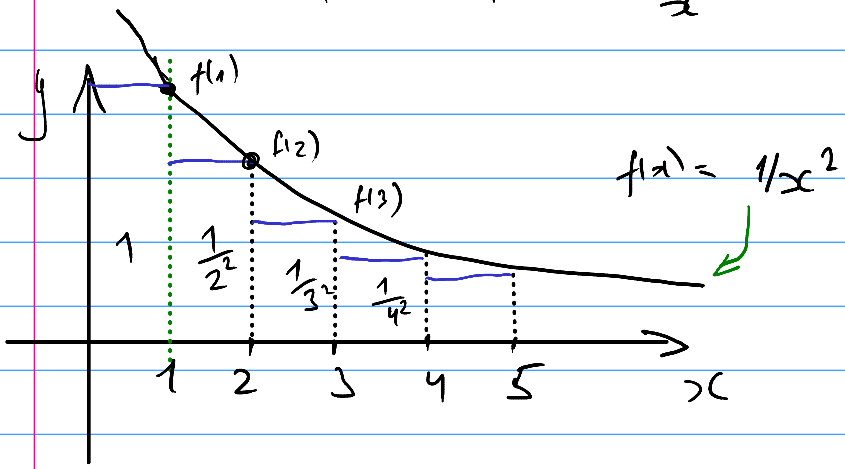
Section 11.3: Integral test

Ex. 1. Consider: $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \underbrace{\frac{1}{3^2}}_{a_3} + \dots$

Then for $f(x) = \frac{1}{x^2}$, $\quad f(k) = a_k$



$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \boxed{\text{Area of rectangles}} \leq \underbrace{1}_{\text{1st rectangle}} + \int_1^{\infty} \frac{1}{x^2}$$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^2}}_{a_k} \leq 1 + \int_1^{\infty} x^{-2}\,dx = 1 + \left(-x^{-1}\right)\Big|_1^{\infty}$$

$$= 2.$$

So, any partial sum satisfies

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$$

$\Rightarrow$ The sequence of partial sums $\{S_n\}$ is bounded:

$$0 < S_n \leq 2$$

But the a_k are nonnegative, so

$$S_{n+1} - S_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1} > 0$$

$\Rightarrow S_{n+1} > S_n$.

The sequence of partial sums $\{S_n\}$ is increasing and bounded $\Rightarrow$ convergent!

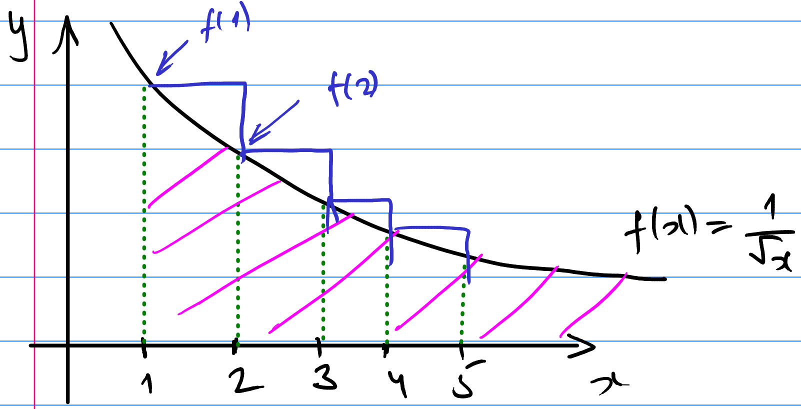
$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

(not to 2 though).

Ex. 2 Consider:

$$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

Introduce $f(x) = \frac{1}{\sqrt{x}}$, then $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = +\infty$



$$\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \boxed{\text{Area of rectangles}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}} dx =$$

$$= 2\sqrt{x}\Big|_1^{\infty} = \lim_{t \to \infty} 2\sqrt{x}\Big|_1^{t} = +\infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = +\infty, \qquad \text{divergent.}$$

The integral test for convergence.

Suppose $f(x)$ is continuous, positive, decreasing on $[1,\infty)$, and

$$f(k) = a_k, \qquad k \geq 1,$$

then

$$\sum_{k=1}^{\infty} a_k \text{ is convergent} \iff \int_1^{\infty} f(x)\,dx \text{ is convergent.}$$

Remark: these properties only need to hold for $k \geq k_0$.