REVIEW QUESTIONS FOR TEST 2

Arc length

1) Write down the arc length formula, for integration both in terms of x and y. How are the limits of integration determined in it? Which expression can be interpreted as the element of arc length ds?
2) Find the length of the following curves:
   (a) $36y^2 = (x^2 - 4)^3, \quad 2 \le x \le 3, \quad y \ge 0$
   (b) $x = \dfrac{y^5}{10} + \dfrac{1}{6y^3}, \quad 1 \le y \le 3$
   (c) $y = \ln(\cos x), \quad 0 \le x \le \pi/6$
   (d) $y = \dfrac{1}{2}(e^x + e^{-x}), \quad 0 \le x \le 1$
   (e) $y = \sqrt{x - x^2} + \arcsin(\sqrt{x}), \quad 0 \le x \le 1$
   (f) $x = \ln(1 - y^2), \quad 0 \le y \le \dfrac{1}{2}$.

Areas of surfaces of revolution

3) Write down the formulas for the area of a surface of revolution. What changes when the axis of rotation is the y-axis? Write down the formula for ds (element of arc length) for integration in x. The same for integration in y.
4) Find area of the surface obtained by rotating the following curves about the x-axis:
   (a) $y = \cos(3x), \quad 0 \le x \le \pi/6$
   (b) $x = \dfrac{1}{3}(y^2 + 2)^{3/2}, \quad 1 \le y \le 2$
   (c) $y = \sqrt{1 + e^x}, \quad 0 \le x \le 1$.
5) Find area of the surface obtained by rotating the following curves about the y-axis:
   (a) $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x, \quad 1 \le x \le 2$
   (b) $x^{2/3} + y^{2/3} = 1, \quad 0 \le y \le 1$
   (c) $y = \dfrac{1}{3}x^{3/2}, \quad 0 \le x \le 6$.

Applications to physics

6) Write down the formula for hydrostatic pressure at the depth d. Write down formulas for the coordinates of a centroid of a flat region confined between two curves; assume that the density is constant. What changes when the region lies between a curve and the x-axis? How are the limits of integration determined in the relevant integrals? What can be said about the location of a centroid of a convex shape? A shape that has an axis of symmetry?
7) See # 5–8, 10–12, and # 15 in Section 8.3.
   (a) A vertical plate is submerged in water, as shown in Figures 1–3. Explain how to approximate the hydrostatic force acting on one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.

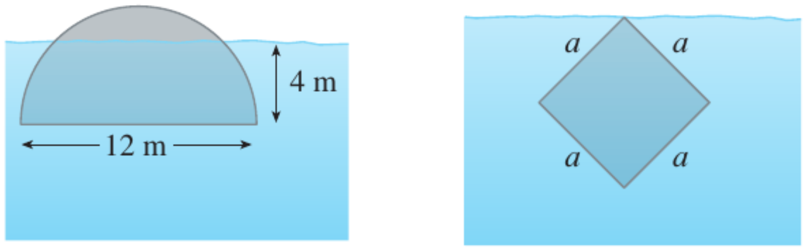

FIGURE 1

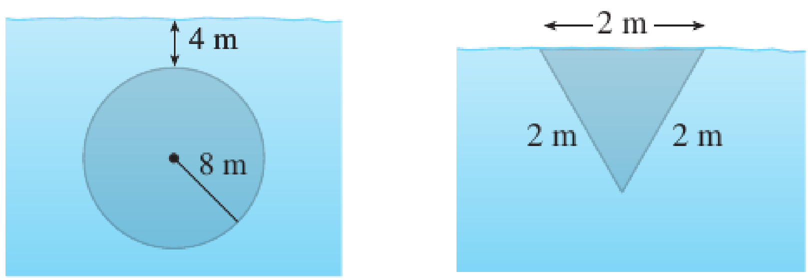

FIGURE 2

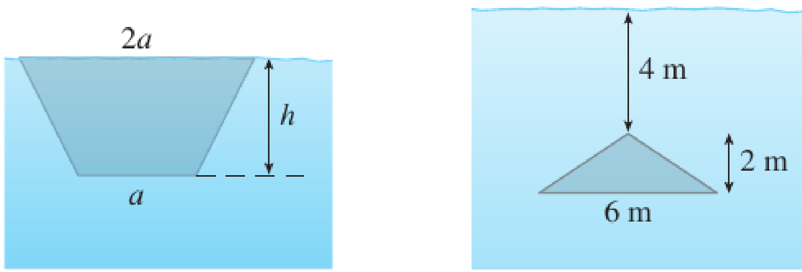

FIGURE 3

(b) A cube with 20-cm-long sides is sitting on the bottom of an aquarium in which the water is one meter deep. Find the hydrostatic force on (i) the top of the cube and (ii) one of the sides of the cube.

8) Find centroids of the regions bounded by the given curves:
   (a) $y = \sin x,\ \ y = 0,\ \ 0 \le x \le \pi$
   (b) $y = 2 - x^2,\ \ y = x^2$
   (c) $y = \sin x,\ \ y = \cos x,\ \ x = 0,\ \ x = \pi/4$
   (d) $x + y = 2,\ \ x = y^2$
   (e) $y = x^3,\ \ y = x^5,\ x \ge 0$
   (f) $y = (e^x + e^{-x})/2,\ y = 4$
   (g) $x = 0,\ y = 1,\ y = 1/\sqrt[3]{x}$.

Separable differential equations

9) Explain what is a separable equation. Write down the general algorithm that should be applied to separable differential equations. In which case is it possible to express y as

an explicit function of x, assuming the equation contains dy/dx? What determines the value of the constant of integration in a particular solution?

What is an orthogonal trajectory? Why are orthogonal trajectories found using a differential equation? Write down the algorithm for finding the equation of orthogonal trajectories to a given family of curves.

10) Solve the differential equations:

(a) $y' = x\sqrt{y}$
(b) $y' + xe^x = 0$
(c) $\dfrac{d\theta}{dt} = \dfrac{t\sec\theta}{\theta e^{t^2}}$
(d) $\dfrac{dz}{dt} + e^{t+z} = 0$
(e) $y' = \sin^3 x$
(f) $y' = x^2 e^x$
(g) $y' = \ln x + 1$
(h) $(x+2)e^y + y\sqrt{x+1}\,\dfrac{dy}{dx} = 0$
(i) $y' = \dfrac{e^x}{x}$
(j) $\dfrac{1}{\sqrt{1-x^2}} + \dfrac{dy}{dx}\cdot\dfrac{1}{\sqrt{1-y^2}} = 0$
(k) $y' = \dfrac{\sqrt{y}}{\sqrt{x}}$.

11) Solve the differential equations and find solutions satisfying the given initial condition:
(a) $y' = 2\sqrt{y}, \quad y(-1) = 1$
(b) $y' = 2x\sqrt{1-y^2}, \quad y(0) = 0$
(c) $\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$
(d) $x\ln x = y(1+\sqrt{3+y^2})y', \quad y(1) = 1$
(e) $y'\cot x + y = 2, \quad y(\pi/3) = 0.$

12) Find orthogonal trajectories of the given family of curves, indexed by the constant k:
(a) $x^2 + 3y^2 = k^2$
(b) $y = \dfrac{k}{x}$.

For the following families of curves, find the equation of the orthogonal trajectory passing through the point $(1, 2)$.
(c) $y^2 = kx^3$.
(d) $y = \dfrac{1}{x+k}$.

13) See # 46, 47 in Section 9.3:
(a) The air in a room with volume 180 m^3 contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m^3/min and the mixed air flows out at the same rate. Find the percentage of carbon dioxide in the room as a function of time. What happens in the long run?
(b) A vat with 500 gallons of beer contains 4% alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate of 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

14) Using the Euler's method, estimate the value of the function satisfying the indicated initial condition and the given differential equation. If possible, find the precise solution of the equation and compare its value to the approximation.
(a) Use step size $\Delta x = 0.1$ to estimate $y(0.5)$ assuming the function y satisfies
$y' = y + xy, \quad y(0) = 1.$
(b) Use step size $\Delta x = 0.2$ to estimate $y(0.6)$ when y satisfies
$y' = \sin(x+y), \quad y(0) = 0.$

Approximate methods for integration and differential equations

15) What are the three rules for approximate integration we discussed? Write down formulas for M_n, T_n, and S_n. Which rule is the most precise, when using the same number of intervals n? (let n be even for simplicity, as required in S_n) Formulate the main steps of Euler's method. How can one increase precision of the approximate solution resulting from this method?
16) Write down approximations for $n = 4$, using all the three rules: M_4, T_4, S_4, for the function $f(x) = e^{-x^2}$ on the interval $[a, b] = [0, 8]$. There is no need to evaluate or simplify any of the sums.
$$\int_0^8 e^{-x^2}\,dx \approx \{M_4, T_4, S_4\}\,.$$
17) For the same function $f(x) = e^{-x^2}$, determine the number of subintervals n necessary to achieve the accuracy of 10^{-5}, using the three rules M_n, T_n, S_n (for the Simpson's rule keep in mind that n must be even).

Parametric curves

18) What is a parametric curve? How and when can one obtain the Cartesian equation of a parametric curve? How to compute the slope of the tangent to a parametric curve? Write down the expressions for the area under a parametric curve, and the arc length of a parametric curve. What is the equation for the area of the surface obtained by rotation of a parametric curve?
19) Eliminate the parameter from the given parametric curve and sketch its graph. Show the direction of motion as the parameter increases.
    (a) $x = \sin t,\ y = \csc t, \quad 0 < t < \pi/2$
    (b) $x = e^t,\ y = e^{-2t}, \quad -\infty < t < \infty.$
20) Find the equation of the tangents to the following curves for the given value of the parameter t or at the given point (x, y):
    (a) $x = t^3 + 1,\ y = t^4 + t, \quad t = -1$
    (b) $x = t\cos t,\ y = t\sin t, \quad t = \pi$
    (c) $x = 1 + \ln t,\ y = t^2 + 2, \quad (1, 3)$
    (d) $x = t^2 - t,\ y = t^2 + t + 1, \quad (0, 3).$
21) Find the length of the given curve:
    (a) $x = 1 + 3t^2,\ y = 4 + 2t^3, \quad 0 \le t \le 1$
    (b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 2$
    (c) $x = t\sin t,\ y = t\cos t, \quad 0 \le t \le 1.$
22) Find the area of surfaces obtained by rotating the following curves about the x-axis:
    (a) $x = t^3,\ y = t^2, \quad 0 \le t \le 1$
    (b) $x = 2\cos^3 t,\ y = 2\sin^3 t, \quad 0 \le t \le \pi/2.$
23) Find the area of surfaces obtained by rotating the following curves about the y-axis:
    (a) $x = 8\sqrt{t},\ y = 2t^2 + 1/t, \quad 1 \le t \le 3$
    (b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 1.$

Answer key

2) (a) $\dfrac{13}{6}$
   (b) $\dfrac{9866}{405}$
   (c) $\dfrac{1}{2}\ln 3$
   (d) $\dfrac{e-e^{-1}}{2}$
   (e) 2
   (f) $-\log(3)+\dfrac{1}{2}$.

4) (a) $\dfrac{\pi}{3}\sqrt{10}+\dfrac{\pi}{9}\ln\left(3+\sqrt{3^2+1}\right)$
   (b) $\dfrac{21}{2}\pi$
   (c) $\pi(e+1)$.

5) (a) $\dfrac{10\pi}{3}$
   (b) $\dfrac{6\pi}{5}$
   (c) $\dfrac{8}{15}\pi\left(25\sqrt{10}+16\right)$.

7) (a) Hydrostatic forces acting on the plates in Figure 1: $\rho g\left(144\arcsin\dfrac{2}{3}+\dfrac{176}{3}\sqrt{5}-144\right)$ and $\dfrac{\rho g\, a^3}{\sqrt{2}}$, respectively.
   (b) (i) 313.6 N; (ii) 352.8 N.

8) (a) $\left(\dfrac{\pi}{2},\dfrac{\pi}{8}\right)$
   (b) $(0,1)$
   (c) $\left(\dfrac{\sqrt{2}\pi-4}{4\left(\sqrt{2}-1\right)},\dfrac{1}{4\left(\sqrt{2}-1\right)}\right)$
   (d) $(8/5,-1/2)$.

10) (a) $y(x)=\left(\dfrac{1}{4}x^2+C\right)^2$
   (b) $y(x)=-(x-1)e^x+C$
   (c) $\theta(t)\sin(\theta(t))+\cos(\theta(t))=C-\dfrac{1}{2}e^{-t^2}$
   (d) $z(t)=-\ln\left(e^t+C\right)$
   (e) $y(x)=\dfrac{\cos(x)^3}{3}-\cos(x)+C$
   (f) $y(x)=\left(x^2-2x+2\right)e^x+C$
   (g) $y(x)=x\log(x)+C$
   (h) $(y(x)+1)e^{-y(x)}=\dfrac{2}{3}(x+4)\sqrt{x+1}+C$

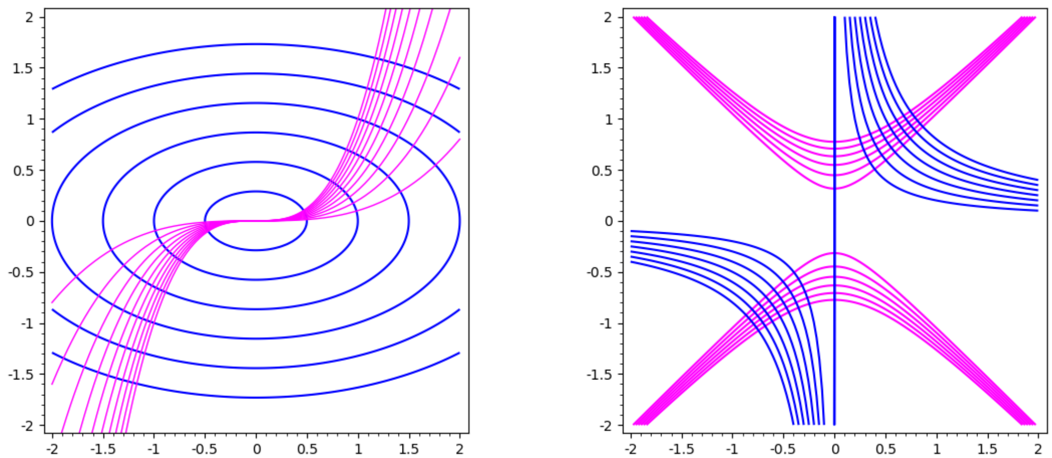

FIGURE 4. Orthogonal trajectories in questions 12) (a)–(b), in magenta; the original families are shown in blue.

(i) $y(x) = \int_0^x \frac{e^t}{t}\, dt$

This $y(x)$ cannot be expressed through elementary functions, much like $\int e^{-x^2}\, dx$.

(j) $y(x) = \sin(-\arcsin(x) + C)$

(k) $y(x) = (\sqrt{x} + C)^2$.

11) (a) $y(x) = (x+2)^2$

(b) $y(x) = \sin(x^2)$

(c) $P(t) = \left(\frac{t^{3/2}}{3} + \sqrt{2} - \frac{1}{3}\right)^2$

(d) $\frac{1}{2}\, y(x)^2 + \frac{1}{3}\left(y(x)^2 + 3\right)^{\frac{3}{2}} = \frac{1}{2}\, x^2 \ln(x) - \frac{1}{4}\, x^2 + \frac{41}{12}$

(e) $y(x) = -2(2\cos(x) - 1)$.

12) (a) $y(x) = Cx^3$

(b) $\frac{1}{2}\, y(x)^2 = \frac{1}{2}\, x^2 + C$.

(c) $x^2 + \frac{3}{2} y^2 = 7$

(d) $x = \frac{y^3}{3} - \frac{8}{3} + 1$

13) (a) Volume of carbon dioxide in the room at time t:
$v(t) = 0.09 + 0.18e^{-t/90}$ m^3,
fraction of the total volume: $v(t)/180$. After a long time the volume of CO_2 is leveled out at 0.09m^3.

(b) Volume of the alcohol contained in the vat at time t:
$v(t) = 30 - 10e^{-t/100}$ gal
fraction of the total volume: $v(t)/500$. After an hour the fraction of alcohol is $v(60)/500 \approx 0.049$.

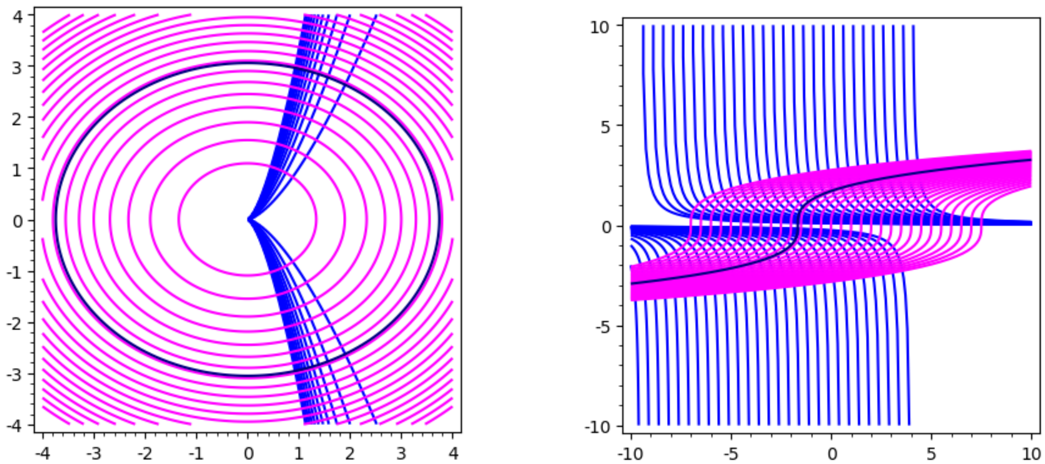

Figure 5. Orthogonal trajectories in questions 12) (c)–(d), in magenta; the original families are shown in blue. The trajectory passing through $(1, 2)$ is shown in dark blue.

16) $M_4 = 2\left(e^{-1^2} + e^{-3^2} + e^{-5^2} + e^{-7^2}\right)$

$T_4 = 1\left(e^{-0^2} + 2e^{-2^2} + 2e^{-4^2} + 2e^{-6^2} + e^{-8^2}\right)$

$S_4 = \dfrac{2}{3}\left(e^{-0^2} + 4e^{-2^2} + 2e^{-4^2} + 4e^{-6^2} + e^{-8^2}\right)$

19) (a) $y = 1/x, \quad 0 < x < 1$

(b) $y = 1/x^2, \quad x > 0.$

20) (a) $y = -x$

(b) $y = \pi(x + \pi)$

(c) $y - 3 = 2(x - 1)$

(d) $y - 3 = 3x.$

21) (a) $4\sqrt{2} - 2$

(b) $e^2 + 1$

(c) $\dfrac{1}{2}\sqrt{2} + \dfrac{1}{2}\ln(1 + \sqrt{2}).$

22) (a) $\dfrac{2}{1215}\pi\left(247\sqrt{13} + 64\right)$

(b) $\dfrac{24}{5}\pi.$

23) (a) $\dfrac{32}{15}\pi\left(103\sqrt{3} + 3\right)$

(b) $\pi\left((e + 1)^2 - 7\right).$