Week 13: Discrete scalar dynamical systems

May 27, 2024

An exercise

In the previous lecture we solved the following exercise

Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x} \; .$$

We found that f has a unique fixed point, $\eta^* = \sqrt{5}$, which is an attractor. More precisely, we found $f'(\sqrt{5}) = 0$, which is the smallest constant λ that satisfy the condition $|\lambda| < 1$. Moreover, using the cobweb diagram we developed the intuition that the basin of attraction of the fixed point $\eta^* = \sqrt{5}$ is the whole interval $(0, \infty)$ and that just few steps are needed to arrive very close to $\sqrt{5}$.

On the other hand, note that $f(x) \in \mathbb{Q}$ for any $x \in \mathbb{Q}$. Recall that a rational number (from $\mathbb{Q}$) can be written as a fraction (of two natural numbers) and, after division, it has a finite number of decimals or repeating decimals.

Rational approximations of $\sqrt{5}$ =2.23606 79774 99789 69640 91736 68731 27623 54406 18359 61152 57242 7089...

$$f : (0,\infty) \to \mathbb{R}, \quad f(x) = \frac{x^2+5}{2x} \ .$$

$$f(\sqrt{5}) = \sqrt{5}, \quad f'(\sqrt{5}) = 0.$$

We compute now the iterations of f starting with $x_0 = 2$. We have

$$\begin{aligned} x_1 &= f(2) = \frac{9}{4} = \color{red}{2.2}5, \\ x_2 &= f(9/4) = \frac{161}{72} = \color{red}{2.236}(1), \\ x_3 &= f(161/72) = \frac{51841}{23184} = \color{red}{2.236067977}92.... \end{aligned}$$

The basin of attraction of $\sqrt{5}$ (the proof)

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x} \quad f'(x) = \frac{x^2 - 5}{2x^2}\ .$$

We prove that: The sequence $x_k = f^k(\eta)$ converges to $\sqrt{5}$ for any $\eta > 0$.

We have that f is decreasing on $(0, \sqrt{5})$ and increasing on $(\sqrt{5}, \infty)$, $\lim_{x \searrow 0} f(x) = +\infty$, $f(\sqrt{5}) = \sqrt{5}$, $\lim_{x \to \infty} f(x) = +\infty$.

Then $f(x) \in (\sqrt{5}, \infty)$ for any $x \in (0, \infty)$. This assures that it is sufficient to study the restriction of f to $(\sqrt{5}, \infty)$. On the other hand it can be easily seen that $f(x) < x$ for all $x \in (\sqrt{5}, \infty)$.

Fix $\eta \in (\sqrt{5}, \infty)$. Then the sequence $(f^k(\eta))_{k \geq 0}$ is decreasing and belongs to the interval $(\sqrt{5}, \eta)$. Thus, it is convergent.

As we know, the only possible limit of a sequence of iterates is a fixed point of f. We reach the conclusion by recalling that $\sqrt{5}$ is the only fixed point of f.

The Newton-Raphson method

We consider now the map

$$g : (0, \infty) \to \mathbb{R}, \quad g(x) = x^2 - 5.$$

Of course, $g(\sqrt{5}) = 0$, i.e. $\sqrt{5}$ is a zero of g. Using the graph of g, we present a graphical method to find again a sequence that converges to $\sqrt{5}$.

Start with $x_0 = \eta > 0$. For $k \in \mathbb{N}$ do the following.
Find x_{k+1} such that the point $(x_{k+1}, 0)$ belongs to the tangent to the graph of g in the point $(x_k, g(x_k))$.

In order to find the formula generated by this method we write first the equation of the tangent

$$y - g(x_k) = g'(x_k)(x - x_k).$$

Since $(x_{k+1}, 0)$ belongs to it, we have $-g(x_k) = g'(x_k)(x_{k+1} - x_k)$, which gives

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}.$$

The Newton-Raphson method

The sequence (x_k) given by

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)} \tag{1}$$

is the sequence of iterates of the function $f(x) = x - \frac{g(x)}{g'(x)}$.
Note that for $g(x) = x^2 - 5$ we have $f(x) = x - \frac{x^2-5}{2x} = \frac{x^2+5}{2x}$. Since we already studied the dynamic of the map f we are convinced that the Newton-Raphson method is very efficient for the map $g(x) = x^2 - 5$.
What about for other maps g? We have

Theorem (The Newton-Raphson method)

Let $V \subset \mathbb{R}$ be a nonempty open interval and the C^2 map $g : V \to \mathbb{R}$. Assume that $g'(x) \neq 0$ for any $x \in V$ and there exists $\eta^* \in V$ such that $g(\eta^*) = 0$. Then there exists $\rho > 0$ such that whenever $|x_0 - \eta^*| < \rho$ we have $\lim_{k\to\infty} x_k = \eta^*$, where (x_k) is defined by (1).

Proof of the Newton-Raphson theorem

Using the remark written in the previous slide and the definition of the attracting fixed point, the conclusion is equivalent to the following statement

η^* is an attracting fixed point of

$$f : V \to \mathbb{R}, \quad f(x) = x - \frac{g(x)}{g'(x)}.$$

In order to prove this, we just have to compute $f(\eta^*)$ and $f'(\eta^*)$.

We have $f(\eta^*) = \eta^* - \frac{g(\eta^*)}{g'(\eta^*)} = \eta^*$. Thus, η^* is a fixed point of f.

Since $f'(x) = 1 - \frac{g'(x)g'(x) - g(x)g''(x)}{[g'(x)]^2}$ for all $x \in V$, we have $f'(\eta^*) = 1 - 1 = 0$ (a very good value, the best one again).

Since $|f'(\eta^*)| < 1$ we deduce that η^* is an attractor for the map f.

Newton fractal

Theorem (The Newton-Raphson method for complex maps)

Let $V \subset \mathbb{C}$ be an open disk and the C^2 map $g : V \to \mathbb{C}$. Assume that $g'(z) \neq 0$ for any $z \in V$ and there exists $\eta^* \in V$ such that $g(\eta^*) = 0$. Then there exists $\rho > 0$ such that whenever $|z_0 - \eta^*| < \rho$ we have $\lim_{k\to\infty} z_k = \eta^*$, where (z_k) is defined by $z_{k+1} = z_k - \frac{g(z_k)}{g'(z_k)}$.

We consider just an example, $g(z) = z^3 - 1$. We see that g has 3 zeros, the roots of order 3 of the unity: $\eta_1^* = 1$, $\eta_2^* = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $\eta_3^* = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Check the hypotheses of the theorem: We have $g'(z) = 3z^2$ which takes the value 0 just in 0. Thus there are disks $V_1, V_2, V_3 \subset \mathbb{C}$ such that $\eta_1^* \in V_1$, $\eta_2^* \in V_2$, $\eta_3^* \in V_3$ and $g'(z) \neq 0$ for any $z \in V_1 \cup V_2 \cup V_3$.

The theorem assures the convergence of the Newton's method to one of the η^* at least if we start sufficiently close to η^*. Of course, the actual basin of attraction can be larger. So, let us denote the basin of attraction of the Newton's method corresponding to η^* by A_1, A_2, A_3.

Newton fractal for $g(z) = z^3 - 1$

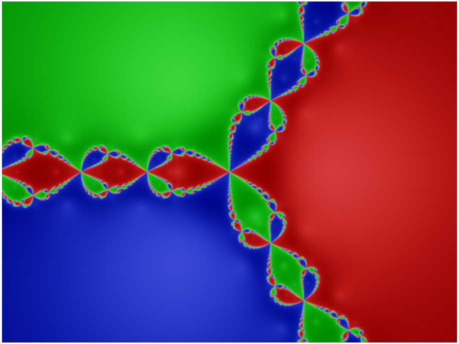

Figure: The basins of attraction A_1, A_2, A_3

The boundary of each basin of attraction is a fractal.

Construction of the figure - the algorithm

Let $g : \mathbb{C} \to \mathbb{C}$. Fix a very small constant $\varepsilon > 0$.

Step 1. Compute $g'(z)$ and $f(z) = z - \frac{g(z)}{g'(z)}$.

Step 2. Compute the roots of $g(z)$.

Step 3. Pick an initial point and calculate the distance between the point and each of the roots of g. If the distance is less then ε, color the point with the color chosen for the respective root.

Step 4. If not, iterate f until the distance between the iterate and one of the roots of g is less then ε. Color the original point with the color chosen for the respective root.

Step 5. Repeat for many points.

You can find information and other pictures on the internet.
For example, here:
https://www.chiark.greenend.org.uk/∼sgtatham/newton/
This is a nice video:
https://www.youtube.com/watch?v=-RdOwhmqP5s&t=968s

Chaos in $f(x) = 4x(1-x)$

Start with $x_0 = \eta \in [0,1]$ and let $x_k = f^k(\eta)$ for all $k \geq 1$, i.e. $x_{k+1} = 4x_k(1-x_k)$ for all $k \geq 0$.

A simple formula for x_k. We have

$$x_k = \sin^2(2^k\theta)$$

where $\theta \in \mathbb{R}$ is such that $x_0 = \sin^2\theta$.

Proof of the formula.
$x_{k+1} = \sin^2(2 \cdot 2^k\theta) = 4\sin^2(2^k\theta)\cos^2(2^k\theta) = 4x_k(1-x_k)$. $\square$

Chaos in $f(x) = 4x(1-x)$

For example, when $x_0 = 0.67$ we take $\theta = \arcsin\left(\sqrt{0.67}\right)$, thus

$$x_{40} = \sin^2\left(2^{40}\arcsin\left(\sqrt{0.67}\right)\right).$$

This is a representation of the exact value. As we have seen in the lab, this is computationally challenging! Let us look at the sequence (x_k) using the cobweb diagram here

https://www.geogebra.org/m/gHYqKMSJ

Main features of this dynamic:

1) There are p-cycles for any $p \geq 1$.

2) The butterfly effect.

3) A dense orbit.

Cycles of any period

The fixed points: 0 and 0.75 (found by solving $x = 4x(1-x)$) .

Recall that $x_k = \sin^2(2^k\theta)$ and that $\sin(x+\pi) = -\sin(x)$.

First let us find a cycle of period 3, i.e. we look for a value x_0 such that $x_3 = x_0$, $x_2 \neq x_0$ and $x_1 \neq x_0$. In other words, we look for a value θ such that $0 < \theta < 2\theta < 4\theta < 8\theta = \theta + \pi$. Then $\theta = \frac{\pi}{7}$.
It is clear that $x_k = \sin^2\left(2^k\frac{\pi}{7}\right)$ is a cycle of period 3.

There is an article by Li and Yorke published in 1975 called *Period three implies chaos*. One of the theorems proved in it assures that, for any map, if there exists a cycle of period 3, then there exists a cycle of any period. Anyway, for our particular example we can also prove this like we proved for period three.

Indeed, for an arbitrary $p \geq 2$ take $\theta = \frac{\pi}{2^p-1}$ (found such that $2^p\theta = \theta + \pi$). Then $x_k = \sin^2\left(2^k\frac{\pi}{2^p-1}\right)$ is a cycle of period p.

The butterfly effect

Given $\eta \in [0,1]$ and $\delta > 0$, there exist $K \geq 1$ and $\tilde{\eta} \in [0,1]$ such that $|\eta - \tilde{\eta}| < \delta$ and $|f^K(\eta) - f^K(\tilde{\eta})| \geq \frac{1}{2}$.

Proof. Write $\eta = \sin^2(\theta)$. Recall that $x_k = f^k(\eta) = \sin^2(2^k\theta)$.

Take $K \geq 1$ such that $\frac{\pi}{2^K} < \delta$.

Take $\zeta \in [0.\pi]$ such that

$$|\sin^2(2^k\theta) - \sin^2(2^k\theta + \zeta)| \geq \frac{1}{2}. \tag{2}$$

Now take $\tilde{\eta} = \sin^2(\theta + \frac{\zeta}{2^K})$. Then, using that

$$|\sin^2(\theta_1) - \sin^2(\theta_2)| \leq |\theta_1 - \theta_2|,$$

(which can be proved using the mean value theorem and $(\sin^2\theta)' = \sin 2\theta$), we obtain $|\eta - \tilde{\eta}| = |\sin^2(\theta) - \sin^2(\theta + \frac{\zeta}{2^K})| \leq |\frac{\zeta}{2^K}| \leq \frac{\pi}{2^K} < \delta$.

Also, from (2) we have $|f^K(\eta) - f^K(\tilde{\eta})| \geq \frac{1}{2}$. $\square$

A dense orbit

There exists $\eta \in [0,1]$ such that $\{f^k(\eta) \,:\, k \geq 0\}$ is dense in $[0,1]$.

This means that for each $x \in [0,1]$ there exists a sub-sequence of $(f^k(\eta))$ which converges to x. Equivalently, for each $x \in [0,1]$ there exists $K \geq 1$ such that $f^K(\eta)$ is arbitrarily close to x. In other words, each $x \in [0,1]$ is as close as we want to a term of the sequence $(f^k(\eta))$.

[1] R. L. Devaney, An Introduction to Chaotic Dynamical Systems, 2nd ed. Reading, MA: Addison-Wesley, 1989.

A map with the three properties

1) There are p-cycles for any $p \geq 1$.

2) The butterfly effect.

3) A dense orbit.

is said to be chaotic (see [1]).

The pendulum equation

A nice video:

https://www.youtube.com/watch?v=p_di4Zn4wz4&t=916s