Section 11.9: Representing functions as power series

Recall:

$$\sum_{k=0}^{\infty} x^k = \underbrace{\frac{1}{1-x}}_{f(x)} \qquad |x|<1$$

Ex. 1 Express $\frac{1}{1-x^2}$ as a power series.

$$\frac{1}{1-x^2} = \underbrace{\frac{1}{1-(x^2)}}_{f(x^2)} = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k}$$

Replaced $x \longmapsto x^2$.

Converges when $|x^2| < 1$

$|x|^2 < 1$ $\quad\sqrt{\cdot}$

$|x| < 1$

Ex. 2 Express as a power series: $\frac{1}{x+2}$

Know: $\frac{1}{1-x} = f(x)$

$$\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} =$$

$$= \frac{1}{2} \cdot \underbrace{\frac{1}{1-\left(-\frac{x}{2}\right)}}_{f\left(-\frac{x}{2}\right)} = \frac{1}{2} \sum_{k=0}^{\infty} \underbrace{\left(-\frac{x}{2}\right)^k}_{\text{geometric series, converges when } \left|-\frac{x}{2}\right|<1} =$$

$$= \frac{1}{2} \cdot \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^k}{2^k} = \underbrace{\sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^k}{2^{k+1}}}_{= \frac{1}{x+2}}$$

$$\left| -\frac{x}{2} \right| < 1$$

$$\left| \frac{x}{2} \right| < 1 \qquad \cdot 2$$

$$|x| < 2$$

Then, the series is convergent when $|x| < 2$.

Ex. 3 Obtain the power series for $\frac{x^3}{x+2}$

Know: $$\frac{1}{x+2} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^k}{2^{k+1}} \qquad \times x^3$$

Then
$$\frac{x^3}{x+2} = x^3 \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^{k+1}}$$

$$= \sum_{x=0}^{\infty} (-1)^k \frac{x^{k+3}}{2^{k+1}}.$$

Converges when the series for $\frac{1}{x+2}$ does; for $|x| < 2$.

Differentiation and integration of power series.

Thm. If $\sum_{k=0}^{\infty} c_k (x-a)^k$ has the radius of convergence $R>0$, then the function it defines:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

is differentiable and continuous, and, moreover, there holds:

1). $$f'(x) = \sum_{k=0}^{\infty} k c_k (x-a)^{k-1} = c_1 + 2c_2 (x-a) + \dots$$

2). $$\int f(x) = \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1} = c_0 (x-a) + c_1 \cdot \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$

In addition, the radii of convergence of the series from 1). and 2). are equal to R. Their centers are still a.

In other words:
The intervals of convergence of the series in 1). and 2). are $(a-R, a+R)$, up to the endpoints.

Thus, differentiation and integration of a convergent power series **does not** change the radius of convergence.

Ex. 4 Express $\dfrac{1}{(1-x)^2}$ $= g(x)$ as a power series.

Recall: We know how to represent $\dfrac{1}{1-x}$:

$$\underbrace{\frac{1}{1-x}}_{f(x)} = \sum_{k=0}^{\infty} x^k$$

$$f'(x) = \left((1-x)^{-1}\right)' = (-1)(1-x)^{-2} \cdot (-1)$$
$$= (1-x)^{-2} = g(x).$$

$$g(x) = f'(x) = \left(\sum_{k=0}^{\infty} x^k\right)' \overset{\text{Thm}}{=} \sum_{k=0}^{\infty} k \cdot x^{k-1}$$

We know: the radius of convergence is preserved under differentiation, so $\sum_{k=0}^{\infty} k \cdot x^{k-1}$ converges on $(0-1, 0+1)$

This is not saying anything about the convergence at the endpoints, $x = \pm 1$.

Exercise: does this series converge at ± 1?

Remark about indexing:

$$\sum_{k=0}^{\infty} k \cdot x^{k-1} = \sum_{k=1}^{\infty} k \cdot x^{k-1}$$

$$= 1 \cdot 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 + \dots$$

By making the change $l = k-1$, $(k = l+1)$ the above series is: $\sum_{l=0}^{\infty} (l+1) \cdot x^l$

Ex. 5 Find the power series for $\ln(1+x)$, determine its radius of convergence.

$f(x) = \ln(1+x)$, Then

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \underbrace{\sum_{k=0}^{\infty} (-x)^k}_{\text{convergent when } |-x|<1,\ |x|<1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot x^k$$

$$f'(x) = \sum_{k=0}^{\infty} (-1)^k x^k \quad \text{when } |x| < 1$$

Integrate both sides:

$$f(x) = \int \frac{1}{1+x}\, dx = \ln(1+x)$$

$$= \int \sum_{k=0}^{\infty} (-1)^k x^k \, dx$$

$$= \sum_{k=0}^{\infty} \int (-1)^k x^k \, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \underline{\int x^k \, dx}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} + C$$

↳ Convergent for $|x| < 1$ (since integration preserves $R = 1$)

Note: the endpoints $x = \pm 1$ need to be checked separately

$f(x) = \ln(1+x) = \left(\text{let } \ell = k+1;\ k = \ell - 1 \right)$

$$= \sum_{\ell=1}^{\infty} \underbrace{(-1)^{\ell+1}} \frac{x^{\ell}}{\ell} + C$$

$(-1)^k = (-1)^{\ell-1} = (-1)^{\ell+1} = (-1)^{\ell-1} \cdot (-1)^2$

Let us determine C.

$$f(x) = \ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} + C$$

Set $x = 0$

$$\ln(1+0) = 0 = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{0^k}{k} + C$$

$$0 = 0 + C \Rightarrow C = 0$$

Answer: $\boxed{\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k},}$ $|x| < 1$

Example 6: Obtain the power series for $f(x) = \arctan(x)$.

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \overset{\frac{1}{1-x}}{=} \underbrace{\sum_{k=0}^{\infty} (-x^2)^k}_{\text{converges when}}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot x^{2k}$$

converges when

$|-x^2| < 1$

$|x|^2 < 1$

$|x| < 1$

Integrate both sides of

$$f'(x) = \sum_{k=0}^{\infty} (-1)^k \cdot x^{2k} \qquad |x| < 1$$

$$f(x) = \arctan x = \int \sum_{k=0}^{\infty} (-1)^k x^{2k} \, dx$$

$$= \sum_{k=0}^{\infty} \int (-1)^k x^{2k} \, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \int x^{2k} \, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{2k+1} + C$$

Converges for $|x| < 1$ (because integration preserves R).

Note: we are not saying anything about the endpoints of the convergence interval: $x = \pm 1$.

To determine C, set $x = 0$

$$f(0) = \underbrace{\arctan(0)}_{0} = \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{0^{2k+1}}{2k+1}}_{0} + C$$

$$0 = 0 + C \implies C = 0$$

$$\boxed{\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}} \qquad |x|<1$$

$$\underbrace{\arctan(x)}_{\substack{\text{odd function:} \\ f(-x) = -f(x)}} = x \underbrace{- \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}}_{\text{odd powers}} + \dots$$

$$\boxed{\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}} \qquad |x|<1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

Ex. 7 Expand $\int \frac{1}{1+x^7}\,dx$ into a power series.

$$\frac{1}{1+x^7} = \frac{1}{1-(-x^7)} \overset{\frac{1}{1-x}}{=\!=} \underbrace{\sum_{k=0}^{\infty} (\underbrace{-x^7}_{(-1)x^7})^k}_{\text{converges when}}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot x^{7k}$$

converges when

$|-x^7| < 1$

$|x^7| < 1$

$|x|^7 < 1$

$|x| < 1.$

$$\frac{1}{1+x^7} = \sum_{k=0}^{\infty} (-1)^k x^{7k}, \quad |x|<1$$

Integrate both sides of this equality:

$$\int \frac{dx}{1+x^7} = \int \sum_{k=0}^{\infty} (-1)^k x^{7k} \, dx, \quad |x|<1$$

$$= \sum_{k=0}^{\infty} \int (-1)^k x^{7k} \, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \int x^{7k} \, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{7k+1}}{7k+1} + C$$

- converges when $|x|<1$, since integration preserves the radius, so $R=1$.

The endpoints of the interval of convergence can change after integration.

Exercise: show that the series for $\int \frac{dx}{1+x^7}$ converges at $x=1$, diverges at $x=-1$.

Answer (to Ex. 7):

$$\int \frac{dx}{1+x^7} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{7k+1}}{7k+1} + C, \quad |x|<1.$$