CS 715

HOMEWORK 2

Noah Cohen Kalafut
Computer Science Doctoral Student
University of Wisconsin-Madison
`nkalafut@wisc.edu`
`https://github.com/Oafish1/CSC-715`

April 1, 2021

1 Problem A

We are given

$$\begin{cases} \partial_{tt} u = \triangle u + f & \text{in } \Omega \times J \\ u(\mathbf{x}, t) = 0 & \text{on } \partial\Omega \\ u(\mathbf{x}, t) = e^{-\frac{|x|^2}{.05}} & x \in \Omega \\ \partial_t u(\mathbf{x}, t) = 0 & x \in \Omega \end{cases} \tag{1}$$

Time is partitioned in a regular grid $\{t_n\}_{n=0}^{N_T}$, $t_n = n \cdot \Delta t$, which is important for our $\partial_{tt} u$ discretization.

1.1 Part a

Because our grid is regularly spaced in time, we can discretize the LHS of our ODE from 1 to obtain

$$\frac{u_{n-1} - 2u_n + u_{n+1}}{(\Delta t)^2} = \triangle u + f$$

This provides the natural update rule

$$u_{n+1} = 2u_n - u_{n-1} + (\Delta t)^2 (\triangle u + f)$$

1.2 Part b

From our result in Part a, we obtain the following by multiplying by a test vector v that vanishes at $\partial\Omega$ and integrating

$$\int_\Omega u_{n+1} v \, dx - \int_\Omega 2u_n v \, dx + \int_\Omega u_{n-1} v \, dx - (\Delta t)^2 \left(\int_\Omega \triangle u v \, dx + \int_\Omega f v \, dx \right) = 0$$

Through integration by parts, we then obtain our variational formulation

$$\int_\Omega u_{n+1} v \, dx - \int_\Omega 2u_n v \, dx + \int_\Omega u_{n-1} v \, dx - (\Delta t)^2 \left(\triangle u \nabla v \Big|_{\partial\Omega} - \int_\Omega \nabla u \nabla v \, dx + \int_\Omega f v \, dx \right) = 0$$

$$\int_\Omega u_{n+1} v \, dx - \int_\Omega 2u_n v \, dx + \int_\Omega u_{n-1} v \, dx + (\Delta t)^2 \left(\int_\Omega \nabla u \nabla v \, dx - \int_\Omega f v \, dx \right) = 0$$

We can use the space $v \in V_0$, which is defined for ∇v and v and disappears at the boundaries. This is more formally defined as $V_0 = \{v : ||\nabla v|| + ||v|| < \infty, v|_{\partial\Omega} = 0\}$. This provides us with the final formulation

$$\int_\Omega u_{n+1} v \, dx - \int_\Omega 2u_n v \, dx + \int_\Omega u_{n-1} v \, dx + (\Delta t)^2 \left(\int_\Omega \nabla u \nabla v \, dx - \int_\Omega f v \, dx \right) = 0 \qquad \forall v \in V_0, t \in J \tag{2}$$

Notice that the first term is a mass matrix.

1.3 Part c

We have the variational formulation

```
var_form = u*v*dx - 2*u_n*v*dx + u_n_1*v*dx + \
    dt**2*(dot(grad(u),grad(v))*dx - f*v*dx)
```

1.4 Part d

The solution at $t = 5$ using $\Delta t = .02$ from our variational formulation can be seen in figure 1. It appears that the wave propagation has created negative-pressure zones near the originating area.

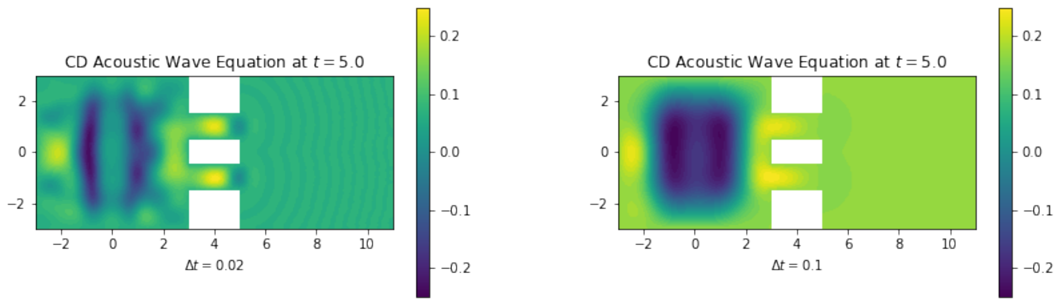

Figure 1: Our solution to system 1 using variational formulation 2 at $t = 5$ using varying Δt

1.5 Part e

As we increase Δt, our approximation appears to blot together, becoming less accurate. Recall that we are using a second-order accurate derivative approximation. So, multiplying our previous Δt by 5, for example, leads to a $25\times$ magnification in (the most significant lower-order) error. This can be seen in figure 1. This error tends to compound over time, as can be seen by the differing values in the right-most areas of our solutions. Also notice that values appear to be, generally, much higher with our larger step. A scheme such as Crank-Nicolson might be more stable due to it being an energy conserving scheme, keeping constant total energy/magnitude.

1.6 Part f

To implement Crank-Nicolson, we can split our ODE in two.

$$\partial_{tt}u = \triangle u + f \sim \begin{cases} \partial_t u = w \\ \partial_t w = \triangle u + f \end{cases}$$

We can then apply our scheme

$$\frac{u_{n+1} - u_n}{\Delta t} = \frac{w_{n+1} + w_n}{2}$$
$$\frac{w_{n+1} - w_n}{\Delta t} = \frac{\triangle u_{n+1} + \triangle u_n}{2} + \frac{f_{n+1} + f_n}{2}$$

Multiplying by v and integrating, we get

$$\int_\Omega v_1(u_{n+1} - u_n)\, dx = \frac{\Delta t}{2} \int_\Omega v_1(w_{n+1} + w_n)\, dx$$
$$\int_\Omega v_2(w_{n+1} - w_n)\, dx = \frac{\Delta t}{2} \left(\int_\Omega v_2(\triangle u_{n+1} + \triangle u_n)\, dx + \int_\Omega v_2(f_{n+1} + f_n)\, dx \right)$$

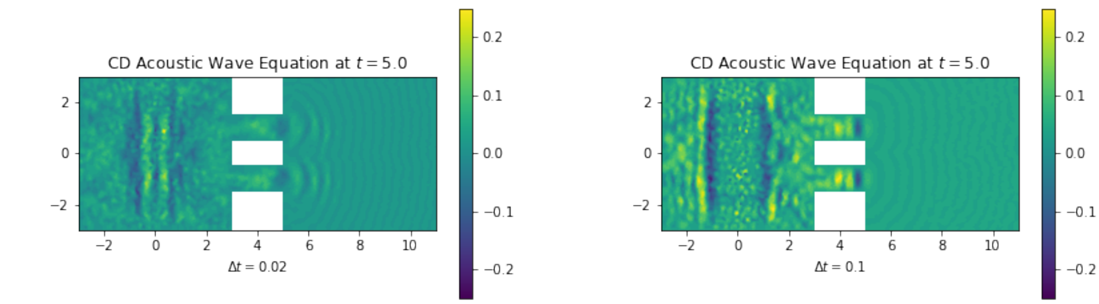

Figure 2: Our solution to system 1 using variational formulation 3 (Crank-Nicolson) at $t = 5$ using varying Δt

Finally, using the fact that v disappears on $\partial\Omega$ and integrating by parts, we achieve the variational formulations

$$\int_\Omega v_1(u_{n+1} - u_n)\,dx = \frac{\Delta t}{2}\int_\Omega v_1(w_{n+1} + w_n)\,dx \tag{3}$$

$$\int_\Omega v_2(w_{n+1} - w_n)\,dx = \frac{\Delta t}{2}\left(\int_\Omega v_2(f_{n+1} + f_n)\,dx - \int_\Omega \nabla v_2(\nabla u_{n+1} + \nabla u_n)\,dx\right) \tag{4}$$

$\forall v_1, v_2 \in V_0, t \in J$.

In our implementation, we can concatenate these two equations through simple addition and solve using two trial and test functions in the mixed space (V_h, V_h).

1.7 Part g

Using our Crank-Nicolson formulation 3, we compute the solution shown in figure 2 at $t = 5$.

1.8 Part h

Increasing Δt still yields a decrease in the fineness of the calculation. This can be seen in figure 2. However, notice that the total energy within the system is more stable than with our previous formulation 2.

If we compare solvers, we can see that, no matter the resolution of the mesh, LU factorization with a pre-factorized matrix provides the fastest solution. This is true for both our explicit discretization 2 and our implicit discretization 3. These can both be seen in figure 3. It can also be seen that our Crank-Nicolson formulation 3 is generally much more computationally expensive than 2.

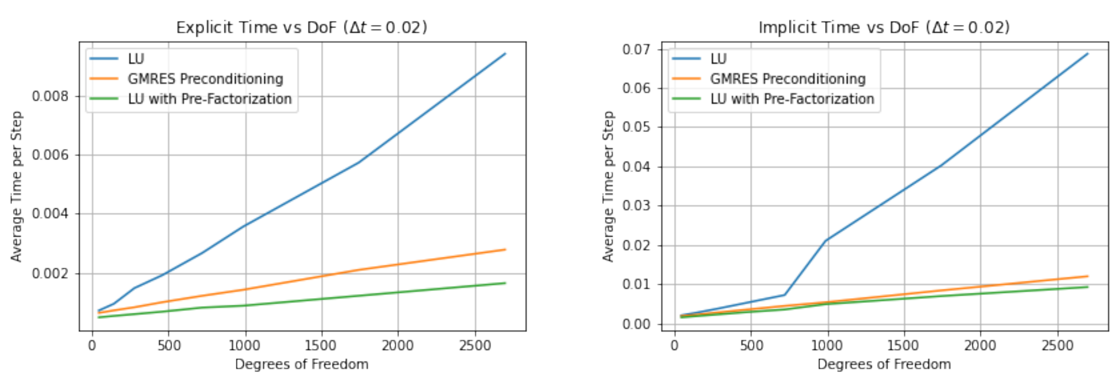

Figure 3: Comparison of average step runtime by degrees-of-freedom and solver

1.9 Part i

As can be seen from the use of higher degree polynomials in our standard formulation 2 (figure 4) and our Crank-Nicolson formulation 3 (figure 5), these higher-order polynomials seemed to destabilize the solutions. In general, qualitatively, the change brought about 'sharper' edges after reflection or deformation of a wave. There were also far more extreme changes to the surrounding environment and the right-most section of the domain during runtime, which remained largely homogeneous or without a clear pattern – much more so than when using linear polynomials. The stability region for the formulation seems to shrink with higher order polynomials.

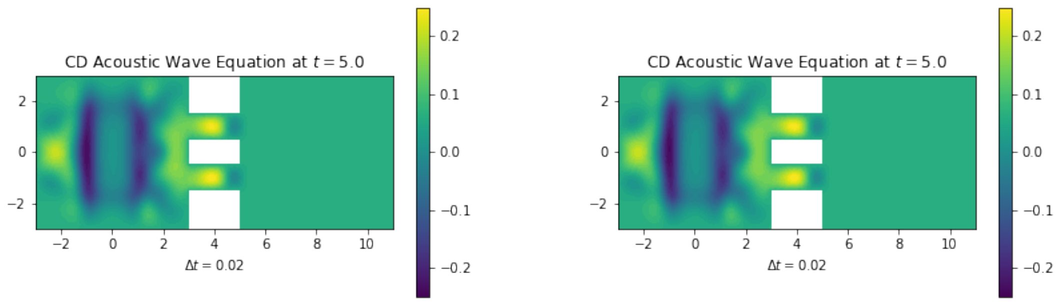

Figure 4: Standard formulation used with polynomial degrees 2 (left) and 3 (right)

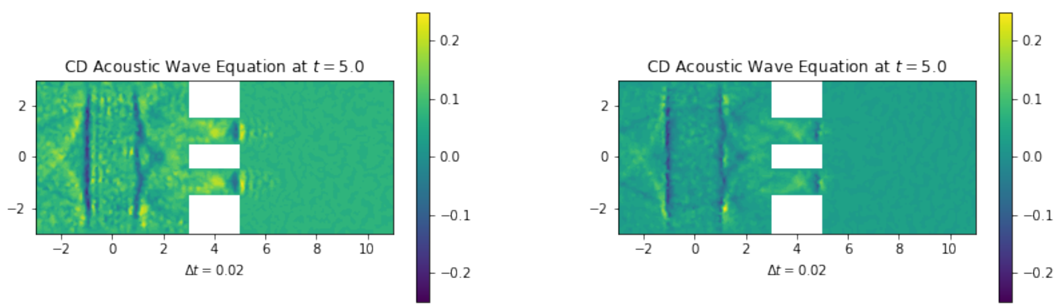

Figure 5: Crank-Nicolson formulation used with polynomial degrees 2 (left) and 3 (right)

2 Problem B

We are given

$$a(u, v) = v^T Au \quad l(v) = v^T b \quad V = \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$$

And assume that $||\cdot||_V$ is the Euclidean norm.

2.1 Part a

If $a(\cdot,\cdot)$ is coercive, $\exists c > 0, \forall x \in V \neq \vec{0} : a(x, x) \geq c||x||^2$. Then,

$$x^T Ax \geq c||x||^2$$

Since we are dealing with the euclidean norm, $x \neq \vec{0}$ implies $||x||^2 > 0$. We already know $c > 0$. Thus, $x^T Ax \geq c||x||^2 > 0$ and A is positive definite.

Recall the definition of an eigenvalue for $x \in V$: $Ax = \lambda x$. Then, $x^T Ax = \lambda x^T x$.

We know $x^T x = ||x||^2 > 0$ and $x^T Ax > 0$ by definition. Thus, $\lambda > 0$.

So, if $a(\cdot, \cdot)$ is coercive, all eigenvalues of A are strictly positive.

2.2 Part b

Suppose that A is not full rank. Then, there must be at least two column vectors that are not linearly independent. So, $\exists c_1, c_2, i, j : c_1 A_{\cdot,i} + c_2 A_{\cdot,j} = 0$. Then, it naturally follows that $\exists \vec{c} \neq \vec{0} : Ac = 0c$. Thus, if A is not full rank, it has at least one 0 eigenvalue.

Given our result from Part a, we know that $\lambda \neq 0$ for all eigenvalues λ of A. As such, A must be full rank.

Following this, since A is a square matrix, we know that A forms a basis over V. Thus, for our equation $Au = b$, by the definition of a basis, there exists some bijective mapping from $b \in V$ to $u \in V$ for all b. In other words, we can express b as a linear combination $A_{\cdot,1} u_1 + A_{\cdot,2} u_2 \cdots = b$.[1]

Thus, there exists a unique solution $u \in V$ to the linear system $Au = b$ given that the bilinear form $a(u, v) = v^T Au$ is coercive.

3 Problem C

We are given the variational problem

$$\begin{cases} -\nabla \cdot k\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ \partial_n u = g & \text{on } \Gamma_2 \end{cases} \tag{5}$$

We know the following about the space

$$\partial\Omega = \bigcup_i \Gamma_i \quad \Omega = \bigcup_i \Omega_i \quad \forall i : \Gamma_i \subset \Omega_i \quad k(x) = \begin{cases} \kappa_1 & x \in \Omega_1 \\ \kappa_2 & x \in \Omega_2 \end{cases} \quad \forall i : \kappa_i \geq 0$$

Where S is the boundary between spaces Ω_1 and Ω_2. Keep in mind that $S \in \partial\Omega_i$ but $S \notin \Gamma_i$

We can begin by multiplying by our test function $v \in V$ and integrating. Now we define $V = \{v : ||\nabla v|| + ||v|| < \infty, v|_{\partial\Omega} = 0\}$. Notice that this is a Hilbert space. As clarification, v disappears at the boundary $\partial\Omega$. We produce

$$-\int_\Omega v\nabla \cdot k\nabla u \, dx = \int_\Omega fv \, dx$$

We can decompose our integral into the subdomains Ω_i

$$\int_{\Omega_1} v\nabla \cdot \kappa_1 \nabla u \, dx + \int_{\Omega_2} v\nabla \cdot \kappa_2 \nabla u \, dx = -\int_\Omega fv \, dx$$

We'll use Green's formula

$$\int_{\partial\Omega_1} \kappa_1 v \partial_n u \, ds + \int_{\partial\Omega_2} \kappa_2 v \partial_n u \, ds - \int_{\Omega_1} \kappa_1 \nabla v \nabla u \, dx - \int_{\Omega_2} \kappa_2 \nabla v \nabla u \, dx = -\int_\Omega fv \, dx$$

Recall that $\partial_n u$ is the derivative of u in the outward normal direction.

Assuming that S is sufficiently smooth, notice that u for the trace $\partial\Omega_i$ is bounded on S because $u \in V \subset H^1$. This is shown by the trace inequality $||u||_{L^2(\partial\Omega_i)} \leq C||u||_{H^1(\Omega_i)}$, which provides an upper bound for $\sqrt{\int_{\partial\Omega_i} u^2}$.

[1] As another method of thinking, since A is full-rank, each equation in our system ($A_{i,1} u_1 + A_{i,2} u_2 \cdots = b_i$) forms a hyperplane with a unique normal. Then, there must be some unique intersection point between all of them.

We know that v vanishes on $\partial\Omega$. Notice that we can only say this of the Γ_i subsections of the boundaries.[2]

$$\int_{S_{\Omega_1}} \kappa_1 v\partial_n u\, ds + \int_{S_{\Omega_2}} \kappa_2 v\partial_n u\, ds - \int_{\Omega_1} \kappa_1 \nabla v \nabla u\, dx - \int_{\Omega_2} \kappa_2 \nabla v \nabla u\, dx = -\int_{\Omega} fv\, dx$$

We can subtract the variational formulation of our original ODE without decomposition. Then, $u \in V$ satisfies our variational problem 5 if and only if

$$\int_{S_{\Omega_1}} \kappa_1 v\partial_n u\, ds = -\int_{S_{\Omega_2}} \kappa_2 v\partial_n u\, ds$$

Since the outward normals of S_{Ω_1} and S_{Ω_2} are precisely opposite for any given point $x \in S$, we can say this condition is only met when

$$\kappa_1 \partial_n u_1 = \kappa_2 \partial_n u_2$$

where $\partial_n u_2$ refers to the derivative of u approaching S in the direction of the normal n (the same normal for both Ω_1 and Ω_2).

Given k is a piecewise constant function, we can say that our variational problem 5 is satisfied if and only if

$$\begin{cases} -\kappa_i \triangle u = f & \text{in } \Omega_i \\ u = 0 & \text{on } \Gamma_1 \\ \partial_n u = g & \text{on } \Gamma_2 \\ \kappa_1 \partial_n u_1 = \kappa_2 \partial_n u_2 & \text{on } S \end{cases}$$

4 Problem D

4.1 Proving our Statement

We are given the domain Ω, a square with boundary Γ. We want to show that some constant C exists such that

$$\left(\int_\Gamma v^2\, ds\right)^{1/2} \leq C||v||_{H^1(\Omega)} \quad \forall v \in H^1(\Omega) \tag{6}$$

Recall the definition of the H^1 norm

$$||v||_{H^1(\Omega)} = (||v||^2_{L^2(\Omega)} + ||\nabla v||^2_{L^2(\Omega)})^{1/2}$$

Assume Γ is sufficiently smooth. Recall the trace inequality for Hilbert spaces

$$||v||_{L^2(\Gamma)} \leq C(||v||^2_{L^2(\Omega)} + ||\nabla v||^2_{L^2(\Omega)})^{1/2}$$

Then, we have

$$\left(\int_\Gamma v^2\, ds\right)^{1/2} = ||v||_{L^2(\Gamma)} \leq C(||v||^2_{L^2(\Omega)} + ||\nabla v||^2_{L^2(\Omega)})^{1/2} = C||v||_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

for some constant C.

Thus, statement 6 must be true.

4.2 Application of our Inequality

We are given the linear form $L : H^1(\Omega) \to \mathbb{R}$

$$L(v) = \int_\Gamma gv\, ds$$

[2] S_{Ω_i} denotes the boundary S approached from within Ω_i.

Using the Cauchy-Schwarz inequality, we can say

$$L(v) = |\langle g, v\rangle_{L^2(\Gamma)}| \leq ||g||_{L^2(\Gamma)}||v||_{L^2(\Gamma)}$$

If $g \in L^2(\Gamma)$, that is

$$||g||^2_{L^2(\Gamma)} < \infty$$

then $||g||_{L^2(\Gamma)}$ is bounded on both sides.

By our previously proved statement 6, since $v \in H^1(\Omega)$, $||v||_{L^2(\Gamma)}$ must be bounded as well.

Thus, our linear form L is bounded for $g \in L^2(\Gamma)$ and is therefore continuous.

$$L(v) = |\langle g, v\rangle_{L^2(\Gamma)}| \leq ||g||_{L^2(\Gamma)}||v||_{L^2(\Gamma)} < \infty$$