Section 7.5: Summary of the integration techniques

1. Simplify $\int \sqrt{x}\,(1+\sqrt{x})\,dx$ $\quad \int \frac{\tan x}{\sec^2 x}\,dx = \int \sin x \cdot \cos x\,dx$

2. Check for simple u-subs $\quad \int \frac{x}{x^2-1}\,dx$

3. Classify the integrand:
   - (a) trig functions, products of sin, cos, etc.
   - (b) rational — " —
   - (c) integration by parts
     $\{\text{polynomial}\} \times \{e^x, \sin x, \cos x\}$
     $\arctan x, \quad \arcsin x$
     $\int \arctan x\,dx$
   - (d) Radicals
     - $\sqrt{\pm x^2 \pm a^2} \longrightarrow$ trig subs
     - $\sqrt[n]{ax+b} \longrightarrow u = \sqrt[n]{ax+b}$

     $u^n = ax+b$
     $n u^{n-1}\,du = a\,dx$

     (may work also for $u = \sqrt[n]{g(x)}$, any $g(x)$)

4. Fundamentally only 2 methods:
   u-sub / int by parts

   - (a) u-sub
   - (b) by parts
   - (c) algebraic transformations
   - (d) reduce to known integrals
   - (e) combine

Evaluate the integrals

Ex. 1

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx \nearrow \int \tan^3 x \sec^3 x\,dx$$

$$\searrow \int \sin^3 x \cdot \cos^{-6} x\,dx$$

$$\int \tan^3 x \cdot \sec^3 x\,dx = \int \tan^2 x \cdot \sec^2 x \cdot \underbrace{\sec x \tan x\,dx}_{du}$$

$$= \int (\sec^2 x - 1)\sec^2 x \cdot \sec x \tan x\,dx$$

$$= \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x\,dx \end{array} \right| = \int (u^2-1)\,u^2\,du = \int (u^4 - u^2)\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{(\sec x)^5}{5} - \frac{(\sec x)^3}{3} + C$$

Ex. 2

$$\int e^{\sqrt{x}}\,dx = \left| \begin{array}{l} u = \sqrt{x} \\ u^2 = x \\ 2u\,du = dx \end{array} \right| = \int e^u \cdot 2u\,du$$

$$= 2\int \underbrace{u}_{w}\,\underbrace{e^u\,du}_{dv} = 2\left(u \cdot e^u - \int e^u\,du\right)$$

$$= 2\,(u e^u - e^u) + C$$

$$= 2\,(\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}}) + C$$

Ex. 3

$2\,d(x^3-3x^2-10x)$

$u = x^3 - 3x^2 - 10x$

$$\int \frac{6x^2-12x-20}{\underbrace{x^3-3x^2-10x}_{\text{improper rational function}}}\,dx$$

$\rightarrow$ long division $\rightarrow$

$\rightarrow$ apply PFD to the proper rational function

Ex. 4

$$\int \boxed{\frac{dx}{x}\,}\frac{1}{\sqrt{\ln x}}$$

possible: $u = \sqrt{\ln x}$

$u^2 = \ln x$

$e^{u^2} = x$

$2u\,e^{u^2}du = dx$

$$\left| \begin{array}{l} u = \ln x \\ du = \dfrac{dx}{x} \end{array} \right| \rightarrow \int \frac{du}{\sqrt{u}} =$$

$$= \int u^{-1/2}\,du$$

Ex. 5

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx \quad \times \frac{\sqrt{1-x}}{\sqrt{1-x}}$$

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \underbrace{\int \frac{dx}{\sqrt{1-x^2}}}_{\arcsin x} - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$

$$= \arcsin x + \sqrt{1-x^2} + c$$

$(c = -c_0)$

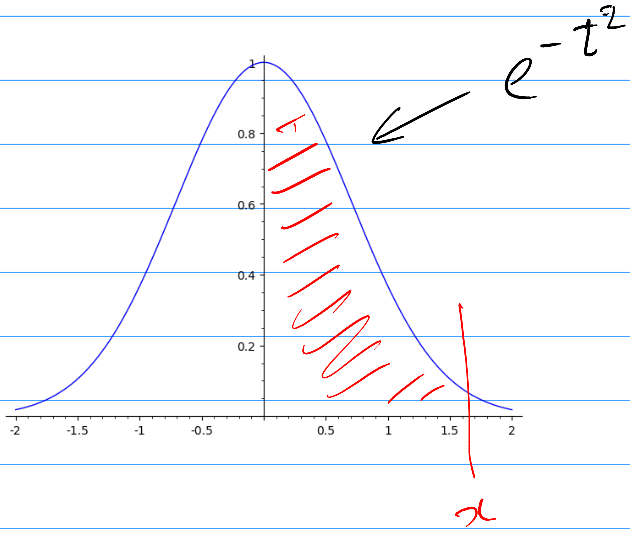
$$\int \frac{\boxed{x}}{\sqrt{1-x^2}}\boxed{dx} = \left| \begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \\ -\frac{1}{2}du &= \underline{x\,dx} \end{aligned} \right| = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + C_0$$

$$= -\sqrt{1-x^2} + c_0$$

Conclusion: Not all functions have antiderivatives expressible in elementary functions (polynomials, trig, inverse trig, etc).

$$\int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x)$$



$$\left.\begin{aligned} &\int \sqrt{x^3+1}\,dx \\ &\int \frac{dx}{\ln x} \end{aligned}\right\}$$ not expressible in elementary functions