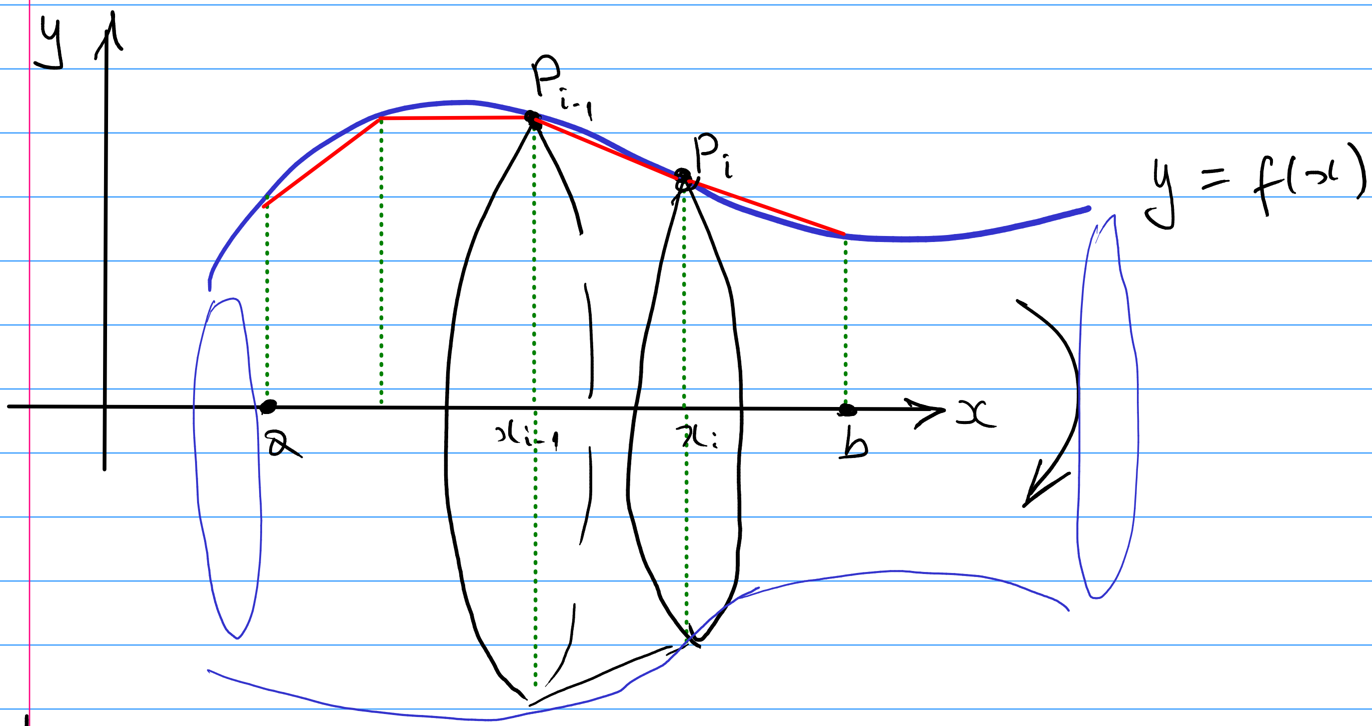
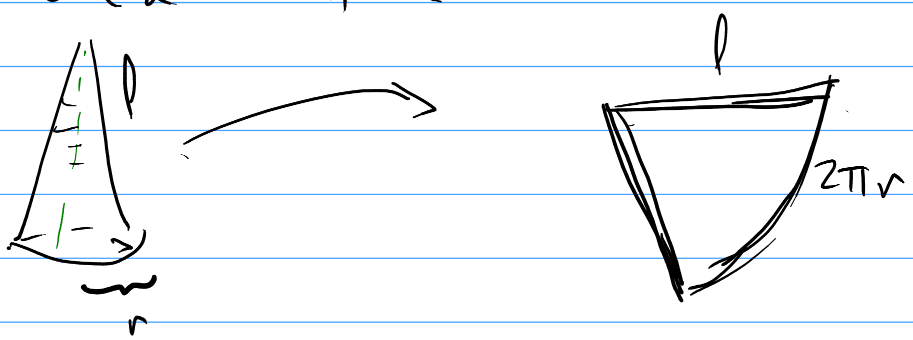
Section 8.2: Areas of surfaces of revolution



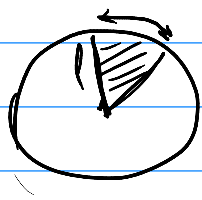
Side area of a cone

$\pi r l$



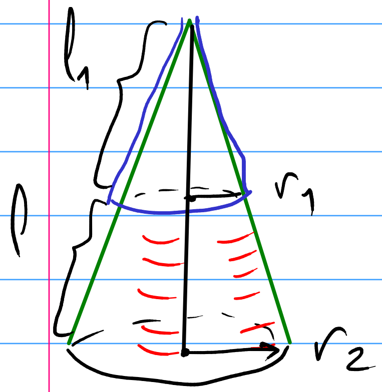
Area $\dfrac{2\pi r}{2} \cdot l = \pi r l$

Area of a sector of the unit disk is proportional to the arc:



$$A_{sector} = \frac{arc}{2\pi l} \cdot \pi l^2 = \frac{arc}{2} l$$

Side area of a conic prism



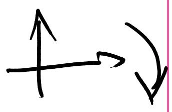
$$A = \pi r_2 (l + l_1) - \pi r_1 l_1$$
$$= \pi \left(r_2 l + (r_2 - r_1) l_1\right)$$
$$= \pi (r_2 l + r_1 l)$$
$$= \boxed{2 \pi r l} \qquad r = \frac{r_1 + r_2}{2}$$

To compute l_1, use similar triangles:

$$\frac{l_1}{l_1 + l} = \frac{r_1}{r_2} \Rightarrow l_1 r_2 = l_1 r_1 + l r_1$$
$$l_1 (r_2 - r_1) = l r_1$$

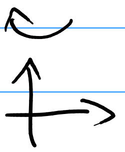
$$S = \lim_{n \to \infty} \sum_{i=1}^{n} \text{Area of the i-th prism}$$
$$= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \cdot r_i l_i$$
$$= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) |P_{i-1} P_i|$$
$$= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \cdot \underbrace{\sqrt{1 + f'(x_i^*)^2}\, \Delta x}_{\text{from arc length}}$$

$$S = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2}\, dx$$



$$S_x = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$

– area of a surface of revolution, obtained by rotating $y=f(x)$ about the x-axis.



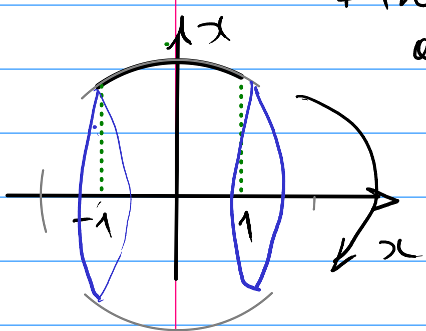
$$S_y = 2\pi \int_a^b x\sqrt{1+f'(x)^2}\,dx$$

— " — about the y-axis.

Ex. 1 $\quad y=\sqrt{4-x^2}, \qquad -1 \le x \le 1$

is an arc of $x^2+y^2=4$

Find the area after rotating about x-axis



$$y' = \frac{1}{2}\cdot(4-x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$$S = 2\pi\int_{-1}^{1}\sqrt{4-x^2}\cdot\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4-x^2} \cdot \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$$

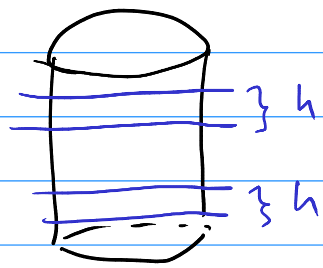
$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2} \cdot \sqrt{\frac{4}{4-x^2}}\, dx$$

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

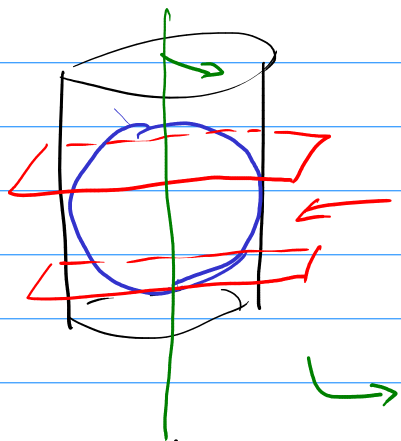
$$= 2\pi \int_{-1}^{1} \cancel{\sqrt{4-x^2}} \cdot \frac{\sqrt{4}}{\cancel{\sqrt{4-x^2}}}\, dx$$

$$= 4\pi \int_{-1}^{1} 1\, dx = 8\pi.$$

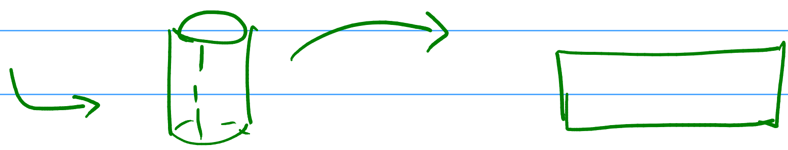
$$S = 4\pi \int_{x_1}^{x_2} dx$$



A digression: cylindrical projection.

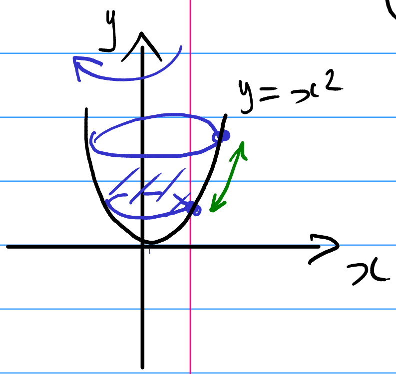


areas coincide!



Cylindrical projection preserves areas.

Ex. 2 The arc of $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis



$$S_y = \int 2\pi r \, dl$$

$$= \int 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int 2\pi \, x(y) \sqrt{1+\left(\frac{dx}{dy}\right)^2} \, dy$$

1). Integrate in x

$$\frac{dy}{dx} = 2x$$

$$S = \int_1^2 2\pi \cdot x \sqrt{1+(2x)^2} \, dx =$$

$$= \pi \int_1^2 \underline{2x} \sqrt{1+4x^2} \, \underline{dx} = \left| \begin{array}{l} u = 1+4x^2 \\ du = 8x\,dx \\ \frac{1}{4} du = \underline{2x\,dx} \end{array} \right.$$

$$= \pi \int_5^{17} \sqrt{u} \, \frac{1}{4} \, du = \frac{\pi}{4} \left. \frac{u^{3/2}}{3/2} \right|_5^{17}$$

$$= \frac{\pi}{2 \cdot 3} \left(17^{3/2} - 5^{3/2} \right).$$

2). Integrate in y.

$$x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$S = \int_1^4 2\pi \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \int_1^4 2\pi \sqrt{y} \cdot \sqrt{1 + \frac{1}{4y}}\, dy$$

$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$

$$= \int_1^4 2\pi \sqrt{y + \frac{1}{4}}\, dy = \left| \begin{array}{l} u = y + \frac{1}{4} \\ du = dy \end{array} \right|$$

$$= \int_{5/4}^{17/4} 2\pi \cdot \sqrt{u}\, du = 2\pi \cdot \frac{u^{3/2}}{3/2} \Bigg|_{5/4}^{17/4}$$

$$= \frac{4}{3}\pi \left(\left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2} \right)$$

$$= \frac{4}{3}\pi \left(\frac{17^{3/2}}{8} - \frac{5^{3/2}}{8} \right)$$

$$= \frac{\pi}{2 \cdot 3} \left(17^{3/2} - 5^{3/2} \right).$$