Measure and Integration I (MAA5616), Fall 2020
Homework 3, due Thursday, Sep. 24

1. A finitely additive measure μ that is continuous below is countably additive. Prove it.
Note: If $\mu(X) < \infty$, continuity above also implies countable additivity.
Note: Compare this problem to #4 in HW2.

2. Let $\mathcal{A} \subset 2^X$ be an infinite σ-algebra, that is, $\operatorname{card}(\mathcal{A}) \geq \operatorname{card}(\mathbb{N})$. Verify the following properties.

- $\mathcal{A}$ contains an infinite sequence of disjoint sets.
- $\operatorname{card}(\mathcal{A}) \geq \mathfrak{c} = \operatorname{card}(\mathbb{R})$.

3. Let $(X, \mathcal{A})$ be a measurable space, so that $X \neq \emptyset$ and $\mathcal{A}$ is a σ-algebra. A mapping $f : X \to Y$ is given, $Y \neq \emptyset$. Verify the following properties.

- The collection of sets $\{E \subset Y : f^{-1}(E) \in \mathcal{A}\}$ is a σ-algebra.
  Note: we already encountered this statement when discussing product spaces.
- If $\mathcal{E} \subset 2^Y$ and $f^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{E}$, then also $f^{-1}(F) \in \mathcal{A}$ for all $F \in \sigma(\mathcal{E})$.

4. Verify that an open set in $\mathbb{R}^n$ is represented as a countable union of disjoint dyadic cubes. Conclude that $\mathcal{B}_{\mathbb{R}^n} = \sigma(\{\text{dyadic cubes in } \mathbb{R}^n\})$.

5. Verify that a dyadic cube in $\mathbb{R}$ of the form

$$\left[\frac{a_l}{2^k}, \frac{a_l + 1}{2^k}\right), \qquad a_l, k \in \mathbb{Z}$$

contains exactly all the numbers in $\mathbb{R}$ with binary expansions prescribed up to the k-th place. For example, if $a_l = k = 0$, we have

$$[0, 1) = \{x \in \mathbb{R} : x = \overline{0.b_1 b_2 b_3 \ldots}\},$$

with b_i denoting the digits in binary expansion. Note that we prohibit periodic $\overline{1}$, which causes the cube to be half-open.

6. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$\left(\overline{\ldots a_{-1} a_0 . a_1 a_2 a_3 \ldots}, \ \overline{\ldots b_{-1} b_0 . b_1 b_2 b_3 \ldots}\right) \mapsto \overline{\ldots a_{-1} b_{-1} a_0 b_0 . a_1 b_1 a_2 b_2 \ldots},$$

where the binary expansion is used and there are infinitely many zero digits on the left. We also prohibit periodic $\overline{1}$.

- Verify that f is injective but not surjective.
- Verify that preimage of a 1-dimensional dyadic cube is either one or two 2-dimensional cubes. (Use #5.)
- Conclude from #3 and #4 that preimage of a Borel set under f is a Borel set.