Avoiding zero probability events when computing Value at Risk contributions

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Introduction

- Let us consider $\boldsymbol{X} = (X_1, ..., X_d)$ the losses (negative of the returns) of d different assets in a portfolio.
- For a **linear portfolio** with unitary exposure to each asset, the portfolio-wide loss is defined as $X = \sum_{i=1}^{d} X_i$
- Risk measures:
  - $\mathbf{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq \alpha\}$
  - $\mathbf{ES}_\alpha(X) = \dfrac{1}{1-\alpha} \displaystyle\int_\alpha^1 \mathrm{VaR}_{1-\beta}(X) d\beta$

Introduction

- After a risk measure of the portfolio is computed, one is usually interested in understanding how much each asset contributes to the overall portfolio risk, in a process known as **risk allocation**.
- **Euler allocation**: by how much the risk is increased if we increase the exposure to one asset by a small amount
- Proposed in Tasche (1999) and discussed in several other papers
  - **Theoretical:** Denault (2001), Kalkbrener (2005), Buch and Dorfleitner (2008)
  - **Empirical:** Tasche (1999), Glasserman (2005), Brownlees and Engle (2012), Mainik and Schaanning (2014), Tasche (2008).

VaR x ES

- **Expected Shortfall**
  - Used in the **SST** for capital calculations
  - Fundamental review of the trading book
  - Euler allocations:

$$\mathscr{C}_i^\alpha = \mathbb{E}[X_i \,|\, X \geq \mathrm{VaR}_\alpha(X)]$$

- **VaR**
  - Used in **Solvency II** for capital calculations
  - Euler allocations:

$$\mathscr{C}_i^\alpha = \mathbb{E}[X_i \,|\, X = \mathrm{VaR}_\alpha(X)]$$

- VaR x ES: Embrechts et al. (2014), Emmer et al. (2015)

Objective

- Our aim: To compute Euler allocations for **Value at Risk**
- Rarely available in closed form
- Given a distribution for $\boldsymbol{X}$, we need to estimate

$$\mathscr{C}_i^{\alpha} = \mathbb{E}[X_i \,|\, X = \mathrm{VaR}_\alpha(X)]$$

- Exact **Monte Carlo:** Needs samples from $\boldsymbol{X} \mid X = \mathrm{VaR}_\alpha(X)$
- Baseline estimator[1]:
  - Sample from $\boldsymbol{X} \mid X \in [\mathrm{VaR}_{\alpha-\delta}(X), \mathrm{VaR}_{\alpha+\delta}(X)]$
  - **δ-allocations**: $\mathscr{C}_i^{\alpha,\delta} = \mathbb{E}[X_i \,|\, X \in [\mathrm{VaR}_{\alpha-\delta}(X), \mathrm{VaR}_{\alpha+\delta}(X)]]$

[1] Glasserman (2005)

Literature review

- Exact VaR allocations via Monte Carlo
  - MCMC: Koike and Minami (2019) and Koike and Hofert (2020)
  - Conditional MC: Fu et al. (2009)
- Kernel estimator: Gouriéroux et al. (2000), Tasche (2008), Liu and Hong (2009)
- Infinitesimal perturbation (IPA): Hong (2009)
- Fourier Transform MC: Siller (2013)

Literature review

- Remember ES allocations are easier: $\mathbb{E}[X_i \mid X \geq \mathrm{VaR}_\alpha(X)]$
- Roudu's talk[2]: Probability Equivalent Level of VaR-ES (PELVE)
- The PELVE is the c such that $\mathrm{ES}_{1-c\varepsilon}(X) = \mathrm{VaR}_{1-\varepsilon}(X)$
- In principle one could use the allocations for $\mathrm{ES}_{1-c\varepsilon}$, but it wouldn't be the same as the allocations for $\mathrm{VaR}_{1-\varepsilon}$.

[2] Li and Wang (2019)

Literature review

- In Asimit et al. (2019) (predecessor to PELVE) the idea is to find α^* such that

$$\mathrm{ES}_{\alpha^*}(X) = \mathrm{VaR}_{\alpha}(X)$$

- And then compute the $\mathrm{ES}_{\alpha^*}(X)$ allocations
- For **elliptical distributions**, the allocations based on ES_{α^*} are the same as the allocations based on VaR_{α}
- Conditions are provided for when the two allocations are close to each other (for large α)

Our proposal

- We also rewrite the VaR allocations as something close to ES allocations
- We identify a **model** by a function of uniform random variables
$$\boldsymbol{X} = g(\boldsymbol{U}) = (g_1(\boldsymbol{U}), \ldots, g_d(\boldsymbol{U})),$$
where $\boldsymbol{U} \sim U[0,1]^k$ and $g \in C^1([0,1]^k; \mathbb{R}^d)$
- So, we express a d-dimensional random vector using k-uniform random variables

Our proposal

Theorem

Assume $\boldsymbol{X} = g(\boldsymbol{U})$ *and that* $\exists f_i \in C^1([0,1]^k;\mathbb{R}^k)$ *s.t., for* $\boldsymbol{u} \in [0,1]^k$,

$$\begin{bmatrix} \nabla g_i(\boldsymbol{u}) \\ \sum_{j\neq i} \nabla g_j(\boldsymbol{u}) \end{bmatrix} f_i(\boldsymbol{u}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then, the marginal risk allocation for the VaR is

$$\mathscr{C}_i^\alpha = \frac{\mathbb{E}\left[X_i\ \pi_i \mid X \geq VaR_\alpha(X)\right]}{\mathbb{E}\left[\pi_i \mid X \geq VaR_\alpha(X)\right]},$$

where $\pi_i = Tr(\nabla f_i(\boldsymbol{U}))$ *is the weight,* $Tr(A)$ *is the trace operator of a matrix* A *and* ∇f_i *is the Jacobian matrix of* f_i.

Our proposal

- The **model** definition is encompassed into the function g (we'll see examples soon)
- Given a model g, one only needs to compute the **weights** π_i
- Compare this new representation with the ES allocations:

$$\frac{\mathbb{E}\left[X_i\ \pi_i \mid X \geq \mathrm{VaR}_\alpha(X)\right]}{\mathbb{E}\left[\pi_i \mid X \geq \mathrm{VaR}_\alpha(X)\right]} \quad \text{vs} \quad \mathbb{E}[X_i \mid X \geq \mathrm{VaR}_\alpha(X)]$$

- Same conditioning events $\Rightarrow$ **variance reduction**' techniques for ES allocations work here as well
  - Targino et al. (2015), Peters et al. (2017), Koike and Minami (2019), Koike and Hofert (2020)

Numerical examples

- We now present the new representation of the VaR allocations for several models
- We also compare the precision of two Monte Carlo estimators for the allocations
  1. The new representation $\mathscr{C}_i^\alpha = \frac{\mathbb{E}\left[X_i\ \pi_i \mid X \geq \mathrm{VaR}_\alpha(X)\right]}{\mathbb{E}\left[\pi_i \mid X \geq \mathrm{VaR}_\alpha(X)\right]}$
  2. The δ-allocation $\mathscr{C}_i^{\alpha,\delta} = \mathbb{E}[X_i \,|\, X \in [\mathrm{VaR}_{\alpha-\delta}(X), \mathrm{VaR}_{\alpha+\delta}(X)]]$
- We use the same MC sample $\boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{(N)}$ for both methods and a pre-computed VaR
- We want to empirically assess the impact of N and δ for $\alpha = 0.5, 0.9$ and 0.99

Independent marginals

- $X_j = \varphi_j(U_j)$, with $U_1, \ldots, U_d \overset{iid}{\sim} U[0,1]$ and φ_j may be an inverse cdf with differentiable density p_j

- Thus, $g_i(\boldsymbol{u}) = \varphi_i(u_i)$ and

$$\nabla g_i(\boldsymbol{u}) = (0, \ldots, 0, \varphi_i'(u_i), 0, \ldots, 0),$$

  where the non-zero entry is in the i-th position.

- Hence, the following f satisfies the condition in the Theorem

$$f_i(\boldsymbol{u}) = \frac{1}{d-1}\left(\frac{1}{\varphi_1'(u_1)}, \ldots, \frac{1}{\varphi_{i-1}'(u_{i-1})}, 0, \frac{1}{\varphi_{i+1}'(u_{i+1})}, \ldots, \frac{1}{\varphi_d'(u_d)}\right)$$

- Therefore,

$$\mathrm{Tr}(\nabla f_i(\boldsymbol{u})) = -\frac{1}{d-1}\sum_{j\neq i}\frac{\varphi_j''(u_j)}{(\varphi_j'(u_j))^2} \Longrightarrow \pi_i = \sum_{j\neq i}\frac{\varphi_j''(U_j)}{(\varphi_j'(U_j))^2} = \sum_{j\neq i}\frac{p_j'(X_j)}{p_j(X_j)}.$$

Independent marginals

Name	**Marginal**	$p_j'(x)/p_j(x)$
Log-Normal	$LN(0,\sigma_j^2)$	$\dfrac{Z_j+\sigma_j}{\sigma_j X_j}$, where $Z_j = \dfrac{1}{\sigma_j}\log X_j$
Exponential	$Exp(\lambda_j)$	λ_j
Gamma	$Gamma(\alpha_j,\beta_j)$	$\left(\dfrac{\alpha_j-1}{X_j}-\beta_j\right)$
Gaussian	$N(0,\sigma_j^2)$	$-\dfrac{x}{\sigma_j^2}$
Generalized Pareto	$GPD(\xi_j,\beta_j)$	$-\dfrac{1+\xi_j}{\beta_j}\left(1+\xi_j\dfrac{x}{\beta_j}\right)^{-1}$, for $x \geq 0$

Independent Log-Normals

- $X_i \overset{ind}{\sim} LN(0, \sigma_i)$
- $\sigma_1 = 0.5$, $\sigma_2 = 1$ and $\sigma_3 = 2$

Independent Log-Normals

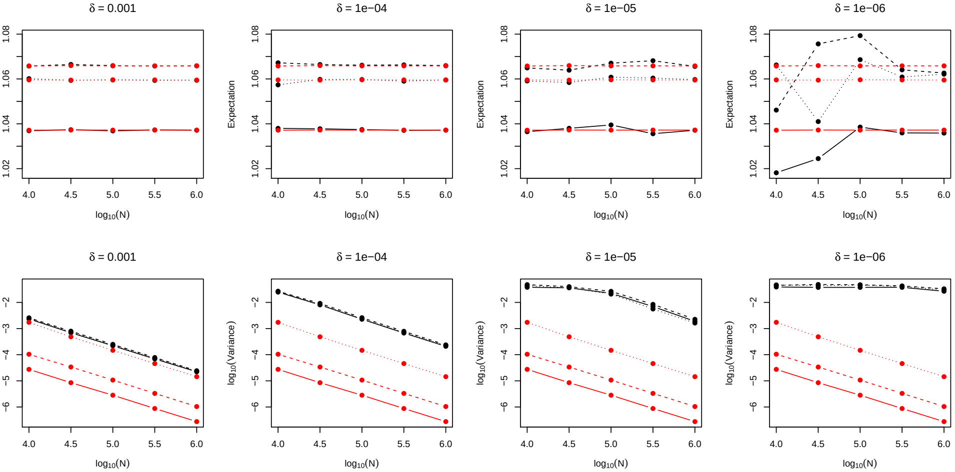

Figure: Mean (top) and variance (bottom) of the δ-estimator (black) and the new (red) for $\mathscr{C}_1^{\alpha}$. Line types (solid, dashed and dotted): different values of α. Columns: different values for δ.

Elliptical Distributions

- Elliptical distributions:

$$\boldsymbol{X} = \mu + RL\boldsymbol{S}$$

  - $\boldsymbol{S}$ is uniformly distributed in the the sphere in $\mathbb{R}^d$
  - L is a $d \times d$ full-rank, lower triangular matrix
  - R is an one-dim. radial random variable independent of $\boldsymbol{S}$.

- Gaussian (special case)

$$\boldsymbol{X} = \mu + L\boldsymbol{Z}.$$

Elliptical Distributions

- The weights for a general elliptical distribution are computed in the preprint.
- **Multivariate Gaussian**: $\boldsymbol{X} = \mu + L\boldsymbol{Z}$
  - $\boldsymbol{Z} \sim N(\mathbf{0}_k, \mathbf{I}_k)$, with $k \geq d$
  - L is a $d \times k$ full-rank, lower triangular matrix.
  - ℓ_i the i-th row of the matrix L
- The weights are $\pi_{\mathrm{i}} = f_i \cdot \boldsymbol{Z}$ where f_i is a solution for

$$\begin{bmatrix} \ell_i \\ \sum_{j \neq i} \ell_j \end{bmatrix} f_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- This linear system has **infinitely many solutions**

Multivariate Gaussian

- $\boldsymbol{X} = (X_1, \ldots, X_d) \sim N(0, \Sigma)$
- VaR contributions can be computed in **closed form**

$$\mathscr{C}_i^\alpha = \Phi^{-1}(\alpha)\frac{(\Sigma\lambda)_i^T}{\sqrt{\lambda^T\Sigma\lambda}}$$

- We also have that $\mathrm{VaR}_\alpha(X) = \Phi^{-1}(\alpha)\sqrt{\lambda^T\Sigma\lambda}$.
- For the example:
  - $\mu = 0$
  - $\Sigma = LL^T$
  - $d = 3$
  - $L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.7 & 0 \\ 1 & 0.8 & 1.1 \end{bmatrix}$
  - Variances: $1.0, 0.74$ and 2.85
  - Correlations ranging from 0.58 to 0.72.

Multivariate Gaussian

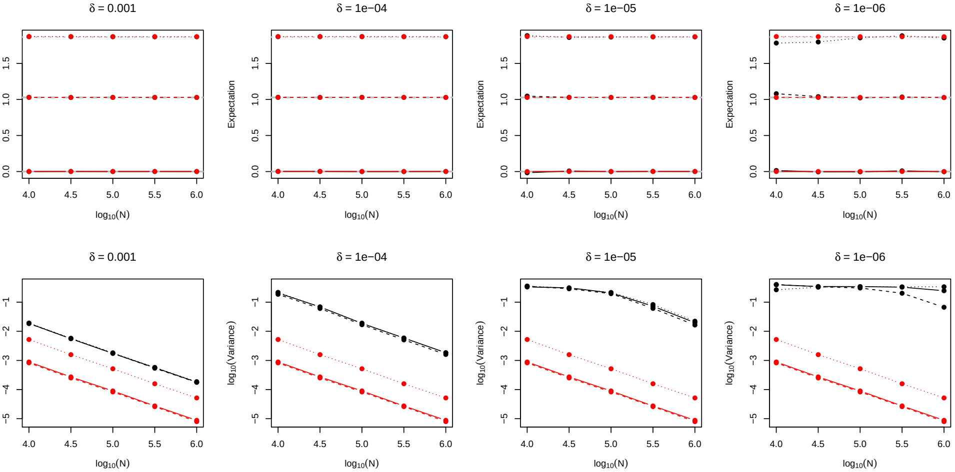

Figure: Mean (top) and variance (bottom) of the δ-estimator (black) and the new (red) for $\mathscr{C}_1^{\alpha}$. Line types (solid, dashed and dotted): different values of α. Columns: different values for δ.

Archimedean copulas

- C_ψ is an **Archimedean copula** with generator ψ and $\boldsymbol{X} = (X_1, \ldots, X_d)$ has joint cdf

$$F(x_1, \ldots, x_d) = C_\psi(F_1(x_1), \ldots, F_d(x_d))$$

- To generate one sample from $\boldsymbol{X}$ we
  1. Sample $\mathcal{V} \sim \mathcal{F} = \mathcal{LS}^{-1}(\psi)$
  2. Sample $U_i \overset{iid}{\sim} U[0,1]$, $i = 1, \ldots, d$
  3. Define $\mathcal{U}_i = \psi(-\log(U_i)/\mathcal{V})$, $i = 1, \ldots, d$
  4. Define $X_i = F_i^{-1}(\mathcal{U}_i)$

- Notation/hipothesis:
  - C_ψ is an Archimedean copula with generator ψ
  - $\mathcal{F} = \mathcal{LS}^{-1}(\psi)$ the inverse Laplace-Stieltjes transform of ψ
  - Both $\mathcal{F}$ and the marginals F_i are absolutely continuous
  - p_i (density of F_i) is differentiable

Archimedean copulas

- Notation:
  - $\mathcal{H} = \mathcal{F}^{-1}$
  - $\phi_j(\boldsymbol{u}) = -\log(u_j)/\mathcal{H}(u_k)$
  - $\gamma_j(\boldsymbol{u}) = \dfrac{\mathcal{H}(u_k)}{\psi'\left(\phi_j(\boldsymbol{u})\right)} + \dfrac{\psi''\left(\phi_j(\boldsymbol{u})\right)}{\psi'\left(\phi_j(\boldsymbol{u})\right)^2}$
- For Archimedean copulas the weights are given by

$$\pi_i = \sum_{j \neq i,k} p_j(X_j)\gamma_j(\boldsymbol{U}) - \frac{p_j'(X_j)}{p_j(X_j)}$$

- For **survival** Archimedean copulas,

$$\pi_i = \sum_{j \neq i,k} p_j(X_j)\gamma_j(\boldsymbol{U}) + \frac{p_j'(X_j)}{p_j(X_j)},$$

Archimedean copulas

Copula		$\gamma_j(u)$
Clayton	$\psi(t) = (1+t)^{-1/\vartheta}$ $\mathcal{V} \sim \Gamma(1/\vartheta, 1)$	$\frac{1}{\psi(\phi_j(\boldsymbol{U}))}\left(-\vartheta(\mathcal{V} - \log U_j) + \vartheta + 1\right)$
Gumbel	$\psi(t) = e^{-t^{1/\vartheta}}$ $\mathcal{V} \sim S(\frac{1}{\vartheta}, 1, c, 0; 1)$ $c = (\cos(0.5\pi/\vartheta))^{\vartheta}$	$\frac{1}{\psi(\phi_j(\boldsymbol{U}))}\left(-\vartheta\mathcal{V}\phi_j(\boldsymbol{U})^{1-\frac{1}{\vartheta}} + (\vartheta - 1)\phi_j(\boldsymbol{U})^{-\frac{1}{\vartheta}} + 1\right)$

Survival Clayton with GPD marginals[3]

- Survival Clayton copula with parameter $\theta = 2$
- Kendall's tau $\tau = 0.5$
- $d = 3$
- $X_i \sim GPD(\xi_i, \beta_j)$
- $\xi_i = 0.3$ (moments up to order 3 are finite)
- $\beta_i = 1$

[3]Model M1 from Koike and Hofert (2020)

Survival Clayton with GPD marginals

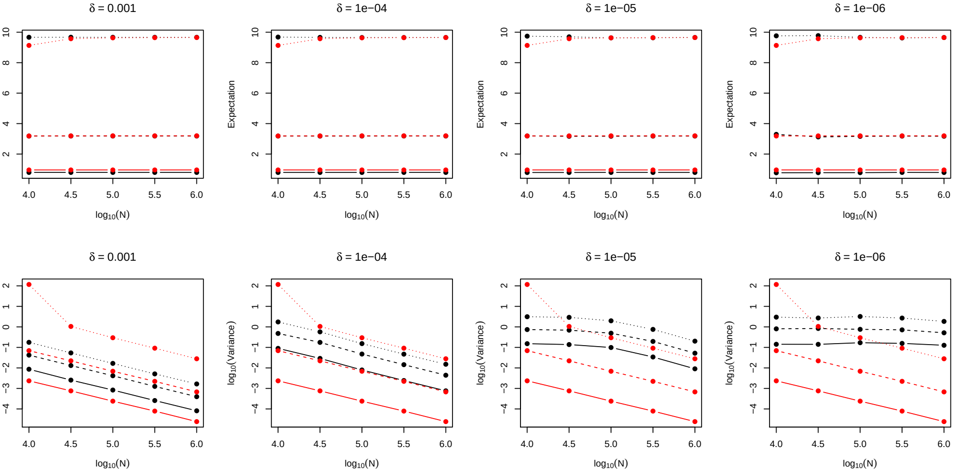

Figure: Mean (top) and variance (bottom) of the δ-estimator (black) and the new (red) for $\mathscr{C}_1^{\alpha}$. Line types (solid, dashed and dotted): different values of α. Columns: different values for δ.

Conclusions

- We are able to derive a **novel expression** for the Value-at-Risk contributions
- We go from an expectation conditional to a zero probability event in the usual representation, to a ratio of expectations conditional to **events of positive probability**
- The new formulation is amenable to Monte Carlo simulation with mild hypothesis on the multivariate models and the precise formulas are provided for a wide range of models
- The new representation shows promising results when compared to a simple estimator
- As the expectations in the proposed formulation **resemble the Expected Shortfall allocations** from which algorithms could be adapted for further computational gains.

Do we have time for Math?

YES! :)

no :(

Appendix

- The main theorem was presented for models of the form $\boldsymbol{X} = g(\boldsymbol{U})$.
- Without loss of generality, we abuse the notation and discuss the proof when $\boldsymbol{X} = g(\boldsymbol{Z})$
- The proof uses **Malliavin calculus**
- A less technical proof using only **integration by parts** may also be possible
- We explain later why we decided to use Malliavin calculus instead of integration by parts

Appendix

- Malliavin calculus is a differential calculus for functionals of the **Brownian motion**
- **Notation:**
  - $(W_t)_{t\in[0,T]}$: k-dimensional Brownian motion,
  - $W_t = (W_t^1, \dots, W_t^k)$
  - $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by $(W_t)_t$
  - $\mathbb{D}^{1,2}$: space of r.v.'s in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ that are differentiable in the Malliavin sense

Appendix

- A very important subspace of $\mathbb{D}^{1,2}$ is the space of **smooth random variables**

$$F = g\left(\int_0^T h_1(s)dW_s, \dots, \int_0^T h_n(s)dW_s\right),$$

with $g \in C_c^\infty(\mathbb{R}^n)$ and $h \in L^2([0,T]; \mathbb{R}^k)$.

- In this case, the **Malliavin derivative** at time $t \leq T$, which is denoted by D_t, is given by

$$D_t F = \sum_{k=1}^{n} \partial_k g\left(\int_0^T h_1(s)dW_s, \dots, \int_0^T h_n(s)dW_s\right) h_k(t),$$

where $\partial_k g$ is the derivative of g with respect to the kth variable.

Appendix

- An important case for our application is $F = g(W_T^1, \ldots, W_T^k)$, where $g \in C^1(\mathbb{R}^k)$
- In this case,
$$D_t F = \nabla g(W_T^1, \ldots, W_T^k)$$
- In the multivariate case where $F = (F^1, \ldots, F^m)$, the Malliavin derivative $D_t F$ is a $m \times k$ matrix where the jth row is given by $D_t F^j$

Appendix

- The adjoint operator of D, denoted by δ and called **Skorokhod integral**, is defined by the integration-by-parts formula:

$$\mathbb{E}[F\delta(v)] = \mathbb{E}\left[\int_0^T D_t F \cdot v_t dt\right], \ \forall\ F \in \mathbb{D}^{1,2}$$

- The domain of δ is characterized by the $\mathbb{R}^k$-valued stochastic processes $v = (v_t)_{t\in[0,T]}$ (**not necessarily adapted** to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$) such that

$$\left|\mathbb{E}\left[\int_0^T D_t F \cdot v_t dt\right]\right| \le C\|F\|_2, \ \forall\ F \in \mathbb{D}^{1,2},$$

where $C > 0$ might depend on v and $\|F\|_2 = \mathbb{E}[|F|^2]^{1/2}$

Appendix

- **Important:** For F_j a smooth random variable and $h_j \in L^2([0,T];\mathbb{R}^k)$, $j = 1, \dots, m$,

$$\delta\left(\sum_{j=1}^m F_j h_j\right) = \sum_{j=1}^m \left(F_j \int_0^T h_j(t) dW_t - \int_0^T D_t F_j \cdot h_j(t) dt\right).$$

- For smooth r.v.'s the Skorohod integral can be computed in terms of **Ito** and **Riemman integrals**

Appendix

- The cornerstone of our result is the following theorems from Ewald (2005) and Fournié et al. (2001)

Appendix

Theorem

Let $F, G \in \mathbb{D}^{1,2}$ such that F is $\mathbb{R}^m$-valued, G is $\mathbb{R}$-valued with $D_t G$ non-degenerate. Assume there exists a process v in the domain of δ and

$$\mathbb{E}\left[\int_0^T D_t G \cdot v_t dt \;\middle|\; F, G \right] = 1.$$

Assume further that $\phi \in C^1(\mathbb{R})$. Then

$$\mathbb{E}[\phi(F) \mid G = 0] = \frac{\mathbb{E}\left[\phi(F)\delta(v)H(G) - \phi'(F)H(G) \int_0^T D_t F v_t dt \right]}{\mathbb{E}[\delta(v)H(G)]},$$

where $H(x) = 1_{x \geq 0}$ is the Heaviside function.

Appendix

Theorem

Additionally to the assumptions of the theorem above, assume

$$\mathbb{E}\left[\int_0^T D_t F \cdot v_t dt \,\middle|\, F, G\right] = 0_{\mathrm{m}},$$

where 0_{m} *is the m-dimensional zero vector. Then, for any Borel measurable function* ϕ *with at most linear growth at infinity,*

$$\mathbb{E}[\phi(F) \mid G = 0] = \frac{\mathbb{E}[\phi(F)\delta(v)H(G)]}{\mathbb{E}[\delta(v)H(G)]}.$$

Thank you for your attention!

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