C are the set of points given by $(x(t), y(t))$.

(1) Graphing
(2) Convert from and to parametric equations to and from Cartesian.
(3) Slope of a parametric curve can be determined from

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

(4) The second derivative is obtained similarly:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}.$$

(5) Use $\int y\,dx = \int y(t)\cdot x'(t)\,dt$, $\int x\,dy = \int x(t)\cdot y'(t)\,dt$ to find areas bounded by parametric curves.
(6) Use $ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ to find lengths of curves and the element of arc length for surface areas.

5.2. Polar coordinates.

Conversion formulas:

$$x = r\cos\theta \qquad\qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad\qquad \tan\theta = y/x$$

The last equation gives $\theta = \arctan(y/x)$, possibly $+\pi$.

Slope for a given value of θ:

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

Areas. The area of the region within the sector $a \le \theta \le b$ enclosed by the curve $r = r(\theta)$ is given by

$$\int_a^b \frac{1}{2} r^2\,d\theta.$$

Element of arc length. Use

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

to find lengths of curves and surface areas.

5.3. **Conic sections in Cartesian coordinates.**

(1) $y^2 = 4px$ – parabola, focus $(p, 0)$, directrix $x = -p$.

(2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a \geq b > 0$ – ellipse, foci $(\pm c, 0)$ where $c^2 = a^2 - b^2$, vertices $(\pm a, 0)$.

(3) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ – hyperbola, foci $(\pm c, 0)$ where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, asymptotes $y/b = \pm\, x/a$.

Interchanging x and y in the above equations results in the corresponding conic section changing orientation from horizontal to vertical (equivalently, the graph is reflected with respect to the line $y = x$).