ASSIGNMENT ON RATE-DISTORTION BOUND

DUE NOVEMBER 3, 2017

The purpose of this assignment is to practice the material on minimal-distortion steganography. In particular, you are asked to draw the rate–distortion bound for a *given image* for an additive distortion function and binary embedding. The assignment is written as a tutorial, guiding you through the steps.

Assignment of costs. You will be working with grayscale images $\mathbf{x} = (x_{ij})$, $i = 1 : M$, $j = 1 : N$, stored as an array of *doubles* in Matlab. Compute the costs of making an embedding change at pixel ij as

$$\begin{aligned} \rho_{ij} &= \frac{1}{1+R_{ij}}, \quad i = 2 : M-1,\ j = 2 : N-1, \text{ where} \\ R_{ij} &= \frac{1}{4}\left(|x_{ij} - x_{i-1,j}| + |x_{ij} - x_{i+1,j}| + |x_{ij} - x_{i,j-1}| + |x_{ij} - x_{i,j+1}|\right). \end{aligned}$$

This cost assignment makes intuitive sense. In smooth regions, the residual R_{ij} is likely to be small and thus the cost ρ_{ij} large, while in textured regions and around edges, R_{ij} is large and the cost ρ_{ij} is small. Indeed, it should be more costly to modify a pixel in a smooth region than in a highly textured region – the textured regions are harder to model well.

Note: I am asking you to obtain the costs only for the internal pixels not lying on the boundary so that you do not have to deal with issues at the boundary of the image. This way, you will essentially end up analyzing a slightly smaller image of dimensions $(M-2) \times (N-2)$ instead of $M \times N$.

R-D bound. The relative payload (in bits per pixel or bpp) and the expected per-pixel distortion are:

$$\alpha(\lambda) = \frac{1}{(M-2)(N-2)} \sum_{i=2}^{M-1} \sum_{j=2}^{N-1} h\left(\frac{1}{1+e^{-\lambda\rho_{ij}}}\right),$$

$$d(\lambda) = \frac{1}{(M-2)(N-2)} \sum_{i=2}^{M-1} \sum_{j=2}^{N-1} \rho_{ij} \frac{e^{-\lambda\rho_{ij}}}{1+e^{-\lambda\rho_{ij}}}.$$

Note: In order to have the payload computed in bits, you need to use $\log_2$ (log2 in Matlab) for the binary entropy function $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

Tasks. You will be working with two grayscale 512×512 test images: '130.bmp' and '293.bmp'. Both images can be downloaded from Blackboard.

1. Compute the costs ρ_{ij} for both images. Order them from the smallest to the largest and plot them – plot(sort(ρ(:))). Include both plots in your submission.

2. Draw the rate–distortion bound in the form of $d(\lambda)$ as a function of $\alpha(\lambda)$ – the graph should show α on the x-axis and $d(\lambda)$ on the y-axis. Use the following 51 values of the parameter $\lambda = (1.2)^{-30}, (1.2)^{-29}, \ldots, (1.2)^{20}$ to draw the R-D bound. Include both images in your submission. Note that you obtain a different bound for

each image. The bound is determined by the costs, which are in turn determined by the image content.

You might want to implement the binary entropy function $h(x)$ so that you can give it a vector (matrix) parameter to avoid having to loop over the pixels. Use this Matlab code – it will not give you troubles with the logarithm (avoiding Matlab warnings of taking log of zero):

`function` $y = h(x)$
$y = \text{zeros}(\text{size}(x));$
$I = \text{abs}(x - 0.5) < 0.4999999;$
$y(I) = -x(I). * \log 2(x(I)) - (1 - x(I)). * \log 2(1 - x(I));$

3. Answer the following questions. The first three questions can be answered simply by reading the values from the rate–distortion plots that you just obtained.

- What relative payload can you embed in image '130.bmp' and in '293.bmp' with per-pixel distortion bounded by $d = 0.05$? This is the distortion–limited sender (DLS). You need to supply two numbers here.
- Why is the payload so much smaller for '293.bmp' than for '130.bmp'? Include an explanation/interpretation.
- Assume a payload–limited sender (PLS) who wants to embed the payload of 0.5 bpp (bits per pixel) in both images. What distortion per pixel will be approximately imposed on each image? You need to supply two numbers here.
- Produce a black-and-white image of the same dimensions as ρ_{ij} showing in white 50,000 pixels with the highest probability of being modified during embedding, while the rest of the pixels should be black. You can compute the probabilities for any value of λ – the 50,000 most likely modified pixels will be always the same pixels. Note that you will compute the probabilities already in Task 2 when drawing the rate–distortion bound.

Example output

To give you an idea of what I expect, I supplied the following example.

FIGURE 1. The original grayscale image "Flower" also available from the Blackboard as 'flower_gray.bmp'.

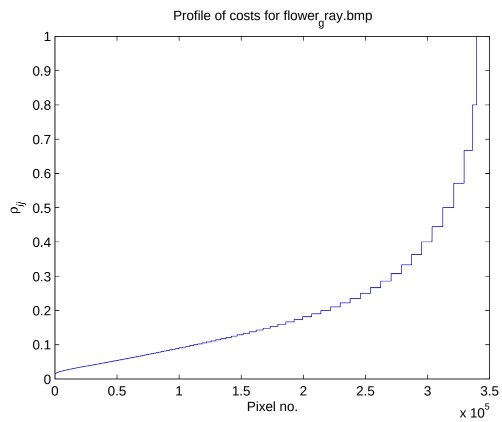

FIGURE 2. The embedding cost profile for "Flower."

(1) What relative payload can you embed in "Flower" with per-pixel distortion bounded by $d = 0.05$? This is the distortion–limited sender or DLS. The payload is slightly below 0.9, approximately ~ 0.88 bpp.
(2) Assume a payload–limited sender (PLS) who wants to embed the payload of 0.5 bpp (bits per pixel) in 'Flower.bmp'. What distortion will be approximately imposed? From the rate–distortion bound, we can see that this payload can be embedded with expected distortion of about 0.013.

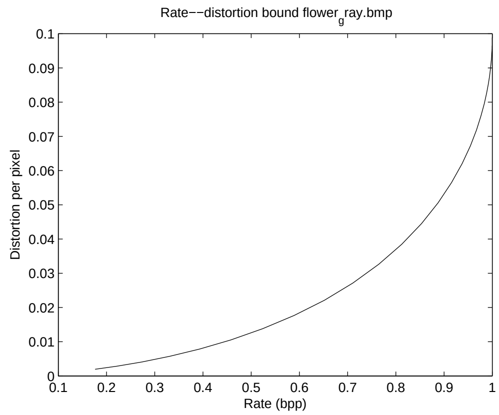

FIGURE 3. The rate–distortion bound for "Flower."

FIGURE 4. In white are all 50,000 pixels with the highest probability of being modified during embedding.

You can earn three additional letter grades by working out the following optional problems. Again, I will count these only if the grade from this optional problem is better or equal to the grade from the main problem above.

OPTIONAL PROBLEM I

Let $\mathcal{M}$ be a binary steganographic method that minimizes distortion with pixel costs ρ_i, $i = 1, \ldots, n$.

(1) Consider a binary distortion-minimizing steganographic method $\mathcal{M}'$ that differs from $\mathcal{M}$ only in the cost assignment – all costs are multiplied by a positive constant, $a > 0$, $\rho'_i = a\rho_i$. Which scheme will be more secure and why?
(2) Consider a binary distortion-minimizing steganographic method $\mathcal{M}'$ that differs from $\mathcal{M}$ only in the cost assignment – all costs are increased by a

positive constant, $\rho_i' = a + \rho_i$. Are $\mathcal{M}$ and $\mathcal{M}'$ the same or different schemes and why?

(3) How would your answer to Question 2 change if there was a fixed positive cost, $\rho_0 > 0$, of *not* making a change?

Optional problem II

Consider the following generalization of LSB matching (also called ± 1 embedding), which is called $\pm k$ embedding, where k is a positive integer. In $\pm k$ embedding, each pixel is allowed to be modified by up to $\pm k$ (it can be left unchanged or modified by $\pm 1, \ldots, \pm k$). For simplicity, ignore the boundary issues when a pixel cannot actually be modified by those amounts. Assume uniform costs: the cost of changing x_i to one of the $2k+1$ values is 1 independently of i with a zero cost for no embedding. Note that with uniform equal costs, the relative distortion is the same as the change rate, β.

Derive the rate–distortion bound – the relationship between the maximal relative message α and the change rate β and the bound on embedding efficiency e as a function of α.

Hint: Start by writing down the probability of changing a pixel and the expected per-pixel distortion β. Then, write down the entropy per pixel (payload per pixel or α) and substitute in it β and simplify. You should be getting this: $\alpha = h_{2k+1}(\beta)$, where $h_q(x) = h_2(x) + x \log_2(q-1)$ is the q-ary entropy function, a generalization of the binary entropy function $h_2(x) = -x \log_2 x - (1-x)\log_2(1-x)$.

Optional problem III

If you look at your RD bounds, they appear to have the the following property: The derivative of the relative distortion d with respect to the relative payload α, $\frac{d}{d\alpha} d(\alpha)$, seems to be zero at $\alpha = 0$ and infinity at $\alpha = 1$ (the maximal payload). Prove that this is indeed the case for an arbitrary cost profile $\rho_i > 0$.