Section 11.1: Sequences

$a_1, \quad a_2, \quad a_3, \quad a_4 \ldots$ } list of numbers

$\{a_n\}, \qquad \{a_n\}_{n=1}^{\infty}$

Ex. 1

(a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}, \qquad a_n = \frac{n}{n+1}, \qquad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

(b) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}, \qquad a_n = \frac{(-1)^n(n+1)}{3^n}, \qquad \left\{-\frac{2}{3}, \frac{3}{3^2}, -\frac{4}{3^3}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}, \qquad a_n = \sqrt{n-3}, \qquad \left\{0, \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\right\}$

Sequence = a function from $\{1, 2, 3, \ldots\}$ to real numbers.

Ex. 2

(a) $\{p_n\}_{n=1800}^{2000}$ $\quad p_n =$ Earth population in year n, on Dec 31 at midnight GMT

(b) $\{a_n\}_{n=0}^{\infty}, \qquad a_n = n$-th digit of e.

Def $\{a_n\}$ has the limit L if a_n become arbitrarily close to L when n is sufficiently large.

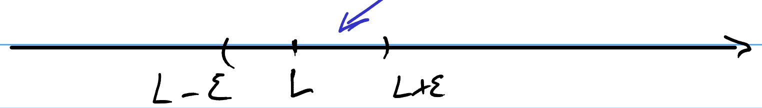
Notation: $\lim_{n\to\infty} a_n = L, \qquad a_n \to L, \quad n \to \infty.$

$\{a_n\}$ is convergent if the lim exists, is finite, divergent otherwise.

Def $\{a_n\}$ has the limit L, if for any $\varepsilon > 0$, there exists an N (integer) s.t.

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n \geq N.$$

a_n, starting from $n = N$



Thm If $\lim\limits_{x \to \infty} f(x) = L$, then for the sequence $\{a_n\}$, $a_n = f(n)$ (n – integer), there holds $\lim\limits_{n \to \infty} a_n = L$.

Ex. 3 $\lim\limits_{x \to \infty} \dfrac{1}{x^p} = 0$, $p > 0$, so $\lim\limits_{n \to \infty} \dfrac{1}{n^p} = 0$.

Def $\lim\limits_{n \to \infty} a_n = \infty$, if for any positive M. there holds $a_n > M$ whenever n is sufficiently.

Limit laws:

Suppose $\{a_n\}$, $\{b_n\}$ – convergent, c – constant, then:

1) $\lim\limits_{n \to \infty} (a_n + b_n) = \lim\limits_{n \to \infty} a_n + \lim\limits_{n \to \infty} b_n$

2) $\lim\limits_{n \to \infty} c \cdot a_n = c \cdot \lim\limits_{n \to \infty} a_n$

$\lim\limits_{n \to \infty} c = c$

3). $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim_{n\to\infty} b_n \neq 0$.

Squeeze thm:
If $a_n \leq b_n \leq c_n$ for $n \geq n_0$, and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

Corollary: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Pf: $\underset{\downarrow \atop 0}{-|a_n|} \leq \underset{\downarrow \text{Sq. thm.} \atop 0}{a_n} \leq \underset{\downarrow \atop 0}{|a_n|}$

Ex. 4 Find $\lim_{n\to\infty} \underbrace{\frac{n^2+3}{3n^2+5}}_{a_n} \cdot \frac{/n^2}{/n^2} = \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{3+\frac{5}{n^2}}$ ($\frac{3}{n^2} \to 0$, $\frac{5}{n^2} \to 0$)

$= \frac{1+0}{3+0} = \frac{1}{3}$.

Ex. 5 Let $a_n = \frac{n}{\sqrt{10+n}}$; is $\{a_n\}$ convergent?

$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} \frac{/n}{/n} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{n}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} =$ ($\frac{10}{n^2} \to 0+$, $\frac{1}{n} \to 0+$)

$= +\infty$.

Divergent.

Ex. 6 Calculate $\lim\limits_{n\to\infty} \underbrace{\frac{\ln n}{n}}_{} \; = a_n$

Consider $f(x) = \frac{\ln x}{x}$, $x > 0$.

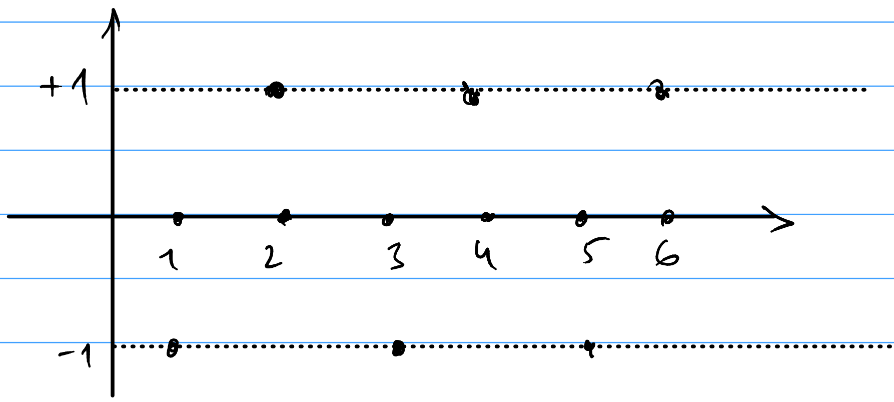
Then $a_n = f(n)$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{1/x}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

It follows $\lim\limits_{n\to\infty} f(n) = 0$.

Ex. 7 Is $a_n = (-1)^n$ convergent?



Divergent.

Ex. 8 $a_n = \frac{(-1)^n}{n}$, $n \geq 1$

$$\underset{\substack{\downarrow\, n\to\infty \\ 0}}{-\frac{1}{n}} \leq \underset{\substack{\downarrow\, n\to\infty \\ 0}}{a_n} \leq \underset{\substack{\downarrow\, n\to\infty \\ 0}}{\frac{1}{n}} \implies \lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Thm. If $\{a_n\}$ is such that $\lim_{n\to\infty} a_n = L$, $f(x)$ is continuous at $x=L$, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Ex. 9 $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right)$

$\sin x$ is continuous at 0

$\lim_{n\to\infty} \frac{\pi}{n} = 0$

By the Thm,

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin(0) = 0.$$

Ex. 10 $a_n = n!/n^n$ (compute the $\lim_{n\to\infty}$)

$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-2)(n-1)n$

So,

$1! = 1$

$2! = 1.2 = 2$

$3! = 1\cdot 2\cdot 3 = 6$

$4! = 1\cdot 2\cdot 3\cdot 4 = 24$

$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n}{n\cdot n\cdot n\cdot \ldots \cdot n\cdot n}$$

$a_n \geq 0$

$$= \lim_{n\to\infty} \frac{1}{n}\cdot \underbrace{\frac{2}{n}}_{\leq 1}\cdot \underbrace{\frac{3}{n}}_{\leq 1}\cdot \ldots \cdot \underbrace{\frac{n-1}{n}}_{\leq 1}\cdot \underbrace{\frac{n}{n}}_{\leq 1} = 0.$$

$$0 \le \underbrace{\overbrace{\frac{1}{n}\cdot \underbrace{\frac{2}{n}}_{\le 1}\cdot \underbrace{\frac{3}{n}}_{\le 1}\cdot \ldots \cdot \underbrace{\frac{n-1}{n}}_{\le 1}\cdot \underbrace{\frac{n}{n}}_{\le 1}}^{a_n}}_{\xrightarrow{n\to\infty}\ 0} \le \frac{1}{n}$$

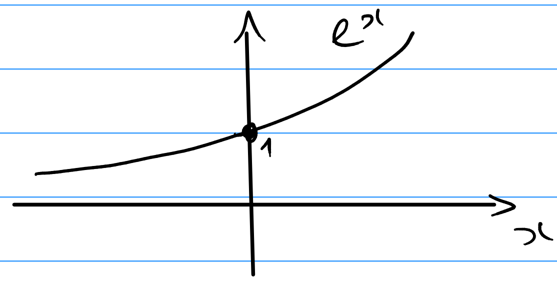
$$0 \to 0 \qquad \frac{1}{n} \xrightarrow{n\to\infty} 0$$

Ex. 11 For which r is $a_n = r^n$ convergent?

Let $r > 0$. Then $a_n = r^n = \left(e^{\ln r}\right)^n =$

$= e^{n \cdot \ln r}$

$$\lim_{n\to\infty} e^{n\cdot \ln r}$$



1). $\ln r > 0$

$r > 1$

$n \cdot \ln r \to +\infty, \quad n \to \infty$

$$\lim_{n\to\infty} a_n = +\infty$$

2). $\ln r < 0$

$0 < r < 1$

$n \cdot \ln r \longrightarrow -\infty, \quad n \to \infty.$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n = 0$$

3). $\ln r = 0$

$r = 1$

$$\lim_{n\to\infty} 1^n = 1$$

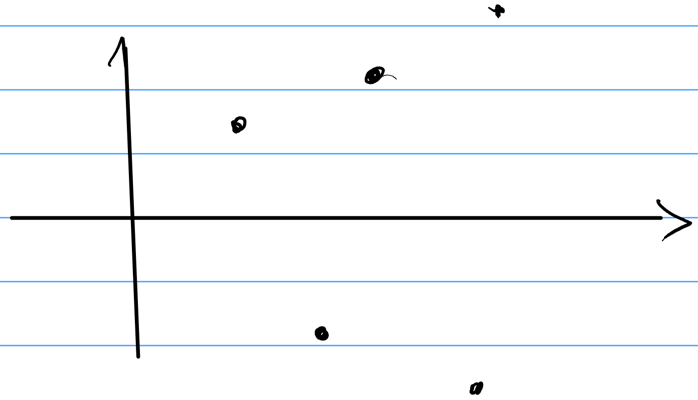
$r < 0 \Rightarrow r = (-1) \cdot \underbrace{|r|}_{>0}$

$a_n = r^n = ((-1)|r|)^n = (-1)^n \cdot |r|^n$

4). $\ln |r| > 0$

$|r| > 1$

$a_n = (-1)^n \cdot \underbrace{|r|^n}_{\to +\infty}$



DNE

5). $\ln |r| < 0$

$$\underset{\downarrow \atop 0}{-|r|^n} \le \underbrace{a_n = (-1)^n \cdot |r|^n}_{\xrightarrow{\text{Sq. thm}} 0} \le \underset{\downarrow \atop 0}{|r|^n}$$

6). $\ln |r| = 0 \Rightarrow r = -1$

$a_n = (-1)^n$

$\lim\limits_{n \to \infty} a_n$ DNE

$a_n = r^n$ is $\begin{cases} \text{convergent} & \text{if } -1 < r \le 1 \\ \text{divergen} & \text{otherwise} \end{cases}$

$$\lim_{n \to \infty} r^n = \begin{cases} 0, & |r| < 1 \\ 1, & r = 1 \end{cases}$$

Def $\{a_n\}$ is increasing [decreasing],

if $a_{n+1} \geq a_n$ $\quad [a_{n+1} \leq a_n]$

$\{a_n\}$ is monotonic if increasing or decreasing.

Ex. 12 Verify: $\left\{\frac{3}{n+5}\right\}$ is decreasing

Need: $a_{n+1} \leq a_n$

$$\frac{3}{(n+1)+5} \leq \frac{3}{n+5}$$

$$\frac{3}{n+6} \leq \frac{3}{n+5} \qquad 1/$$

$$\frac{n+6}{3} \geq \frac{n+5}{3} \qquad \times 3$$

$$n+6 \geq n+5$$

Ex. 13 Show: $a_n = \frac{n}{n^2+1}$ is decreasing

Consider $f(x) = \frac{x}{x^2+1}$; then $a_n = f(n)$.

$$f'(x) = \frac{1\cdot(x^2+1) - x\cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} =$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$f'(x) < 0$, $x > 1$, decreasing,

so $\{f(n)\} = \{a_n\}$ is also decreasing

Def $\{a_n\}$ is bounded above [below],

if for some M [for some m]

there holds $M \geq a_n$ [$m \leq a_n$].

$n \geq 1$.

If $\{a_n\}$ is bounded above, below $\Rightarrow$

$\Rightarrow$ bounded.

Thm. (Monotonic sequence thm)

Every bounded, monotonic sequence is convergent.

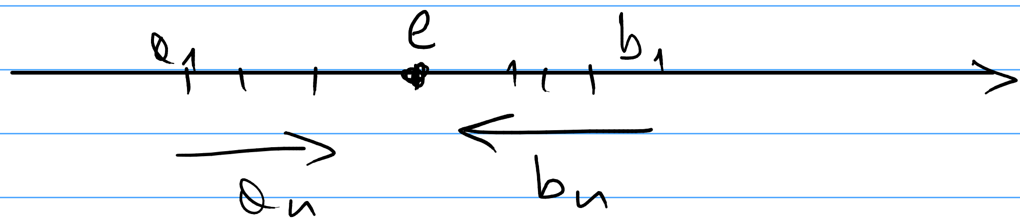
For example: $\left(1+\frac{1}{n}\right)^n \to e$

Showing that $\left(1+\frac{1}{n}\right)^n$ is convergent

can be done like so:

$a_n = \left(1+\frac{1}{n}\right)^n$ is increasing

$b_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is decreasing, and both are bounded:



$$a_n \leq b_n .$$