Measure and Integration I (MAA5616), Fall 2020
Homework 8, due Thursday, Oct. 29

We write $\int f\,dx = \int f$ for the Lebesgue integral on $\mathbb{R}$.

1. Suppose $\{f_n\}, f \in L_+$. If $f_n \to f$, $n \to \infty$, and $f_n \le f$ a.e., prove that $\int f_n \to \int f$.

Notice that we do not assume monotone convergence.

2. Compute the given limit. Justify your answer.

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin(x/n)\,dx.$$

3. Show: $\int_0^\infty x^n e^{-x}\,dx = n!$ by taking the n-th derivative of both sides of the equality $\int_0^\infty e^{-tx}\,dx = 1/t$.

4. An easy example of nowhere continuous function that is not (locally) Riemann integrable is $1_{\mathbb{Q}}$. However, by modifying this function on a Lebesgue null set gives a constant. For $g(x)$ in this example such modification no longer exists.

Denote $f(x) = 1_{(0,1)}\, x^{-1/2}$, and let

$$g(x) = \sum_1^\infty \frac{1}{2^n} f(x - r_n),$$

where $\{r_n\}_1^\infty$ is an enumeration of $\mathbb{Q}$. Prove:

- $g \in L^1(\mathbb{R}, \lambda)$, in particular $g < \infty$ a.e. (Use Theorem 2.25.)
- For any nonempty $(a,b) \subset \mathbb{R}$, $\sup_{(a,b)} g(x) = \infty$.
- Function g is discontinuous at every $x \in \mathbb{R}$.
- The two previous properties hold also for any g_0, such that $g_0 = g$ a.e. (So modifying g on a null set does not give a nice function.)
- The function g^2 is not integrable on any $(a,b) \subset \mathbb{R}$. (Even though it is finite a.e.)

5.
- Prove that the cardinality of continuous functions on $\mathbb{R}$ is $\mathfrak{c}$.
- Prove that the cardinality of monotonic functions on $\mathbb{R}$ is $\mathfrak{c}$.

Hint: Continuous functions are determined by their values on $\mathbb{Q}$. Then use # 4 from HW 1.
Show that monotonic functions have at most countably many discontinuities. Then prove that a monotonic function is determined by its values on $\mathbb{Q}$ and the positions and sizes of its jumps.

The set of *all* functions $\mathbb{R} \to \mathbb{R}$ of course has higher cardinality than $\mathfrak{c}$, because it contains indicators.