

REVIEW QUESTIONS FOR POWER SERIES

Power series

- 1) Give a definition of the power series. What is the radius of convergence / interval of convergence of a power series? Explain how to compute them for a given series.
- 2) Determine the radius and interval of convergence for the following series:

(a) $\sum_{n=1}^{\infty} \frac{x^n}{3n+1}$

(b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 4^n}$

(c) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3+1}$

(d) $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^n}$

(e) $\sum_{n=1}^{\infty} \frac{n}{2^n(n^3+1)} x^n$

(f) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n 7^n} x^n$

(g) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

(h) $\sum_{n=1}^{\infty} 3^n (2x-1)^n.$

Representing functions by power series

- 3) Explain how to differentiate / integrate a power series. What is the impact of these operations on the radius of convergence?
- 4) Give a power series representations of the following functions:

(a) $f(x) = \frac{1}{1-x}$

(b) $f(x) = \frac{4}{2x+1}$

(c) $f(x) = \frac{x}{1-x}$

(d) $f(x) = \frac{x}{3x^2-1}$

(e) $f(x) = \frac{1}{x^2+2x+2}$

(f) $f(x) = \frac{4x+1}{x^2+6x+10}.$

In the following questions use partial fractions decomposition first, then obtain the expansions of the resulting fractions:

(g) $f(x) = \frac{2x-4}{x^2-4x+3}$

(h) $f(x) = \frac{2x+3}{x^2+3x+2}.$

(i) $f(x) = \frac{3x^2-5x+5}{(x-2)(x^2+3)}$

(j) $f(x) = \frac{3x^2+2x+1}{(x+1)(x^2+x+2)}.$

Taylor and Maclaurin series

- 5) Write down the formula for Taylor series. What needs to be changed to obtain Maclaurin series? What is the expression for $T_n(x)$, the n -th degree Taylor polynomial? What is the Taylor inequality, and how is it used to estimate the error in approximating $f(x)$ with its Taylor polynomial $T_n(x)$? Write down the important Maclaurin series you know.
- 6) Obtain Maclaurin series for the following functions, using the definition or any other convenient method. Do not show that $R_n(x) \rightarrow 0$.

- | | |
|-----------------------------|---|
| (a) $f(x) = (1 - x)^{-2}$ | (h) $f(x) = \sqrt[4]{(1 - x)}$ |
| (b) $f(x) = (x + 3)^2$ | (i) $f(x) = (2 + x)^{-2/3}$ |
| (c) $f(x) = \cos x$ | (j) $f(x) = x \cos 3x$ |
| (d) $f(x) = \sin 2x$ | (k) $f(x) = \begin{cases} \frac{1 - \cos x}{x^2}, & x \neq 0 \\ 1/2, & x = 0 \end{cases}$ |
| (e) $f(x) = \ln(1 + x)$ | (l) $f(x) = \frac{x^2}{\sqrt{2 + x}}$ |
| (f) $f(x) = x^2 \ln(1 + x)$ | |
| (g) $f(x) = e^{x^3}$ | |

7) Compute the Taylor series of the following functions centered at the specified a . Do not show that $R_n(x) \rightarrow 0$.

- | | |
|--|--------------------------------------|
| (a) $f(x) = x^3 + 4x^2 + x + 3, \quad a = 2$ | (d) $f(x) = \sqrt{x}, \quad a = 9$ |
| (b) $f(x) = \ln x, \quad a = 1$ | (e) $f(x) = \cos x, \quad a = \pi/4$ |
| (c) $f(x) = e^{3x}, \quad a = 2$ | (f) $f(x) = \sin x, \quad a = \pi/6$ |

8) Use series to compute the given limits:

- | | |
|---|---|
| (a) $\lim_{x \rightarrow 0} \frac{x - \ln(1 + x)}{x^2}$ | (c) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{1 + x - e^x}$ |
| (b) $\lim_{x \rightarrow 0} \frac{\sqrt{1 + x} - 1 - x/2}{x^2}$ | (d) $\lim_{x \rightarrow 0} \frac{\sin x - x + x^3/6}{x^5}$ |

Applications of Taylor polynomials

- 9) Explain the method for approximating functions with Taylor polynomials, and its purposes. How to choose the center of such a polynomial? Explain how the singularities of a function influence the radius of convergence of its Taylor polynomial.
- 10) Find the Taylor polynomial T_3 of degree 3 for the following functions, centered at the given a :

- | | |
|---------------------------------------|--|
| (a) $f(x) = e^x, \quad a = 1$ | (e) $f(x) = x^2 \ln(1 + x), \quad a = 0$ |
| (b) $f(x) = \sin x, \quad a = \pi/6$ | (f) $f(x) = x \sin x, \quad a = 0$ |
| (c) $f(x) = \cos 2x, \quad a = \pi/4$ | (g) $f(x) = \sqrt{x}, \quad a = 4$ |
| (d) $f(x) = e^x \sin x, \quad a = 0$ | (h) $f(x) = e^{x^2}, \quad a = 0$ |

- 11) For the functions of the previous question, estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $|x - a| \leq 1/2$. Use the formula

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1},$$

which holds when $|f^{(n+1)}(x)| \leq M$.

Answer key.

- 2) (a) $R = 1, [-1, 1]$ (e) $R = 2, [-2, 2]$
 (b) $R = 4, [1 - 4, 1 + 4]$ (f) $R = 7, (-7, 7]$
 (c) $R = 1, [2 - 1, 2 + 1]$ (g) $R = +\infty, (-\infty, \infty)$
 (d) $R = +\infty, (-\infty, \infty)$ (h) $R = 1/6, (1/2 - 1/6, 1/2 + 1/6)$

- 4) (a) $f(x) = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$
 (b) $f(x) = \sum_{n=0}^{\infty} 4(-2)^n x^n, \quad |x| < 1/2$
 (c) $f(x) = \sum_{n=0}^{\infty} x^{n+1}, \quad |x| < 1$
 (d) $f(x) = \sum_{n=0}^{\infty} -3^n x^{2n+1}, \quad |x| < 1/\sqrt{3}$
 (e) $f(x) = \sum_{n=0}^{\infty} (-1)^n (x+1)^{2n}, \quad |x+1| < 1$
 (f) $f(x) = \sum_{n=0}^{\infty} 4(-1)^n (x+3)^{2n+1} - \sum_{n=0}^{\infty} 11(-1)^n (x+3)^{2n}, \quad |x+3| < 1$
 (g) $f(x) = \sum_{n=0}^{\infty} -2(x-2)^{2n+1}, \quad |x-2| < 1$
 (h) $f(x) = \sum_{n=0}^{\infty} -8 \cdot 4^n (x+3/2)^{2n+1}, \quad |x+3/2| < 1/2$
 (i) $f(x) = \sum_{n=0}^{\infty} \left(\frac{(-1)^{n+1}}{3^{n+1}} - \frac{1}{2^{2n+1}} \right) x^{2n} + \sum_{n=0}^{\infty} \left(\frac{2(-1)^n}{3^{n+1}} - \frac{1}{2^{2n+2}} \right) x^{2n+1}, \quad |x| < 1/2$
 (j) $f(x) = \frac{3x^2 + 2x + 1}{(x+1)(x^2 + x + 2)}.$
 (k)

$$f(x) = \sum_{n=0}^{\infty} \left(2^{2n+1} + 2(-1)^{n+1} \left(\frac{4}{7} \right)^{n+1} \right) (x+1/2)^{2n} \\ + \sum_{n=0}^{\infty} \left(-2^{2n+2} + 2(-1)^n \left(\frac{4}{7} \right)^{n+1} \right) (x+1/2)^{2n+1}, \quad |x+1/2| < 1/2.$$

- 6) (a) $f(x) = \sum_{n=0}^{\infty} (n+1)x^n$ (d) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$
 (b) $f(x) = x^2 + 6x + 9$ (e) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$
 (c) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ (f) $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+3}}{n+1}$

$$\begin{aligned}
\text{(g)} \quad f(x) &= \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} & \text{(j)} \quad f(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n} x^{2n+1}}{(2n)!} \\
\text{(h)} \quad f(x) &= \sum_{n=0}^{\infty} (-1)^n \binom{1/4}{n} x^n & \text{(k)} \quad f(x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{(2n)!} \\
\text{(i)} \quad f(x) &= \sum_{n=0}^{\infty} 2^{-n-2/3} \binom{-2/3}{n} x^n & \text{(l)} \quad f(x) &= \sum_{n=0}^{\infty} 2^{-n-1/2} \binom{-1/2}{n} x^{n+2}.
\end{aligned}$$

$$\begin{aligned}
7) \quad \text{(a)} \quad f(x) &= 29 + 29(x-2) + 10(x-2)^2 + (x-2)^3 \\
\text{(b)} \quad f(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \\
\text{(c)} \quad f(x) &= \sum_{n=0}^{\infty} \frac{e^6 3^n (x-2)^n}{n!} \\
\text{(d)} \quad f(x) &= \sum_{n=0}^{\infty} 3^{1-2n} \binom{1/2}{n} (x-9)^n \\
\text{(e)} \quad f(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{2}} \frac{(x-\pi/4)^{2n}}{(2n)!} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{2}} \frac{(x-\pi/4)^{2n+1}}{(2n+1)!} \\
\text{(f)} \quad f(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{2}} \frac{(x-\pi/6)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{3}{2}} \frac{(x-\pi/6)^{2n+1}}{(2n+1)!}.
\end{aligned}$$

$$\begin{aligned}
8) \quad \text{(a)} \quad 1/2 & & \text{(c)} \quad -1 \\
\text{(b)} \quad -1/8 & & \text{(d)} \quad 1/120.
\end{aligned}$$

$$\begin{aligned}
10) \quad \text{(a)} \quad T_3(x) &= e + e(x-1) + \frac{e(x-2)^2}{2} + \frac{e(x-1)^3}{6} \\
\text{(b)} \quad T_3(x) &= \frac{1}{2} + \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{6}\right) - \frac{\left(x - \frac{\pi}{6}\right)^2}{4} - \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{6}\right)^3 \\
\text{(c)} \quad T_3(x) &= -2 \left(x - \frac{\pi}{4}\right) + \frac{4 \left(x - \frac{\pi}{4}\right)^3}{3} \\
\text{(d)} \quad T_3(x) &= x + x^2 + \frac{x^3}{3} \\
\text{(e)} \quad T_3(x) &= x^3 \\
\text{(f)} \quad T_3(x) &= x^2 \\
\text{(g)} \quad T_3(x) &= 2 + \frac{x-4}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} \\
\text{(h)} \quad T_3(x) &= 1 + x^2.
\end{aligned}$$

11) In what follows we estimate the remainder $R_n(x) = f(x) - T_n(x)$. We used the assumption that $|x - a| \leq 1/2$.

$$\begin{aligned}
\text{(a)} \quad |R_3(x)| &\leq \frac{e^{3/2}}{4!} \left(\frac{1}{2}\right)^4 \text{ in view of } |f^{(4)}(x)| = |e^x| \leq e^{3/2} \text{ for } |x-1| \leq 1/2. \\
\text{(b)} \quad |R_3(x)| &\leq \frac{1}{4!} \left(\frac{1}{2}\right)^4 \text{ in view of } |f^{(4)}(x)| = |\sin(x)| \leq 1 \text{ for } |x - \pi/6| \leq 1/2.
\end{aligned}$$

- (c) $|R_3(x)| \leq \frac{2^4}{4!} \left(\frac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |2^4 \cos(x)| \leq 2^4$ for $|x - \pi/4| \leq 1/2$.
- (d) $|R_3(x)| \leq \frac{4e^{1/2}}{4!} \left(\frac{1}{2}\right)^4$ in view of $|f^{(4)}(x)| = |-4e^x \sin x| \leq 4e^{1/2}$ for $|x| \leq 1/2$.
- (e) $|R_3(x)| \leq \frac{136}{4!} \left(\frac{1}{2}\right)^4$ in view of

$$\begin{aligned}
 |f^{(4)}(x)| &= \left| -\frac{6x^2}{(x+1)^4} + \frac{16x}{(x+1)^3} - \frac{12}{(x+1)^2} \right| \\
 &\leq \left| \frac{6x^2}{(x+1)^4} \right| + \left| \frac{16x}{(x+1)^3} \right| + \left| \frac{12}{(x+1)^2} \right| \\
 &\leq \left| \frac{6 \cdot (1/2)^2}{(-1/2+1)^4} \right| + \left| \frac{16 \cdot 1/2}{(-1/2+1)^3} \right| + \left| \frac{12}{(-1/2+1)^2} \right| \\
 &= 6 \cdot 4 + 16 \cdot 4 + 12 \cdot 4 = 136 \quad \text{for } |x| \leq 1/2.
 \end{aligned}$$

Here the absolute values of the fractions in the second line are estimated as the largest value in the numerator over the smallest value in the denominator.

- (f) $R_3(x) \leq \frac{9/2}{4!} \left(\frac{1}{2}\right)^4$ in view of

$$\begin{aligned}
 |f^{(4)}(x)| &= |x \sin(x) - 4 \cos(x)| \\
 &\leq |x \sin(x)| + 4|\cos(x)| \\
 &\leq \frac{1}{2} \cdot 1 + 4 = 9/2 \quad \text{for } |x| \leq 1/2.
 \end{aligned}$$

- (g) $R_3(x) \leq \frac{15}{343\sqrt{14}} \cdot \frac{1}{4!} \left(\frac{1}{2}\right)^4$ in view of

$$|f^{(4)}(x)| = \left| -\frac{15}{16x^{7/2}} \right| \leq \frac{15}{343\sqrt{14}} \quad \text{for } |x-4| \leq 1/2.$$

We used here that the absolute value of $f^{(4)}$ is decreasing, and so the maximum is attained at the left endpoint of $|x-4| \leq 1/2$.

- (h) $R_3(x) \leq \frac{25e^{1/4}}{4!} \left(\frac{1}{2}\right)^4$ in view of

$$\begin{aligned}
 |f^{(4)}(x)| &= \left| 48e^{x^2}x^2 + 12e^{x^2} + 16e^{x^2}x^4 \right| \\
 &\leq 48e^{1/2}(1/2)^2 + 12e^{(1/2)^2} + 16e^{(1/2)^2}(1/2)^4 \\
 &= 25e^{1/4} \quad \text{for } |x| \leq 1/2.
 \end{aligned}$$