REVIEW QUESTIONS FOR POWER SERIES

Power series

1) Give a definition of the power series. What is the radius of convergence / interval of convergence of a power series? Explain how to compute them for a given series.
2) Determine the radius and interval of convergence for the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 4^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n^3+1)} x^n$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n7^n} x^n$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

(h) $\displaystyle\sum_{n=1}^{\infty} 3^n(2x-1)^n.$

Representing functions by power series

3) Explain how to differentiate / integrate a power series. What is the impact of these operations on the radius of convergence?
4) Give a power series representations of the following functions:

(a) $f(x) = \dfrac{1}{1-x}$

(b) $f(x) = \dfrac{4}{2x+1}$

(c) $f(x) = \dfrac{x}{1-x}$

(d) $f(x) = \dfrac{x}{3x^2-1}$

(e) $f(x) = \dfrac{1}{x^2+2x+2}$

(f) $f(x) = \dfrac{4x+1}{x^2+6x+10}.$

In the following questions use partial fractions decomposition first, then obtain the expansions of the resulting fractions:

(g) $f(x) = \dfrac{2x-4}{x^2-4x+3}$

(h) $f(x) = \dfrac{2x+3}{x^2+3x+2}.$

(i) $f(x) = \dfrac{3x^2-5x+5}{(x-2)\,(x^2+3)}$

(j) $f(x) = \dfrac{3x^2+2x+1}{(x+1)\,(x^2+x+2)}.$

Taylor and Maclaurin series

5) Write down the formula for Taylor series. What needs to be changed to obtain Maclaurin series? What is the expression for $T_n(x)$, the n-th degree Taylor polynomial? What is the Taylor inequality, and how is it used to estimate the error in approximating $f(x)$ with its Taylor polynomial $T_n(x)$? Write down the important Maclaurin series you know.
6) Obtain Maclaurin series for the following functions, using the definition or any other convenient method. Do not show that $R_n(x) \to 0$.

(a) $f(x) = (1-x)^{-2}$
(b) $f(x) = (x+3)^2$
(c) $f(x) = \cos x$
(d) $f(x) = \sin 2x$
(e) $f(x) = \ln(1+x)$
(f) $f(x) = x^2 \ln(1+x)$
(g) $f(x) = e^{x^3}$
(h) $f(x) = \sqrt[4]{(1-x)}$
(i) $f(x) = (2+x)^{-2/3}$
(j) $f(x) = x \cos 3x$
(k) $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ 1/2, & x = 0 \end{cases}$
(l) $f(x) = \dfrac{x^2}{\sqrt{2+x}}$.

7) Compute the Taylor series of the following functions centered at the specified a. Do not show that $R_n(x) \to 0$.

(a) $f(x) = x^3 + 4x^2 + x + 3, \qquad a = 2$
(b) $f(x) = \ln x, \qquad a = 1$
(c) $f(x) = e^{3x}, \qquad a = 2$
(d) $f(x) = \sqrt{x}, \qquad a = 9$
(e) $f(x) = \cos x, \qquad a = \pi/4$
(f) $f(x) = \sin x, \qquad a = \pi/6$.

8) Use series to compute the given limits:

(a) $\displaystyle\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2}$
(b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$
(c) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{1 + x - e^x}$
(d) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$.

Applications of Taylor polynomials

9) Explain the method for approximating functions with Taylor polynomials, and its purposes. How to choose the center of such a polynomial? Explain how the singularities of a function influence the radius of convergence if its Taylor polynomial.

10) Find the Taylor polynomial T_3 for the following functions, centered at the given a:

(a) $f(x) = e^x, \qquad a = 1$
(b) $f(x) = \sin x, \qquad a = \pi/6$
(c) $f(x) = \cos 2x, \qquad a = \pi/4$
(d) $f(x) = e^x \sin x, \qquad a = 0$
(e) $f(x) = x^2 \ln(1+x), \qquad a = 0$
(f) $f(x) = x \sin x, \qquad a = 0$
(g) $f(x) = \sqrt{x}, \qquad a = 4$
(h) $f(x) = e^{x^2}, \qquad a = 0$.

11) For the functions of the previous question, estimate the accuracy of the approximation $f(x) \approx T_3(x)$ when $|x - a| < 0.5$.