Conic sections

quadratic → degenerate

parabola → a line → $y=0$

ellipse → a point → $x^2+y^2=0$

hyperbola → 2 lines → $y^2-m^2x^2=0$

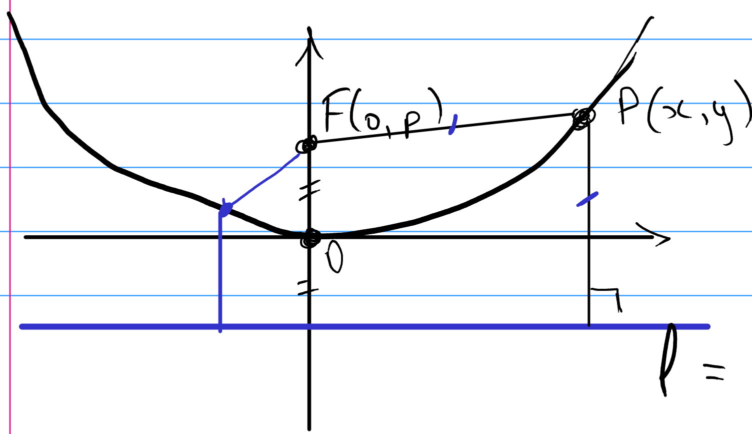
$$x^2=4py \quad \text{— parabola}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{— ellipse}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{— hyperbola}$$

Parabola

Set of points equidistant (= at the equal distance) from a fixed point F (focus), and a fixed line ℓ (directrix)



$\ell = \quad y=-p$

For any point $P(x,y)$ that lies on the parabola,

$$|PF| = |Pl|$$

$$\sqrt{(x-0)^2+(y-p)^2} = |y-(-p)|$$

$$\sqrt{x^2+(y-p)^2} = |y+p|$$

To simplify: square both sides

$(\ldots)$

$$x^2 = 4p \cdot y$$

(parabola with focus $(0,p)$ directrix $y=-p$)

Interchanging $x \leftrightarrow y$:

$$y^2 = 4p \cdot x$$

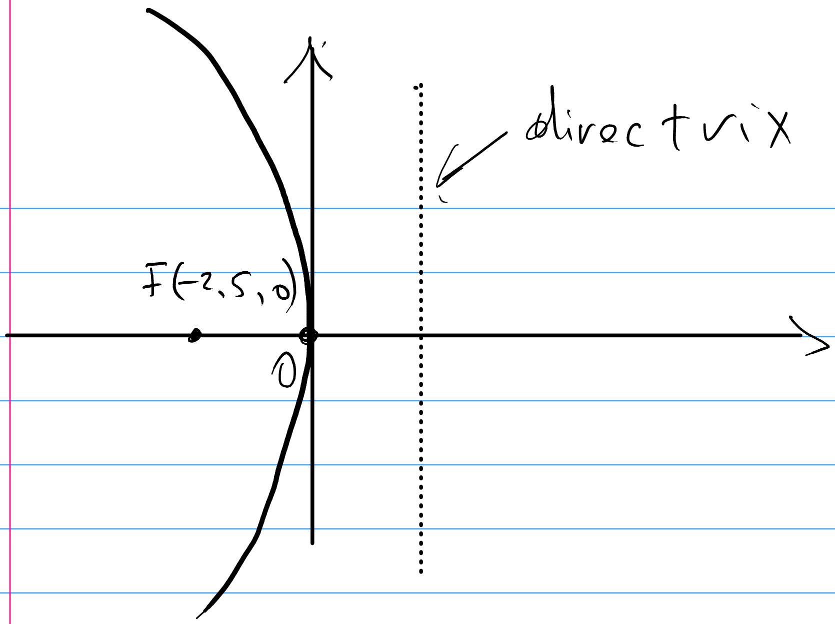
(parabola with focus $(p,0)$, directrix $x=-p$)

Ex. 1 Find the focus, directrix for:

$$y^2+10x=0$$

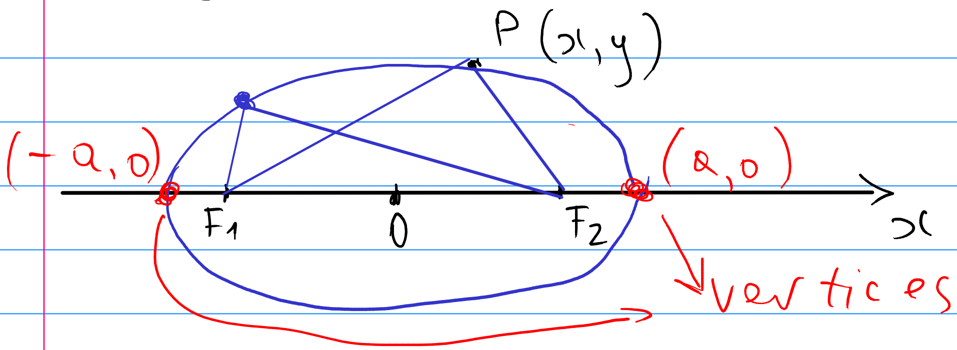
$$y^2 = \underbrace{-10}_{4p} x$$

parabola - horizontal w/ focus $(p,0) = (-2.5, 0)$, directrix $x = 2.5$



Ellipse:

Set of points in the plane, such that the sum of distances from them to two fixed point F_1, F_2 (foci) is constant.



$F_1(-c, 0)$
$F_2(c, 0)$

$$\underbrace{|PF_1| + |PF_2| = 2a}_{\text{using the distance formula:}} \qquad a>0, \text{ const}$$

$$\sqrt{(x+c)^2+(y-0)^2} + \underbrace{\sqrt{(x-c)^2+(y-0)^2}} = 2a$$

+ Square both sides, simplify:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$ denote $a^2 - c^2 = b^2$

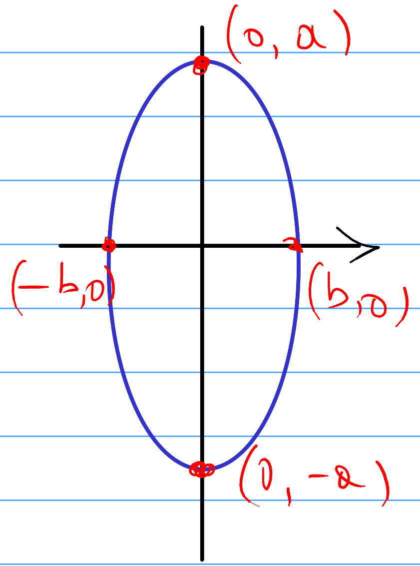
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

ellipse w/
foci $(\pm c, 0)$, $c^2 = a^2 - b^2$
vertices $(\pm a, 0)$
horizontally oriented

Interchanging $x \leftrightarrow y$:

$$\boxed{\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1}$$

ellipse w/
foci $(0, \pm c)$, $c^2 = a^2 - b^2$
vertices $(0, \pm a)$
vertically oriented



see here

Ex. 2 Sketch $9x^2 + 16y^2 = 144$, locate its foci, vertices.

$9x^2 + 16y^2 = 144$ $\quad / \, 144$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

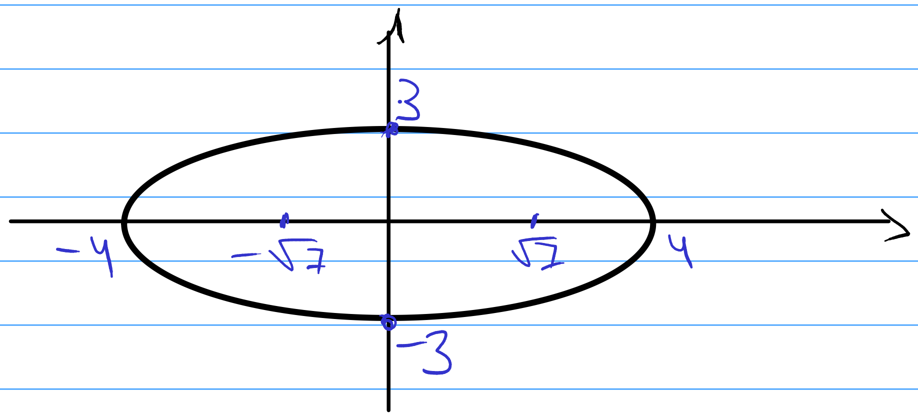
$16 \leftarrow a^2$, $9 \leftarrow b^2$ (a is bigger than b)

$a = 4$, $b = 3$ $\Rightarrow$ $c = \sqrt{16-9} = \sqrt{7}$

(horizontally oriented ellipse)

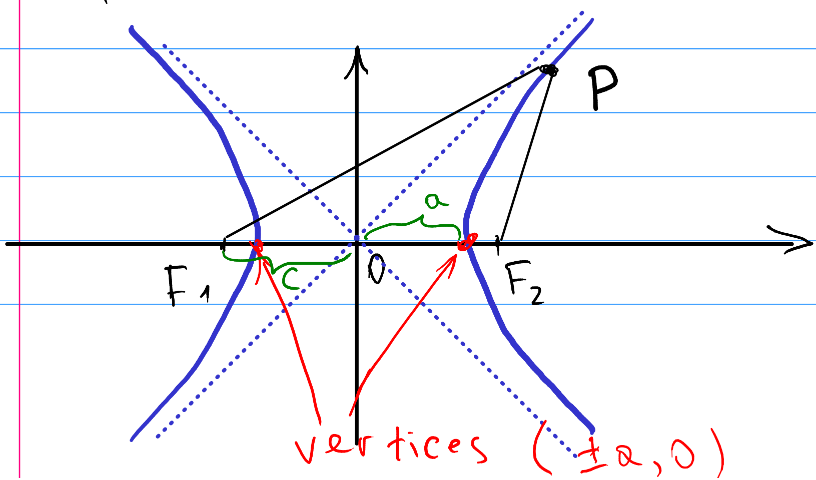
$\Rightarrow$ Foci $(\pm\sqrt{7}, 0)$

Vertices $(\pm 4, 0)$



Hyperbola

Set of points, such that the difference of distances to two fixed points F_1, F_2 (foci) is constant



$F_1(-c, 0)$ $\quad c > 0$

$F_2(c, 0)$

$$|PF_1| - |PF_2| = \pm 2a \qquad a > 0, \text{ constant}$$

Using the distance formula just as for ellipse gives:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{where } c^2 = a^2 + b^2$$

Hyperbola, oriented horizontally, w/
foci $(\pm c, 0)$
vertices $(\pm a, 0)$

The asymptotes correspond to large x and y. This means, in

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

1 is much smaller than the other terms,

1 is negligible $\Rightarrow$ $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2}$ } equation of the asymptotes

Indeed: solving it gives

$$\frac{y}{b} = \pm \frac{x}{a} \Rightarrow y = \pm \frac{b}{a} x$$

Summary:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

} horizontally oriented hyperbola w/
foci $(\pm c, 0)$
vertices $(\pm a, 0)$
$c^2 = a^2 + b^2$

asymptotes
$$y = \pm \frac{b}{a} x$$

Interchanging $x \rightleftharpoons y$ gives

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

} **vertically** oriented hyperbola w/
foci $(0, \pm c)$
vertices $(0, \pm a)$
$c^2 = a^2 + b^2$

asymptotes
$$y = \pm \frac{a}{b} x$$

Ex. 3 Find the foci and equation of the hyperbola w/ vertices $(0, \pm 1)$, asymptote $y = 2x$.

Asymptotes: $y = \pm 2x = \pm \frac{a}{b} x$

Vertically oriented hyperbola

$\frac{a}{b} = 2$ also: $a = 1$, since vertices $(0, \pm a)$

Thus: $b = 1/2$, $c^2 = a^2 + b^2$

$c^2 = 1 + \frac{1}{4}$

$c = \frac{\sqrt{5}}{2}$

We can determine the foci now:

$(0, \pm c) = (0, \pm \frac{\sqrt{5}}{2})$.

Furthermore, the equation is:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{1^2} - \frac{x^2}{(1/2)^2} = 1$$

$$y^2 - 4x^2 = 1$$