Measure and Integration II (MAA5617), Spring 2021
Homework 2, due Thursday, Feb 4

1. Prove: for functions $F, G \in BV$ there holds
- $T_{F+G}(a,b) \leq T_F(a,b) + T_G(a,b)$ with any $a < b \in \mathbb{R}$;
- $\alpha F + \beta G \in BV$, with any coefficients $\alpha, \beta \in \mathbb{R}$.

2. Prove that for $F = x\sin(1/x)$ and any $\epsilon > 0$, $F \notin BV([-\epsilon, \epsilon])$.

3. Prove that for $G = x^2 \sin(1/x)$, $G \in BV([-1,1])$

4. Consider the following pair of measures: λ, the Lebesgue measure on $\mathbb{R}$; μ, the counting measure on subsets of $\mathbb{R}$. That is,

$$\mu(A) = \begin{cases} \operatorname{card}(A), & A \text{ is finite,} \\ +\infty, & \text{otherwise.} \end{cases}$$

- Verify that $\lambda \ll \mu$.
- Does there exist a function f for which $d\lambda = f d\mu$? How does this agree with the Radon-Nikodym theorem?

5. Let f be the function defined by

$$f(x) = \begin{cases} \sum_{i \geq 1} \frac{t_i/2}{2^i}, & x \in C, \\ f\left(\max\left\{y < x,\ y \in C\right\}\right), & \text{otherwise,} \end{cases}$$

where C is the 1/3 Cantor set and t_i are the digits in the ternary expansion of x (it was introduced in #3 from HW 5 in the fall semester). Verify that $\mu_f \perp \lambda$.

6. We know that for an increasing function, the set of its discontinuities is at most countable. For a given countable $A \subset \mathbb{R}$, present a bounded increasing $F : \mathbb{R} \to \mathbb{R}$, discontinuous at the points of A, and only at them.