Measure and Integration II (MAA5617), Spring 2021
Homework 4, due Thursday, Feb 25

Below L^1, L^1_{loc} refer to the measure spaces with $\lambda = \lambda^n$ as the underlying measure.

1. For $f \in L^1_{loc}$, prove that the average $A_r f(x)$ is jointly continuous as a function of $(x, r) \in \mathbb{R}^n \times (0, \infty)$. Hint: this is a lemma in Folland.

2. Given an $f \in L^1$ and $\epsilon > 0$, construct a continuous function g, for which

$$\int |f - g| \, d\lambda < \epsilon.$$

Show that the same result holds if in addition $g \in C^\infty(\mathbb{R}^n)$, that is, g and all of its partial derivatives are continuous.

- Because f can be approximated with simple functions, it suffices to assume f is an indicator of a rectangle (why?).
- For two functions f_1, f_2 on $\mathbb{R}^n$, define their convolution as

$$f_1 * f_2(x) := \int f_1(x - y) f_2(y) \, dy.$$

- Construct a continuous function $\phi \in C^\infty(\mathbb{R}^n)$ supported on $B(\bar{0}, 1)$. Hint: e^{-1/t^2} (extended to $t = 0$ by continuity) is infinitely differentiable at $t = 0$.
- Using the dominated convergence theorem, show that

$$f * r^{-n} \phi(x/r)$$

  converges to f in L^1 for $r \downarrow 0$, and is infinitely differentiable, as desired.

3. If μ_1, μ_2 are positive Borel measures on $\mathbb{R}^n$, $\mu_1 \perp \mu_2$, and $\mu_1 + \mu_2$ is regular, then so is each μ_i, $i = 1, 2$.

4. Suppose $E \in \mathcal{B}_{\mathbb{R}^n}$. Using the Lebesgue differentiation theorem, show that

$$D_E(x) = \lim_{r \downarrow 0} \frac{\lambda(E \cap B(x, r))}{\lambda(B(x, r))}$$

is equal to 1 for λ-a.e. $x \in E$, and to 0 for λ-a.e. $x \notin E$.

5. As in the previous problem, give an example of $E \in \mathcal{B}_{\mathbb{R}}$ and $x \in E$, for which $D_E(x) = \alpha$ for a given $\alpha \in (0, 1)$. Hint: use a nested sequence of fat Cantor sets with positive measures.

6. Construct an $E \in \mathcal{B}_{\mathbb{R}}$, such that $0 < \lambda(E \cap I) < \lambda(I)$ for any interval $I \subset [0, 1]$. (Use fat Cantor sets again).