Supplementary Material

Below are more details that are omitted in the main text due to the space limitation.

Proof of Theorem 4.2

Theorem 4.2 *The estimate* $\hat{q}_{ru}(G)$ *produced by* $\mathbf{Local_{ru}}$ *is unbiased, i.e.,* $\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G)$.

Before beginning with the detailed proof, we need to introduce a lemma as below.

Lemma 1 ([Kozubowski and Nadarajah(2010)])**.** *Given a random variable* $X \sim Lap(x, b)$*, then*

$$\mathbb{E}(X^r) = \sum_{k=0}^{r} \left\{ \frac{1}{2} \left[1 + (-1)^k \right] \frac{r!}{(r-k)!} b^k x^{r-k} \right\}. \tag{A.1}$$

Now, let us give the proof of Theorem 4.2.

Proof. It is clear to see that $\tilde{a}_{i,j}$ is in fact a Bernoulli random variable and $\tilde{d}_i \sim \text{Lap}\left(d_i, \frac{1}{\varepsilon_2}\right)$. Then, we have

$$\mathbb{E}(\tilde{a}_{i,j}) = a_{i,j}(1-p) + (1-a_{i,j})p, \tag{A.2a}$$

$$\mathbb{E}\left(\tilde{d}_i^2\right) = d_i^2 + \frac{2}{\varepsilon_2^2}. \quad \text{(by Lemma 1)} \tag{A.2b}$$

$$\mathbb{E}\left(\tilde{d}_i^4\right) = d_i^4 + \frac{12}{\varepsilon_2^2} d_i^2 + \frac{24}{\varepsilon_2^4} \quad \text{(by Lemma 1)} \tag{A.2c}$$

Next, we move to the proof of unbiasedness of $\hat{q}_{ru}(G)$. First, we obtain

$$\begin{aligned}
\mathbb{E}(X_1) &= \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{(E(\tilde{a}_{i,j}) - p) E\left(\tilde{d}_i\right) E\left(\tilde{d}_j\right)}{1-2p} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{[a_{i,j}(1-p) + (1-a_{i,j})p - p] d_i d_j}{1-2p} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i-1} a_{i,j} d_i d_j = \sum_{(v_i, v_j) \in E} d_i d_j.
\end{aligned} \tag{A.3}$$

Similarly, it is easy to check

$$
\begin{aligned}
\mathbb{E}\left(Y_1\right) &= \mathbb{E}\left[\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right)^2 - \frac{5n+4}{\varepsilon_2^4}\right] \\
&= \mathbb{V}\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right) - \frac{5n+4}{\varepsilon_2^4} \\
&\quad + \left[\mathbb{E}\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right)\right]^2 \\
&= \frac{1}{4}\sum_{i=1}^{n}\left[\mathbb{E}\left(\tilde{d}_i^4\right) - \left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\right] - \frac{5n+4}{\varepsilon_2^4} \\
&\quad + \left[\frac{1}{2}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^2\right) - \frac{n+2}{\varepsilon_2^2}\right]^2 \\
&= \frac{1}{4}\sum_{i=1}^{n}\left(\frac{8}{\varepsilon_2^2}d_i^2 + \frac{20}{\varepsilon_2^4}\right) - \frac{5n+4}{\varepsilon_2^4} \\
&\quad + \left[\frac{1}{2}\sum_{i=1}^{n}\left(\frac{2}{\varepsilon_2^2} + d_i^2\right) - \frac{n+2}{\varepsilon_2^2}\right]^2 \\
&= \frac{2}{\varepsilon_2^2}\sum_{i=1}^{n}d_i^2 + \frac{5n}{\varepsilon_2^4} + \left[\frac{1}{2}\sum_{i=1}^{n}d_i^2 - \frac{2}{\varepsilon_2^2}\right]^2 - \frac{5n+4}{\varepsilon_2^4} \\
&= \left[\sum_{(v_i,v_j)\in E}\frac{1}{2}\left(d_i+d_j\right)\right]^2 .
\end{aligned}
\tag{A.4}
$$

To sum up, we verify that

$$
\mathbb{E}\left[\hat{q}_{ru}\left(G\right)\right] = \frac{\mathbb{E}\left(X_1\right)}{M} - \frac{\mathbb{E}\left(Y_1\right)}{M^2} = q_{ru}\left(G\right). \tag{A.5}
$$

$\square$

Proof of Theorem 4.3

Theorem 4.3 *When ε_1, ε_2 are constants, the estimate $\hat{q}_{ru}(G)$ produced by* $\mathbf{Local_{ru}}$ *provides the following utility guarantee:*

$$\mathrm{MSE}\left(\frac{n^3 d_{\max}^2 + n^2 d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^6}{M^4}\right). \tag{B.1}$$

Let us introduce a lemma to succeed in verifying Theorem 4.3.

Lemma 2 ([Imola et al.(2022)]). *Let x_1, x_2 be two random variables, then* $\mathbb{V}(x_1 + x_2) \leq 4\max\{\mathbb{V}(x_1), \mathbb{V}(x_2)\}$.

From now on, we show the detailed proof of Theorem 4.3 as follows.

Proof. Due to $\mathbb{V}(a_{i,j}) = \mathbb{E}\left(a_{i,j}^2\right) - \left[\mathbb{E}(a_{i,j})\right]^2 = p(1-p)$, we can obtain

$$\begin{aligned}\mathbb{E}\left(\hat{a}_{i,j}^2\right) &= \mathbb{V}(\hat{a}_{i,j}) + \left[\mathbb{E}(\hat{a}_{i,j})\right]^2 \\ &= \frac{p(1-p)}{(1-2p)^2} + a_{i,j}^2 \\ &= \frac{p(1-p)}{(1-2p)^2} + a_{i,j}.\end{aligned} \tag{B.2}$$

From Theorem 4.2, it is clear to the eye that the estimate $\hat{q}_{ru}(G)$ produced by $\mathbf{Local_{ru}}$ is unbiased. By the bias-variance decomposition [Murphy(2012)], the mean squared error (MSE) of $\hat{q}_{ru}(G)$ is equal to its variance. Let $P = \frac{X_1}{M}$ and $Q = -\frac{Y_1}{M^2}$, then

$$\begin{aligned}\mathrm{MSE}(\hat{q}_{ru}(G)) = \mathbb{V}(\hat{q}_{ru}(G)) &= \mathbb{V}(P+Q) \\ &\leq 4\max\{\mathbb{V}(P), \mathbb{V}(Q)\}. \\ &\quad \text{(by Lemma 2)}\end{aligned} \tag{B.3}$$

Now, our task is to calculate $\mathbb{V}(P)$ and $\mathbb{V}(Q)$ separately.

For $\mathbb{V}(P)$, since $\mathbb{V}(P) = M^{-2}\mathbb{V}(X_1)$, we need to focus on calculation of $\mathbb{V}(X_1)$. For ease of presentation, we define $\tilde{B}_{i,j} = \frac{(\tilde{a}_{i,j}-p)\tilde{d}_i\tilde{d}_j}{1-2p}$, then

$$\begin{aligned}
\mathbb{V}\left(X_1\right) &= \mathbb{V}\left(\sum_{i=2}^{n}\sum_{j=1}^{i-1}\tilde{B}_{i,j}\right) \\
&= \sum_{i=2}^{n}\mathbb{V}\left(\sum_{j=1}^{i-1}\tilde{B}_{i,j}\right) + \sum_{2\le k,l\le n,k\ne l}\mathrm{Cov}\left(\sum_{j=1}^{k-1}\tilde{B}_{k,j},\sum_{j=1}^{l-1}\tilde{B}_{l,j}\right) \\
&= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\mathbb{V}\left(\tilde{B}_{i,j}\right) + \sum_{i=2}^{n}\sum_{1\le k,l\le i-1,k\ne l}\mathrm{Cov}\left(\tilde{B}_{i,k},\tilde{B}_{i,l}\right) \\
&\quad + \sum_{2\le k,l\le n,k\ne l}\sum_{j=1}^{k-1}\sum_{t=1}^{l-1}\mathrm{Cov}\left(\tilde{B}_{k,j},\tilde{B}_{l,t}\right) \\
&= P_1 + P_2 + P_3,
\end{aligned} \tag{B.4}$$

where $P_1 = \sum_{i=2}^{n}\sum_{j=1}^{i-1}\mathbb{V}\left(\tilde{B}_{i,j}\right)$, $P_2 = \sum_{i=2}^{n}\sum_{1\le k,l\le i-1,k\ne l}\mathrm{Cov}\left(\tilde{B}_{i,k},\tilde{B}_{i,l}\right)$ and $P_3 = \sum_{2\le k,l\le n,k\ne l}\sum_{j=1}^{k-1}\sum_{t=1}^{l-1}\mathrm{Cov}\left(\tilde{B}_{k,j},\tilde{B}_{l,t}\right)$.

Next, we calculate P_1, P_2 and P_3 respectively, and obtain

$$\begin{aligned}
P_1 &= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\mathbb{E}\left[\left(\tilde{B}_{i,j}\right)^2\right] - \left[\mathbb{E}\left(\tilde{B}_{i,j}\right)\right]^2\right] \\
&= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\left(\frac{p\left(1-p\right)}{\left(1-2p\right)^2} + a_{i,j}\right)\left(\frac{2}{\varepsilon_2^2} + d_i^2\right)\left(\frac{2}{\varepsilon_2^2} + d_j^2\right)\right] - \sum_{i=2}^{n}\sum_{j=1}^{i-1} a_{i,j} d_i^2 d_j^2 \\
&= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\frac{p\left(1-p\right)}{\left(1-2p\right)^2}\left(\frac{2}{\varepsilon_2^2} + d_i^2\right)\left(\frac{2}{\varepsilon_2^2} + d_j^2\right)\right] \\
&\quad + \sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{2}{\varepsilon_2^2} a_{i,j}\left(\frac{2}{\varepsilon_2^2} + d_i^2 + d_j^2\right) \\
&\le \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\frac{p\left(1-p\right)}{\left(1-2p\right)^2}\left(\frac{2}{\varepsilon_2^2} + d_{\max}^2\right)\left(\frac{2}{\varepsilon_2^2} + d_{\max}^2\right)\right] \\
&\quad + \sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{2}{\varepsilon_2^2}\left(\frac{2}{\varepsilon_2^2} + 2d_{\max}^2\right) \\
&= O\left(n^2 d_{\max}^4\right),
\end{aligned} \tag{B.5a}$$

$$
\begin{aligned}
P_2 &= \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\ne l} \operatorname{Cov}\left(\tilde{B}_{i,k}, \tilde{B}_{i,l}\right) \\
&= \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\ne l} \left[\mathbb{E}\left(\tilde{B}_{i,k}\tilde{B}_{i,l}\right) - \mathbb{E}\left(\tilde{B}_{i,k}\right)\mathbb{E}\left(\tilde{B}_{i,l}\right)\right] \\
&= \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\ne l} \left[a_{i,k}a_{i,l}\left(\frac{2}{\varepsilon_2^2} + d_i^2\right) d_k d_l\right] - \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\ne l} a_{i,k}a_{i,l}d_i^2 d_k d_l \\
&= \frac{2}{\varepsilon_2^2} \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\ne l} a_{i,k}a_{i,l}d_k d_l \\
&\le \frac{2}{\varepsilon_2^2} \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\ne l} d_{\max}^2 \\
&= O\left(n^3 d_{\max}^2\right),
\end{aligned}
\tag{B.5b}
$$

$$
\begin{aligned}
P_3 &= \sum_{2\le k,l\le n,k\ne l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} \operatorname{Cov}\left[\tilde{B}_{k,j}, \tilde{B}_{l.t}\right] \\
&= \sum_{2\le k,l\le n,k\ne l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} \mathbb{E}\left(\tilde{B}_{k,j}\tilde{B}_{l.t}\right) - \sum_{2\le k,l\le n,k\ne l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1}\mathbb{E}\left(\tilde{B}_{k,j}\right)\mathbb{E}\left(\tilde{B}_{l.t}\right) \\
&= \sum_{2\le k,l\le n,k\ne l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} a_{k,j}a_{l,t}\mathbb{E}\left(\tilde{d}_k\tilde{d}_j\tilde{d}_l\tilde{d}_t\right) - \sum_{2\le k,l\le n,k\ne l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1}a_{k,j}d_k d_j a_{l,t}d_l d_t \\
&= 2\sum_{2\le k<l\le n,k\ne l} \sum_{j=1}^{k-1} \left[\frac{2}{\varepsilon_2^2}a_{k,j}a_{l,t}d_k d_l + \frac{2}{\varepsilon_2^2}a_{k,j}a_{l,t}d_j d_l\right] \\
&= \frac{4}{\varepsilon_2^2} \sum_{2\le k<l\le n,k\ne l} \sum_{j=1}^{k-1} \left[a_{k,j}a_{l,t}d_k d_l + a_{k,j}a_{l,t}d_j d_l\right] \\
&\le \frac{4}{\varepsilon_2^2} \sum_{2\le k<l\le n,k\ne l} \sum_{j=1}^{k-1} 2d_{\max}^2 \\
&= O\left(n^3 d_{\max}^2\right).
\end{aligned}
\tag{B.5c}
$$

Thus,

$$\mathbb{V}\left(X_1\right) = O\left(n^3 d_{max}^3 + n^2 d_{max}^4\right). \tag{B.6}$$

This leads to the following expression

$$\mathbb{V}\left(P\right) = O\left(\frac{n^3 d_{max}^3 + n^2 d_{max}^4}{M^2}\right). \tag{B.7}$$

Below we move on to the calculation of $\mathbb{V}\left(Q\right)$. By Lemma 1, we first derive

$$\mathbb{E}\left(\tilde{d}_i^6\right) = d_i^6 + \frac{30}{\varepsilon_2^2} d_i^4 + \frac{360}{\varepsilon_2^4} d_i^2 + \frac{720}{\varepsilon_2^6}, \tag{B.8a}$$

$$\mathbb{E}\left(\tilde{d}_i^8\right) = d_i^8 + \frac{56}{\varepsilon_2^2} d_i^6 + \frac{1680}{\varepsilon_2^4} d_i^4 + \frac{20160}{\varepsilon_2^6} d_i^2 + \frac{40320}{\varepsilon_2^8}. \tag{B.8b}$$

To make further progress, we write

$$\begin{aligned} \mathbb{V}\left(\tilde{d}_i^2\right) &= \mathbb{E}\left(\tilde{d}_i^4\right) - \left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2 \\ &= \left(d_i^4 + \frac{12}{\varepsilon_2^2} d_i^2 + \frac{24}{\varepsilon_2^4}\right) - \left(d_i^2 + \frac{2}{\varepsilon_2^2}\right)^2 \\ &= \frac{8}{\varepsilon_2^2} d_i^2 + \frac{20}{\varepsilon_2^4}, \end{aligned} \tag{B.9}$$

and

$$\begin{aligned} \mathbb{V}\left(\tilde{d}_i^4\right) &= \mathbb{E}\left(\tilde{d}_i^8\right) - \left[\mathbb{E}\left(\tilde{d}_i^4\right)\right]^2 \\ &= \frac{32}{\varepsilon_2^2} d_i^6 + \frac{1488}{\varepsilon_2^4} d_i^4 + \frac{19584}{\varepsilon_2^6} d_i^2 + \frac{39744}{\varepsilon_2^8}. \end{aligned} \tag{B.10}$$

Armed with the consequences above, let us derive quantity $\mathbb{V}(Q)$ as below

$$\begin{aligned}
&\mathbb{V}(Q) = \mathbb{V}\left(-\frac{Y_1}{M^2}\right) = \frac{1}{M^4}\mathbb{V}(Y_1) \\
&= \frac{1}{M^4}\mathbb{V}\left[\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right)^2 - \frac{5n+4}{\varepsilon_2^4}\right] \\
&= \frac{1}{M^4}\mathbb{V}\left[\frac{1}{4}\left(\sum_{i=1}^{n}\tilde{d}_i^2\right)^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right] \\
&= \frac{1}{M^4}\mathbb{V}\left[\frac{1}{4}\left(\sum_{i=1}^{n}\tilde{d}_i^4 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2\right) - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right] \\
&= \frac{1}{M^4}\mathbb{E}\left[\left(\frac{1}{4}\sum_{i=1}^{n}\tilde{d}_i^4 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right] \\
&\quad - \frac{1}{M^4}\left[\mathbb{E}\left(\frac{1}{4}\sum_{i=1}^{n}\tilde{d}_i^4 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right)\right]^2 \\
&= \frac{1}{M^4}Q_1 - \frac{1}{M^4}Q_2,
\end{aligned} \tag{B.11}$$

where

$$\begin{aligned}
Q_1 &= \mathbb{E}\left[\left(\frac{1}{4}\sum_{i=1}^{n}\tilde{d}_i^4 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right], \\
Q_2 &= \left[\mathbb{E}\left(\frac{1}{4}\sum_{i=1}^{n}\tilde{d}_i^4 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right)\right]^2.
\end{aligned}$$

Analogously, we need to calculate Q_1 and Q_2 separately. In essence, it is easy to derive

$$
\begin{aligned}
Q_1 &= \mathbb{E}\left[\left(\frac{1}{4}\sum_{i=1}^{n}\tilde{d}_i^4 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{1}{4}\sum_{i=1}^{n}\tilde{d}_i^4 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right] \\
&= \frac{1}{16}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^8\right) + \frac{3}{16}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_j^4\right) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^6\right)\mathbb{E}\left(\tilde{d}_j^2\right) \\
&\quad + \frac{3}{8}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right) \qquad \text{(B.12a)} \\
&\quad + \frac{1}{16}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{k}\sum_{s=1,s\neq i,j,k}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right)\mathbb{E}\left(\tilde{d}_s^2\right) \\
&\quad + \frac{(n+2)^2}{\varepsilon_2^4}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^4\right) + \frac{(n+2)^2}{\varepsilon_2^4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right) \\
&\quad - \frac{n+2}{2\varepsilon_2^2}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^6\right) - \frac{3(n+2)}{2\varepsilon_2^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_j^2\right) \\
&- \frac{n+2}{2\varepsilon_2^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right),
\end{aligned}
$$

$$
\begin{aligned}
Q_2 &= \left[\mathbb{E}\left(\frac{1}{4}\sum_{i=1}^{n}\tilde{d}_i^4 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\tilde{d}_i^2\tilde{d}_j^2 - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2\right)\right]^2 \\
&= \left[\frac{1}{4}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^4\right) + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right) - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2 \\
&= \frac{1}{16}\sum_{i=1}^{n}\left[\mathbb{E}\left(\tilde{d}_i^4\right)\right]^2 + \frac{1}{16}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_j^4\right) \\
&\quad + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right) \\
&\quad + \frac{1}{8}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right) \\
&\quad + \frac{1}{8}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\left[\mathbb{E}\left(\tilde{d}_j^2\right)\right]^2 \\
&\quad + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right) \\
&\quad + \frac{1}{16}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{k}\sum_{s=1,s\neq i,j,k}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right)\mathbb{E}\left(\tilde{d}_s^2\right) \\
&\quad + \frac{(n+2)^2}{\varepsilon_2^4}\sum_{i=1}^{n}\left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2 + \frac{(n+2)^2}{\varepsilon_2^4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right) \\
&\quad - \frac{n+2}{2\varepsilon_2^2}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_i^2\right) - \frac{n+2}{2\varepsilon_2^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_j^2\right) \\
&\quad - \frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right) \\
&\quad - \frac{n+2}{2\varepsilon_2^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\mathbb{E}\left(\tilde{d}_i^2\right)\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right).
\end{aligned}
\tag{B.12b}
$$

From Eqs.(B.12a) and (B.12b), we obtain

$$
\begin{aligned}
\mathbb{V}\left(Q\right) &= \frac{1}{16M^4}\sum_{i=1}^{n}\left[\mathbb{E}\left(\tilde{d}_i^8\right)-\left[\mathbb{E}\left(\tilde{d}_i^4\right)\right]^2\right] \\
&+\frac{1}{4M^4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left[\mathbb{E}\left(\tilde{d}_i^6\right)-\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_i^2\right)\right]\mathbb{E}\left(\tilde{d}_j^2\right) \\
&+\frac{1}{8M^4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left[\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_j^4\right)-\left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\left[\mathbb{E}\left(\tilde{d}_j^2\right)\right]^2\right] \\
&+\frac{1}{4M^4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,j}^{n}\left[\mathbb{E}\left(\tilde{d}_i^4\right)-\left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\right]\mathbb{E}\left(\tilde{d}_j^2\right)\mathbb{E}\left(\tilde{d}_k^2\right) \\
&+\frac{(n+2)^2}{\varepsilon_2^4M^4}\sum_{i=1}^{n}\left[\mathbb{E}\left(\tilde{d}_i^4\right)-\left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\right] \\
&-\frac{n+2}{2\varepsilon_2^2M^4}\sum_{i=1}^{n}\left[\mathbb{E}\left(\tilde{d}_i^6\right)-\mathbb{E}\left(\tilde{d}_i^4\right)\mathbb{E}\left(\tilde{d}_i^2\right)\right] \\
&-\frac{n+2}{\varepsilon_2^2M^4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left[\mathbb{E}\left(\tilde{d}_i^4\right)-\left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\right]\mathbb{E}\left(\tilde{d}_j^2\right) \\
&= O\left(\frac{n^3 d_{\max}^6}{M^4}\right).
\end{aligned} \tag{B.13}
$$

Finally, from Eqs.(B.7) and (B.13) it follows that

$$
\mathrm{MSE} = O\left(\frac{n^3 d_{\max}^2 + n^2 d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^6}{M^4}\right). \tag{B.14}
$$

□

Proof of Theorem 4.6

Theorem 4.6 *The estimate* $\hat{q}_{ru}(G)$ *produced by* $\textbf{Shuffle}_{\textbf{ru}}$ *satisfies* $\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G)$.

Proof. First, we prove that X_2 is unbiased, i.e.,

$$\begin{aligned}
\mathbb{E}(X_2) &= \mathbb{E}\left(\sum_{i=2}^{n} \hat{r}_i\right) \\
&= \mathbb{E}\left[\sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} \frac{(\tilde{a}_{i,j} - p)\,\tilde{d}_j}{1-2p}\right] \\
&= \sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} \mathbb{E}\left(\frac{\tilde{a}_{i,j} - p}{1-2p}\right) \mathbb{E}\left(\tilde{d}_j\right) \qquad \text{(C.1)} \\
&= \sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} a_{ij} d_j \\
&= \sum_{(v_i,v_j)\in E} d_i d_j.
\end{aligned}$$

We now consider Y_2. Since Y_2 has the similar form as Y_1, and the unbiasedness of Y_1 has been already proven in the proof of Theorem 4.2, it follows that Y_2 is also unbiased. Thus, we have

$$\mathbb{E}(Y_2) = \left[\sum_{(v_i,v_j)\in E} \frac{1}{2}(d_i + d_j)\right]^2. \qquad \text{(C.2)}$$

Armed with the results above, we come to

$$\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G). \qquad \text{(C.3)}$$

□

Proof of Theorem 4.7

Theorem 4.7 *When* ε, δ *are constants,* $\alpha \in (0,1)$, $\varepsilon_0 = \log(n) + O(1)$, *the estimate* $\hat{q}_{ru}(G)$ *produced by* $\mathbf{Shuffle_{ru}}$ *provides the following utility guarantee:*

$$\mathrm{MSE}\left(\hat{q}_{ru}(G)\right) = O\left(\frac{n^{1+\alpha}d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^2}{(\log n)^2 M^2} + \frac{n^3 d_{\max}^6}{(\log n)^2 M^4}\right). \tag{D.1}$$

Proof. Let $U = \frac{X_2}{M}$ and $W = -\frac{Y_2}{M^2}$, then the MSE of $\hat{q}_{ru}(G)$ by $\mathbf{Shuffle_{ru}}$ can be written as follows

$$\begin{aligned} \mathrm{MSE}\left(\hat{q}_{ru}(G)\right) &= \mathbb{V}\left(\hat{q}_{ru}(G)\right) = \mathbb{V}(U+W) \\ &\leq 4\max\left\{\mathbb{V}(U), \mathbb{V}(W)\right\}. \quad \text{(by Lemma 2)} \end{aligned} \tag{D.2}$$

We now need to calculate $\mathbb{V}(U)$ and $\mathbb{V}(W)$ separately. Since the expression for W is similar to that of Q in the proof of Theorem 4.3, this leads to $\mathbb{V}(W) \leq O\left(\frac{n^3 d_{\max}^6}{(\log n)^2 M^4}\right)$. Next, we only need to compute $\mathbb{V}(U)$ to establish the upper bound of $\mathrm{MSE}\left(\hat{q}_{ru}(G)\right)$.

$$
\begin{aligned}
\mathbb{V}(U) &= M^{-2}\mathbb{V}\left(\sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} \frac{(\tilde{a}_{i,j}-p)\,\tilde{d}_j}{1-2p}\right) \\
&= M^{-2}\sum_{i=2}^{n} d_i^2 \mathbb{V}\left(\sum_{j=1}^{i-1} \frac{(\tilde{a}_{i,j}-p)\,\tilde{d}_j}{1-2p}\right) \\
&\quad + 2M^{-2}\sum_{2\le k<l\le n} \mathrm{Cov}\left(\sum_{j=1}^{k-1} \frac{(\tilde{a}_{k,j}-p)\,d_k\tilde{d}_j}{1-2p}, \sum_{h=1}^{l-1} \frac{(\tilde{a}_{l,h}-p)\,d_l\tilde{d}_h}{1-2p}\right) \\
&= M^{-2}\sum_{i=2}^{n} \frac{d_i^2}{(1-2p)^2} \sum_{j=1}^{i-1} \mathbb{V}\left[(\tilde{a}_{i,j}-p)\,\tilde{d}_j\right] \\
&\quad + \frac{2M^{-2}}{(1-2p)^2} \sum_{2\le k<l\le n} \sum_{j=1}^{k-1}\sum_{h=1}^{l-1} d_k d_l \mathrm{Cov}\left((\tilde{a}_{k,j}-p)\,\tilde{d}_j, (\tilde{a}_{l,h}-p)\,\tilde{d}_h\right) \\
&= M^{-2}\sum_{i=2}^{n} \frac{d_i^2}{(1-2p)^2} \sum_{j=1}^{i-1} \left[p(1-p)\,d_j^2 + \frac{2(1-2p)^2 a_{i.j}}{\alpha^2\varepsilon_0^2} + \frac{2p(1-p)}{\alpha^2\varepsilon_0^2}\right] \\
&\quad + \frac{4M^{-2}}{\alpha^2\varepsilon_0^2} \sum_{2\le k<l\le n} \sum_{j=1}^{k-1} a_{k,j}a_{l,j}d_k d_l \\
&\le \frac{p(1-p)}{(1-2p)^2 M^2} \sum_{i=2}^{n}\sum_{j=1}^{i-1} d_{\max}^4 + \frac{4}{\alpha^2\varepsilon_0^2 M^2} \sum_{2\le k<l\le n} (k-1)\, d_{\max}^2 \\
&= \frac{p(1-p)\,n(n-1)}{2(1-2p)^2 M^2} d_{\max}^4 + \frac{2n(n-1)(n-2)}{3\alpha^2\varepsilon_0^2 M^2} d_{\max}^2 \\
&= \frac{n^{\alpha}(n-1)}{M^2} d_{\max}^4 + \frac{n(n-1)(n-2)}{(\log n)^2 M^2} d_{\max}^2 \\
&= O\left(\frac{n^{1+\alpha} d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^2}{(\log n)^2 M^2}\right).
\end{aligned}
\tag{D.3}
$$

Therefore, we gain

$$
\mathrm{MSE}\left(\hat{q}_{ru}(G)\right) = O\left(\frac{n^{1+\alpha} d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^2}{(\log n)^2 M^2} + \frac{n^3 d_{\max}^6}{(\log n)^2 M^4}\right). \tag{D.4}
$$

□

Proof of Theorem 4.10

Theorem 4.10 *The estimate* $\hat{q}_{ru}(G)$ *produced by* **Decentral$_{\mathbf{ru}}$** *satisfies* $\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G)$.

Proof. Since $\tilde{T}_i \sim \mathrm{Lap}\left(\frac{\hat{d}_{max}}{\varepsilon_2}\right)$, by Lemma 1, we have

$$\mathbb{E}\left(\tilde{T}_i\right) = T_i + \frac{2\hat{d}_{max}^2}{\varepsilon_2^2}. \tag{E.1}$$

Then, we can easily obtain

$$\begin{aligned}\mathbb{E}(X_2) &= \mathbb{E}\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i\tilde{T}_i\right)\\ &= \frac{1}{2}\sum_{i=1}^{n}d_iT_i\\ &= \sum_{(v_i,v_j)\in E}d_id_j,\end{aligned} \tag{E.2}$$

Following a similar derivation as in Eq.(A.4), we have

$$\mathbb{E}(Y_2) = \left[\sum_{(v_i,v_j)\in E}\frac{1}{2}(d_i+d_j)\right]^2. \tag{E.3}$$

Finally, we show that

$$\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G). \tag{E.4}$$

□

Proof of Theorem 4.11

Theorem 4.11 *When ε_1, ε_2 are constants, the estimate $\hat{q}_{ru}(G)$ produced by* $\textbf{Decentral}_{\textbf{ru}}$ *provides the following utility guarantee:*

$$\mathrm{MSE}\left(\hat{q}_{ru}\left(G\right)\right) = O\left(\frac{n^3 d_{\max}^6}{M^4}\right). \tag{F.1}$$

Proof. Since $\tilde{T}_i \sim \mathrm{Lap}\left(T_i, \frac{\hat{d}_{\max}}{\varepsilon_2}\right)$, then by Lemma 1, we can get

$$\mathbb{E}\left(\tilde{T}_i^2\right) = T_i^2 + \frac{2\hat{d}_{\max}^2}{\varepsilon_2^2}, \tag{F.2}$$

Below we calculate the MSE of $q_{ru}\left(G\right)$ produced by $\textbf{Decentral}_{\textbf{ru}}$. Let $H = \frac{X_3}{M}$ and $S = -\frac{Y_3}{M^2}$, then

$$\begin{aligned}\mathrm{MSE}\left(\hat{q}_{ru}\left(G\right)\right) &= \mathbb{V}\left(\hat{q}_{ru}\left(G\right)\right) = \mathbb{V}\left(H+S\right)\\ &\leq 4\max\left\{\mathbb{V}\left(H\right), \mathbb{V}\left(S\right)\right\}. \quad \text{(by Lemma 2)}\end{aligned} \tag{F.3}$$

Next, we calculate $\mathbb{V}\left(H\right)$ and $\mathbb{V}\left(S\right)$ respectively.

$$\begin{aligned}\mathbb{V}\left(H\right) &= \mathbb{V}\left(\frac{X_3}{M}\right) = M^{-2}\mathbb{V}\left(X_3\right)\\ &= \frac{1}{4M^2}\sum_{i=1}^{n}\mathbb{V}\left(\hat{d}_i\hat{T}_i\right)\\ &= \frac{1}{4M^2}\sum_{i=1}^{n}\left[\mathbb{E}\left(\hat{d}_i^2\hat{T}_i^2\right) - \mathbb{E}\left(\hat{d}_i\hat{T}_i\right)^2\right]\\ &= \frac{1}{4M^2}\sum_{i=1}^{n}\left[\mathbb{E}\left(\hat{d}_i^2\right)\mathbb{E}\left(\hat{T}_i^2\right) - \mathbb{E}\left(\hat{d}_i\right)^2\mathbb{E}\left(\hat{T}_i\right)^2\right]\\ &= \frac{1}{4M^2}\sum_{i=1}^{n}\left[\left(d_i^2 + \frac{2}{\varepsilon_1^2}\right)\left(T_i^2 + \frac{2(d_{[1]}^* + d_{[2]}^*)^2}{\varepsilon_2^2}\right) - d_i^2 T_i^2\right]\\ &= \frac{1}{4M^2}\sum_{i=1}^{n}\frac{2\varepsilon_2^2 T_i^2 + 2\varepsilon_1^2 d_i^2 (d_{[1]}^* + d_{[2]}^*)^2 + 4(d_{[1]}^* + d_{[2]}^*)^2}{\varepsilon_1^2\varepsilon_2^2}\\ &\leq \frac{1}{4M^2}\sum_{i=1}^{n}\frac{2\varepsilon_2^2 d_{max}^4 + 2\varepsilon_1^2 d_{max}^2 (d_{[1]}^* + d_{[2]}^*)^2 + 4(d_{[1]}^* + d_{[2]}^*)^2}{\varepsilon_1^2\varepsilon_2^2}\\ &= O\left(\frac{n d_{max}^2 (d_{[1]}^* + d_{[2]}^*)^2}{M^2}\right).\end{aligned} \tag{F.4}$$

Since the expression for S is similar to the expression for Q in the proof of Theorem 4.3, we obtain $\mathbb{V}(S) \leq O\left(\frac{n^3 d_{\max}^6}{M^4}\right)$.

To sum up, we have

$$\mathrm{MSE}\left(\hat{q}_{ru}\left(G\right)\right) = O\left(\frac{n^3 d_{\max}^6}{M^4}\right). \tag{F.5}$$

□

References

[Imola et al.(2022)] Jacob Imola, Takao Murakami, and Kamalika Chaudhuri. 2022. Differentially private triangle and 4-cycle counting in the shuffle model. In *Proceedings of the 2022 ACM SIGSAC Conference on Computer and Communications Security*. 1505–1519.

[Kozubowski and Nadarajah(2010)] Tomasz J Kozubowski and Saralees Nadarajah. 2010. Multitude of Laplace distributions. *Statistical Papers* 51 (2010), 127–148.

[Murphy(2012)] Kevin P Murphy. 2012. *Machine learning: a probabilistic perspective*. MIT press.