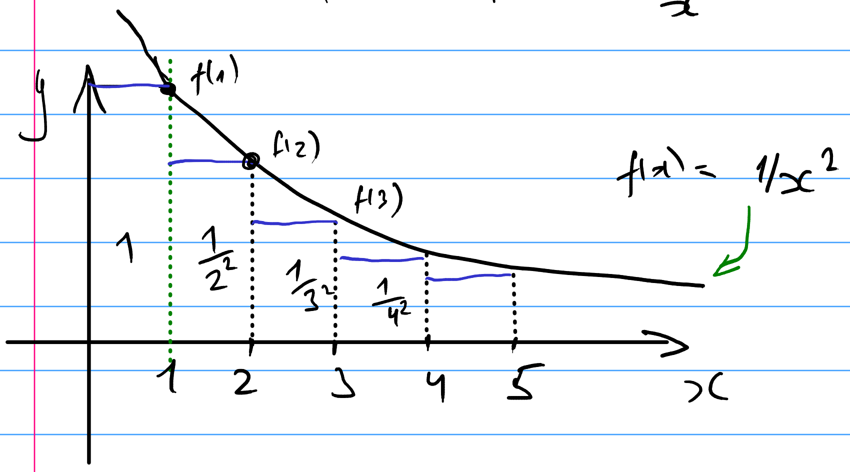
# Section 11.3: Integral test

Ex. 1. Consider: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \underbrace{\frac{1}{3^2}}_{a_3} + \dots$

Then for $f(x) = \frac{1}{x^2}$, $\quad f(k) = a_k$




$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \boxed{\text{Area of rectangles}} \leq \underbrace{1}_{\text{1st rectangle}} + \int_1^{\infty} \frac{1}{x^2}$$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^2}}_{a_k} \leq 1 + \int_1^{\infty} x^{-2}\, dx = 1 + \left(-x^{-1}\right)\Big|_1^{\infty} = 2.$$

So, any partial sum satisfies

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$$

$\Rightarrow$ The sequence of partial sums $\{S_n\}$ is bounded:

$$0 < S_n \leq 2$$

But the $a_k$ are nonnegative, so

$$s_{n+1} - s_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1} > 0$$

$\Rightarrow s_{n+1} > s_n$.

The sequence of partial sums $\{s_n\}$ is increasing and bounded $\Rightarrow$ convergent!

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

(not to 2 though).

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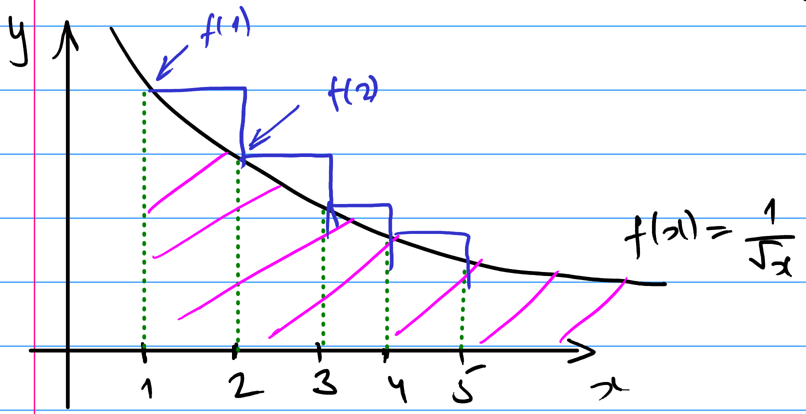
Ex. 2 Consider:

$$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

Introduce $f(x) = \frac{1}{\sqrt{x}}$, then $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = +\infty$




$$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \boxed{\text{Area of rectangles}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}} dx =$$

$$= 2\sqrt{x}\Big|_1^{\infty} = \lim_{t \to \infty} 2\sqrt{x}\Big|_1^{t} = +\infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{x^{1/2}} = +\infty, \quad \text{divergent.}$$

## The integral test for convergence.

Suppose $f(x)$ is continuous, positive, decreasing on $[1, \infty)$, and

$$f(k) = a_k, \qquad k \geq 1,$$

then

$$\sum_{k=1}^{\infty} a_k \text{ is convergent} \iff \int_1^{\infty} f(x)\,dx \text{ is convergent.}$$

Remark: these properties only need to hold for $k \geq k_0$.

Ex. 3 Test for convergence: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

Consider $f(x) = \dfrac{1}{x^2+1}$ $\Big\}$ positive, continuous, decreasing

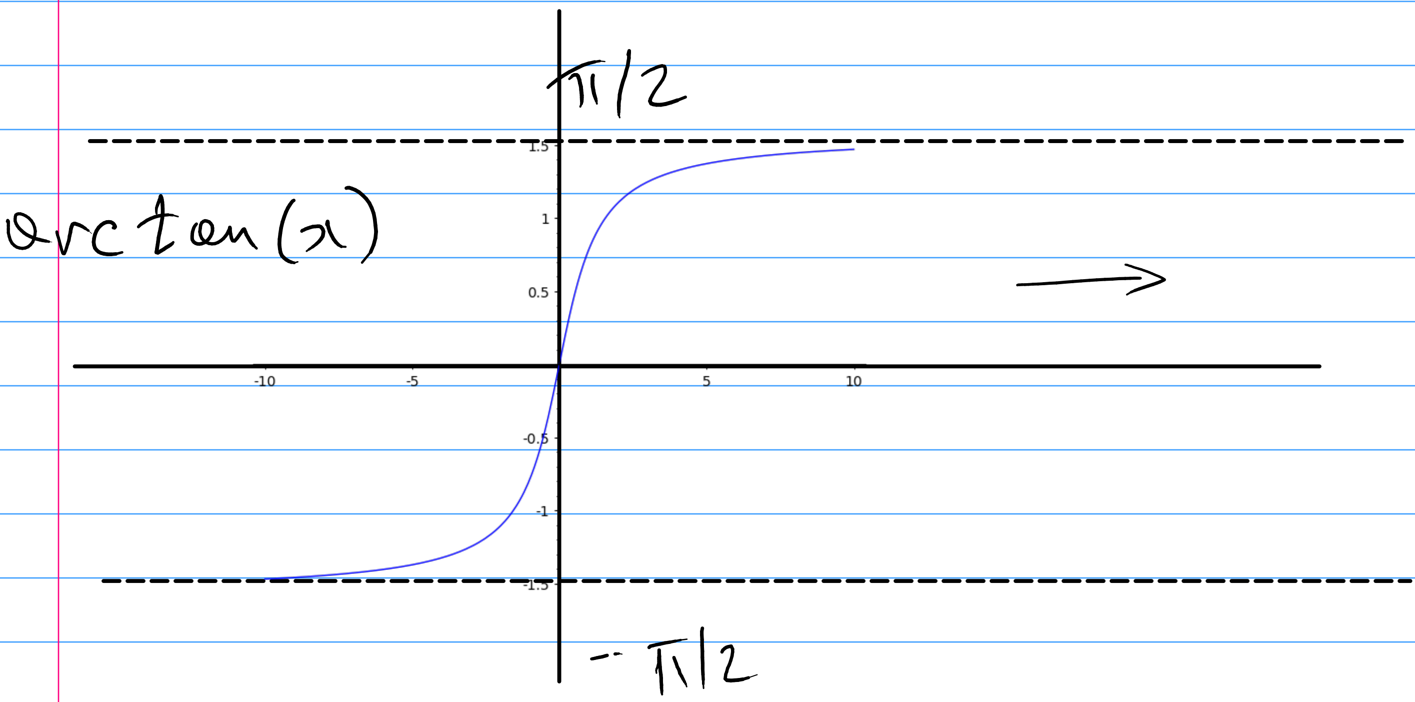
on $[1, \infty)$.

By the integral test, the given series is convergent if and only if the improper integral is convergent:

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \arctan x \Big|_1^t$$

$$= \lim_{t \to \infty} (\arctan t - \arctan 1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \Big\} \text{ Convergent}$$




$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2+1} \quad \text{is} \quad \underline{\text{convergent}}.$$

Ex. 4 For which $p$ is $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^p}}_{p-\text{series}}$ convergent?

For $p \leq 0$, $\quad a_k = \frac{1}{k^p} \not\to 0, \; k \to \infty$

$\Rightarrow$ the series is <u>divergent</u> by the divergence test.

For $p>0$, consider $f(x)=\frac{1}{x^p}$ for $x$ in $[1,\infty)$,

this $f(x)$ is $\begin{cases} \text{positive} \\ \text{continuous} \\ \text{decreasing} \end{cases}$

$\Rightarrow$ the integral test applies,

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent $\Leftrightarrow$

$\Leftrightarrow$ $\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent $\Leftrightarrow$

$\Leftrightarrow$ $p>1$.

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent $\Leftrightarrow$ $p>1$.