Section 11.6 Absolute convergence, ratio and root tests

Given a series $\sum_k a_k$, consider $\sum_k |a_k|$

<u>Def.</u> A series $\sum_k a_k$ is called absolutely convergent, if $\sum_k |a_k|$ is convergent.

Ex. 1 $\underbrace{\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2}}_{\text{absolutly convergent}}$, as $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\text{convergent}}$

Ex. 2 Alternating harmonic series: $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}$

is not absolutely convergent, because

$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k}}_{\text{divergent}}$.

<u>Def</u> A series $\sum_k a_k$ is conditionally convergent, if $\sum a_k$ is convergent, but not absolutely convergent.

<u>Thm</u> If $\sum_k a_k$ is absolutely convergent, then it is convergent.

Pf. Notice: $a_k \leq |a_k|$

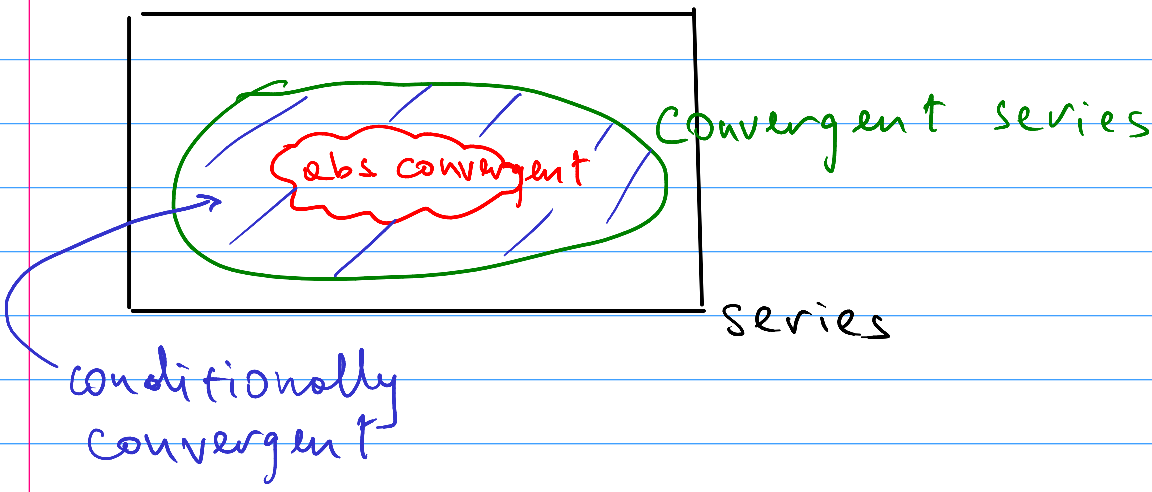
$$0 \leq a_k + |a_k| \leq 2|a_k|$$

Now, $\sum_k (a_k + |a_k|)$ has nonnegative terms

$\Rightarrow$ by the comparison test, $\sum (a_k + |a_k|)$ is convergent.

Finally $\sum a_k = \underbrace{\sum_k (a_k + |a_k|)}_{\text{convergent}} - \underbrace{\sum_k |a_k|}_{\text{convergent}}$

$\Rightarrow$ convergent.



Ex. 3 Test for convergence: $\sum_{k=1}^{\infty} \underbrace{\frac{\cos k}{k^2}}_{a_k}$

Consider

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{\overbrace{|\cos k|}^{\leq 1}}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \Big\} \text{ convergent, p-series, } p=2$$

$\Rightarrow \sum_k |a_k|$ is convergent by the comparison test.

$\Rightarrow$ the original series is absolutely convergent

$\Rightarrow$ convergent.

Ratio test

Given a series $\sum a_k$, if:

i). $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L < 1$, then $\sum a_k$ is absolutely convergent

ii). $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L > 1$ (in particular, can be $L = +\infty$)

then $\sum a_k$ is divergent

iii). $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$, then the test is inconclusive.

For iii): consider $\sum_{k=1}^{\infty} \frac{1}{k}$ $\quad \sum_{k=1}^{\infty} \frac{1}{k^2}$

For both,

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1}{k+1} \Big/ \frac{1}{k} = \lim_{k\to\infty} \frac{k}{k+1} = 1.$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} \frac{/k^2}{/k^2} = \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^2} = 1.$$

Ex.4 Test for convergence: $\sum_{k=1}^{\infty} (-1)^k \frac{k^3}{3^k}$ $\Big\}$ convergent

Ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left| \cancel{(-1)^{k+1}} \frac{(k+1)^3}{3^{k+1}} \Big/ \cancel{(-1)^k} \frac{k^3}{3^k} \right|$$

$$= \frac{(k+1)^3}{\cancel{3^{k+1}}} \cdot \frac{\cancel{3^k}}{k^3} = \frac{(k+1)^3}{3 \cdot k^3} = \frac{1}{3} \cdot \left(1+\frac{1}{k}\right)^3$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{1}{3}\cdot\overbrace{\left(1+\frac{1}{k}\right)^3}^{\to 1} = \frac{1}{3} < 1$$

$\Rightarrow$ by the ratio test, absolutely convergent series

Ex. 5 Test for convergence: $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{k!}$

Ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^{k+1}}{(k+1)!} \Big/ \frac{k^k}{k!}\right| = \frac{(k+1)^{k+1}}{(k+1)!}\cdot\frac{\cancel{k!}}{k^k}$$

$$= \frac{(k+1)^{k+1}}{\cancel{k+1}}\cdot\frac{1}{k^k} = \left(\frac{k+1}{k}\right)^k = \left(1+\frac{1}{k}\right)^k$$

$$\left(1+\frac{1}{k}\right)^k \to \underbrace{e}_{>1} \qquad k\to\infty.$$

$\Rightarrow$ by the ratio test, the series is **divergent**

Note: we could show $a_k \not\to 0$, $k\to\infty$, conclude divergence from the divergence test.

Indeed,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{\overbrace{k\cdot k\cdot\ldots\cdot k}^{k^k}}{1\cdot 2\cdot\ldots\cdot k} = \lim_{k\to\infty}\overbrace{\frac{k}{1}}^{k}\cdot\overbrace{\frac{k}{2}}^{\geq 1}\cdot\ldots\cdot\overbrace{\frac{k}{k}}^{\geq 1}$$

$$\geq \lim_{k\to\infty} k = +\infty.$$

Root test.

Given a series $\sum_k a_k$. If:

i). $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L < 1$, then the series is absolutely convergent

ii). $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L > 1$ (possibly $L = +\infty$), then the series is divergent

iii). $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$, the test is inconclusive.

Ex. 6 Test for convergence $\sum_{k=1}^{\infty} \left(\frac{2k+3}{3k+2}\right)^k$ } convergent

Root test:

$$\sqrt[k]{|a_k|} = \sqrt[k]{\left(\frac{2k+3}{3k+2}\right)^k} = \frac{2k+3}{3k+2}$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{2k+3}{3k+2} = \frac{2}{3} < 1$$

$\Rightarrow$ by the root test, absolutely convergent series.