Section 11.2: Series

Previously: sequence = an infinite list of numbers

series = a sum of an infinite sequence of numbers.

Consider e, the base of the natural log

$$\underbrace{e = 2.718281828459045\ldots}_{\text{shorthand for:}}$$

$$e = \underbrace{2 + 7\cdot\frac{1}{10} + 1\cdot\frac{1}{100} + 8\cdot\frac{1}{1000} + 2\cdot\frac{1}{10\,000} + \ldots}_{\text{infinite sum = a series}}$$

Given a sequence $\{a_n\}$, $a_1, a_2, a_3, \ldots$
we will write

$$\sum_{n=1}^{\infty} a_n \qquad \sum_n a_n$$

When is it meaningful to consider infinite sums?

A partial sum:

$$S_n = \sum_{k=1}^{n} a_k$$

Def For a series $\sum_{k=1}^{\infty} a_k$, let $s_n = \sum_{k=1}^{n} a_k$.

If the sequence $\{s_n\}$ is convergent,

$$\lim_{n\to\infty} s_n = s \qquad \text{(a finite number)}$$

then the series is said to be convergent, and

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = s$$

s - sum of the series.

If the limit DNE or is $\pm\infty$, the series is divergent.

Sum of a series $\overset{\text{def}}{=}$ limit of of partial sums

$$\sum_{k=1}^{\infty} a_k \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Ex. 1 Suppose $s_n = \sum_{k=1}^{n} a_k = \frac{2n}{3n+5}$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

$$= \lim_{n\to\infty} \frac{2n}{3n+5} \frac{/n}{/n} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}.$$

$$s_n = \sum_{k=1}^{n} a_k = \frac{2n}{3n+5}$$

$$S_1 = a_1 = \frac{2\cdot 1}{3\cdot 1+5} = \frac{1}{4}$$

$$S_2 = a_1 + a_2 = \frac{2\cdot 2}{3\cdot 2+5}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{2\cdot 3}{3\cdot 3+5}$$

$$\dots$$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_n$$

$$S_{n+1} = \sum_{k=1}^{n+1} a_k = a_1 + a_2 + \dots + a_n + a_{n+1}$$

$$S_{n+1} - S_n = a_{n+1}$$

$$\boxed{S_n - S_{n-1} = a_n}$$

$$\frac{2n}{3n+5} - \frac{2(n-1)}{3(n-1)+5} = a_n$$

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{2k}{3k+5} - \frac{2k-2}{3k+2} \right)$$

Ex. 2 Compute the sum of

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k(k+1)}}_{a_k}$$

} telescoping series

Consider a_k:

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_1 = a_1 = \frac{1}{1} - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)$$

$$= 1 - \frac{1}{3}$$

$$S_3 = \underbrace{a_1 + a_2}_{S_2} + a_3 = S_2 + a_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}$$

$$= 1 - \frac{1}{4}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right)$$

$$= 1 - \frac{1}{5}.$$

$$S_n = a_1 + a_2 + \dots + a_n$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

Ex. 3 The geometric series:

$$\sum_{k=1}^{\infty} a r^{k-1} \qquad a \neq 0$$

r - common ratio

$$\sum_{k=1}^{\infty} a r^{k-1} = a + ar + ar^2 + ar^3 + \dots$$

(Previously: $\{r^n\}$)

Suppose $r = 1$, then

$$\sum_{k=1}^{\infty} a = a + a + a + \dots$$

$$S_n = n \cdot a$$

$$\lim_{n\to\infty} S_n = +\infty \cdot \frac{a}{|a|}$$

Now let $r \neq 1$.

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$r \cdot S_n = \quad ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$(1-r)\,S_n = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} = \sum_{k=1}^{n} a r^{k-1}.$$

$$\sum_{k=1}^{\infty} a r^{k-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} \lim_{n\to\infty} (1-r^n) =$$

Recall that $\lim\limits_{n\to\infty} r^n = \begin{cases} 1, & r=1 \\ 0, & |r|<1 \\ \text{divergent} & \text{otherwise} \end{cases}$

Thus,

$$= \frac{a}{1-r}\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} r^n\right) = \frac{a}{1-r}.$$

To summarize:

Geometric series

$$\boxed{\begin{array}{l} \displaystyle\sum_{k=1}^{\infty} a\cdot r^{k-1} = \frac{a}{1-r}, \quad \text{if } |r|<1 \quad \text{(convergent)} \\[2ex] \text{divergent otherwise (for all other } r). \end{array}}$$