Measure and Integration I (MAA5616), Fall 2020
Homework 9, due Thursday, Nov. 12

1. If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf_n \int f_n$. Prove it.

2. If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure. Prove it.

3. Suppose $f \in L^1(\mu)$.

- Given an $\epsilon > 0$, find a simple function $\phi = \sum_j a_j 1_{E_j}$, such that

$$\int |f - \phi| < \epsilon.$$

This shows, simple functions are dense in L^1 with the usual L^1 distance.

- For $\mu = \lambda$, show that it can be assumed $E_j = (a_j, b_j)$, a collection of open intervals.
- For $\mu = \lambda$ and an $\epsilon > 0$, construct a continuous function g with compact support, such that

$$\int |f - g|\, d\lambda < \epsilon.$$

4. (Luzin's theorem) For a Lebesgue-measurable $f : [a, b] \to \mathbb{R}$ and any given $\epsilon > 0$, there exists a compact $E \subset [a, b]$ with $\lambda(E) > b - a - \epsilon$, such that $f|_E$ is continuous.

Use Yegorov's theorem together with a sequence of continuous functions g_n, constructed as in the previous problem, corresponding to $\epsilon_n \downarrow 0$.

This result extends to general topological spaces X, if the measure μ on X is finite and satisfies regularity conditions: $\mu(E) = \sup\{\mu(F) : F \subset E,\ F \text{ compact}\}$ and $\mu(E) = \inf\{\mu(G) : G \supset E,\ G \text{ open}\}$.

5. (Strong Luzin's theorem) For $f,\ \epsilon$ as above, there exists a compact $E \subset [a, b]$ with $\lambda(E) > b - a - \epsilon$ and a continuous $g : [a, b] \to \mathbb{R}$, such that $f|_E = g|_E$.

This statement also holds for the general functions $f : X \to Y$ with X as above, possibly assuming that Y is a connected metric space. You should think about what breaks when the space X is connected, but Y is not; consider $Y = [a, b] \cup [c, d]$ with $b < c$ for instance.