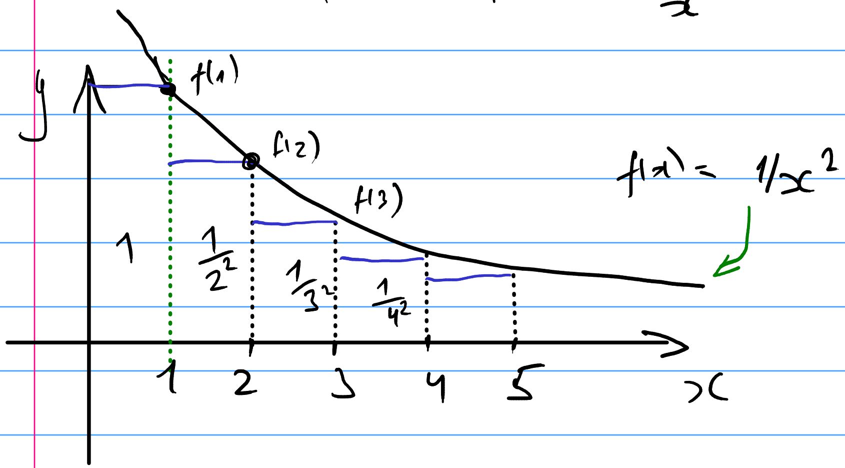
Section 11.3: Integral test

Ex. 1. Consider: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \overbrace{\frac{1}{3^2}}^{a_3} + \dots$

Then for $f(x) = \dfrac{1}{x^2}$, $\quad f(k) = a_k$



$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \boxed{\text{Area of rectangles}} \leq \underbrace{1}_{\text{1st rectangle}} + \int_1^{\infty} \frac{1}{x^2}$$

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^2}}_{a_k} \leq 1 + \int_1^{\infty} x^{-2}\,dx = 1 + \left(-x^{-1}\right)\Big|_1^{\infty}$$

$$= 2.$$

So, any partial sum satisfies

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$$

$\Rightarrow$ The sequence of partial sums $\{S_n\}$ is bounded:

$$0 < S_n \leq 2$$

But the a_k are nonnegative, so

$$s_{n+1} - s_n = \sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1} > 0$$

$\Rightarrow s_{n+1} > s_n$.

The sequence of partial sums $\{s_n\}$ is increasing and bounded $\Rightarrow$ convergent!

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

(not to 2 though).