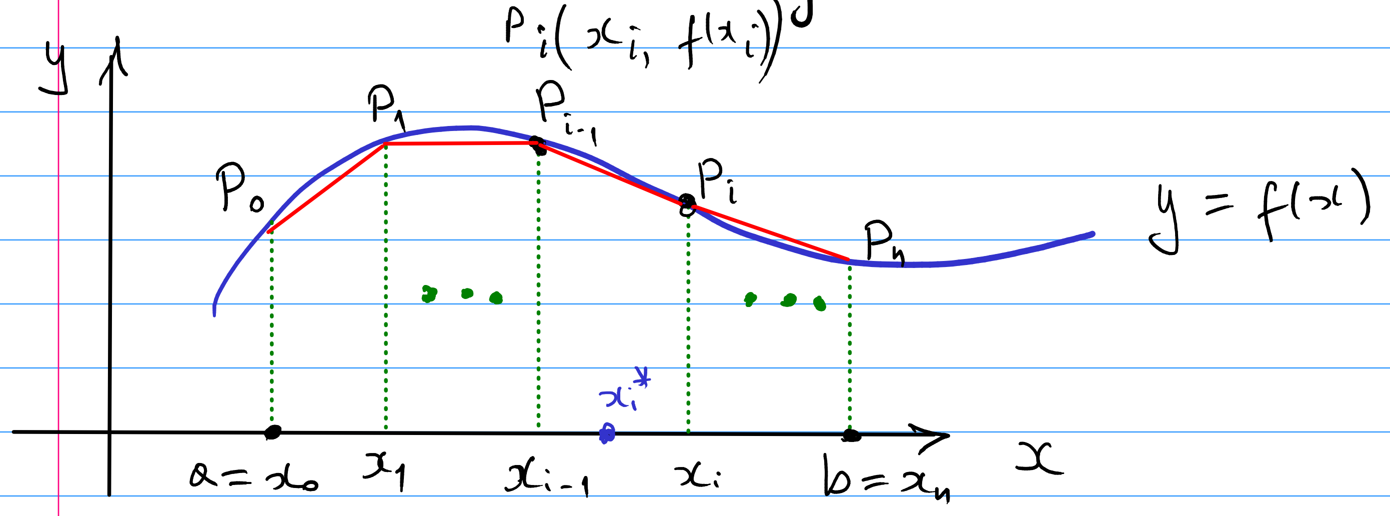
Section 8.1: Arc length



$$\underbrace{L}_{\text{length of the curve}} = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i| \quad \text{(if this exists)}$$

$|P_{i-1}P_i|$ = distance between
$P_{i-1}(x_{i-1}, f(x_{i-1}))$
$P_i(x_i, f(x_i))$

From the distance formula:

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$\Delta x = \frac{b-a}{n}$

$$= \sqrt{(\Delta x)^2 + (f'(x_i^*) \cdot \Delta x)^2} \quad =$$

(from the MVT: $f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$)

$$|P_{i-1}P_i| = \sqrt{1 + f'(x_i^*)^2}\,\Delta x$$

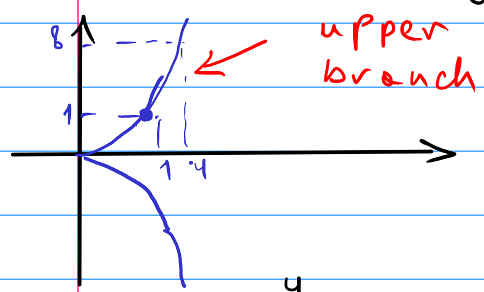
$$L = \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2}\,\Delta x}_{\text{Riemman}}$$

$$\boxed{L = \int_a^b \sqrt{1 + f'(x)^2}\,dx}$$

↳ arc length for the curve $y = f(x)$, $a \le x \le b$.

Ex. 1 Find length of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.

Solve for y: $y = \pm x^{3/2}$



$$y = x^{3/2} \qquad 1 \le x \le 4,$$

$$y' = \frac{3}{2} x^{1/2}$$

$$L = \int_1^4 \sqrt{1 + (y')^2}\,dx = \int_1^4 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\,dx$$

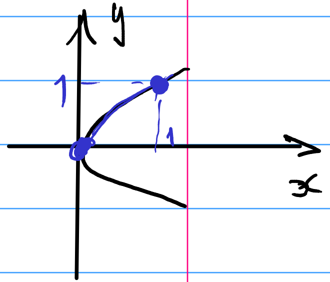
$$= \int_1^4 \sqrt{1+\frac{9}{4}x}\, dx = \left| \begin{array}{l} u = 1+\frac{9}{4}x \\ du = \frac{9}{4}dx \end{array} \right|$$

$$\frac{4}{9}du = dx$$

$$= \int_{13/4}^{10} \sqrt{u}\ \frac{4}{9}du = \frac{4}{9} \cdot \frac{u^{3/2}}{3/2} \Big|_{13/4}^{10} =$$

$$= \frac{8}{27} u^{3/2} \Big|_{13/4}^{10} = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right).$$

Ex. 2 Find length of $y^2 = x$ from $(0,0)$ to $(1,1)$.



$$x = y^2 \qquad 0 \le y \le 1$$

$$\frac{dx}{dy} = 2y$$

$$L = \int_0^1 \sqrt{1+4y^2}\, dy = \left| \begin{array}{l} 2y = \tan\theta \\ dy = \frac{\sec^2\theta}{2} d\theta \end{array} \right|$$

$$= \int_0^{\theta_2} \sqrt{1+\tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\theta_2} \sec^3\theta\, d\theta$$

$$= \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\theta_2}$$

$$\left(\tan\theta_2 = 2, \qquad \sec\theta_2 = \sqrt{\tan^2\theta_2 + 1} = \sqrt{5}\right)$$

$$= \frac{1}{4}\left(\sqrt{5}\cdot 2 + \ln|2+\sqrt{5}|\right).$$

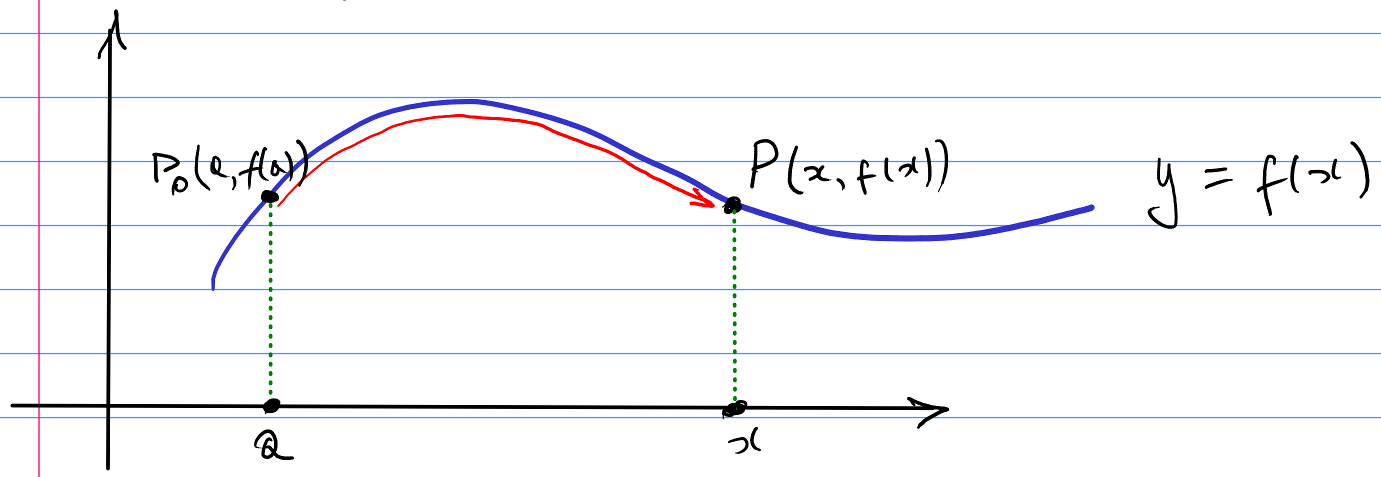
$$\int \sec^3\theta\, d\theta = \left| \begin{array}{ll} u = \sec\theta & dv = \sec^2\theta\, d\theta \\ du = \sec\theta\tan\theta\, d\theta & v = \tan\theta \end{array} \right| = \sec\theta\tan\theta -$$

$$- \int \sec\theta\,(\sec^2\theta - 1)\, d\theta$$

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \int \sec\theta\, d\theta\right)$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$

Arc length function



$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt \quad \Big\} \text{ arc length function}$$

Ex. 3 Find the arc length function for

$y = x^2 - \frac{1}{8}\ln x$ $\qquad P_0(1,1)$

$$y'(x) = 2x - \frac{1}{8x}$$

$$1 + (y'(x))^2 = 1 + \left(2x - \frac{1}{8x}\right)^2 =$$

$$= 1 + (2x)^2 - 2\cdot 2x\cdot\frac{1}{8x} + \left(\frac{1}{8x}\right)^2$$

$$= 1 + (2x)^2 \overbrace{- \frac{1}{2}}^{2\cdot\text{cross prod}} + \left(\frac{1}{8x}\right)^2$$

$$= (2x)^2 \underbrace{+ \frac{1}{2}}_{2\cdot\text{cross prod}} + \left(\frac{1}{8x}\right)^2$$

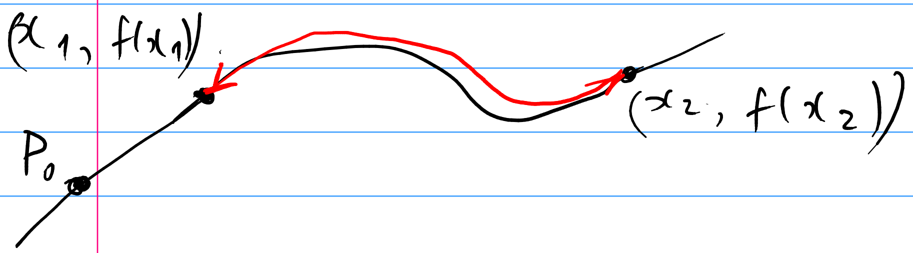
$$= \left(2x + \frac{1}{8x}\right)^2$$

$$s(x) = \int_1^x \sqrt{\left(2t + \frac{1}{8t}\right)^2}\, dt$$

$x > 0$

$$= \int_1^x \left(2t + \frac{1}{8t}\right) dt = \left(t^2 + \frac{1}{8}\ln t\right)\Big|_1^x$$

$$= x^2 + \frac{1}{8}\ln x - 1.$$



arc length $= s(x_2) - s(x_1)$

Summary

Arc length for $y = f(x)$ between (a, c) and (b, d):

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$