SOME IMPORTANT MACLAURIN SERIES AND A COMMENT ON BINOMIAL COEFFICIENTS

The **Maclaurin series** of a function f is the power series expansion of f(x) about 0 in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

It is a special case of Taylor series with a = 0. As usual, we assume that 0! = 1.

Several important Maclaurin series are given below. You need to know them for Test 4.

$$\begin{array}{lll} e^x & = \displaystyle \sum_{k=0}^\infty \frac{x^k}{k!} & = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots & R = +\infty \\ \sin x & = \displaystyle \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!} & = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \dots & R = +\infty \\ \cos x & = \displaystyle \sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{(2k)!} & = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \dots & R = +\infty \\ (1+x)^p & = \displaystyle \sum_{k=0}^\infty \binom{p}{k} x^k & = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 \dots & R = 1 \\ \frac{1}{1-x} & = \displaystyle \sum_{k=0}^\infty x^k & = 1 + x + x^2 + x^3 + x^4 + \dots & R = 1 \\ \ln(1+x) & = \displaystyle \sum_{k=0}^\infty (-1)^k \frac{x^{k+1}}{k+1} & = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots & R = 1 \\ \arctan x & = \displaystyle \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1} & = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots & R = 1 \end{array}$$

Observe that the series for $\ln(1+x)$ and $\arctan x$ can be obtained from the series for 1/(1+x) and $1/(1+x^2)$ respectively, by integration. Binomial coefficients are given by the formula

$$\binom{p}{k} = \frac{p(p-1)(p-2)\dots(p-k+1)}{k!},$$

where p is any real number; k is a positive integer.

Binomial coefficients.

To begin, let's write out what different powers of $(a+b)^p$ expand to, for integer $p=0,1,2,3,\ldots$, like so:

$$(a+b)^{0} = 1$$

$$(a+b)^{1} = a+b$$

$$(a+b)^{2} = a^{2} + 2ab + b^{2}$$

$$(a+b)^{3} = a^{3} + 3a^{2}b + 3ab^{2} + b^{3}.$$

For larger positive integers p = 4, 5, 6, ..., we could similarly expand $(a + b)^p$. The expression for $(a - b)^p$ follows from $(a + b)^p$ by replacing b with -b and keeping track of the sign of

 $(-b)^k = (-1)^k b^k$. The coefficients next to $a^l b^k$ in the expansion on the right-hand side can be written out in the so-called *Pascal's triangle*:

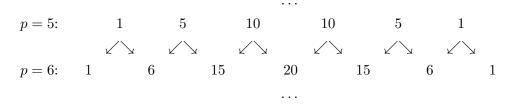
Turns out that the k-th number, with k running from 0 to p, in the p-th line above is given by what we called the binomial coefficient:

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = \frac{p(p-1)(p-2)\dots(p-k+2)(p-k+1)}{k!}.$$

It is also called p choose k, because it is the number of ways of choosing k objects among pdifferent objects (think about choosing k marbles among p different marbles numbered $1, 2, \ldots, p$).

Let's take the third line above for example, where p=3. The number of ways to choose 0 objects among 3 is 1 — choosing 0 objects means choosing nothing, and you have only one way of doing that. That's why this row starts with 1. Next, the number of ways to choose 1 object among 3 objects is 3: you can choose the first, or the second, or the third one. Then, there is 3 ways to choose 2 objects among 3 objects — you can take all but the first, or all but the second, or all but the third. Finally, there is only 1 way to choose 3 objects among 3 objects — you have to take all of them. So, there we have the row for p=3:

The rest of the rows of the above table can be given meaning in the same way. And, there is a neat way to compute the numbers in the table: each number is the sum of the two from the previous row, sitting diagonally to the left and to the right of it, like so:



Even though it is called *Pascal's triangle*, it (as is the case with many discoveries) was known long before being found by the person it was named after.