Recall:

the binomial series

$$f(x) = (1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k \qquad |x| < 1$$

p - any real number

"p choose k" = binomial coefficient

Here $\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}$

Ex. Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{4-x}}$, determine R.

(*): $(a\cdot b)^p = a^p \cdot b^p$ a, b

use (*)

$$\frac{1}{\sqrt{4-x}} = (4-x)^{-1/2} = \left(4\left(1-\frac{x}{4}\right)\right)^{-1/2} = 4^{-1/2} \cdot \left(1-\frac{x}{4}\right)^{-1/2}$$
$$= \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-1/2}$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-1/2} \overset{\text{binomial series}}{=} \underbrace{\frac{1}{2}\sum_{k=0}^{\infty}\binom{-1/2}{k}\left(-\frac{x}{4}\right)^k}$$

converges when $\left|-\frac{x}{4}\right| < 1$

when $|x| < 4$

$|x-0| < 4$

$|x-a| < R$

$R = 4.$

Now, let's simplify that series:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\sum_{k=0}^{\infty}\binom{-1/2}{k}\left(-\frac{x}{4}\right)^k \ (=)$$

Recall that:
$$\binom{-1/2}{k} = \frac{-1/2\,(-1/2-1)(-1/2-2)\cdots(-1/2-k+1)}{k!} = \frac{-\frac{1}{2}\left(-\frac{1}{2}-\frac{2}{2}\right)\left(-\frac{1}{2}-\frac{4}{2}\right)\cdots\left(-\frac{1}{2}-\frac{2k-2}{2}\right)}{k!}$$

$$(=)\ \frac{1}{2}\sum_{k=0}^{\infty}\frac{\cancel{(-1)^k}\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)}{2^k\cdot k!}\cdot\cancel{(-1)^k}\cdot\frac{x^k}{4^k}$$

$2^k \cdot (2^2)^k = 2^{3k}$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{2^{3k}\cdot k!}x^k$$

Conclusion:
$$\frac{1}{\sqrt{4-x}} = \sum_{k=0}^{\infty}\frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{2^{3k+1}\cdot k!}\cdot x^k \quad \text{when} \quad |x| < 4.$$

Ex. Find the Taylor series for $f(x) = \sin x$ at $a = \pi/3$. Then, find R.

Taylor series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \qquad a = \pi/3$$

$f^{(0)}(x) = \sin x$ $\qquad f^{(0)}\left(\frac{\pi}{3}\right) = \sqrt{3}/2$

$f'(x) = \cos x$ $\qquad f'(\pi/3) = 1/2$

$f''(x) = -\sin x$ $\qquad f''(\pi/3) = -\sqrt{3}/2$

$f'''(x) = -\cos x$ $\qquad f'''(\pi/3) = -1/2$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}(\pi/3) = \sqrt{3}/2$

$\dots \qquad \dots$

Taylor series for $f(x)$:

$$f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f^{(''')}(\pi/3)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \dots$$

$$= \frac{\sqrt{3}}{2} + \frac{1/2}{1!}\left(x-\frac{\pi}{3}\right) + \frac{-\sqrt{3}/2}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{-1/2}{3!}\left(x-\frac{\pi}{3}\right)^3 + \dots$$

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \cdot \frac{\sqrt{3}}{2(2k)!}\left(x-\frac{\pi}{3}\right)^{2k} + \sum_{k=0}^{\infty}(-1)^k \frac{\left(x-\frac{\pi}{3}\right)^{2k+1}}{2(2k+1)!}$$

$R = +\infty$, shown just as for the MacLaurin series of $\sin x$ (apply the ratio test)

Applications of Taylor/Maclaurin series to limits

Ex. Evaluate $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2} =$ $\qquad \frac{0}{0}$

$$= \lim_{x\to 0} \frac{\left(\cancel{1} + \cancel{\frac{x}{1!}} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \cancel{1} - \cancel{x}}{x^2} =$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \dots}{x^2} = \lim_{x\to 0}\left(\frac{1}{2!} + \frac{x}{3!} + \dots\right)$$

contain x

$$= \frac{1}{2!} = \frac{1}{2}.$$

Multiplication and division of power series

Ex. Find the first three nonzero terms in the Maclaurin series for $f(x) = e^x \cdot \sin x$

$$f(x) = e^x \cdot \sin x$$

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right)\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$= 1\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$+ \frac{x}{1!}\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$+ \frac{x^2}{2!}\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$+ \dots$ high degree terms there

$$= x + \frac{x^2}{1!} - \frac{x^3}{3!} + \frac{x^3}{2} + \underbrace{\dots}_{\text{higher degree}}$$

The three lowest degree terms:

$$x + x^2 - \frac{x^3}{6} + \frac{x^3}{2} = x + x^2 + \frac{x^3}{3}$$

Ex. Find the first three nonzero of the Maclaurin series for $\tan x$.

$$\tan x = \frac{\sin x}{\cos x} =$$

$$= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}$$

Long division, as if these infinite power series were polynomials!

$$\begin{array}{r|l}
 & x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \\
1 - \frac{x^2}{2} + \frac{x^4}{24} & \cancel{x} - \frac{x^3}{6} + \frac{x^5}{120} \\
 & -\ \cancel{x} - x^3/2 + x^5/24 \\
\hline
 & \frac{1}{3}\cancel{x^3} - \frac{1}{30}x^5 \\
 & -\ \frac{1}{3}\cancel{x^3} - \frac{1}{6}x^5 + \frac{1}{72}x^7 \\
\hline
 & \frac{2}{15}x^5 + \dots
\end{array}$$

$$\frac{1}{2} - \frac{1}{6}$$

$$\frac{1}{120} - \frac{5}{120} = -\frac{4}{120}$$

$$\frac{1}{6} - \frac{1}{30} = \frac{4}{30}$$

$$\underbrace{\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots}$$

This can be used to compute limits with $\tan x$, when $x \to 0$