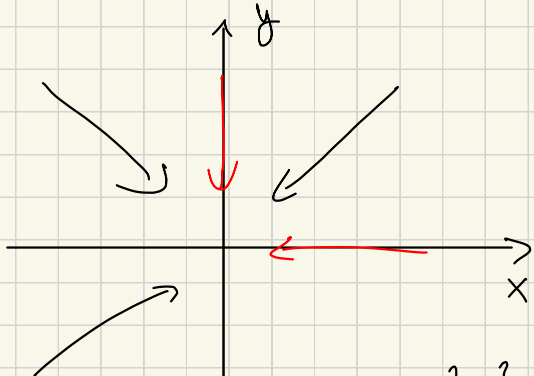
1) a) $\dfrac{x^2-y^2}{x^2+y^2}$



$y = mx$, $\dfrac{x^2-y^2}{x^2+y^2} =$

$= \dfrac{x^2 - m^2x^2}{x^2+m^2x^2} = \dfrac{1-m^2}{1+m^2}$ which depends on m $\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

or take $y=0$, $\lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$ (left/right)

$\neq$

$x=0$, $\lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$ (up/down) $\Rightarrow$

$\Rightarrow$ the way I approach the origin will change my limit $\Rightarrow$

$\Rightarrow \nexists \lim$.

b) $\dfrac{x+y}{x^2+y^2}$

$y=0 \Rightarrow \lim\limits_{x\to 0} \dfrac{x}{x^2} = \lim\limits_{x\to 0} \dfrac{1}{x}$ doesn't exist $\Rightarrow$

$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+y^2}$

c) $\dfrac{x^3+y^3}{x^2+y^2}$

$y=0$, $\lim\limits_{x\to 0} \dfrac{x^3}{x^2} = \lim\limits_{x\to 0} x = 0$, the same for $x=0$.

$\Rightarrow$ we could have a limit. $=0$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \underset{\to 0}{} \le \underbrace{|x|+|y|}_{\to 0}$$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x|^3+|y|^3}{x^2+y^2} = \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2}$$

$$= |x|\cdot\frac{x^2}{x^2+y^2} + |y|\cdot\frac{y^2}{x^2+y^2}$$

$$\le |x|+|y|$$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x - \sin y}{x-y} =$

$$\lim_{(x,y)\to(0,0)} \frac{\cos\left(\frac{x+y}{2}\right)\cdot \sin\left(\frac{x-y}{2}\right)}{\frac{x-y}{2}} = \cos 0 = 1$$

$\to 1$

$\dfrac{\sin t}{t} \to 1$, as $t\to 0$

a) $f(x,y) = e^{-(x^2+y^2)}$

$$\frac{\partial f}{\partial x} = e^{-(x^2+y^2)} \cdot \frac{\partial}{dx}\left[-(x^2+y^2)\right] = e^{-(x^2+y^2)} \cdot (-2x)$$

$$\frac{df}{dy} = e^{-(x^2+y^2)} \cdot \frac{d}{dy}\left[-(x^2+y^2)\right] = e^{-(x^2+y^2)} \cdot (-2y)$$

Defined $\forall (x,y) \in \mathbb{R}^2$

b) • $f(x,y) = \cos(x+y)$

$$\frac{df}{dx} = -\sin(x+y)$$

$$\frac{df}{dy} = -\sin(x+y)$$

• $\frac{df}{dx} = -\sin x \cos y - \cos x \sin y = -\sin(x+y)$

Defined $\forall (x,y) \in \mathbb{R}^2$

$$\frac{df}{dy} = -\cos x \cdot \sin y - \sin x \cdot \cos y = -\sin(x+y)$$

c) $\frac{df}{dx} = \left(\sqrt{x^2+y^2}\right)'_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{df}{dy} = \frac{y}{\sqrt{x^2+y^2}}$, defined $\forall (x,y) \neq (0,0)$

d) $\frac{df}{dx} = 2xyz$, $\frac{df}{dy} = x^2 z + e^z$

$\frac{df}{dz} = x^2 y + y e^z$ Defined $\forall (x,y,z) \in \mathbb{R}^3$

$$3)\ \underbrace{L = \lim_{(x,y)\to(0,0)} \frac{\overbrace{f(x,y) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(x-x_0,y-y_0)}^{(x-x_0)(y-y_0)}}{\|(x-x_0,y-y_0)\|} = 0}_{\text{We need to prove this}}$$

$f(x,y) - f(x_0,y_0) - (y_0,x_0)\cdot(x-x_0,y-y_0) =$

$= xy - x_0y_0 - y_0(x-x_0) - x_0(y-y_0)$

$= xy - \cancel{x_0y_0} - y_0x + \cancel{x_0y_0} - x_0y + x_0y_0$

$$= xy - xy_0 - x_0 y + x_0 y_0$$

$$= (x - x_0)(y - y_0)$$

$$\| (x - x_0, y - y_0) \| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$L = \lim_{(x,y)\to(x_0,y_0)} \frac{(x-x_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \lim_{(u,v)\to(0,0)} \frac{uv}{\sqrt{u^2+v^2}} = 0$$

$$uv = \frac{1}{2}\left((u+v)^2 - (u^2+v^2)\right)$$

$$\cdot \quad uv = \frac{1}{2}\left(\frac{(u+v)^2}{\sqrt{u^2+v^2}} - \sqrt{u^2+v^2}\right) \_ \_ \_ \_ \_$$

$$\cdot \ \text{Q} \quad u^2 + v^2 \geq 2 \cdot |uv|, \quad \frac{1}{\sqrt{u^2+v^2}} \leq \frac{1}{\sqrt{2} \cdot \sqrt{|uv|}} \Rightarrow$$

$$\Rightarrow 0 \leq \frac{|uv|}{\sqrt{u^2+v^2}} \leq \frac{\sqrt{|uv|}}{\sqrt{2}} \longrightarrow 0$$

4) Continuity: $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0) \Longrightarrow$

$\Longrightarrow$ cont. at $(0,0)$

$$\frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{|xy|}}{\sqrt{2}} \to 0$$

Partial derivatives at $(0,0)$:

$$\frac{df}{dx}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{df}{dy}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = 0$$

- assume f is diff at $(0,0)$, then $Df(0,0) = \nabla f(0,0) =$

$$= \left(\frac{df}{dx}(0,0), \frac{df}{dy}(0,0)\right) = (0,0)$$

$$\underbrace{\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \overbrace{f(0,0)}^{=0} - \overbrace{(0,0)\cdot(x,y)}^{0}}{\|x,y\|}}_{\text{definitions.}} \cancel{\neq} 0 \Rightarrow f \text{ is not diff. at the origin}$$

$$\Leftrightarrow \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|x,y\|} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \not\exists$$

Take $y = mx$, $\lim_{x\to 0} \frac{m\cancel{x^2}}{(1+m^2)\cancel{x^2}} = \frac{m}{1+m^2}$ - depends on $m \Rightarrow$

$\Rightarrow \lim \not\exists$.

5) a) $\frac{df}{dx} = -e^{-x}\cdot \sin(x+2y) + e^{-x}\cdot \cos(x+2y)$

$\frac{df}{dy} = 2\cdot e^{-x}\cdot \cos(x+2y)$

$\frac{df}{dx}(0, \frac{\pi}{4}) = -\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = -1$

$$\frac{df}{dy}\Big(0,\frac{\pi}{4}\Big) = 2\cdot \cos\frac{\pi}{2} = 0.$$

$$\Rightarrow \nabla f\Big(0,\frac{\pi}{4}\Big) = (-1, 0)$$

b) $$\frac{df}{dx} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\Big(\frac{-y}{x^2}\Big) \qquad (\arctan u)' = \frac{1}{1+u^2}\cdot u'$$

$$= \frac{\cancel{x^2}}{x^2+y^2}\cdot\frac{-y}{\cancel{x^2}} = \frac{-y}{x^2+y^2}$$

$$\frac{df}{dy} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\Big(\frac{y}{x}\Big)'_y = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x^{\cancel{2}}}{x^2+y^2}\cdot\frac{1}{\cancel{x}} =$$

$$= \frac{x}{x^2+y^2}$$

$$\nabla f(1,1) = \Big(\frac{-1}{2}, \frac{1}{2}\Big)$$

c) $$\frac{df}{dx} = yz\cdot e^{xyz} \qquad \frac{df}{dz} = xy\cdot e^{xyz}$$

$$\frac{df}{dy} = xz\, e^{xyz}$$

$\nabla f(0,0,0) = (0,0,0)$

d) $\frac{df}{dx} = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $\frac{df}{dy} = \frac{y}{\sqrt{x^2+y^2+z^2}}$

$\frac{df}{dz} = \frac{z}{\sqrt{x^2+y^2+z^2}}$, $\nabla f(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$