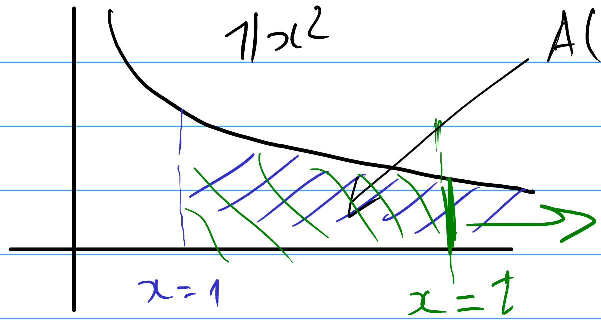
Section 7.8: Improper integrals

Previously: in $\int_a^b f(x)\,dx$ $\begin{cases} a, b - \text{finite} \\ f(x) - \text{finite} \end{cases}$

Now: in an **improper** integral: either range or $f(x)$ is infinite.

Type I : infinite interval
Type II : infinite function

Ex. consider area under $1/x^2$, $x \geq 1$



$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -x^{-1}\Big|_1^t$$

$$= 1 - \frac{1}{t}.$$

We define $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} A(t)$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left(1 - \underbrace{\frac{1}{t}}_{\downarrow 0}\right) = 1.$$

Def An improper integral of type I:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Similarly,

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

We say that an improper integral is **convergent**, if the corresponding limit exists and is finite, and **divergent** otherwise.

Ex. 1 Determine the convergence of

$$\int_1^\infty 1/x \, dx = \lim_{t\to\infty} \int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln|x| \Big|_1^t$$

$$= \lim_{t\to\infty} \ln|t| = +\infty$$ } the integral is divergent

Ex. 2 Determine, for which p

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent.}$$

Let $p \neq 1$.

$$\int_1^\infty x^{-p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t$$

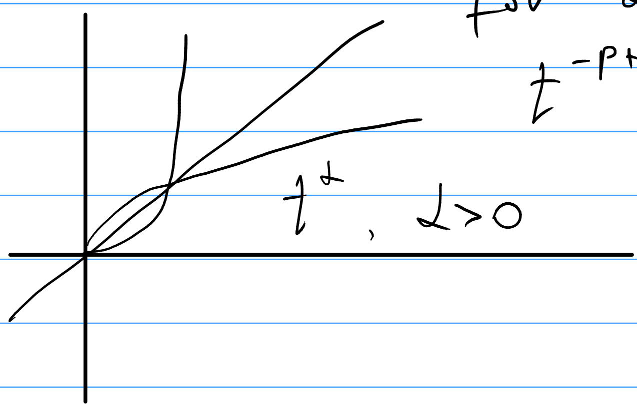
$$= \lim_{t\to\infty} \frac{1}{-p+1}\left(t^{-p+1} - 1\right)$$

a). $-p+1>0$ $\quad p<1$ $\quad \}$ divergent

$$\frac{1}{-p+1}\lim_{t\to\infty}\left(t^{-p+1}-1\right) = +\infty.$$

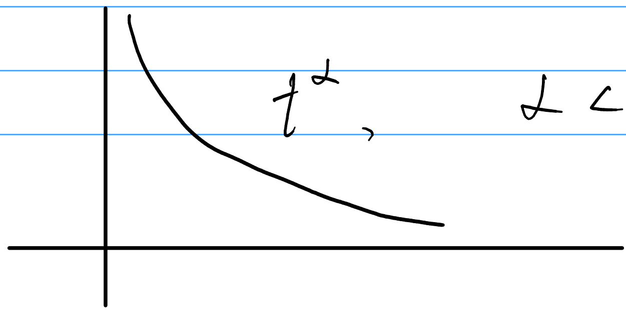
for a positive power,

$t^{-p+1} \to +\infty, \quad t \to +\infty$



b). $-p+1<0$ $\quad p>1$ $\quad \}$ convergent

$$\frac{1}{-p+1}\lim_{t\to\infty}\left(\underbrace{t^{-p+1}}_{\to 0} - 1\right) = \frac{1}{p-1}.$$



To summarize:

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is} \quad \begin{cases} \text{convergent}, & p>1 \\ \text{divergent}, & p\le 1. \end{cases}}$$

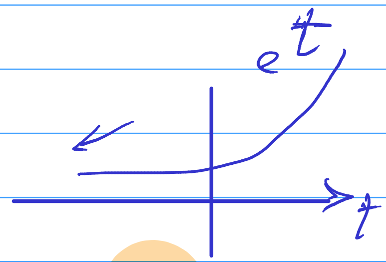
Ex. 3

$$\int_{-\infty}^{0} x e^{x}\,dx = \lim_{t\to-\infty} \int_t^0 x e^x\,dx =$$

$$= \left| \begin{matrix} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{matrix} \right| =$$

$$\lim_{t\to-\infty} \left(x e^x \Big|_t^0 - \int_t^0 e^x\,dx \right) =$$

$$= \lim_{t\to-\infty} \left(-t e^t - e^x \Big|_t^0 \right)$$

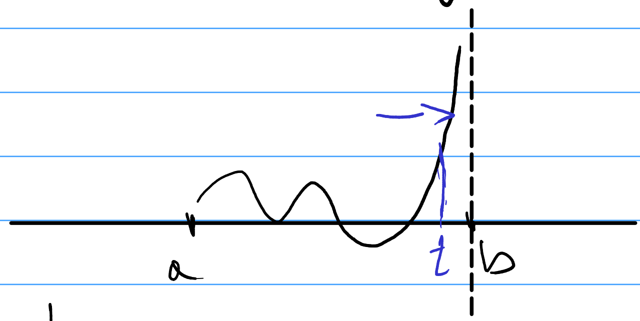


$$= \lim_{t\to-\infty} \Big(\underbrace{-t e^t}_{\downarrow\, 0} - (\underbrace{1}_{\downarrow\, 1} - \underbrace{e^t}_{\downarrow\, 0}) \Big) = -1.$$

$$\lim_{t\to-\infty} \underbrace{t}_{\downarrow\, -\infty} \underbrace{e^t}_{\to 0} = \lim_{t\to-\infty} \frac{t}{e^{-t}} \overset{l'H}{=} \lim_{t\to-\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t\to-\infty} -e^t = 0$$

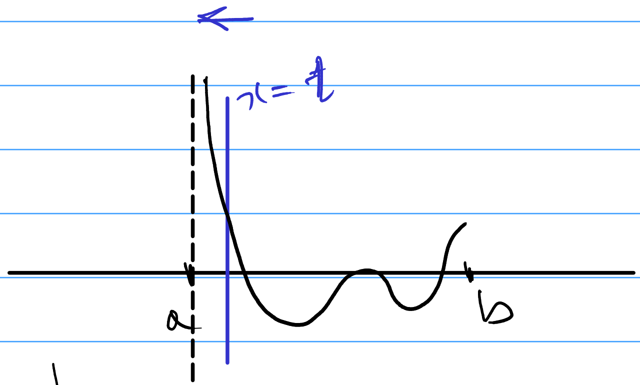
Improper integrals of type II.



Def

$$\int_a^b f(x)\,dx = \lim_{t \to b-} \int_a^t f(x)\,dx$$

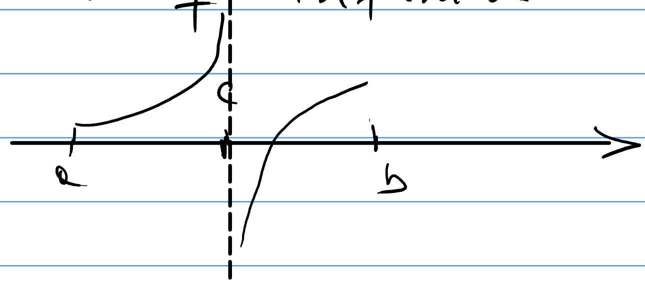
(f is infinite at b).



$$\int_a^b f(x)\,dx = \lim_{t \to a+} \int_t^b f(x)\,dx$$

(f is infinite at a)

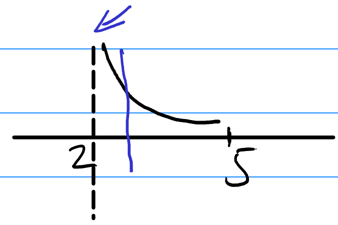
For f infinite inside $[a, b]$:



$$\int_a^b f(x)\,dx = \left(\int_a^c + \int_c^b\right) f(x)\,dx$$

Ex. 4 $\displaystyle\int_2^5 (x-2)^{-1/2}\,dx = \int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

$\displaystyle = \lim_{t\to 2+} \int_t^5 (x-2)^{-1/2}\,dx$



$\displaystyle = \lim_{t\to 2+} \left.\frac{(x-2)^{1/2}}{1/2}\right|_t^5$

$\displaystyle = \lim_{t\to 2+} 2\left((5-2)^{1/2} - \underbrace{(t-2)^{1/2}}_{\to 0}\right)$

$= 2\cdot\sqrt{3}.$

Comparison theorem

Suppose $f(x) \geq g(x) \geq 0$, continuous for $x \geq a$

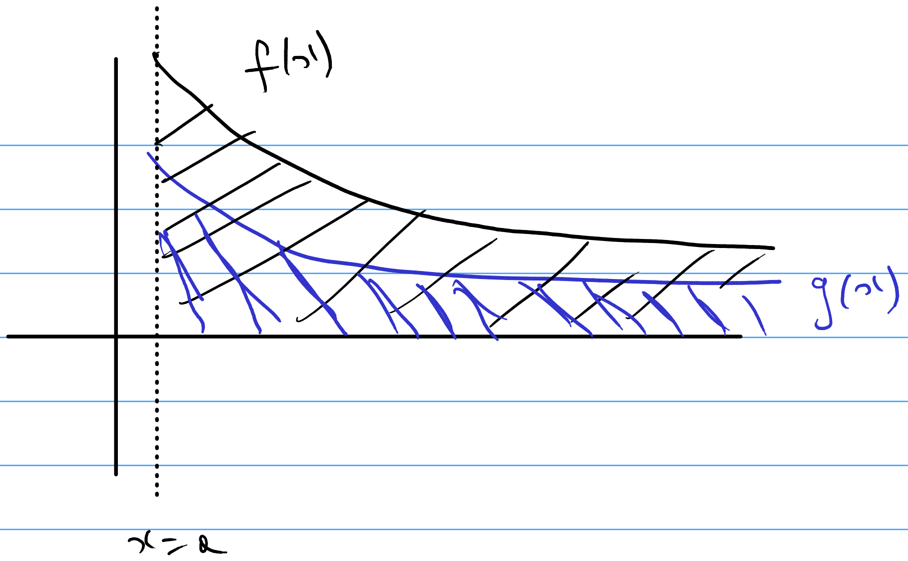
Then

a). If $\displaystyle\int_a^\infty f(x)\,dx$ is convergent,

so is $\displaystyle\int_a^\infty g(x)\,dx$

b). If $\displaystyle\int_a^\infty g(x)\,dx$ is divergent,

so is $\displaystyle\int_a^\infty f(x)\,dx$



Exercise: Show that

$$\int_1^\infty e^{-x^2}\,dx \quad \text{is convergent}$$

Hint: use that $e^{-x} \geq e^{-x^2}$, $x \geq 1$, and the comparison theorem.