Section 11.6 : Power series

Def : a power series is:

$$\sum_{k=0}^{\infty} \underbrace{c_k}_{\text{coefficients}} \cdot x^{\overset{\text{variable}}{k}} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Power series is a function!

$$\overset{x}{\underset{\text{input}}{\longrightarrow}} \overset{\text{function}}{\boxed{\text{power series}}} \overset{\text{number series}}{\underset{\text{output}}{\longrightarrow}}$$

Ex. 1 Let $c_k = 1$, $k \geq 0$. Then:

$$\underbrace{\sum_{k=0}^{\infty} x^k}_{\substack{\text{geometric series:} \\ r = x \\ a = 1}} = 1 + \underbrace{x + x^2}_{\frac{x^2}{x} = x} + \underbrace{x^3}_{\frac{x^3}{x^2} = x} + \dots$$

This power series converges if and only if $|r| = |x| < 1$.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad |x| < 1.$$

Def

$$\sum_{k=0}^{\infty} c_k (x-a)^k =$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

— power series, centered at a.

Ex. 2 Let $c_k = 0$, $k \geq 3$;

$c_0 = 5$

$c_1 = 0$

$c_2 = 4$

Consider a power series, centered at $a = 2$, with these c_k.

$$\sum_{k=0}^{\infty} c_k (x-2)^k = c_0 + c_1(x-2) + c_2(x-2)^2$$

$$= 5 + \qquad\qquad + 4(x-2)^2$$

$$= 4(x-2)^2 + 5$$

Power series $\simeq$ infinite polynomials.

Ex. 3 For which x is

$$\sum_{k=1}^{\infty} \underbrace{\frac{(x-3)^k}{k}}_{} = a_k \quad \text{– convergent?}$$

$a = 3$

$c_k = \frac{1}{k} \qquad k \geq 1$

$c_0 = 0$

Apply the ratio test: $\left(a_k = \frac{(x-3)^k}{k} \right)$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \underbrace{\frac{(x-3)^{k+1}}{k+1}}_{a_{k+1}} \Big/ \underbrace{\frac{(x-3)^k}{k}}_{a_k} \right|$$

$$\lim_{k\to\infty} \left| \frac{\cancel{(x-3)}^{k+1}}{k+1} \cdot \frac{k}{\cancel{(x-3)}^k} \right| =$$

$$\lim_{k\to\infty} \left| (x-3) \cdot \underbrace{\frac{k}{k+1}}_{} \right| =$$

$$\lim_{k\to\infty} \frac{k}{k+1} \cdot |x-3| \overset{>0}{=} |x-3| \cdot \lim_{k\to\infty} \frac{k}{k+1} \to 1$$

$$= |x-3|.$$

The given series is absolutely convergent, when $|x-3| < 1$.

It is divergent when $|x-3| > 1$.

Finally, when $|x-3| = 1 \Rightarrow$ the ratio test is inconclusive.

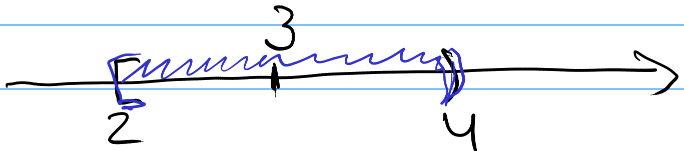
When $|x-3|=1$: $x=4$, or $x=2$.

For $x=4$:

$$\sum_{k=1}^{\infty}\frac{(x-3)^k}{k}=\sum_{k=1}^{\infty}\frac{1}{k}\ \Big\}\ \text{divergent: as a } p\text{-series, } p=1.$$

For $x=2$:

$$\sum_{k=1}^{\infty}\frac{(x-3)^k}{k}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\ \Big\}\ \text{convergent by the alt series test.}$$



Answer: this series is convergent for x in $\underbrace{[2,4)}_{\text{domain of the function, given by the power series}}$.

Ex. 4 For which x is

$0! = 1$ $\quad \sum_{k=0}^{\infty} \underbrace{k!\cdot x^k}_{=a_k}$ — convergent?

Apply the ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{(k+1)!\cdot \cancel{x^{k+1}}}{k!\cdot\cancel{x^k}}\right|=$$

$$\lim_{k\to\infty}\left|\frac{(k+1)!\cdot x}{k!}\right|=\lim_{k\to\infty}|x|\cdot\frac{(k+1)!}{k!}$$

$$\lim_{k\to\infty} |x| \cdot \frac{1\cdot 2\cdot 3\cdot \ldots \cdot k\,(k+1)}{1\cdot 2\cdot 3\cdot \ldots \cdot k} =$$

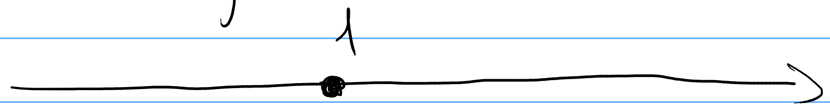
$$\lim_{k\to\infty} |x|\cdot(k+1) = |x|\cdot \lim_{k\to\infty} k+1$$

$$= |x|\cdot(+\infty)$$

$$= \underbrace{+\infty}_{>1}, \quad \text{if } |x| \neq 0$$

Divergent for $x \neq 0$!

Convergent for $x = 0$ only

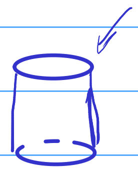


↳ domain of $\sum_{k=0}^{\infty} k!\cdot x^k$

Ex. 5 Find the domain of the J_0 Bessel function, given by

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2}$$

describes the shape of the vibrating drum membrane



That is, we have to find x for which the series of J_0 converges.

Apply the test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} \frac{x^{2(k+1)}}{2^{2(k+1)}((k+1)!)^2}}{(-1)^k \frac{x^{2k}}{2^{2k}(k!)^2}} \right|$$

$$\lim_{k\to\infty}\left|\frac{x^{2k+2}}{2^{2k+2}((k+1)!)^2}\Big/\frac{x^{2k}}{2^{2k}\cdot(k!)^2}\right| =$$

$$\lim_{k\to\infty}\left|\frac{x^2}{4\,(\underbrace{k!\,(k+1)}_{(k+1)!\,=\,k!\,(k+1)})^2}\cdot\frac{(k!)^2}{1}\right| =$$

$(ab)^2 = a^2 b^2$

$$\lim_{k\to\infty}\left|\frac{x^2}{4}\cdot\frac{\cancel{(k!)^2}}{\cancel{(k!)^2}\cdot(k+1)^2}\right| =$$

$$= \lim_{k\to\infty}\left|\frac{x^2}{4\,(k+1)^2}\right| = \lim_{k\to\infty}\frac{x^2}{4\,(k+1)^2} =$$

$$\frac{x^2}{4}\cdot\underbrace{\lim_{k\to\infty}\frac{1}{(k+1)^2}}_{=\,0} = \frac{x^2}{4}\cdot 0 = 0. < 1$$

By the Ratio test, the series for J_0 is absolutely convergent for all x! $\Rightarrow$ domain of J_0 is $(-\infty,\infty)$.

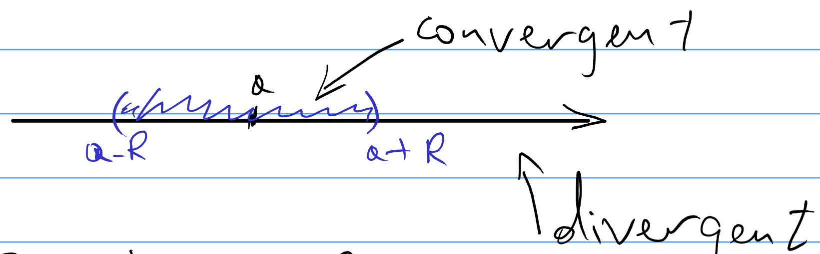
↑ domain of J_0.

Thm: For any power series $\sum_{k=0}^{\infty} c_k (x-a)^k$, there are only the following possibilities:

i) The series converges only at $x = a$.

ii) — " — — " — if $|x-a| < R$ for some positive R, and it is divergent $|x-a| > R$

iii) It is convergent for all $x \in (-\infty, \infty)$.



For i), $R = 0$

iii), $R = +\infty$.

R = the radius of convergence

The set of x, on which the series is convergent = the interval of convergence.

Series	Radius of convergence	Interval of conv.
$\sum_{k=0}^{\infty} x^k$	$R = 1$	$\|x\| < 1$ $\quad (-1, 1)$
$\sum_{k=0}^{\infty} k! \cdot x^k$	$R = 0$	$x = 0$
$\sum_{k=1}^{\infty} \frac{(x-3)^k}{k}$	$R = 1$	$[2, 4)$
J_0	$R = +\infty$	$(-\infty, \infty)$

To determine the interval of convergence:

- apply the ratio test to find R
- inspect the endpoints of the convergence interval.

Ex. 6 Find radius/interval of convergence:

$$\sum_{k=0}^{\infty} \frac{(-3)^k \cdot x^k}{\sqrt{k+1}}$$

Ratio test:

$$\lim_{k\to\infty} \left| \underbrace{\frac{(-3)^{k+1} \cdot x^{k+1}}{\sqrt{(k+1)+1}}}_{a_{k+1}} \Big/ \underbrace{\frac{(-3)^k \cdot x^k}{\sqrt{k+1}}}_{a_k} \right| =$$

$= (-1)^k \cdot 3^k$

$$\lim_{k\to\infty} \left| \frac{\cancel{3^{k+1}} \cdot \cancel{x^{k+1}}}{\sqrt{k+2}} \cdot \frac{\sqrt{k+1}}{\cancel{3^k} \cdot \cancel{x^k}} \right| =$$

$$\lim_{k\to\infty} \left| \frac{3x}{\sqrt{k+2}} \cdot \sqrt{k+1} \right| = \lim_{k\to\infty} |3x| \cdot \frac{\sqrt{k+1}}{\sqrt{k+2}}$$

$$= |3x| \cdot \lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k+2}} \qquad \sqrt{\frac{k+1}{k+2}} \to 1$$

$$= |3x|.$$

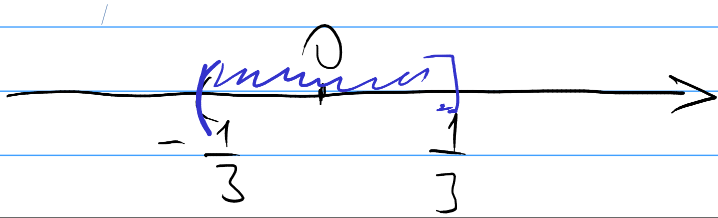
$|3x| < 1 \quad /3$

$|x| < \frac{1}{3}$

$|x - a| < R$

$|x - 0| < \frac{1}{3}$

$R = \frac{1}{3}.$



At the endpoints:

$x = \frac{1}{3}$

$$\sum_{k=0}^{\infty} \frac{(-3)^k \cdot x^k}{\sqrt{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \cancel{3^k} \cdot \cancel{\left(\frac{1}{3}\right)^k}}{\sqrt{k+1}}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{\sqrt{k+1}}$$ convergent by the alt. series test.

$x = -\frac{1}{3}$

$((-1)^2)^k = 1$

$$\sum_{k=0}^{\infty} \frac{(-3)^k \cdot x^k}{\sqrt{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \cancel{3^k} \cdot (-1)^k \cdot \cancel{\left(\frac{1}{3}\right)^k}}{\sqrt{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}}$$ divergent - by the lim comparison test, $b_k = \frac{1}{\sqrt{k}}$ $\quad \sum \frac{1}{\sqrt{k}}$ divergent as a p-series.

A: $\left(-\frac{1}{3}; \frac{1}{3}\right]$

$R = \frac{1}{3}.$