Measure and Integration II (MAA5617), Spring 2021
Practice problems

1. Recall the notation
$$M_q(g) = \sup\left\{\left|\int fg\, d\mu\right| : f \text{ simple and } \|f\|_p = 1\right\}.$$
Suppose μ is semifinite, $q < \infty$, $M_q(g) < \infty$. Prove:
$$\{x : |g(x)| > 1/n\}$$
has finite measure μ for every $n \geq 1$.

2. Recall that "separable" = "containing a dense countable subset". Prove:
- $L^p(\mathbb{R}^n, \lambda^n)$ is separable for $1 \leq p < \infty$ (polynomials are dense on compact sets; use σ-finiteness).
- $L^\infty(\mathbb{R}^n, \lambda^n)$ is not separable (prove that it contains an uncountable subset without points of concentration).

3. Let $0 < p < q < \infty$. Prove:
- $L^p \not\subset L^q \iff X$ contains sets of arbitrarily small positive measure.
- $L^q \not\subset L^p \iff X$ contains sets of arbitrarily large finite measure.

4. (Jensen's inequality) Suppose $(\Omega, \mathcal{S}, \mathbb{P})$ is a measure space with $\mathbb{P}(\Omega) = 1$, $X : \Omega \to \mathbb{R}$ a measurable function, and $\phi : \mathbb{R} \to \mathbb{R}$ a convex function. Prove:
$$\phi\left(\int X\, d\mathbb{P}(\omega)\right) \leq \int \phi(X)\, d\mathbb{P}(\omega).$$
(Use that every point $(x, \phi(x))$ of the graph of ϕ has a support plane, that is, $\phi(y) - \phi(x) \geq \alpha(y - x)$ for some number α.)

5. Show that the L^p-norm is not induced by an inner product for $p \neq 2$.

6. Let $\alpha_n \geq 0$ and $\sum_1^\infty \alpha_n = \infty$. Prove that for some suitable sequence of $c_n \geq 0$, $\sum_1^\infty \alpha_n c_n = \infty$, but $\sum_1^\infty \alpha_n c_n^2 < \infty$.

7. For a pair of functions f, g in $L^1(\mathbb{R})$, prove that their convolution
$$f * g(x) = \int f(y) g(x - y)\, d\lambda(y)$$
is measurable and in $L^1(\mathbb{R})$ as well.

8. Suppose (X, μ) is a measure space with $\mu(X) = 1$. Prove that for $f \in L^r(\mu)$,

$$s \mapsto \|f\|_s$$

is an increasing function on $(0, r]$.

Let V be a vector space, V^* its dual. We will say that $\{f_n\} \subset V$ converges to an $f \in V$ *weakly*, if

$$\lim_{n\to\infty} \phi(f_n) = \phi(f), \qquad \forall \phi \in V^*.$$

9. Let $1 < p < \infty$. Prove that if $\{f_n\}$ is bounded in $L^p([0,1], \lambda)$ and converges to f in measure, then it converges to f weakly in $L^p([0,1], \lambda)$.

10. In the same L^p as above, if $\{f_n\}$ converges to f weakly, then $\{\|f_n\|_p\}$ is bounded.

11. In general, weak convergence does not imply convergence in measure (take $X = \mathbb{R}$).

12. Let L be the vector space of bounded functions from the normalized unit circle $\mathbb{T}$ (equivalently: periodized [0,1)) to $\mathbb{R}$. The functions are not assumed to be measurable. Let for $f \in L$

$$p(f) = \inf M(f; a_1, \ldots, a_n),$$

where the inf is over all $n \in \mathbb{N}$ and all finite collections $\{a_i\} \subset \mathbb{T}$, and where

$$M(f; a_1, \ldots, a_n) = \sup_{t\in\mathbb{T}} \frac{1}{n} \sum_{1}^{n} f(t + a_i).$$

The addition is understood $\mod 1$.

Constructed in this way p is a sublinear functional (check this; also see the link below). By the Hahn-Banach theorem, there exists a linear functional $\mathcal{F}$ such that

$$\mathcal{F}(f) \le p(f).$$

Denoting

$$\int f(t)\,dt := \frac{1}{2}[\mathcal{F}(f(t)) + \mathcal{F}(f(1-t))]$$

, one has a functional that to each element of L puts in correspondence the number $\int f(t)\,dt$, so that for all functions f, h and scalars α, β:

- $\int [\alpha f + \beta h]\,dt = \int \alpha f\,dt + \int \beta h\,dt$
- $\int f\,dt \ge 0$ when $f \ge 0$
- $\int f(t+\alpha)\,dt = \int f(t)\,dt$
- $\int f(1-t)\,dt = \int f(t)\,dt$
- $\int 1\,dt = 1$

Integrating indicators with respect to this notion of integral gives a finitely additive measure, defined on all sets. Compare to the Vitali counterexample, showing the nonexistence of countably additive measures on all sets. The argument above is due to Banach: `http://matwbn.icm.edu.pl/ksiazki/or/or2/or213.pdf#page=14`.