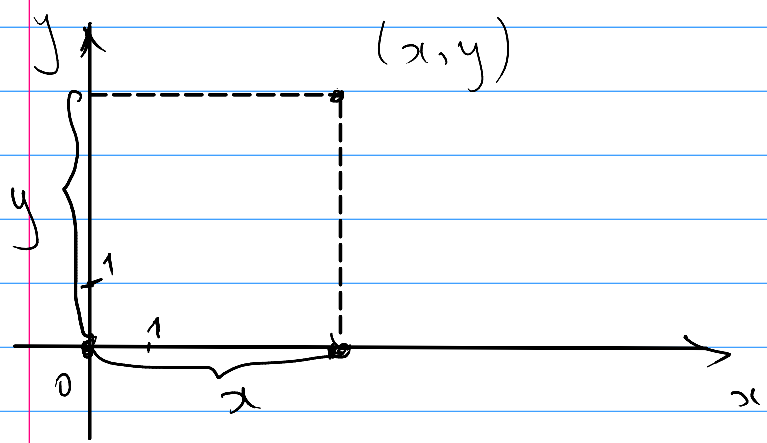
Polar coordinates

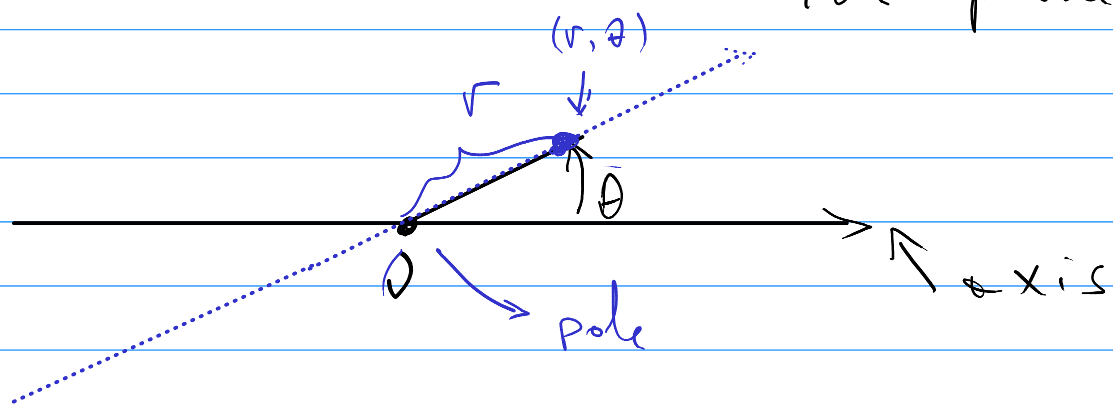
$(x,y) \rightleftarrows$ points in the plane

Cartesian coordinates

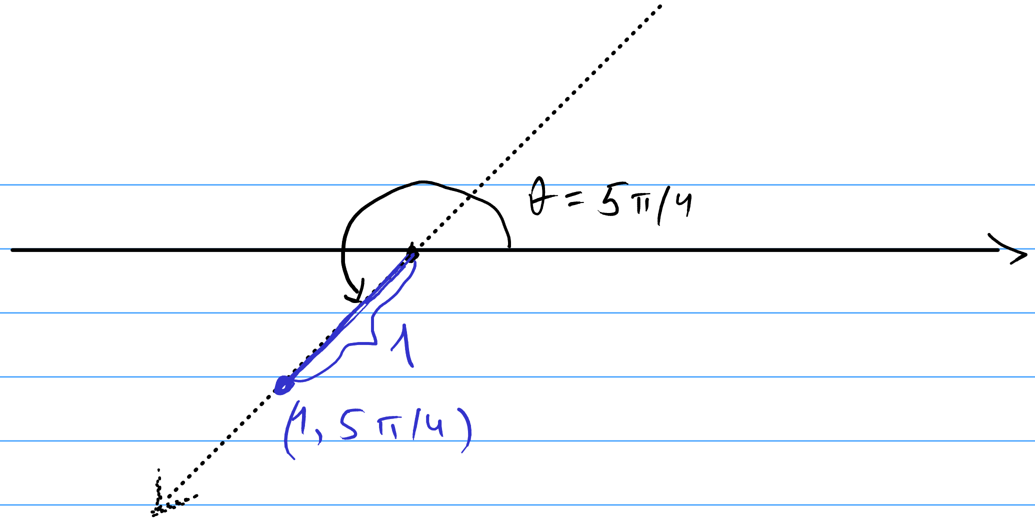


Polar coordinates

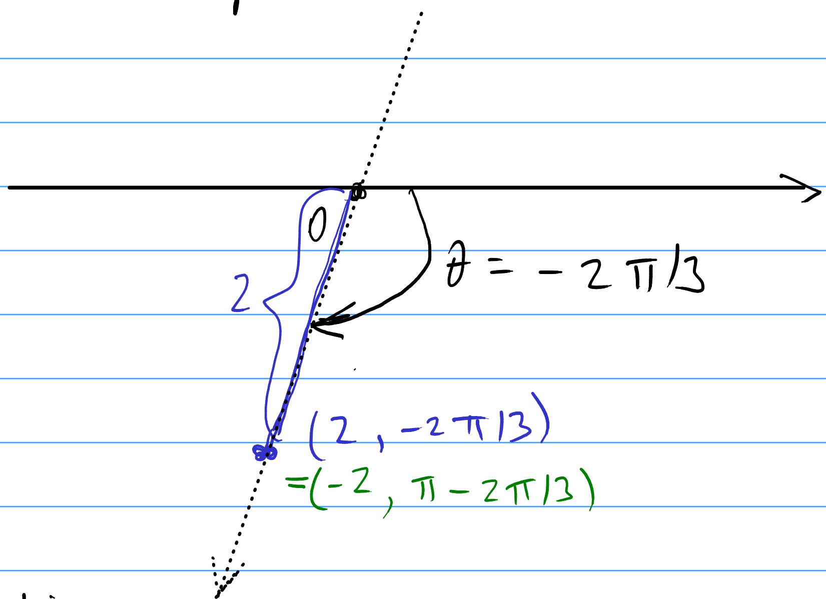
$(r,\theta) \rightleftarrows$ points in the plane



Ex. 1 Find the point $(\underbrace{1}_{r}, \underbrace{5\pi/4}_{\theta})$ in polar coordinates



Ex. 2 Find the point $(\underbrace{2}_{r}, \underbrace{-2\pi/3}_{\theta})$ in polar coordinates:



integer

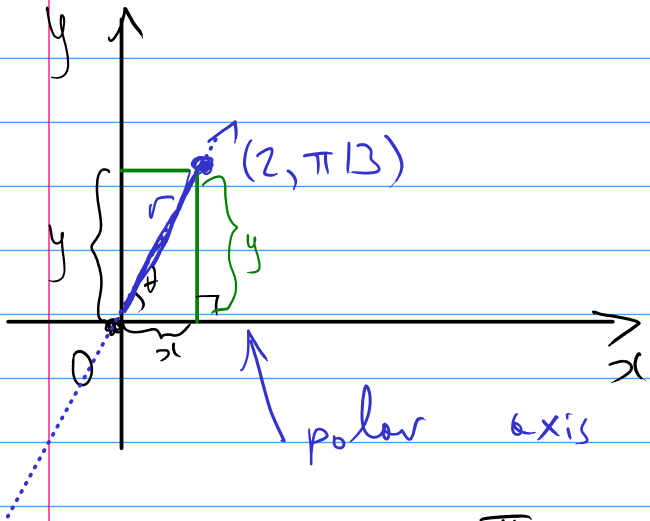
Notice:

1). $(r, \theta) = (r, \theta + 2\pi k)$

$\quad\quad = (-r, \theta + (2k+1)\pi)$

2). $\underbrace{(0,0)}_{\text{origin}} = (0, \theta)$ for any θ

Ex. 3 Convert $\overbrace{(\underbrace{2}_{r}, \underbrace{\pi/3}_{\theta})}^{\text{polar coords}}$ to Cartesian coordinates:



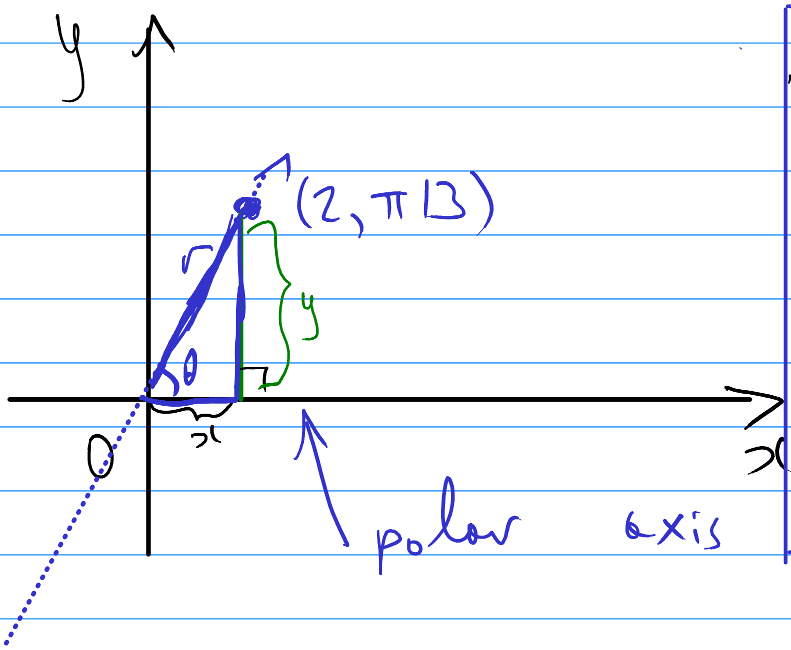
$$\boxed{\begin{aligned} x &= r \cdot \cos\theta \\ y &= r \cdot \sin\theta \end{aligned}}$$

$$x = 2 \cdot \cos\frac{\pi}{3} = 1$$

$$y = 2 \cdot \sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Polar coords $(2, \pi/3) \longrightarrow (1, \sqrt{3})$ Cartesian coords

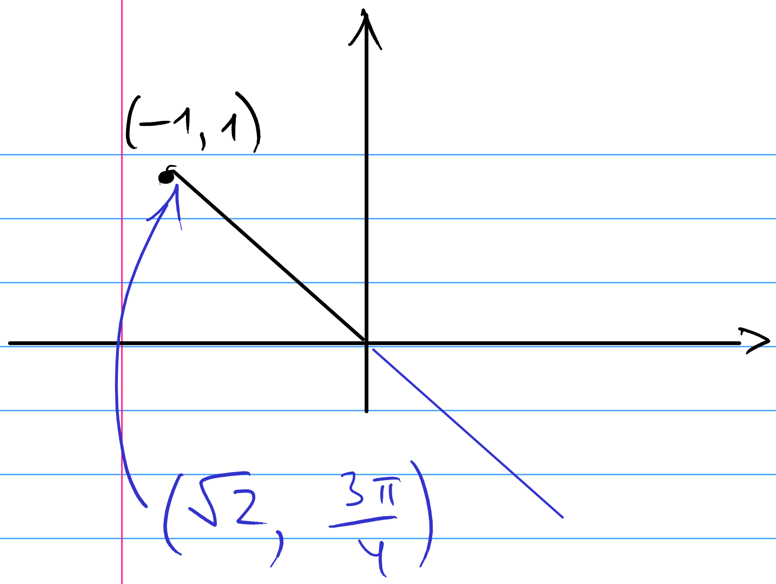
Ex. 4 Convert $\overbrace{(-1, 1)}^{\text{Cartesian coords}}$ to polar coordinates



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\frac{y}{x}$$

$+\pi$ (depending on the quadrant)



$$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+1^2} = \sqrt{2}$$

$$\theta = \arctan\left(\frac{y}{x}\right) + \pi = \arctan\left(\frac{1}{-1}\right) + \pi = \arctan(-1) + \pi = -\pi/4 + \pi$$

Conversion formulas:

$(r,\theta) \to (x,y)$	$(x,y) \to (r,\theta)$
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\theta = \arctan\left(\frac{y}{x}\right)$
	$(+\pi)$

Polar curves

Def: A polar curve is a set of points (in polar coordinates) that satisfy the equation:

$$\boxed{r = f(\theta)}$$

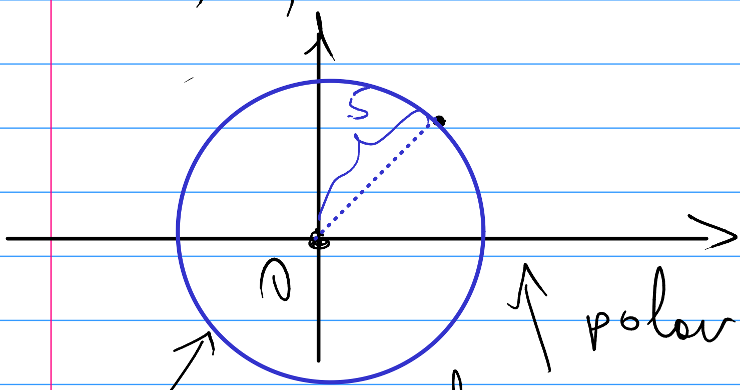
(Compare to Cartesian curves:

$$y = f(x)\ )$$

Ex. 5 Which curve is given by $r = 5$?

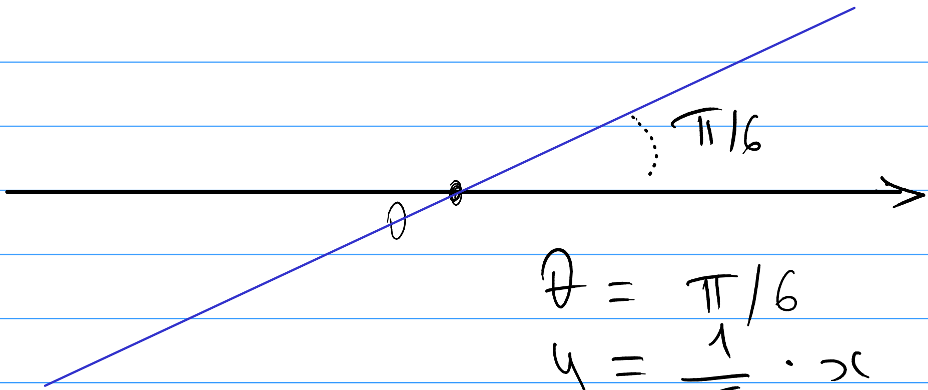
(r, θ) where $r = 5$

$$x^2 + y^2 = 25$$



circle of radius 5, centered at the origin

Ex. 6 Identify the curve for $\theta = \frac{\pi}{6}$.



$$\left(r, \frac{\pi}{6}\right)$$

$$\theta = \pi/6$$
$$y = \frac{1}{\sqrt{3}} \cdot x$$

$$\left(m = \text{slope} = \tan\theta = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\right)$$

Ex. 7 Identify: $\underbrace{r = 2\cos\theta}_{\text{circle of } R=1 \text{ centered at } (1,0)}$ $\quad 0 \le \theta \le 2\pi$

Recall:

$$r = 2\cos\theta \quad \cdot r$$

$$x = r\cdot\cos\theta$$
$$y = r\cdot\sin\theta$$
$$r = \sqrt{x^2+y^2}$$

conversion formulas

$$\underbrace{r^2}_{x^2+y^2} = 2\underbrace{r\cos\theta}_{x}$$

$$\underbrace{x^2+y^2}=2x$$

Complete the square:

$$x^2-2x+y^2=0$$

$$(x^2-2x+1)-1+y^2=0$$

$$(x-1)^2+y^2=1$$

circle, centered at $(1,0)$ with $R=1$

Equation of a circle in Cartesian coordinates:

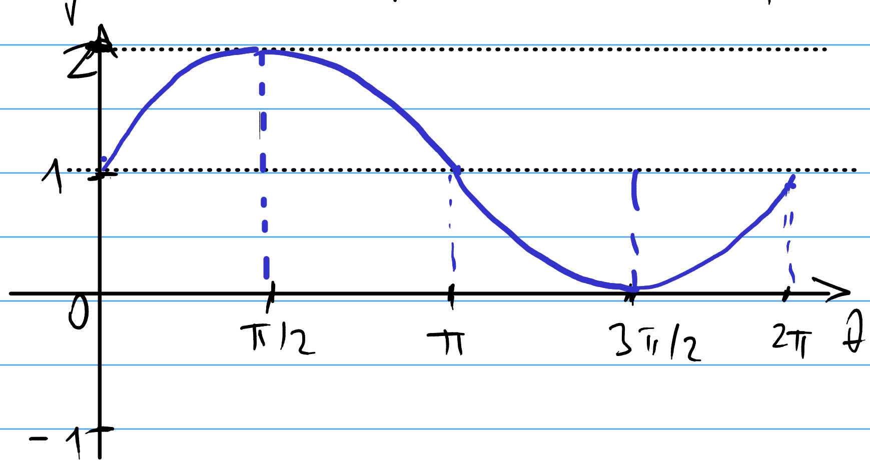
$$(x-a)^2+(y-b)^2=R^2$$

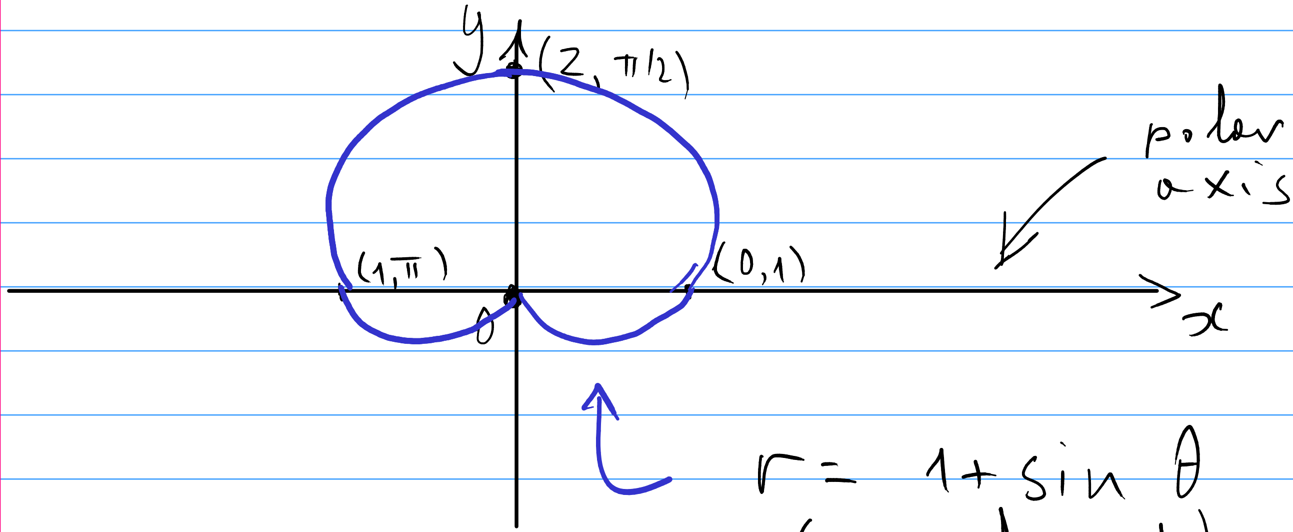
– centered at (a,b), radius R.

Exercise: identify $r = 4\sin\theta$ in polar coordinates.

Ex. 8 Sketch the curve $r = 1 + \sin\theta$
$0 \le \theta \le 2\pi$

Draw the dependence of r on θ.





$r = 1 + \sin\theta$
(cardioid)

Exercise: Sketch the graph of $r = 1 - \cos\theta$

Polar curves as parametric curves

Recall (conversion formulas):

$$r = f(\theta)$$

$$x = r \cdot \cos\theta = f(\theta) \cdot \cos\theta$$
$$y = r \cdot \sin\theta = f(\theta) \cdot \sin\theta$$

$(x(\theta), y(\theta))$ — a point on the curve

Our polar curve $r = f(\theta)$ is parametric with parameter θ:

$$\boxed{\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}}$$

Thus, for example, the slope of a polar curve is given by:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r' \cdot \sin\theta + r \cdot \cos\theta}{r' \cdot \cos\theta - r \cdot \sin\theta}$$

Note: for the origin, $r = 0$, so

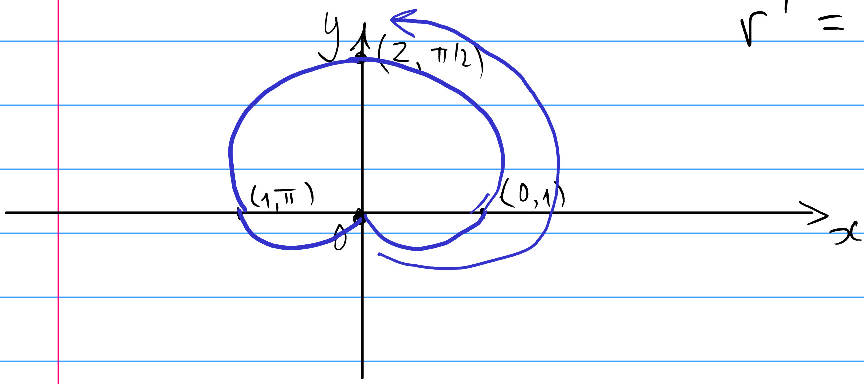
$$\frac{dy}{dx}(\text{origin}) = \frac{\sin\theta}{\cos\theta} = \tan\theta, \quad \text{if } r' \neq 0$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = (\text{simplify})$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex. Find the arc length of the cardioid: $r = 1 + \sin\theta$

$r' = \cos\theta$



$$L = 2\int_{-\pi/2}^{\pi/2} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{1 + 2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{1}}\, d\theta$$

See the digression on p. 11 for what happened here

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{2(1+\sin\theta)}\, d\theta$$

$$= 2\int_{-\pi/2}^{-\pi/2} \sqrt{2\left(1+\cos\left(\frac{\pi}{2}-\theta\right)\right)}\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{2 \cdot 2\cos^2\left(\frac{\frac{\pi}{2}-\theta}{2}\right)}\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{4 \cdot \cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} 2 \cdot \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) d\theta = \left| \begin{array}{l} u = \frac{\pi}{4} - \frac{\theta}{2} \\ du = -\frac{d\theta}{2} \\ d\theta = -2du \end{array} \right.$$

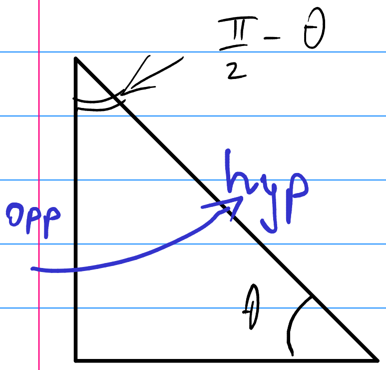
$$= 4 \int_{\pi/2}^{0} \cos(u) \, -2 \, du$$

$$= 8 \int_{0}^{\pi/2} \cos u \, du = 8 \cdot \sin u \Big|_{0}^{\pi/2}$$

$$= 8.$$

A digression about half-angle formulas
and
properties of trigonometric functions

First, let's learn to express $\sin\theta$ as a $\cos$(another angle)



$$\boxed{\sin\theta = \frac{opp}{hyp} = \cos\left(\frac{\pi}{2}-\theta\right)} \qquad (1)$$

Then, recall the half-angle formulas:

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

The first gives:

$$(2) \quad \boxed{1+\cos 2A = 2\cos^2 A} \qquad \text{(for any } A)$$

We apply this with $A = \frac{\pi}{2}-\theta$:

$$1+\sin\theta \overset{(1)}{=} 1+\cos\left(\frac{\pi}{2}-\theta\right) \overset{(2)}{=} 2\cos^2\left(\frac{\frac{\pi}{2}-\theta}{2}\right)$$