Measure and Integration II (MAA5617), Spring 2021
Homework 5, due Thursday, Apr 1

Below $T : \mathcal{X} \to \mathcal{Y}$ is a linear map between normed vector spaces; $\mathcal{Z}$ is also a normed vector space.

1. Prove that
$$\rho(x, y) = \frac{\|x - y\|}{1 + \|x - y\|}$$
defines a metric on $\mathcal{X}$ that does not correspond to any norm.

2. Prove the equivalence of different ways to define operator norm:
$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq \vec{0}\right\} \\ &= \inf\{C > 0 : \|Tx\| \leq C\|x\| \ \forall x \in \mathcal{X}\}.\end{aligned}$$

3. Prove that operator norm is a norm on $L(\mathcal{X}, \mathcal{Y})$.

4. Let $B \in L(\mathcal{Y}, \mathcal{Z})$. Show that
$$\begin{gathered}\phi_B : L(\mathcal{X}, \mathcal{Y}) \to L(\mathcal{X}, \mathcal{Z}), \\ \phi_B(A) = B \circ A,\end{gathered}$$
satisfies $\|\phi_B\| \leq \|B\|$ (these are operator norms in different spaces!).

5. Prove that $T = \frac{d}{dt} : \mathbb{R}[t] \to \mathbb{R}[t]$ is an unbounded linear operator. Here $\mathbb{R}[t]$ are polynomials in t over $[a, b]$, equipped with uniform norm.

Constructing an unbounded operator on a complete space is more tricky, and is usually done using the axiom of choice, see Folland p. 179.

6. Suppose A is a symmetric $n \times n$ matrix over $\mathbb{R}$. Prove that its operator norm $\|A\|$ is equal to the absolute value of its largest eigenvalue. (Use Lagrange's method to diagonalize A.)

7. Prove that any two norms on $\mathbb{R}^n$ are equivalent.

Show that any given norm $\|\cdot\|$ is continuous with respect to the Euclidean norm $\|\cdot\|_2$. This will imply that it achieves its maximum and minimum on the unit sphere $\{x : \|x\|_2 = 1\}$, and they must both be finite and positive, implying equivalence.

Continuity can be obtained like so: given an x with $\|x\|_2 = 1$ and the standard basis $\{e_i\}_1^n$, there holds

$$\|x\| = \left\| \sum_i \alpha_i e_i \right\| \le \sum_i \|e_i\| = \|x\|_2 \cdot \sum_i \|e_i\| =: C\|x\|_2.$$

Rescaling x gives for *any* vector in $\mathbb{R}^n$

$$\|x\| \le C\|x\|_2,$$

which is Lipschitz continuity:

$$\|v_1 - v_2\| \le C\|v_1 - v_2\|_2,$$

for any pair of vectors $v_1, v_2 \in \mathbb{R}^n$.

8. Give a counterexample to the parallelogram law in $C([0,1])$ with the uniform norm.

Recall that a set is called *convex* if it contains every line segment $[x, y]$ connecting a pair of its points x, y.

9. Prove that any closed convex set in a Hilbert space has a unique element of smallest norm. (Argue as in the first part of Theorem 5.24.)

10. For any subset E of a Hilbert space $\mathcal{H}$, prove that $(E^\perp)^\perp$ is the smallest closed vector subspace of $\mathcal{H}$, containing E.

11. Let $\mathcal{H} = l^2(\mathbb{N})$, the space of square-summable sequences $x = (x_1, x_2 \ldots)$ with elements from $\mathbb{R}$, with the norm

$$\|x\|^2 = \sum_1^\infty x_i^2,$$

as defined in class.

- Construct a bounded (=of bounded norm) sequence $\{x_i\}$ of elements from $\mathcal{H}$, such that $\|x_i - x_j\| \ge 1$.
- Prove that the unit ball in $\mathcal{H}$ is not compact.