MAC 2312: Calculus w/ analytic geometry II

— integration techniques;

— applications:

i). geometric

ii). diff. eqns

— series:

i). number series

ii). power series

— conics

i). cartesian coords

ii). polar —"—

iii). planetary motion.

Section 7.1 : Integration by parts

Chain rule $\longmapsto$ Substitution rule

Product rule $\longmapsto$ Integration by parts

Chain rule:

$F' = f$

$G = g$

$\int dx$

$$\left(F(G(x))\right)' = F'(G(x)) \cdot G'(x)$$

Substitution rule:

$$\int f(g(x)) \cdot g'(x)\,dx = \left| \begin{array}{l} u = g(x) \\ du = g'(x)\,dx \end{array} \right|$$

$$= \int f(u)\,du$$

Product rule:

$$\left(f(x) \cdot g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$$

$$\int \left(f(x) \cdot g(x)\right)' dx = \int f'(x)\,g(x) + f(x)\,g'(x)\,dx$$

$$f(x) \cdot g(x) = \int \left(f'(x)\,g(x) + f(x)\,g'(x)\right) dx$$

$$\underbrace{f(x) \cdot g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx}_{\text{integration by parts}}$$

$$\int f(x)\,\underbrace{g'(x)\,dx} = f(x)\cdot g(x) - \int \underbrace{f'(x)}\,g(x)\,\underbrace{dx}$$

$f(x) = u$ $\qquad du = \underbrace{f'(x)\,dx}$

$g(x) = v$ $\qquad dv = \underbrace{g'(x)\,dx}$

$$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$

} formula for int by parts

Ex. 1 Find $\int \underbrace{x}_{u}\,\underbrace{\sin x\,dx}_{dv} = x(-\cos x) - \int \underbrace{-\cos x}_{v}\,\underbrace{dx}_{du} =$

$u = x \qquad dv = \sin x\,dx$

$du = dx \qquad v = -\cos x$

$= -x\cos x + \underbrace{\int \cos x\,dx}$

$= -x\cos x + \sin x + C$

$$\int x\sin x\,dx = -x\cos x + \sin x + C.$$

Note:

$$\int \overset{dv}{x}\cdot\underbrace{\sin x}_{u}\,dx = \sin x\cdot\frac{x^2}{2} - \overbrace{\int \frac{x^2}{2}\cdot\cos x\,dx}^{\text{worse expression}}$$

$u = \sin x \qquad dv = x\,dx$

$du = \cos x\,dx \qquad v = \frac{x^2}{2}$

$$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$

Ex 2 Evaluate

$$\int \underbrace{t^2}_{u}\underbrace{e^t\,dt}_{dv} = \left| \begin{array}{ll} u = t^2 & dv = e^t\,dt \\ du = 2t\,dt & v = e^t \end{array} \right|$$

$$= t^2\cdot e^t - \int \underbrace{e^t}_{v}\cdot\underbrace{2t\,dt}_{du}$$

$$= t^2\cdot e^t - 2\int te^t\,dt$$

$$\int \underbrace{t}_{u}\underbrace{e^t\,dt}_{dv} = \left| \begin{array}{ll} u = t & dv = e^t\,dt \\ du = dt & v = e^t \end{array} \right| =$$

$$= t\cdot e^t - \underbrace{\int e^t\,dt}$$

$$= t\cdot e^t - e^t + C$$

$$\int t^2e^t\,dt = t^2e^t - 2\left(t\cdot e^t - e^t + C\right)$$

$$= t^2e^t - 2t\cdot e^t + 2e^t \underbrace{-2C}_{C_1}$$

$$= t^2e^t - 2te^t + 2e^t + C_1.$$

Ex. 3 Evaluate $\int \ln x \, dx = x \ln x - x + C$

$$\int \underbrace{\ln x}_{u} \, \underbrace{dx}_{dv} = \left| \begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{dx}{x} & v = x \end{array} \right|$$

$$= \ln x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \ln x \cdot x - \int 1 \, dx$$

$$= x \cdot \ln x - x + C$$

Ex. 4 Compute

$$\int_0^1 \underbrace{\arctan x}_{u} \, \underbrace{dx}_{dv} = \left| \begin{array}{ll} u = \arctan x & dv = dx \\ du = \frac{1}{1+x^2} dx & v = x \end{array} \right|$$

$$\left(\int_a^b f(x) \cdot g'(x) \, dx = f(x) \cdot g(x) \Big|_a^b - \int_a^b g(x) f'(x) \, dx \right)$$

$$= x \cdot \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx$$

$$= \left(1 \cdot \frac{\pi}{2} - 0 \cdot 0 \right) - \underbrace{\int_0^1 \frac{x}{1+x^2} \, dx} = \frac{\pi}{2} - \frac{\ln 2}{2}.$$

$$\int_0^1 \frac{x}{1+x^2} \, dx = \left| \begin{array}{l} u = 1 + x^2 \\ du = 2x \cdot dx \\ \frac{1}{2} du = x \, dx \end{array} \right|$$

$$= \int_1^2 \frac{\frac{1}{2} du}{u} = \frac{1}{2} \int_1^2 \frac{du}{u} = \frac{1}{2} \ln |u| \Big|_1^2 = \frac{\ln 2}{2}.$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ex.5 Compute $\int \underbrace{e^t}_{u} \cdot \underbrace{\sin t\,dt}_{dv}$

$$\int \underbrace{e^t}_{u} \cdot \underbrace{\sin t\,dt}_{dv} = \left| \begin{array}{ll} u = e^t & dv = \sin t\,dt \\ du = e^t\,dt & v = -\cos t \end{array} \right|$$

$$= e^t(-\cos t) - \int \underbrace{(-\cos t)}_{v}\underbrace{e^t\,dt}_{du}$$

$$= -e^t \cos t + \int e^t \cos t\,dt$$

$$\int e^t \cos t\,dt = \left| \begin{array}{ll} u = e^t & dv = \cos t\,dt \\ du = e^t\,dt & v = \sin t \end{array} \right|$$

$$= e^t \sin t - \int \sin t \cdot e^t\,dt$$

$$\int e^t \sin t\,dt = -e^t \cos t + e^t \sin t - \int \sin t\, e^t\,dt$$

$$2\int e^t \sin t\,dt = -e^t \cos t + e^t \sin t \quad /2$$

$$\int e^t \sin t\,dt = \frac{1}{2}\left(-e^t \cos t + e^t \sin t\right) + C$$

Ex. 6 Prove the reduction formula (for sines)

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \cdot \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

($n \geq 1$, integer)

Recall:

$\cos^2 x + \sin^2 x = 1$ (Pythagorean identity)

$\cos^2 x = 1 - \sin^2 x$

Suppose $n \geq 2$.

$$\int \sin^n x\,dx = \int \underbrace{\overbrace{(\sin x)\cdot(\sin x)\cdot \ldots \cdot}^{u}(\sin x)}_{n \text{ factors}}\overbrace{\,dx}^{dv}$$

$$= \left|\begin{array}{ll} u = (\sin x)^{n-1} & dv = \sin x\,dx \\ du = (n-1)(\sin x)^{n-2}\cdot \cos x\,dx & v = -\cos x \end{array}\right|$$

$$= (\sin x)^{n-1}\cdot(-\cos x) - \int (-\cos x)(n-1)(\sin x)^{n-2}\cos x\,dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \cos^2 x \cdot \sin^{n-2} x\,dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx$$

$$= -\sin^{n-1} x\,\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx \qquad /n$$

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{x} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$