Section 7.4: Integrating rational functions via partial fractions.

Rational functions:

$$f(x) = \frac{P(x)}{Q(x)} \qquad P, Q \text{ - polynomials}$$

$$\left\{\begin{matrix}\text{rational} \\ \text{function}\end{matrix}\right\} \longmapsto \left\{\begin{matrix}\text{sum of} \\ \text{partial} \\ \text{fractions}\end{matrix}\right\}$$

$$\underbrace{\frac{x^3 - x^2 + 8x}{(x^2+1)(x-2)^2}}_{\text{rational function}} = \underbrace{\frac{x-1}{x^2+1}}_{} + \underbrace{\frac{4}{(x-2)^2}}_{}$$

partial fractions

$$f(x) = \frac{P(x)}{Q(x)}$$

f is proper, if $\deg P < \deg Q$

f is improper, if $\deg P \geq \deg Q$.

Partial fraction decomposition applies only to proper rational functions!

For an improper f: apply the long division:

$$f(x) = \frac{P(x)}{Q(x)} \overset{\text{long division}}{=} S(x) + \underbrace{\frac{R(x)}{Q(x)}}_{\text{proper rational function}}$$

Outline of the PFD:

Take $f = \dfrac{P(x)}{Q(x)}$ – proper

1. Factor $Q(x)$ → 2 types of irreducible factors: $(ax+b)$ OR (ax^2+bx+c) w/ $b^2-4ac<0$

2. Make a sum of partial fractions: for each factor of the form in the left column, add terms from the right column

Terms of $Q(x)$	Terms in PFD
$(ax+b)$	$\dfrac{A}{ax+b}$
$(ax+b)^r$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \dots + \dfrac{A_r}{(ax+b)^r}$
(ax^2+bx+c) w/ $b^2-4ac<0$	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^r$ w/ $b^2-4ac<0$	$\dfrac{A_1x+B_1}{(ax^2+bx+c)} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \dfrac{A_rx+B_r}{(ax^2+bx+c)^r}$

3. Equate
$$f(x) = PFD,$$
then determine the undefined coefficients

Evaluate integrals:

Ex. 1

$$\int \underbrace{\frac{x^3+x}{x-1}}_{\text{improper rational f-n}} dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx =$$

Perform the long division

$$\begin{array}{r|l} & x^2+x+2 \\ \hline x-1 & x^3+x \\ & -(x^3-x^2) \\ \hline & x^2+x \\ & -(x^2-x) \\ \hline & 2x \\ & -(2x-2) \\ \hline & 2 \end{array}$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C$$

Ex. 2

$$\int \underbrace{\frac{x^2+2x-1}{2x^3+3x^2-2x}}_{\text{proper rational function}} dx$$

Case I: unique linear factors

$$\begin{aligned} 2x^3+3x^2-2x &= x(2x^2+3x-2) \\ &= x\cdot 2(x-(-2))\left(x-\frac{1}{2}\right) \\ &= x\cdot(x+2)(2x-1) \end{aligned}$$

$$2x^2+3x-2=0$$

$$x_{1,2}=\frac{-3\pm\sqrt{9+16}}{4}=\frac{-3\pm 5}{4}$$

$$x_1=-2$$

$$x_2=\frac{1}{2}$$

$$\frac{x^2+2x-1}{x(x+2)(2x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}$$

Multiply both sides by $x(x+2)(2x-1)$:

$$x^2+2x-1 = A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2)$$

For unique linear factors: plug in roots of the denominator

$x=-2$:

$$-1 = B(-2)(-5) \Rightarrow B = -\frac{1}{10}$$

$x=0$:

$$-1 = A\cdot 2(-1) \Rightarrow A = \frac{1}{2}$$

$x=\frac{1}{2}$:

$$\frac{1}{4} = C\cdot\frac{1}{2}\cdot\frac{5}{2} \Rightarrow C = \frac{1}{5}$$

$$\int\left(\frac{1}{2x} - \frac{1}{10(x+2)} + \frac{1}{5(2x-1)}\right)dx$$

$$= \frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| + K$$

Ex. 3 $\displaystyle\int \underbrace{\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}}_{\text{improper rational function}}\,dx$ } Case II: repeated linear factors

Long division:

$$
\begin{array}{r}
x+1 \\
x^3-x^2-x+1 \,\big)\, \overline{\cancel{x^4}-2x^2+4x+1} \\
-\;\underline{\cancel{x^4}-x^3-x^2+x} \\
x^3-x^2+3x+1 \\
-\;\underline{x^3-x^2-x+1} \\
4x
\end{array}
$$

$$\int \left(x+1+\underbrace{\frac{4x}{x^3-x^2-x+1}}_{\text{proper}}\right) dx$$

Factor the denominator:

$$
\begin{aligned}
x^3-x^2-x+1 &= (x^3-x^2)-(x-1) \\
&= x^2(x-1)-(x-1) \\
&= (x-1)(\underline{x^2-1}) = (x-1)\,\underline{(x-1)(x+1)} \\
&= (x-1)^2(x+1)
\end{aligned}
$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Multiply by the common denominator:

$$4x = A\underbrace{(x-1)^2}_{x^2-2x+1} + B\underbrace{(x-1)(x+1)}_{x^2-1} + C(x+1)$$

Equate the coefficients at correspon-
ding powers of x:

x^2:
$$0 = A + B \implies \underline{A = -B}$$

x:
$$\underline{4 = -2A + C}$$

x^0:
$$\underline{0 = A - B + C}$$

$$4 = -2A + C$$
$$0 = A - (-A) + C$$

$$\begin{cases} 4 = -2A + C \\ 0 = 2A + C \end{cases} \implies 4 = 2C$$

$$C = 2$$
$$A = -1$$
$$B = 1$$

$$\int \left(x + 1 + \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} \right) dx$$

$$= \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + K$$

Ex. 4 Compute

$$\underbrace{\int \frac{2x^2 - x + 4}{x^3 + 4x}}_{\text{proper rational function}} dx$$

Case III: unique (unrepeated) irreducible quadratic factors

Factor the denominator:

$$x^3 + 4x = x\underbrace{(x^2+4)}$$

$$D = 0^2 - 4 \cdot 4 = -16 < 0$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

Multiply by the common denominator:

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

Equating the coefficients:

x^2:

$$2 = A + B \implies B = 1$$

x^1:

$$-1 = C \implies C = -1$$

x^0:

$$4 = 4A \implies A = 1$$

$$\int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx = \int \left(\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right) dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K$$

$$\int \frac{x}{x^2+4}\,dx = \left| \begin{array}{l} u = x^2+4 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \end{array} \right| = \int \frac{\frac{1}{2}\,du}{u} =$$

$$= \frac{1}{2}\ln|u| + K_1 = \frac{1}{2}\ln|x^2+4| + K_1$$

$$\int \frac{1}{x^2+a^2}\,dx = \int \frac{1}{a^2\left(\left(\frac{x}{a}\right)^2+1\right)}\,dx$$

$$= \frac{1}{a^2}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,dx = \left| \begin{array}{l} u = \frac{x}{a} \\ du = dx/a \\ a\,du = dx \end{array} \right| =$$

$$= \frac{1}{a^2}\int \frac{1}{u^2+1}\,a\,du = \frac{1}{a}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{a}\arctan(u) + K_2 = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + K_2$$

Ex 5.

$$\underbrace{\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}}_{\text{proper rational function}}$$

Case IV: repeated irreducible quadratic factors

$$\frac{-x^3+2x^2-x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiply by the common denominator:

$$-x^3+2x^2-x+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$x=0$:

$$1 = A \quad \Rightarrow \quad A = 1$$

Equate coefficients at similar powers of x:

x^4:

$$0 = A + B \Rightarrow B = -1$$

x^3:

$$-1 = C \Rightarrow C = -1$$

x^2:

$$2 = 2A + B + D \Rightarrow D = 1$$

x^1:

$$-1 = C + E = E = 0$$

x^0:

$$1 = A \Rightarrow A = 1$$

$$\int \left(\frac{1}{x} + \frac{-1 \cdot x - 1}{x^2+1} + \frac{1 \cdot x}{(x^2+1)^2} \right) dx$$

$$= \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx$$

$$= \ln|x| - \frac{1}{2} \ln|x^2+1| - \arctan x - \frac{1}{2(x^2+1)} + K$$

$$\int \frac{x}{(x^2+1)^2} dx = \left| \begin{array}{l} u = x^2+1 \\ du = 2x\,dx \\ \frac{1}{2} du = x\,dx \end{array} \right|$$

$$= \int \frac{\frac{1}{2} du}{u^2} = \frac{1}{2} \int u^{-2} du = \frac{1}{2} \frac{u^{-1}}{-1} + K_0$$

$$= -\frac{1}{2} \frac{1}{x^2+1} + K_0$$

Ex. 5'

$$\int \frac{1}{(x^2+1)^2}\,dx = \left| \begin{array}{l} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{array} \right|$$

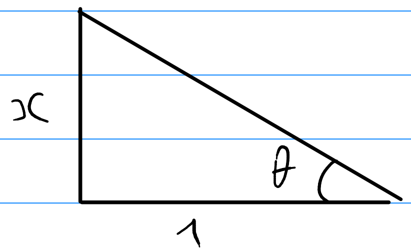
$$= \int \frac{1}{(\underbrace{\tan^2\theta+1}_{\sec^2\theta})^2} \cdot \sec^2\theta\, d\theta = \int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2}$$

$$= \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta\, d\theta = \left| \begin{array}{l} \text{half-angle} \\ \text{formula} \end{array} \right|$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + K$$

$$= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + K$$

$$\sin 2A = 2\sin A\cos A$$



$$\theta = \arctan x$$

$$x = 1\cdot\tan\theta$$

$$\text{opp} = \text{adj}\cdot\tan\theta$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+1}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + K.$$

Ex. 6 $\displaystyle\int \underbrace{\frac{4x^2-3x+2}{4x^2-4x+3}}_{\text{improper}}\,dx = \int \frac{(4x^2-4x+3)+x-1}{4x^2-4x+3}\,dx$

$$= \int \Bigl(1+\underbrace{\frac{x-1}{4x^2-4x+3}}_{\text{proper}}\Bigr)\,dx \;\overset{(=)}{}$$

$$D = (-4)^2 - 4\cdot 4\cdot 3 = -2\cdot 16 = -32 < 0$$

$$\frac{x-1}{4x^2-4x+3} = \frac{Ax+B}{4x^2-4x+3}$$

$$\int \frac{x-1}{4x^2-4x+3}\,dx \underset{\text{the square}}{\overset{\text{complete}}{=}} \int \frac{x-1}{(2x)^2-2\cdot 2x+1^2+2}\,dx$$

$$= \int \frac{x-1}{(2x-1)^2+2}\,dx = \left| \begin{array}{l} u = 2x-1 \\ du = 2\,dx \\ \frac{1}{2}du = dx \\ x = \frac{u+1}{2} \end{array} \right|$$

$$= \int \frac{\frac{u+1}{2}-1}{u^2+2}\cdot\frac{1}{2}\,du = \frac{1}{4}\int \frac{u-1}{u^2+2}\,du$$

$$= \frac{1}{4}\int \frac{u}{u^2+2}\,du - \frac{1}{4}\int \frac{du}{u^2+2}$$

$$= \frac{1}{8}\ln|u^2+2| - \frac{1}{4}\,\frac{1}{\sqrt{2}}\arctan\Bigl(\frac{u}{\sqrt{2}}\Bigr) + K$$

$$= \frac{1}{8}\ln|(2x-1)^2+2| - \frac{1}{4\sqrt{2}}\arctan\Bigl(\frac{u}{\sqrt{2}}\Bigr) + K.$$

$$(=)\quad x + \frac{1}{8}\ln|(2x-1)^2+2| - \frac{1}{4\sqrt{2}}\arctan\Bigl(\frac{u}{\sqrt{2}}\Bigr) + K.$$

Rationalizing substitutions

$\sqrt[n]{g(x)} \longmapsto$ apply $u = \sqrt[n]{g(x)}$
(guaranteed to work for $g(x) = ax + b$)

Ex. 7 $\displaystyle\int \frac{\sqrt{x+3}}{x}\,dx = \left| \begin{array}{l} u = \sqrt{x+3} \\ u^2 = x+3 \\ 2u\,du = dx \\ x = u^2 - 3 \end{array} \right| = \int \frac{u}{u^2-3}\, 2u\,du$

$$= 2\int \frac{u^2}{u^2-3}\,du = 2\int \frac{(u^2-3)+3}{u^2-3}\,du = 2\int \Big(1 + \underbrace{\frac{3}{u^2-3}}_{\text{proper}}\,du\Big)$$

$$\frac{3}{(u-\sqrt{3})(u+\sqrt{3})} = \frac{A}{u-\sqrt{3}} + \frac{B}{u+\sqrt{3}}$$

Multiply by the common denominator:

$$3 = A(u+\sqrt{3}) + B(u-\sqrt{3})$$

$u = \sqrt{3}$:

$$3 = 2A\sqrt{3} \implies A = \frac{\sqrt{3}}{2}$$

$u = -\sqrt{3}$:

$$3 = -2\sqrt{3}B \implies B = -\frac{\sqrt{3}}{2}.$$

$$2\int \left(1 + \frac{\sqrt{3}}{2}\left(\frac{1}{u-\sqrt{3}} - \frac{1}{u+\sqrt{3}}\right)\right) du$$

$$= 2u + \sqrt{3}\left(\ln|u-\sqrt{3}| - \ln|u+\sqrt{3}|\right) + K$$

$$= 2u + \sqrt{3}\ln\left|\frac{u-\sqrt{3}}{u+\sqrt{3}}\right| + K$$

$$= 2\sqrt{x+3} + \sqrt{3}\ln\left|\frac{\sqrt{x+3}-\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}\right| + K.$$