Section 11.11:

Approximating functions by Taylor polynomials

Suppose that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \Big\} \text{ Taylor series } f(x)$$

Recall:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \Big\} \text{ Taylor polynomial of degree } n \text{ of } f(x).$$

We know: $T_n(x) \to f(x)$, $n \to \infty$, then we can approximate $f(x) \approx T_n(x)$. The error we make in this approximation is estimated by the Taylor's inequality:

if $|f^{(n+1)}(x)| \leq M$, then

$$|R_n(x)| = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Ex. 1. (a) Approximate $f(x) = \sqrt[3]{x}$ about $a = 8$ by the Taylor polynomial of degree 2

(b) Estimate the accuracy of this approximation.

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = \frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot x^{-5/3}$

$f'''(x) = \frac{1}{3}\cdot\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdot x^{-8/3}$

$f(8) = 2$

$f'(8) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$

$f''(8) = -\frac{2}{9}\cdot\frac{1}{32} = -\frac{1}{9\cdot 16}$

$= -\frac{1}{144}.$

We have:

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$$= 2 + \frac{1}{12}(x-8) + \left(-\frac{1}{144}\right)\cdot\frac{1}{2}(x-8)^2$$

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

To summarize:

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2,$$ when x is close to 8

(b) Using the Taylor's inequality:

$$|f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1},$$

where M is a constant such that $|f^{(n+1)}(x)| \le M$

We will approximate $f(x) = \sqrt[3]{x}$ when $|x-8| \le 1$.

$n = 2$ $(7 \le x \le 9)$

$$f'''(x) = \frac{1}{3}\cdot\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\cdot x^{-8/3} = \frac{10}{27}\cdot x^{-8/3}$$

Want: bound $|f'''(x)| \le M$ when $7 \le x \le 9$.

$$|f'''(x)| = \left|\frac{10}{27}x^{-8/3}\right| = \frac{10}{27}\cdot\frac{1}{x^{8/3}}$$

Since $x^{8/3}$ is increasing, the denominator is smallest when $x = 7$, so that's when the right-hand side is the largest.

$$|f'''(x)| \le \frac{10}{27}\cdot\frac{1}{7^{8/3}} \le \frac{10}{27}\cdot\frac{1}{7^2} = \underbrace{\frac{10}{1323}}_{M}.$$

$$|f(x)-T_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1},$$

$n=2$

$$|\underbrace{f(x)}_{\sqrt[3]{x}} - T_2(x)| \le \frac{10}{1323 \cdot 3!} \cdot |x-8|^3$$

$7 \le x \le 9$

$$\le \frac{10}{1323 \cdot 6} = \frac{10}{7938} \approx .00126$$

Ex. 2 Consider: $F(x) = \int e^{-x^2}\,dx$

Goal: obtain the Maclaurin series for $F(x)$.

Let $f(x) = e^{-x^2}$.

Recall: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, for all x $(R=+\infty)$

Substitute: $x \longmapsto (-x^2)$

$$f(x) = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

$R = +\infty$ (is not changed)

So, we have:

$$F(x) = \int e^{-x^2}\,dx = \int f(x)\,dx$$

$$= \int \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}\,dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \int \frac{x^{2k}}{k!} \, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{k!} \cdot \frac{x^{2k+1}}{2k+1} + C$$

Here also: $R = +\infty$

Conclusion:

$$F(x) = \int e^{-x^2} dx = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)\, k!} + C$$

for all x $\quad (R = +\infty)$

Let's pick a specific antiderivative:

$F(x) = \int e^{-x^2} dx$ with $F(0) = 0$

In the above series, $C = 0$

$$F(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)\, k!}$$

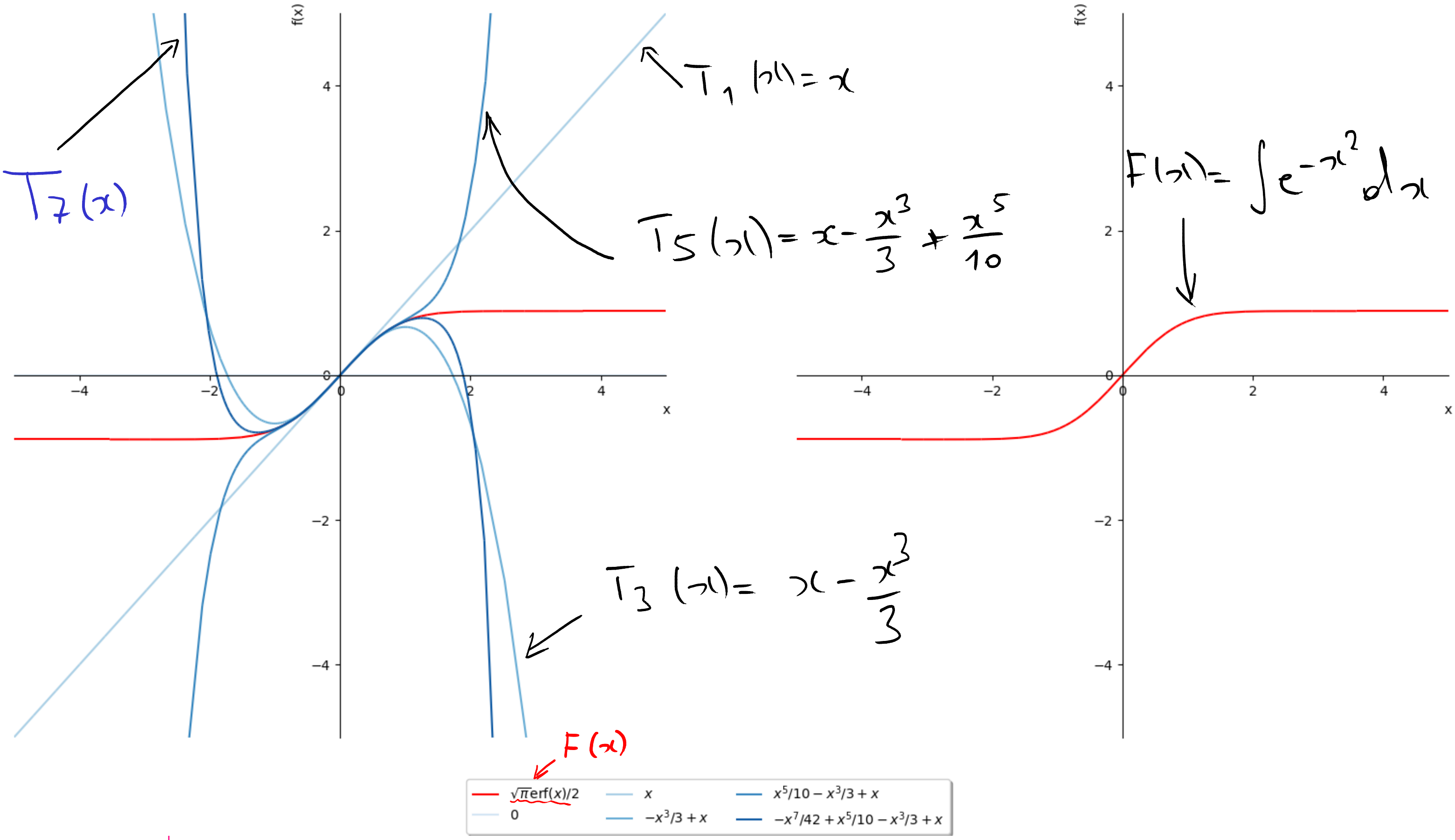
$T_0(x) = 0$ — constant polynomial app.

$$T_1(x) = (-1)^0 \frac{x^1}{1 \cdot 0!} = x$$

$$T_3(x) = \sum_{k=0}^{1} (-1)^k \frac{x^{2k+1}}{(2k+1)\, k!} = x - \frac{x^3}{3}$$

$$T_5(x) = \sum_{k=0}^{2} (-1)^k \frac{x^{2k+1}}{(2k+1)k!} = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2}$$

$$T_7(x) = \sum_{k=0}^{3} (-1)^k \frac{x^{2k+1}}{(2k+1)k!} = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}$$



Recall: Taylor's inequality:

$$|F(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - 0|^{n+1}$$

$M =$ constant such that

$$|F^{(n+1)}(x)| \leq M$$

Exercise: compute $F^{(2)}$, $F^{(4)}$, find explicit values for M