SUMMARY OF CALC2 CONCEPTS

Contents

1. Methods of Integration

1.1. Integration formulas refresher.

$$(1) \quad \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1 \qquad\qquad (11) \quad \int \sec x \, dx = \ln |\sec x + \tan x|$$

$$(2) \quad \int \frac{1}{x} \, dx = \ln |x| \qquad\qquad (12) \quad \int \csc x \, dx = \ln |\csc x - \cot x|$$

$$(3) \quad \int e^x \, dx = e^x \qquad\qquad (13) \quad \int \sec^2 x \, dx = \tan x$$

$$(4) \quad \int b^x \, dx = \frac{1}{\ln b} b^x \qquad\qquad (14) \quad \int \csc^2 x \, dx = -\cot x$$

$$(5) \quad \int \ln x \, dx = x \ln x - x \qquad\qquad (15) \quad \int \sec x \tan x \, dx = \sec x$$

$$(6) \quad \int \log_b x \, dx = \frac{1}{\ln b}(x \ln x - x) \qquad\qquad (16) \quad \int \csc x \cot x \, dx = -\csc x$$

$$(7) \quad \int \sin x \, dx = -\cos x \qquad\qquad (17) \quad \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$(8) \quad \int \cos x \, dx = \sin x \qquad\qquad (18) \quad \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right|$$

$$(9) \quad \int \tan x \, dx = \ln |\sec x| \qquad\qquad (19) \quad \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}$$

$$(10) \quad \int \cot x \, dx = -\ln |\csc x| \qquad\qquad (20) \quad \int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln |x + \sqrt{x^2 \pm a^2}|$$

1.2. Integration by parts.

$$\int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx$$

$$\int u \, dv = uv - \int v \, du$$

1.3. Trigonometric integrals.

Odd positive powers of sine and cosine. Substitute for the cofunction, save one factor of the function, and use the identity $\sin^2 x + \cos^2 x = 1$ to get everything in terms of the cofunction.

Only even positive powers of sine and cosine. Use the half-angle identities to reduce powers:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Power of secant is even or power of tangent is odd. If Secant is even save a factor of $\sec^2 x$, let $u = \tan x$. If tangent is odd save a factor of $\sec x \tan x$, substitute for secant. Then use the identity $\tan^2 x + 1 = \sec^2 x$ to get everything in terms of u.

Power of secant is odd and power of tangent is even. Use $\tan^2 x + 1 = \sec^2 x$ to rewrite in terms of secant.

When the argument is not the same in a product. Reduce to the same angle by using the sum formulas, or use product to sum identities:

$$\sin A \cos B = \frac{1}{2}\left(\sin(A - B) + \sin(A + B)\right)$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A - B) - \cos(A + B)\right)$$

$$\cos A \cos B = \frac{1}{2}\left(\cos(A - B) + \cos(A + B)\right).$$

1.4. **Trigonometric substitution.**
Integrals involving the form $a^2 - x^2$.
Substitute $x = a\sin\theta$ $(-\pi/2 \le \theta \le \pi/2)$ and use the identity $\sin^2\theta + \cos^2\theta = 1$
Integrals involving the form $a^2 + x^2$.
Substitute $x = a\tan\theta$ $(-\pi/2 < \theta < \pi/2)$ and use the identity $\tan^2\theta + 1 = \sec^2\theta$
Integrals involving the form $x^2 - a^2$.
Substitute $x = a\sec\theta$ $(0 \le \theta < \pi/2)$ and use the identity $\tan^2\theta + 1 = \sec^2\theta$
Completing the square.
$a(x^2 + bx) = a(x + (b/2))^2 - a(b/2)^2$

1.5. **Partial fractions. Decomposing rational functions.**

Step1. Use long division if the degree of the numerator is greater than or equal to the degree of the denominator, i.e. rewrite as follows

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}, \qquad deg(R(x)) < deg(g(x)).$$

Step2. Completely factor the denominator into linear and irreducible (with negative discriminant) quadratic factors: $(ax + b)^n$ and/or $(ax^2 + bx + c)^m$.

Step3. For each distinct linear factor, $(ax + b)^n$, the partial fraction decomposition will include the sum

$$\frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_n}{(ax + b)^n}.$$

Step4. For each distinct irreducible quadratic factor, $(ax^2 + bx + c)^m$ with $b^2 - 4ac < 0$, the partial fraction decomposition will include the sum

$$\frac{A_1 x + B_1}{(ax^2 + bx + c)} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3 x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_m x + B_m}{(ax^2 + bx + c)^m}$$

Rationalizing substitution.
If the integrand involves a higher-order root $\sqrt[n]{f(x)}$, make a substitution of the form $u = \sqrt[n]{f(x)}$ to change the integrand into a rational function of u.

1.6. **Integration tables.** Generally one has to rewrite the integrand or make a substitution before an integral from the tables in the back of the text may be applied. On an exam you may be given a selection of integrals from the tables and you will have to integrate using the appropriate one.

1.7. **Approximate integration. Midpoint rule.** Use a Riemann sum where the x_i^* are chosen to be the midpoints of the respective subintervals:

$$\int_a^b f(x)\,dx \approx \Delta x[f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)].$$

Trapezoidal rule. Rather than rectangles, the area is approximated using trapezoids.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)],$$

$$x_i = a + i\Delta x.$$

Simpson's rule. For Simpson's Rule quadratic approximations for the curve in each subinterval are used to approximate the integral. Number n must be even for this method.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Midpoint rule error.

$$|error| \leq \frac{K(b-a)^3}{24n^2}$$

where $|f''(x)| \leq K$ for all $a \leq x \leq b$.

Trapezoid rule error.

$$|error| \leq \frac{K(b-a)^3}{12n^2}$$

where $|f''(x)| \leq K$ for all $a \leq x \leq b$.

Simpson's rule error.

$$|error| \leq \frac{K(b-a)^5}{180n^4}$$

where $|f^{(4)}(x)| \leq K$ for all $a \leq x \leq b$.

1.8. **Improper integrals.** We define an improper integral using limits and say that an improper integral is *convergent* if the limit exists (is a real number). Otherwise, it is *divergent*.

One or both limits of integration are infinite.

(1) $\int_a^\infty f(x)\,dx = \lim_{n\to\infty} \int_a^n f(x)\,dx$
(2) $\int_{-\infty}^b f(x)\,dx = \lim_{n\to-\infty} \int_n^b f(x)\,dx$
(3) $\int_{-\infty}^\infty f(x)\,dx = \lim_{m\to-\infty} \int_m^a f(x)\,dx + \lim_{n\to\infty} \int_a^n f(x)\,dx$ (if both limits exist)

Discontinuities in the limits of integration.

(1) $f(x)$ has a discontinuity at $x = a$. Then $\int_a^b f(x)\,dx = \lim_{n\to a^+} \int_n^b f(x)\,dx$
(2) $f(x)$ has a discontinuity at $x = b$. Then $\int_a^b f(x)\,dx = \lim_{n\to b^-} \int_a^n f(x)\,dx$
(3) $f(x)$ has a discontinuity at $x = c$ and $a < c < b$. Then

$$\int_a^b f(x)\,dx = \lim_{m\to c^-} \int_a^m f(x)\,dx + \lim_{n\to c^+} \int_n^b f(x)\,dx$$

(if both limits exist)

Inverse powers $\frac{1}{x^p}$**.** In what follows a is a positive finite number.

(1) If $p > 1$ then $\int_a^\infty \frac{dx}{x^p}$ is convergent.
(2) If $p \leq 1$ then $\int_a^\infty \frac{dx}{x^p}$ is divergent.

(3) If $p \geq 1$ then $\int_0^a \dfrac{dx}{x^p}$ is divergent.

(4) If $p < 1$ then $\int_0^a \dfrac{dx}{x^p}$ is convergent.

Comparison theorem.

Suppose $0 \leq f(x) \leq g(x) \leq h(x)$ for all x in the interval (a, b), where a and/or b may be infinite.

(1) If $\int_a^b f(x)\, dx$ is divergent, then $\int_a^b g(x)\, dx$ is divergent.

(2) If $\int_a^b h(x)\, dx$ is convergent, then $\int_a^b g(x)\, dx$ is convergent.

2. Geometric applications

2.1. Arc length.

Function of x. If $y = f(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$ on $a \leq x \leq b$, is $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

Function of y. If $x = g(y)$ is continuous on $[c, d]$, then the length of the curve $x = g(y)$ on $c \leq y \leq d$, is $L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$.

Arc length function. Let f be a smooth function on $[a, b]$. The *arc length function s* is defined by $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$

Arc length notation.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

2.2. Surface area of a solid of revolution.

$$A = \int_\alpha^\beta 2\pi r\, ds,$$

where r is the radius of the surface, ds is the element of arc length, see above. If the axis of rotation is the x or y axis then the radius is one of x, y, $f(x)$, or $f(y)$. See also section 4.1 below, for the formula of ds for parametric curves.

3. Physics applications

3.1. **Hydrostatic force and pressure.** The force exerted by a fluid on a thin plate is $F = mg = \rho g d$, where ρ is the density of the fluid, g acceleration of the free fall, d depth at which the plate is submerged. The pressure P on the plate is defined to be the force per unit area: $P = F/A = \rho g d$. To find hydrostatic force when the depth is not constant express the approximate force on a thin slice to express the force as an integral of the variable pressure over the entire shape.

3.2. **Center of mass. Moments.** If we have a system of n particles with masses $m_1, m_2, \ldots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ in the xy-plane, then the *moment of the system about the y-axis* is defined by

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the *moment of the system about the x-axis* is defined by

$$M_x = \sum_{i=1}^{n} m_i y_i.$$

Center of Mass. The *center of mass* is given by the coordinates

$$(\overline{x}, \overline{y}) = (M_y/m, M_x/m),$$

where $m = \sum m_i$.

Centroid. The center of mass of a solid plate is called the *centroid* of the region it occupies. If $\mathcal{R}$ is the region in the xy-plane bounded above by $y = f(x)$ and below by $y = g(x)$ over the interval $[a, b]$, then the centroid of $\mathcal{R}$ is

$$(\overline{x}, \overline{y}) = (\frac{1}{A} \int_a^b x[f(x) - g(x)] \, dx, \frac{1}{2A} \int_a^b [(f(x))^2 - (g(x))^2] \, dx),$$

where A is the area of $\mathcal{R}$.

3.3. **Separable differential equations.** A *separable equation* is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. In other words, it has the form

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

This allows to rewrite such an equation as follows:

$$\frac{dy}{g(y)} = f(x) dx,$$

assuming $g(y) \neq 0$. Then integration on both sides gives:

$$\int \frac{dy}{g(y)} = \int f(x) dx.$$

Solving for y the last equality gives it as a function $y(x)$. The constant of integration is found from the initial conditions, if available; otherwise the solution of the original equation is a family of functions.

Orthogonal Trajectory. An *orthogonal trajectory* of a family of curves $\{\mathcal{C}_k\}$ indexed by k is a curve $\mathcal{C}$ that intersects each curve of the family orthogonally (at a right angle). This means, if
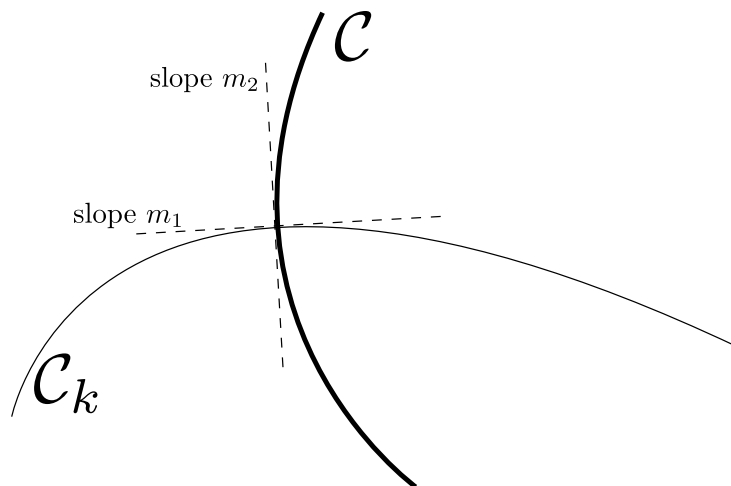
FIGURE 1. Orthogonal trajectory $\mathcal{C}$ intersecting a curve from the given family $\mathcal{C}_k$.

m_1 is the slope of the tangent to a curve $\mathcal{C}_k$ from the family at a point $P(x, y)$, and m_2 is the slope of the tangent to the orthogonal trajectory $\mathcal{C}$ at the point P, then

$$m_1 \cdot m_2 = -1.$$

3.4. **Exponential growth and decay.** There are many applications where a function is proportional to its first derivative. In other words, $\frac{dy}{dx} = ky$. This differential equation is called the *natural law of growth ($k > 0$) or decay ($k < 0$).*

Other population models do not need to be memorized. It is however required to be able to solve the relevant separable equations.

4. Polar coordinates and conic sections

4.1. Parametric equations. A set of equations that are defined using a single independent variable are called *parametric equations*. Typically, we use t, called the *parameter*, as the independent variable to define the functions $x(t)$ and $y(t)$ (and perhaps $z(t)$). These equations define a *parametric curve*, C, in the plane (or in 3-space if $z(t)$ is given) such that the points on C are the set of points given by $(x(t), y(t))$.

(1) Graphing
(2) Convert from and to parametric equations to and from Cartesian.
(3) Slope of a parametric curve can be determined from

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

(4) The second derivative is obtained similarly:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{(dx/dt)}.$$

(5) Use $\int y\,dx = \int y(t)\cdot x'(t)\,dt$, $\int x\,dy = \int x(t)\cdot y'(t)\,dt$ to find areas bounded by parametric curves.

(6) Use $ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ to find lengths of curves and the element of arc length for surface areas.

4.2. Polar coordinates.

(1) $x = r\cos\theta$
(2) $y = r\sin\theta$
(3) $r^2 = x^2 + y^2$
(4) $\tan\theta = y/x$.
(5)
$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

(6) The area of the region within the sector $a \le \theta \le b$ enclosed by the curve $r = r(\theta)$ is given by $\int_a^b \frac{1}{2}r^2\,d\theta$.

(7) Use $ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ to find lengths of curves and surface areas.

4.3. Conic sections in Cartesian coordinates.

(1) $y^2 = 4px$ – parabola, focus $(p, 0)$, directrix $x = -p$.
(2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a \ge b > 0$ – ellipse, foci $(\pm c, 0)$ where $c^2 = a^2 - b^2$, vertices $(\pm a, 0)$.
(3) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ – hyperbola, foci $(\pm c, 0)$ where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, asymptotes $y/b = \pm x/a$.

Interchanging x and y in the above equations results in the corresponding conic section changing orientation from horizontal to vertical (equivalently, the graph is reflected with respect to the axis $y = x$).

4.4. **Conic sections in polar coordinates.**

Theorem 4.4.1. *A polar equation of the form below represents a conic section with eccentricity e. The conic section is an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$.*

Equation	Focus	Directrix $(d > 0)$	Eccentricity	Vertices
$r = \dfrac{ed}{1 \pm e\cos\theta}$	$(0,0)$	$x = \pm d$	e	on the x-axis
$r = \dfrac{ed}{1 \pm e\sin\theta}$	$(0,0)$	$y = \pm d$	e	on the y-axis

5. Series

5.1. Sequences.

(1) A sequence is a function whose domain is the set of positive integers.
   A sequence is often thought of as a list of numbers where the order is important and each number is called a term.
(2) A *recursive* sequence uses previous terms to define later terms.
(3) *If* $\lim_{n \to \infty} a_n$ is a real number then we say the sequence is *convergent.* If the limit is not a real number then we say the sequence is *divergent.*
(4) A sequence of the form $\{ar^{n-1}\}$ is called a *geometric sequence.*
(5) A sequence that alternates signs is called an *alternating sequence.*
(6) A sequence, a_n, is *bounded above* if there is a number M such that $a_n \leq M$ for all n.
(7) A sequence, a_n, is *bounded below* if there is a number M such that $a_n \geq M$ for all n.
(8) A sequence, a_n, is *bounded* if it is both bounded above and below.
(9) A sequence, a_n, is *increasing* if $a_n \leq a_{n+1}$ for all n.
(10) A sequence, a_n, is *decreasing* if $a_n \geq a_{n+1}$ for all n.
(11) A sequence, a_n, is *monotonic* if it is either increasing or decreasing.

Theorem 5.1.1. *A geometric sequence is convergent if and only if* $-1 < r \leq 1$.

Theorem 5.1.2. *If* $a_n \leq b_n \leq c_n$ *for all* $n > K$ *where* K *is some fixed integer, then*

$$\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \leq \lim_{n \to \infty} c_n.$$

Corollary 5.1.2.1. *(1)* $\lim_{n \to \infty} -|a_n| \leq \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} |a_n|$
(2) Suppose a_n *is an alternating sequence. If* $\lim_{n \to \infty} |a_n| = 0$, *then* $\lim_{n \to \infty} a_n = 0$. *If* $\lim_{n \to \infty} |a_n| \neq 0$, *then* $\lim_{n \to \infty} a_n$ *diverges.*

Theorem 5.1.3. *A bounded monotonic function is convergent.*

(1) If $\{a_n\}_{n=1}^{\infty}$ is a sequence, then $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \sum a_n$ is an *(infinite) seres.*
(2) The *n-th partial sum* is $s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$
(3) The *sum* of the *series* is

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} s_n.$$

(4) The series is called *convergent* if the limit defining the sum is a finite real number and this number is then the *sum of the series.* If the limit is not a finite number (does not exist or is infinite), the series is called *divergent.*
(5) The *n-th remainder term* of the series $\sum_{k=1}^{\infty} a_k$ is $r_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots = \sum_{k=n+1}^{\infty} a_k$.
(6) To conclude, $\sum_{k=1}^{\infty} a_k = s_n + r_n$.
(7) A series of the form $\sum_{n=1}^{\infty} ar^{n-1}$ is called a *geometric series.* Number r is called the *common ratio* of the series.
(8) If cancellation occurs in consecutive terms of a series, the series is called *telescoping.*
(9) A series $\sum a_n$ is called *absolutely convergent* if the series $\sum |a_n|$ converges.
(10) A series $\sum a_n$ is called *conditionally convergent* if the series $\sum a_n$ converges, but the series $\sum |a_n|$ diverges.

5.1.1. *Tests to determine convergence of series.*

Geometric series. If $\sum_{n=1}^{\infty} ar^{n-1}$ is a geometric series, then

(1) $s_k = \frac{a(1-r^n)}{(1-r)}$

(2) $\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

Telescoping. Middle terms in the partial sums cancel so one can use the definition to find the sum.

Test for divergence. If the series $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. The contrapositive is (and is also true): If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Harmonic series. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

p-series. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Comparison theorem. If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences with $0 \leq a_n \leq b_n \leq c_n$ then

(1) If $\sum_{n=1}^{\infty} a_n$ is divergent then $\sum_{n=1}^{\infty} b_n$ also is divergent.
(2) If $\sum_{n=1}^{\infty} c_n$ is convergent then $\sum_{n=1}^{\infty} b_n$ also is convergent.

The integral test. Assume f is a continuous, positive, decreasing function on $[K, \infty)$ such that $a_n = f(n)$ for all integers $n \geq K$. Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_K^{\infty} f(x)\, dx$ is convergent

Limit comparison test. Suppose $a_n \geq 0$, $b_n \geq 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a real number with $c > 0$. Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Alternating series test. Suppose $\sum_{n=1}^{\infty} a_n$ is an alternating series. If

(1) $|a_{n+1}| \leq |a_n|$ (the sequence $\{|a_n|\}$ is decreasing), and
(2) $\lim_{n\to\infty} |a_n| = 0$.

then the series is convergent.

Absolute convergence. If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Ratio test. Let $\sum a_n$ be a series with $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ (L is not necessarily a real number)

(1) If $0 \leq L < 1$, then $\sum a_n$ is absolutely convergent.
(2) If $L > 1$ or $L = \infty$, then $\sum a_n$ is divergent.
(3) If $L = 1$, then the test fails (is inconclusive).

Root test. Let $\sum a_n$ be a series with $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ (L is not necessarily a real number)

(1) If $0 \leq L < 1$, then $\sum a_n$ is absolutely convergent.
(2) If $L > 1$ or $L = \infty$, then $\sum a_n$ is divergent.
(3) If $L = 1$, then the test fails (is inconclusive).

Note that the ratio and root test always fail simultaneously, so after one fails applying the other is redundant.

Theorem 5.1.4. *If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then*

(1) $\displaystyle \sum_{n=1}^{\infty} [a_n + b_n] = \left(\sum_{n=1}^{\infty} a_n \right) + \left(\sum_{n=1}^{\infty} b_n \right)$

(2) $\displaystyle \sum_{n=1}^{\infty} [a_n - b_n] = \left(\sum_{n=1}^{\infty} a_n \right) - \left(\sum_{n=1}^{\infty} b_n \right)$

(3) $\displaystyle \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$, *where c is a constant*

If either series is divergent, then the above equalities do not necessarily hold.

5.1.2. *Finding the exact sum of a series.* The *sum* of the *series* $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$.

Geometric Series. If $\sum_{n=1}^{\infty} ar^{n-1}$ is a geometric series, then

(1) $s_k = \frac{a(1-r^n)}{(1-r)}$

(2) $\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

Telescoping. Middle terms in the partial sums cancel so one can use the definition to find the sum.

5.1.3. *Estimating the sum of a series with s_n.*

Estimate and error. The error in using the partial sum s_n to estimate the infinite sum is the remainder term, R_n.

Error based on integral test. Assume f is a continuous, positive, decreasing function on $[n, \infty)$ such that $a_k = f(k)$ for all integers $k \geq n$. Also assume $\sum_{n=1}^{\infty} a_n$ is convergent. Then $\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_n^{\infty} f(x)\, dx$

Error based on comparison test. If we know $0 \leq a_n \leq b_n$, then $0 \leq \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} b_k$

Error based when working with an alternating series. Suppose $\sum_{n=1}^{\infty} a_n$ is an alternating series. If

(1) $|a_{n+1}| \leq |a_n|$ (the sequence $\{|a_n|\}$ is decreasing), and
(2) $\lim_{n \to \infty} |a_n| = 0$.

then

$$|R_n| \leq |a_{n+1}|.$$

5.2. **Power series.** A series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n$$

where c_n, and a are constants and x is a variable, is called a *power series in* $(x - a)$, *or centered at a, or about a.*

Theorem 5.2.1. *For a given power series $\sum c_n (x - a)^n$, there are three possibilities.*

(1) *The series converges for $x = a$ only. In this case we say the radius of convergence of the series is $R = 0$ and the interval of convergence is $\{a\}$.*
(2) *The series converges for all x. In this case we say the radius of convergence of the series is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.*
(3) *There is a positive real number R where the series converges for all x with $|x - a| < R$ and diverges for all x with $|x - a| > R$. In this case we say the radius of convergence of the series is R. The interval of convergence is one of the intervals $(a - R, a + R), (a - R, a + R], [a - R, a + R),$ or $[a - R, a + R]$.*

5.3. **Representing functions using power series. Formula for Geometric Series.**

$$\sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \qquad \text{when} \quad |x| < 1$$

Theorem 5.3.1. *Let $f(x) = \sum c_n (x - a)^n$ be a power series with radius of convergence $R > 0$. Then*

(1) *f is continuous on the interval $(a - R, a + R)$*
(2) *f is differentiable on the interval $(a - R, a + R)$ and its derivative is*

$$f'(x) = \frac{d}{dx} \sum c_n (x - a)^n = \sum \frac{d}{dx} c_n (x - a)^n = \sum n c_n (x - a)^{n-1}.$$

(3) f may be integrated on a closed interval contained in $(a - R, a + R)$ and

$$\int f(x)\, dx = \int \sum c_n (x - a)^n\, dx = \sum \int c_n (x - a)^n\, dx = \sum \frac{c_n}{n + 1}(x - a)^{n+1}.$$

5.4. Taylor and Maclaurin series.

Definition 5.4.1. *The Taylor Series of a function f centered at a is the power series expansion of $f(x)$ about a and is in the form*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Definition 5.4.2. *The Maclaurin Series of a function f is the power series expansion of $f(x)$ about 0 and is in the form*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

As usual, we assume that $0! = 1$.

Some important Maclaurin series:

(1) e^x $= \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $= 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dots$ $R = +\infty$

(2) $\sin x$ $= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $= x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040} + \dfrac{x^9}{362880} - \dots$ $R = +\infty$

(3) $\cos x$ $= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $= 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \dfrac{x^8}{40320} - \dots$ $R = +\infty$

(4) $(1+x)^k$ $= \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ $= 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 \dots$ $R = 1$

(5) $\dfrac{1}{1-x}$ $= \displaystyle\sum_{n=0}^{\infty} x^n$ $= 1 + x + x^2 + x^3 + x^4 + \dots$ $R = 1$

(6) $\ln(1+x)$ $= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ $= x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dots$ $R = 1$

(7) $\arctan x$ $= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ $= x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dots$ $R = 1$

Observe that the series for $\ln(1 + x)$ and $\arctan x$ can be obtained from the series for $1/(1 + x)$ and $1/(1 + x^2)$ respectively, by integration. Binomial coefficients are given by the formula

$$\binom{k}{n} = \frac{k(k-1)(k-2)\dots(k-n+1)}{n!},$$

where k is any real number; n is a positive integer.

5.5. Approximating functions using polynomials.
Suppose f is a function with Taylor series about a given by $\sum a_n (x - a)^n$. We call the n-th partial sum of the series the *n-th degree Taylor polynomial.*

We denote the n-th degree Taylor polynomial by $T_n(x)$, so that

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Theorem 5.5.1. *Let T_n denote the n-th degree Taylor polynomial of a function, f, about a and let $R_n = f - T_n$. If $\lim_{n \to \infty} R_n(x) = 0$ for $|x - a| < R$, then f is equal to the sum of the Taylor series when $|x - a| < R$.*

Theorem 5.5.2 (Taylor's Inequality). *If $|f^{(n+1)}(x)| \leq M$ for all x with $|x - a| \leq d$, then the remainder $R_n(x)$ if the Taylor series satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \text{ for } |x - a| \leq d.$$