Section 11.6: Power series

Def: a power series is:

$$\sum_{k=0}^{\infty} \underbrace{c_k}_{\text{coefficients}} \cdot \overset{\text{variable}}{x^k} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Power series is a function!

$$\underset{\text{input}}{\xrightarrow{x}} \overset{\text{function}}{\boxed{\text{power series}}} \underset{\text{output}}{\xrightarrow{\text{number series}}}$$

Ex. 1 Let $c_k = 1$, $k \geq 0$. Then:

$$\underbrace{\sum_{k=0}^{\infty} x^k}_{\substack{\text{geometric series:} \\ r = x \\ a = 1}} = 1 + x + \underbrace{x^2}_{\frac{x^2}{x} = x} + \underbrace{x^3}_{\frac{x^3}{x^2} = x} + \dots$$

This power series converges if and only if $|r| = |x| < 1$.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad |x| < 1.$$

Def

$$\sum_{k=0}^{\infty} c_k (x-a)^k =$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

— power series, centered at a.

Ex. 2 Let $c_k = 0$, $k \geq 3$;

$c_0 = 5$

$c_1 = 0$

$c_2 = 4$

Consider a power series, centered at $a = 2$, with these c_k.

$$\sum_{k=0}^{\infty} c_k (x-2)^k = c_0 + c_1(x-2) + c_2(x-2)^2$$

$$= 5 + \qquad + 4(x-2)^2$$

$$= 4(x-2)^2 + 5$$

Power series $\simeq$ infinite polynomials.

Ex. 3 For which x is

$$\sum_{k=1}^{\infty} \underbrace{\frac{(x-3)^k}{k}}_{} = a_k \quad - \text{convergent?}$$

$a = 3$

$c_k = \frac{1}{k} \quad k \geq 1$

$c_0 = 0$

Apply the ratio test: $\left(a_k = \frac{(x-3)^k}{k}\right)$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left| \underbrace{\frac{(x-3)^{k+1}}{k+1}}_{a_{k+1}} \Big/ \underbrace{\frac{(x-3)^k}{k}}_{a_k} \right|$$

$$\lim_{k\to\infty} \left| \frac{\cancel{(x-3)}^{k+1}}{k+1} \cdot \frac{k}{\cancel{(x-3)^k}} \right| =$$

$$\lim_{k\to\infty} \left| (x-3) \cdot \underbrace{\frac{k}{k+1}}_{} \right| =$$

$$\lim_{k\to\infty} \frac{k}{k+1} \cdot |x-3| \overset{>0}{=} |x-3| \cdot \lim_{k\to\infty} \underbrace{\frac{k}{k+1}}_{\to 1}$$

$$= |x-3|.$$

The given series is absolutely convergent, when $|x-3| < 1$.

It is divergent when $|x-3| > 1$.

Finally, when $|x-3| = 1 \Rightarrow$ the ratio test is inconclusive.

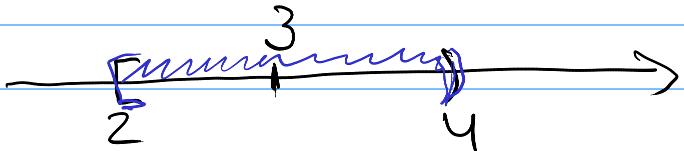
When $|x-3|=1$: $x=4$, or $x=2$.

For $x=4$:

$$\sum_{k=1}^{\infty} \frac{(x-3)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \Bigg\}$$ divergent: as a p-series, $p=1$.

For $x=2$:

$$\sum_{k=1}^{\infty} \frac{(x-3)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \Bigg\}$$ convergent by the alt series test.



Answer: this series is convergent for x in $\underbrace{[2, 4)}_{\text{domain of the function, given by the power series}}$.

Ex. 4 For which x is $\sum_{k=0}^{\infty} \underbrace{k! \cdot x^k}_{=a_k}$ — convergent?

$0! = 1$

Apply the ratio test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)! \cdot \cancel{x^{k}}^{k+1}}{k! \cdot \cancel{x^k}}\right| =$$

$$\lim_{k\to\infty} \left|\frac{(k+1)! \cdot x}{k!}\right| = \lim_{k\to\infty} |x| \cdot \frac{(k+1)!}{k!}$$

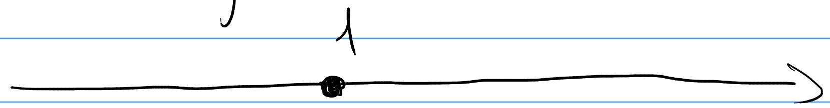
$$\lim_{k\to\infty} |x| \cdot \frac{\cancel{1\cdot 2\cdot 3\cdot \ldots \cdot k}\,(k+1)}{\cancel{1\cdot 2\cdot 3\cdot \ldots \cdot k}} =$$

$$\lim_{k\to\infty} |x|\cdot (k+1) = |x| \cdot \lim_{k\to\infty} k+1$$

$$= |x| \cdot (+\infty)$$

$$= \underbrace{+\infty}_{>1}, \quad \text{if } |x| \neq 0$$

Divergent for $x \neq 0$!
Convergent for $x = 0$ only

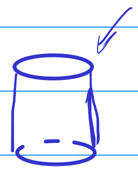


↳ domain of $\sum_{k=0}^{\infty} k! \cdot x^k$

Ex. 5 Find the domain of the J_0 Bessel function, given by:

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2}$$

describes the shape of the vibrating drum membrane



That is, we have to find x for which the series of J_0 converges.

Apply the Ratio test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} \dfrac{x^{2(k+1)}}{2^{2(k+1)}((k+1)!)^2}}{(-1)^k \dfrac{x^{2k}}{2^{2k}(k!)^2}} \right|$$

$$\lim_{k\to\infty}\left|\frac{\cancel{x^{2k}}x^{2}}{\cancel{2^{2k}}2^{2}((k+1)!)^2}\Big/\frac{\cancel{x^{2k}}}{\cancel{2^{2k}}\cdot(k!)^2}\right| =$$

$$\lim_{k\to\infty}\left|\frac{x^2}{4(\underbrace{(k+1)k!}_{(k+1)!=k!(k+1)})^2}\cdot\frac{(k!)^2}{1}\right| =$$

$(ab)^2 = a^2b^2$

$$\lim_{k\to\infty}\left|\frac{x^2}{4}\cdot\frac{\cancel{(k!)^2}}{\cancel{(k!)^2}\cdot(k+1)^2}\right| =$$

$$= \lim_{k\to\infty}\left|\frac{x^2}{4(k+1)^2}\right| = \lim_{k\to\infty}\frac{x^2}{4(k+1)^2} =$$

$$\frac{x^2}{4}\cdot\underbrace{\lim_{k\to\infty}\frac{1}{(k+1)^2}}_{=0} = \frac{x^2}{4}\cdot 0 = \underbrace{0 < 1}_{\text{independent of } x}$$

By the Ratio test, the series for J_0 is absolutely convergent for all x $\Rightarrow$ domain of J_0 is $(-\infty,\infty)$.

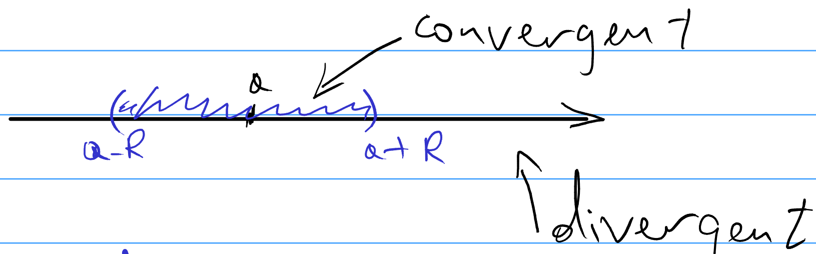
domain of J_0.

Thm: For any power series $\sum_{k=0}^{\infty} c_k (x-a)^k$, there are only the following possibilities for convergence:

i). The series converges only for $x = a$

ii). The series converges for $|x-a| < R$
diverges for $|x-a| > R$,
for a positive R

iii). The series converges for all x in $(-\infty, \infty)$.



R = the radius of convergence

In i)., $R = 0$
iii)., $R = +\infty$.

The set of x, on which the series is convergent = the interval of convergence.

Series	Radius of convergence	Interval of conv.
$\sum_{k=0}^{\infty} x^k$	$R = 1$	$\lvert x\rvert < 1$ $(-1, 1)$
$\sum_{k=0}^{\infty} k! \cdot x^k$	$R = 0$	$x = 0$
$\sum_{k=1}^{\infty} \frac{(x-3)^k}{k}$	$R = 1$	$[2, 4)$
J_0	$R = +\infty$	$(-\infty, \infty)$

To determine the interval of convergence:

- apply the ratio test to find R
- inspect the endpoints of the convergence interval.

Ex. 6 Find radius/interval of convergence:

$$\sum_{k=0}^{\infty} \frac{(-3)^k \cdot x^k}{\sqrt{k+1}}$$

Ratio test:

$$\lim_{k\to\infty} \left| \underbrace{\frac{(-3)^{k+1} \cdot x^{k+1}}{\sqrt{(k+1)+1}}}_{a_{k+1}} \Big/ \underbrace{\frac{(-3)^k \cdot x^k}{\sqrt{k+1}}}_{a_k} \right| =$$

$= (-1)^k \cdot 3^k$

$$\lim_{k\to\infty} \left| \frac{\cancel{3^{k+1}} \cdot \cancel{x^{k+1}}}{\sqrt{k+2}} \cdot \frac{\sqrt{k+1}}{\cancel{3^k} \cdot \cancel{x^k}} \right| =$$

$$\lim_{k\to\infty} \left| \frac{3x}{\sqrt{k+2}} \cdot \sqrt{k+1} \right| = \lim_{k\to\infty} |3x| \cdot \frac{\sqrt{k+1}}{\sqrt{k+2}}$$

$$= |3x| \cdot \lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k+2}}$$

$$\sqrt{\frac{k+1}{k+2}} \to 1$$

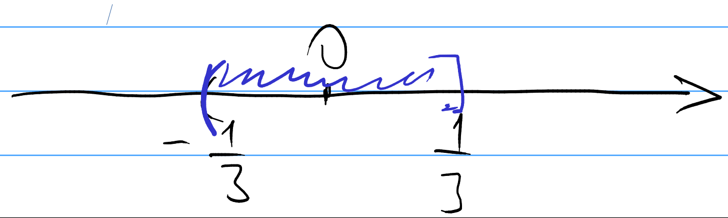
$$= |3x|.$$

$$|3x| < 1 \qquad /3$$

$$|x| < \frac{1}{3}$$

$$|x - 0| < \frac{1}{3} \qquad |x - a| < R$$

$$R = \frac{1}{3}.$$



At the endpoints:

$x = \frac{1}{3}$

$$\sum_{k=0}^{\infty} \frac{(-3)^k \cdot x^k}{\sqrt{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \cancel{3^k} \cdot \cancel{\left(\frac{1}{3}\right)^k}}{\sqrt{k+1}}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{\sqrt{k+1}}$$ } convergent by the alt. series test.

$x = -\frac{1}{3}$

$$\left(\left(-1\right)^2\right)^k = 1$$

$$\sum_{k=0}^{\infty} \frac{(-3)^k \cdot x^k}{\sqrt{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \cancel{3^k} \cdot (-1)^k \cdot \cancel{\left(\frac{1}{3}\right)^k}}{\sqrt{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}}$$ } divergent - by the lim comparison test, $b_k = \frac{1}{\sqrt{k}}$, $\sum \frac{1}{\sqrt{k}}$ divergent as a p-series.

A: $\left(-\frac{1}{3}; \frac{1}{3}\right]$

$$R = \frac{1}{3}.$$