Sections 7.7 and 9.2:

Approximate methods

Goal: $$\int_a^b f(x)\,dx = F(b) - F(a)$$

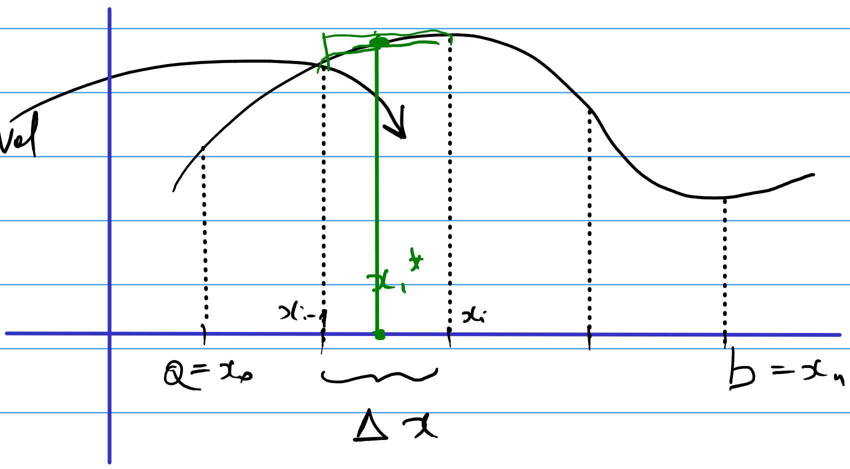
This formula can be difficult to use, recall for example

$$\int e^{-x^2}\,dx$$

To circumvent this problem, recall

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot \Delta x$$

i-th subinterval



$$\Delta x = \frac{b-a}{n}$$

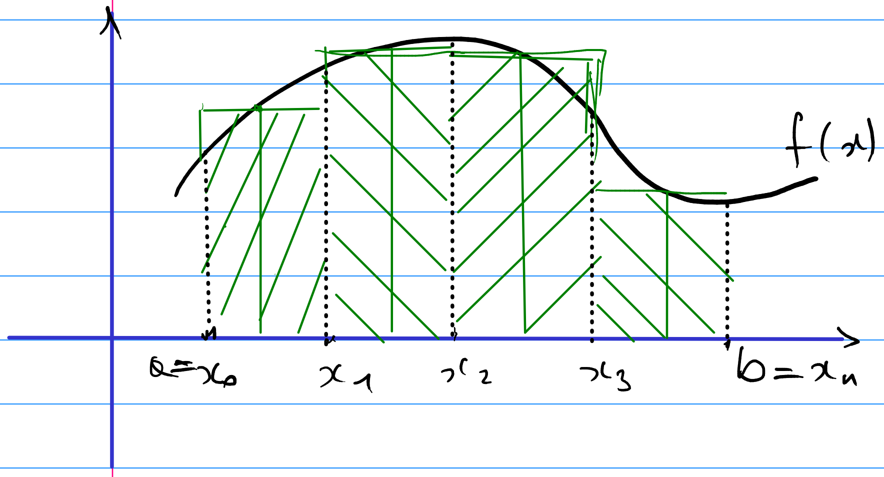
$$x_{i-1} \le x_i^* \le x_i$$

$$x_i = a + i\cdot \Delta x$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Approximate integration rules

Midpoint rule



$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + \Delta x \cdot i$$

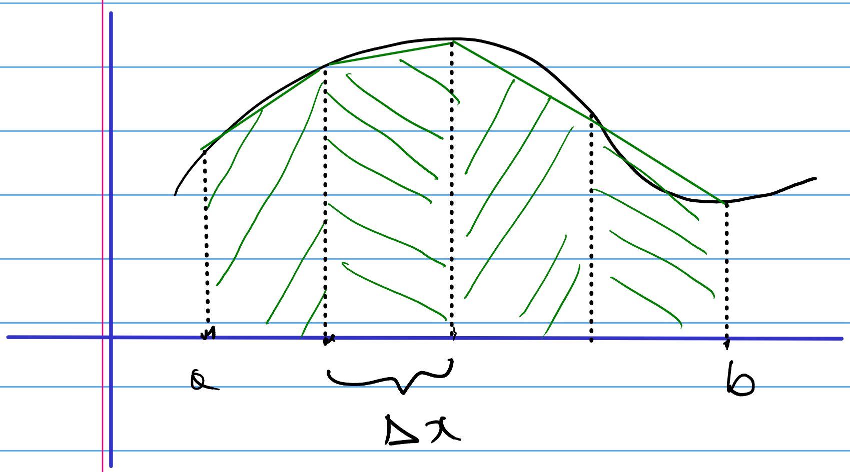
i-th subinterval has endpoints x_{i-1} and x_i

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x =$$

here $\bar{x}_i = \dfrac{x_{i-1}+x_i}{2} = a + \left(i - \frac{1}{2}\right)\Delta x$

$$\boxed{M_n = \Delta x\left(f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\right)}$$

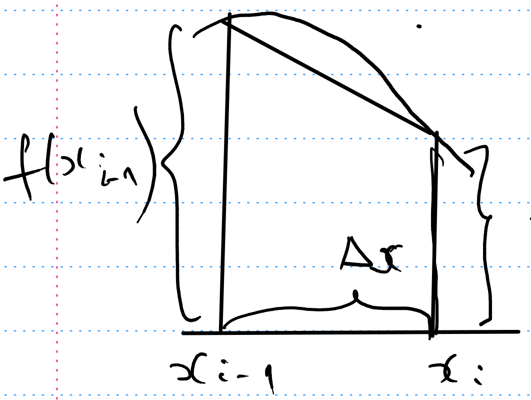
Trapezoidal rule



$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + \Delta x \cdot i$$

Area of the i-th trapezoid:

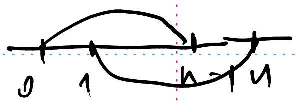


$$A_i = \frac{f(x_{i-1}) + f(x_i)}{2} \cdot \Delta x$$

$$\int_a^b f(x)\,dx \simeq T_n = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \cdot \Delta x$$

$$= \frac{\Delta x}{2} \sum_{i=1}^{n} \left(f(x_{i-1}) + f(x_i) \right)$$

$$= \frac{\Delta x}{2} \left(\sum_{i=1}^{n} f(x_{i-1}) + \sum_{i=1}^{n} f(x_i) \right)$$



$i-1 \to i$

$$= \frac{\Delta x}{2} \left(\sum_{i=0}^{n-1} f(x_i) + \sum_{i=1}^{n} f(x_i) \right)$$

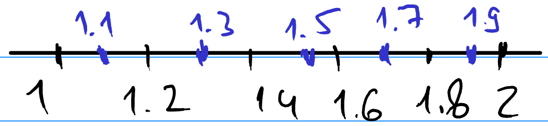
$$= \frac{\Delta x}{2} \left(f(x_0) + \underbrace{\sum_{i=1}^{n-1} f(x_i) + \sum_{i=1}^{n-1} f(x_i)} + f(x_n) \right)$$

$$= \frac{\Delta x}{2} \left(f(x_0) + \sum_{i=1}^{n-1} 2f(x_i) + f(x_n) \right)$$

$$T_n = \frac{\Delta x}{2} \left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n) \right)$$

Comparison of midpoint and trapezoidal rules

$$I = \int_1^2 \frac{dx}{x} = \ln|x| \Big|_1^2 = \ln 2 \approx 0.6931$$



$$\Delta x = \frac{2-1}{2} = 0.2$$

$$x_i = a + i \cdot \Delta x$$

$$n = 5$$

a). $$T_5 = \frac{1}{2} \cdot \frac{2-1}{5} \Big(f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2) \Big)$$

$$= 0.1 \left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2} \right) \approx$$

$$\approx 0.6956$$

b). $$M_5 = \Delta x \left(f(\bar{x}_1) + \dots + f(\bar{x}_5) \right)$$

$$= 0.2 \left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} \cdot \frac{1}{1.9} \right) \approx$$

$$\simeq 0.6919$$

$$E_T = \underline{I} - T_5 = -0.0025$$

$$E_M = \underline{I} - M_5 = 0.0012$$

Simpson's rule

$$S_{2n} = \frac{T_n + 2M_n}{3}$$

Error bounds for T_n, M_n

Suppose $|f''(x)| \le K$, $a \le x \le b$; (const)
for E_T and E_M defined by

$$E_T = \int_a^b f(x)\,dx - T_n$$

$$E_M = \int_a^b f(x)\,dx - M_n,$$

there holds

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

These estimates can be used to answer: how large the n must be in order to E_T or E_M be smaller than a certain number?

E.g. for $|E_T| < 10^{-5}$ in $\displaystyle\int_1^2 \frac{1}{x}\,dx$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = +\frac{2}{x^3} \Big\}\ \text{decreasing on } [1,2]$$

$$\Rightarrow |f''(x)| = f''(x) \le f''(1) = 2 =: K$$

We must have:

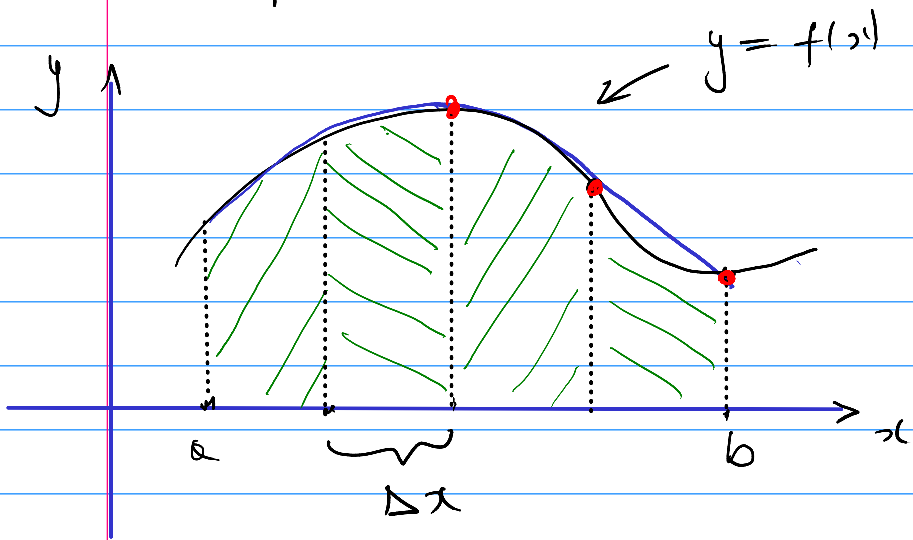
$$\frac{2 \cdot 1^3}{12 n^2} < 10^{-5} \implies \text{solve for } n$$

$$n^2 > \frac{2 \cdot 10^5}{12} = \frac{1}{6} \cdot 10^5$$

$$n > \sqrt{\frac{10^5}{6}} \approx 129.01$$

$$\implies n \geq 130$$

Simpson's rule



$n = 2k$ even

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i \cdot \Delta x$$

$$\int_a^b f(x)\,dx \simeq S_n = \frac{\Delta x}{3} \Big(\overbrace{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}^{n+1 = 2k+1 \text{ terms}} \Big)$$

1 4 2 4 2 4 1

Observe: $S_{2k} = \dfrac{T_k + 2M_k}{3}$

Error bound for S_n:

if $|f^{(4)}(x)| \leq K$, $a \leq x \leq b$, then

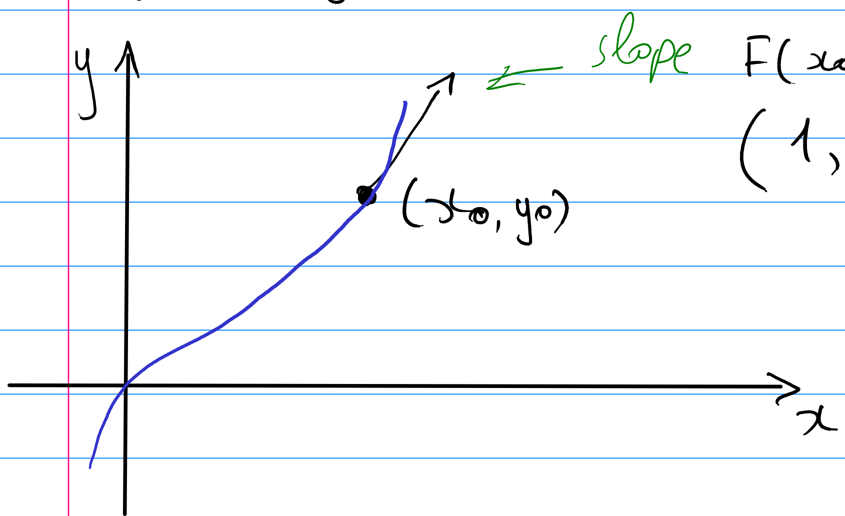
$$|E_S| \leq \frac{K(b-a)^5}{180\, n^4}$$

where $E_S = \int_a^b f(x)\, dx - S_n$.

Euler's method

$$\boxed{\frac{dy}{dx} = F(x,y)} \qquad (*)$$

$$\frac{dy}{dx} = y \iff y = C \cdot e^x$$



$F(x_0, y_0)$

$(1, F(x_0, y_0))$

This gives a way to approximate solutions to $(*)$ as follows:

Suppose an initial condition is given:
$y(x_0) = y_0$

Fix $\Delta x > 0$ - small step

Construct an approximate solution to $(*)$ by repeating the procedure:

- Given a point (x_k, y_k) on $y = y(x)$, solution to $(*)$, construct the next point on the graph as follows:

$$(x_k, y_k) \longmapsto (x_{k+1}, y_{k+1})$$

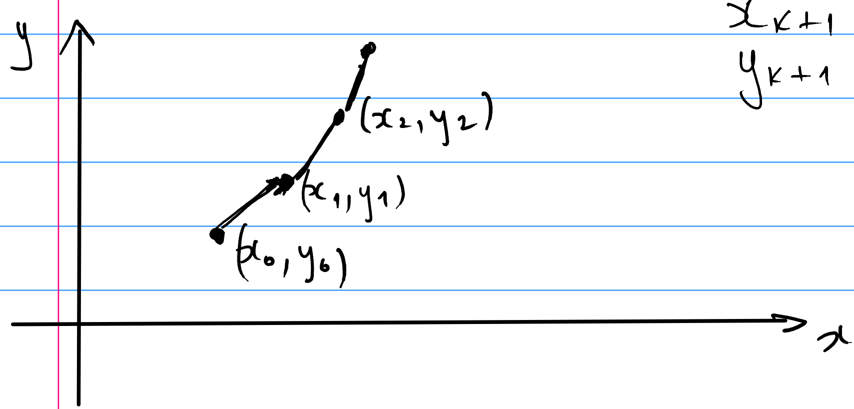
$$x_{k+1} = x_k + \Delta x$$
$$y_{k+1} = y_k + F(x_k, y_k) \cdot \Delta x$$

- The obtained sequence of points (x_k, y_k) approximates the graph of a solution $y(x)$

We are using that $y'(x_k) = F(x_k, y_k)$, so

$$\Delta y = y_{k+1} - y_k \approx y'(x_k) \cdot \Delta x = F(x_k, y_k) \Delta x$$

This approximation becomes precise for $\Delta x \to 0$



$$x_{k+1} = x_k + \Delta x$$
$$y_{k+1} = y_k + F(x_k, y_k) \cdot \Delta x$$