Measure and Integration II (MAA5617), Spring 2021
Homework 4, due Thursday, Feb 25

Below L^1, L^1_{loc} refer to the measure spaces with $\lambda = \lambda^n$ as the underlying measure.

1. For $f \in L^1_{loc}$, prove that the average $A_r f(x)$ is jointly continuous as a function of $(x, r) \in \mathbb{R}^n \times (0, \infty)$. Hint: this is a lemma in Folland.

2. Given an $f \in L^1$ and $\epsilon > 0$, construct a continuous function g, for which

$$\int |f - g| \, d\lambda < \epsilon.$$

Show that the same result holds if in addition $g \in C^\infty(\mathbb{R}^n)$, that is, g and all of its partial derivatives are continuous.

- Because f can be approximated with simple functions, it suffices to assume f is an indicator of a rectangle (why?).
- For two functions f_1, f_2 on $\mathbb{R}^n$, define their convolution as

$$f_1 * f_2(x) := \int f_1(x - y) f_2(y) \, dy.$$

- Construct a continuous function $\phi \in C^\infty(\mathbb{R}^n)$ supported on $B(\bar{0}, 1)$. Hint: e^{-1/t^2} (extended to $t = 0$ by continuity) is infinitely differentiable at $t = 0$.
- Using the dominated convergence theorem, show that

$$f * r^{-n} \phi(x/r)$$

  converges to f in L^1 for $r \downarrow 0$, and is infinitely differentiable, as desired.

3. If μ_1, μ_2 are positive Borel measures on $\mathbb{R}^n$, $\mu_1 \perp \mu_2$, and $\mu_1 + \mu_2$ is regular, then so is each μ_i, $i = 1, 2$.

4. Suppose $E \in \mathcal{B}_{\mathbb{R}^n}$. Using the Lebesgue differentiation theorem, show that

$$D_E(x) = \lim_{r \downarrow 0} \frac{\lambda(E \cap B(x, r))}{\lambda(B(x, r))}$$

is equal to 1 for λ-a.e. $x \in E$, and to 0 for λ-a.e. $x \notin E$.

5. As in the previous problem, give an example of $E \in \mathcal{B}_{\mathbb{R}}$ and $x \in E$, for which $D_E(x) < 1$. Hint: use a nested sequence of fat Cantor sets with positive measures.

6. Construct an $E \in \mathcal{B}_{\mathbb{R}}$, such that $0 < \lambda(E \cap I) < \lambda(I)$ for any interval $I \subset [0, 1]$. (Use fat Cantor sets again).

Comments

5. A somewhat nicer approach than taking care of the ratios for Cantor sets is to consider some (arguably even simpler) fractals. Pick an $\alpha \in (0,1)$; it will be our target density. That is, we will improve the statement of the problem by constructing a set with density α at a certain point. For a set $E \subset [0,1]$ with $0 \in E$, we will denote

$$\rho(x,E) := \frac{\lambda(E \cap [0,x])}{x}.$$

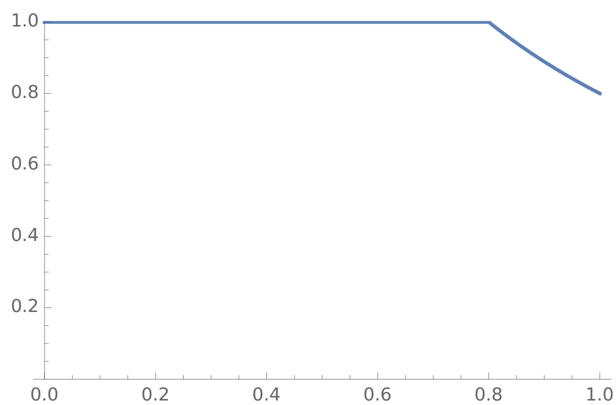
You should check that

$$D_E(0) = \lim_{x \downarrow 0} \rho(x,E),$$

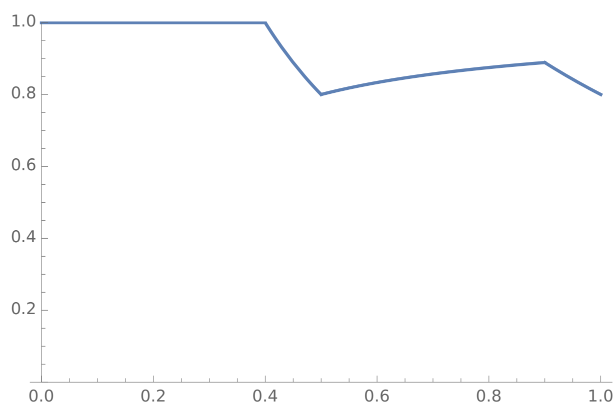
if this limit exists. Our objective is to construct an E with the density α at 0.
To begin with, note that

$$\rho_1(x) := \rho(x,[0,\alpha]) = \begin{cases} 1, & 0 \le x \le \alpha, \\ \alpha/x, & \alpha \ge x \end{cases}$$

and looks like this:



so $\rho(x,E)$ can deviate from α (0.8 in this graph) a lot. Let's say we rescale $[0,\alpha]$ and put two of its copies in $[0,1/2]$ and $[1/2,1]$. Then $\rho(x,\ [0,\alpha/2] \cup [1/2, 1/2+\alpha/2])$ becomes



which deviates from 0.8 less than ρ_1 on $[1/2,1]$:

$$\rho_2(x) := \begin{cases} \rho_1(2x), & 0 \le x \le 1/2, \\ \frac{\alpha}{2} + \frac{\rho_1(2(x-1/2))}{2}, & 1/2 \le x \le 1 \end{cases}$$

(the last equation generalizes to subsequent iterations and all ρ_j, $j \geq 2$).
Repeating this process by putting rescaled copies of the set $[0, \alpha]$ in each of $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 4]$, etc, we will get increasingly good approximations of α on $[1/2, 1]$ (show that

$$\max_{[1/2,1]} |\rho_j(x) - \alpha|$$

is halved with every iteration).
To complete the proof, let E_k be the set obtained as a union of 2^k copies of $[0, \alpha]$, placed in the intervals $[i/2^j, (i+1)/2^j]$. Then place a rescaled copy of E_k in $[1/2^{k+1}, 1/2^k]$; denote it by F_k. Verify that

$$\bigcup_{k \geq 1} F_k$$

has density α at 0, as desired. Now, you should think about how to make this construction more elegant.

6. Let $C_\alpha \subset [0, 1]$ be a fixed fat Cantor set of positive measure. Recall that it is nowhere dense, that is, for every interval $I \subset [0, 1]$ there exists an interval $J \subset I$, disjoint from C_α. Color the left half of C_α in *blue*, the right one in *red*. Then repeat:

- The colored sets constructed up to now are nowhere dense is $[0, 1]$.
- The union of the colored sets constructed up to now is a closed set, so its complement is open, and has a countable number of connected components.
- By the above, it is possible to place a rescaled copy of C_α in each connected component of the open uncolored set, and color this copy like so: the left half in blue, the right one in red.

This gives a countable union of rescaled copies of C_α. Each interval contains both red- and blue-colored subsets (why?), which proves the desired statement, if E is taken to be the red-colored part of $[0, 1]$.