Section 11.5 Alternating series

An a.s is a series $\sum_{k=1}^{\infty} a_k$, in which

$a_k = (-1)^{k-1} b_k$, or $a_k = (-1)^k b_k$

with $b_k > 0$.

Examples:

1. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

2). $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \dots$

Alternating series test

If the alternating series

$\sum_{k=1}^{\infty} (-1)^{k-1} b_k$, $\quad b_k > 0$,

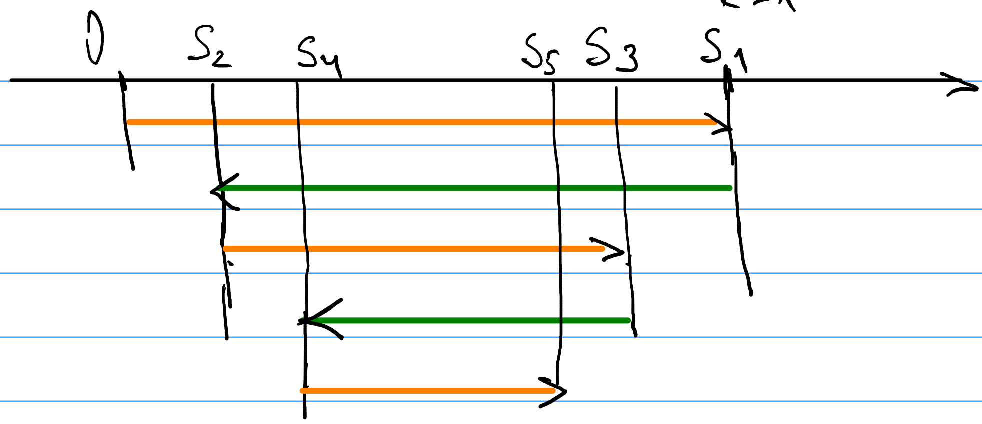
satisfies:

i). $b_{k+1} \leq b_k$, $\quad k \geq 1$ $\qquad (|a_{k+1}| \leq |a_k|)$

ii). $\lim_{k \to \infty} b_k = 0$,

then it is convergent.

Partial sums of $\sum_{k=1}^{\infty} (-1)^{k-1} b_k$:



Ex. 1 Alternating harmonic series:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

i). $b_{k+1} \le b_k$, $\quad \frac{1}{k+1} \le \frac{1}{k}$ -

true for $k \geq 1$.

ii). $\lim_{k\to\infty} b_k = 0$, indeed $\lim_{k\to\infty} \frac{1}{k} = 0$

$\Rightarrow$ this series is convergent by the alternating series test.

Ex. 2 Test for convergence: $\sum_{k=1}^{\infty} \underbrace{(-1)^{k+1} \frac{k^2}{k^3+1}}_{a_k}$

i). To check this assumption: need to check that $b_k = \frac{k^2}{k^3+1}$ is decreasing in k.

Consider $f(x) = \frac{x^2}{x^3+1}$ for $x > 0$; we will show that f is decreasing.

$$f'(x) = \frac{2x(x^3+1) - x^2 \cdot 3x^2}{(x^3+1)^2} = \frac{-x^4 + 2x}{(x^3+1)^2}$$

$$= \frac{x\,(2-x^3)}{(x^3+1)^2} < 0, \quad \text{when } 2 < x^3, \; x > \underbrace{\sqrt[3]{2}}_{<2}$$

$f(x)$ is decreasing, $x \geq 2$.

$\Rightarrow b_{k+1} \leq b_k, \quad k \geq 2.$

ii). $\lim_{k\to\infty} b_k = 0$; indeed, $\lim_{k\to\infty} \frac{k^2 /k^3}{(k^3+1)/k^3}$

$$= \lim_{k\to\infty} \frac{\overset{\to 0}{1/k}}{1 + \underbrace{1/k^3}_{\downarrow 0}} = \frac{0}{1+0} = 0.$$

$\Rightarrow$ by the AST, this series is convergent.

Remainder estimation with AST

Let $s = \sum_{k=1}^{\infty} (-1)^{k-1} b_k$, $b_k > 0$, where

i). $b_{k+1} \leq b_k$ ii). $\lim_{k\to\infty} b_k$.

Then $|R_n| = |s - s_n| = \left| \sum_{k=n+1}^{\infty} a_k \right| \leq b_{n+1}.$

For example, for the alternating harmonic series: $\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right)$

$$|R_{10}| \leqslant b_{11} = \frac{1}{11}.$$