Section 7.4: Integrating rational functions via partial fractions.

Rational functions:

$$f(x) = \frac{P(x)}{Q(x)}$$ P, Q - polynomials

$$\begin{Bmatrix} \text{rational} \\ \text{function} \end{Bmatrix} \longmapsto \begin{Bmatrix} \text{sum of} \\ \text{partial} \\ \text{fractions} \end{Bmatrix}$$

$$\underbrace{\frac{x^3 - x^2 + 8x}{(x^2+1)(x-2)^2}}_{\text{rational function}} = \underbrace{\frac{x-1}{x^2+1}}_{} + \underbrace{\frac{4}{(x-2)^2}}_{\text{partial fractions}}$$

$$f(x) = \frac{P(x)}{Q(x)}$$

f is proper, if $\deg P < \deg Q$

f is improper, if $\deg P \geq \deg Q$.

Partial fraction decomposition applies only to proper rational functions!

For an improper f: apply the long division:

$$f(x) = \frac{P(x)}{Q(x)} \overset{\text{long division}}{=} S(x) + \underbrace{\frac{R(x)}{Q(x)}}_{\text{proper rational function}}$$

Outline of the PFD:

Take $f = \dfrac{P(x)}{Q(x)}$ — proper

1. Factor $Q(x)$ $\to$ 2 types of irreducible factors: $(ax+b)$ OR (ax^2+bx+c) w/ $b^2-4ac<0$

2. Make a sum of partial fractions: for each factor of the form in the left column, add terms from the right column

Terms of $Q(x)$	Terms in PFD
$(ax+b)$	$\dfrac{A}{ax+b}$
$(ax+b)^r$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \dots + \dfrac{A_r}{(ax+b)^r}$
(ax^2+bx+c) w/ $b^2-4ac<0$	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^r$ w/ $b^2-4ac<0$	$\dfrac{A_1x+B_1}{(ax^2+bx+c)} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \dfrac{A_rx+B_r}{(ax^2+bx+c)^r}$

3. Equate $f(x) = \text{PFD}$, then determine the undefined coefficients

Evaluate integrals:

Ex. 1 $\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx \stackrel{}{=}$

$\underbrace{\frac{x^3+x}{x-1}}_{\text{improper rational f-n}}$

Perform the long division

$$\begin{array}{r|l} & x^2+x+2 \\ \hline x-1 & x^3+x \\ & -\ x^3-x^2 \\ \hline & \quad x^2+x \\ & -\ x^2-x \\ \hline & \qquad 2x \\ & -\ 2x-2 \\ \hline & \qquad\ 2 \end{array}$$

$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C$

Ex. 2 $\int \underbrace{\frac{x^2+2x-1}{2x^3+3x^2-2x}}_{\text{proper rational function}}\,dx$ } Case I: unique linear factors

$$\begin{aligned} 2x^3+3x^2-2x &= x(2x^2+3x-2) \\ &= x\cdot 2(x-(-2))(x-\tfrac{1}{2}) \\ &= x\cdot(x+2)(2x-1) \end{aligned}$$

$2x^2+3x-2=0$

$x_{1,2}=\frac{-3\pm\sqrt{9+16}}{4}=\frac{-3\pm 5}{4}$

$x_1=-2$

$x_2=\frac{1}{2}$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}$$

Multiply both sides by $x(x+2)(2x-1)$:

$$x^2+2x-1 = A(x+2)(2x-1) + Bx(2x-1)$$
$$+ Cx(x+2)$$

For unique linear factors: plug in roots of the denominator

$x=-2$:

$$-1 = B(-2)(-5) \Rightarrow B = -\frac{1}{10}$$

$x=0$:

$$-1 = A\cdot 2(-1) \Rightarrow A = \frac{1}{2}$$

$x=\frac{1}{2}$:

$$\frac{1}{4} = C\cdot\frac{1}{2}\cdot\frac{5}{2} \Rightarrow C = \frac{1}{5}$$

$$\int\left(\frac{1}{2x} - \frac{1}{10(x+2)} + \frac{1}{5(2x-1)}\right)dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| + K$$

Ex. 3 $$\underbrace{\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx}_{\text{improper rational function}}$$ } Case II: repeated linear factors

Long division:

$$\begin{array}{r|l}
 & x+1 \\
x^3-x^2-x+1 & x^4-2x^2+4x+1 \\
- & \cancel{x^4}-x^3-x^2+x \\
\hline
 & x^3-x^2+3x+1 \\
- & x^3-x^2-x+1 \\
\hline
 & 4x
\end{array}$$

$$\int \Big(x+1+\underbrace{\frac{4x}{x^3-x^2-x+1}}_{\text{proper}}\Big)\,dx$$

Factor the denominator:

$$\begin{aligned}
x^3-x^2-x+1 &= (x^3-x^2)-(x-1)\\
&= x^2(x-1)-(x-1)\\
&= (x-1)\underbrace{(x^2-1)}=(x-1)\underbrace{(x-1)(x+1)}\\
&= (x-1)^2(x+1)
\end{aligned}$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Multiply by the common denominator:

$$4x = A\underbrace{(x-1)^2}_{x^2-2x+1} + B\underbrace{(x-1)(x+1)}_{x^2-1} + C(x+1)$$

Equate the coefficients at correspon-
ding powers of x:

x^2:
$$0 = A + B \implies \underline{A = -B}$$
x:
$$\underline{4 = -2A + C}$$
x^0:
$$\underline{0 = A - B + C}$$

$$4 = -2A + C$$
$$0 = A - (-A) + C$$

$$\begin{cases} 4 = -2A + C \\ 0 = 2A + C \end{cases} \implies 4 = 2C$$
$$C = 2$$
$$A = -1$$
$$B = 1$$

$$\int \left(x + 1 + \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx$$
$$= \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + K$$