Section 11.10 Taylor and Maclaurin series

Q: What functions have power series representations, and how to find these representations?

Assume: $f(x)$ can be represented as a power series.

Thm. If f can be represented as
$$f(x) = \sum_k c_k (x-a)^k, \qquad |x-a| < R,$$
(for $R>0$)

then
$$c_k = \frac{f^{(k)}(a)}{k!} \qquad k \geq 0$$

(here $f^{(k)}$ – k-th derivative,
$f^{(0)} = f$;
$0! = 1$)

Substitute the expression for c_k into the power series:

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k, \qquad |x-a|<R}$$

– the Taylor series of f at $x=a$.

When $a=0$,

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k \qquad |x|<R}$$

– Maclaurin series of f.

Note: this formula holds, if $f(x)$ can be represented as a power series.

Ex. 1 Compute the Maclaurin series for $f(x) = e^x$, determine R. (Assuming that it has a power series representation).

$$f(x) = e^x \implies \begin{aligned} f^{(0)}(x) &= f(x) = e^x \\ f^{(1)}(x) &= f'(x) = e^x \\ f^{(2)}(x) &= e^x \\ &\dots \\ f^{(k)}(x) &= e^x \end{aligned}$$

Thus: $f^{(k)}(0) = e^0 = 1$

$$\boxed{e^x = \underbrace{\sum_{k=0}^{\infty} \frac{x^k}{k!}}_{\text{Maclaurin series}}, \qquad \text{for all } x. \ (R = +\infty)}$$

Determine R: (Applying the ratio test)

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \underbrace{\frac{x^{k+1}}{(k+1)!}}_{a_{k+1}} \Big/ \underbrace{\frac{x^k}{k!}}_{a_k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x}{\underbrace{(k+1)!}_{=(k+1)\cdot k!}} \cdot \frac{k!}{1} \right| = \lim_{k\to\infty} \left| \frac{x}{(k+1)\cancel{k!}} \cdot \cancel{k!} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x}{k+1} \right| = \lim_{k\to\infty} \frac{|x|}{k+1} = |x| \cdot \lim_{k\to\infty} \frac{1}{k+1}$$

$$= |x| \cdot 0 = 0 < 1$$

$\implies$ the series for e^x is (abs.) convergent for all x, $R = +\infty$.

In Ex1: $\boxed{\text{f(x) has power series rep}}$ $\longrightarrow$ $\boxed{\text{the power series}}$

Usually: $\boxed{\text{Start with } f(x)}$ $\xrightarrow[\;]{\text{use the formula from the thm: } c_k = \frac{f^{(k)}(a)}{k!}}$ $\boxed{\text{Compute a Taylor expansion}}$

$\boxed{\text{Show that these two are equal}}$

We will discuss, how to show that a function is equal to Taylor series (and therefore can be represented as a power series).

Let

$$\underbrace{T_n(x)}_{\text{Taylor polynomial}} = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}}_{\text{n-th partial sum of the Taylor series}} (x-a)^k$$

$$f(x) = \underbrace{T_n(x)}_{\text{Taylor polynomial}} + \underbrace{R_n(x)}_{\text{remainder}}$$

Note:

$$R_n(x) = f(x) - T_n(x)$$

If we showed that $R_n(x) \to 0$, $n \to \infty$, we would know that $T_n(x) \to f(x)$,

Remainder estimate (Taylor inequality)

Goal: to show that $f(x) - T_n(x)$ is „small" when $n \to \infty$

If $|f^{(n+1)}(x)| \le \overbrace{M}^{\text{constant}}$ for $|x-a| \le d$, then

$$|f(x) - T_n(x)| = |R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

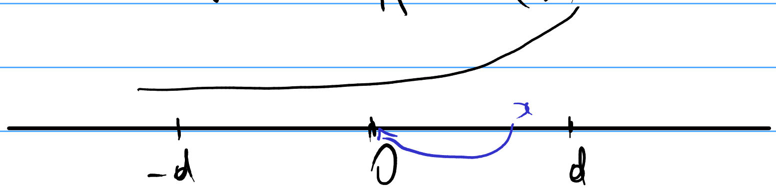
for $|x-a| \le d$.

Ex. 2 Prove that e^x is equal to its Maclaurin series.

Let

$f^{(n+1)}(x) = e^x$, then on $|x| \le d$,

we have $|f^{(n+1)}(x)| = |e^x| = \underbrace{e^d}_{M}$



Then, by the remainder estimate,

$$|f(x) - T_n(x)| = |R_n(x)| \le \frac{e^d}{(n+1)!} \cdot |x-0|^{n+1}$$

$\underline{|x-0| \le d}$

We need to show that $R_n \to 0$, $n \to \infty$, since then $T_n \to f$, and we are done.

We have

$$|f(x) - T_n(x)| = |e^x - T_n(x)| \le \frac{e^d}{(n+1)!} \cdot d^{n+1}$$

It remains to show:

$$\lim_{n\to\infty} \frac{e^{d}}{(n+1)!} \cdot d^{n+1} = 0.$$

Note: d is any fixed positive number.

$$\lim_{n\to\infty} \frac{d^{n+1}}{(n+1)!} = \lim_{n\to\infty} \overbrace{\frac{\boxed{d}\cdot\boxed{d}\cdot \ldots \boxed{d}}{\boxed{1}\cdot\boxed{2}\cdot \ldots \cdot \boxed{(n+1)}}}^{n+1 \text{ factor}} =$$

$$\lim_{n\to\infty} \frac{d}{1}\cdot\frac{d}{2}\cdot \ldots \cdot \frac{d}{n+1} = \left\{ \begin{array}{l} \text{let's say,} \\ d < n_0 \text{ integer} \end{array}\right.$$

$$\lim_{n\to\infty} \underbrace{\frac{d}{1}\cdot\frac{d}{2}\cdot \ldots \cdot \frac{d}{2n_0}}_{2n_0 \text{ factors} \; = A} \cdot \underbrace{\frac{d}{2n_0+1}\cdot\frac{d}{2n_0+2} \cdots \frac{d}{n+1}}_{n-2n_0 \text{ factor}}$$

$$\lim_{n\to\infty} A \underbrace{\underbrace{\frac{d}{2n_0+1}}_{<\frac{1}{2}} \cdot \underbrace{\frac{d}{2n_0+2}}_{<\frac{1}{2}} \cdot \ldots \cdot \underbrace{\frac{d}{n+1}}_{<\frac{1}{2}}}_{\leq \left(\frac{1}{2}\right)^{n-2n_0} \searrow 0} = 0.$$

By the remainder estimate, $T_n(x) \to \overbrace{f(x)}^{e^x}$ for $|x| \leq d$, for any positive d!

We have showed:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(previously it was assumed e^x has power series expansion, we have now justified it)

Ex 3. Find the Maclaurin series for $\sin x$, show that it converges to $\sin x$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k \qquad |x| \leq R$$

$$\begin{aligned}
f^{(0)}(x) &= \sin x & \qquad f^{(0)}(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \\
&\dots & &\dots
\end{aligned}$$

Maclaurin series:

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \dots$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

To establish the above equality, we use the Taylor inequality (remainder estimate)

We need to establish that

$$|f^{(n+1)}(x)| \leq M, \qquad |x - 0| \leq d$$

then:

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - 0|^{n+1} \qquad |x - 0| \leq d$$

Then, we will show that the right-hand side goes to 0 when $n \to \infty$.

We have: $f^{(n+1)}(x) = \sin^{(n+1)}(x) \nearrow \pm\sin x$, $\searrow \pm\cos x$

$|f^{(n+1)}(x)| = |\sin^{(n+1)}(x)| \le M$ $\quad |x| \le d$

Then, $M = 1$ works!

By the remainder estimate,

$$|\sin x - T_n(x)| \le \frac{1}{(n+1)!} \cdot |x|^{n+1} \qquad |x| \le d$$

$$\le \frac{d^{n+1}}{(n+1)!} \longrightarrow 0, \quad n \to \infty$$

(for all d)

This shows: $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \to \sin x$

for all x!

$$\boxed{\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

(for all x,)
$R = +\infty$.

Ex. 4 Find the Maclaurin series for $\cos x$.

We will differentiate the series for $\sin x$:

$$(\sin x)' = \cos x = \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right)'$$

$$= \sum_{x=0}^{\infty} (-1)^k \cdot \frac{(2k+1) \cdot x^{2k}}{\underbrace{(2k+1)!}_{(2k)! \cdot (2k+1)}}$$

$(n+1)! = n! \, (n+1)$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\cancel{(2k+1)}\, x^{2k}}{(2k)! \cdot \cancel{(2k+1)}}$$

$$\boxed{\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}$$

(for all x, $R = +\infty$)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Ex. 5 Find the Maclaurin series for $f(x) = x \cdot \cos(x^3)$, determine R.

$$f(x) = x \cdot \cos(x^3) = x \cdot \sum_{k=0}^{\infty} (-1)^k \frac{(x^3)^{2k}}{(2k)!}$$

$$= x \cdot \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k}}{(2k)!}$$

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+1}}{(2k)!} \qquad R = +\infty.$$

$R = +\infty$ because:

1) the series for $\cos(x)$ converges for all x

$\Rightarrow$ series for $\cos(x^3)$ converges for all x as well

2). multiplication by x does not change R.

Ex.6 Find the Maclaurin series for

$f(x) = (1+x)^p$, $\quad p$ - any fixed real number, $p \neq 0$

Then, find R.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k$$

$$\begin{aligned} f^{(0)}(x) &= (1+x)^p & f^{(0)}(0) &= 1 \\ f'(x) &= p(1+x)^{p-1} & f'(0) &= p \\ f''(x) &= p(p-1)(1+x)^{p-2} & f''(0) &= p(p-1) \\ f'''(x) &= p(p-1)(p-2)(1+x)^{p-3} & f'''(0) &= p(p-1)(p-2) \\ &\dots & &\dots \\ f^{(k)}(x) &= p(p-1)(p-2)\dots(p-k+1)\cdot(1+x)^{p-k} & f^{(k)}(0) &= p(p-1)\dots(p-k+1) \end{aligned}$$

Maclaurin series for $f(x) = (1+x)^p$:

$$\left.\sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\dots(p-k+1)}{k!} \cdot x^k \quad \right\} \text{binomial series}$$

For $p = n$ (integer), the above sum has only terms up to degree n:

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot x^k$$

When $k = n+1$, $\quad (n-(n+1)+1) = 0$

$$\underbrace{(1+x)^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot x^k}_{\text{binomial formula}} = \sum_{k=0}^{n} \underbrace{\frac{n!}{k!(n-k)!}}_{\text{binomial coeff.}} x^k$$

binomial coefficient

$$(1+x)^p = \sum_{k=0}^{\infty} \overbrace{\binom{p}{k}} x^k$$

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdot\ldots\cdot(p-k+1)}{k!}$$

p - real
k - integer

,, p choose k "

$$\binom{p}{0} = 1$$

Ratio test to determine R:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left| \frac{\dfrac{\cancel{p(p-1)(p-2)\cdot\ldots\cdot(p-k+1)}(p-(k+1)+1)}{(k+1)!}\cancel{x^{k+1}}}{\dfrac{\cancel{p(p-1)(p-2)\cdot\ldots\cdot(p-k+1)}}{k!}\cdot\cancel{x^k}} \right|$$

$$= \left| \frac{(p-k)}{(k+1)!}\cdot x \Big/ \frac{1}{k!} \right| = \left| \frac{(p-k)}{(k+1)\cancel{k!}}\cdot x \cdot \cancel{k!} \right|$$

$$= \left|\frac{p-k}{k+1}\right| \cdot |x|$$

$$\lim_{k\to\infty} \left|\frac{p-k}{k+1}\right| \cdot |x| = \lim_{k\to\infty} \left| \frac{\overset{\to 0}{p/k} - 1}{1 + \underset{\to 0}{1/k}} \right| \cdot |x| =$$

$$\lim_{k\to\infty} |-1| \cdot |x| = |x|$$

The series converges when $|x| < 1$
$|x - 0| < R$

$R = 1$

To summarize:

Binomial series

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \qquad |x| < 1$$

p – any real

where $\binom{p}{k} = \frac{p(p-1)(p-2)\cdot\ldots\cdot(p-k+1)}{k!}, \quad k \geq 1$

$\binom{p}{0} = 1.$

Proof of the Thm about Taylor series.

Suppose that $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$, $(|x-a|<R)$

(*) $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$

Let us determine the c_k:

When $x = a$,
$f(a) = c_0$

Since we can differentiate power series term by term (we have a Thm about this!), let's differentiate (*), both sides of it, with respect to x:

(**) $f'(x) = c_1 + c_2 2(x-a) + c_3 \cdot 3(x-a)^2 + \dots$ $|x-a|<R$

$f'(a) = c_1 \implies c_1 = \frac{f'(a)}{1}$

Let's differentiate both sides of (**) with respect to x:

(***) $f''(x) = 2c_2 + c_3 \cdot 3 \cdot 2(x-a) + \dots$ $|x-a|<R$

$f''(a) = 2c_2 \implies c_2 = \frac{f''(a)}{2}$

Differentiate (***) with respect to x:

$$f'''(x) = c_3 \cdot \underbrace{3 \cdot 2 \cdot 1}_{3!} + \overbrace{\ldots}^{(x-a)}$$

$$f'''(a) = 3! \cdot c_3 \implies c_3 = \frac{f'''(a)}{3!}$$

Differentiating $f(x)$ and its power series k times, we will obtain

$$c_k = \frac{f^{(k)}(a)}{k!}$$ } formula from the Thm.

in exactly the same way as above!