Measure and Integration I (MAA5616), Fall 2020
Homework 6, due Thursday, Oct. 15

1. Using #3–4 from HW5, construct a Lebesgue-measurable function g and a continuous h such that $g \circ h$ is not measurable.

In HW5 we proved that $x + f(x)$ is continuous.

- Check that when j is Borel-measurable, the composition $j \circ h$ is Borel-measurable for any Borel-measurable h (in particular, continuous h works). Conclude that the g you constructed is not Borel-measurable.
- Prove that any monotone function $h : \mathbb{R} \to \mathbb{R}$ is Borel-measurable.
- Using the g you constructed, give an example of a Lebesgue- but not Borel-measurable set.

2. For a sequence of measures $\{\mu\}_n$ defined on $(X, \mathcal{M})$, such that $\mu_n(E) \leq \mu_{n+1}(E)$ for every $E \in \mathcal{M}$, prove that μ given by

$$\mu(E) = \sup_n \mu_n(E), \qquad E \in \mathcal{M},$$

is also a measure on $\mathcal{M}$.

- Give a counterexample showing that the monotonicity assumption is necessary.
- Is there an analogous result for decreasing sequences of measures?

  Recall the assumptions necessary for continuity of a measure from above.

In the following problems, $(X, \mathcal{M}, \mu)$ is a measure space.

3. From #3, conclude that for $f \in L_+$, the map $A \mapsto \int_A f$ is a measure on $\mathcal{M}$.

4. Given a function $f \in L_+$ such that $\int f < \infty$, prove that $\{x \in X : f(x) = \infty\}$ is a null set. Also, that $\{x \in X : f(x) > 0\}$ is σ-finite (a countable union of sets with finite measure μ).

Compare this problem to #1 in HW2.

5. (Borel-Cantelli lemma) Suppose $\mu(X) = 1$, $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$, and consider the set of $x \in X$ that belong to infinitely many A_n:

$$B = \limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.$$

Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then $\mu(B) = 0$.

For any $k \geq 1$, $B \subset B_k = \bigcup_{n=k}^{\infty} A_n$, and $\mu(B_k) \leq \sum_{n=k}^{\infty} \mu(A_n)$ by subadditivity of μ.