REVIEW QUESTIONS FOR TEST 4

Geometric series, divergence test, telescoping series

1) What is an infinite series; how are the partial sums of a series defined? How is the sum of a series related to the limit of the sequence of partial sums? Write down the formula for the sum of geometric series; when does a geometric series converge? What is a telescoping series, and how to compute its sum? Formulate the divergence test. Which arithmetic operations can be performed on convergent series?
2) For the following series, determined by the sequences of partial sums, compute the value of the infinite sum or show that the series diverges:

   (a) $s_n = 2 + (-1)^n$, $n \geq 1$

   (b) $s_n = 3 + \dfrac{(-1)^n}{n}$, $n \geq 1$

   (c) $s_n = \sin(4^{-n})$, $n \geq 1$

   (d) $s_n = \dfrac{n+1}{n}$, $n \geq 1$.

3) Determine whether the following series converge; if they do, determine their sums:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\pi^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{(-2)^{3n+1}}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{3n+1}}{7^n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} 3^{3n+1} 4^{-2n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{2})^{-n}$.

4) Compute the indicated partial sum of the given series:

   (a) s_5 for $\displaystyle\sum_{n=1}^{\infty} 3^n$

   (b) s_7 for $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5^n}$

   (c) s_4 for $\displaystyle\sum_{n=1}^{\infty} 2 \cdot 7^{-n+1} 3^{n-2}$

   (d) s_3 for $\displaystyle\sum_{n=1}^{\infty} \frac{9}{16 \cdot 14^n}$

5) Test the following series for convergence:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2-n}{1+3n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (\sin 100)^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+(2/3)^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{4^n} + \frac{3}{n}\right)$.

   (f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3^n} + \frac{5}{e^n}\right)$.

6) Determine convergence of the given series by expressing the n-th partial sum s_n as a telescoping sum:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ *Hint:* use partial fraction decomposition.

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

(e) $\displaystyle\sum_{n=1}^{\infty} \left(e^{3/n} - e^{3/(n+1)}\right)$.

Integral test and sum estimates

7) Write down the integral test. Which assumptions on $f(x)$ must be satisfied? Can the integral test be applied if these assumptions hold only for $n \geq N$, where N is a positive integer? Write down the general form of the p-series; when does a p-series converge? Explain how the integral test can be used for remainder estimates.

8) Determine if the given series are convergent:

(a) $\dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{11} + \dfrac{1}{15} + \dfrac{1}{19} + \dots,$ assuming that the given pattern continues.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n-1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$

(d) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} n^{-\sqrt{3}}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n+2}$

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

9) Explain why the integral test cannot be applied to the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$.

10) For the given series, estimate the indicated remainder $R_n = s - s_n$ from below and from above, using the integral test. Namely, use the formula

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$, R_5

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$, R_{10}

(c) $\displaystyle\sum_{k=1}^{\infty} e^{-k}$, R_8

(d) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$, R_6.

11) For the series in the previous question, determine the number of terms n that are necessary to guarantee that $|R_n| < 10^{-5}$.

Comparison tests

12) Write down the statement of the comparison test; what types of series can it be applied to? What known series are usually used for comparison? State the limit comparison test; describe its common applications.
13) Determine whether the given series converge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3+8}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+3n+4}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{11^n+6}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{11^n-6}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt[3]{n^2+n+1}}$

(h) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n^4+3n^3+3n^2+1}{n^2+4}}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{n^2e^n+1}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{n+3^n}$

(k) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n)$

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$.

Alternating series

14) Define alternating series. Write down the statement of the alternating series test. Why is the second assumption of the test necessary for the series to be convergent? Explain how to estimate the remainder of the n-th partial sum, $R_n = s - s_n$, for alternating series.
15) Test the given series for convergence:

(a) $\displaystyle\frac{1}{\ln 2} - \frac{1}{\ln 4} + \frac{1}{\ln 6} - \frac{1}{\ln 8} + \frac{1}{\ln 10} - \ldots,$ assuming that the given pattern continues.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4+7n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{\sqrt{n+1}}$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{3})^{-n}$

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+5n+1}$

(f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \arctan n$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$

(h) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

(i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

(j) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

(k) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$.

16) For the following series, estimate $|R_n|$ for the indicated n. Use the inequality

$$|R_n| \leq b_{n+1}.$$

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}, \quad R_7$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1/3)^n}{n}, \quad R_3$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 2^n}, \quad R_3$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)^n, \quad R_6.$

17) For every series in the previous question, determine the number of terms n that are necessary to guarantee that $|R_n| < 10^{-5}$.

Absolute convergence, ratio and root tests

18) Write down the definition of absolute convergence; does the usual convergence imply absolute convergence? When a series is called conditionally convergent? Does absolute convergence imply the usual convergence? Write down the statement of the ratio test; explain when it is inconclusive. Formulate the root test; give examples of series for which it should be used. Suppose using the ratio test for a certain series is inconclusive — should you apply the root test then?

19) Apply the ratio test to the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n n^3}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/4)}{n!}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{200}}{n!}.$

20) Apply the root test to the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{4n^2+8n+4}\right)^n.$

21) Try using either ratio or root test to determine whether the given series converge. If the result is inconclusive, can you think of a different test that may be applied? What can you say about the absolute convergence of the series?

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(b) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{n}{\ln n}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{1+n\sqrt{n}}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n5^{2n}}{10^{n+1}}$.

General strategy for testing convergence

22) List all the convergence tests you know. In what cases the choice of the test is the most straightforward? Explain how to reduce series with terms containing polynomial factors to p-series. Can comparison tests be applied to alternating series? Which tests should be used to determine absolute convergence? When is it appropriate to use the integral test?

23) Determine whether the given series converge using the tests you know and their combinations:

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

(k) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{3n}}{(1+2n^2)^n}$

(l) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{3n+1}\right)$

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{5^n n!}$

(n) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1}$

(o) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

(p) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n^{-3}$.

Answer key.

2) (a) Divergent.
   (b) $s = 3$
   (c) $s = 0$
   (d) $s = 1$.

3) (a) Convergent; $1/7$.
   (b) Divergent.
   (c) Convergent; $\dfrac{5/\pi}{1 - 1/\pi}$.
   (d) Convergent; $9/22$.
   (e) Divergent.
   (f) Divergent.
   (g) Convergent; $\dfrac{1}{\sqrt{2}(1 - 1/\sqrt{2})}$.

4) (a) 363
   (b) $39062/78125$
   (c) $1160/1029$
   (d) $1899/43904$

5) (a) Divergent.
   (b) Divergent.
   (c) Convergent.
   (d) Divergent.
   (e) Divergent.
   (f) Convergent.

6) (a) Convergent.
   (b) Divergent.
   (c) Convergent.
   (d) Convergent.
   (e) Convergent.

8) (a) Divergent.
   (b) Divergent.
   (c) Convergent.
   (d) Convergent.
   (e) Convergent.
   (f) Convergent.
   (g) Convergent.
   (h) Convergent.
   (i) Convergent.

9) (a) The function $f(x) = \dfrac{2 + \cos x}{x}$ is not decreasing on any interval $[M\,,\infty)$.
   (b) The terms of this series are not all of the same sign.

10) (a) $1/648 \le R_5 \le 1/375$.
   (b) $\pi/2 - \arctan(11) \le R_{10} \le \pi/2 - \arctan(10)$.
   (c) $e^{-9} \le R_8 \le e^{-8}$.
   (d) $1/\ln(7) \le R_6 \le 1/\ln(6)$.

11) (a) $n \ge 33$.
   (b) $n \ge 100\,000$.
   (c) $n \ge 12$.
   (d) $n > e^{10^5} \approx 2.8066633604 \times 10^{43\,429}$.
   Note: this means that the given series converges *extremely* slowly.

13) (a) Convergent.
   (b) Divergent.
   (c) Convergent.
   (d) Convergent.
   (e) Divergent.
   (f) Convergent.
   (g) Divergent.
   (h) Divergent.
   (i) Convergent.
   (j) Convergent.
   (k) Divergent.
   (l) Divergent.

15) (a) Convergent.
(b) Convergent.
(c) Convergent.
(d) Convergent.
(e) Divergent.
(f) Divergent.
(g) Convergent.
(h) Convergent.
(i) Divergent.
(j) Convergent.
(k) Divergent.

16) (a) $|R_7| \leq 1/8^6$.
(b) $|R_3| \leq 1/(4 \cdot 3^4)$.
(c) $|R_3| \leq 1/(4^3 \cdot 2^4)$.
(d) $|R_6| \leq 1/7^7$.

17) (a) $n \geq 6$.
(b) $n \geq 8$.
(c) $n \geq 7$.
(d) $n \geq 6$.

19) (a) Convergent.
(b) Divergent.
(c) Convergent.
(d) Convergent.
(e) Divergent.
(f) Convergent.
(g) Convergent.

20) (a) Divergent.
(b) Convergent.
(c) Divergent.
(d) Convergent.

21) (a) Convergent conditionally.
(b) Divergent.
(c) Convergent absolutely.
(d) Convergent conditionally.
(e) Convergent absolutely.
(f) Divergent.

23) (a) Convergent.
(b) Convergent.
(c) Divergent.
(d) Convergent.
(e) Divergent.
(f) Convergent.
(g) Divergent.
(h) Divergent.
(i) Divergent.
(j) Convergent.
(k) Divergent.
(l) Divergent.
(m) Convergent.
(n) Convergent.
(o) Convergent.
(p) Convergent.