REVIEW QUESTIONS FOR TEST 2

Arc length

1) Write down the arc length formula, for integration both in terms of x and y. How are the limits of integration determined in it? Which expression can be interpreted as the element of arc length ds?
2) Find the length of the following curves:
   (a) $36y^2 = (x^2-4)^3, \quad 2 \le x \le 3, \quad y \ge 0$
   (b) $x = \dfrac{y^5}{10} + \dfrac{1}{6y^3}, \quad 1 \le y \le 3$
   (c) $y = \ln(\cos x), \quad 0 \le x \le \pi/6$
   (d) $y = \dfrac{1}{2}(e^x + e^{-x}), \quad 0 \le x \le 1$
   (e) $y = \sqrt{x - x^2} + \arcsin(\sqrt{x}), \quad 0 \le x \le 1$
   (f) $x = \ln(1-y^2), \quad 0 \le y \le \dfrac{1}{2}$.

Areas of surfaces of revolution

3) Write down the formulas for the area of a surface of revolution. What changes when the axis of rotation is the y-axis? Write down the formula for ds (element of arc length) for integration in x. The same for integration in y.
4) Find area of the surface obtained by rotating the following curves about the x-axis:
   (a) $y = \cos(3x), \quad 0 \le x \le \pi/6$
   (b) $x = \dfrac{1}{3}(y^2+2)^{3/2}, \quad 1 \le y \le 2$
   (c) $y = \sqrt{1+e^x}, \quad 0 \le x \le 1$.
5) Find area of the surface obtained by rotating the following curves about the y-axis:
   (a) $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x, \quad 1 \le x \le 2$
   (b) $x^{2/3} + y^{2/3} = 1, \quad 0 \le y \le 1$
   (c) $y = \dfrac{1}{3}x^{3/2}, \quad 0 \le x \le 6$.

Applications to physics

6) Write down the formula for hydrostatic pressure at the depth d. Write down formulas for the coordinates of a centroid of a flat region confined between two curves; assume that the density is constant. What changes when the region lies between a curve and the x-axis? How are the limits of integration determined in the relevant integrals? What can be said about the location of a centroid of a convex shape? A shape that has an axis of symmetry?
7) See # 5–8, 10–12, and # 15 in Section 8.3.
   (a) A vertical plate is submerged in water, as shown in Figures 1–3. Explain how to approximate the hydrostatic force acting on one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.

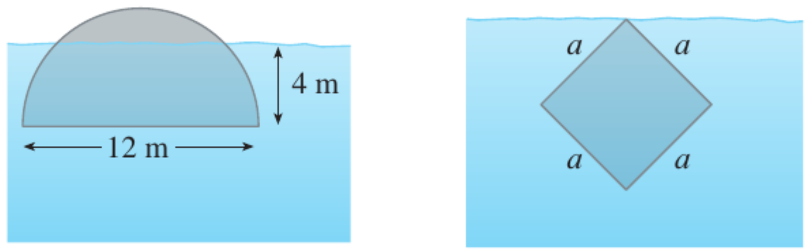

FIGURE 1

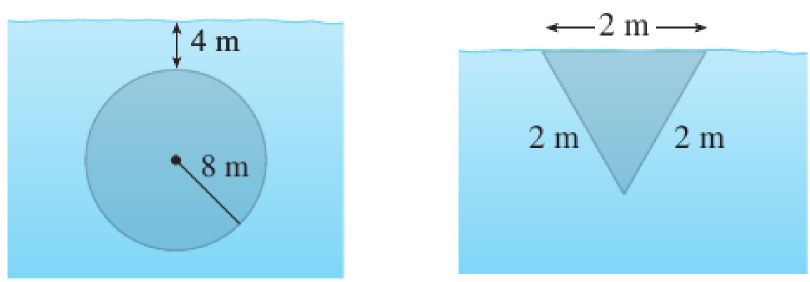

FIGURE 2

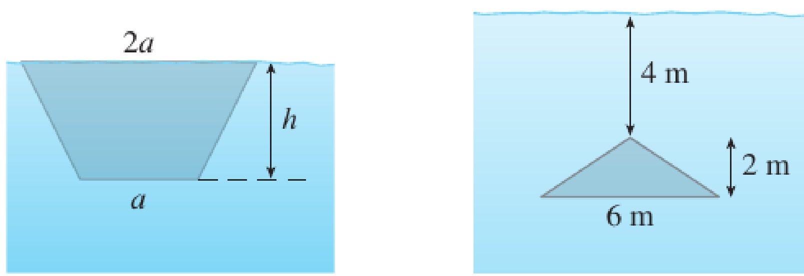

FIGURE 3

(b) A cube with 20-cm-long sides is sitting on the bottom of an aquarium in which the water is one meter deep. Find the hydrostatic force on (i) the top of the cube and (ii) one of the sides of the cube.

8) Find centroids of the regions bounded by the given curves:
(a) $y = \sin x,\ \ y = 0,\ \ 0 \le x \le \pi$
(b) $y = 2 - x^2,\ \ y = x^2$
(c) $y = \sin x,\ \ y = \cos x,\ \ x = 0,\ \ x = \pi/4$
(d) $x + y = 2,\ \ x = y^2$
(e) $y = x^3,\ \ y = x^5,\ x \ge 0$
(f) $y = (e^x + e^{-x})/2,\ y = 4$
(g) $x = 0,\ y = 1,\ y = 1/\sqrt[3]{x}$.

Separable differential equations

9) Explain what is a separable equation. Write down the general algorithm that should be applied to separable differential equations. In which case is it possible to express y as

an explicit function of x, assuming the equation contains dy/dx? What determines the value of the constant of integration in a particular solution?

What is an orthogonal trajectory? Why are orthogonal trajectories found using a differential equation? Write down the algorithm for finding the equation of orthogonal trajectories to a given family of curves.

10) Solve the differential equations:

(a) $y' = x\sqrt{y}$
(b) $y' + xe^x = 0$
(c) $\dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}}$
(d) $\dfrac{dz}{dt} + e^{t+z} = 0$
(e) $y' = \sin^3 x$
(f) $y' = x^2 e^x$
(g) $y' = \ln x + 1$
(h) $(x+2)e^y + y\sqrt{x+1}\,\dfrac{dy}{dx} = 0$
(i) $y' = \dfrac{e^x}{x}$
(j) $\dfrac{1}{\sqrt{1-x^2}} + \dfrac{dy}{dx}\cdot\dfrac{1}{\sqrt{1-y^2}} = 0$
(k) $y' = \dfrac{\sqrt{y}}{\sqrt{x}}$.

11) Solve the differential equations and find solutions satisfying the given initial condition:
(a) $y' = 2\sqrt{y}, \quad y(-1) = 1$
(b) $y' = 2x\sqrt{1-y^2}, \quad y(0) = 0$
(c) $\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$
(d) $x\ln x = y(1+\sqrt{3+y^2})y', \quad y(1) = 1$
(e) $y'\cot x + y = 2, \quad y(\pi/3) = 0.$

12) Find orthogonal trajectories of the given family of curves, indexed by the constant k:
(a) $x^2 + 3y^2 = k^2$
(b) $y = \dfrac{k}{x}$.

For the following families of curves, find the equation of the orthogonal trajectory passing through the point $(1, 2)$.
(c) $y^2 = kx^3$.
(d) $y = \dfrac{1}{x+k}$.

13) See # 46, 47 in Section 9.3:
(a) The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?
(b) A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

Sequences

14) What is a sequence? Explain when a sequence $\{a_n\}$ converges to a number L (has limit L). State the limit laws for sequences. State the Squeeze theorem. What is the limit of a composition of a continuous function with a converging sequence? What are the assumptions necessary to interchange a function with a limit? State the Monotonic sequence theorem.

15) Determine whether the given sequence converges; if it does, find the limit. We assume that $n \geq 1$ in all the sequences.

(a) $a_n = \dfrac{3+5n^2}{n+n^2}$
(b) $a_n = \dfrac{3+5n^2}{1+n}$
(c) $a_n = 3^n 7^{-n}$
(d) $a_n = \dfrac{(-1)^n n}{n+\sqrt{n}}$
(e) $a_n = 1 + \left(-\dfrac{1}{2}\right)^n$
(f) $a_n = \sqrt[n]{2^{1+3n}}$
(g) $a_n = n\sin(1/n)$
(h) $a_n = \sin n$
(i) $a_n = \dfrac{n^2}{\sqrt{n^3+4n}}$
(j) $a_n = \ln(n+1) - \ln n$
(k) $a_n = \arctan(\ln n)$
(l) $a_n = \left(1+\dfrac{2}{n}\right)^n$
(m) $a_n = \dfrac{n!}{2^n}$, where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.
(n) $a_n = n - \sqrt{n+1}\sqrt{n+3}$.
Hint: multiply and divide by the conjugate expression.

16) Determine whether the following sequences are bounded, and whether they are eventually increasing or decreasing:

(a) $a_n = \cos n$
(b) $a_n = \dfrac{1}{2n+3}$
(c) $a_n = \dfrac{1-n}{2+n}$
(d) $a_n = n^3 - 11n + 3$
(e) $a_n = 2 + \dfrac{(-1)^n}{n}$
(f) $a_n = \ln\left(1+\dfrac{1}{n}\right)$
(g) $a_n = n^{1/n}$.

Answer key

2) (a) $\frac{13}{6}$
   (b) $\frac{9866}{405}$
   (c) $\frac{1}{2}\ln 3$
   (d) $\frac{e-e^{-1}}{2}$
   (e) 2
   (f) $-\log(3)+\frac{1}{2}$.

4) (a) $\frac{\pi}{3}\sqrt{10}+\frac{\pi}{9}\ln\left(3+\sqrt{3^2+1}\right)$
   (b) $\frac{21}{2}\pi$
   (c) $\pi(e+1)$.

5) (a) $\frac{10\pi}{3}$
   (b) $\frac{6\pi}{5}$
   (c) $\frac{8}{15}\pi\left(25\sqrt{10}+16\right)$.

7) (a) Hydrostatic forces acting on the plates in Figure 1:
   $\rho g\left(144\arcsin\frac{2}{3}+\frac{176}{3}\sqrt{5}-144\right)$ and $\frac{\rho g\, a^3}{\sqrt{2}}$, respectively.
   (b) (i) 313.6 N; (ii) 352.8 N.

8) (a) $\left(\frac{\pi}{2},\frac{\pi}{8}\right)$
   (b) $(0,1)$
   (c) $\left(\frac{\sqrt{2}\pi-4}{4\left(\sqrt{2}-1\right)},\frac{1}{4\left(\sqrt{2}-1\right)}\right)$
   (d) $(8/5,-1/2)$.

10) (a) $y(x)=\left(\frac{1}{4}x^2+C\right)^2$
   (b) $y(x)=-(x-1)e^x+C$
   (c) $\theta(t)\sin(\theta(t))+\cos(\theta(t))=C-\frac{1}{2}e^{-t^2}$
   (d) $z(t)=-\ln\left(e^t+C\right)$
   (e) $y(x)=\frac{\cos(x)^3}{3}-\cos(x)+C$
   (f) $y(x)=\left(x^2-2x+2\right)e^x+C$
   (g) $y(x)=x\log(x)+C$
   (h) $(y(x)+1)e^{-y(x)}=\frac{2}{3}(x+4)\sqrt{x+1}+C$

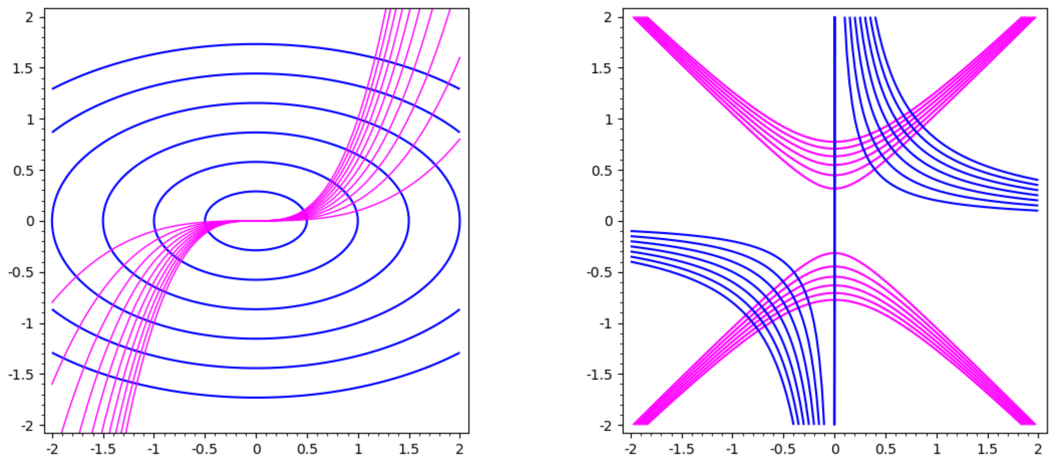

FIGURE 4. Orthogonal trajectories in questions 12) (a)–(b), in magenta; the original families are shown in blue.

(i) $y(x) = \displaystyle\int_0^x \frac{e^t}{t}\,dt$

This $y(x)$ cannot be expressed through elementary functions, much like $\displaystyle\int e^{-x^2}\,dx$.

(j) $y\,(x) = \sin(-\arcsin\,(x) + C)$

(k) $y\,(x) = (\sqrt{x} + C)^2$.

11) (a) $y(x) = (x+2)^2$

(b) $y(x) = \sin(x^2)$

(c) $P(t) = \left(\dfrac{t^{3/2}}{3} + \sqrt{2} - \dfrac{1}{3}\right)^2$

(d) $\dfrac{1}{2}\,y\,(x)^2 + \dfrac{1}{3}\left(y\,(x)^2 + 3\right)^{\frac{3}{2}} = \dfrac{1}{2}\,x^2 \ln\,(x) - \dfrac{1}{4}\,x^2 + \dfrac{41}{12}$

(e) $y(x) = -2(2\cos(x) - 1)$.

12) (a) $y(x) = Cx^3$

(b) $\dfrac{1}{2}\,y\,(x)^2 = \dfrac{1}{2}\,x^2 + C$.

(c) $x^2 + \dfrac{3}{2}y^2 = 7$

(d) $x = \dfrac{y^3}{3} - \dfrac{8}{3} + 1$

13) (a) Volume of carbon dioxide in the room at time t:
$v(t) = 0.09 + 0.18e^{-t/90}$ m^3,
fraction of the total volume: $v(t)/180$. After a long time the volume of CO_2 is leveled out at 0.09m^3.

(b) Volume of the alcohol contained in the vat at time t:
$v(t) = 30 - 10e^{-t/100}$ gal
fraction of the total volume: $v(t)/500$. After an hour the fraction of alcohol is $v(60)/500 \approx 0.049$.

15) (a) Converges to 5.

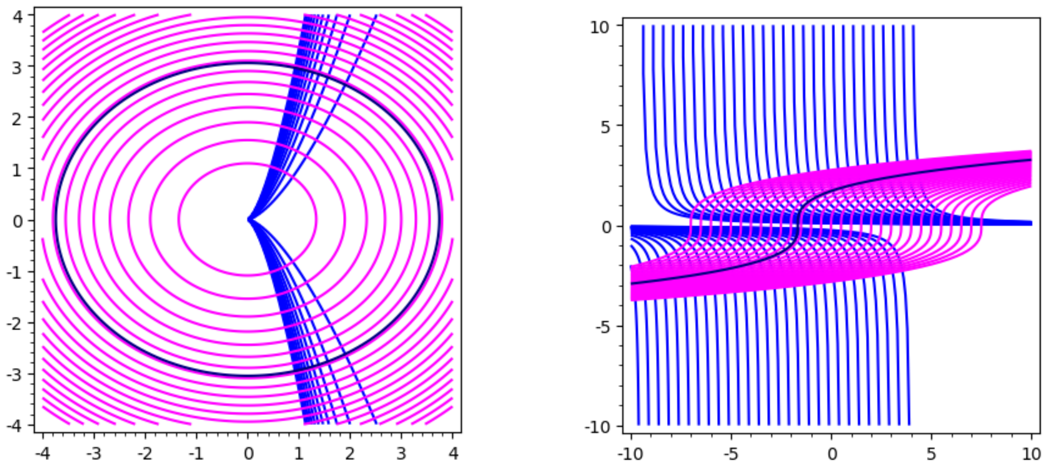

FIGURE 5. Orthogonal trajectories in questions 12) (c)–(d), in magenta; the original families are shown in blue. The trajectory passing through $(1,2)$ is shown in dark blue.

(b) Diverges to $+\infty$.
(c) Converges to 0.
(d) Diverges.
(e) Converges to 1.
(f) Converges to 8.
(g) Converges to 1.
(h) Diverges.
(i) Diverges to $+\infty$.
(j) Converges to 0.
(k) Converges to $\pi/2$.
(l) Converges to e^2.
(m) Diverges to $+\infty$.
(n) Converges to -2.

16) (a) Bounded.
(b) Bounded, decreasing.
(c) Bounded, decreasing.
(d) Unbounded, eventually increasing.
(e) Bounded.
(f) Bounded, decreasing.
(g) Bounded, decreasing.