Measure and Integration I (MAA5616), Fall 2020
Homework 10, due Thursday, Nov. 19

We refer to the two parts of Theorem 2.37 as Tonelli's and Fubini's theorem, respectively.

1. Recall that one of the assumptions on the factor spaces X, Y in Tonelli's theorem was σ-finiteness. In this example, we will see that this assumption is indeed necessary.

A counterexample can be constructed as follows. Consider $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ with $X = Y = [0,1]$; $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$ (intersection of the Borel σ-algebra with the interval); $\mu = \lambda$ (Lebesgue measure), $\nu =$ counting measure on Y. Let $D = \{(x,x) : x \in [0,1]\}$ be the diagonal in $X \times Y$. We now proceed to show that the iterated integrals of the indicator 1_D are not equal to its integral with respect to $\mu \times \nu$.

- Show that
$$\iint 1_{x=y}\, d\mu(x) d\nu(y) = \int \left(\int 1_{x=y}\, d\mu(x) \right) d\nu(y) = 0.$$
- Show that
$$\iint 1_{x=y}\, d\nu(y) d\mu(x) = \int \left(\int 1_{x=y}\, d\nu(y) \right) d\mu(x) = 1.$$
- Finally, check
$$\int 1_{x=y}\, d(\mu \times \nu) = +\infty.$$
For this, use that the product measure $\mu \times \nu$ is the restriction of the outer measure generated by rectangles to its measurable σ-algebra. I.e.,
$$\mu \times \nu(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i) : E \subset \bigcup_{i=1}^{\infty} A_i \times B_i \right\}.$$
- Conclude that σ-finiteness is necessary in Tonelli's theorem.

2. On the other hand, when $Y = \mathbb{N}$ with the counting measure (so, σ-countable in a nice way) and X is *any measure space*, Fubini-Tonelli theorem (both parts of it) holds. To prove this, use Theorems 2.15 and 2.25.

3. Take $X = Y = \mathbb{N}$ with $\mathcal{M} = \mathcal{N} = 2^{\mathbb{N}}$ and $\mu = \nu =$ counting measure. For $(x,y) \in \mathbb{N}^2$, let
$$f(x,y) = \begin{cases} 1, & x = y, \\ -1, & x = y+1, \\ 0, & \text{otherwise}. \end{cases}$$
Then $\int |f|\, d(\mu \times \nu) = +\infty$, and the two iterated integrals are unequal (compute them!). Thus, Fubini's theorem fails here.

You may find it convenient to think about this function as a table of numbers. One iterated integral corresponds to adding up the numbers in this table row by row, the other — column by column.