REVIEW QUESTIONS FOR TEST 3

Polar coordinates

1) Explain how a point in the plane is determined by its polar coordinates. Write down the conversion formulas between the Cartesian and polar coordinates. How does the quadrant containing a certain point determines its polar angle?
2) Plot the point with the given polar coordinates and find its Cartesian coordinates:
   (a) $(1, \pi/3)$
   (b) $(-8, 7\pi/4)$
   (c) $(3, -5\pi/2)$
   (d) $(0, -71\pi/4)$
   (e) $(2\sqrt{2}, 3\pi/4)$.
3) For the given Cartesian coordinates of a point, find its polar coordinates (r, θ) with $r \geq 0$ and $0 \leq \theta < 2\pi$. Then, find the expression of polar coordinates with $r \leq 0$ and $0 \leq \theta < 2\pi$:
   (a) $(0, 5)$
   (b) $(5\sqrt{3}, -5)$
   (c) $(2\sqrt{2}, -2\sqrt{2})$
   (d) $(1, \sqrt{3})$.

Polar curves

4) What is a polar curve? How to sketch the graph of a polar curve, given its equation? How to determine the slope of a polar curve at a given point?
5) Identify the curve:
   (a) $r^3 = 125$
   (b) $\theta = \pi/4$
   (c) $r = 4 \sec\theta$
   (d) $r = -2\sec\theta$
   (e) $r = 3\csc\theta$
   (f) $r^2 \cos 2\theta = 1$.
6) Find a polar equation for the curve given in Cartesian coordinates:
   (a) $y = 2$
   (b) $y = x$
   (c) $x^2 + y^2 = 2x$
   (d) $4y^2 = x$
   (e) $x^2 - y^2 = 4$.
7) Find the slope of the tangent line to the given polar curve at the point corresponding to the specified value of θ:
   (a) $r = 2\cos\theta, \quad \theta = \pi/3$
   (b) $r = 1/\theta, \quad \theta = \pi$
   (c) $r = 2 + \sin 3\theta, \quad \theta = \pi/4$
   (d) $r = \cos(\theta/3), \quad \theta = \pi$.

8) Find points on the given curve where the tangent line is horizontal or vertical:
   (a) $r = 3\cos\theta$
   (b) $r = 1 + \cos\theta$
   (c) $r = e^{\theta}$
   (d) $r = 1 - \sin\theta$.
9) Sketch the polar curve:
   (a) $r = \theta, \quad \theta \geq 0$
   (b) $r = 1 + \sin\theta$
   (c) $r = 2 + \sin 3\theta$
   (d) $r^2 = \cos 4\theta$
   (e) $r = 2\cos(\theta/2)$.

Areas and lengths for polar curves

10) Write down the formula for the area enclosed by a polar curve $r = f(\theta)$, as well as for the area between a pair of polar curves $r = f(\theta)$ and $r = g(\theta)$. Explain how to determine the range of integration in the corresponding integrals. How to compute the arc length of a polar curve?
11) Find the area of the region that lies inside the first curve and outside the second curve:
   (a) $r = 4\sin\theta, \quad r = 2$
   (b) $r = 1 - \sin\theta, \quad r = 1$
   (c) $r = 3\cos\theta, \quad r = 1 + \cos\theta$
   (d) $r^2 = 8\cos 2\theta, \quad r = 2$.
12) Find all the points of intersection of the given curves:
   (a) $r = \sin\theta, \quad r = 1 - \sin\theta$
   (b) $r = 1 + \cos\theta, \quad r = 1 - \sin\theta$
   (c) $r = \sin\theta, \quad r = \sin 2\theta$
   (d) $r = 2\sin 2\theta, \quad r = 1$.
13) Find the length of the polar curve:
   (a) $r = 2\cos\theta, \quad 0 \leq \theta \leq \pi$
   (b) $r = \theta^2, \quad 0 \leq \theta \leq 2\pi$
   (c) $r = 2(1 + \cos\theta)$.

Conic sections

14) Give the geometric definitions of parabola, ellipse, hyperbola. Explain why these curves are called "conic sections". Write down their equations in Cartesian coordinates and explain how to determine the geometric characteristics (foci, vertices, asymptotes) from an equation. What is the equation of a conic shifted by the vector (v_1, v_2)?
15) Identify the conic by its equation in Cartesian coordinates, find the foci and vertices. For a hyperbola, determine also the asymptotes:
   (a) $4x^2 = y^2 + 4$
   (b) $4y^2 = x + 4$
   (c) $3x^2 - 6x - 2y = 1$
   (d) $x^2 - 2x + 2y^2 - 8y + 7 = 0$
   (e) $-9y^2 - 54y + 25x^2 - 200x + 94 = 0$
   (f) $y^2 + 2\,y + 4\,x^2 + 16\,x + 1 = 0$
   (g) $-9\,y^2 + 18\,y + 4\,x^2 - 24\,x - 9 = 0$

(h) $-2\,y^2 - 16\,y + 3\,x^2 + 6\,x - 35 = 0$
(i) $16\,y - x^2 + 6\,x - 9 = 0$.

16) Find an equation in Cartesian coordinates for the conic satisfying the given conditions:
(a) parabola, vertex $(0, 0)$, focus $(1, 0)$
(b) parabola, focus $(0, 0)$, directrix $x = -3$
(c) parabola, focus $(2, 3)$, vertex $(2, -3)$
(d) ellipse, foci $(\pm 2, 3)$, vertices $(\pm 4, 3)$
(e) ellipse, foci $(-7, 6)$ and $(-7, 2)$, vertex $(-7, 1)$
(f) ellipse, center $(4, -1)$, vertex $(0, -1)$, focus $(6, -1)$
(g) hyperbola, vertices $(0, \pm 3)$, foci $(0, \pm 4)$
(h) hyperbola, vertices $(\pm 2, 0)$, asymptotes $y = \pm 3x$
(i) hyperbola, vertices $(-1, 2)$ and $(7, 2)$, foci $(-2, 2)$ and $(8, 2)$.

Conic sections in polar coordinates

17) Write down the equation of conic sections in polar coordinates. Which values of eccentricity correspond to ellipse/parabola/hyperbola? What changes in the equation when the directrix is $x = -d$? And when the directrix is parallel to the polar axis: $y = \pm d$?

18) Find equation of a conic with the focus at the origin and satisfying the given conditions:
(a) parabola, directrix $y = -5$
(b) parabola, directrix $x = 8$
(c) ellipse, eccentricity $\frac{1}{3}$, directrix $x = 2$
(d) ellipse, eccentricity $\frac{1}{5}$, directrix $x = -7$
(e) hyperbola, eccentricity 3, directrix $y = 12$
(f) hyperbola, eccentricity 7, directrix $y = -3$.

19) Find the eccentricity, identify the curve, give equation of the directrix, sketch the graph of the following conics:
(a) $r = \dfrac{4}{5 - 4\sin\theta}$
(b) $r = \dfrac{2}{3 + 3\sin\theta}$
(c) $r = \dfrac{1}{3 - 4\sin\theta}$
(d) $r = \dfrac{4}{2 + 3\cos\theta}$
(e) $r = \dfrac{4}{2 - 4\cos\theta}$
(f) $r = \dfrac{2}{3 + \sin\theta}$.

Sequences

20) What is a sequence? Explain when a sequence $\{a_n\}$ converges to a number L (has limit L). State the limit laws for sequences. State the Squeeze theorem. What is the limit of a composition of a continuous function with a converging sequence? What are the assumptions necessary to interchange a function with a limit? State the Monotonic sequence theorem.

21) Determine whether the given sequence converges; if it does, find the limit. We assume that $n \geq 1$ in all the sequences.

(a) $a_n = \dfrac{3+5n^2}{n+n^2}$

(b) $a_n = \dfrac{3+5n^2}{1+n}$

(c) $a_n = 3^n 7^{-n}$

(d) $a_n = \dfrac{(-1)^n n}{n+\sqrt{n}}$

(e) $a_n = 1 + \left(-\dfrac{1}{2}\right)^n$

(f) $a_n = \sqrt[n]{2^{1+3n}}$

(g) $a_n = n \sin(1/n)$

(h) $a_n = \sin n$

(i) $a_n = \dfrac{n^2}{\sqrt{n^3+4n}}$

(j) $a_n = \ln(n+1) - \ln n$

(k) $a_n = \arctan(\ln n)$

(l) $a_n = \left(1 + \dfrac{2}{n}\right)^n$

(m) $a_n = \dfrac{n!}{2^n}$, where $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

(n) $a_n = n - \sqrt{n+1}\sqrt{n+3}$. *Hint:* multiply and divide by the conjugate.

22) Determine whether the following sequences are bounded, and whether they are eventually increasing or decreasing:

(a) $a_n = \cos n$

(b) $a_n = \dfrac{1}{2n+3}$

(c) $a_n = \dfrac{1-n}{2+n}$

(d) $a_n = n^3 - 11n + 3$

(e) $a_n = 2 + \dfrac{(-1)^n}{n}$

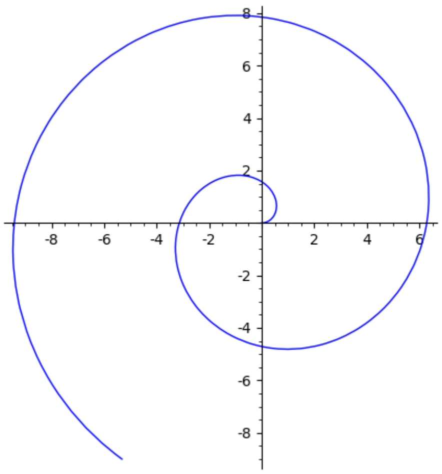
(f) $a_n = \ln\left(1 + \dfrac{1}{n}\right)$

(g) $a_n = n^{1/n}$.

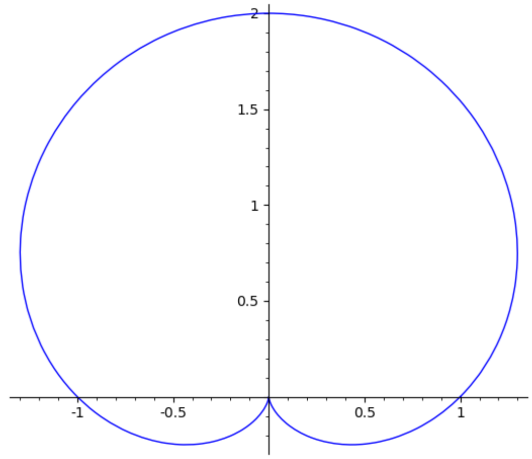
Answer key

2) (a) $(1/2, \frac{\sqrt{3}}{2})$
   (b) $(-4\sqrt{2}, 4\sqrt{2})$
   (c) $(0, -3)$
   (d) $(0, 0)$
   (e) $(-2, 2)$.
3) (a) $(5, \pi/2)$; $(-5, 3\pi/2)$
   (b) $(10, 11\pi/6)$; $(-10, 5\pi/6)$
   (c) $(4, 7\pi/4)$; $(-4, 3\pi/4)$
   (d) $(2, \pi/3)$; $(-2, 4\pi/3)$.
5) (a) Circle $x^2 + y^2 = 25$.
   (b) Line $y = x$.
   (c) Line $x = 4$.
   (d) Line $x = -2$.
   (e) Line $y = 3$.
   (f) Hyperbola $x^2 - y^2 = 1$.
6) (a) $r = 2\csc\theta$
   (b) $\theta = \pi/4$
   (c) $r = 2\cos\theta$
   (d) $r = \frac{1}{4}\csc\theta\cot\theta$
   (e) $r^2\cos 2\theta = 4$.
7) (a) $\frac{1}{3}\sqrt{3}$
   (b) $-\pi$
   (c) $-\frac{\sqrt{2}-1}{\sqrt{2}+2}$
   (d) $-\sqrt{3}$.
8) The answers are given in polar coordinates; only unique points are included. For example, in (a), the point $(-3/\sqrt{2}, 5\pi/4)$ also has a horizontal tangent, but it coincides with $(3/\sqrt{2}, \pi/4)$, and so is omitted.
   (a) Horizontal: $(3/\sqrt{2},\ \pi/4), (-3/\sqrt{2},\ 3\pi/4)$;
   vertical: $(3,\ 0), (0, 0)$.
   (b) Horizontal: $(3/2, \pi/3), (0, \pi), (3/2, 5\pi/3)$;
   vertical: $(2, 0), (1/2,\ 2\pi/3), (1/2,\ 4\pi/3)$.
   (c) Horizontal: $(e^{3\pi/4+k\pi},\ 3\pi/4 + k\pi)$, k–integer;
   vertical: $(e^{\pi/4+k\pi},\ \pi/4 + k\pi)$, k–integer.
   (d) Horizontal: $(1/2,\ \pi/6), (2,\ 3\pi/2), (1/2,\ 5\pi/6)$;
   vertical: $(0, \pi/2), (3/2,\ 7\pi/6), (3/2,\ 11\pi/6)$.

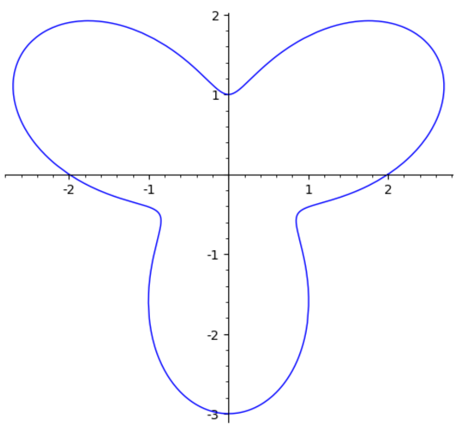
9) See the graphs below.



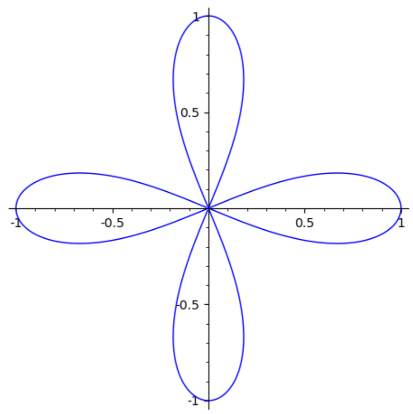
(a) $r = \theta$



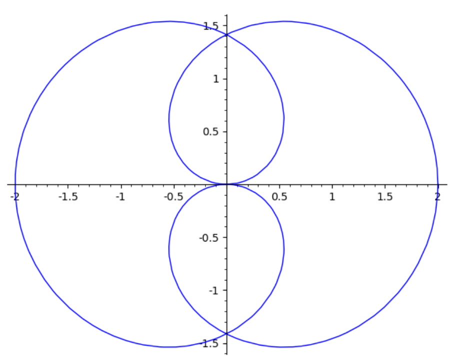
(b) $r = 1 + \sin\theta$



(c) $2 + \sin 3\theta$



(d) $r^2 = \cos 4\theta$



(e) $r = 2\cos(\theta/2)$

11) (a) $\dfrac{4}{3}\pi + 2\sqrt{3}$

(b) $\frac{1}{4}\pi + 2$
(c) π
(d) $2\left(-\frac{2}{3}\pi + 2\sqrt{3}\right)$.

12) The answers are given in polar coordinates; only unique points are included.
(a) $(1/2, \pi/6), (1/2, 5\pi/6)$
(b) $(0,0), (1-1/\sqrt{2}, 3\pi/4), (1+1/\sqrt{2}, 7\pi/4)$
(c) $(0,0), (\sqrt{3}/2, \pi/3), (\sqrt{3}/2, 2\pi/3)$
(d) $(1, \pi/12)$, $(1, 5\pi/12)$, $(1, 7\pi/12)$, $(1, 11\pi/12)$, $(1, 13\pi/12)$, $(1, 17\pi/12)$, $(1, 19\pi/12)$, $(1, 23\pi/12)$.

13) (a) 2π
(b) $\frac{8}{3}(\pi^2+1)^{3/2} - \frac{8}{3}$
(c) 16.

15) (a) Hyperbola $x^2 - y^2/4 = 1$; foci $(\pm\sqrt{5}/2, 0)$, vertices $(\pm 1, 0)$, asymptotes $y = \pm 2x$.
(b) Parabola $y^2 = \frac{1}{4}(x+4)$; focus $(-4+1/16, 0)$, vertex $(-4, 0)$.
(c) Parabola $(x-1)^2 = \frac{2}{3}(y+2)$; focus $(1, -2+1/6)$, vertex $(1, -2)$.
(d) Ellipse $\frac{(x-1)^2}{2} + \frac{(y-2)^2}{1} = 1$; foci $(1\pm\sqrt{3}, 2)$, vertices $(1\pm\sqrt{2}, 2)$.
(e) Hyperbola $\frac{(x-4)^2}{9} - \frac{(y+3)^2}{25} = 1$; foci $(4\pm\sqrt{34}, -3)$, vertices $(4\pm 3, 3)$, asymptotes $y+3 = \pm\frac{5}{3}(x-4)$.
(f) Ellipse $\frac{(y+1)^2}{16} + \frac{(x+2)^2}{4} = 1$; foci $(-2, -1\pm\sqrt{12})$, vertices $(-2, -1\pm 4)$.
(g) Hyperbola $\frac{(x-3)^2}{9} - \frac{(y-1)^2}{4} = 1$; foci $(3\pm\sqrt{13}, 1)$, vertices $(3\pm 3, 1)$, asymptotes $y-1 = \pm\frac{2}{3}(x-3)$.
(h) Hyperbola $\frac{(x+1)^2}{2} - \frac{(y+4)^2}{3} = 1$; foci $(-1\pm\sqrt{5}, -4)$, vertices $(-1\pm\sqrt{2}, -4)$, asymptotes $y+4 = \pm\sqrt{\frac{3}{2}}(x+1)$.
(i) Parabola $(x-3)^2 = 16y$, focus $(3, 4)$, vertex $(3, 0)$.

16) (a) $x^2 = 4y$
(b) $y^2 = 6(x+3/2)$
(c) $(x-2)^2 = 8(y+3)$
(d) $\frac{x^2}{16} + \frac{(y-3)^2}{12} = 1$
(e) $\frac{(y-4)^2}{9} + \frac{(x+7)^2}{5} = 1$
(f) $\frac{(x-4)^2}{16} + \frac{(y+1)^2}{12} = 1$
(g) $\frac{y^2}{9} - \frac{x^2}{7} = 1$

(h) $\dfrac{x^2}{4} - \dfrac{y^2}{36} = 1$

(i) $\dfrac{(x-3)^2}{16} - \dfrac{(y-2)^2}{9} = 1.$

18) (a) $r = \dfrac{5}{1-\sin\theta}$

(b) $r = \dfrac{8}{1+\cos\theta}$

(c) $r = \dfrac{4/3}{1+(2/3)\cos\theta}$

(d) $r = \dfrac{7/5}{1-(1/5)\cos\theta}$

(e) $r = \dfrac{36}{1+3\sin\theta}$

(f) $r = \dfrac{21}{1-7\sin\theta}.$

19) See the graphs on the next page.

(a) Ellipse, $E = 4/5$, $y = -1$.

(b) Parabola, $y = 2/3$.

(c) Hyperbola, $E = 4/3$, $y = -1/4$.

(d) Hyperbola, $E = 3/2$, $x = 4/3$.

(e) Hyperbola, $E = 2$, $x = -1$.

(f) Ellipse, $E = 1/3$, $y = 2$.

21) (a) Converges to 5.

(b) Diverges to $+\infty$.

(c) Converges to 0.

(d) Diverges.

(e) Diverges.

(f) Converges to 8.

(g) Converges to 1.

(h) Diverges.

(i) Diverges to $+\infty$.

(j) Converges to 0.

(k) Converges to $\pi/2$.

(l) Converges to e^2.

(m) Diverges to $+\infty$.

(n) Converges to 2.

22) (a) Bounded.

(b) Bounded, decreasing.

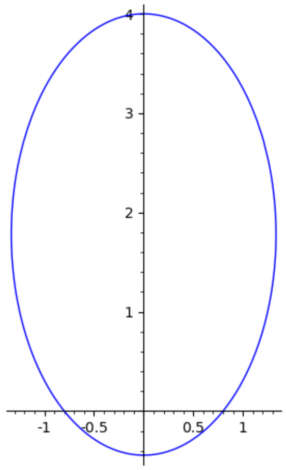
(c) Bounded, decreasing.

(d) Unbounded, eventually increasing.

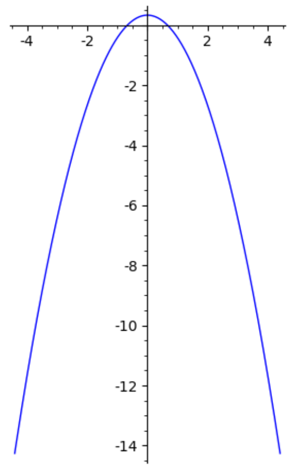
(e) Bounded.

(f) Bounded, decreasing.

(g) Bounded, decreasing.



19 (a)



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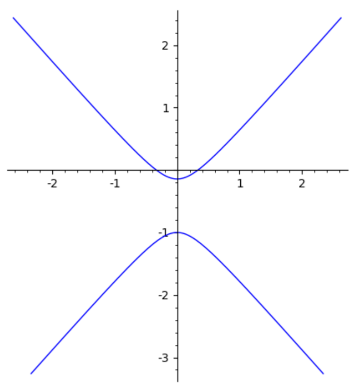
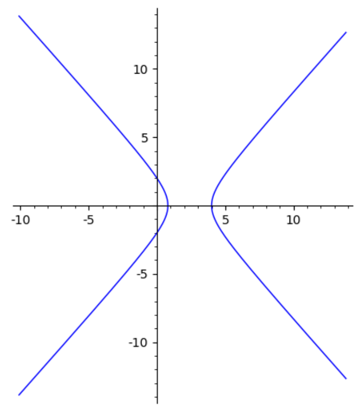


(19 c)



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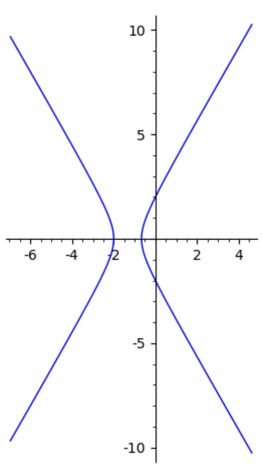
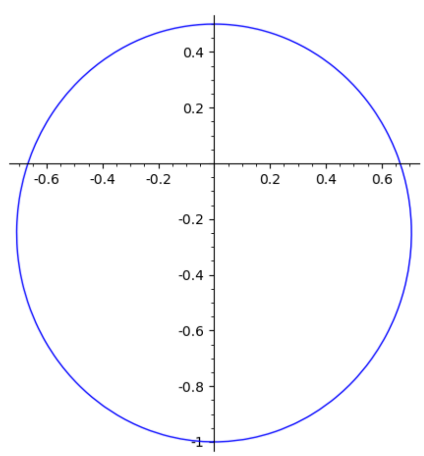


(19 e)



(19 f)