REVIEW QUESTIONS FOR TEST 4

Power series

1) Give a definition of the power series. How does a power series define a function; what is the domain of such a function? What are the different possibilities for convergence of a power series? How are the radius of convergence and interval of convergence defined? Explain how to compute them for a given series.
2) Determine the radius and interval of convergence for the following series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{3k+1}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2 4^k}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^3+1}$

(d) $\displaystyle\sum_{k=1}^{\infty} k^k x^k$

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k}{k^k}$

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k(k^3+1)} x^k$

(g) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k 7^k} x^k$

(h) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{k!}$

(i) $\displaystyle\sum_{k=1}^{\infty} 3^k (2x-1)^k$

(j) $\displaystyle\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{k^4} (x+3)^k$

(k) $\displaystyle\sum_{k=2}^{\infty} \frac{(x-2)^k}{3^k \ln k}$

(l) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{4k}}{k(\ln k)^2}$.

3) Suppose that the power series $\displaystyle\sum_{k=0}^{\infty} c_k x^k$ converges for $x=3$ and $x=-4$. What can be said about the convergence of the following number series:

(a) $\displaystyle\sum_{k=1}^{\infty} c_k(-3)^k$

(b) $\displaystyle\sum_{k=1}^{\infty} c_k$

(c) $\displaystyle\sum_{k=1}^{\infty} c_k 2^k$

(d) $\displaystyle\sum_{k=1}^{\infty} c_k 4^k$

Representing functions by power series

4) Give an example of applying algebraic manipulations to represent a function by power series. Explain how to differentiate / integrate a power series. What is the impact of these operations on the radius of convergence? Can the convergence at the endpoints change after one of these operations? Explain how to obtain the power series expansions of $\ln(1+x)$ and $\arctan x$. Give an example of representing an antiderivative by a power series.
5) Give power series representations of the following functions. Choose a convenient center a for your expansion.

(a) $f(x) = \dfrac{1}{1-x}$

(b) $f(x) = \left(\dfrac{x}{2-x}\right)^2$

(c) $f(x) = \dfrac{4}{2x+1}$

(d) $f(x) = \ln(1+2x^3)$

(e) $f(x) = \dfrac{x}{1-x}$

(f) $f(x) = \dfrac{x}{3x^2-1}$

(g) $f(x) = \arctan(2x)$

(h) $f(x) = \dfrac{1}{x^2+2x+2}$

(i) $f(x) = \dfrac{4x+1}{x^2+6x+10}$.

In the following questions use partial fractions decomposition first, then obtain the expansions of the resulting fractions:

(j) $f(x) = \dfrac{2x-4}{x^2-4x+3}$

(k) $f(x) = \dfrac{2x+3}{x^2+3x+2}$.

(l) $f(x) = \dfrac{3x^2-5x+5}{(x-2)(x^2+3)}$

(m) $f(x) = \dfrac{3x^2+2x+1}{(x+1)(x^2+x+2)}$.

6) Represent the following antiderivatives by power series; determine the radii of convergence.

(a) $\displaystyle\int \frac{\arctan(x^3)}{x^2}\,dx$

(b) $\displaystyle\int x^2 \ln(1+x)\,dx$

(c) $\displaystyle\int \frac{x}{1+x^4}\,dx$

(d) $\displaystyle\int \frac{4x^3}{3-x^5}\,dx$

For the last two functions, is this approach easier than using partial fraction decomposition?

Taylor and Maclaurin series

7) Write down the formula for the Taylor series of a function $f(x)$. What value of a must be used to obtain its Maclaurin series? What is the expression for $T_n(x)$, the n-th degree Taylor polynomial? How is it different from the n-th partial sum of the Taylor series? What is the Taylor inequality, and how is it used to estimate the error in approximating $f(x)$ with its Taylor polynomial $T_n(x)$? Why do we need to show that $R_n(x)$, the n-th remainder of the Taylor series, goes to 0? Write down Maclaurin series for the functions you know: e^x, $\sin x$, $\cos x$, $(1+x)^p$, $\ln(1+x)$, $\arctan x$; determine their intervals of convergence.

8) Obtain Maclaurin series for the following functions, using the definition or any other convenient method. Do not show that $R_n(x) \to 0$.

(a) $f(x) = (x+3)^2$

(b) $f(x) = \cos x$

(c) $f(x) = \sin 2x$

(d) $f(x) = \ln(1+x)$

(e) $f(x) = (1-x)^{-2}$

(f) $f(x) = x^2 \ln(1+x)$

(g) $f(x) = e^{x^3}$

(h) $f(x) = x^2 e^{3x}$

(i) $f(x) = \sqrt[4]{(1-x)}$

(j) $f(x) = (2+x)^{-2/3}$

(k) $f(x) = x\cos 3x$

(l) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(m) $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ 1/2, & x = 0 \end{cases}$

(n) $f(x) = \dfrac{x^2}{\sqrt{2+x}}$. (o) $f(x) = \dfrac{5}{\sqrt[3]{3-x^2}}$.

9) In problems (a)–(d) from the previous question, show that $R_n(x) \to 0$, where as usual

$$R_n(x) = f(x) - T_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

To this end, use the formula

$$|R_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1},$$

which holds when $|f^{(n+1)}(x)| \le M$. (*Example (d) is more difficult.*)

10) Compute the Taylor series of the following functions centered at the specified a. Do not show that $R_n(x) \to 0$.

(a) $f(x) = x^3 + 4x^2 + x + 3, \quad a = 2$
(b) $f(x) = \ln x, \quad a = 1$
(c) $f(x) = e^{3x}, \quad a = 2$
(d) $f(x) = \sqrt{x}, \quad a = 9$
(e) $f(x) = \cos x, \quad a = \pi/4$
(f) $f(x) = \sin x, \quad a = \pi/6$.

11) Use series to compute the given limits:

(a) $\displaystyle\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2}$
(b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$
(c) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x}$
(d) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$.

How does this approach compare to directly applying l'Hôpital's rule?

Applications of Taylor polynomials

12) Explain the method for approximating functions with Taylor polynomials, and its purposes. How to choose the center of such a polynomial? For example, if you had to compute $\sin(0.1)$, what would be a good choice for a? What about $\sin(63)$?

Explain how the singularities of a function influence the radius of convergence of its Taylor polynomial. For example, do you know of a quick explanation for why the radius of convergence of the Taylor series of $f(x) = \dfrac{1}{2-x}$ at $a = -1$ is 3? (*Hint:* $f(x)$ has a singularity at $x = 2$, and we are putting the center of the series a at the distance 3 away from this singularity.)

13) Find the Taylor polynomial T_3 of degree 3 for the following functions, centered at the given a:

(a) $f(x) = e^x, \quad a = 1$
(b) $f(x) = \sin x, \quad a = \pi/6$
(c) $f(x) = \cos 2x, \quad a = \pi/4$
(d) $f(x) = e^x \sin x, \quad a = 0$
(e) $f(x) = x^2 \ln(1+x), \quad a = 0$
(f) $f(x) = x \sin x, \quad a = 0$
(g) $f(x) = \sqrt{x}, \quad a = 4$
(h) $f(x) = e^{x^2}, \quad a = 0$.

14) For the functions of the previous question, estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $|x-a| \leq 1/2$. Use the formula

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1},$$

which holds when $|f^{(n+1)}(x)| \leq M$.

Answer key.

2) (a) $R = 1$, $[-1, 1)$
(b) $R = 4$, $[1-4, 1+4]$
(c) $R = 1$, $[2-1, 2+1]$
(d) $R = 0$, $\{0\}$
(e) $R = +\infty$, $(-\infty, \infty)$
(f) $R = 2$, $[-2, 2]$
(g) $R = 7$, $(-7, 7]$
(h) $R = +\infty$, $(-\infty, \infty)$
(i) $R = 1/6$, $(1/2 - 1/6\,, 1/2 + 1/6)$
(j) $R = 0$, $\{-3\}$
(k) $R = 1/3$, $[-1, 5]$
(l) $R = 1$, $[-1, 1]$

3) (a) Absolutely convergent.
(b) Absolutely convergent.
(c) Absolutely convergent.
(d) Unknown (impossible to determine from the given information).

4) A good example is $\dfrac{x}{1+x^2}$. To obtain the power series representation, we first take the expansion for the geometric series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x| < 1.$$

Then, using the substitution $x \mapsto (-x^2)$ gives

$$\frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-x^2)^k \qquad |x| < 1.$$

Multiplying both sides by x then results in the representation for $\dfrac{x}{1+x^2}$.

5) (a) $f(x) = \displaystyle\sum_{k=0}^{\infty} x^k, \qquad |x| < 1$

(b) $f(x) = \displaystyle\sum_{k=0}^{\infty} (k+1)\; 2^{-k-2}\, x^{k+2}, \qquad |x| < 2$

(c) $f(x) = \displaystyle\sum_{k=0}^{\infty} 4(-2)^k x^k, \qquad |x| < 1/2$

(d) $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-2)^{k+1}\; x^{3\,k}}{k}, \qquad |x| < \sqrt[3]{2}$

(e) $f(x) = \displaystyle\sum_{k=0}^{\infty} x^{k+1}, \qquad |x| < 1$

(f) $f(x) = \displaystyle\sum_{k=0}^{\infty} -3^k x^{2k+1}, \qquad |x| < 1/\sqrt{3}$

(g) $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\; 2^{2\,k+1}\, x^{2\,k+1}}{2\,k+1}, \qquad |x| < 1/2$

(h) $f(x) = \displaystyle\sum_{k=0}^{\infty} (-1)^k (x+1)^{2k}, \qquad |x+1| < 1$

(i) $f(x) = \sum_{k=0}^{\infty} 4(-1)^k (x+3)^{2k+1} - \sum_{k=0}^{\infty} 11(-1)^k (x+3)^{2k}, \qquad |x+3| < 1$

(j) Taylor series at $a = 2$:

$f(x) = \sum_{k=0}^{\infty} -2(x-2)^{2k+1}, \qquad |x-2| < 1.$

Maclaurin series:

$f(x) = \sum_{k=0}^{\infty} \left(-1 - 3^{-1-k}\right) x^k, \qquad |x| < 1.$

Note that for both series the radii of convergence are 1 because the function has a singularity at $x = 1$, the midpoint between $a = 0$ and $a = 2$.

(k) $f(x) = \sum_{k=0}^{\infty} -8 \cdot 4^k (x + 3/2)^{2k+1}, \qquad |x + 3/2| < 1/2$

(l) $f(x) = \sum_{k=0}^{\infty} \left(\frac{(-1)^{k+1}}{3^{k+1}} - \frac{1}{2^{2k+1}} \right) x^{2k} + \sum_{k=0}^{\infty} \left(\frac{2(-1)^k}{3^{k+1}} - \frac{1}{2^{2k+2}} \right) x^{2k+1}, \qquad |x| < 1/2$

(m) Note that the second factor in denominator can be written as $x^2 + x + 2 = (x + 1/2)^2 + 3/4$, whereas the first is $(x + 1/2) + 1/2$. We can therefore expand the given function about $a = -1/2$; equivalently, in terms of powers of $(x+1/2)$. The resulting Taylor series has different expressions for the even and odd coefficients:

$$f(x) = \sum_{k=0}^{\infty} \left(2^{2k+1} + 2(-1)^{k+1} \left(\frac{4}{7}\right)^{k+1} \right) (x + 1/2)^{2k}$$
$$+ \sum_{k=0}^{\infty} \left(-2^{2k+2} + 2(-1)^k \left(\frac{4}{7}\right)^{k+1} \right) (x + 1/2)^{2k+1}, \qquad |x + 1/2| < 1/2.$$

6) (a) $\sum_{k=0}^{\infty} \frac{(-1)^k \, x^{6k+2}}{(2k+1)\,(6k+2)}, \qquad R = 1$

(b) $-\sum_{k=1}^{\infty} \frac{(-1)^k \, x^{k+3}}{k\,(k+3)}, \qquad R = 1$

(c) $\sum_{k=0}^{\infty} \frac{(-1)^k \, x^{4k+2}}{4k+2}, \qquad R = 1$

(d) $4 \sum_{k=0}^{\infty} \frac{3^{-k-1} \, x^{5k+4}}{5k+4}, \qquad R = \sqrt[5]{3}$

8) (a) $f(x) = x^2 + 6x + 9$

(b) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

(c) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1} x^{2k+1}}{(2k+1)!}$

(d) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$

(e) $f(x) = \sum_{k=0}^{\infty} (k+1) x^k$

(f) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+3}}{k+1}$

(g) $f(x) = \sum_{k=0}^{\infty} \frac{x^{3k}}{k!}$

(h) $f(x) = x^2 \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} = \sum_{k=0}^{\infty} \frac{3^k x^{k+2}}{k!}$

(i) $f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{1/4}{k} x^k$

(j) $f(x) = \sum_{k=0}^{\infty} 2^{-k-2/3} \binom{-2/3}{k} x^k$

(k) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{3^{2k} x^{2k+1}}{(2k)!}$

(l) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}$

(m) $f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-2}}{(2k)!}$

(n) $f(x) = \sum_{k=0}^{\infty} 2^{-k-1/2} \binom{-1/2}{k} x^{k+2}$.

(o) $f(x) = \sum_{k=0}^{\infty} \frac{5 \cdot 3^{-\frac{k}{2}-\frac{1}{3}} \left(\frac{k}{2}-\frac{2}{3}\right)!}{\left(-\frac{2}{3}\right)! \frac{k}{2}!} x^{2k}$.

9) Computing the derivatives for the functions in (a)–(d) of question 8), we have:

(a) $|f^{(n+1)}(x)| = 0$, $n \geq 2$

(b) $|f^{(n+1)}(x)| \leq 1$

(c) $|f^{(n+1)}(x)| \leq 2^{n+1}$

(d) $|f^{(n+1)}(x)| = \frac{n!}{|1+x|^{n+1}}$

We can therefore take the bound M on the absolute value $|f^{(n+1)}(x)|$ as follows:
(a) $M = 0, n \geq 2$
(b) $M = 1$
(c) $M = 2^{n+1}$
(d) Since $|1+x| \geq |1-|x||$, when $|x| \leq d < 1$ we have $|1+x| \geq 1-d$, and so one can take $M = \frac{n!}{(1-d)^{n+1}}$

This gives, by the formula for R_n, that for $|x| \leq d$ there holds
(a) $|R_n(x)| = 0$, $n \geq 2$
(b) $|R_n(x)| \leq \frac{1}{(n+1)!} d^{n+1}$
(c) $|R_n(x)| \leq \frac{2^{n+1}}{(n+1)!} d^{n+1} = \frac{(2d)^{n+1}}{(n+1)!}$
(d) $|R_n(x)| \leq \frac{1}{(n+1)|1-d|^{n+1}} d^{n+1} = \frac{1}{n+1} \cdot \left(\frac{d}{1-d}\right)^{n+1}$.

Recall (see page 5 in the notes `here`) that for any fixed number A,

$$\lim_{n\to\infty} \frac{A^{n+1}}{(n+1)!} = 0.$$

It follows that in (a)–(c), $R_n(x) \to 0$ for all x, and in (d), $R_n(x) \to 0$ holds for $d \leq 1/2$.

Note that in (d) we can prove convergence of the power series to the function on the entire interval $|x| < 1$ and not just $|x| < 1/2$ by integrating $1/(1-x)$ term by term. The biggest interval on which this convergence holds is indeed $|x| < 1$, because the radius of convergence of the series in (d) is $R = 1$.

10) (a) $f(x) = 29 + 29(x-2) + 10(x-2)^2 + (x-2)^3$

(b) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x-1)^{k+1}}{k+1}$

(c) $f(x) = \sum_{k=0}^{\infty} \frac{e^6 3^k (x-2)^k}{k!}$

(d) $f(x) = \sum_{k=0}^{\infty} 3^{1-2k} \binom{1/2}{k} (x-9)^k$

(e) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{2}} \frac{(x-\pi/4)^{2k}}{(2k)!} - \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{2}} \frac{(x-\pi/4)^{2k+1}}{(2k+1)!}$

(f) $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{2}} \frac{(x-\pi/6)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} (-1)^k \sqrt{\frac{3}{2}} \frac{(x-\pi/6)^{2k+1}}{(2k+1)!}$.

11) (a) $1/2$
(b) $-1/8$
(c) -1
(d) $1/120$.

13) (a) $T_3(x) = e + e(x-1) + \frac{e(x-2)^2}{2} + \frac{e(x-1)^3}{6}$
(b) $T_3(x) = \frac{1}{2} + \frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)}{2} - \frac{\left(x-\frac{\pi}{6}\right)^2}{4} - \frac{\sqrt{3}\left(x-\frac{\pi}{6}\right)^3}{12}$
(c) $T_3(x) = -2\left(x-\frac{\pi}{4}\right) + \frac{4\left(x-\frac{\pi}{4}\right)^3}{3}$
(d) $T_3(x) = x + x^2 + \frac{x^3}{3}$
(e) $T_3(x) = x^3$
(f) $T_3(x) = x^2$
(g) $T_3(x) = 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512}$
(h) $T_3(x) = 1 + x^2$.

14) In what follows we estimate the remainder $R_n(x) = f(x) - T_n(x)$. We used the assumption that $|x-a| \le 1/2$.
(a) $|R_3(x)| \le \frac{e^{3/2}}{4!}\left(\frac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |e^x| \le e^{3/2}$ for $|x-1| \le 1/2$.
(b) $|R_3(x)| \le \frac{1}{4!}\left(\frac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |\sin(x)| \le 1$ for $|x-\pi/6| \le 1/2$.
(c) $|R_3(x)| \le \frac{2^4}{4!}\left(\frac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |2^4\cos(x)| \le 2^4$ for $|x-\pi/4| \le 1/2$.
(d) $|R_3(x)| \le \frac{4e^{1/2}}{4!}\left(\frac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |-4e^x \sin x| \le 4e^{1/2}$ for $|x| \le 1/2$.

(e) $|R_3(x)| \leq \frac{136}{4!}\left(\frac{1}{2}\right)^4$ in view of

$$\begin{aligned}
|f^{(4)}(x)| &= \left| -\frac{6x^2}{(x+1)^4} + \frac{16x}{(x+1)^3} - \frac{12}{(x+1)^2} \right| \\
&\leq \left| \frac{6x^2}{(x+1)^4} \right| + \left| \frac{16x}{(x+1)^3} \right| + \left| \frac{12}{(x+1)^2} \right| \\
&\leq \left| \frac{6 \cdot (1/2)^2}{(-1/2+1)^4} \right| + \left| \frac{16 \cdot 1/2}{(-1/2+1)^3} \right| + \left| \frac{12}{(-1/2+1)^2} \right| \\
&= 6 \cdot 4 + 16 \cdot 4 + 12 \cdot 4 = 136 \qquad \text{for } |x| \leq 1/2.
\end{aligned}$$

Here the absolute values of the fractions in the second line are estimated as the largest value in the numerator over the smallest value in the denominator.

(f) $R_3\,(x) \leq \frac{9/2}{4!}\left(\frac{1}{2}\right)^4$ in view of

$$\begin{aligned}
|f^{(4)}(x)| &= |x\sin(x) - 4\cos(x)| \\
&\leq |x\sin(x)| + 4|\cos(x)| \\
&\leq \frac{1}{2} \cdot 1 + 4 = 9/2 \qquad \text{for } |x| \leq 1/2.
\end{aligned}$$

(g) $R_3\,(x) \leq \frac{15}{343\sqrt{14}} \cdot \frac{1}{4!}\left(\frac{1}{2}\right)^4$ in view of

$$|f^{(4)}(x)| = \left| -\frac{15}{16x^{7/2}} \right| \leq \frac{15}{343\sqrt{14}} \qquad \text{for } |x-4| \leq 1/2.$$

We used here that the absolute value of $f^{(4)}$ is decreasing, and so the maximum is attained at the left endpoint of $|x-4| \leq 1/2$.

(h) $R_3\,(x) \leq \frac{25e^{1/4}}{4!}\left(\frac{1}{2}\right)^4$ in view of

$$\begin{aligned}
|f^{(4)}(x)| &= \left| 48e^{x^2}x^2 + 12e^{x^2} + 16e^{x^2}x^4 \right| \\
&\leq 48e^{1/2}(1/2)^2 + 12e^{(1/2)^2} + 16e^{(1/2)^2}(1/2)^4 \\
&= 25e^{1/4} \qquad \text{for } |x| \leq 1/2.
\end{aligned}$$