REVIEW QUESTIONS FOR PARAMETRIC CURVES, POLAR COORDINATES, AND CONIC SECTIONS

Parametric curves

1) What is a parametric curve? How and when can one obtain the Cartesian equation of a parametric curve? How to compute the slope of the tangent to a parametric curve? Write down the expressions for the area under a parametric curve, and the arc length of a parametric curve. What is the equation for the area of the surface obtained by rotation of a parametric curve?
2) Eliminate the parameter from the given parametric curve and sketch its graph. Show the direction of motion as the parameter increases.
   (a) $x = \sin t,\ y = \csc t, \quad 0 < t < \pi/2$
   (b) $x = e^t,\ y = e^{-2t}, \quad -\infty < t < \infty.$
3) Find the equation of the tangents to the following curves for the given value of the parameter t or at the given point (x, y):
   (a) $x = t^3 + 1,\ y = t^4 + t, \quad t = -1$
   (b) $x = t\cos t,\ y = t\sin t, \quad t = \pi$
   (c) $x = 1 + \ln t,\ y = t^2 + 2, \quad (1, 3)$
   (d) $x = t^2 - t,\ y = t^2 + t + 1, \quad (0, 3).$
4) Find the length of the given curve:
   (a) $x = 1 + 3t^2,\ y = 4 + 2t^3, \quad 0 \le t \le 1$
   (b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 2$
   (c) $x = t\sin t,\ y = t\cos t, \quad 0 \le t \le 1.$
5) Find the area of surfaces obtained by rotating the following curves about the x-axis:
   (a) $x = t^3,\ y = t^2, \quad 0 \le t \le 1$
   (b) $x = 2\cos^3 t,\ y = 2\sin^3 t, \quad 0 \le t \le \pi/2.$
6) Find the area of surfaces obtained by rotating the following curves about the y-axis:
   (a) $x = 8\sqrt{t},\ y = 2t^2 + 1/t, \quad 1 \le t \le 3$
   (b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 1.$

Polar coordinates

7) Explain how a point in the plane is determined by its polar coordinates. Write down the conversion formulas between the Cartesian and polar coordinates. How does the quadrant containing a certain point determines its polar angle?
8) Plot the point with the given polar coordinates and find its Cartesian coordinates:
   (a) $(1, \pi/3)$
   (b) $(-8, 7\pi/4)$
   (c) $(3, -5\pi/2)$
   (d) $(0, -71\pi/4)$
   (e) $(2\sqrt{2}, 3\pi/4).$
9) For the given Cartesian coordinates of a point, find its polar coordinates (r, θ) with $r \ge 0$ and $0 \le \theta < 2\pi$. Then, find the expression of polar coordinates with $r \le 0$ and $0 \le \theta < 2\pi$:
   (a) $(0, 5)$

(b) $(5\sqrt{3}, -5)$
(c) $(2\sqrt{2}, -2\sqrt{2})$
(d) $(1, \sqrt{3})$.

Polar curves

10) What is a polar curve? How to sketch the graph of a polar curve, given its equation? How to determine the slope of a polar curve at a given point?
11) Identify the curve:
(a) $r^3 = 125$
(b) $\theta = \pi/4$
(c) $r = 4\sec\theta$
(d) $r = -2\sec\theta$
(e) $r = 3\csc\theta$
(f) $r^2\cos 2\theta = 1$.
12) Find a polar equation for the curve given in Cartesian coordinates:
(a) $y = 2$
(b) $y = x$
(c) $x^2 + y^2 = 2x$
(d) $4y^2 = x$
(e) $x^2 - y^2 = 4$.
13) Find the slope of the tangent line to the given polar curve at the point corresponding to the specified value of θ:
(a) $r = 2\cos\theta, \quad \theta = \pi/3$
(b) $r = 1/\theta, \quad \theta = \pi$
(c) $r = 2 + \sin 3\theta, \quad \theta = \pi/4$
(d) $r = \cos(\theta/3), \quad \theta = \pi$.
14) Find points on the given curve where the tangent line is horizontal or vertical:
(a) $r = 3\cos\theta$
(b) $r = 1 + \cos\theta$
(c) $r = e^\theta$
(d) $r = 1 - \sin\theta$.
15) Sketch the polar curve:
(a) $r = \theta, \quad \theta \geq 0$
(b) $r = 1 + \sin\theta$
(c) $r = 2 + \sin 3\theta$
(d) $r^2 = \cos 4\theta$
(e) $r = 2\cos(\theta/2)$.

Areas and lengths for polar curves

16) Write down the formula for the area enclosed by a polar curve $r = f(\theta)$, as well as for the area between a pair of polar curves $r = f(\theta)$ and $r = g(\theta)$. Explain how to determine the range of integration in the corresponding integrals. How to compute the arc length of a polar curve?
17) Find the area of the region that lies inside the first curve and outside the second curve:
(a) $r = 4\sin\theta, \quad r = 2$
(b) $r = 1 - \sin\theta, \quad r = 1$

(c) $r = 3\cos\theta, \quad r = 1 + \cos\theta$
(d) $r^2 = 8\cos 2\theta, \quad r = 2.$

18) Find all the points of intersection of the given curves:
(a) $r = \sin\theta, \quad r = 1 - \sin\theta$
(b) $r = 1 + \cos\theta, \quad r = 1 - \sin\theta$
(c) $r = \sin\theta, \quad r = \sin 2\theta$
(d) $r = 2\sin 2\theta, \quad r = 1.$

19) Find the length of the polar curve:
(a) $r = 2\cos\theta, \quad 0 \le \theta \le \pi$
(b) $r = \theta^2, \quad 0 \le \theta \le 2\pi$
(c) $r = 2(1 + \cos\theta).$

Conic sections

20) Give the geometric definitions of parabola, ellipse, hyperbola. Explain why these curves are called "conic sections". Write down their equations in Cartesian coordinates and explain how to determine the geometric characteristics (foci, vertices, asymptotes) from an equation. What is the equation of a conic shifted by the vector (v_1, v_2)?

21) Identify the conic by its equation in Cartesian coordinates, find the foci and vertices. For a hyperbola, determine also the asymptotes:
(a) $4x^2 = y^2 + 4$
(b) $4y^2 = x + 4$
(c) $3x^2 - 6x - 2y = 1$
(d) $x^2 - 2x + 2y^2 - 8y + 7 = 0$
(e) $-9y^2 - 54y + 25x^2 - 200x + 94 = 0$
(f) $y^2 + 2\,y + 4\,x^2 + 16\,x + 1 = 0$
(g) $-9\,y^2 + 18\,y + 4\,x^2 - 24\,x - 9 = 0$
(h) $-2\,y^2 - 16\,y + 3\,x^2 + 6\,x - 35 = 0$
(i) $16\,y - x^2 + 6\,x - 9 = 0.$

22) Find an equation in Cartesian coordinates for the conic satisfying the given conditions:
(a) parabola, vertex $(0, 0)$, focus $(1, 0)$
(b) parabola, focus $(0, 0)$, directrix $x = -3$
(c) parabola, focus $(2, 3)$, vertex $(2, -3)$
(d) ellipse, foci $(\pm 2, 3)$, vertices $(\pm 4, 3)$
(e) ellipse, foci $(-7, 6)$ and $(-7, 2)$, vertex $(-7, 1)$
(f) ellipse, center $(4, -1)$, vertex $(0, -1)$, focus $(6, -1)$
(g) hyperbola, vertices $(0, \pm 3)$, foci $(0, \pm 4)$
(h) hyperbola, vertices $(\pm 2, 0)$, asymptotes $y = \pm 3x$
(i) hyperbola, vertices $(-1, 2)$ and $(7, 2)$, foci $(-2, 2)$ and $(8, 2)$.

Answer key

3) (a) $y = 1/x, \quad 0 < x < 1$
(b) $y = 1/x^2, \quad x > 0.$

4) (a) $y = -x$
(b) $y = \pi(x + \pi)$
(c) $y - 3 = 2(x - 1)$
(d) $y - 3 = 3x.$

5) (a) $4\sqrt{2} - 2$
(b) $e^2 + 1$
(c) $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1 + \sqrt{2}).$

6) (a) $\frac{2}{1215}\pi\left(247\sqrt{13} + 64\right)$
(b) $\frac{24}{5}\pi.$

7) (a) $\frac{32}{15}\pi\left(103\sqrt{3} + 3\right)$
(b) $\pi\left((e+1)^2 - 7\right).$

8) (a) $(1/2, \frac{\sqrt{3}}{2})$
(b) $(-4\sqrt{2}, 4\sqrt{2})$
(c) $(0, -3)$
(d) $(0, 0)$
(e) $(-2, 2).$

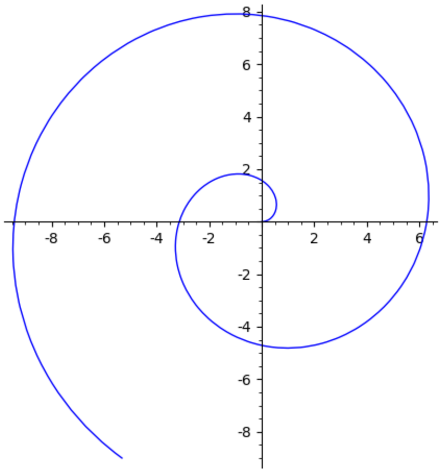
9) (a) $(5, \pi/2); \quad (-5, 3\pi/2)$
(b) $(10, 11\pi/6); \quad (-10, 5\pi/6)$
(c) $(4, 7\pi/4); \quad (-4, 3\pi/4)$
(d) $(2, \pi/3); \quad (-2, 4\pi/3).$

11) (a) Circle $x^2 + y^2 = 25.$
(b) Line $y = x.$
(c) Line $x = 4.$
(d) Line $x = -2.$
(e) Line $y = 3.$
(f) Hyperbola $x^2 - y^2 = 1.$

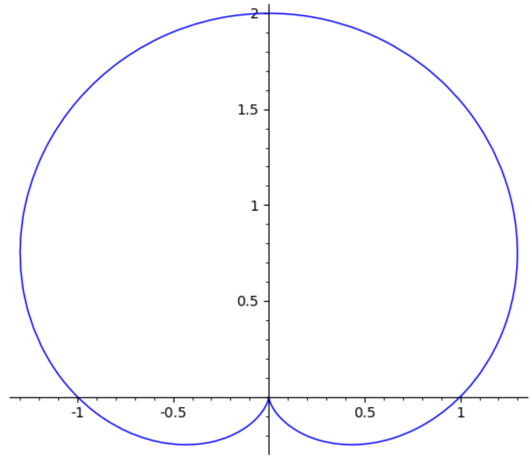
12) (a) $r = 2\csc\theta$
(b) $\theta = \pi/4$
(c) $r = 2\cos\theta$
(d) $r = \frac{1}{4}\csc\theta\cot\theta$
(e) $r^2\cos 2\theta = 4.$

13) (a) $\frac{1}{3}\sqrt{3}$
(b) $-\pi$
(c) $-\frac{\sqrt{2} - 1}{\sqrt{2} + 2}$
(d) $-\sqrt{3}.$

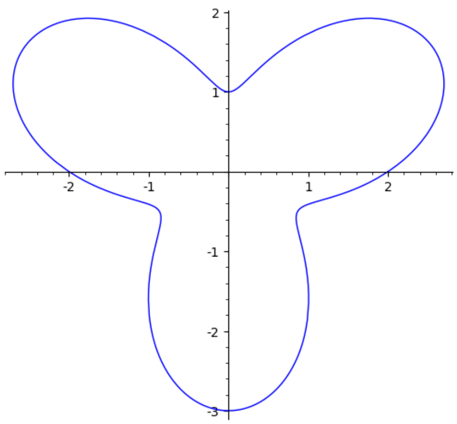
14) The answers are given in polar coordinates; only unique points are included. For example, in (a), the point $(-3/\sqrt{2}, 5\pi/4)$ also has a horizontal tangent, but it coincides with $(3/\sqrt{2}, \pi/4)$, and so is omitted.
(a) Horizontal: $(3/\sqrt{2},\ \pi/4), (-3/\sqrt{2},\ 3\pi/4)$;
vertical: $(3,\ 0), (0, 0)$.
(b) Horizontal: $(3/2, \pi/3), (0, \pi), (3/2, 5\pi/3)$;
vertical: $(2, 0), (1/2,\ 2\pi/3), (1/2,\ 4\pi/3)$.
(c) Horizontal: $(e^{3\pi/4 + k\pi},\ 3\pi/4 + k\pi)$, k–integer;
vertical: $(e^{\pi/4 + k\pi},\ \pi/4 + k\pi)$, k–integer.
(d) Horizontal: $(1/2,\ \pi/6), (2,\ 3\pi/2), (1/2,\ 5\pi/6)$;
vertical: $(0, \pi/2), (3/2,\ 7\pi/6), (3/2,\ 11\pi/6)$.



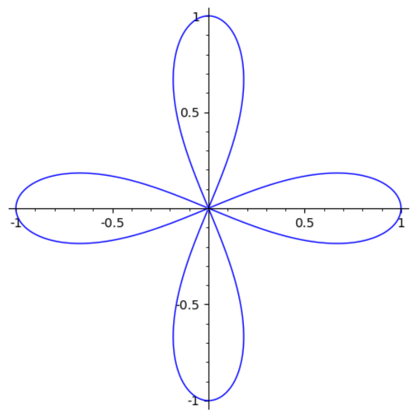
(a) $r = \theta$



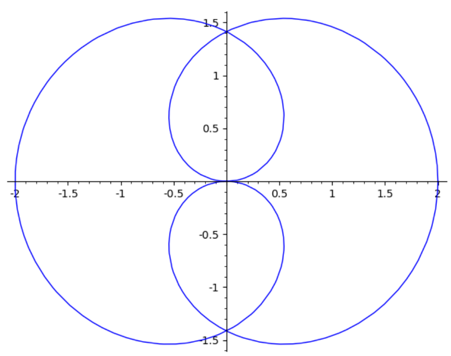
(b) $r = 1 + \sin\theta$



(c) $2 + \sin 3\theta$



(d) $r^2 = \cos 4\theta$



(e) $r = 2\cos(\theta/2)$

15) See the graphs below.

17) (a) $\dfrac{4}{3}\pi + 2\sqrt{3}$

(b) $\dfrac{1}{4}\pi + 2$

(c) π

(d) $2\left(-\frac{2}{3}\pi + 2\sqrt{3}\right)$.

18) The answers are given in polar coordinates; only unique points are included.
   (a) $(1/2, \pi/6), (1/2, 5\pi/6)$
   (b) $(0,0), (1-1/\sqrt{2}, 3\pi/4), (1+1/\sqrt{2}, 7\pi/4)$
   (c) $(0,0), (\sqrt{3}/2, \pi/3), (\sqrt{3}/2, 2\pi/3)$
   (d) $(1, \pi/12)$, $(1, 5\pi/12)$, $(1, 7\pi/12)$, $(1, 11\pi/12)$, $(1, 13\pi/12)$, $(1, 17\pi/12)$, $(1, 19\pi/12)$, $(1, 23\pi/12)$.

19) (a) 2π
   (b) $\frac{8}{3}(\pi^2+1)^{3/2} - \frac{8}{3}$
   (c) 16.

21) (a) Hyperbola $x^2 - y^2/4 = 1$; foci $(\pm\sqrt{5}, 0)$, vertices $(\pm 1, 0)$, asymptotes $y = \pm 2x$.
   (b) Parabola $y^2 = \frac{1}{4}(x+4)$; focus $(-4+1/16, 0)$, vertex $(-4, 0)$.
   (c) Parabola $(x-1)^2 = \frac{2}{3}(y+2)$; focus $(1, -2+1/6)$, vertex $(1, -2)$.
   (d) Ellipse $\frac{(x-1)^2}{2} + \frac{(y-2)^2}{1} = 1$; foci $(1\pm 1, 2)$, vertices $(1\pm\sqrt{2}, 2)$.
   (e) Hyperbola $\frac{(x-4)^2}{9} - \frac{(y+3)^2}{25} = 1$; foci $(4\pm\sqrt{34}, -3)$, vertices $(4\pm 3, -3)$, asymptotes $y+3 = \pm\frac{5}{3}(x-4)$.
   (f) Ellipse $\frac{(y+1)^2}{16} + \frac{(x+2)^2}{4} = 1$; foci $(-2, -1\pm\sqrt{12})$, vertices $(-2, -1\pm 4)$.
   (g) Hyperbola $\frac{(x-3)^2}{9} - \frac{(y-1)^2}{4} = 1$; foci $(3\pm\sqrt{13}, 1)$, vertices $(3\pm 3, 1)$, asymptotes $y-1 = \pm\frac{2}{3}(x-3)$.
   (h) Hyperbola $\frac{(x+1)^2}{2} - \frac{(y+4)^2}{3} = 1$; foci $(-1\pm\sqrt{5}, -4)$, vertices $(-1\pm\sqrt{2}, -4)$, asymptotes $y+4 = \pm\sqrt{\frac{3}{2}}(x+1)$.
   (i) Parabola $(x-3)^2 = 16y$, focus $(3, 4)$, vertex $(3, 0)$.

22) (a) $y^2 = 4x$
   (b) $y^2 = 6(x+3/2)$
   (c) $(x-2)^2 = 24(y+3)$
   (d) $\frac{x^2}{16} + \frac{(y-3)^2}{12} = 1$
   (e) $\frac{(y-4)^2}{9} + \frac{(x+7)^2}{5} = 1$
   (f) $\frac{(x-4)^2}{16} + \frac{(y+1)^2}{12} = 1$
   (g) $\frac{y^2}{9} - \frac{x^2}{7} = 1$
   (h) $\frac{x^2}{4} - \frac{y^2}{36} = 1$

(i) $\dfrac{(x-3)^2}{16} - \dfrac{(y-2)^2}{9} = 1.$