Measure and Integration I (MAA5616), Fall 2020
Homework 5, due Thursday, Oct. 8

1. Consider sequences $\{a_j\}_{j=1}^{\infty} \subset (0,1)$.

- Prove $\prod_1^\infty (1-a_j) > 0$ iff $\sum_1^\infty a_j < \infty$.
  (Apply log to the product and use a comparison test for $\sum_j \log(1-a_j)$ and $-\sum_j a_j$.)
- Given a $b \in [0,1)$, construct a sequence $\{a_j\}_{j=1}^{\infty} \subset (0,1)$ for which $\prod_1^\infty (1-a_j) = b$.

Consider the construction of the 1/3-Cantor set. The j-th step of it involves removing the open middle 1/3 subinterval from each connected component $[a_{k,j}, b_{k,j}]$, obtained on the $j-1$-st step. The removed interval is $(\frac{2a}{3} + \frac{b}{3}, \frac{a}{3} + \frac{2b}{3})$. There is a single interval before the first step: $[0,1]$.

Removing 1/3 of length on each step resulted in $\lambda(C) = 0$. Removing any fixed fraction of the length on each step gives the same effect.

2. Construct Cantor-like sets of positive Lebesgue measure.

(a) Using #1, verify that for any $b \in (0,1)$, removing $a_j \in (0,1)$ of length on the j-th step of the construction results in a closed set of Lebesgue measure b, for an appropriate sequence $\{a_j\}_j$.

(b) Show that removing middle subintervals of length α^j (**not** fraction of length) for $0 < \alpha < 1/3$ from each connected component on the j-th step also results in a closed set of positive Lebesgue measure.

3. As shown in HW4, the 1/3-Cantor set C consists precisely of the elements of [0,1] which have only the digits 0 and 2 in their ternary expansion, $C = \left\{x : x = \sum_{i\ge1} \frac{t_i}{3^i},\ t_i \in \{0,2\}\right\}$. Consider the following function:

$$f(x) = \begin{cases} \sum_{i\ge1} \frac{t_i/2}{2^i}, & x \in C, \\ f\left(\max\{y < x,\ y \in C\}\right), & \text{otherwise.} \end{cases}$$

Prove:

- f is increasing on [0,1];
- $f([0,1]) = [0,1]$;
- conclude that f is continuous;
- prove that $x + f(x)$ is a homeomorphism (continuous 1-1 function with a continuous inverse).
  Both f and $f(x) + x$ map C to a set of positive measure! By the next problem, this gives a homeomorphism between a measurable subset of C and a non-measurable set.

4. Suppose $E \subset [0,1)$ is a Lebesgue-measurable set. Recall the set $V \subset [0,1)$ we constructed using the axiom of choice. This V must be non-measurable, otherwise we would have a contradiction to the definition of measure. Prove:

- If $E \subset V$, $\lambda(E) = 0$.
- If $\lambda(E) > 0$, E contains a non-measurable set.
  We have $E = \bigcup_r E \cap V_r$, where $V_r = V + r \mod 1$ and $r \in [0,1) \cap \mathbb{Q}$. If all $E \cap V_r$ are measurable, $\lambda(E) = 0$ because r runs over a countable set, a contradiction.