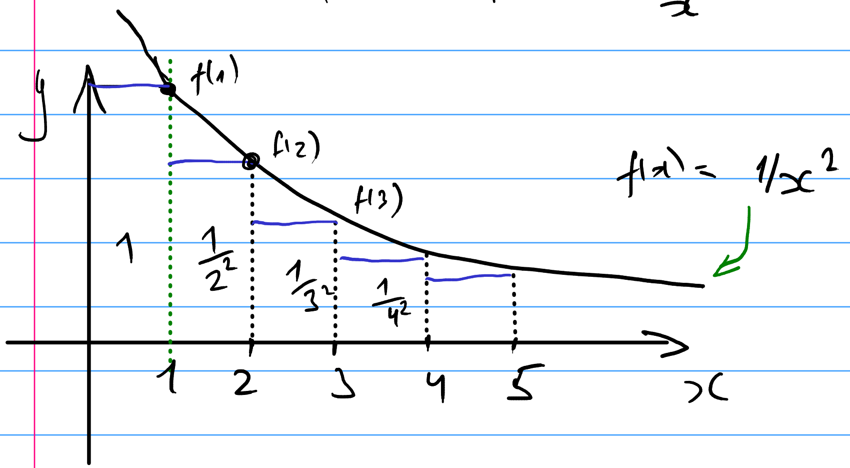
Section 11.3: Integral test

Ex. 1 Consider: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \underbrace{\frac{1}{3^2}}_{a_3} + \dots$

Then for $f(x) = \dfrac{1}{x^2}$, $\quad f(k) = a_k$



$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \boxed{\text{Area of rectangles}} \leq \underbrace{1}_{\text{1st rectangle}} + \int_1^{\infty} \frac{1}{x^2}$$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^2}}_{a_k} \leq 1 + \int_1^{\infty} x^{-2}\,dx = 1 + \left(-x^{-1}\right)\Big|_1^{\infty} = 2.$$

So, any partial sum satisfies

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$$

$\Rightarrow$ The sequence of partial sums $\{S_n\}$ is bounded:

$$0 < S_n \leq 2$$

But the a_k are nonnegative, so

$$S_{n+1} - S_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1} > 0$$

$\Rightarrow S_{n+1} > S_n$.

The sequence of partial sums $\{S_n\}$ is increasing and bounded $\Rightarrow$ convergent!

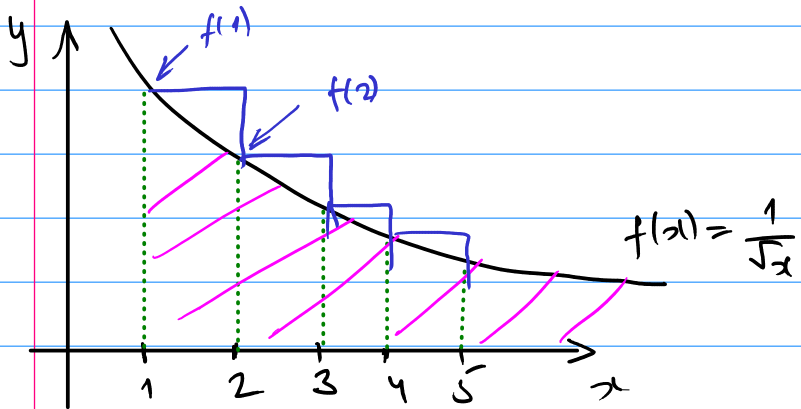
$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

(not to 2 though).

Ex. 2 Consider:

$$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

Introduce $f(x) = \frac{1}{\sqrt{x}}$, then $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = +\infty$



$$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \boxed{\text{Area of rectangles}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}} dx =$$

$$= 2\sqrt{x}\Big|_1^{\infty} = \lim_{t \to \infty} 2\sqrt{x}\Big|_1^{t} = +\infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{x^{1/2}} = +\infty, \quad \text{divergent.}$$

The integral test for convergence.

Suppose $f(x)$ is continuous, positive, decreasing on $[1,\infty)$, and

$$f(k) = a_k, \qquad k \geq 1,$$

then

$$\sum_{k=1}^{\infty} a_k \text{ is convergent} \iff \int_1^{\infty} f(x)\,dx \text{ is convergent.}$$

Remark: these properties only need to hold for $k \geq k_0$.

Ex. 3 Test for convergence: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

Consider $f(x) = \dfrac{1}{x^2+1}$ } positive, continuous, decreasing

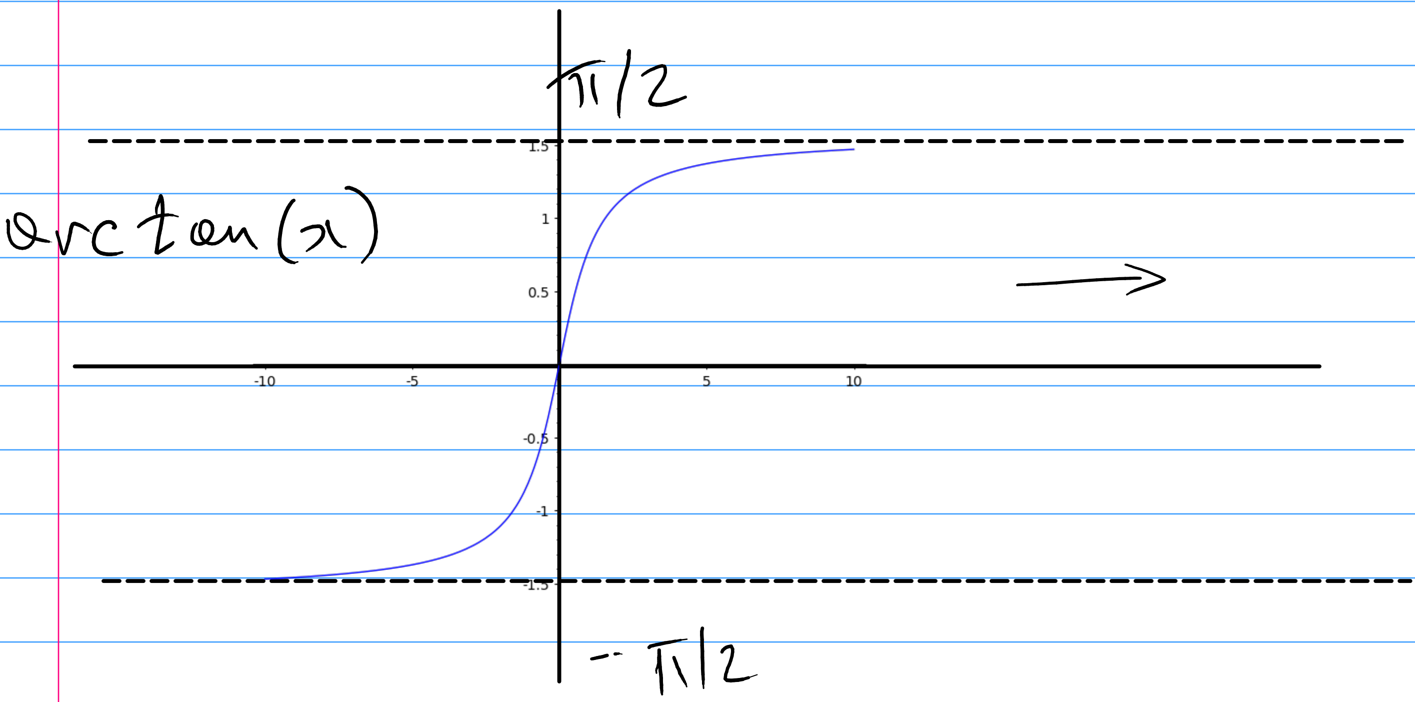
on $[1,\infty)$.

By the integral test, the given series is convergent if and only if the improper integral is convergent:

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_1^{t} \frac{dx}{x^2+1} = \lim_{t\to\infty} \arctan x \Big|_1^t$$

$$= \lim_{t\to\infty} (\arctan t - \arctan 1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \Big\} \text{ Convergent}$$



$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2+1} \quad \text{is} \quad \underline{\text{convergent}}.$$

Ex. 4 For which p is $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^p}}_{p\text{-series}}$ convergent?

For $p \le 0$, $\quad a_k = \frac{1}{k^p} \not\to 0, \; k \to \infty$

$\Rightarrow$ the series is <u>divergent</u> by the divergence test.

For $p > 0$, consider $f(x) = \frac{1}{x^p}$
for x in $[1, \infty)$,

this $f(x)$ is $\begin{cases} \text{positive} \\ \text{continuous} \\ \text{decreasing} \end{cases}$

$\Rightarrow$ the integral test applies,

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent $\Leftrightarrow$

$\Leftrightarrow$ $\int_1^{\infty} \frac{1}{x^p} \, dx$ is convergent $\Leftrightarrow$

$\Leftrightarrow$ $p > 1$.

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent $\Leftrightarrow$ $p > 1$.

Ex. 5 Test for convergence:

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} \quad \} \text{ divergent.}$$

$$\underbrace{f(x)}_{} = \frac{\ln x}{x} \quad \text{for} \quad \underline{x \geq 1}.$$

positive

continuous

but not decreasing!

$$f'(x) = \left(\frac{\ln x}{x}\right)' = \frac{1/x \cdot x - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2} < 0, \quad \text{when} \quad \ln x > 1,$$

or $x > e$.

$\Rightarrow$ the integral test applies to

$$\sum_{k=3}^{\infty} \frac{\ln k}{k}$$

$\Rightarrow$ the series is convergent if and only if so is the integral

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{t \to \infty} \int_1^{t} \frac{\ln x}{x}\, dx$$

$$= \left| \begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix} \right| = \lim_{t \to \infty} \int_0^{\ln t} u \, du$$

$$= \lim_{t \to \infty} \frac{u^2}{2} \Big|_0^{\ln t} = \lim_{t \to \infty} \frac{(\ln t)^2}{2} = +\infty$$

$\Rightarrow$ the series is divergent

Estimating the sum of a series

We have seen: $\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$.

In fact $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (Euler, uses Fourier expansions)

We can use the ideas of the Integral test to estimate, how well the partial sums of a series approximate the infinite sum:

Given a series $\sum_{k=1}^{\infty} a_k$, suppose $a_k \geq 0$, $\sum_{k=1}^{\infty} = S$.

Consider $\overbrace{R_n}^{\text{remainder}} = S - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k$

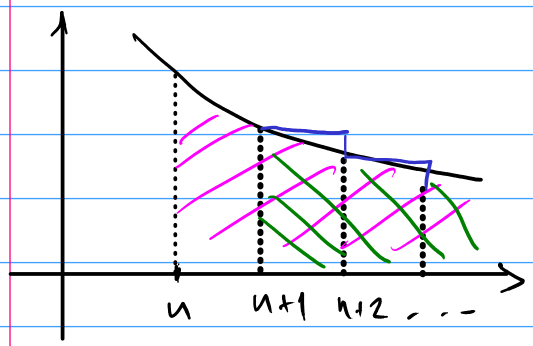
$$R_n = \sum_{k=n+1}^{\infty} a_k$$

Applying the area arguments from the integral test, we can estimate R_n:

Remainder estimate. Suppose $f(x)$ is continuous, positive, decreasing on $x \geq n \geq 1$, and $a_k = f(k)$ $k \geq n$.

Then for $R_n = \sum_{k=n+1}^{\infty} a_k$ there holds

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$



Ex. 6 (a) Approximate $\sum_{k=1}^{\infty} \frac{1}{k^3}$ by s_{10}. Estimate the error.

(b) How many terms are necessary for the precision of 10^{-5}?

(a) $$S - s_{10} = R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \left.\frac{x^{-2}}{-2}\right|_{10}^{t}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left(\frac{1}{100} - \frac{1}{t^2}\right) = \frac{1}{200} = 0.005$$

$$s_{10} = \sum_{k=1}^{10} \frac{1}{k^3} = 1.1975$$

(b) $s - s_n = R_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2} \cdot \frac{1}{n^2}$

To guarantee that $R_n \leq 10^{-5}$, we take n so large that $\frac{1}{2n^2} \leq 10^{-5}$

$\Rightarrow R_n \leq \frac{1}{2n^2} \leq 10^{-5}$

Solve for n:

$$\frac{1}{2n^2} \leq 10^{-5} = \frac{1}{10^5} \Rightarrow 2n^2 > 10^5$$

$$n \geq \sqrt{\frac{10^5}{2}} \approx 223.6$$

$\Rightarrow n \geq 224$