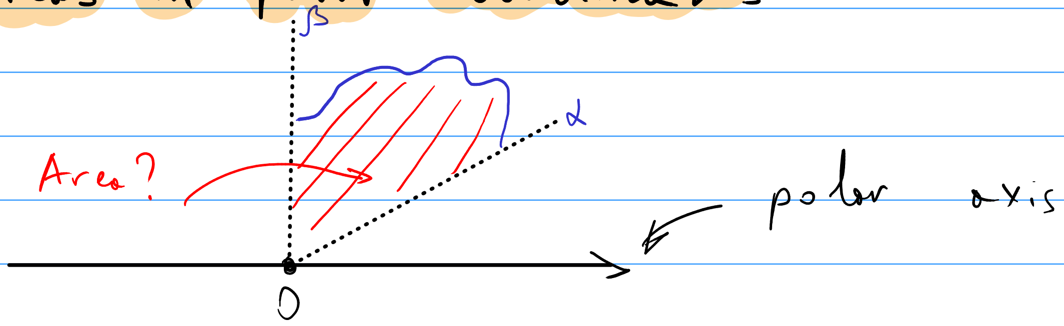
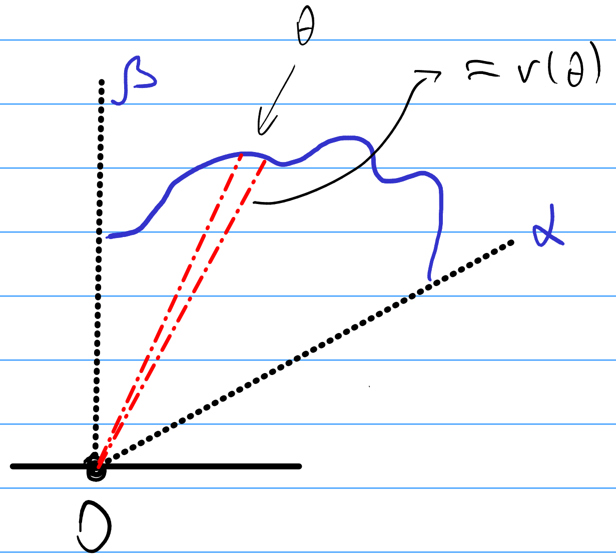
Areas in polar coordinates

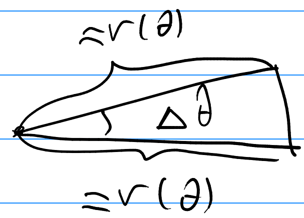


$r = r(\theta) \qquad \alpha \le \theta \le \beta$

Area bounded by this curve:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2 \, d\theta$$





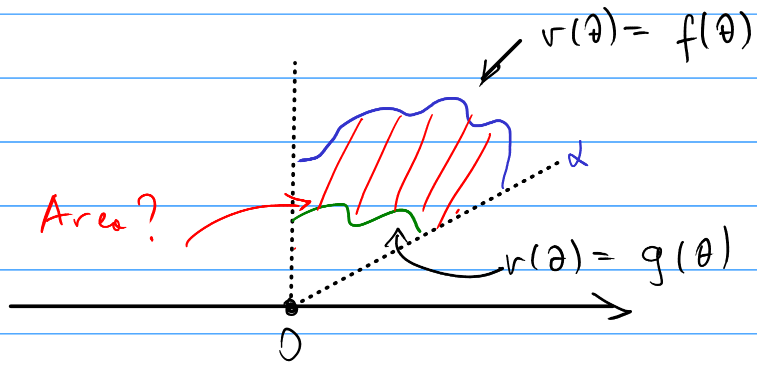
$$\Delta A = \frac{1}{2} ab \cdot \sin(\Delta\theta)$$

$$\Delta A \approx \frac{1}{2} r(\theta)^2 \, \Delta\theta$$

$$dA = \frac{1}{2} r(\theta)^2 \, d\theta$$

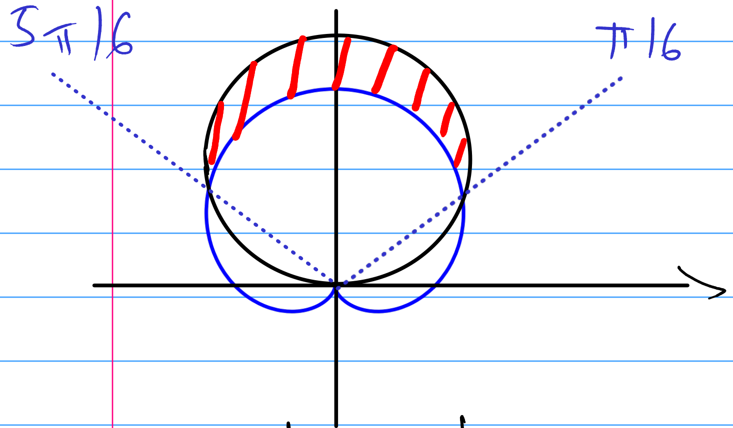
Recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$\sin x \simeq x, \quad x \to 0$



$$A = \int_{\alpha}^{\beta} \frac{1}{2} \left(f(\theta)^2 - g(\theta)^2 \right) d\theta$$

Ex 1 Find the area between the circle $r = 3\sin\theta$ and the cardioid $r = 1+\sin\theta$ (inside the circle, outside the cardioid)



Find the points (angles) of intersection:

$$3\sin\theta = 1+\sin\theta \implies 2\sin\theta = 1$$

$$\implies \sin\theta = \frac{1}{2} \qquad (\theta \text{ in } [0, 2\pi])$$

$$\implies \theta_{1,2} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta$$

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left((3\sin\theta)^2 - (1+\sin\theta)^2\right) d\theta$$

$$= 2 \cdot \int_{\pi/6}^{\pi/2} \frac{1}{2}\left(9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(8 \sin^2\theta - 2\sin\theta - 1\right) d\theta$$

$$\left(\text{Half-angle formula: } \quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right)$$

$$= \int_{\pi/6}^{\pi/2} \left(8 \cdot \frac{1}{2}(1-\cos 2\theta) - 2\sin\theta - 1\right) d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left(3 - 4\cos 2\theta - 2\sin\theta\right) d\theta$$

$$= \left(3\theta - 4 \cdot \frac{\sin 2\theta}{2} - 2(-\cos\theta)\right)\Bigg|_{\pi/6}^{\pi/2}$$

$$= \left(3\theta - 2\sin 2\theta + 2\cos\theta\right)\Bigg|_{\pi/6}^{\pi/2}$$

$$= \frac{3\pi}{2} - \frac{\pi}{2} - 2\left(0 - \frac{\sqrt{3}}{2}\right) + 2\left(0 - \frac{\sqrt{3}}{2}\right)$$

$$= \pi.$$