Seminar 1

1. Find the lower and the upper bounds, then sup, inf, max, min for each of the following:

   (a) $[-3, 2) \cup \{3\}$.
   (b) $(-1, 1] \cup (2, \infty)$.
   (c) $(-5, 5) \cap \mathbb{Z}$.
   (d) $\emptyset$.

2. Find the sup, inf, max, min for each of the following sets:

   (a) $\{x \in \mathbb{Q} \mid x^2 < 3\}$.
   (b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$.
   (c) $\{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.
   (d) $\{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}\}$.

3. Suppose that S is nonempty and bounded above. Show that the set $-S := \{-x \mid x \in S\}$ is bounded below and $\inf(-S) = -\sup(S)$.

4. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on a nonempty set D. Prove that

   $$\inf_{x\in D} \big(f(x) + g(x)\big) \geq \inf_{x\in D} f(x) + \inf_{x\in D} g(x) \quad \text{and} \quad \sup_{x\in D} \big(f(x) + g(x)\big) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x).$$

   Give examples where the above inequalities are strict.

5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

   $$\sup_{x\in S}(ax + b) = a \sup(S) + b.$$

6. Which of the following sets are neighborhoods of 0?

   $$[-1, 1] \cup \{2\}; \quad (-1, 1) \cap \mathbb{Q}; \quad \bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}].$$

7. Let $x \in \mathbb{R}$ and $U, V \in \mathcal{V}(x)$. Prove that $U \cap V \in \mathcal{V}(x)$.

8. ★ Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$.

9. Find the interior and the closure for each of the following sets:

   (a) $(1, 2]$.
   (b) $[-3, 2) \cup \{3\}$.
   (c) $(-1, 1] \cup (2, \infty)$.
   (d) $(-5, 5) \cap \mathbb{Z}$.

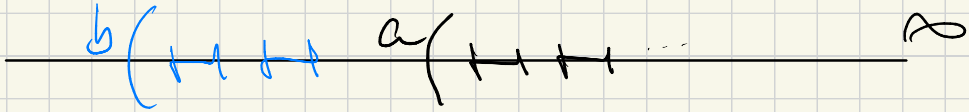
10. ★ Let $A = (0, 1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\mathrm{int}A = \emptyset$ and $\mathrm{cl}A = [0, 1]$.

Homework questions are marked with ★.
Solutions should be uploaded on Teams before the next lecture.

10.10.2023

V is a nbhood of ∞ if $\exists\, a \in \mathbb{R}$ s.t

$$(a, \infty) \subset V$$



$$V = (b, +\infty)$$

$$\boxed{\text{there is plenty of room at } \infty}$$

$$V = (b, +\infty) \cap \{ [b_1, b_2] \cup [b_3, b_4] \ldots \cup$$

$$\cup\, (a_1, +\infty) \}$$

with sequences

V nbh of ∞ if $\exists \{v_k\}_{k \in \mathbb{N}^*} \subset V$, $v_k \xrightarrow{k \to \infty} \infty$

sequence

Why fancy defs?

CAUCHY & WEIERSTRASS ~1820

$1, -1, 1, -1, \dots$

$$\underbrace{1 + (-1)}_{0} + \underbrace{1 + (-1)}_{0} + \dots$$

$$1 + \overbrace{(-1) + 1}^{0} + \overbrace{(-1) + \dots}^{0}$$

$$= \begin{cases} 0 \\ 1 \end{cases}$$

Def: limit of a sequence: ε-defs

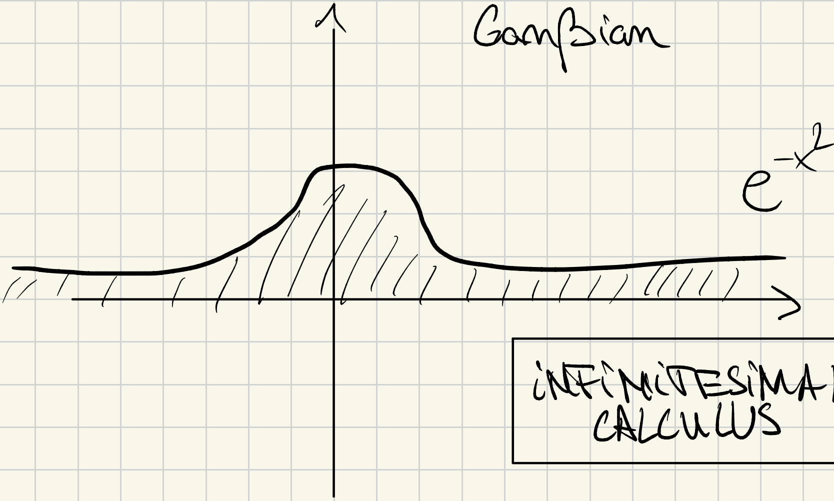
WHY? Differential / Integral (INFINITESIMAL) CALCULUS (Engl. term)

VS.

MATH. ANALYSIS (French term)

c. 1644 l'Hospital

Gaußian



$\boxed{\text{INFINITESIMAL CALCULUS}}$

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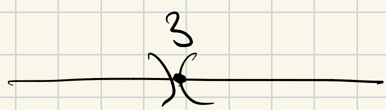
closure

A is closed $\Leftrightarrow \mathbb{R} \setminus A$ is open

ex $A = \{3\}$, $\mathbb{R} \setminus A =$



singleton is closed

9) d) $A = (-5, 5) \cap \mathbb{Z} = \{-4, -3, \dots, 0, \dots, 4\}$

$cl(A) = \{-4, 3, \dots, 0, 1, \dots, 4\}$

! ex $[0,1)$

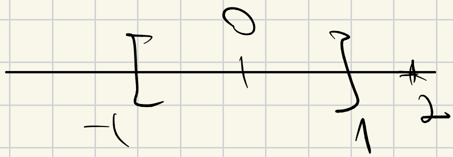
$\underbrace{\{1-\frac{1}{n}\}_{n=1,2,\dots,\infty}}_{x_n}$

$x_n \to 1 \Rightarrow cl[0,1) = [0,1]$

include all limits of sequences possible

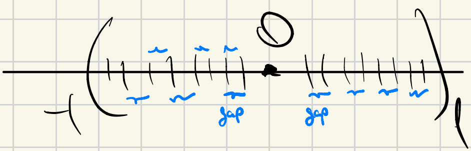
6) nbh of 0?

• $[-1,1]\cup\{2\}$



YES

• $(-1,1)\cap\mathbb{Q} = B$



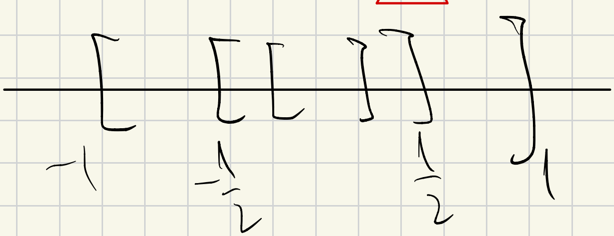
NO

$\forall a\in\mathbb{Q}\setminus\{0\}$

$(a,0)\not\subset B$

• $\bigcap_{n=0}^{\infty}\left[-\frac{1}{n},\frac{1}{n}\right] = \{0\}$ ⚠

NO

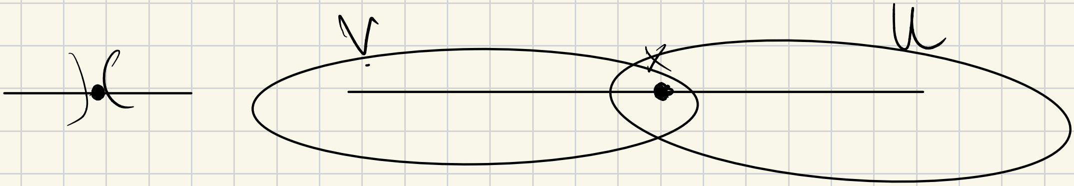


7) $U, V \in \mathcal{U}_{(x)}$, $x \in \mathbb{R}$

? $U \cap V \in \mathcal{U}_{(x)}$

def of $\mathcal{U}_{(x)}$: $x \in U \cap V$

$\exists (x-\varepsilon_i , x+\varepsilon_i) \subset U \cap V$



$U \subset \mathcal{U}_{(x)}$

$\exists \varepsilon_l^u > 0$ s.t. $(x-\varepsilon_l^u , x] \subset U$

$V \subset \mathcal{U}_{(x)}$

$\exists \varepsilon_l^v > 0$ s.t. $(x-\varepsilon_l^v , x] \subset V$

take $\min\{\varepsilon_l^u , \varepsilon_l^v\} = \varepsilon_l$

and $(\varepsilon , x] \subset U \cap V$