CS 715

EIT

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https://github.com/0afish1/CSC-715

May 4, 2021

1 Problem a

We are given

$$\begin{cases} -\nabla \cdot (\sigma \nabla u) = 0 & \Omega \\ u = \phi & \partial\Omega \end{cases} \tag{1}$$

We can expand on Ω

$$-\sigma \triangle u - \nabla\sigma\nabla u = 0$$

We introduce the test function v, which disappears on $\partial\Omega$. Multiplying by v and integrating we get

$$-\int_\Omega v\sigma \triangle u \, dx - \int_\Omega v\nabla\sigma\nabla u \, dx = 0$$

We can then utilize Green's formula, keeping in mind that v disappears on $\partial\Omega$

$$-(\int_{\partial\Omega} v\sigma\partial_n u \, ds - \int_\Omega \nabla(v\sigma)\nabla u \, dx) - \int_\Omega v\nabla u\nabla\sigma \, dx = 0$$
$$\int_\Omega \nabla(v\sigma)\nabla u \, dx = \int_\Omega v\nabla\sigma\nabla u \, dx$$
$$\int_\Omega \sigma\nabla v\nabla u \, dx = 0$$

Keeping our boundary condition in mind, we introduce the space,

$$V = \{v : ||v|| + ||\nabla v|| < \infty\}$$

propose the trial space,

$$V_\phi = \{v : v \in V, v|_{\partial\Omega} = \phi\}$$

and the test space

$$V_0 = \{v : v \in V, v|_{\partial\Omega} = 0\}$$

This provides us with the variational formulation: For some $\sigma \in \mathcal{C}^1$ find $u \in V_\phi$ such that

$$\int_\Omega \sigma\nabla v\nabla u \, dx = 0 \quad \forall v \in V_0 \tag{2}$$

2 Problem b

Let $\mathscr{K}$ be a mesh on Ω. Let $K \in \mathscr{K}$ be a triangle.

We introduce the space of linear functions on K.

$$\mathscr{P}(K) = \{v : v = c_0 + c_1 x_1 + c_2 x_2, (x_1, x_2) \in K, c_0, c_1, c_2 \in \mathbb{R}\}$$

Further, we have the space of continuous ($\mathscr{C}^0$), piecewise-linear polynomials

$$V_h = \{v : v \in \mathscr{C}^0(\Omega), v|_K \in \mathscr{P}(K)\ \forall K \in \mathscr{K}\}$$

We propose the function spaces

$$V_{h,\phi} = \{v : v \in V, v|_{\partial\Omega} = \phi\}$$
$$V_{h,0} = \{v : v \in V, v|_{\partial\Omega} = 0\}$$

Notice that this implies the nodal basis

$$\lambda_i(N_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

where N_j is one of three points defining a triangle K and $\lambda_i(N_j) \in V_h$.

We then have the discrete variational formulation: For some $\sigma \in \mathcal{C}^1$ find $u \in V_{h,\phi}$ such that

$$\int_\Omega \sigma \nabla v \nabla u_h \, dx = 0 \quad \forall v \in V_{h,0} \tag{3}$$

2.1 Part i

We use the hat functions $\varphi_i \in V_h$, defined as

$$\varphi_i(N_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

where N_j is any node that composes mesh $\mathscr{K}$. We can uniquely define any function $f \in V_h$ as a linear combination of these hat functions.

Since $u_h \in V_h$, we can define

$$u_h = \sum_{j=1}^{n_p} \xi_j \varphi_j$$

where n_p is the total number of nodes in the mesh $\mathscr{K}$. We further define n_b and n_i, which are the number of boundary and internal nodes, respectively.

With (3) and the ansatz above, we can say

$$\int_\Omega \sigma \nabla \varphi_i \nabla \left(\sum_{j=1}^{n_p} \xi_j \varphi_j \right) \, dx = \sum_{j=1}^{n_p} \xi_j \int_\Omega \sigma \nabla \varphi_i \nabla \varphi_j \, dx = 0 \quad i = 1, \ldots, n_p$$

This provides the stiffness matrix and load vectors

$$A_{ij} = \int_\Omega \sigma \nabla \varphi_i \nabla \varphi_j \, dx \quad i, j = 1, \ldots, n_p$$
$$b = \vec{0}$$

Our variational form is now equivalently expressed as

$$A\xi = b \tag{4}$$

2.2 Part ii

We can partition our stiffness matrix into the following block matrix

$$\begin{bmatrix} A_{ii} & A_{ib} \\ A_{bi} & A_{bb} \end{bmatrix}$$

with

$$(A_{jk})_{lm} = \int_\Omega \sigma \nabla \varphi_l \nabla \varphi_m \, dx \quad l = N_j, m = N_k$$

where $N_i = 1, \ldots, n_i; N_b = n_i + 1, \ldots, n_p$. Note that matrix A_{jk} is of dimension $n_j \times n_k$. We also define i and b as representing interior and exterior node partitions, respectively. Notice that A_{bi} will have no non-zero values and A_{bb} will only have non-zero values on the diagonal. Inserting our boundary conditions, the problem can be represented as

$$\begin{bmatrix} A_{ii} & A_{ib} \\ 0 & I_{n_b} \end{bmatrix} \begin{bmatrix} \xi_i \\ \xi_b \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \end{bmatrix} \tag{5}$$

3 Problem c

We define the map $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

$$\Lambda_\sigma \phi := \sigma \partial_n u$$

We define $\Psi|_{\partial\Omega} = \psi$ and start from the inner product

$$\langle \Lambda_\sigma \phi, \psi \rangle_{L^2(\partial\Omega)}$$

Recall that, by Green's formula,

$$\int_\Omega \sigma \Psi \triangle u = \int_{\partial\Omega} \sigma \Psi \nabla u - \int_\Omega \nabla(\sigma\Psi) \nabla u$$

Expanding our inner product using Green's formula, we get

$$\int_{\partial\Omega} \psi \Lambda_\sigma \phi = \int_{\partial\Omega} \psi \sigma \partial_n u = \int_\Omega \Psi(\sigma \triangle u + \nabla\sigma\nabla u) + \int_\Omega \sigma \nabla u \nabla \Psi$$

Using (1), we can simplify

$$\langle \Lambda_\sigma \phi, \psi \rangle_{L^2(\partial\Omega)} = \int_\Omega \sigma \nabla u \nabla \Psi \tag{6}$$

4 Problem d

We are given Dirichlet data $\{u^i_{\partial\Omega}\}_{i=1}^{n_b}$ encoded in the columns of a matrix D, from (5). We have

$$\begin{bmatrix} A_{ii} & A_{ib} \\ 0 & I \end{bmatrix} \begin{bmatrix} U_I \\ U_B = D \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{\phi} \end{bmatrix}$$

since ξ_j controls the abundance of each hat function with magnitude 1. Since we are solving for multiple problems at once, our load vector has number of columns matching that of D. Note that this implies $\vec{0}$ and $\vec{\phi}$ are row vectors.

We then need to solve the equation

$$A_{ii} U_I + A_{ib} D = \vec{0}$$

for U_I.

This produces

$$U_I = -(A_{ii})^{-1} A_{ib} D$$

Thus, for given Dirichlet data $\{u^i_{\partial\Omega}\}_{i=1}^{n_b}$ encoded in the columns of a matrix D, we can solve all problems (1) at the same time with

$$U = \begin{pmatrix} U_I \\ U_B \end{pmatrix} = \begin{pmatrix} -(A_{ii})^{-1} A_{ib} D \\ D \end{pmatrix} \tag{7}$$

5 Problem e

Let's say we have our boundary conditions $\Phi = \{\varphi_i\}_{i=n_i+1}^{n_b}$. Notice that this collection of bases is the identity matrix. We have corresponding solution $U_I = \{u_i\}_{i=n_i+1}^{n_b}$ where u_i refers to the solution to (1) for inhomogeneous boundary condition φ_i. Then, from (4) we have

$$\begin{bmatrix} A_{ii} & A_{ib} \\ A_{bi} & A_{bb} \end{bmatrix} \begin{bmatrix} U_I \\ \Phi \end{bmatrix}$$

This produces an expression

$$A_{bi}U_I + A_{bb}\Phi$$

Let's examine this expression for some u_i, φ_i pair. For row k of the output,

$$\sum_m u_{i,m} \int_\Omega \sigma \nabla\varphi_m \nabla\varphi_k + \sum_m \varphi_{i,m} \int_\Omega \sigma \nabla\varphi_m \nabla\varphi_k = \left\{ \int_\Omega \sigma \nabla u_m \nabla \varphi_k \right\}_{m=1}^{n_p} \begin{bmatrix} u_i \\ \varphi_i \end{bmatrix}_m$$

From (7), we discovered

$$U_I = -(A_{ii})^{-1} A_{ib} \Phi$$

Combining this with our initial expression, we get an alternate version of the operator

$$(A_{bi}(A_{ii})^{-1}(-A_{ib}) + A_{bb})\Phi$$

Notice that this is the Schur complement of A_{bb} in our stiffness matrix multiplied by the boundary vectors.
Let's define

$$(\Lambda_\sigma^h)_{ij} = \langle \Lambda_\sigma \varphi_i, \varphi_j \rangle_{L^2(\partial\Omega)}$$

Based on (6), we can expand

$$(\Lambda_\sigma^h)_{ij} = \int_\Omega \sigma \nabla u_i \nabla \varphi_j$$

Notice that we used the relationship $\Lambda_\sigma \varphi_i = \sigma \partial_n u_i$.
Also notice that our two operators are functionally equivalent. So, we have $\Lambda_\sigma^h = A_{bi}(A_{ii})^{-1}(-A_{ib}) + A_{bb}$.

6 Problem f

We begin with

$$J(\sigma) = \frac{1}{2} ||\sigma \partial_n u - d||^2_{L^2(\partial\Omega)}$$

Getting into the form we want by derivating in the direction h, keeping in mind that, for fixed σ on $\partial\Omega$, $h|_{\partial\Omega} = 0$

$$\begin{aligned}
\nabla_\sigma J(\sigma) \cdot h &= \nabla_\sigma \frac{1}{2} ||\sigma \partial_n u - d||^2_{L^2(\partial\Omega)} \cdot h \\
&= \nabla_\sigma \frac{1}{2} \langle \sigma \partial_n u - d, \sigma \partial_n u - d \rangle_{L^2(\partial\Omega)} \cdot h \\
&= \langle \sigma \partial_n u - d, h \cdot \partial_n u \rangle_{L^2(\partial\Omega)} \\
&= \langle (\partial_n u)(\sigma \partial_n u - d), h \rangle_{L^2(\Omega)}
\end{aligned}$$

We now have the adjoint $\nabla_\sigma \partial_n u$. We use the extended Lagrangian method. Note that $\mathscr{L}(\sigma, u, \lambda, \mu) = J(\sigma)$.

$$\begin{aligned}
\mathscr{L}(\sigma, u, \lambda, \mu) &= J(\sigma) - \langle \lambda, \nabla \cdot (\sigma \nabla u) \rangle_{L^2(\Omega)} - \langle \mu, (\phi - u) \rangle_{L^2(\partial\Omega)} \\
&= J(\sigma) - \int_\Omega \lambda \nabla \cdot (\sigma \nabla u)\, dx - \int_{\partial\Omega} \mu(\phi - u)\, ds
\end{aligned}$$

We can use Green's theorem to expand the second term.

$$\begin{aligned}
\mathscr{L}(\sigma,u,\lambda,\mu) &= J(\sigma) + \int_\Omega \sigma\nabla\lambda\nabla u\,dx - \int_{\partial\Omega}\lambda\cdot\sigma\partial_n u\,ds - \int_{\partial\Omega}\mu(\phi-u)\,ds \\
\frac{\partial}{\partial u}\mathscr{L}(\sigma,u,\lambda,\mu)\cdot g &= \int_{\partial\Omega}(\sigma\partial_n g)(\sigma\partial_n u - d) + \int_\Omega \sigma\nabla\lambda\nabla g\,dx - \int_{\partial\Omega}\lambda\cdot\sigma\partial_n g\,ds - \int_{\partial\Omega}\mu g\,ds \\
\frac{\partial}{\partial u}\mathscr{L}(\sigma,u,\lambda,\mu)\cdot g &= \int_\Omega \sigma\nabla\lambda\nabla g\,dx + \int_{\partial\Omega}((\sigma\partial_n u - d)-\lambda)\cdot\sigma\partial_n g\,ds - \int_{\partial\Omega}\mu g\,ds
\end{aligned}$$

We suppose that our inputs are correct and $\frac{\partial}{\partial u}\mathscr{L}(\sigma,u,\lambda,\mu)=0$. Then we have the formulation: $\forall g\in V_0$

$$\frac{\partial}{\partial u}\mathscr{L}(\sigma,u,\lambda,\mu)\cdot g = \int_\Omega \sigma\nabla\lambda\nabla g\,dx + \int_{\partial\Omega}((\sigma\partial_n u - d)-\lambda)\cdot\sigma\partial_n g\,ds$$

We can choose the adjoint state λ that satisfies

$$\begin{cases} -\nabla\cdot(\sigma\nabla\lambda) = 0 & \Omega \\ \lambda = \sigma\partial_n u - d & \partial\Omega \end{cases} \tag{8}$$

This is why our adjoint is so similar to our initial problem (1).

Derivating with respect to σ in the direction of h provides

$$\begin{aligned}
\nabla_\sigma J(\sigma)\cdot h &= \frac{\partial}{\partial\sigma}\mathscr{L}(\sigma,u,\lambda,\mu)\cdot h \\
\nabla_\sigma J(\sigma)\cdot h &= \int_{\partial\Omega}(h\partial_n u)(\sigma\partial_n u - d)\,ds + \int_\Omega h\nabla\lambda\nabla u\,dx - \int_{\partial\Omega}\lambda\cdot h\partial_n u\,ds
\end{aligned}$$

Supposing that σ is fixed on the boundary, $h|_{\partial\Omega}=0$ and we can eliminate the first and third terms.

$$\nabla_\sigma J(\sigma)\cdot h = \langle\nabla\lambda\nabla u, h\rangle_{L^2(\Omega)} = \int_\Omega h\nabla\lambda\nabla u\,dx$$

Thus,

$$\nabla_\sigma J(\sigma) = \nabla\lambda\nabla u$$

7 Problem g

We want to take the derivative of

$$J(\sigma) := \frac{1}{2}\sum_{i=1}^{n_b}||\sigma\partial_n u_i - d_i||^2_{L^2(\partial\Omega)}$$

where u_i is the solution satisfying (1) for boundary conditions φ_i.

We can begin by applying regular calculus, keeping in mind that, for fixed σ on $\partial\Omega$, $h|_{\partial\Omega}=0$

$$\begin{aligned}
\nabla_\sigma J(\sigma)\cdot h &= \nabla_\sigma\frac{1}{2}||\sigma\partial_n u_i - d_i||^2_{L^2(\partial\Omega)}\cdot h \\
&= \nabla_\sigma\frac{1}{2}\langle\sigma\partial_n u_i - d_i, \sigma\partial_n u_i - d_i\rangle_{L^2(\partial\Omega)}\cdot h \\
&= \langle\sigma\partial_n u_i - d_i, h\cdot\partial_n u_i\rangle_{L^2(\partial\Omega)} \\
&= \langle(\partial_n u_i)(\sigma\partial_n u_i - d_i), h\rangle_{L^2(\Omega)}
\end{aligned}$$

Let's define

$$J_i(\sigma) = \langle(\partial_n u_i)(\sigma\partial_n u_i - d_i), h\rangle_{L^2(\Omega)} \quad J(\sigma) = \sum_{i=1}^{n_b}J_i(\sigma)$$

Once again we use the extended Lagrangian

$$\begin{aligned}\mathscr{L}_i(\sigma, u_i, \lambda_i, \mu_i) &= J_i(\sigma) - \langle \lambda_i, \nabla \cdot (\sigma \nabla u_i) \rangle_{L^2(\Omega)} - \langle \mu_i, (\varphi_i - u_i) \rangle_{L^2(\partial\Omega)} \\ &= J_i(\sigma) - \int_\Omega \lambda_i \nabla \cdot (\sigma \nabla u_i)\, dx - \int_{\partial\Omega} \mu_i (\varphi_i - u_i)\, ds\end{aligned}$$

We use Green's theorem to simplify

$$\mathscr{L}_i(\sigma, u_i, \lambda_i, \mu_i) = J_i(\sigma) + \int_\Omega \sigma \nabla \lambda_i \nabla u_i\, dx - \int_{\partial\Omega} \lambda_i \cdot \sigma \partial_n u_i\, ds - \int_{\partial\Omega} \mu_i (\varphi_i - u_i)\, ds$$

$$\frac{\partial}{\partial u_i} \mathscr{L}_i(\sigma, u_i, \lambda_i, \mu_i) \cdot g = \int_\Omega \sigma \nabla \lambda_i \nabla g\, dx + \int_{\partial\Omega} ((\sigma \partial_n u_i - d_i) - \lambda_i) \cdot \sigma \partial_n g\, ds - \int_{\partial\Omega} \mu_i g\, ds$$

We suppose that our inputs are correct and $\frac{\partial}{\partial u} \mathscr{L}(\sigma, u, \lambda, \mu) = 0$. Then we have the formulation: $\forall g \in V_0$

$$\frac{\partial}{\partial u_i} \mathscr{L}_i(\sigma, u_i, \lambda_i, \mu_i) \cdot g = \int_\Omega \sigma \nabla \lambda_i \nabla g\, dx + \int_{\partial\Omega} ((\sigma \partial_n u_i - d_i) - \lambda_i) \cdot \sigma \partial_n g\, ds$$

We can choose the adjoint state λ_i that satisfies

$$\begin{cases} -\nabla \cdot (\sigma \nabla \lambda_i) = 0 & \Omega \\ \lambda_i = \sigma \partial_n u_i - d_i & \partial\Omega \end{cases} \tag{9}$$

Derivating with respect to σ in the direction of h provides

$$\begin{aligned}\nabla_\sigma J_i(\sigma) \cdot h &= \frac{\partial}{\partial \sigma} \mathscr{L}_i(\sigma, u_i, \lambda_i, \mu_i) \cdot h \\ \nabla_\sigma J_i(\sigma) \cdot h &= \int_{\partial\Omega} (h \partial_n u_i)(\sigma \partial_n u_i - d_i)\, ds + \int_\Omega h \nabla \lambda_i \nabla u_i\, dx - \int_{\partial\Omega} \lambda_i \cdot h \partial_n u_i\, ds\end{aligned}$$

Supposing that σ is fixed on the boundary, $h|_{\partial\Omega} = 0$ and we can eliminate the first and third terms.

$$\nabla_\sigma J_i(\sigma) \cdot h = \langle \nabla \lambda_i \nabla u_i, h \rangle_{L^2(\Omega)} = \int_\Omega h \nabla \lambda_i \nabla u_i\, dx$$

So,

$$\nabla_\sigma J_i(\sigma) = \nabla \lambda_i \nabla u_i$$

Using our previous definition $J(\sigma) = \sum_{i=1}^{n_b} J_i(\sigma)$,

$$\nabla_\sigma J(\sigma) = \sum_{i=1}^{n_b} \nabla \lambda_i \nabla u_i$$

8 Problem i

Done.

9 Problem j

Done. test_derivative errors higher than expected.

10 Problem k

Done.

11 Problem l

Done.

12 Problem m

Done. test-gradient error relatively high, but likely because of the high error in test_derivative.