Measure and Integration II (MAA5617), Spring 2021
Homework 3, due Thursday, Feb 11

1. Suppose ν is a finite signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

- Verify $|\nu(E)| \leq |\nu|(E)$.
- Verify that $\nu \ll |\nu|$ and $\left|\frac{d\nu}{d|\nu|}\right| = 1$ holds $|\nu|$-a.e.
- Verify that

$$|\nu|(E) = \sup\left\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N},\ E = \bigsqcup_1^n E_j\right\}.$$

2. For $F, \{F_j\}_j \subset NBV$ such that $F_j \to F$ pointwise on $\mathbb{R}$, show $T_F \leq \liminf_j T_{F_j}$.

3. Give an example of a function f, such that $f \notin BV([a,b])$ for any $a < b \in \mathbb{R}$.

4. Because the function you constructed in the previous problem was likely discontinuous, let's consider the following modification of the famous example of Weierstrass, due to van der Waerden. Let $f_0 = |x|$ on $[-1/2, 1/2]$, extended to $\mathbb{R}$ by periodicity. Let further

$$f_j(x) = \frac{f_0(3^j x)}{3^j}, \qquad j \geq 1,$$

and

$$f(x) = \sum_{j=0}^{\infty} f_j(x).$$

- Verify that the series for f converges uniformly on $\mathbb{R}$. Conclude that f is everywhere continuous.
- For $x \in \mathbb{R}$, let $h_n = \pm 3^{-n-1}$, where the sign is chosen so that $|f_n(x+h_n) - f_n(x)| = h_n$ (check how this works for $n = 0$). Then

$$|f_j(x+h_n) - f_j(x)| = h_n, \qquad 0 \leq j \leq n,$$

  and

$$|f_j(x+h_n) - f_j(x)| = 0, \qquad j > n.$$

  Using this, show that

$$\frac{f(x+h_n) - f(x)}{h_n} = \sum_{j=0}^{n} \frac{f_j(x+h_n) - f_j(x)}{h_n}$$

  takes different values depending on the parity of n.
- Conclude that $f'(x)$ does not exist.
- This implies $f \notin BV([a,b])$ for any interval $[a,b]$. Why?