Measure and Integration I (MAA5616), Fall 2020
Homework 2, **postponed to** Tuesday, Sep. 15

1. Suppose $x_\alpha \geq 0$, $\alpha \in A$, and
$$\sum_{\alpha \in A} x_\alpha < +\infty.$$
Prove that at most countably many of x_α are strictly positive.
Note: this is the reason we only consider at most countable additivity for measures.

2. A dyadic cube in $\mathbb{R}^n$ is defined as
$$\prod_{l=1}^{n} \left[\frac{a_l}{2^k}, \frac{a_l+1}{2^k}\right), \qquad a_l, k \in \mathbb{Z}.$$
Show: the class of unions of disjoint dyadic cubes is closed under differences and finite intersections (an empty union of cubes is the empty set).
Note: another way of phrasing this is to say that dyadic cubes form a semiring.

3. Verify that the collection of countable and co-countable subsets of an uncountable X forms a σ-algebra.

We say that $\mathcal{M} \subset 2^X$ is a *monotone class*, if it is closed under monotone unions and intersections: for $A_i, B_i \in \mathcal{M}$, $i \geq 1$, such that $A_1 \subset A_2 \subset \ldots$ and $B_1 \supset B_2 \supset \ldots$,
$$\bigcup_{i \geq 1} A_i \in \mathcal{M}, \qquad \bigcap_{i \geq 1} B_i \in \mathcal{M}.$$
For any family of monotone classes, their intersection is also a monotone class. Since 2^X is trivially a monotone class, for any nonempty $\mathcal{E} \subset 2^X$ there exists a unique minimal monotone class, containing $\mathcal{E}$. We denote it by $\mathcal{M}(\mathcal{E})$.

4. Let $\mathcal{A} \subset 2^X$ be an algebra of sets. Prove that
$$\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A}).$$

5. Verify that $(X, 2^X, \mu)$ with
$$\mu(E) = \begin{cases} 0, & \operatorname{card}(E) \leq \operatorname{card}\mathbb{N}, \\ +\infty, & \text{otherwise}, \end{cases}$$
is a measure space (that is, μ is countably additive on the sets from the σ-algebra 2^X).

6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Prove
- if $E, F \in \mathcal{A}$ are such that $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$;
- if $E \sim F \overset{\text{def}}{\iff} \mu(E \triangle F) = 0$, then $\sim$ is an equivalence relation on $\mathcal{A}$;
- if $\rho(E, F) = \mu(E \triangle F)$, then $\rho(E, F) \leq \rho(E, G) + \rho(G, F)$ for any $G \in \mathcal{A}$, and so ρ is a metric on the equivalence classes of $\sim$.