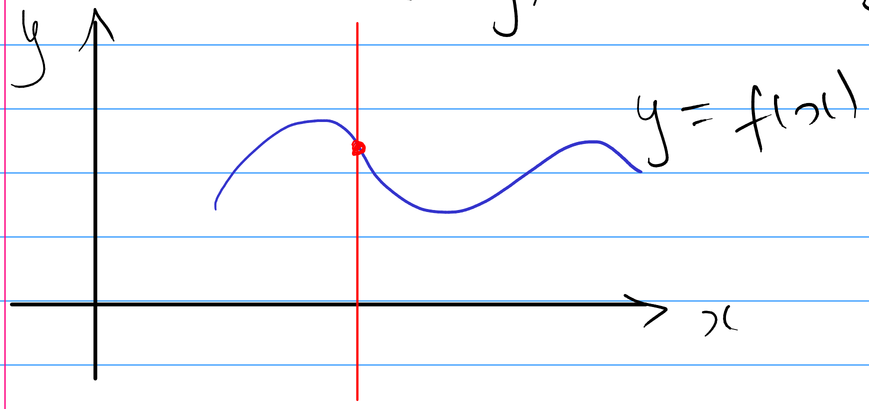
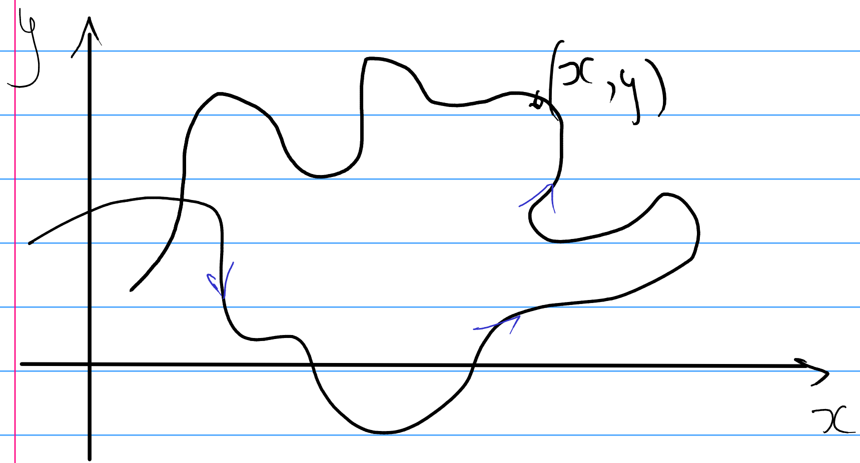
Parametric curves

Graph: (x, y) : $y = f(x)$



vertical line test

On the other hand:



does not pass the vertical line test

We would like to handle curves that describe positions of an object in the plane depending on time.

(x, y) — coordinates of the particle

think of x, y as functions of time: $(x(t), y(t))$

We will say that a pair of functions $(x(t), y(t))$ defines a <u>parametric curve</u>.

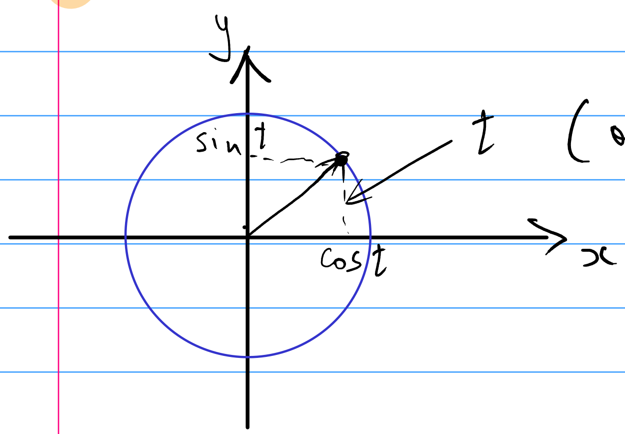
Note: any usual curve is also parametric

Take $y = f(x)$ – usual curve

Then we can think of it as a parametric curve:

$$\begin{cases} x = t \\ y = f(t) \end{cases} \quad - \text{ parametric curve}$$

Ex. 1. Consider the unit circle:



t (angle, or time)

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad -\infty < t < \infty$$

(definition of the unit circle)

Verify that the point $(\underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)})$ is on the unit circle:

$$x^2(t) + y^2(t) = (\cos t)^2 + (\sin t)^2 = 1$$

$$\left.x^2 + y^2 = 1\right\} \text{ unit circle}$$

So: $(x(t), y(t)) = (\cos t, \sin t)$

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \quad -\infty < t < +\infty$$

$\hookrightarrow$ Parametric equation of the unit circle

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad -\infty < t < \infty$$

— parametric equation of the unit circle

Ex. 2 Identify the curve

$$\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases} \quad -\infty < t < \infty$$

Parameter elimination technique.

1) Express t through y (easier than t through x)

$$t = y - 1$$

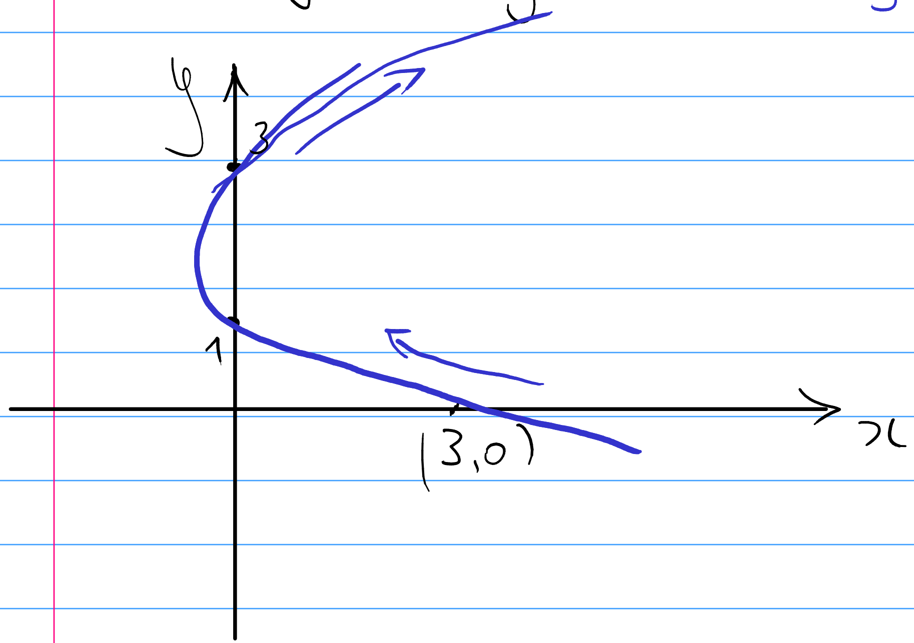
2) Substitute this expression into

the first equation:

$$x = t^2 - 2t = (y-1)^2 - 2(y-1)$$

$$= y^2 - 2y + 1 - 2y + 2$$

$$x = y^2 - 4y + 3 \quad \Big\} \ \underline{\text{parabola}}$$



Ex. 3 Identify the curve

$$\begin{cases} x = t^2 \\ y = \ln t \end{cases} \qquad t > 0$$

Eliminating t:

1) From $x = t^2 \implies t = \sqrt{x}$.

Substitute into $y = \ln t = \ln\sqrt{x} = \ln x^{1/2}$

$= \frac{1}{2} \ln x$

$$y = \frac{1}{2} \ln x \qquad x > 0$$

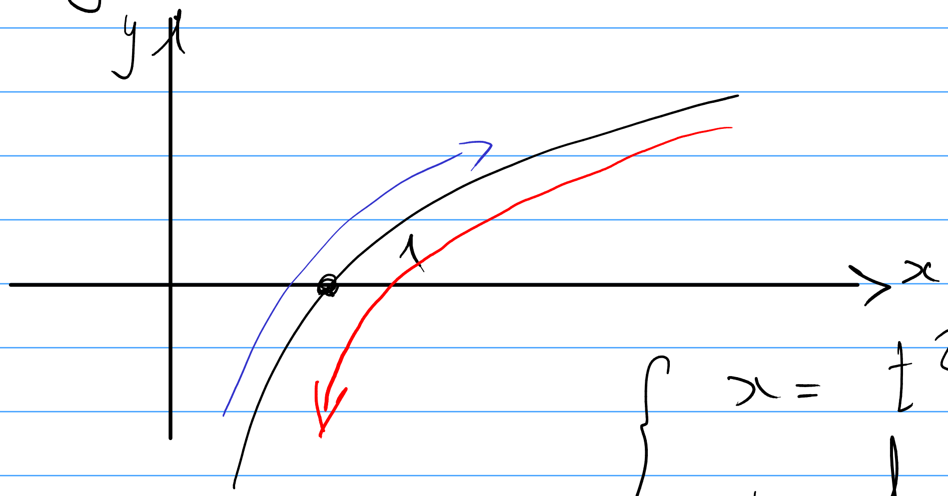
2). From $y = \ln t \Rightarrow e^y = e^{\ln t} = t$

$t = e^y$

Substitute into $x = t^2 = (e^y)^2$

$x = e^{2y}$

$y = \frac{1}{2} \ln x$



$$\begin{cases} x = t^2 \\ y = \ln t \end{cases} \quad t > 0$$

For comparison: $t \mapsto (-t)$

$$\begin{cases} x = (-t)^2 = t^2 \\ y = \ln(-t) \end{cases}$$

$t < 0$

t in $(-\infty, 0)$

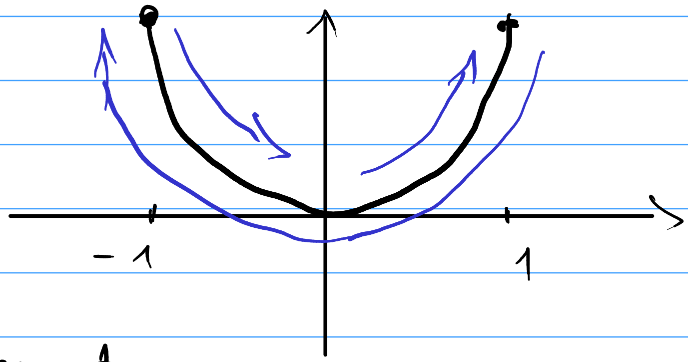
Ex. 4 Identify the parametric curve:

$$\begin{cases} x = \sin t \\ y = \sin^2 t \end{cases} \quad -\infty < t < \infty$$

Parameter elimination:

$y = \sin^2 t = (\sin t)^2 = x^2$

$y = x^2, \qquad x \text{ in } [-1, 1]$



$$\begin{cases} x = \sin t \\ y = (\sin t)^2 \end{cases}$$

$t = \arcsin x$

$y = (\sin(\arcsin x))^2$

Ex. 5 Identify the parametric curve:

$$\begin{cases} x = \tan^2 \theta \\ y = \sec \theta \end{cases} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

It is easier to make x the function of y, not vice versa, because of the square.

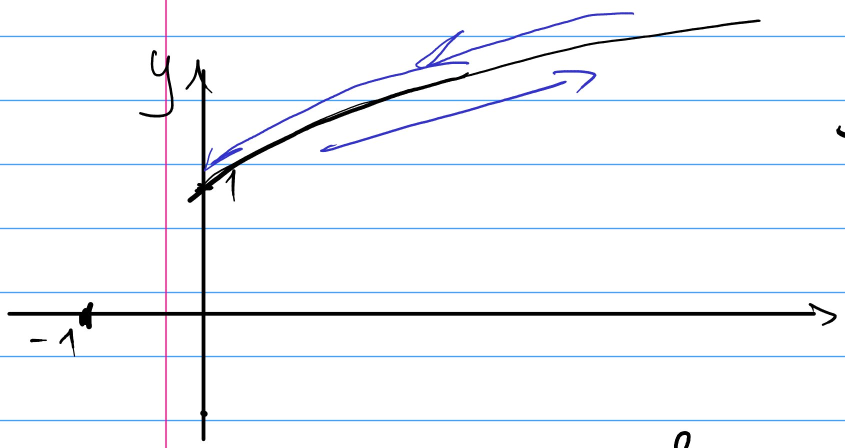
Recall: $\sec^2\theta - \tan^2\theta = 1$

$y^2 - x = 1$

$x = y^2 - 1 \qquad y \geq 1$

When $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$,

$$y = \sec\theta = \frac{1}{\cos\theta} \geq 1$$



} piece of parabola $x = y^2 - 1$ (traversed twice)

$$\begin{cases} x = \tan^2\theta \\ y = \sec\theta \end{cases} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Tangents to parametric curves

For a parametric curve

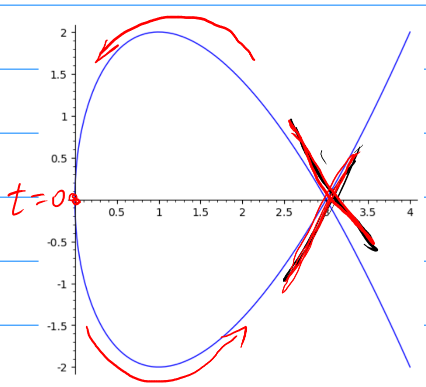
$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad a \leq t \leq b$$

To find equation of the tangent at a point (x_0, y_0) we need the slope of the curve at this point. Then the tangent is given by

$$y - y_0 = m(x - x_0) \qquad \begin{matrix} x_0 = x(t_0) \\ y_0 = y(t_0) \end{matrix}$$

$$m = \frac{dy}{dx}(t_0) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}(t_0)$$

Ex. 1 For the parametric curve
$x = t^2$
$y = t^3 - 3t$
Find the two tangents at $(3, 0)$.



$$\begin{cases} x = 3 = t^2 \\ y = 0 = t^3 - 3t \end{cases}$$

$$\Rightarrow t = \pm\sqrt{3}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t} = \frac{3}{2}\,\frac{t^2-1}{t}$$

$$m_{1,2} = \frac{3}{2}\cdot\frac{(\pm\sqrt{3})^2-1}{\pm\sqrt{3}} = \frac{3}{2}\cdot\frac{3-1}{\pm\sqrt{3}}$$

$$= \frac{3}{\pm\sqrt{3}} = \pm\sqrt{3}$$

Tangents:

$$y - 0 = \pm\sqrt{3}(x-3)$$

$$y = \pm\sqrt{3}(x-3)$$

Arc length of parametric curves

Recall: for $y = f(x)$, $a \le x \le b$,

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Consider now the curve

$x = x(t)$

$y = y(t)$ $\quad \alpha \le t \le \beta$

Then, the arc length of this curve is:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If we recall: any regular curve is parametric with

$$\begin{cases} x = t \\ y = y(t) = y(x) \end{cases} \quad \text{regular curve}$$

$$L = \int_{\alpha}^{\beta} \sqrt{\underbrace{\left(\frac{dx}{dt}\right)^2}_{1} + \underbrace{\left(\frac{dy}{dt}\right)^2}_{\frac{dy}{dx}}}\, dx$$

Ex. 2 Compute the arc length of the unit circle:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \qquad 0 \leq t \leq 2\pi$$

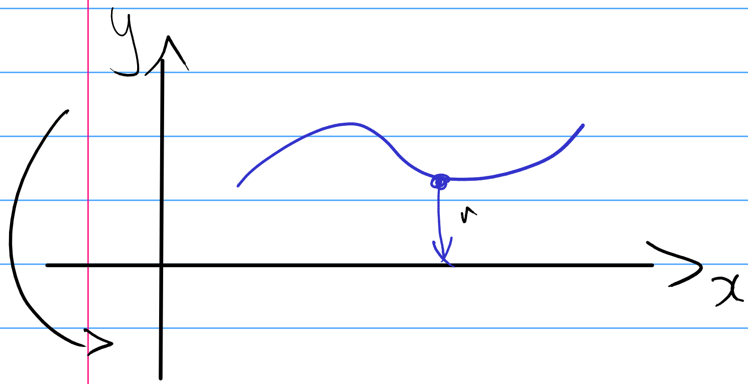
$$L = \int_{0}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{0}^{2\pi} \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{1}}\, dt$$

$$= \int_0^{2\pi} \sqrt{1}\, dt = 2\pi.$$

$$= \text{circumference} = 2\pi \overbrace{r}^{1} = 2\pi.$$

Area of the surface of revolution for parametric curves



$$S = \int 2\pi \cdot r \cdot ds$$

We have already learned that

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

For rotation about the x-axis:

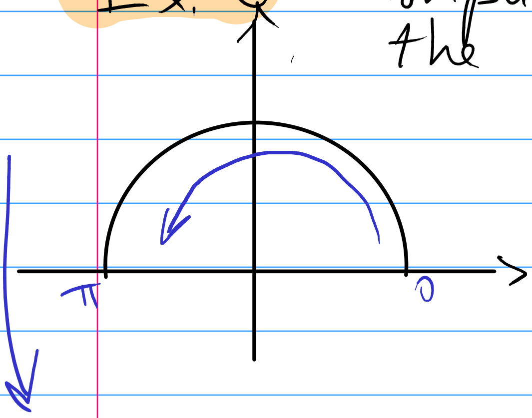
$$S = \int_\alpha^\beta 2\pi \cdot y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

—— " —— about the y-axis

$$S = \int_\alpha^\beta 2\pi \cdot x(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

area of the surface of revolution for the curve $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $\alpha \le t \le \beta$

Ex. 3 Compute the surface area for the sphere of radius r.



$$\underbrace{\begin{cases} x = r\cdot\cos t \\ y = r\cdot\sin t \end{cases}}_{\text{Circle of radius } r.} \quad 0 \le t \le \pi$$

$$S = \int_0^\pi 2\pi \cdot y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^\pi r\cdot\sin t \cdot \sqrt{\underbrace{(-r\sin t)^2 + (r\cos t)^2}_{r^2\sin^2 t + r^2\cos^2 t}}\, dt$$

$$= 2\pi \int_0^\pi r\cdot\sin t \cdot r\, dt$$

$$= 2\pi r^2 \int_0^\pi \sin t\, dt$$

$$= 2\pi\cdot r^2 \cdot (-\cos t)\Big|_0^\pi$$

$$= 2\pi\cdot r^2 \cdot \left(-(-1) - (-1)\right)$$

$$= 4\pi r^2$$