Measure and Integration I (MAA5616), Fall 2020
Homework 11, due Sunday, Dec 6 **at 7pm**

1. Use Theorem 2.44 to prove that Lebesgue measure λ^n is invariant under rotations (T is a rotation if it is inverse to its transpose: $TT^* = \mathrm{Id}$).

2. Construct a subset of $\mathbb{R}^2$ which is in $\mathcal{L}^2 \setminus \mathcal{B}_{\mathbb{R}^2}$.

Use the set in $\mathcal{L}^1 \setminus \mathcal{B}_{\mathbb{R}^1}$, constructed in #3 from HW 5.

3. Prove Tonelli's theorem with the assumption $f \in L^1(\mu \times \nu)$ instead of assuming that μ and ν are σ-finite.

Use that $f \in L^1 \cap L_+$ implies $\{(x,y) : f(x,y) > 0\}$ is a σ-finite set, by #4 in HW 6.

4. For $X = [0,1]^2$ (unit square in $\mathbb{R}^2$), check which of the following integrals exist and are equal:

$$\int f(x,y)\, d\lambda^2(x,y), \qquad \iint f(x,y)\, d\lambda(x) d\lambda(y), \qquad \iint f(x,y)\, d\lambda(y) d\lambda(x),$$

where

- $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$;
- $f(x,y) = \begin{cases} (x-1/2)^{-3}, & 0 < y < |x-1/2|, \\ 0, & \text{otherwise.} \end{cases}$

For the first function, it is useful to consider its antisymmetry with respect to the line $x = y$; for the second, draw the region $\{(x,y) : 0 \le x \le 1,\ 0 < y < |x-1/2|\}$ and also investigate its symmetries.

5. Consider $f(x) = \frac{\sin x}{x}$, defined by continuity on $[0,\infty)$.

- Show $\int_0^\infty |f|\, d\lambda = \infty$.
- Show $\lim_{b\to\infty} \int_0^b f\, d\lambda = \pi/2$ by integrating $e^{-xy}\sin x$ with respect to x and y.

This is an example of function not in L^1, for which the improper Riemann integral converges.

6. Given a σ-finite measure space $(X, \mathcal{M}, \mu)$ and a measurable $f : X \to [0,\infty]$, show that

$$\int f\, d\mu = \int_0^\infty \mu(\{x : f(x) > t\})\, d\lambda(t).$$

Think of the integrand in RHS as measure of sections of the subgraph of f. The LHS is the measure of this subgraph in $X \times [0,\infty]$.

7. Let $f : [0,1]^2 \to \mathbb{R}$ be $\mathcal{L}^2$-measurable, and for λ-a.e. x and λ-a.e. y, functions f_x and f^y are constant. Show that for some constant c, $f(x,y) = c$ for λ^2-a.e. (x,y).

Otherwise, there is an $r \in \mathbb{R}$ for which both $\{(x,y) : f(x,y) < r\}$ and $\{(x,y) : f(x,y) \geq r\}$ have positive λ^2-measure. By the assumption, together with a point (a,b) these sets contain the entire lines $x = a$, $y = b$, and so must intersect. This gives a contradiction.

8. Let $T = \{(x,y) \in [0,1]^2 : x - y \in \mathbb{Q}\}$. Show that $\lambda^2(T) = 0$, but $T \cap (A \times A) \neq \emptyset$ for any $A \subset [0,1]$ such that $\lambda(A) > 0$.

Use #4 from HW 7, by which $A - A$ contains an interval. With that, the tricky part is to see why T is measurable.