REVIEW QUESTIONS FOR TEST 1

Integration by parts

1) Write down the formula for integration by parts. How should u and dv be selected? What class of functions requires using this technique?
2) Evaluate the integrals:
   (a) $\displaystyle\int x^2 \cos x\, dx$
   (b) $\displaystyle\int x \sin 6x\, dx$
   (c) $\displaystyle\int e^{-\theta} \cos 2\theta\, d\theta$
   (d) $\displaystyle\int x^5 \ln x\, dx$
   (e) $\displaystyle\int_0^{1/\sqrt{2}} \sin^{-1} x\, dx$
   (f) $\displaystyle\int x \ln(2x+5)\, dx.$

Trigonometric integrals

3) Products of which functions comprise this type of integrals? Which two main strategies we used? Write down Pythagorean identities for sin and cos, and also for sec and tan. How are they used? Write down the half-angle formulas. In what cases they should be used?
4) Evaluate the integrals:
   (a) $\displaystyle\int \sin^4 x \cos^3 x\, dx$
   (b) $\displaystyle\int \sin^2 x\, dx$
   (c) $\displaystyle\int \cos^4 x\, dx$
   (d) $\displaystyle\int_0^{\pi/3} \tan^5 x \sec^3 x\, dx$
   (e) $\displaystyle\int \tan^2 x\, dx$
   (f) $\displaystyle\int \tan x (\tan x \sec x)^4\, dx$
   (g) $\displaystyle\int \sec 3x\, dx.$

Trigonometric substitution

5) What are the three types of trigonometric substitutions? What kinds of expressions are they used for? How are Pythagorean identities involved in the substitution? Write down the transformations inside the radical after a substitution has been performed, for all the three substitutions.
6) Evaluate the integrals:
   (a) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$
   (b) $\displaystyle\int_0^2 \frac{1}{\sqrt{4+x^2}}\,dx$
   (c) $\displaystyle\int \sqrt{25-9x^2}\,dx$
   (d) $\displaystyle\int x\sqrt{x^2+2x}\,dx$
   (e) $\displaystyle\int_{2/3}^{4/3} \frac{x}{\sqrt{9x^2-4}}\,dx$
   (f) $\displaystyle\int \frac{x}{\sqrt{x^2+7}}\,dx.$

Integration of rational functions

7) Explain what is a rational function. What is a proper rational function? Using long division, express
$$\frac{3x^4-x^3+11x+3}{x^2+5x+1}$$
as a sum of a polynomial and a proper fraction.

   What are the four types of factors in the denominator $Q(x)$? Write down partial fractions that corresponds to unique linear factors of $Q(x)$; the same for repeated linear factors. What is the partial fraction corresponding to irreducible quadratic factors; to such repeated factors? Write down your answers.
8) Write down the partial fraction decomposition for:
$$\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}.$$
9) Evaluate the integrals:
   (a) $\displaystyle\int_{-1}^0 \frac{x^3-4x+1}{x^2-3x+2}\,dx$
   (b) $\displaystyle\int \frac{x}{x^3+x^2+x+1}\,dx$
   (c) $\displaystyle\int \frac{x^3+6x-2}{x^4+6x^2}\,dx$
   (d) $\displaystyle\int \frac{1}{2\sqrt{x+3}+x}\,dx$
   (e) $\displaystyle\int \frac{\ln x+5}{x(\ln^2 x+4)^2}\,dx$
   (f) $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx.$

Strategies for integration

10) What are the four classes of functions for which we have developed special approaches? What two fundamental techniques underlie most integration strategies?
11) See # 13, 25, 41, 63, 70, 75 in Section 7.5.

Improper integrals

12) What two types of improper integrals do you know? Explain, how the value of an improper integral is approximated by proper integrals in each case. For what values of p is the integral
$$\int_1^\infty \frac{1}{x^p}\,dx$$
convergent? State the comparison theorem and explain how to determine convergence of an improper integral by comparing it to another improper integral.
13) Determine convergence of the following integrals:

(a) $\displaystyle\int_0^\infty \cos 2x\,dx$

(b) $\displaystyle\int_0^\infty \frac{dx}{\sqrt[4]{1+3x}}$

(c) $\displaystyle\int_{20}^\infty \frac{dx}{9x^2-42x+49}$

(d) $\displaystyle\int_0^\infty \cos^2 x\,dx$

(e) $\displaystyle\int_3^5 \frac{dx}{x-3}$

(f) $\displaystyle\int_0^2 \frac{dx}{\sqrt{x}}$.

14) Determine convergence of the following integrals and evaluate the convergent ones:

(a) $\displaystyle\int_0^\infty e^{-\sqrt{x}}\,dx$

(b) $\displaystyle\int_1^\infty xe^{-x^2}\,dx$

(c) $\displaystyle\int_0^\infty \frac{dx}{4x+10^{100}}$

(d) $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2}$

(e) $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-1}}$.

(f) $\displaystyle\int_1^3 \frac{dx}{\sqrt[3]{6-2x}}$.

Answer key

2) (a) $2\,x\cos(x) + \left(x^2 - 2\right)\sin(x) + C$
   (b) $-\frac{1}{6}\,x\cos(6\,x) + \frac{1}{36}\,\sin(6\,x) + C$
   (c) $-\frac{1}{5}\,(\cos(2\theta) - 2\,\sin(2\theta))e^{-\theta} + C$
   (d) $\frac{1}{6}\,x^6\log(x) - \frac{1}{36}\,x^6 + C$
   (e) $\frac{1}{8}\,\sqrt{2}\pi + \frac{1}{2}\,\sqrt{2} - 1$
   (f) $-x^2/4 + 5x/4 + \frac{1}{8}\left(4\,x^2 - 25\right)\log(2\,x + 5) + C$.

4) (a) $-\frac{1}{7}\,\sin(x)^7 + \frac{1}{5}\,\sin(x)^5 + C$
   (b) $\frac{1}{2}\,x - \frac{1}{4}\,\sin(2\,x) + C$
   (c) $\frac{3}{8}\,x + \frac{1}{32}\,\sin(4\,x) + \frac{1}{4}\,\sin(2\,x) + C$
   (d) $848/105$
   (e) $-x + \tan(x) + C$
   (f) $\frac{1}{8}\,\sec(x)^8 - \frac{1}{3}\,\sec(x)^6 + \frac{1}{4}\,\sec(x)^4 + C = \frac{1}{8}\,\tan(x)^8 + \frac{1}{6}\,\tan(x)^6 + C$
   (g) $\frac{1}{3}\,\log|\sec(3\,x) + \tan(3\,x)| + C$.

6) (a) $-\frac{1}{2}\,\sqrt{-x^2 + 16}x + 8\,\arcsin\left(\frac{1}{4}\,x\right) + C$
   (b) $\log\left(\sqrt{2} + 1\right)$
   (c) $\frac{1}{2}\,\sqrt{-9\,x^2 + 25}x + \frac{25}{6}\,\arcsin\left(\frac{3}{5}\,x\right) + C$
   (d) $\frac{1}{3}\left(x^2 + 2\,x\right)^{\frac{3}{2}} - \frac{1}{2}\,\sqrt{x^2 + 2\,x}(x + 1) + \frac{1}{2}\,\log\left(2\,x + 2\,\sqrt{x^2 + 2\,x} + 2\right) + C$
   (e) $\frac{2}{9}\,\sqrt{3}$
   (f) $\sqrt{x^2 + 7} + C$.

7) $\dfrac{3x^4 - x^3 + 11x + 3}{x^2 + 5x + 1} = 3\,x^2 - 16\,x + 77 - \dfrac{2\,(179\,x + 37)}{x^2 + 5\,x + 1}$

8) $\dfrac{x^4 - 2x^3 + x^2 + 2x - 1}{x^2 - 2x + 1} = x^2 + \dfrac{2}{x - 1} + \dfrac{1}{(x - 1)^2}$

9) (a) $-\log(3) - \log(2) + \frac{5}{2}$
   (b) $\frac{1}{2}\,\arctan(x) + \frac{1}{4}\,\log\left(x^2 + 1\right) - \frac{1}{2}\,\log(x + 1) + C$
   (c) $\frac{1}{18}\,\sqrt{6}\arctan\left(\frac{1}{6}\,\sqrt{6}x\right) + \frac{1}{3\,x} + \log(x) + C$
   (d) $\frac{3}{2}\,\log\left(\sqrt{x + 3} + 3\right) + \frac{1}{2}\,\log\left(\sqrt{x + 3} - 1\right) + C$

(e) $\dfrac{5\log(x)-4}{8\left(\log(x)^2+4\right)}+\dfrac{5}{16}\arctan\left(\dfrac{1}{2}\log(x)\right)+C$

(f) $2\log(e^x+2)-\log(e^x+1)+C$.

13) (a) Divergent.
(b) Divergent.
(c) Convergent.
(d) Divergent.
(e) Divergent.
(f) Convergent.

14) (a) Convergent to 2.
(b) Convergent to $1/2e$.
(c) Divergent.
(d) Convergent to 1.
(e) Convergent to 6.
(f) Convergent to $\dfrac{3}{2^{2/3}}$.