CS 714

HOMEWORK 3

Noah Cohen Kalafut
Computer Science Doctoral Student
University of Wisconsin-Madison
nkalafut@wisc.edu
https://github.com/Oafish1/CSC-714

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1 Problem A

1.1 Part a

1.1.1 Part i

Let's take the derivative of $u(x(t), t)$ with respect to t. We get

$$\frac{d}{dt}u(x(t), t) = \partial_t u(x(t), t) + \partial_x u(x(t), t)x'(t)$$

Since u is constant in the characteristic curve, we know $\frac{d}{dt}u(x(t), t) = 0$. Thus,

$$\partial_t u(x(t), t) + \partial_x u(x(t), t)x'(t) = 0$$

1.1.2 Part ii

Using the advection equation and our result from Part i, we have

$$\partial_t u(x(t), t) + a\partial_x u(x(t), t) = 0$$
$$\partial_t u(x(t), t) + \partial_x u(x(t), t)x'(t) = 0$$

Then, subtracting the two equations

$$x'(t) - a = 0$$
$$x'(t) = a$$

1.1.3 Part iii

We now have

$$\frac{d}{dt}u(x(t), t) = 0$$
$$u(x, 0) = u_0(x)$$

So, $u(x(t), t)$ does not change with t and

$$u(x(t), t) = u_0(x_0)$$

1.1.4 Part iv

We now know the solution to the function along the characteristic curve for any x_0. So, if we can solve for x_0, we can solve for $u(x, t) = u_0(x_0)$.

We know that, for the advection equation, $a = x'(t)$ is a constant. So, $x_0 = x - at$ and

$$u(x, t) = u_0(x - at)$$

1.2 Part b

1.2.1 Part i

In this case, $u_0(x)$ should be absolutely or square integrable, L^1 and/or L^2.

As is common with functions involving variable powers of e, we have regularity $\hat{u}_0(x) \in \mathcal{C}^\infty$.

As the magnitude of ξ approaches infinity, if $u_0(x)$ is absolutely integrable, $\hat{u}(\xi) \to 0$ by the *Riemann-Lebesgue Lemma*. If $u_0(x) \in L^2$, $\hat{u}_0(\xi) \to 0$ as $\xi \to \infty$ as well.

1.2.2 Part ii

We have

$$v(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{v}(\xi) d\xi$$

We can take the derivative

$$\begin{aligned} v'(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{d}{dx} e^{i\xi x} \right) \hat{v}(\xi) + e^{i\xi x} \left(\frac{d}{dx} \hat{v}(\xi) \right) d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} i\xi e^{i\xi x} \hat{v}(\xi) d\xi \end{aligned}$$

Thus,

$$v'(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} i\xi \hat{v}(\xi) d\xi$$

1.2.3 Part iii

We can say

$$\begin{aligned} \mathcal{F}[v(x+h)](\xi) &= \int_{\mathbb{R}} e^{-i\xi(x+h)} v(x+h) dx \\ &= e^{-i\xi h} \int_{\mathbb{R}} e^{-i\xi x} v(x+h) dx \\ &= e^{-i\xi h} \hat{v}(\xi) \end{aligned}$$

The last step above is due to the integration. Since the function is evaluated for $\mathbb{R}$, translation does not affect the result.

Thus,

$$\mathcal{F}[v(\cdot + h)](\xi) = e^{-i\xi h} \hat{v}(\xi)$$

1.2.4 Part iv

Define

$$\hat{u}(\xi, t) = \int_{\mathbb{R}} e^{-i\xi x} u(x,t) dx$$

Then,

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \hat{u}(\xi, t) dx$$

Using the same method as in Part ii, we can obtain

$$\partial_x u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} i\xi e^{i\xi x} \hat{u}(\xi, t) d\xi$$

Note $\partial_t u(x,t) = -a \partial_x u(x,t)$. Then,

$$\partial_t u(x,t) = \frac{-1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} a i\xi \hat{u}(\xi, t) d\xi$$

Inverting this, by our definition,

$$\mathcal{F}[\partial_t u(x,t)](\xi,t) = -ai\xi\hat{u}(\xi,t)$$

Notice $\mathcal{F}[\partial_x v(x)](\xi) = \partial_x \hat{v}(x)(\xi)$.
Thus,

$$\partial_t \hat{u}(\xi,t) = -ai\xi\hat{u}(\xi,t)$$

1.2.5 Part v

Since a is constant, we know $x = x_0 + at$. Then, by Part iii, we can say

$$\mathcal{F}[u(x_0+at)](\xi) = e^{-ai\xi t}\mathcal{F}[u(x_0)](\xi)$$
$$\hat{u}(\xi,t) = e^{-ai\xi t}\hat{u}_0(\xi)$$

1.2.6 Part vi

If $u_0(\xi) \in L^2(\mathbb{R})$, we can determine $\hat{u}(\xi,t) = e^{-ai\xi t}\hat{u}_0(\xi)$ for any fixed t. Notice $\hat{u}(\xi,t) \in L^2(\mathbb{R})$.
Then, by the definition of Lebesgue spaces,

$$\int_{\mathbb{R}} |\hat{u}(\xi,t)|^2 d\xi < \infty$$

By Plancherel's theorem, we then know

$$\int_{\mathbb{R}} |u(x,t)|^2 dx < \infty$$

1.2.7 Part vii

We have

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(x-at)}\hat{u}_0(\xi)d\xi$$

Which we can simplify using our result in Part v

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(x-at)}\hat{u}_0(\xi)d\xi &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}e^{-i\xi at}\hat{u}_0(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\hat{u}(\xi,t)d\xi \\ &= u(x,t)\end{aligned}$$

Thus,

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(x-at)}\hat{u}_0(\xi)d\xi$$

1.2.8 Part iix

We can see, by our definition of the Fourier Transform,

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi(x-at)}\hat{u}_0(\xi)d\xi = u_0(x-at)$$

Then, from our result in Part vii,

$$u(x,t) = u_0(x-at)$$

2 Problem B

2.1 Part a

We are given

$$u_t = \pm u_{xxxx}$$

Which provides

$$\hat{u}_t(\xi, t) = \pm i^4 \xi^4 \hat{u}(\xi, t) = \pm \xi^4 \hat{u}(\xi, t)$$

Notice that, if we choose $+$, the magnitude for frequency ξ will continue to grow over time for all $|\xi| > 1$. So, we need to choose $-$ for stable solutions. For an example, look below.

2.1.1 Example

Consider the Explicit Euler

$$\hat{u}(\xi, t + \Delta t) = \hat{u}(\xi, t) \pm h\xi^4 \hat{u}(\xi, t)$$

Then,

$$\hat{u}(\xi, t + \Delta t) = (1 \pm \Delta t \xi^4)\hat{u}(\xi, t)$$

Therefore, we want $|1 \pm \Delta t \xi^4| \leq 1$ for stability. We know $\Delta t \xi^4 \geq 0$ for all ξ. Thus, we want to choose $u_t = -u_{xxxx}$ as our PDE.

Keep in mind that it is impossible to have $|1 \pm \Delta t \xi^4| \leq 1$ for all ξ as ξ is unbounded by nature. However, also keep in mind that u_{xxxx} is not discretized here and that higher frequencies are often ignored for high Δx. Then, when Δx shrinks, so should Δt so that $|1 \pm \Delta t \xi^4| \leq 1$ for relevant ξ. For an illustration of this, take a look at Part b and Part c which document the relationship in more detail.

2.2 Part b

With the Explicit Euler Method, we acquire

$$u^{t+1} = u^t - (\Delta t)u_{xxxx}$$

For some finite difference discretization, we will obtain

$$U^{t+1} = U^t - \left(\frac{\Delta t}{\Delta x^4}\right) AU^t$$

Which becomes

$$U^{t+1} = \left(I - \left(\frac{\Delta t}{\Delta x^4}\right) A\right) U^t$$
$$U^t = B^t U^0$$

Suppose B has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, v_2, \ldots, v_n$ that form a basis.

Then, given $E^t = B^t E^0$ from above,

$$E^0 = \sum_i^n c_i v_i$$
$$E^t \leq \rho(B)^t E^0$$

We then want $\rho(B) \leq 1$. Note that the eigenvalues of B are the eigenvalues of A scaled by $-\left(\frac{\Delta t}{\Delta x^4}\right)$ plus 1. Say A has minimal and maximal eigenvalues γ, Γ. Then

$$\rho(B) \leq 1$$
$$|1 - \frac{\Delta t}{\Delta x^4}\gamma|, |1 - \frac{\Delta t}{\Delta x^4}\Gamma| \leq 1$$
$$0 \leq \frac{\Delta t}{\Delta x^4}\gamma, \frac{\Delta t}{\Delta x^4}\Gamma \leq 2$$

Which implies

$$\frac{\Delta t}{\Delta x^4} \leq \min(\frac{2}{\gamma}, \frac{2}{\Gamma}) \tag{1}$$

Note, for either $\gamma, \Gamma = 0$, the condition on γ, Γ is met. Thus, 1 implies stability. Notice that this is actually equivalent to the CFL condition.

2.3 Part c

Given

$$\frac{\Delta t}{\Delta x^4} \leq \min(\frac{2}{\gamma}, \frac{2}{\Gamma})$$

We know, in general, if we scale Δx by q, then we need to scale Δt by q^4; this is impractical.

To remedy this, we can use a method with a larger stability region to make the stability condition more lenient. Examples include the Trapezoidal Rule and Backward Euler.

3 Problem C

3.1 Part a

For the 2D wave equation discretized with a second-order accurate scheme in time alone, we have

$$\begin{gathered}
u_{tt} = \Delta u \\
u^{t+1} - 2u^t + u^{t-1} = h_t^2 \Delta u^t \\
(u + u_t h_t + u_{tt}\frac{h_t^2}{2} + \dots) - 2u + (u - u_t h_t + u_{tt}\frac{h_t^2}{2} + \dots) = h_t^2 \Delta u \\
u_{tt} + u_{tttt}\frac{h_t^2}{12} + O(h_t^4) = \Delta u
\end{gathered}$$

For u sufficiently smooth we can say (*Piazza @59*)

$$\begin{aligned}
u_{tttt} &= \partial_{tt} u_{tt} \\
&= \partial_{tt}(u_{xx} + u_{yy}) \\
&= (\partial_{xx} + \partial_{yy}) u_{tt} \\
&= (\partial_{xx} + \partial_{yy}) \Delta u \\
&= \Delta^2 u
\end{aligned}$$

Then,

$$u_{tt} + \Delta^2 u \frac{h_t^2}{12} + O(h_t^4) = \Delta u$$

So,

$$u_{tt} = \Delta u + \Delta^2 u \frac{h_t^2}{12}$$

is our modified equation. By adding $\Delta^2 u \frac{h_t^2}{12}$ to the right-hand side of the equation, we are making our schemes fourth-order accurate. While we will be using a Chebyshev grid for the actual implementation, the general process that shows the accuracy of the scheme can be seen in the below example.

3.1.1 Example

Using a second-order time discretization on our new PDE, we will have

$$\begin{gathered}
u_{tt} + \Delta^2 u \frac{h_t^2}{12} + O(h_t^4) = \Delta u + \Delta^2 u \frac{h_t^2}{12} \\
u_{tt} + O(h_t^4) = \Delta u
\end{gathered}$$

Then, for some approximation of Δu and $\Delta^2 u$, we can come up with the method

$$\hat{u}_{tt} = \hat{\Delta}u + \frac{h_t^2}{12}\hat{\Delta}^2 u$$
$$h_t^2\hat{u}_{tt} = h_t^2\hat{\Delta}u + \frac{h_t^4}{12}\hat{\Delta}^2 u$$
$$h_t^2\hat{u}_{tt} = h_t^2\hat{\Delta}u + \frac{h_t^4}{12}\hat{\Delta}^2 u$$

To use this, suppose $h_t^2\hat{u}_{tt} = u^{t+1} - 2u^t + u^{t-1}$. Solving for u^{t+1} provides the method that will be used in Part c,

$$u^{t+1} = 2u^t - u^{t-1} + h_t^2\hat{\Delta}u + \frac{h_t^4}{12}\hat{\Delta}^2 u$$

3.2 Part b

We can create a fourth-order forward difference approximation of u^t for $t = 0$ and solve for u^1.

Let's start by solving a five-point stencil.

$$\frac{\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4}{\Delta t}$$

Looking at the Taylor Series we get

$$u_t = u_0 + u_0'(t\Delta t) + u_0''\frac{(t\Delta t)^2}{2} + u_0'''\frac{(t\Delta t)^3}{6} + u_0''''\frac{(t\Delta t)^4}{24} + O(\Delta t^5)$$

We want

$$u_0'(\Delta t) + O(\Delta t^5)$$

So, we then need to solve

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$
$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 1$$
$$\alpha_1 + 4\alpha_2 + 9\alpha_3 + 16\alpha_4 = 0$$
$$\alpha_1 + 8\alpha_2 + 27\alpha_3 + 64\alpha_4 = 0$$
$$\alpha_1 + 16\alpha_2 + 81\alpha_3 + 256\alpha_4 = 0$$

Solving this provides

$$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 = -\frac{25}{12}, 4, -3, \frac{4}{3}, -\frac{1}{4}$$

We can then solve for $t = 1$ by solving for u_1 in $\frac{\partial}{\partial t}u(x, y, 0) = f(x)f(y)$ using our approximation.

$$u_1 = \frac{1}{4}\left(\frac{25}{12}u_0 + 3u_2 - \frac{4}{3}u_3 + \frac{1}{4}u_4 + f(x)f(y)\Delta t\right)$$

Implementing this will involve initializing a 3D matrix with all zeros, applying the above, then applying our scheme from Part a. This will be repeated until convergence.

3.3 Part c

In solving our modified PDE, using a Chebyshev grid, we can use a differentiation matrix [1] or spectral differentiation [2]. In this case, the results are indistinguishable. The methods used for computing second derivatives are taken from the cited chapters.

The log-log plot of N vs Max Norm of the Error can be seen in figure 1. This does appear to be at least fourth-order accurate, if not better. Keep in mind that, in this implementation, $h_t \propto h^2$. Some samples can also be seen in figure 2.

Figure 1: N vs Max Norm of the Error. Note that the lesser N are *very* small (*Spectral_Heat.m*).

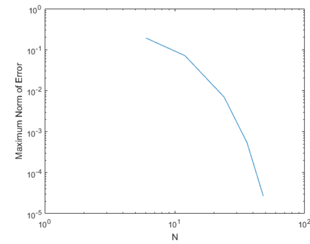
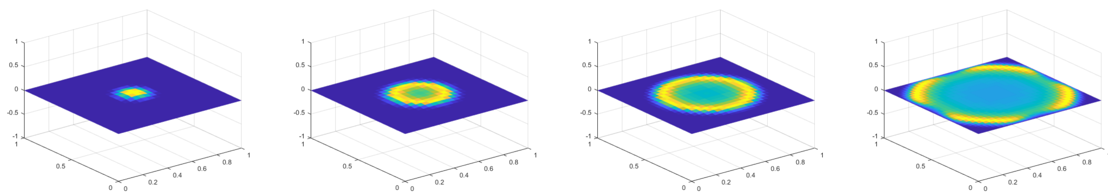


Figure 2: Samples for $N = 36$ at $t = .16, .39, .63, 1.18$



3.4 Part d

From the general equation

$$y''(t) = \gamma y$$

We were able to determine in HW2 that

$$\pi(\xi, z) = \rho(\xi) - \lambda(\Delta t)^2\sigma(\xi) = \xi^2 - (2+z)\xi + 1 \text{ for } z = \lambda(\Delta t)^2$$

Solving for ξ, we determined

$$\xi = \frac{2 + z \pm \sqrt{z^2 + 4z}}{2}$$

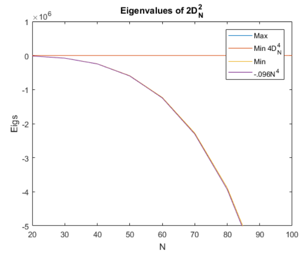
And $|\xi| \leq 1$, $|\xi| < 1$ for non-simple roots, implies absolute stability. This lends the region shown in figure 3.

Figure 3: Region of absolute stability on the complex plane (*Stability.m*)

So, $\lambda(\Delta t)^2 \in (-4, 0) \in \mathbb{R}$. Notice the open interval. This is because the boundary values of z produce identical, non-simple roots.

The eigenvalues of our second differentiation matrix are strictly negative and real with the largest (in magnitude) being proportional to N^4. In particular, $-.048N^4$ [3]. However, note that we are actually using this matrix twice in calculating the laplacian, so we have $\approx -.096N^4$ as our maximal eigenvalue. Keep in mind that, in calculating the Biharmonic/Square Laplacian, we are also using $\hat{D}_N^4$ which has negligible eigenvalues in comparison, even though it is used four times per calculation. All of this can be seen in figure 4.

Figure 4: N vs Minimal and Maximal Eigenvalues (*Cheb_Eig.m*). Max and $4D_N^4$ overlap on the plot.



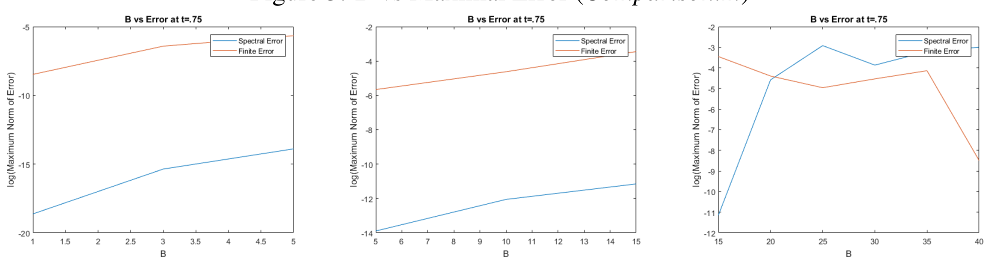
Combining this with our previous calculation provides our CFL condition.

$$-.096N^4(\Delta t)^2 \underset{\sim}{\geq} -4$$
$$\frac{\Delta t}{(\Delta x, y)^2} \underset{\sim}{\leq} 6.45$$

3.5 Part e

The comparison was performed with $N = 32$ and $N_{\text{fine}} = 64$. Each method was compared to its respective fine counterpart for every given B. If this was not done, the program might favor one algorithm's accuracy over the other, even if interpolation is used. The results can be seen in figure 5

Figure 5: B vs Maximal Error (*Comparison.m*)



Spectral accuracy was noticeably affected by the change in B. This makes sense, since, for high frequency $\sin$ waves, the lack of coverage in the middle of the graph when using a chebyshev grid could be a problem – especially considering the max norm. FD accuracy became unstable for larger B, perhaps this is a result of N_{fine} needing to be larger than for spectral accuracy.

The graph suggests that, for accuracy of 10^{-3}, spectral methods require roughly $\frac{N}{25} \approx 1.28$ average points per wavelength while FD methods require $\frac{N}{15} \approx 2.14$.

4 Problem D

To see why this is true, consider

$$|\hat{f}(\xi)| = ||f||_{TV}|\xi|^{-1}$$

for some frequency ξ. Then, the magnitude of ξ is equivalent to the total variance of the function divided by the frequency ξ. So, ξ accounts for all the variance in f.

Now, consider

$$|\hat{f}(\xi)| \geq ||f||_{TV}|\xi|^{-1}$$

This then implies that the variance in ξ exceeds that in f. This is impossible since the wave with frequency ξ is a component of f by the definition of the Fourier Transform. So,

$$|\hat{f}(\xi)| \leq ||f||_{TV}|\xi|^{-1}$$

References

[1] *Chebyshev Differentiation Matrices*, chapter 6, pages 51–59.
[2] *Chebyshev Series and the FFT*, chapter 8, pages 75–86.
[3] *Time-Stepping and Stability Regions*, chapter 10, pages 101–114.