Measure and Integration I (MAA5616), Fall 2020
Homework 7, due Thursday, Oct. 22

1. For $\{f\}, \{f_n\} \subset L_+$ such that $f_n \searrow f$ pointwise and $\int f_1 < \infty$, prove $\int f = \lim_n \int f_n$.

2. Suppose $\{f\}, \{f_n\} \subset L_+$ such that $f_n \to f$ pointwise.

- Prove that $\int_E f = \lim_n \int_E f_n$ for all $E \in \mathcal{M}$ if $\int f = \lim_n \int f_n < \infty$.
- Prove that $\int_E f = \lim_n \int_E f_n$ is not necessarily true for all $E \in \mathcal{M}$, if $\int f = \lim_n \int f_n = \infty$.

3. Suppose $X = [0, 1]$. Let $f : X \to [0, \infty)$ be Lebesgue-measurable and $\int f \, d\lambda < \infty$. Prove: if

$$\int f^n \, d\lambda = \int f \, d\lambda, \qquad n \geq 1,$$

then there exists a measurable set $E \subset [0, 1]$ such that $f(x) = 1_E(x)$ for λ-a.e. $x \in [0, 1]$.

4. Prove: if E is Lebesgue-measurable and $\lambda(E) > 0$, the set

$$E - E = \{x - y : x, y \in E\}$$

contains a neighborhood of zero.

Hint: let $G \supset E$ such that $\lambda(E) > \frac{3}{4}\lambda(G)$. There must exist a connected component (an interval) $I \subset G$ for which $\lambda(E \cap I) > \frac{3}{4}\lambda(I)$. Show that for any $x \in (-\delta, \delta)$ with a sufficiently small δ, the interval $I \cup (x + I)$ is shorter than $\frac{3}{2}\lambda(I)$, but contains both $E \cap I$ and $x + (E \cap I)$, which must therefore intersect. Then $x \in E - E$.

The beginning of the hint shows also that for any $a \in (0, 1)$, there exists an open interval I for which $\lambda(E \cap I) > a\,\lambda(I)$.

Recall that $C - C = [-1, 1]$, so positivity of the measure is not a necessary condition.