Measure and Integration I (MAA5616), Fall 2020
Homework 4, due Thursday, Oct. 1

In the following problems, $\lambda = \mu_x$ is the outer measure on $\mathbb{R}$, corresponding to the function $F(x) = x$ (Lebesgue measure), and $\mathcal{M}$ is the σ-algebra of measurable sets for λ.

1. Prove: for an $E \in \mathcal{M}$,

$$\lambda(E) = \inf \left\{ \sum_{j=1}^{\infty} (b_j - a_j) \ : \ E \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.$$

For a general μ_F, the above infimum must be taken of $\sum_j \mu_F((a_j, b_j))$ instead.

2. Prove: for an $E \in \mathcal{M}$,

$$\lambda(E) = \inf\{\lambda(G) : G \supset E,\ G \text{ is open }\} = \sup\{\lambda(F) : F \subset E,\ F \text{ is closed }\}.$$

(The first equality uses #1, the second follows by complementation.)

3. Using the above, prove that any $E \in \mathcal{M}$ can be represented i) as union of an F_σ set and a null set; ii) as difference of a G_δ set and a null set.
(An F_σ set is a countable union of closed sets; a G_δ set is a countable intersection of open sets).

4. Show that the result of #2 can serve as the definition of measurability. That is, prove: a set $A \subset \mathbb{R}$ is λ-measurable iff for any $\epsilon > 0$, there exist a closed F and an open G such that $F \subset A \subset G$ and $\lambda(G \setminus F) < \epsilon$.
(The closed sets F and G^c are disjoint, and a sufficiently short interval will intersect only one of them. Any covering of A can be subdivided into a covering with sufficiently short intervals. Using this, conclude that $\lambda(E) \geq \lambda(E \cap F) + \lambda(E \cap G^c)$, which then gives the measurability equation.)

The *1/3-Cantor set* C is obtained by removing the interval $(1/3, 2/3)$ from $[0, 1]$, then removing the middle 1/3 open subinterval from each of the two resulting closed intervals, etc. On the n-th step, remove the middle 1/3 open subinterval from each of the 2^{n-1} closed intervals obtained on the previous step.

5.
- Verify that C is given precisely by the elements of $[0, 1]$ that have ternary expansions containing only 0 and 2. (Here we do allow representations ending with infinite ternary $\overline{2}$.)
- Conclude that C has the cardinality $\mathfrak{c}$.
- Check that $\lambda(C) = 0$.

To summarize, C is uncountable, and yet has Lebesgue measure zero.