# Supplementary Material

Below are more details that are omitted in the main text due to the space limitation.

## Proof of Theorem 4.2

**Theorem 4.2** *The estimate $\hat{q}_{ru}(G)$ produced by* **Local$_{\mathbf{ru}}$** *is unbiased, i.e.,* $\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G)$.

Before beginning with the detailed proof, we need to introduce a lemma as below.

**Lemma 1** ([Kozubowski and Nadarajah(2010)]). *Given a random variable $X \sim Lap(x, b)$, then*

$$\mathbb{E}(X^r) = \sum_{k=0}^{r} \left\{ \frac{1}{2} \left[ 1 + (-1)^k \right] \frac{r!}{(r-k)!} b^k x^{r-k} \right\}. \tag{A.1}$$

Now, let us give the proof of Theorem 4.2.

*Proof.* It is clear to see that $\tilde{a}_{i,j}$ is in fact a Bernoulli random variable and $\tilde{d}_i \sim \text{Lap}\left(d_i, \frac{1}{\varepsilon_2}\right)$. Then, we have

$$\mathbb{E}(\tilde{a}_{i,j}) = a_{i,j}(1-p) + (1-a_{i,j})p, \tag{A.2a}$$

$$\mathbb{E}\left(\tilde{d}_i^2\right) = d_i^2 + \frac{2}{\varepsilon_2^2}. \quad \text{(by Lemma 1)} \tag{A.2b}$$

Next, we move to the proof of unbiasedness of $\hat{q}_{ru}(G)$. First, we obtain

$$\begin{aligned}
\mathbb{E}(X_1) &= \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{(E(\tilde{a}_{i,j}) - p) E\left(\tilde{d}_i\right) E\left(\tilde{d}_j\right)}{1-2p} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{[a_{i,j}(1-p) + (1-a_{i,j})p - p] d_i d_j}{1-2p} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{i-1} a_{i,j} d_i d_j \\
&= \sum_{(v_i, v_j) \in E} d_i d_j.
\end{aligned} \tag{A.3}$$

Similarly, it is easy to check

$$
\begin{aligned}
\mathbb{E}\left(Y_1\right) &= \mathbb{E}\left[\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right)^2 - \frac{5n+4}{\varepsilon_2^4}\right] \\
&= \mathbb{V}\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right) - \frac{5n+4}{\varepsilon_2^4} \\
&\quad + \left[\mathbb{E}\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right)\right]^2 \\
&= \frac{1}{4}\sum_{i=1}^{n}\left[\mathbb{E}\left(\tilde{d}_i^4\right) - \left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2\right] - \frac{5n+4}{\varepsilon_2^4} \\
&\quad + \left[\frac{1}{2}\sum_{i=1}^{n}\mathbb{E}\left(\tilde{d}_i^2\right) - \frac{n+2}{\varepsilon_2^2}\right]^2 \qquad (A.4) \\
&= \frac{1}{4}\sum_{i=1}^{n}\left(\frac{8}{\varepsilon_2^2}d_i^2 + \frac{20}{\varepsilon_2^4}\right) - \frac{5n+4}{\varepsilon_2^4} \\
&\quad + \left[\frac{1}{2}\sum_{i=1}^{n}\left(\frac{2}{\varepsilon_2^2} + d_i^2\right) - \frac{n+2}{\varepsilon_2^2}\right]^2 \\
&= \frac{2}{\varepsilon_2^2}\sum_{i=1}^{n}d_i^2 + \frac{5n}{\varepsilon_2^4} + \left[\frac{1}{2}\sum_{i=1}^{n}d_i^2 - \frac{2}{\varepsilon_2^2}\right]^2 - \frac{5n+4}{\varepsilon_2^4} \\
&= \left[\sum_{(v_i,v_j)\in E}\frac{1}{2}\left(d_i + d_j\right)\right]^2.
\end{aligned}
$$

To sum up, we verify that

$$
\mathbb{E}\left[\hat{q}_{ru}\left(G\right)\right] = \frac{\mathbb{E}\left(X_1\right)}{M} - \frac{\mathbb{E}\left(Y_1\right)}{M^2} = q_{ru}\left(G\right). \qquad (A.5)
$$

□

## Proof of Theorem 4.3

**Theorem 4.3** *When $\varepsilon_1$, $\varepsilon_2$ are constants, the estimate $\hat{q}_{ru}(G)$ produced by* **Local$_{\mathbf{ru}}$** *provides the following utility guarantee:*

$$\mathrm{MSE}\left(\frac{n^3 d_{\max}^2 + n^2 d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^6}{M^4}\right). \tag{B.1}$$

Let us bring two lemmas in order to succeed in verifying Theorem 4.3.

**Lemma 2** ([Imola et al.(2022)])**.** *Let $x_1, x_2$ be two random variables, then $\mathbb{V}(x_1 + x_2) \leq 4\max\{\mathbb{V}(x_1), \mathbb{V}(x_2)\}$.*

**Lemma 3.** *Given constants $c_i, x_i, b > 0$, and random variables $X_i \sim Lap(x_i, b)$, $i = 1, 2, \ldots, n$, then*

$$\mathrm{Cov}\left[\sum_{i=1}^{n} c_i X_i^2, \left(\sum_{i=1}^{n} c_i X_i^2\right)^2\right] > 0. \tag{B.2}$$

*Proof.*

$$\begin{aligned}
&\mathrm{Cov}\left[\sum_{i=1}^{n} c_i X_i^2, \left(\sum_{i=1}^{n} c_i X_i^2\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{n} c_i X_i^2\right) \cdot \left(\sum_{i=1}^{n} c_i X_i^2\right)^2\right] - \mathbb{E}\left(\sum_{i=1}^{n} c_i X_i^2\right) \mathbb{E}\left[\left(\sum_{i=1}^{n} c X_i^2\right)^2\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\mathbb{E}\left(c_i X_i^2 \cdot c_j X_j^2 \cdot c_k X_k^2\right) - \mathbb{E}\left(c_i X_i^2\right)\mathbb{E}\left(c_j X_j^2 \cdot c_k X_k^2\right)\right] \\
&\triangleq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} E_{i,j,k},
\end{aligned}$$

Below we analyze the sign of $E_{i,j,k}$ for different values of $i, j$ and $k$.

**Case 1:** $i = j = k$.

$$\begin{aligned}
E_{i,j,k} &= \mathbb{E}\left(c_i^3 X_i^6\right) - \mathbb{E}\left(c_i X_i^2\right)\mathbb{E}\left(c_i^2 X_i^4\right) \\
&= c_i^3 \mathbb{E}\left(X_i^6\right) - c_i^3 \mathbb{E}\left(X_i^2\right)\mathbb{E}\left(X_i^4\right) \\
&= c_i^3\left(x_i^6 + 30b^2 x_i^4 + 360b^4 x_i^2 + 720b^6\right) \\
&\quad - c_i^3\left(x_i^2 + 2b^2\right)\left(x_i^4 + 12b^2 x_i^2 + 24b^4\right) \quad \text{(by Lemma 1)} \\
&= c_i^3\left(16b^2 x_i^4 + 312b^4 x_i^2 + 672b^6\right) > 0.
\end{aligned}$$

**Case 2:** $i = j \neq k$ or $i = k \neq j$. [1]

$$\begin{aligned}
E_{i,j,k} &= \mathbb{E}\left(c_i^2 X_i^4 \cdot c_s X_s^2\right) - \mathbb{E}\left(c_i X_i^2\right)\mathbb{E}\left(c_i X_i^2 \cdot c_s X_s^2\right) \\
&= c_i^2 c_s \mathbb{E}\left(X_i^4\right)\mathbb{E}\left(X_s^2\right) - c_i^2 c_s \mathbb{E}\left(X_i^2\right)\mathbb{E}\left(X_i^2\right)\mathbb{E}\left(X_s^2\right) \\
&= c_i^2 c_s\left[\mathbb{E}\left(X_i^4\right) - \left(\mathbb{E}\left(X_i^2\right)\right)^2\right]\mathbb{E}\left(X_s^2\right) \\
&= c_i^2 c_s \mathbb{V}\left(X_i^2\right)\mathbb{E}\left(X_s^2\right) \geq 0.
\end{aligned}$$

**Case 3:** $j = k \neq i$.

$$E_{i,j,k} = \mathbb{E}\left(c_i X_i^2 \cdot c_j^2 X_j^4\right) - \mathbb{E}\left(c_i X_i^2\right)\mathbb{E}\left(c_j^2 X_j^4\right) = 0.$$

**Case 4:** $i \neq j \neq k$.

$$E_{i,j,k} = \mathbb{E}\left(c_i X_i^2 \cdot c_j X_j^2 \cdot c_k X_k^2\right) - \mathbb{E}\left(c_i X_i^2\right)\mathbb{E}\left(c_j X_j^2 \cdot c_k X_k^2\right) = 0.$$

Therefore,

$$\operatorname{Cov}\left[\sum_{i=1}^n c_i X_i^2, \left(\sum_{i=1}^n c_i X_i^2\right)^2\right] > 0.$$

$\square$

[1] Let $s = \begin{cases} j, & \text{if } i = k \neq j \\ k, & \text{if } i = j \neq k \end{cases}$.

From now on, we show the detailed proof of Theorem 4.3 as follows.

*Proof.* Due to $\mathbb{V}(a_{i,j}) = \mathbb{E}\left(a_{i,j}^2\right) - \left[\mathbb{E}(a_{i,j})\right]^2 = p(1-p)$, we can obtain

$$\begin{aligned}\mathbb{E}\left(\hat{a}_{i,j}^2\right) &= \mathbb{V}(\hat{a}_{i,j}) + \left[\mathbb{E}(\hat{a}_{i,j})\right]^2 \\ &= \frac{p(1-p)}{(1-2p)^2} + a_{i,j}^2 \\ &= \frac{p(1-p)}{(1-2p)^2} + a_{i,j}.\end{aligned} \tag{B.3}$$

From Theorem 4.2, it is clear to the eye that the estimate $\hat{q}_{ru}(G)$ produced by $\mathbf{Local_{ru}}$ is unbiased. By the bias-variance decomposition [Murphy(2012)], the mean squared error (MSE) of $\hat{q}_{ru}(G)$ is equal to its variance. Let $P = \frac{X_1}{M}$ and $Q = -\frac{Y_1}{M^2}$, then

$$\begin{aligned}\mathrm{MSE}\left(\hat{q}_{ru}(G)\right) = \mathbb{V}\left(\hat{q}_{ru}(G)\right) &= \mathbb{V}(P+Q) \\ &\leq 4\max\left\{\mathbb{V}(P), \mathbb{V}(Q)\right\}. \\ &\quad \text{(by Lemma 2)}\end{aligned} \tag{B.4}$$

Now, our task is to calculate $\mathbb{V}(P)$ and $\mathbb{V}(Q)$ separately.

For $\mathbb{V}(P)$, since $\mathbb{V}(P) = M^{-2}\mathbb{V}(X_1)$, we need to focus on calculation of $\mathbb{V}(X_1)$. For ease of presentation, we define $\tilde{B}_{i,j} = \frac{(\tilde{a}_{i,j}-p)\tilde{d}_i\tilde{d}_j}{1-2p}$, then

$$\begin{aligned}&\mathbb{V}(X_1) = \mathbb{V}\left(\sum_{i=2}^{n}\sum_{j=1}^{i-1}\tilde{B}_{i,j}\right) \\ &= \sum_{i=2}^{n}\mathbb{V}\left(\sum_{j=1}^{i-1}\tilde{B}_{i,j}\right) + \sum_{2\leq k,l\leq n, k\neq l}\mathrm{Cov}\left(\sum_{j=1}^{k-1}\tilde{B}_{k,j}, \sum_{j=1}^{l-1}\tilde{B}_{l,j}\right) \\ &= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\mathbb{V}\left(\tilde{B}_{i,j}\right) + \sum_{i=2}^{n}\sum_{1\leq k,l\leq i-1, k\neq l}\mathrm{Cov}\left(\tilde{B}_{i,k}, \tilde{B}_{i,l}\right) \\ &\quad + \sum_{2\leq k,l\leq n, k\neq l}\sum_{j=1}^{k-1}\sum_{t=1}^{l-1}\mathrm{Cov}\left(\tilde{B}_{k,j}, \tilde{B}_{l,t}\right) \\ &= P_1 + P_2 + P_3,\end{aligned} \tag{B.5}$$

where $P_1 = \sum_{i=2}^{n}\sum_{j=1}^{i-1}\mathbb{V}\left(\tilde{B}_{i,j}\right)$, $P_2 = \sum_{i=2}^{n}\sum_{1\le k,l\le i-1,k\neq l}\mathrm{Cov}\left(\tilde{B}_{i,k},\tilde{B}_{i,l}\right)$ and $P_3 = \sum_{2\le k,l\le n,k\neq l}\sum_{j=1}^{k-1}\sum_{t=1}^{l-1}\mathrm{Cov}\left(\tilde{B}_{k,j},\tilde{B}_{l,t}\right)$.

Next, we calculate $P_1$, $P_2$ and $P_3$ respectively, and obtain

$$\begin{aligned}
P_1 &= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\mathbb{E}\left[\left(\tilde{B}_{i,j}\right)^2\right]-\left[\mathbb{E}\left(\tilde{B}_{i,j}\right)\right]^2\right] \\
&= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\left(\frac{p\left(1-p\right)}{\left(1-2p\right)^2}+a_{i,j}\right)\left(\frac{2}{\varepsilon_2^2}+d_i^2\right)\left(\frac{2}{\varepsilon_2^2}+d_j^2\right)\right]-\sum_{i=2}^{n}\sum_{j=1}^{i-1}a_{i,j}d_i^2d_j^2 \\
&= \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\frac{p\left(1-p\right)}{\left(1-2p\right)^2}\left(\frac{2}{\varepsilon_2^2}+d_i^2\right)\left(\frac{2}{\varepsilon_2^2}+d_j^2\right)\right] \\
&\quad+\sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{2}{\varepsilon_2^2}a_{i,j}\left(\frac{2}{\varepsilon_2^2}+d_i^2+d_j^2\right) \\
&\le \sum_{i=2}^{n}\sum_{j=1}^{i-1}\left[\frac{p\left(1-p\right)}{\left(1-2p\right)^2}\left(\frac{2}{\varepsilon_2^2}+d_{\max}^2\right)\left(\frac{2}{\varepsilon_2^2}+d_{\max}^2\right)\right] \\
&\quad+\sum_{i=2}^{n}\sum_{j=1}^{i-1}\frac{2}{\varepsilon_2^2}\left(\frac{2}{\varepsilon_2^2}+2d_{\max}^2\right) \\
&= O\left(n^2d_{\max}^4\right),
\end{aligned} \tag{B.6a}$$

$$
\begin{aligned}
P_2 &= \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\neq l} \operatorname{Cov}\left(\tilde{B}_{i,k}, \tilde{B}_{i,l}\right) \\
&= \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\neq l} \left[\mathbb{E}\left(\tilde{B}_{i,k}\tilde{B}_{i,l}\right) - \mathbb{E}\left(\tilde{B}_{i,k}\right)\mathbb{E}\left(\tilde{B}_{i,l}\right)\right] \\
&= \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\neq l} \left[a_{i,k}a_{i,l}\left(\frac{2}{\varepsilon_2^2} + d_i^2\right) d_k d_l\right] - \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\neq l} a_{i,k}a_{i,l}d_i^2 d_k d_l \\
&= \frac{2}{\varepsilon_2^2} \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\neq l} a_{i,k}a_{i,l}d_k d_l \\
&\le \frac{2}{\varepsilon_2^2} \sum_{i=2}^{n} \sum_{1\le k,l\le i-1,k\neq l} d_{\max}^2 \\
&= O\left(n^3 d_{\max}^2\right),
\end{aligned}
\tag{B.6b}
$$

$$
\begin{aligned}
P_3 &= \sum_{2\le k,l\le n,k\neq l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} \operatorname{Cov}\left[\tilde{B}_{k,j}, \tilde{B}_{l.t}\right] \\
&= \sum_{2\le k,l\le n,k\neq l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} \mathbb{E}\left(\tilde{B}_{k,j}\tilde{B}_{l.t}\right) - \sum_{2\le k,l\le n,k\neq l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} \mathbb{E}\left(\tilde{B}_{k,j}\right)\mathbb{E}\left(\tilde{B}_{l.t}\right) \\
&= \sum_{2\le k,l\le n,k\neq l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} a_{k,j}a_{l,t}\mathbb{E}\left(\tilde{d}_k\tilde{d}_j\tilde{d}_l\tilde{d}_t\right) - \sum_{2\le k,l\le n,k\neq l} \sum_{j=1}^{k-1}\sum_{t=1}^{l-1} a_{k,j}d_k d_j a_{l,t} d_l d_t \\
&= 2 \sum_{2\le k<l\le n,k\neq l} \sum_{j=1}^{k-1} \left[\frac{2}{\varepsilon_2^2} a_{k,j}a_{l,t}d_k d_l + \frac{2}{\varepsilon_2^2} a_{k,j}a_{l,t}d_j d_l\right] \\
&= \frac{4}{\varepsilon_2^2} \sum_{2\le k<l\le n,k\neq l} \sum_{j=1}^{k-1} \left[a_{k,j}a_{l,t}d_k d_l + a_{k,j}a_{l,t}d_j d_l\right] \\
&\le \frac{4}{\varepsilon_2^2} \sum_{2\le k<l\le n,k\neq l} \sum_{j=1}^{k-1} 2d_{\max}^2 \\
&= O\left(n^3 d_{\max}^2\right).
\end{aligned}
\tag{B.6c}
$$

Thus,

$$\mathbb{V}\left(X_1\right) = O\left(n^3 d_{max}^3 + n^2 d_{max}^4\right). \tag{B.7}$$

This leads to the following expression

$$\mathbb{V}\left(P\right) = O\left(\frac{n^3 d_{max}^3 + n^2 d_{max}^4}{M^2}\right). \tag{B.8}$$

Below we move on to the calculation of $\mathbb{V}\left(Q\right)$. By Lemma 1, we first derive

$$\mathbb{E}\left(\tilde{d}_i^4\right) = d_i^4 + \frac{12}{\varepsilon_2^2} d_i^2 + \frac{24}{\varepsilon_2^4}, \tag{B.9a}$$

$$\mathbb{E}\left(\tilde{d}_i^8\right) = d_i^8 + \frac{56}{\varepsilon_2^2} d_i^6 + \frac{1680}{\varepsilon_2^4} d_i^4 + \frac{20160}{\varepsilon_2^6} d_i^2 + \frac{40320}{\varepsilon_2^8}. \tag{B.9b}$$

To make further progress, we write

$$\begin{aligned} \mathbb{V}\left(\tilde{d}_i^2\right) &= \mathbb{E}\left(\tilde{d}_i^4\right) - \left[\mathbb{E}\left(\tilde{d}_i^2\right)\right]^2 \\ &= \left(d_i^4 + \frac{12}{\varepsilon_2^2} d_i^2 + \frac{24}{\varepsilon_2^4}\right) - \left(d_i^2 + \frac{2}{\varepsilon_2^2}\right)^2 \\ &= \frac{8}{\varepsilon_2^2} d_i^2 + \frac{20}{\varepsilon_2^4}, \end{aligned} \tag{B.10}$$

and

$$\begin{aligned} \mathbb{V}\left(\tilde{d}_i^4\right) &= \mathbb{E}\left(\tilde{d}_i^8\right) - \left[\mathbb{E}\left(\tilde{d}_i^4\right)\right]^2 \\ &= \frac{32}{\varepsilon_2^2} d_i^6 + \frac{1488}{\varepsilon_2^4} d_i^4 + \frac{19584}{\varepsilon_2^6} d_i^2 + \frac{39744}{\varepsilon_2^8}. \end{aligned} \tag{B.11}$$

Armed with the consequences above, let us derive quantity $\mathbb{V}(Q)$ as below

$$\begin{aligned}
\mathbb{V}(Q) &= \mathbb{V}\left(-\frac{Y_1}{M^2}\right) \\
&= M^{-4}\mathbb{V}\left[\frac{5n+4}{\varepsilon_2^4} - \left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{n+2}{\varepsilon_2^2}\right)^2\right] \\
&= M^{-4}\mathbb{V}\left[\frac{n+2}{\varepsilon_2^2}\sum_{i=1}^{n}\tilde{d}_i^2 - \frac{1}{4}\left(\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right] \\
&= \frac{(n+2)^2}{M^4\varepsilon_2^4}\mathbb{V}\left(\sum_{i=1}^{n}\tilde{d}_i^2\right) + \frac{1}{16M^4}\mathbb{V}\left[\left(\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right] \\
&\quad - \frac{n+2}{2M^4\varepsilon_2^2}\mathrm{Cov}\left[\sum_{i=1}^{n}\tilde{d}_i^2, \left(\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right] \\
&\leq \frac{(n+2)^2}{M^4\varepsilon_2^4}\mathbb{V}\left(\sum_{i=1}^{n}\tilde{d}_i^2\right) + \frac{1}{16M^4}\mathbb{V}\left[\left(\sum_{i=1}^{n}\tilde{d}_i^2\right)^2\right] \\
&\quad \text{(by Lemma 3)} \\
&= \frac{(n+2)^2}{M^4\varepsilon_2^4}Q_1 + \frac{1}{16M^4}Q_2,
\end{aligned} \tag{B.12}$$

where $Q_1 = \mathbb{V}\left(\sum_{i=1}^n \tilde{d}_i^2\right)$ and $Q_2 = \mathbb{V}\left[\left(\sum_{i=1}^n \tilde{d}_i^2\right)^2\right]$.

Analogously, we need to calculate $Q_1$ and $Q_2$ separately. In essence, it is easy to derive

$$\begin{aligned}
Q_1 = \sum_{i=1}^{n}\mathbb{V}\left(\tilde{d}_i^2\right) &= \sum_{i=1}^{n}\left(\frac{8}{\varepsilon_2^2}d_i^2 + \frac{20}{\varepsilon_2^4}\right) \\
\leq O\left(nd_{\max}^2\right),
\end{aligned} \tag{B.13a}$$

$$
\begin{aligned}
Q_2 &= \mathbb{V}\left[\sum_{i=1}^{n} \tilde{d}_i^4 + \sum_{1\le j,k\le n, j\ne k} \tilde{d}_j^2 \tilde{d}_k^2\right] \\
&\le 4\max\left\{\mathbb{V}\left(\sum_{i=1}^{n} \tilde{d}_i^4\right), \mathbb{V}\left(\sum_{1\le j,k\le n, j\ne k} \tilde{d}_j^2 \tilde{d}_k^2\right)\right\}. \\
&\quad \text{(by Lemma 2)}
\end{aligned}
\tag{B.13b}
$$

In order to obtain an upper bound for $Q_2$, we first define two notations $Q_3 = \mathbb{V}\left(\sum_{i=1}^{n} \tilde{d}_i^4\right)$ and $Q_4 = \mathbb{V}\left(\sum_{1\le j,k\le n, j\ne k} \tilde{d}_j^2 \tilde{d}_k^2\right)$. Then, we start to calculate $Q_3$ and $Q_4$. Obviously, $Q_3$ is expressed as

$$
\begin{aligned}
Q_3 &= \mathbb{V}\left(\sum_{i=1}^{n} \tilde{d}_i^4\right) \\
&= \sum_{i=1}^{n} \mathbb{V}\left(\tilde{d}_i^4\right) \\
&= \sum_{i=1}^{n} \left(\frac{32}{\varepsilon_2^2} d_i^6 + \frac{1488}{\varepsilon_2^4} d_i^4 + \frac{19584}{\varepsilon_2^6} d_i^2 + \frac{39744}{\varepsilon_2^8}\right) \\
&\le O\left(n d_{\max}^6\right).
\end{aligned}
\tag{B.14}
$$

At the same time, we have

$$
\begin{aligned}
Q_4 &\leq \mathbb{E}\left[\left(\sum_{1\leq j,k\leq n, j\neq k} \tilde{d}_j^2 \tilde{d}_k^2\right)^2\right] \\
&\leq \mathbb{E}\left[\left((n-1)\sum_{i=1}^{n} \tilde{d}_i^4\right)^2\right] \\
&= (n-1)^2\, \mathbb{E}\left[\left(\sum_{i=1}^{n} \tilde{d}_i^4\right)^2\right] \\
&= (n-1)^2\left[\mathbb{V}\left(\sum_{i=1}^{n} \tilde{d}_i^4\right) + \left[\mathbb{E}\left(\sum_{i=1}^{n} \tilde{d}_i^4\right)\right]^2\right] \\
&= (n-1)^2\, \mathbb{V}\left(\sum_{i=1}^{n} \tilde{d}_i^4\right) + \left[(n-1)\sum_{i=1}^{n} \mathbb{E}\left(\tilde{d}_i^4\right)\right]^2 \\
&= (n-1)^2\, Q_3 + \left[(n-1)\sum_{i=1}^{n}\left(d_i^4 + \frac{12}{\varepsilon_2^2} d_i^2 + \frac{24}{\varepsilon_2^4}\right)\right]^2 \\
&\leq O\left(n^4 d_{\max}^8\right).
\end{aligned} \tag{B.15}
$$

From Eqs.(B.13b), (B.14) and (B.15), we obtain

$$
Q_2 \leq 4Q_4 \leq O\left(n^4 d_{\max}^8\right). \tag{B.16}
$$

Based on $Q_1 \leq O\left(n d_{\max}^2\right)$ in Eq.(B.13a) and consequence in Eq.(B.16), it is not hard to see

$$
\mathbb{V}(Q) = O\left(\frac{n^3 d_{\max}^6}{M^4}\right). \tag{B.17}
$$

Finally, from Eqs.(B.8) and (B.17) it follows that

$$
\mathrm{MSE} = O\left(\frac{n^3 d_{\max}^2 + n^2 d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^6}{M^4}\right). \tag{B.18}
$$

□

## Proof of Theorem 4.6

**Theorem 4.6** *The estimate* $\hat{q}_{ru}(G)$ *produced by* **Shuffle$_{\mathbf{ru}}$** *satisfies* $\mathbb{E}\left(\hat{q}_{ru}(G)\right) = q_{ru}(G)$.

*Proof.* First, we prove that $X_2$ is unbiased, i.e.,

$$\begin{aligned}
\mathbb{E}\left(X_2\right) &= \mathbb{E}\left(\sum_{i=2}^{n} \hat{r}_i\right) \\
&= \mathbb{E}\left[\sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} \frac{\left(\tilde{a}_{i,j} - p\right)\tilde{d}_j}{1-2p}\right] \\
&= \sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} \mathbb{E}\left(\frac{\tilde{a}_{i,j} - p}{1-2p}\right) \mathbb{E}\left(\tilde{d}_j\right) \\
&= \sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} a_{ij} d_j \\
&= \sum_{(v_i,v_j)\in E} d_i d_j.
\end{aligned} \tag{C.1}$$

We now consider $Y_2$. Since $Y_2$ has the similar form as $Y_1$, and the unbiasedness of $Y_1$ has been already proven in the proof of Theorem 4.2, it follows that $Y_2$ is also unbiased. Thus, we have

$$\mathbb{E}\left(Y_2\right) = \left[\sum_{(v_i,v_j)\in E} \frac{1}{2}\left(d_i + d_j\right)\right]^2. \tag{C.2}$$

Armed with the results above, we come to

$$\mathbb{E}\left(\hat{q}_{ru}(G)\right) = q_{ru}(G). \tag{C.3}$$

□

## Proof of Theorem 4.7

**Theorem 4.7** *When $\varepsilon$, $\delta$ are constants, $\alpha \in (0,1)$, $\varepsilon_0 = \log(n) + O(1)$, the estimate $\hat{q}_{ru}(G)$ produced by* $\mathbf{Shuffle_{ru}}$ *provides the following utility guarantee:*

$$\mathrm{MSE}\left(\hat{q}_{ru}(G)\right) = O\left(\frac{n^{1+\alpha} d_{\max}^4}{M^2} + \frac{n^3 d_{\max}^2}{(\log n)^2 M^2} + \frac{n^3 d_{\max}^6}{(\log n)^2 M^4}\right). \tag{D.1}$$

*Proof.* Let $U = \frac{X_2}{M}$ and $W = -\frac{Y_2}{M^2}$, then the MSE of $\hat{q}_{ru}(G)$ by $\mathbf{Shuffle_{ru}}$ can be written as follows

$$\begin{aligned} \mathrm{MSE}\left(\hat{q}_{ru}(G)\right) &= \mathbb{V}\left(\hat{q}_{ru}(G)\right) = \mathbb{V}(U+W) \\ &\leq 4\max\left\{\mathbb{V}(U), \mathbb{V}(W)\right\}. \quad \text{(by Lemma 2)} \end{aligned} \tag{D.2}$$

We now need to calculate $\mathbb{V}(U)$ and $\mathbb{V}(W)$ separately. Since the expression for $W$ is similar to that of $Q$ in the proof of Theorem 4.3, this leads to $\mathbb{V}(W) \leq O\left(\frac{n^3 d_{\max}^6}{(\log n)^2 M^4}\right)$. Next, we only need to compute $\mathbb{V}(U)$ to establish the upper bound of $\mathrm{MSE}\left(\hat{q}_{ru}(G)\right)$.

$$
\begin{aligned}
\mathbb{V}\left(U\right) &= M^{-2}\mathbb{V}\left(\sum_{i=2}^{n} d_i \sum_{j=1}^{i-1} a_{i,j}\tilde{d}_j\right) \\
&= M^{-2}\sum_{i=2}^{n} d_i^2 \mathbb{V}\left(\sum_{j=1}^{i-1} a_{i,j}\tilde{d}_j\right) \\
&\quad + 2M^{-2}\sum_{2\leq k<l\leq n} \mathrm{Cov}\left(\sum_{j=1}^{k-1} a_{k,j}d_k\tilde{d}_j, \sum_{h=1}^{l-1} a_{l,h}d_l\tilde{d}_h\right) \\
&= M^{-2}\sum_{i=2}^{n} d_i^2 \sum_{j=1}^{i-1} a_{i,j}^2 \mathbb{V}\left(\tilde{d}_j\right) \\
&\quad + 2M^{-2}\sum_{2\leq k<l\leq n} \sum_{j=1}^{k-1}\sum_{h=1}^{l-1} \mathrm{Cov}\left(a_{k,j}d_k\tilde{d}_j, a_{l,h}d_l\tilde{d}_h\right) \\
&= \frac{2}{M^2\varepsilon^2}\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_{ij}d_i^2 \qquad\qquad (D.3) \\
&\quad + 2M^{-2}\sum_{2\leq k<l\leq n} \sum_{j=1}^{k-1} a_{kj}a_{lj}d_kd_l\mathbb{V}\left(\tilde{d}_j\right) \\
&= \frac{2}{M^2\varepsilon^2}\sum_{i=2}^{n}\sum_{j=1}^{i-1} a_{i,j}d_i^2 + \frac{4}{M^2\varepsilon^2}\sum_{2\leq k<l\leq n} \sum_{j=1}^{k-1} a_{k,j}a_{l,j}d_kd_l \\
&\leq \frac{n(n-1)d_{max}^2}{M^2\varepsilon^2} + \frac{4}{M^2\varepsilon^2}\sum_{2\leq k<l\leq n}(k-1)d_{max}^2 \\
&= \frac{n(n-1)d_{max}^2}{M^2\varepsilon^2} + \frac{2n(n-1)(n-2)d_{max}^2}{3M^2\varepsilon^2} \\
&= O\left(\frac{n^{1+\alpha}d_{\max}^4}{M^2} + \frac{n^3d_{\max}^2}{\left(\log n\right)^2 M^2}\right).
\end{aligned}
$$

Therefore, we gain

$$
\mathrm{MSE}\left(\hat{q}_{ru}\left(G\right)\right) = O\left(\frac{n^{1+\alpha}d_{\max}^4}{M^2} + \frac{n^3d_{\max}^2}{\left(\log n\right)^2 M^2} + \frac{n^3d_{\max}^6}{\left(\log n\right)^2 M^4}\right). \qquad (D.4)
$$

□

## Proof of Theorem 4.10

**Theorem 4.10** *The estimate* $\hat{q}_{ru}(G)$ *produced by* $\mathbf{Decentral_{ru}}$ *satisfies* $\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G)$.

*Proof.* Since $\tilde{T}_i \sim \text{Lap}\left(\frac{\hat{d}_{max}}{\varepsilon_2}\right)$, by Lemma 1, we have

$$\mathbb{E}\left(\tilde{T}_i\right) = T_i + \frac{2\hat{d}_{max}^2}{\varepsilon_2^2}. \tag{E.1}$$

Then, we can easily obtain

$$\begin{aligned}\mathbb{E}(X_2) &= \mathbb{E}\left(\frac{1}{2}\sum_{i=1}^{n}\tilde{d}_i\tilde{T}_i\right) \\ &= \frac{1}{2}\sum_{i=1}^{n} d_i T_i \\ &= \sum_{(v_i,v_j)\in E} d_i d_j,\end{aligned} \tag{E.2}$$

Following a similar derivation as in Eq.(A.4), we have

$$\mathbb{E}(Y_2) = \left[\sum_{(v_i,v_j)\in E}\frac{1}{2}(d_i + d_j)\right]^2. \tag{E.3}$$

Finally, we show that

$$\mathbb{E}(\hat{q}_{ru}(G)) = q_{ru}(G). \tag{E.4}$$

□

## Proof of Theorem 4.11

**Theorem 4.11** *When $\varepsilon_1$, $\varepsilon_2$ are constants, the estimate $\hat{q}_{ru}(G)$ produced by* **Decentral$_{\mathbf{ru}}$** *provides the following utility guarantee:*

$$\mathrm{MSE}\left(\hat{q}_{ru}(G)\right) = O\left(\frac{n^3 d_{\max}^6}{M^4}\right). \tag{F.1}$$

*Proof.* Since $\tilde{T}_i \sim \mathrm{Lap}\left(T_i, \frac{\hat{d}_{\max}}{\varepsilon_2}\right)$, then by Lemma 1, we can get

$$\mathbb{E}\left(\tilde{T}_i^2\right) = T_i^2 + \frac{2\hat{d}_{\max}^2}{\varepsilon_2^2}, \tag{F.2}$$

Below we calculate the MSE of $q_{ru}(G)$ produced by **Decentral$_{\mathbf{ru}}$**. Let $H = \frac{X_3}{M}$ and $S = -\frac{Y_3}{M^2}$, then

$$\begin{aligned}\mathrm{MSE}\left(\hat{q}_{ru}(G)\right) &= \mathbb{V}\left(\hat{q}_{ru}(G)\right) = \mathbb{V}(H+S) \\ &\le 4\max\left\{\mathbb{V}(H), \mathbb{V}(S)\right\}. \quad \text{(by Lemma 2)}\end{aligned} \tag{F.3}$$

Next, we calculate $\mathbb{V}(H)$ and $\mathbb{V}(S)$ respectively.

$$\begin{aligned}\mathbb{V}(H) = \mathbb{V}\left(\frac{X_3}{M}\right) &= M^{-2}\mathbb{V}(X_3) \\ &= \frac{1}{2M^2}\sum_{i=1}^{n}\mathbb{V}\left(\hat{d}_i\hat{T}_i\right) \\ &= \frac{1}{2M^2}\sum_{i=1}^{n}\left[\mathbb{E}\left(\hat{d}_i^2\hat{T}_i^2\right) - \mathbb{E}\left(\hat{d}_i\hat{T}_i\right)^2\right] \\ &= \frac{1}{2M^2}\sum_{i=1}^{n}\left[\mathbb{E}\left(\hat{d}_i^2\right)\mathbb{E}\left(\hat{T}_i^2\right) - \mathbb{E}\left(\hat{d}_i\right)^2\mathbb{E}\left(\hat{T}_i\right)^2\right] \\ &= \frac{1}{2M^2}\sum_{i=1}^{n}\left[\left(d_i^2 + \frac{2}{\varepsilon_1^2}\right)\left(T_i^2 + \frac{2\hat{d}_{max}^2}{\varepsilon_2^2}\right) - d_i^2T_i^2\right] \\ &= \frac{1}{2M^2}\sum_{i=1}^{n}\frac{2\varepsilon_2^2T_i^2 + 2\varepsilon_1^2d_i^2\hat{d}_{max}^2 + 4\hat{d}_{max}^2}{\varepsilon_1^2\varepsilon_2^2} \\ &\le \frac{1}{2M^2}\sum_{i=1}^{n}\frac{2\varepsilon_2^2\hat{d}_{max}^4 + 2\varepsilon_1^2\hat{d}_{max}^4 + 4\hat{d}_{max}^2}{\varepsilon_1^2\varepsilon_2^2} \\ &= O\left(\frac{n\hat{d}_{max}^4}{M^2}\right).\end{aligned} \tag{F.4}$$

Since the expression for $S$ is similar to the expression for $Q$ in the proof of Theorem 4.3, we obtain $\mathbb{V}(S) \leq O\left(\frac{n^3 d_{\max}^6}{M^4}\right)$.

To sum up, we have

$$\mathrm{MSE}\left(\hat{q}_{ru}\left(G\right)\right) = O\left(\frac{n^3 d_{\max}^6}{M^4}\right). \tag{F.5}$$

□

# References


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