Section 11.4 Comparison tests

Ex. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

For $k \geq 3 > e$: $\ln k > 1$, and

$$\frac{\ln k}{k} > \frac{1}{k} \qquad k \geq 3$$

Sum both sides for $k \geq 3$:

$$\underbrace{\sum_{k=3}^{\infty} \frac{\ln k}{k}}_{\text{divergent}} \geq \sum_{k=3}^{\infty} \frac{1}{k} = +\infty$$

The comparison test:

Suppose $\sum_k a_k$ and $\sum_k b_k$ are such that

$$0 \leq a_k \leq b_k$$

Then:

a). If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.

b). If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.

Comparison can be to the known series:

- p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$
- geometric series $\sum_{k=1}^{\infty} a r^{k-1}$

Ex. 1 Test for convergence: $\sum_{k=1}^{\infty} \underbrace{\frac{1}{2^k+1}}_{a_k}$

There holds: $a_k = \frac{1}{2^k+1} < \frac{1}{2^k} = b_k$,

and $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$ } convergent.

$\Rightarrow$ by the comparison test, $\sum_{k=1}^{\infty} \frac{1}{2^k+1}$ } convergent

Ex. 2 Test for convergence: $\sum_{k=1}^{\infty} \underbrace{\frac{5}{2k^2+4k+3}}_{a_k}$

$$a_k = \frac{5}{2k^2+4k+3} \leq \frac{5}{2k^2} = b_k$$

$\sum_{k=1}^{\infty} \frac{5}{2k^2} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$ } convergent.

$\Rightarrow \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{2k^2+4k+3}$ is convergent.

Ex. 3 Test for convergence:

$\sum_{k=2}^{\infty} \underbrace{\frac{\ln k}{(k-1)^{1/2}}}_{a_k}$ } divergent.

For $k \geq 3$,

$$a_k = \frac{\ln k}{(k-1)^{1/2}} \geq \frac{1}{(k-1)^{1/2}} \geq \frac{1}{k^{1/2}} = b_k$$

$$\sum_{k=3}^{\infty} \frac{1}{k^{1/2}} \Big\} \text{ divergent, p-series } p = 1/2$$

Note: $$\sum_{k=1}^{\infty} \frac{1}{2^k - 1}$$

Then: $$\frac{1}{2^k - 1} > \frac{1}{2^k} \Big\} \text{ convergent.}$$

So, no conclusion from the comparison test.

The limit comparison test:

Suppose, for the series $\sum_k a_k$ and $\sum_k b_k$:

$a_k > 0$, $b_k > 0$, $k \geq 1$.

If $$\lim_{k \to \infty} \frac{a_k}{b_k} = L > 0 \quad (L - \text{finite}),$$

then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \sum_{k=1}^{\infty} b_k \text{ converges.}$$

Now, for $\sum_{k=1}^{\infty} \underbrace{\frac{1}{2^k - 1}}_{a_k}$: let $b_k = \frac{1}{2^k}$.

Then:

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{1}{2^k - 1} \Big/ \frac{1}{2^k} = \lim_{k \to \infty} \frac{2^k}{2^k - 1}$$

$$= \lim_{k\to\infty} \frac{1}{1-\underbrace{1/2^k}_{\to 0}} = \frac{1}{1-0} = 1 > 0$$

$\Rightarrow$ limit comparison test applies

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^k}$ is convergent $\Rightarrow$

$\sum_{k=1}^{\infty} \frac{1}{2^k-1}$ converges as well.

Ex. 4 Test for convergence: $\underbrace{\sum_{k=1}^{\infty} \frac{2k^2+3k}{\sqrt{5+k^5}}}^{\text{divergent.}}$

$a_k = \frac{2k^2+3k}{\sqrt{5+k^5}}$; take $b_k = \frac{2k^2}{k^{5/2}}$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k^2+3k}{\sqrt{5+k^5}} \Big/ \frac{2k^2}{k^{5/2}} =$$

$$\lim_{k\to\infty} \frac{2k^2+3k}{\sqrt{5+k^5}} \cdot \frac{k^{5/2}}{2k^2} \quad \Big/ \frac{2k^2 \cdot k^{5/2}}{2k^2 \cdot k^{5/2}}$$

$$\lim_{k\to\infty} \frac{1+\overbrace{\frac{3}{2k}}^{\to 0}}{\sqrt{\underbrace{\frac{5}{k^5}}_{\to 0}+1}} = 1 > 0$$

$\Rightarrow$ Limit comparison test applies.

$\sum_k b_k = \sum_{k=1}^{\infty} \frac{2}{k^{1/2}}$ } p-series $p = 1/2$ divergent