Bonus: series and differential equations

#6 on 11.9 (Webassign)

Show: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a solution of the differential equation $f'(x) = f(x)$.

Consider $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(R = +\infty)$

Differentiate both sides with respect to x:

$$f'(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right)' = \sum_{n=0}^{\infty} \frac{n \cdot x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{\cancel{n} \cdot x^{n-1}}{\cancel{n} \cdot (n-1)!}$$

$n! = (n-1)! \cdot n$

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Let: $k = n - 1$

$$f'(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x).$$

here $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

We want to show: $f(x) = e^x$. There are 2 options:

1). Use Maclaurin series (as is done in 11.10).

2). Use the equation $f'(x) = f(x)$ that we just obtained

For $f(x)$ we have:

0-th term

$$f'(x) = f(x) \quad \underline{\text{and}} \quad f(0) = a_0 = \frac{0^0}{0!} = 1$$

Diff. equation: $\frac{dy}{dx} = y(x)$ $\qquad$ $f(0) = 1$ } initial condition

Solution: $y(x) = e^x$

Separate variables and integrate:

$$\int \frac{dy}{y} = \int dx$$

$$\ln|y| = x + C \qquad e^x \text{ both sides}$$

$$|y| = e^C \cdot e^x \qquad \text{set } K = e^C$$

$$y = \pm K e^x \qquad \text{must be } K > 0$$

$$y = K e^x \qquad \text{now only } K \neq 0$$

$$y = K e^x \qquad \text{use the initial condition } y(0) = 1$$

$$y(0) = 1 = K \cdot e^0$$

$$\Rightarrow K = 1$$