Measure and Integration II (MAA5617), Spring 2021
Homework 1, due Thursday, Jan 28

Below ν is a signed measure on a measurable space $(X, \mathcal{M})$.

1. Prove: E is a null set for ν iff $|\nu|(E) = 0$.

2. Let ν_-, ν_+ be the Jordan decomposition of ν. For any $E \in \mathcal{M}$, show:

- $\nu_+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$;
- $\nu_-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$;
- $|\nu|(E) = \sup\{\left|\int_E f\, d\nu\right| : |f| \leq 1\}$, where f are taken to be $\mathcal{M}$-measurable functions on X.

3. Suppose ψ, ξ are positive measures such that $\nu = \psi - \xi$. Show:

$$\psi \geq \nu_+, \qquad \xi \geq \nu_-.$$

4. Using the previous question, show that for finite signed measures ν_1, ν_2 on $(X, \mathcal{M})$ there holds a triangle inequality for total variation:

$$|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|.$$

5. (To be solved after the Radon-Nikodym theorem is proved.) Given a σ-finite measure μ on $(X, \mathcal{M})$, suppose that $\mathcal{N} \subset \mathcal{M}$ is a σ-algebra, and let $\nu := \mu|_{\mathcal{N}}$. Let further $f \in L^1(\mu)$ be given. Show:

- there exists an $\mathcal{N}$-measurable $g \in L^1(\nu)$ such that

$$\int_E f\, d\mu = \int_E g\, d\nu, \qquad \text{for all } E \in \mathcal{N}.$$

- If g_0 is another such function, $g_0 = g$ ν-a.e.

In probability theory, measurable functions are called *random variables*. The random variable g introduced in this problem is known as the *conditional expectation* of random variable f *with respect* to the σ-algebra $\mathcal{N}$.

6. Compute the volume of the unit ball in $\mathbb{R}^n$.

Express this volume as the integral of $n-1$-dimensional volumes of sections orthogonal to a coordinate axis, then obtain a recurrence relation. You will need the following standard identity for the B-function:

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}.$$