Section 11.4 Comparison tests

Ex. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

For $k \geq 3 > e$: $\ln k > 1$, and

$\frac{\ln k}{k} > \frac{1}{k}$ $\qquad k \geq 3$

Sum both sides for $k \geq 3$:

$$\underbrace{\sum_{k=3}^{\infty} \frac{\ln k}{k}}_{\text{divergent}} \geq \sum_{k=3}^{\infty} \frac{1}{k} = +\infty$$

The comparison test:

Suppose $\sum_k a_k$ and $\sum_k b_k$ are such that

$$0 \leq a_k \leq b_k$$

Then:

a). If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.

b). If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.

Comparison can be to the known series:

- p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$
- geometric series $\sum_{k=1}^{\infty} a r^{k-1}$

Ex. 1 Test for convergence: $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{1}{2^k+1}}_{a_k}$

There holds: $a_k = \dfrac{1}{2^k+1} < \dfrac{1}{2^k} = b_k$,

and $\displaystyle\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$ } convergent.

$\Rightarrow$ by the comparison test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k+1}$ } convergent

Ex. 2 Test for convergence: $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{5}{2k^2+4k+3}}_{a_k}$

$a_k = \dfrac{5}{2k^2+4k+3} \leq \dfrac{5}{2k^2} = b_k$

$\displaystyle\sum_{k=1}^{\infty} \frac{5}{2k^2} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$ } convergent.

$\Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{5}{2k^2+4k+3}$ is convergent.

Ex. 3 Test for convergence:

$\displaystyle\sum_{k=2}^{\infty} \underbrace{\frac{\ln k}{(k-1)^{1/2}}}_{a_k}$ } divergent.

For $k \geq 3$,

$a_k = \dfrac{\ln k}{(k-1)^{1/2}} \geq \dfrac{1}{(k-1)^{1/2}} \geq \dfrac{1}{k^{1/2}} = b_k$

$$\sum_{K=3}^{\infty} \frac{1}{K^{1/2}} \Bigg\} \text{ divergent, } \\ \text{p-series} \\ p = 1/2$$