Section 11.2 : Series

Previously: sequence = an infinite list of numbers

series = a sum of an infinite sequence of numbers.

Consider e, the base of the natural log

$$\underbrace{e = 2.718281828459045\ldots}_{\text{shorthand for:}}$$

$$e = \underbrace{2 + 7\cdot\frac{1}{10} + 1\cdot\frac{1}{100} + 8\cdot\frac{1}{1000} + 2\cdot\frac{1}{10\,000} + \ldots}_{\text{infinite sum} = \text{a series}}$$

Given a sequence $\{a_n\}$, $a_1, a_2, a_3, \ldots$

we will write

$$\sum_{n=1}^{\infty} a_n \qquad\qquad \sum_n a_n$$

When is it meaningful to consider infinite sums?

A partial sum:

$$S_n = \sum_{k=1}^{n} a_k$$

Def For a series $\sum_{k=1}^{\infty} a_k$, let $s_n = \sum_{k=1}^{n} a_k$.

If the sequence $\{s_n\}$ is convergent,

$\lim_{n\to\infty} s_n = s$ (a finite number)

then the series is said to be convergent, and

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = s$$

s - sum of the series.

If the limit DNE or is $\pm\infty$, the series is divergent.

Sum of a series $\overset{def}{=}$ limit of of partial sums

$$\sum_{k=1}^{\infty} a_k \overset{def}{=} \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Ex. 1 Suppose $S_n = \sum_{k=1}^{n} a_k = \frac{2n}{3n+5}$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

$$= \lim_{n\to\infty} \frac{2n}{3n+5} \frac{/n}{/n} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}.$$

$$S_n = \sum_{k=1}^{n} a_k = \frac{2n}{3n+5}$$

$$S_1 = a_1 = \frac{2\cdot 1}{3\cdot 1+5} = \frac{1}{4}$$

$$S_2 = a_1 + a_2 = \frac{2\cdot 2}{3\cdot 2+5}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{2\cdot 3}{3\cdot 3+5}$$

$$\ldots$$

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

$$S_{n+1} = \sum_{k=1}^{n+1} a_k = a_1 + a_2 + \ldots + a_n + a_{n+1}$$

$$S_{n+1} - S_n = a_{n+1}$$

$$\boxed{S_n - S_{n-1} = a_n}$$

$$\frac{2n}{3n+5} - \frac{2(n-1)}{3(n-1)+5} = a_n$$

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{2k}{3k+5} - \frac{2k-2}{3k+2} \right)$$

Ex. 2 Compute the sum of

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k(k+1)}}_{a_k}$$

} telescoping series

Consider a_k :

$$a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_1 = a_1 = \frac{1}{1} - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)$$

$$= 1 - \frac{1}{3}$$

$$S_3 = \underbrace{a_1 + a_2}_{S_2} + a_3 = S_2 + a_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}$$

$$= 1 - \frac{1}{4}$$

$$S_3 = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right)$$

$$= 1 - \frac{1}{5}.$$

$$S_n = a_1 + a_2 + \ldots + a_n$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

Ex. 3 The geometric series:

$$\sum_{k=1}^{\infty} a r^{k-1} \qquad a \neq 0$$

r - common ratio

$$\sum_{k=1}^{\infty} a r^{k-1} = a + ar + ar^2 + ar^3 + \dots$$

(Previously: $\{r^n\}$)

Suppose $r=1$, then

$$\sum_{k=1}^{\infty} a = a + a + a + \dots$$

$$S_n = n \cdot a$$

$$\lim_{n\to\infty} S_n = +\infty \cdot \frac{a}{|a|}$$

Now let $r \neq 1$.

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$r \cdot S_n = \quad ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$(1-r)\,S_n = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} = \sum_{k=1}^{n} a r^{k-1}.$$

$$\sum_{k=1}^{\infty} a r^{k-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r} \lim_{n\to\infty} (1-r^n) =$$

Recall that $\lim_{n\to\infty} r^n = \begin{cases} 1, & r=1 \\ 0, & |r|<1 \\ \text{divergent} & \text{otherwise} \end{cases}$

Thus,

$$= \frac{a}{1-r}\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} r^n\right) = \frac{a}{1-r}.$$

To summarize:

Geometric series

$$\boxed{\sum_{k=1}^{\infty} a\cdot r^{k-1} = \frac{a}{1-r}, \quad \text{if } |r|<1 \ \text{(convergent)} \qquad \text{divergent otherwise (for all other } r\text{).}}$$

$$\sum_{k=1}^{\infty} a\cdot r^{k-1} = a + ar + ar^2 + ar^3 + \ldots$$

Ex. 4 Find the sum of the given series:

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots \quad \boxed{= 3}$$

5

$$\frac{a_2}{a_1} = -\frac{10}{3} / 5 = -\frac{2}{3}$$

$$\frac{a_4}{a_3} = -\frac{40}{27} \Big/ \frac{20}{9} = -\frac{40}{27} \cdot \frac{9}{20} = \underbrace{-\frac{2}{3}}_{r}.$$

The sum of this geometric series:

$$\frac{a}{1-r} = \frac{5}{1-(-2/3)} = \frac{5}{5/3} = 3.$$

Ex. 5 Is $\sum_{k=1}^{\infty} \underbrace{2^{2k} 3^{1-k}}_{a_k}$ convergent?

$$\frac{a_{k+1}}{a_k} = 2^{2(k+1)} \cdot 3^{1-(k+1)} \Big/ 2^{2k} \cdot 3^{1-k}$$

$$= 2^{2k+2-2k} \cdot 3^{1-k-1-1+k}$$

$$= 2^2 \cdot 3^{-1} = \underbrace{\frac{4}{3}}_{r} > 1$$

It follows, $|r| \geq 1$, the series is divergent.

a = first term $= a_1 = 2^2 \cdot 3^0 = 4.$

Ex. 6 Write $2.3\overline{17} = 2.3\underbrace{17}\underbrace{17}\underbrace{17}\ldots$ as an irreducible fraction.

$$2.3\overline{17} = 2.3 + \underbrace{\overbrace{\frac{17}{10^3}}^{a} + \frac{17}{10^5} + \frac{17}{10^7} + \ldots}_{\text{geometric series}}$$

$$\left.\begin{array}{l} \dfrac{17}{10^5} \Big/ \dfrac{17}{10^3} = \dfrac{1}{10^2} \\[2ex] \dfrac{17}{10^7} \Big/ \dfrac{17}{10^5} = \dfrac{1}{10^2} \end{array}\right\} r$$

$$2.3\overline{17} = 2.3 + \frac{17/10^3}{1 - \frac{1}{10^2}} \qquad \begin{array}{l} \times 10^3 \\ \times 10^3 \end{array}$$

$$= 2.3 + \frac{17}{1000 - 10} = \frac{23}{10} + \frac{17}{990}$$

$$= \frac{23 \cdot 99 + 17}{990} = \frac{2300 - 23 + 17}{990}$$

$$= \frac{2294}{990} = \frac{1147}{495}.$$

Thm. If the series $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} a_k = 0$.

Pf. If the series converges to a finite S, then

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

We know:

$$S_n - S_{n-1} = a_n.$$

Apply $n\to\infty$:

$$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0 = \lim_{n\to\infty} a_n,$$

So, $\lim_{k\to\infty} a_k = 0$.

$\square$

This gives:

<u>Test for divergence</u>: if in the series $\sum_{k=1}^{\infty} a_k$, there holds $\lim_{k\to\infty} a_k \neq 0$ or DNE, then this series is divergent.

Warning: the opposite isn't true!

I.e., if $\lim_{k\to\infty} a_k = 0 \not\Rightarrow \sum_{k=1}^{\infty} a_k$ converges.

Ex. 7 Show that $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k}}_{\text{harmonic series}}$ is divergent.

Consider some partial sums of this series:

$$S_1 = \frac{1}{1}$$

$$S_2 = \frac{1}{1} + \frac{1}{2}$$

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$$

$$S_4 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$
$$= 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}$$

$$S_8 = \frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{3}}_{>1/4} + \frac{1}{4} + \underbrace{\frac{1}{5}}_{>1/8} + \underbrace{\frac{1}{6}}_{>1/8} + \underbrace{\frac{1}{7}}_{>1/8} + \underbrace{\frac{1}{8}}_{\geq 1/8}$$

$$> 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} = 1 + 3 \cdot \frac{1}{2}$$

Similarly,

$$S_{16} > 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16}$$
$$= 1 + 4 \cdot \frac{1}{2}$$

Moral:

$$S_{2^n} > 1 + n \cdot \frac{1}{2} \longrightarrow +\infty.$$

$\Rightarrow$ This series is divergent

(even though $a_k \to 0$, $k \to \infty$).

Ex. 8 Check $\sum_{k=1}^{\infty} \frac{k^2}{5k^2+1}$ for convergence.

Claim: it is divergent.

Indeed, $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2 \;/k^2}{5k^2+1 \;/k^2}$

$$= \lim_{k\to\infty} \frac{1}{5+1/k^2} = \frac{1}{5} \neq 0$$

$\Rightarrow$ by the test for divergence, we have our **Claim**.

Thm. If $\sum_k a_k$, $\sum_k b_k$ are convergent, then so are $\sum_k c\,a_k$, $\sum_k (a_k+b_k)$, and

i). $\sum_{k=1}^{\infty} c\,a_k = c\sum_{k=1}^{\infty} a_k$ $\qquad c = \text{const}$

ii). $\sum_{k=1}^{\infty} (a_k+b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

iii). $\sum_{k=1}^{\infty} (a_k-b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$

Ex. 9 Evaluate $\sum_{k=1}^{\infty} \left(\frac{3}{k(k+1)} - \frac{1}{2^k} \right) = 2.$

Observe:

$$\sum_{k=1}^{\infty} \overbrace{\frac{1}{k(k+1)}}^{a_k} = 1 \quad \Big\} \text{ convergent}$$

telescoping

$$\underbrace{\sum_{k=1}^{\infty} \overbrace{\frac{1}{2^k}}^{b_k}}_{\text{convergent}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1/2}{1 - 1/2} = 1.$$

geometric

$$\sum_{k=1}^{\infty} \left(\frac{3}{k(k+1)} - \frac{1}{2^k} \right) = \sum_{k=1}^{\infty} (3 \cdot a_k - b_k) \overset{\text{Thm}}{=}$$

$$= 3 \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k = 3 \cdot 1 - 1 = 2.$$

Ex. Find the sum $\sum_{k=0}^{\infty} x^k$, $\quad |x| < 1$

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

Common ratio: $r = x$

First term $a = 1$

$$\underbrace{\sum_{k=0}^{\infty} x^k}_{\text{power series}} = \frac{1}{1-x} \qquad |x| < 1.$$

Here x is a variable!