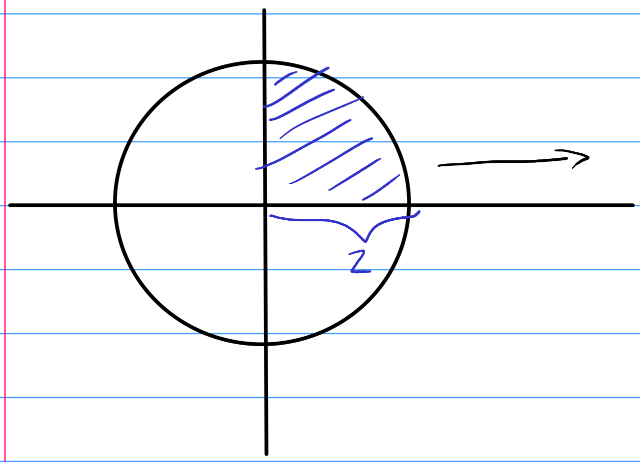
Section 7.3: Trigonometric substitution



$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \sqrt{4 - x^2}$$

$$\int y(x)\,dx = \int \sqrt{4-x^2}\,dx = \left| \begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta\,d\theta \end{array} \right|$$

Recall: Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1$$

$$4\sin^2\theta + 4\cos^2\theta = 4$$

$$4 - 4\sin^2\theta = 4\cos^2\theta = (2\cos\theta)^2$$

$$= \int \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta\,d\theta$$

$$= \int \sqrt{(2\cos\theta)^2} \cdot 2\cos\theta\,d\theta$$

$$= \int |2\cos\theta| \cdot 2\cos\theta\,d\theta$$

$$\overset{?}{=} 4\int \cos^2\theta\,d\theta = \text{trig integral, apply Section 7.2.}$$

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$	$a^2(1-\sin^2\theta) = a^2 \cdot \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$	$a^2(1+\tan^2\theta) = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$, $0 \le \theta < \pi/2$	$a^2(\sec^2\theta - 1) = a^2\tan^2\theta$

Ex. 1 Evaluate $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \left| \begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\, d\theta \end{array} \right|$

$\sin\theta = x/3$

$\theta = \arcsin(x/3)$

$$= \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{9(\underbrace{1-\sin^2\theta}_{\cos^2\theta})}}{9\cdot\sin^2\theta} \cdot 3\cos\theta\, d\theta =$$

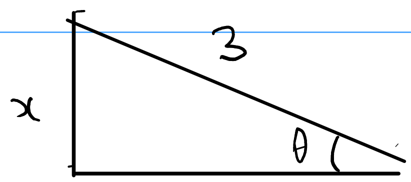
$$\int \frac{\cancel{3}\cos\theta}{\cancel{9}\cdot\sin^2\theta} \cdot \cancel{3}\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta =$$

$$= \int \frac{1-\sin^2\theta}{\sin^2\theta}\, d\theta = \int (\csc^2\theta - 1)\, d\theta$$

$\frac{a \pm b}{c} = \frac{a}{c} \pm \frac{b}{c}$

$$= -\cot\theta - \theta + C$$

To express $\cot\theta$ through x, consider the triangle:



$x = 3 \cdot \sin\theta$

$\text{opp} = \text{hyp} \cdot \sin\theta$

$$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{9-x^2}}{x}$$

$$= -\frac{\sqrt{9-x^2}}{x} - \arcsin(x/3) + C$$

Ex. 2

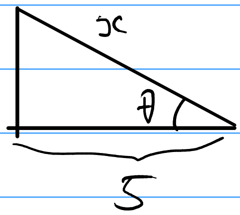
$$\int \frac{dx}{\sqrt{x^2-25}} = \left| \begin{array}{l} x = 5\sec\theta \\ dx = 5\sec\theta\tan\theta\, d\theta \end{array} \right|$$

$$= \int \frac{5\cdot\sec\theta\tan\theta\, d\theta}{\sqrt{25\sec^2\theta - 25}} = \int \frac{5\cdot\sec\theta\tan\theta\, d\theta}{\sqrt{25\tan^2\theta}}$$

$$= \int \frac{5\cdot\sec\theta\tan\theta\, d\theta}{5\cdot\tan\theta} = \int \sec\theta\, d\theta$$

$$= \ln|\underbrace{\sec\theta}_{x/5} + \tan\theta| + C =$$

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\frac{x^2}{5^2} - 1}$$



$$hyp = adj\cdot\sec\theta$$

$$x = 5\cdot\sec\theta$$

$$\tan\theta = \frac{opp}{adj} = \frac{\sqrt{x^2-25}}{5}$$

$$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + C$$

$$= \ln\left|\frac{1}{5}\left(x + \sqrt{x^2-25}\right)\right| + C$$

$$= \ln|x + \sqrt{x^2-25}| \underbrace{- \ln 5 + c}_{C_1}$$

$$= \ln|x + \sqrt{x^2-5}| + C_1$$

Ex. 3 $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{3\sqrt{3}/2} \frac{x^3}{\left(\sqrt{4x^2+9}\right)^3}\,dx$

$$= \int_0^{3\sqrt{3}/2} \frac{x^3}{8\left(\sqrt{x^2+9/4}\right)^3}\,dx = \frac{1}{8}\int_0^{3\sqrt{3}/2} \frac{x^3}{\left(\sqrt{x^2+\underbrace{9/4}_{(3/2)^2}}\right)^3}\,dx$$

$$= \left| \begin{array}{l} x = \frac{3}{2}\tan\theta \\ dx = \frac{3}{2}\sec^2\theta\, d\theta \end{array} \right|$$

$x_2 = \frac{3\sqrt{3}}{2} = \frac{3}{2}\tan\theta_2$

$x_1 = 0 = \frac{3}{2}\tan\theta_1$

$\theta_1 = 0$

$\tan\theta_2 = \sqrt{3}$

$$= \frac{1}{8}\int_0^{\pi/3} \frac{\left(\frac{3}{2}\tan\theta\right)^3}{\left(\sqrt{\left(\frac{3}{2}\tan\theta\right)^2+\left(\frac{3}{2}\right)^2}\right)^3}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\pi/3} \frac{\left(\frac{3}{2}\right)^3\tan^3\theta}{\left(\sqrt{\left(\frac{3}{2}\right)^2\cdot\sec^2\theta}\right)^3}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\pi/3} \frac{\left(\cancel{\frac{3}{2}}\tan\theta\right)^3}{\left(\cancel{\frac{3}{2}}\sec\theta\right)^3}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \left(\frac{\sin\theta}{\cos\theta}\right)^3\cdot\cos\theta\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\, d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{(1-\cos^2\theta)\cdot\sin\theta\, d\theta}{\cos^2\theta}$$

$$= \left| \begin{array}{l} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{array} \right| = -\frac{3}{16}\int_1^{1/2} \frac{(1-u^2)\, du}{u^2}$$

$$= \frac{3}{16}\int_{1/2}^{1} (u^{-2}-1)\, du = \frac{3}{16}\left(\frac{u^{-1}}{-1} - u\right)\Big|_{1/2}^{1}$$

$$= \frac{3}{16}\left(-\frac{1}{u} - u\right)\Big|_{1/2}^{1} = \frac{3}{16}\Big[(-1-1)$$

$$- \left(-2-\frac{1}{2}\right)\Big] = \frac{3}{16}\left(-2+\frac{5}{2}\right)$$

$$= +\frac{3}{32}$$

$$\sqrt{\pm x^2 \pm a^2}$$

Ex. 4 Evaluate $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\, dx$

$$\int \frac{x}{\sqrt{3-(2x+x^2)}}\, dx = \int \frac{x}{\sqrt{3+1-(1+2x+x^2)}}\, dx$$

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$
$$1^2 \quad +2x \quad +x^2$$

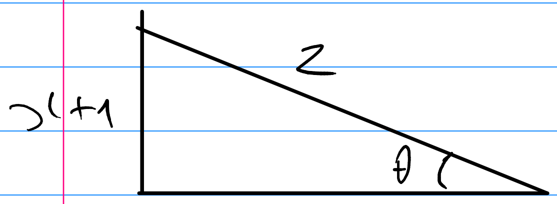
$$= \int \frac{x}{\sqrt{4-(x+1)^2}}\, dx = \left| \begin{array}{l} x+1 = 2\sin\theta \\ dx = 2\cos\theta\, d\theta \end{array} \right|$$

$a^2 - x^2$

$$= \int \frac{2\sin\theta - 1}{\sqrt{4-(2\sin\theta)^2}} \cdot 2\cos\theta \, d\theta = \int \frac{2\sin\theta - 1}{\cancel{2\cos\theta}} \cdot \cancel{2\cos\theta} \, d\theta$$

$$= \int (2\sin\theta - 1)\, d\theta = -2\cos\theta - \theta + C =$$

$$\sin\theta = \frac{x+1}{2} \implies \theta = \arcsin\left(\frac{x+1}{2}\right)$$



$$x+1 = 2\sin\theta$$

$$opp = hyp \cdot \sin\theta$$

$$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{4-(x+1)^2}}{2}$$

$$= -2 \cdot \frac{\sqrt{4-(x+1)^2}}{2} - \arcsin\left(\frac{x+1}{2}\right) + C.$$