Section 7.2: Trigonometric integrals

Ex. 1 Evaluate $\int \cos^3 x \, dx$

$$\int \underbrace{\cos^2 x}\cdot \cos x \, dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right|$$

$$\int (1-\sin^2 x)\cdot \cos x \, dx = \left| \begin{array}{l} u=\sin x \\ du=\cos x\, dx \end{array} \right| = \int (1-u^2)\, du$$

$$= \int 1 \, du - \int u^2 \, du = u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{(\sin x)^3}{3} + C$$

Ex. 2 Find $\int \sin^5 x \cos^2 x \, dx$

$$\int \sin^4 x \cdot \cos^2 x \cdot \sin x \, dx =$$

$$= \int \underbrace{(\sin^2 x)^2}_{1-\cos^2 x} \cdot \cos^2 x \cdot \sin x \, dx = \int (1-\cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$$

$$= \left| \begin{array}{l} u = -\cos x \\ du = \sin x \, dx \end{array} \right| = \int \left(1-(-u)^2\right)^2 \cdot (-u)^2 \cdot du$$

$$= \int (1-u^2)^2 \cdot u^2 \, du = \int (1-2u^2+u^4)\cdot u^2 \, du =$$

$$(a \pm b)^2 = a^2 + 2ab + b^2$$

$$= \int (u^2 - 2u^4 + u^6)\, du = \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{(-\cos x)^3}{3} - \frac{2(-\cos x)^5}{5} + \frac{(-\cos x)^7}{7} + C$$

Half-angle identities:

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

Ex. 3 $\displaystyle\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \frac{1}{2}(1-\cos 2x)\, dx$

$$= \frac{1}{2}\Big(\int_0^{\pi} 1\, dx - \underbrace{\int_0^{\pi} \cos 2x\, dx}_{\substack{u = 2x \\ du = 2dx \\ \frac{1}{2}du = dx}}\Big)$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^{2\pi} \cos u \,\frac{1}{2}\, du$$

$$= \frac{\pi}{2} - \frac{1}{2}\cdot\frac{1}{2}\sin u \Big|_0^{2\pi} = \frac{\pi}{2}.$$

Ex. 4 Evaluate $\displaystyle\int \sin^4 x\, dx$

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx =$$

$$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx = \frac{1}{4}\int (1-\cos 2x)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx$$

$(a-b)^2 = a^2-2ab+b^2$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx$$

$$= \frac{1}{4}\int\left(\frac{3}{2} - 2\overbrace{\cos 2x}^{\frac{\sin 2x}{2}} + \frac{1}{2}\overbrace{\cos 4x}^{\frac{\sin 4x}{4}}\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

Strategy for evaluating $\int \sin^m x \cos^n x \, dx$

(m, n – integer)

(a) If n – odd $\quad n = 2k+1$

$$\int \sin^m x \cdot \cos^{2k+1} x \, dx = \int \sin^m x \cdot \overbrace{(\cos^2 x)^k}^{\cos^{2k} x} \cdot \underbrace{\cos x \, dx}_{d(\sin x)}$$

$$= \int \sin^m x \left(1 - \sin^2 x\right)^k \cdot \cos x \, dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \right|$$

$$= \int u^m (1-u^2)^k \, du = \text{expand, integrate...}$$

(b) If m – odd, $\quad m = 2k+1$

$$\int \sin^{2k+1} x \cdot \cos^n x \, dx = \int (\sin^2 x)^k \cdot \cos^n x \cdot \underbrace{\sin x \, dx}_{-d(\cos x)}$$

$$= \int (1 - \cos^2 x)^k \cdot \cos^n x \cdot \sin x \, dx = \left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \\ -du = \sin x \, dx \end{array} \right.$$

$$= \int (1-u^2)^k \cdot u^n (-du) = -\int (1-u^2)^k \cdot u^n \, du$$

$= \text{expand, integrate...}$

(c) If neither m nor n is odd, apply the half-angle identities:

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A) \qquad \sin^2 A = \frac{1}{2}(1 - \cos 2A).$$

$$\cos^2 x + \sin^2 x = 1$$
$$(\cos x)' = -\sin x$$
$$(\sin x)' = \cos x$$

$$\cos^2 x + \sin^2 x = 1 \qquad / \cos^2 x$$
$$1 + \left(\frac{\sin x}{\cos x}\right)^2 = \left(\frac{1}{\cos x}\right)^2$$
$$1 + \tan^2 x = \sec^2 x$$
$$(\tan x)' = \sec^2 x$$
$$(\sec x)' = \sec x \tan x$$

$$\cos^2 x + \sin^2 x = 1 \qquad / \sin^2 x$$
$$\left(\frac{\cos x}{\sin x}\right)^2 + 1 = \left(\frac{1}{\sin x}\right)^2$$
$$\cot^2 x + 1 = \csc^2 x$$
$$(\cot x)' = -\csc^2 x$$
$$(\csc x)' = -\csc x \cdot \cot x$$

$(\tan x)' = \sec^2 x$; $(\sec x)' = \sec x \tan x$

Ex. 5 $\int \tan^6 x \cdot \sec^4 x \, dx =$

$$\int \tan^6 x \cdot \sec^2 x \cdot \underbrace{\sec^2 x \, dx}_{d(\tan x)}$$

$$= \int \tan^6 x \cdot (1+\tan^2 x) \cdot \sec^2 x \, dx = \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \right.$$

$$= \int u^6 (1+u^2) \, du = \int (u^6 + u^8) \, du = \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{(\tan x)^7}{7} + \frac{(\tan x)^9}{9} + C$$

Ex. 6 $\int \tan^5 x \cdot \sec^7 x \, dx$

$$\int \tan^5 x \cdot \sec^7 x \, dx = \int \underbrace{\tan^4 x}_{(\tan^2 x)^2} \cdot \sec^6 x \cdot \underbrace{\sec x \tan x \, dx}_{d(\sec x)}$$

$$= \int (\sec^2 x - 1)^2 \cdot \sec^6 x \cdot \sec x \tan x \, dx$$

$$= \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right|$$

$$= \int (u^2-1)^2 \cdot u^6 \, du = \int (u^4 - 2u^2 + 1) u^6 \, du$$

$$= \int (u^{10} - 2u^8 + u^6) \, du = \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{(\sec x)^{11}}{11} - \frac{2(\sec x)^9}{9} + \frac{(\sec x)^7}{7} + C.$$

Strategy for evaluating $\int \tan^m x \sec^n x \, dx$

$(m, n$ — positive integers$)$

(a) n - even, $\quad n = 2k, \quad k \geq 1.$

$$\int \tan^m x \cdot \sec^{2k} x \, dx = \int \tan^m x \cdot \sec^{2k-2} x \cdot \underbrace{\sec^2 x \, dx}_{d(\tan x)}$$

$$= \int \tan^m x \cdot (\sec^2 x)^{k-1} \cdot \sec^2 x \, dx$$

$(b^p)^r = b^{p \cdot r}$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \cdot \sec^2 x \, dx$$

$$= \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x \, dx \end{array} \right|$$

$$= \int u^m (1+u^2)^{k-1} \, du$$

$=$ expand, integrate.

(b) m - odd, $\quad m = 2k+1, \quad n \geq 1$

$$\int \tan^{2k+1} x \cdot \sec^n x \, dx =$$

$$= \int (\tan^2 x)^k \cdot \sec^{n-1} x \cdot \underbrace{\sec x \cdot \tan x \, dx}_{d(\sec x)}$$

$$= \int (\sec^2 x - 1)^k \cdot \sec^{n-1} x \cdot \sec x \tan x \, dx$$

$$= \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array} \right|$$

$$= \int (u^2 - 1)^k \cdot u^{n-1} \, du = \text{expand, integrate...}$$

The other cases are handled by integration by parts (reduction formulas) + special cases for small m,n.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right.$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C =$$

Recall: $-\ln t = \ln \frac{1}{t}$

$$= \ln \left| \frac{1}{\cos x} \right| + C = \ln|\sec x| + C.$$

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \left| \begin{array}{l} u = \sec x + \tan x \\ du = \sec x \tan x + \sec^2 x \, dx \end{array} \right.$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

Reduction formulas for $\tan x$, $\sec x$.

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$
$(n \geq 2)$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$
$(n \geq 2)$

$$\int \tan^n x\,dx = \int \tan^{n-2} x \cdot \tan^2 x\,dx$$
$$= \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx$$
$$= \int \tan^{n-2} x \cdot \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$
$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \int \underbrace{\sec^{n-2} x}_{u} \cdot \underbrace{\sec^2 x\,dx}_{dv}$$

$$= \left| \begin{array}{ll} u = \sec^{n-2} x & dv = \sec^2 x\,dx \\ du = (n-2)\sec^{n-3} x \cdot \sec x \cdot \tan x\,dx & v = \tan x \\ \quad = (n-2)\sec^{n-2} \cdot \tan x\,dx & \end{array} \right.$$

$$= \sec^{n-2} x \cdot \tan x - (n-2)\int \tan x \cdot \sec^{n-2} x \cdot \tan x\,dx$$
$$= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x\,dx$$
$$= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$\int \sec^n x \, dx =$$

$$= \sec^{n-2}x \cdot \tan x - (n-2)\int \sec^n x \, dx - (n-2)\int \sec^{n-2} x \, dx$$

$$(n-1)\int \sec^n x \, dx = \sec^{n-2}x \tan x - (n-2)\int \sec^{n-2} x \, dx$$

$$/(n-1)$$