REVIEW QUESTIONS FOR PARAMETRIC CURVES AND POLAR COORDINATES

Parametric curves

1) What is a parametric curve? How and when can one obtain the Cartesian equation of a parametric curve? How to compute the slope of the tangent to a parametric curve? Write down the expressions for the area under a parametric curve, and the arc length of a parametric curve. What is the equation for the area of the surface obtained by rotation of a parametric curve?
2) Eliminate the parameter from the given parametric curve and sketch its graph. Show the direction of motion as the parameter increases.
   (a) $x = \sin t,\ y = \csc t, \quad 0 < t < \pi/2$
   (b) $x = e^t,\ y = e^{-2t}, \quad -\infty < t < \infty$.
3) Find the equation of the tangents to the following curves for the given value of the parameter t or at the given point (x, y):
   (a) $x = t^3 + 1,\ y = t^4 + t, \quad t = -1$
   (b) $x = t\cos t,\ y = t\sin t, \quad t = \pi$
   (c) $x = 1 + \ln t,\ y = t^2 + 2, \quad (1, 3)$
   (d) $x = t^2 - t,\ y = t^2 + t + 1, \quad (0, 3)$.
4) Find the length of the given curve:
   (a) $x = 1 + 3t^2,\ y = 4 + 2t^3, \quad 0 \le t \le 1$
   (b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 2$
   (c) $x = t\sin t,\ y = t\cos t, \quad 0 \le t \le 1$.
5) Find the area of surfaces obtained by rotating the following curves about the x-axis:
   (a) $x = t^3,\ y = t^2, \quad 0 \le t \le 1$
   (b) $x = 2\cos^3 t,\ y = 2\sin^3 t, \quad 0 \le t \le \pi/2$.
6) Find the area of surfaces obtained by rotating the following curves about the y-axis:
   (a) $x = 8\sqrt{t},\ y = 2t^2 + 1/t, \quad 1 \le t \le 3$
   (b) $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 1$.

Polar coordinates

7) Explain how a point in the plane is determined by its polar coordinates. Write down the conversion formulas between the Cartesian and polar coordinates. How does the quadrant containing a certain point determines its polar angle?
8) Plot the point with the given polar coordinates and find its Cartesian coordinates:
   (a) $(1, \pi/3)$
   (b) $(-8, 7\pi/4)$
   (c) $(3, -5\pi/2)$
   (d) $(0, -71\pi/4)$
   (e) $(2\sqrt{2}, 3\pi/4)$.
9) For the given Cartesian coordinates of a point, find its polar coordinates (r, θ) with $r \ge 0$ and $0 \le \theta < 2\pi$. Then, find the expression of polar coordinates with $r \le 0$ and $0 \le \theta < 2\pi$:
   (a) $(0, 5)$

(b) $(5\sqrt{3}, -5)$
(c) $(2\sqrt{2}, -2\sqrt{2})$
(d) $(1, \sqrt{3})$.

Polar curves

10) What is a polar curve? How to sketch the graph of a polar curve, given its equation? How to determine the slope of a polar curve at a given point?
11) Identify the curve:
    (a) $r^3 = 125$
    (b) $\theta = \pi/4$
    (c) $r = 4\sec\theta$
    (d) $r = -2\sec\theta$
    (e) $r = 3\csc\theta$
    (f) $r^2 \cos 2\theta = 1$.
12) Find a polar equation for the curve given in Cartesian coordinates:
    (a) $y = 2$
    (b) $y = x$
    (c) $x^2 + y^2 = 2x$
    (d) $4y^2 = x$
    (e) $x^2 - y^2 = 4$.
13) Find the slope of the tangent line to the given polar curve at the point corresponding to the specified value of θ:
    (a) $r = 2\cos\theta, \quad \theta = \pi/3$
    (b) $r = 1/\theta, \quad \theta = \pi$
    (c) $r = 2 + \sin 3\theta, \quad \theta = \pi/4$
    (d) $r = \cos(\theta/3), \quad \theta = \pi$.
14) Find points on the given curve where the tangent line is horizontal or vertical:
    (a) $r = 3\cos\theta$
    (b) $r = 1 + \cos\theta$
    (c) $r = e^\theta$
    (d) $r = 1 - \sin\theta$.
15) Sketch the polar curve:
    (a) $r = \theta, \quad \theta \geq 0$
    (b) $r = 1 + \sin\theta$
    (c) $r = 2 + \sin 3\theta$
    (d) $r^2 = \cos 4\theta$
    (e) $r = 2\cos(\theta/2)$.

Areas and lengths for polar curves

16) Write down the formula for the area enclosed by a polar curve $r = f(\theta)$, as well as for the area between a pair of polar curves $r = f(\theta)$ and $r = g(\theta)$. Explain how to determine the range of integration in the corresponding integrals. How to compute the arc length of a polar curve?
17) Find the area of the region that lies inside the first curve and outside the second curve:
    (a) $r = 4\sin\theta, \quad r = 2$
    (b) $r = 1 - \sin\theta, \quad r = 1$

(c) $r = 3\cos\theta, \quad r = 1 + \cos\theta$
(d) $r^2 = 8\cos 2\theta, \quad r = 2.$

18) Find all the points of intersection of the given curves:
(a) $r = \sin\theta, \quad r = 1 - \sin\theta$
(b) $r = 1 + \cos\theta, \quad r = 1 - \sin\theta$
(c) $r = \sin\theta, \quad r = \sin 2\theta$
(d) $r = 2\sin 2\theta, \quad r = 1.$

19) Find the length of the polar curve:
(a) $r = 2\cos\theta, \quad 0 \le \theta \le \pi$
(b) $r = \theta^2, \quad 0 \le \theta \le 2\pi$
(c) $r = 2(1 + \cos\theta).$

Answer key

3) (a) $y = 1/x, \quad 0 < x < 1$
   (b) $y = 1/x^2, \quad x > 0.$

4) (a) $y = -x$
   (b) $y = \pi(x + \pi)$
   (c) $y - 3 = 2(x - 1)$
   (d) $y - 3 = 3x.$

5) (a) $4\sqrt{2} - 2$
   (b) $e^2 + 1$
   (c) $\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1 + \sqrt{2}).$

6) (a) $\frac{2}{1215}\pi\left(247\sqrt{13} + 64\right)$
   (b) $\frac{24}{5}\pi.$

7) (a) $\frac{32}{15}\pi\left(103\sqrt{3} + 3\right)$
   (b) $\pi\left((e+1)^2 - 7\right).$

8) (a) $(1/2, \frac{\sqrt{3}}{2})$
   (b) $(-4\sqrt{2}, 4\sqrt{2})$
   (c) $(0, -3)$
   (d) $(0, 0)$
   (e) $(-2, 2).$

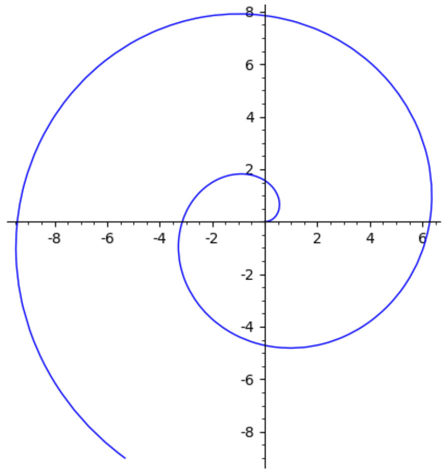
9) (a) $(5, \pi/2); \quad (-5, 3\pi/2)$
   (b) $(10, 11\pi/6); \quad (-10, 5\pi/6)$
   (c) $(4, 7\pi/4); \quad (-4, 3\pi/4)$
   (d) $(2, \pi/3); \quad (-2, 4\pi/3).$

11) (a) Circle $x^2 + y^2 = 25.$
    (b) Line $y = x.$
    (c) Line $x = 4.$
    (d) Line $x = -2.$
    (e) Line $y = 3.$
    (f) Hyperbola $x^2 - y^2 = 1.$

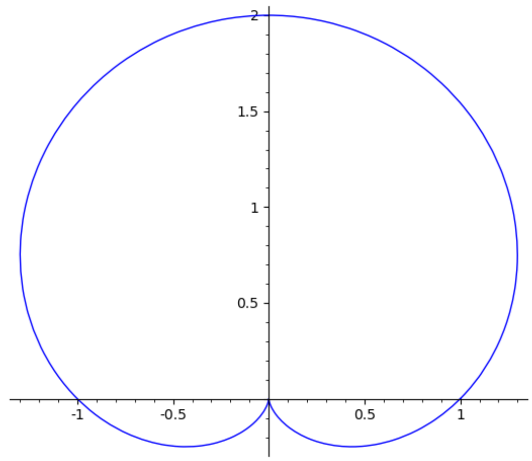
12) (a) $r = 2\csc\theta$
    (b) $\theta = \pi/4$
    (c) $r = 2\cos\theta$
    (d) $r = \frac{1}{4}\csc\theta\cot\theta$
    (e) $r^2\cos 2\theta = 4.$

13) (a) $\frac{1}{3}\sqrt{3}$
    (b) $-\pi$
    (c) $-\frac{\sqrt{2} - 1}{\sqrt{2} + 2}$
    (d) $-\sqrt{3}.$

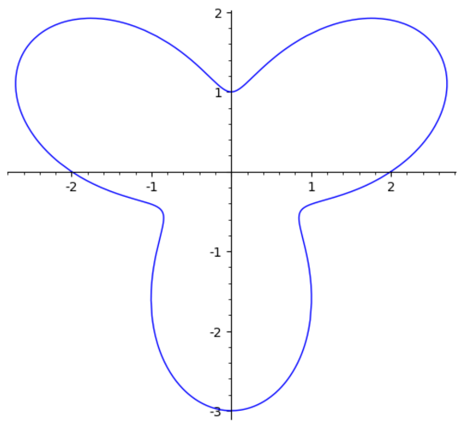
14) The answers are given in polar coordinates; only unique points are included. For example, in (a), the point $(-3/\sqrt{2}, 5\pi/4)$ also has a horizontal tangent, but it coincides with $(3/\sqrt{2}, \pi/4)$, and so is omitted.
    (a) Horizontal: $(3/\sqrt{2},\ \pi/4), (-3/\sqrt{2},\ 3\pi/4)$;
        vertical: $(3,\ 0), (0, 0)$.
    (b) Horizontal: $(3/2, \pi/3), (0, \pi), (3/2, 5\pi/3)$;
        vertical: $(2, 0), (1/2,\ 2\pi/3), (1/2,\ 4\pi/3)$.
    (c) Horizontal: $(e^{3\pi/4 + k\pi},\ 3\pi/4 + k\pi)$, k–integer;
        vertical: $(e^{\pi/4 + k\pi},\ \pi/4 + k\pi)$, k–integer.
    (d) Horizontal: $(1/2,\ \pi/6), (2,\ 3\pi/2), (1/2,\ 5\pi/6)$;
        vertical: $(0, \pi/2), (3/2,\ 7\pi/6), (3/2,\ 11\pi/6)$.



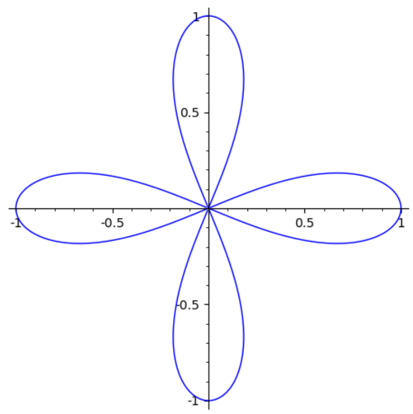
(a) $r = \theta$



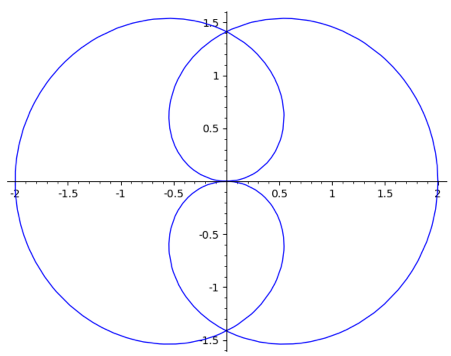
(b) $r = 1 + \sin\theta$



(c) $2 + \sin 3\theta$



(d) $r^2 = \cos 4\theta$



(e) $r = 2\cos(\theta/2)$

15) See the graphs (a)–(e).

17) (a) $\dfrac{4}{3}\pi + 2\sqrt{3}$

(b) $\dfrac{1}{4}\pi + 2$

(c) π

(d) $2\left(-\dfrac{2}{3}\,\pi + 2\sqrt{3}\right)$.

18) The answers are given in polar coordinates; only unique points are included.

(a) $(1/2, \pi/6), (1/2, 5\pi/6)$

(b) $(0,0), (1-1/\sqrt{2}, 3\pi/4), (1+1/\sqrt{2}, 7\pi/4)$

(c) $(0,0), (\sqrt{3}/2, \pi/3), (\sqrt{3}/2, 2\pi/3)$

(d) $(1, \pi/12)$, $(1, 5\pi/12)$, $(1, 7\pi/12)$, $(1, 11\pi/12)$, $(1, 13\pi/12)$, $(1, 17\pi/12)$, $(1, 19\pi/12)$, $(1, 23\pi/12)$.

19) (a) 2π

(b) $\dfrac{8}{3}\,(\pi^2+1)^{3/2} - \dfrac{8}{3}$

(c) 16.