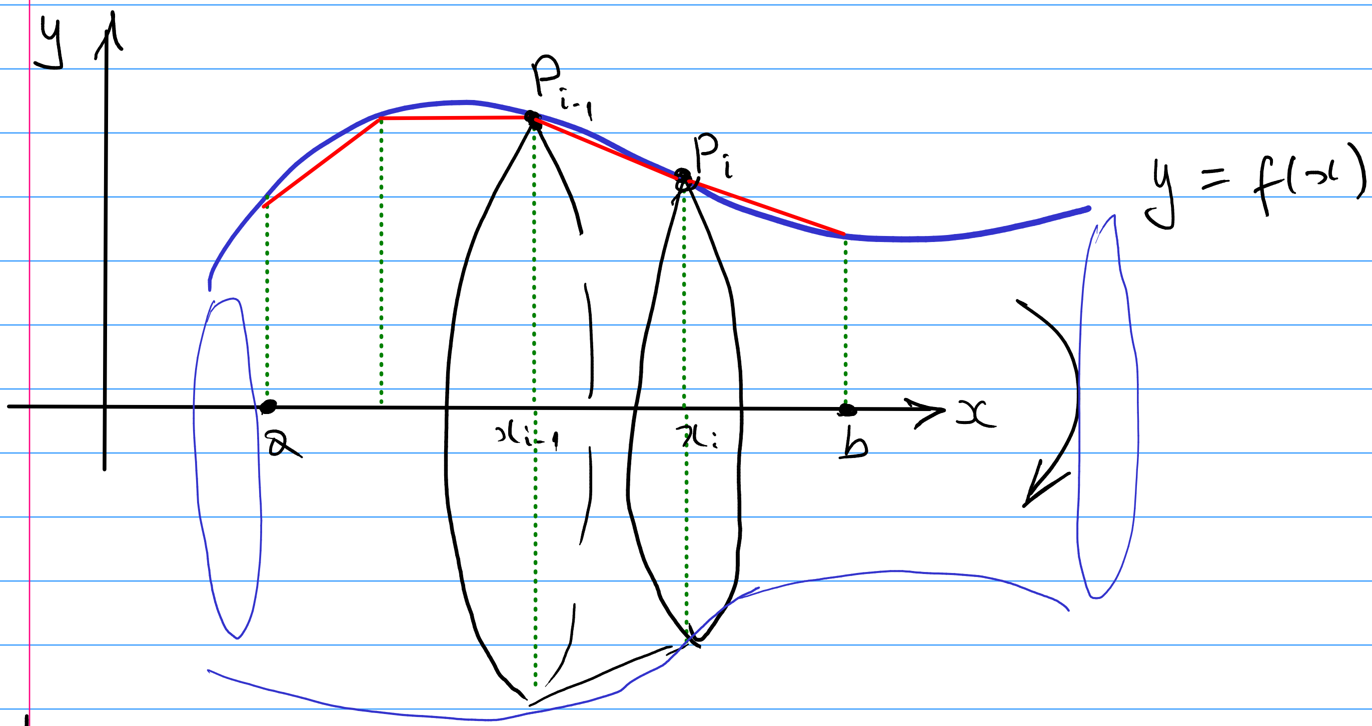
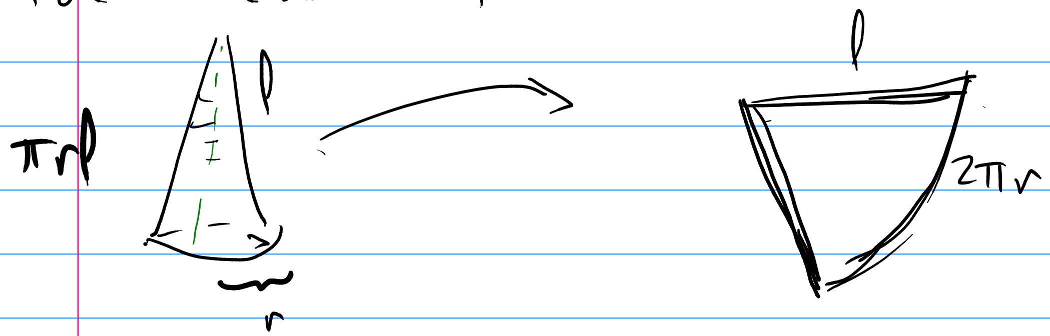
# Section 8.2: Areas of surfaces of revolution

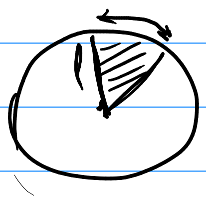



Side area of a cone



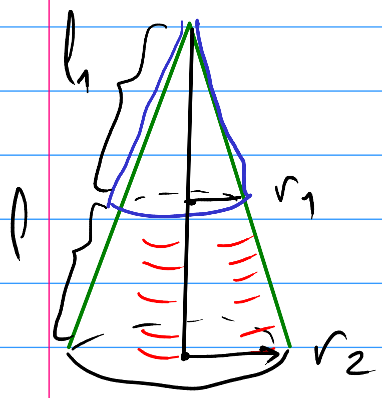

Area $\dfrac{2\pi r}{2} \cdot \ell =$
$= \pi r \ell$

Area of a sector of the unit disk is proportional to the arc:




$$A_{sector} = \frac{arc}{2\pi\ell} \cdot \pi \ell^2$$
$$= \frac{arc}{2}\ell$$

Side area of a conic prism




$$A = \pi r_2 (l + l_1) - \pi r_1 l_1$$
$$= \pi ( r_2 l + \underbrace{(r_2 - r_1) l_1} )$$
$$= \pi ( r_2 l + r_1 l )$$
$$= \boxed{2 \pi r l} \qquad r = \frac{r_1 + r_2}{2}$$

To compute $l_1$, use similar triangles:

$$\frac{l_1}{l_1 + l} = \frac{r_1}{r_2} \Rightarrow l_1 r_2 = l_1 r_1 + l r_1$$
$$\underbrace{l_1 (r_2 - r_1)} = l r_1$$

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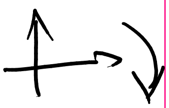
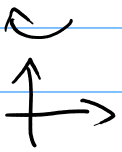
$$S = \lim_{n \to \infty} \sum_{i=1}^{n} \text{Area of the i-th prism}$$
$$= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \cdot r_i l_i$$
$$= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \underbrace{|P_{i-1} \cdot P_i|}$$
$$= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \cdot \underbrace{\sqrt{1 + f'(x_i^*)^2}\, \Delta x}_{\text{from arc length}}$$

$$S = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2}\, dx$$



$$S_x = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2}\, dx$$

– area of a surface of revolution, obtained by rotating $y = f(x)$ about the $x$-axis.



$$S_y = 2\pi \int_a^b x \sqrt{1 + f'(x)^2}\, dx$$

— " — about the $y$-axis.

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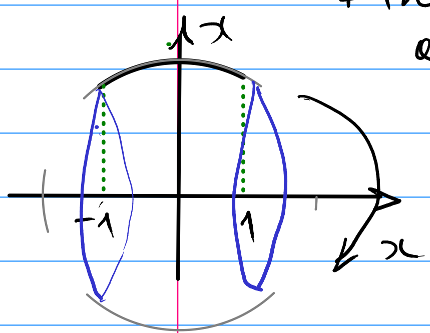
Ex. 1 $\quad y = \sqrt{4 - x^2}$, $\quad -1 \leq x \leq 1$
is an arc of $x^2 + y^2 = 4$
Find the area after rotating about $x$-axis




$$y' = \frac{1}{2} \cdot (4 - x^2)^{-1/2} (-2x)$$

$$= \frac{-x}{\sqrt{4 - x^2}}$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \cdot \sqrt{1 + \left( \frac{-x}{\sqrt{4 - x^2}} \right)^2}\, dx$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4-x^2} \cdot \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2} \cdot \sqrt{\frac{4}{4-x^2}}\, dx$$

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$$= 2\pi \int_{-1}^{1} \cancel{\sqrt{4-x^2}} \cdot \frac{\sqrt{4}}{\cancel{\sqrt{4-x^2}}}\, dx$$

$$= 4\pi \int_{-1}^{1} 1\, dx = 8\pi .$$

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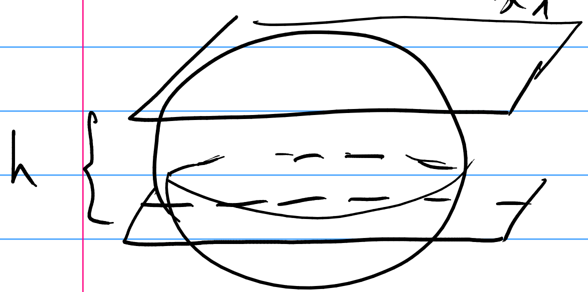
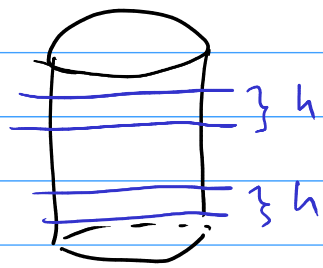
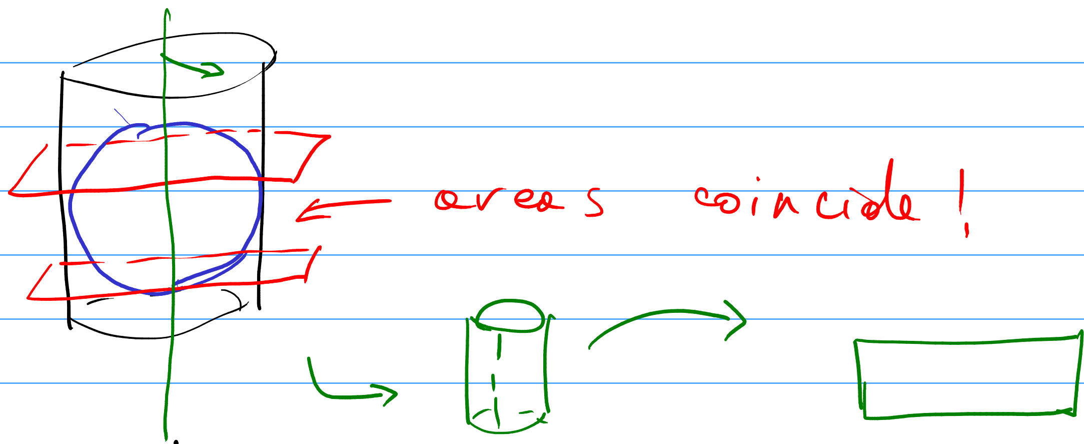
$$S = 4\pi \int_{x_1}^{x_2} dx$$



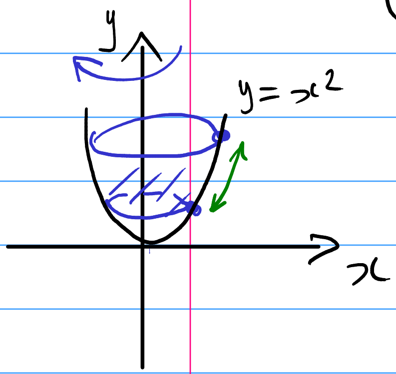




A digression: cylindrical projection.




Cylindrical projection preserves areas.

Ex. 2 The arc of $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the $y$-axis




$$S_y = \int 2\pi r\, dl$$

$$= \int 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int 2\pi\, x(y) \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

1). Integrate in $x$

$$\frac{dy}{dx} = 2x$$

$$S = \int_1^2 2\pi \cdot x \sqrt{1+(2x)^2}\, dx =$$

$$= \pi \int_1^2 \underline{2x} \sqrt{1+4x^2}\, \underline{dx} = \left| \begin{array}{l} u = 1+4x^2 \\ du = 8x\,dx \\ \frac{1}{4} du = \underline{2x\,dx} \end{array} \right.$$

$$= \pi \int_5^{17} \sqrt{u}\, \frac{1}{4}\, du = \frac{\pi}{4} \left. \frac{u^{3/2}}{3/2} \right|_5^{17}$$

$$= \frac{\pi}{2 \cdot 3} \left( 17^{3/2} - 5^{3/2} \right).$$

2). Integrate in $y$.

$$x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$S = \int_1^4 2\pi \sqrt{y} \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \int_1^4 2\pi \sqrt{y} \cdot \sqrt{1 + \frac{1}{4y}}\, dy$$

$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$

$$= \int_1^4 2\pi \sqrt{y + \frac{1}{4}}\, dy = \left| \begin{array}{l} u = y + \frac{1}{4} \\ du = dy \end{array} \right|$$

$$= \int_{5/4}^{17/4} 2\pi \cdot \sqrt{u}\, du = 2\pi \cdot \left. \frac{u^{3/2}}{3/2} \right|_{5/4}^{17/4}$$

$$= \frac{4}{3}\pi \left( \left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2} \right)$$

$$= \frac{4}{3}\pi \left( \frac{17^{3/2}}{8} - \frac{5^{3/2}}{8} \right)$$

$$= \frac{\pi}{2 \cdot 3} \left( 17^{3/2} - 5^{3/2} \right).$$

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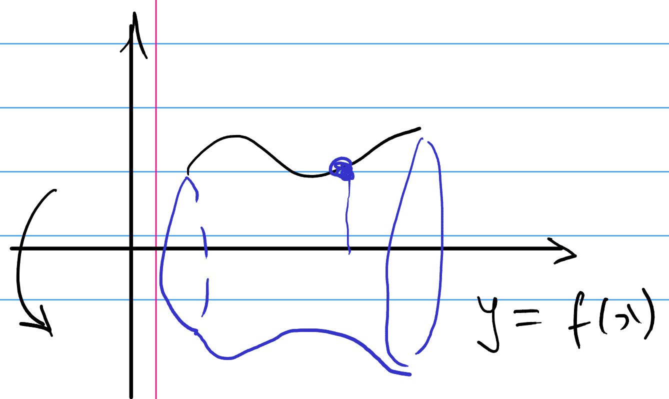
## Summary

$$S = \int 2\pi r\, dl$$

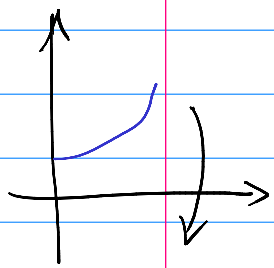



$$S_x = \int 2\pi y \sqrt{dx^2 + dy^2}$$

$$S_y = \int 2\pi x \sqrt{dx^2 + dy^2}$$

→ $y = e^x \qquad 0 \le x \le 1$, about $O_x$:

$$S_x = \int 2\pi r \, dl = \int_0^1 2\pi\, y(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



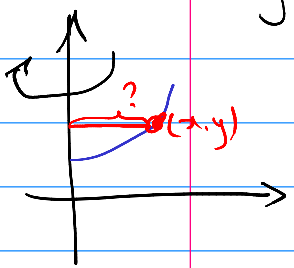
$$= \int_0^1 2\pi e^x \sqrt{1+(e^x)^2}\, dx$$

→ $x = \ln y \qquad 1 \le y \le e$ about $O_x$:

$$S_x = \int_1^e 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^e 2\pi y \sqrt{1+\left(\frac{1}{y}\right)^2}\, dy$$

→ $y = e^x \qquad 0 \le x \le 1$ about $O_y$:




$$S_y = \int 2\pi r \, dl = \int_0^1 2\pi x \sqrt{1+(e^x)^2}\, dx$$

→ $x = \ln y \qquad 1 \le y \le e$

$$S_y = \int_1^e 2\pi\, x(y) \sqrt{1+\left(\frac{1}{y}\right)^2}\, dy$$

$$= \int_1^e 2\pi \ln y \sqrt{1+\left(\frac{1}{y}\right)^2}\, dy$$