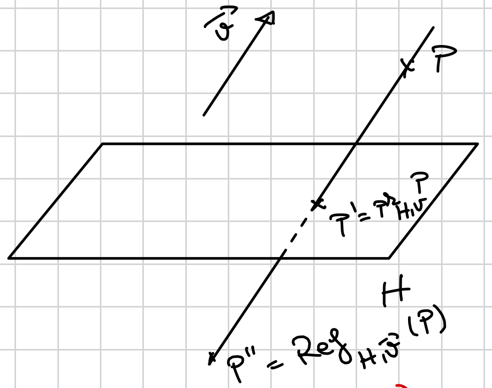
Seria 8: Cap 5 ex 1, 2, 4, 5, 11, 12, 13, 14, 16

5.1. Consider an orthonormal coordinate system K of E^n where $n = 2$ or 3. Starting from the matrix form of the projections and reflexions, deduce the matrices of:

a) the orthogonal projections on the coordinate axes and on the coordinate hyperplanes of K



$$pr_{H,\vec{v}}P = \left(J_n - \frac{\vec{a} \otimes \vec{v}}{\langle \vec{a}, \vec{v} \rangle}\right) \cdot P - \frac{a_{n+1}}{\langle \vec{a}, \vec{v} \rangle} \cdot \vec{a}$$

$\vec{a} \rightarrow$ normal vector

$\vec{v} \rightarrow$ direction vector

hyperplane: $H : a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a_{n+1} = 0$

$$pr_H^{\perp} = \left(J_n - \frac{\vec{a} \otimes \vec{a}}{\langle \vec{a}, \vec{a} \rangle}\right) \cdot P - \frac{a_{n+1}}{\langle \vec{a}, \vec{a} \rangle} \cdot \vec{a}$$

$xOy : z = 0$

$yOz : x = 0$

$xOz : y = 0$

$Ox : \begin{cases} y = 0 \\ z = 0 \end{cases}$

$Oy : \begin{cases} x = 0 \\ z = 0 \end{cases}$

$Oz : \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\vec{a} = (0, 0, 1)$

$$\vec{a} \otimes \vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0\ 0\ 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\langle \vec{a}, \vec{a} \rangle = 1$ $\qquad a_{n+1} = 0$

$$pr_{(xOy),\vec{a}} P = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P$$

$$pr_{(xOy)} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$$

$$pr_{l,w} P = \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle} \cdot P + \left(J_n - \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle} \right) \cdot Q$$

$\vec{v}$ director vector of l

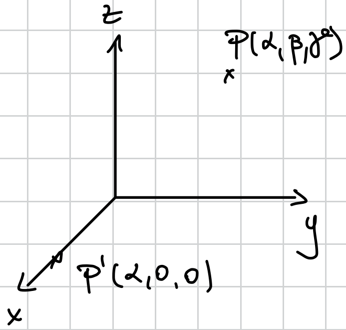
$\vec{a}$ normal vector of w

$w: \ a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a_{n+1} = 0$

Q : point from l

$$pr_{Ox}^{\perp}(P) = \frac{\vec{a} \otimes \vec{a}}{\langle \vec{a}, \vec{a} \rangle} \cdot P$$

We chose $Q(0,0,0)$



$W: \ yOz: \ x = 0 \ \Rightarrow \vec{a}(1,0,0)$

$Ox: \begin{cases} x = \lambda \\ y = 0 \\ z = 0 \end{cases} \Rightarrow \Delta(Ox) = \langle (1,0,0) \rangle$

$\Rightarrow \vec{a}(1,0,0)$

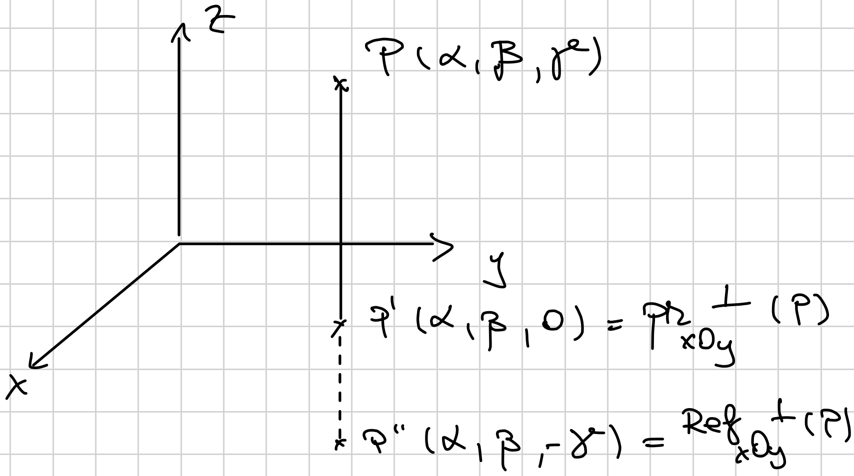
$\langle \vec{a}, \vec{a} \rangle = 1$

$$\vec{a} \otimes \vec{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow pr_{O_x}^{\perp}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P$$

$$Ref_{l,w}(P) = \left(-J_n + 2 \cdot \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle} \right) \cdot P + 2 \left(J_n - \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle} \right) Q$$

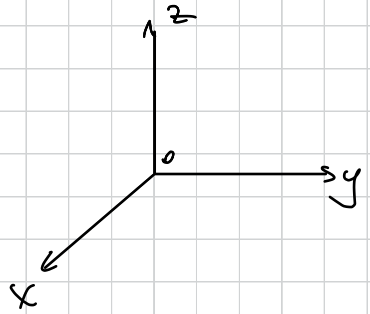
$$Ref_H(P) = \left(J_n - 2 \frac{\vec{v} \otimes \vec{a}}{\langle \vec{v}, \vec{a} \rangle} \right) \cdot P - 2 \frac{a_{n+1}}{\langle \vec{v}, \vec{a} \rangle} \vec{v}$$



$\vec v = \vec a(0,0,1) \Rightarrow Ref^{\perp}_{xOy}(P) = \left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\0&0&2\end{pmatrix}\right)P = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\cdot P$

5.2 Consider the vector $\vec v(2,1,1)$

Give the matrix form for the parallel projection / reflection on the $\pi: z=0$ parallel to $\vec v$



$H: z=0 \Rightarrow \vec a(0,0,1)$ $\quad a_{n+1}=0$

$\vec v(2,1,1)$

$$Pr_{(xOy,\vec v)} P = \left(I_n - \frac{\vec a \otimes \vec v}{\langle \vec a, \vec v\rangle}\right)P - \frac{a_{n+1}}{\langle \vec a, \vec v\rangle}\cdot \vec a$$

$$\vec a \otimes \vec v = \begin{pmatrix}0\\0\\1\end{pmatrix}(2\ 1\ 1) = \begin{pmatrix}0&0&0\\0&0&0\\2&1&1\end{pmatrix}$$

$\langle \vec a, \vec v\rangle = 1$

$$\Rightarrow pr_{xOy,\vec v}(P) = \left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\2&1&1\end{pmatrix}\right)P = \begin{pmatrix}1&0&0\\0&0&0\\-2&-1&0\end{pmatrix}\cdot P$$

$$Ref_{xOy,\vec v}(P) = \left(I_n - 2\frac{\vec v \otimes \vec a}{\langle \vec v, \vec a\rangle}\right)\cdot P - 2\underbrace{\frac{a_{n+1}}{\langle \vec v, \vec a\rangle}}_{0}\vec v$$

$$= \left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&4\\0&0&2\\0&0&2\end{pmatrix}\right)P = \begin{pmatrix}1&0&-4\\0&1&-2\\0&0&-1\end{pmatrix}P$$

$$\vec v \otimes \vec a = \begin{pmatrix}2\\1\\1\end{pmatrix}(0\ 0\ 1) = \begin{pmatrix}0&0&2\\0&0&1\\0&0&1\end{pmatrix}$$

5.4. Det the orthogonal reflexion of $P(6,-5,5)$ in the plane $\pi: 2x-3y+z-4=0$ by det the matrix of reflectio

$$Ref_{\pi}(P) = \left(I_3 - 2\frac{\vec v \otimes \vec a}{\langle \vec v, \vec a\rangle}\right)\cdot P - 2\frac{a_{n+1}}{\langle \vec v, \vec a\rangle}\cdot \vec v$$

$\vec{v} = \vec{a}\,(2, -3, 1) \qquad a_{n+1} = -4$

$$\vec{a} \otimes \vec{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} (2\ \ -3\ \ 1) = \begin{pmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\langle \vec{a}, \vec{a} \rangle = 14$$

$$Ref_{\pi}(P) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix} \cdot \frac{\cancel{2}}{\cancel{14}\ 7} \right) P - \cancel{2} \cdot \frac{(-4)}{\cancel{14}\ 7} \cdot \vec{a}$$

$$= \begin{pmatrix} \frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \\ \frac{6}{7} & -\frac{2}{7} & \frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & \frac{6}{7} \end{pmatrix} \cdot P + \frac{4}{7} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 3 & 6 & -2 \\ 6 & -2 & 3 \\ -2 & 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -5 \\ 5 \end{pmatrix} + \frac{4}{7} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

5.5 Consider an orthonormal coordinate system K of $\mathbb{E}^2$. Starting from the matrix form of the projection and reflection, show that

$$Pr_{\vec{a}}^{\perp}(\vec{b}) = \frac{\langle \vec{a}, \vec{b} \rangle}{\langle \vec{a}, \vec{a} \rangle} \cdot \vec{a}$$

Let $\vec{a}\,(a_1, a_2) \Rightarrow \vec{a}^{\perp}\,(a_2, -a_1) \perp \vec{a}$

$$Pr_{H}^{\perp}(P) = \left(I_n - \frac{\vec{a} \otimes \vec{a}}{|\vec{a}|^2} \right) P - \frac{a_{n+1}}{|\vec{a}|^2} \cdot \vec{a} \qquad a_{n+1} = 0$$

Rolul lui $\vec{a}$ din formulă e jucat în cazul nostru de $\vec{a}^{\perp}$

$$|\vec{a}^{\perp}|^2 = a_2^2 + a_1^2$$

$$\vec{a}^{\perp} \otimes \vec{a}^{\perp} = \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} (a_2\ \ -a_1) = \begin{pmatrix} a_2^2 & -a_1a_2 \\ -a_1a_2 & a_1^2 \end{pmatrix}$$

$$Pr_{H}^{\perp}(P) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a_2^2 & -a_1a_2 \\ -a_1a_2 & a_1^2 \end{pmatrix} \cdot \frac{1}{a_1^2 + a_2^2} \right) \cdot P$$

$$= \frac{1}{a_1^2 + a_2^2} \begin{pmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{pmatrix}$$

$$Pr_{\vec{a}}^{\perp}(\vec{b}) = \frac{1}{a_1^2+a_2^2}\begin{pmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Sem 9: C5 ex: 16, 18, 19, 20.3, 22, 23, 24

5.16. Let F be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a translation with vector $(-2,5)$. Get the inverse transformation, F^{-1}!

$F = Rot_{(-\frac{\pi}{3})} \circ T(-2,5) : \mathbb{E}^2 \to \mathbb{E}^2$ $\qquad (f\circ g)^{-1} = g^{-1}\circ f^{-1}$

$$F^{-1} = \left(Rot(-\tfrac{\pi}{3}) \circ T(-2,5)\right)^{-1} = T(-2,5)^{-1} \circ Rot(-\tfrac{\pi}{3})^{-1} = T(2,-5)\circ Rot(\tfrac{\pi}{3})$$

$$Rot(\tfrac{\pi}{3}) = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

5.18. Get the cosine of the angle of the rotation f

$$f(x) = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\cos\theta = \frac{1}{2}\, tr(\text{lin} f)$$

$$= \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{3}} \cdot \cancel{4}^{2} = \frac{2\sqrt{3}}{3}$$

$f(x) = Ax+b \qquad \tilde{f}(x) = \tilde{A}x + \tilde{b}$

$(f\circ\tilde{f})(x) = A(\tilde{A}x+\tilde{b})+b \qquad$ and $\qquad (\tilde{f}\circ f)(x) = \tilde{A}(Ax+b)+\tilde{b}$

$= A\tilde{A}x + A\tilde{b}+b \qquad\qquad = \tilde{A}Ax + \tilde{A}b+\tilde{b}$

We need $\tilde{f} = f^{-1} \Rightarrow \tilde{f}\circ f = J_2 \Rightarrow \begin{cases} \tilde{A}A = J_2 \\ \tilde{A}b+\tilde{b} = 0 \end{cases}$

$$A = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2\end{bmatrix} \qquad \det A = \frac{1}{13}\cdot 13 = 1$$

$${}^tA = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & 3 \\ -3 & 2\end{bmatrix} \qquad A' = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ -3 & 2\end{pmatrix} = \tilde{A} \Rightarrow \tilde{b} = -\tilde{A}b$$

$$\Rightarrow \tilde{b} = -\frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ -3 & 2\end{pmatrix}\cdot\begin{bmatrix} 1 \\ -2\end{bmatrix}$$

$$= -\frac{1}{\sqrt{13}}\begin{bmatrix} -4 \\ -7\end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 4 \\ 7\end{bmatrix} \Rightarrow \tilde{f} = f^{-1} = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ -3 & 2\end{pmatrix}X + \frac{1}{\sqrt{13}}\begin{bmatrix} +4 \\ +7\end{bmatrix}$$

5.19. Verify that $A, B \in SO(3)$. Det the axis of rotation and the rotation angle.

$$A \in SO(3) \iff \begin{cases} \det(A) = 1 \\ {}^tA\cdot A = I_3\end{cases}$$

$$\cos\theta = \frac{1}{2}\left[tr(\underbrace{lin f_1}_{A}) - 1\right]$$

Let $v_1 = \begin{pmatrix} x \\ y \\ z\end{pmatrix}$ be the axis of rotation $\Rightarrow v_1 = A\cdot v_1 \Rightarrow$ sistem

5.22. Use Euler-Rodrigues formula and write down the matrix form of a rotation around the axis $\mathbb{R}\vec{v}$, where $\vec{v} = (1,1,0)$. Use this matrix form to give a parametrization of a cylinder with axis $\mathbb{R}\vec{v}$ and diameter $\sqrt{2}$.

Euler-Rodrigues: $Rot_{\vec{v},\theta}(P) = \cos(\theta)P + \sin(\theta)(\vec{v}\times P) + (1-\cos(\theta))\cdot\langle\vec{v},P\rangle\vec{v}$

$\vec{v}$ - unit vector

$\theta \in \mathbb{R} \to$ angle

$\mathbb{R}\vec{v} \to$ rotation axis

$$Rot_{\vec{v},\theta}(P) = \cos\theta\begin{bmatrix} x \\ y \\ z\end{bmatrix} + \sin\theta\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ x & y & z\end{bmatrix} + (1-\cos\theta)(x+y)\cdot\underbrace{\vec{v}}_{\begin{bmatrix}1\\1\\0\end{bmatrix}}$$

$$= \cos\theta\begin{bmatrix} x \\ y \\ z\end{bmatrix} + \sin\theta(z\vec{i} - z\vec{j} + (y-x)\vec{k}) + (1-\cos\theta)\cdot\begin{bmatrix} x+y \\ x+y \\ 0\end{bmatrix}$$

$$= \cos\theta\begin{bmatrix} x \\ y \\ z\end{bmatrix} + \sin\theta\begin{bmatrix} z \\ -z \\ y-x\end{bmatrix} + (1-\cos\theta)\begin{bmatrix} x+y \\ x+y \\ 0\end{bmatrix}$$

$$= \begin{bmatrix} x\cos\theta + z\sin\theta + x + y - x\cos\theta - y\cos\theta \\ y\cos\theta - z\sin\theta + x + y - x\cos\theta - y\cos\theta \\ z\cos\theta + y\sin\theta - x\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} x + y + z\sin\theta - y\cos\theta \\ x + y - z\sin\theta - x\cos\theta \\ -x\sin\theta + y\sin\theta + z\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 1-\cos\theta & \sin\theta \\ 1-\cos\theta & 1 & -\sin\theta \\ -\sin\theta & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let $P'(u_p, v_p, w_p)$ $\quad u_p^2 + v_p^2 + w_p^2 = 2$ → sphere

??? $u_p + v_p = 0$

$\Downarrow$

$P'(1,-1,0)$

Let $\Delta \parallel \vec{v}$, $P \in \Delta$ $\Rightarrow$ $\Delta : \begin{cases} x = 1 + \lambda \\ y = -1 + \lambda \\ z = 0 \end{cases}$ $\quad \lambda \in \mathbb{R}$

replace them here

⇒ parametrize idk what

$\varphi : E^n \longrightarrow E^n$ → affine morphism

$\varphi(P) = A \cdot P + b$, $A \in M_n(\mathbb{R})$, $b \in \mathbb{R}^n$

φ isometry $\Leftrightarrow \forall\, P, Q \in E^n$: $\text{dist}(\varphi(P), \varphi(Q)) = \text{dist}(P, Q)$

$\Leftrightarrow A \in O(n) = \{ M \in M_n(\mathbb{R}) \mid M^{-1} = M^T \} \Leftrightarrow A \cdot A^T = I_n$

- $\det(A) = 1 \Rightarrow$ direct isometry
- $\det(A) = -1 \Rightarrow$ indirect isometry

for $n=2$ $\quad \varphi : E^2 \longrightarrow E^2$ → direct isometry:
- identity
- translation: $T_{\vec{v}}$
- rotation around a point C with an angle θ: $Rot_{C,\theta}$

→ indirect isometry: - reflection w.r. to a line ℓ: Ref_ℓ

- glide-reflection: $T_{\vec{v}} \circ Ref_\ell$, $\vec{v} \in \Delta(\ell)$

φ - rotation $\Rightarrow tr A = 2 \cdot \cos\theta$

$Fix(\varphi) = \{ P \in E^n \mid \varphi(P) = P \}$ → fixed points

- identity: all
- translation: none
- $Rot_{C,\theta}$: C
- reflection: line
- glide: none

5.16. F isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a translation with vector $(-2,5)$

Find F^{-1}

$$F = Rot_{(-\frac{\pi}{3})} \circ T_{(-2,5)}$$

$$F^{-1} = T_{(-2,5)}^{-1} \circ Rot_{(-\frac{\pi}{3})}^{-1}$$
$$= T_{(2,-5)} \circ Rot_{(\frac{\pi}{3})}$$

trick: $f(P) = A \cdot P + b$ $\quad \left(\begin{array}{c|c} A & b \\ \hline 0 0 .. 0 & 1 \end{array}\right)$

$$\hat{f}(\hat{P}) = \left(\begin{array}{c|c} A & b \\ \hline 0 0 .. 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c} P \\ \hline 1 \end{array}\right)$$

$$\left[\hat{T}_{(2,-5)} \right] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[\hat{Rot}_{(0,\frac{\pi}{3})} \right] = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[\hat{F}^{-1} \right] = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 2 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow F^{-1}(P) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \cdot P + \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

5.18. $f: E^2 \to E^2$ $\quad f(p) = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 5 \end{pmatrix} \cdot p + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Show that f is a rotation, find its centre and angle

$A = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$ $\qquad {}^tA = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$

$A \cdot {}^tA = \frac{1}{25}\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix} = I_2$

$\det A = \frac{1}{25} \cdot 25 = 1 \Rightarrow f$ direct isometry

$\underbrace{A \in SO(3)}_{\text{special orthogonal}}$

$fix(f) = \{ p \in E^2 \mid f(p) = p \}$

Let $p = \begin{pmatrix} x \\ y \end{pmatrix}$

$f(p) = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\frac{1}{5}\begin{pmatrix} 3x - 4y \\ 4x + 3y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{cases} 3x - 4y = 5x - 5 \\ 4x + 3y = 5y + 10 \end{cases} \Rightarrow \begin{cases} 2x + 4y - 5 = 0 \\ 4x - 2y - 10 = 0 \end{cases}$ $(-)$

$-10y = 0 \Rightarrow y = 0$

$\Rightarrow x = \frac{5}{2}$

$\Rightarrow fix(f) = \{ (\frac{5}{2}, 0) \}$

$\Rightarrow f$ rotation

$\cos\theta = \frac{tr A}{2} = \frac{3}{5}$

Check if f is a rotation $\begin{cases} 1) \text{ check } A \in SO(n) \\ 2) \text{ find } fix(f) \end{cases}$

for $n=3$ $\quad f: E^3 \to E^3$ isometry

→ direct : - identity

- translation $T_{\vec{v}}$

- rotation around the axis l with angle θ : $Rot_{l,\theta}$

- glide-rotation $\quad T_{\vec{v}} \circ Rot_{l,\theta}$

$\vec{v} \in D(l)$

→ indirect : - reflection w.r.t a plane : Ref_{π}

- glide-reflection : $T_{\vec{v}} \circ Ref_{\pi}$

$\vec{v} \in D(\pi)$

- rotation-reflection : $Rot_{l,\theta} \circ Ref_{\pi}$

$l \perp \pi$

$Tr A = 2\cos\theta + 1$ $\qquad Rot_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5.19 Verify that the matrix $A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \in SO(3)$

Find the axis and angle of rotation

$A \cdot A^T = I_3 \qquad \det A = 1 \Rightarrow \quad A \in SO(3)$

$v \cdot A = v \qquad \Rightarrow v = (-\alpha, 0, 2\alpha) \quad \alpha \in \mathbb{R}$

$\Rightarrow l$ is a line $\Rightarrow A$ is a rotation

$\cos\theta = \frac{1}{2}(Tr A - 1)$

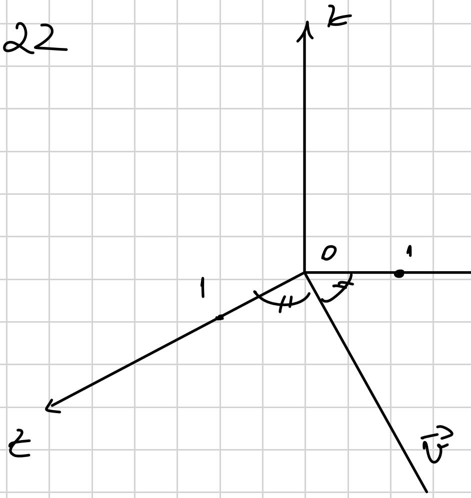
$= \frac{1}{2}\left(-\frac{1}{3} - 1\right) = -\frac{2}{3}$

Euler – Rodrigues : $Rot_{l,\theta}(P) = \cos(\theta) \cdot P + \sin(\theta) \cdot (\vec{v} \times P) + (1-\cos\theta)\langle \vec{v}, P\rangle \vec{v}$

$\vec{v} \in D(l) \qquad \vec{v}$ - unit vector

$|\vec{v}| = 1$

5.22



Normalize

$\vec{v}(1,1,0)$ $\quad |\vec{v}| = \sqrt{2}$

$\vec{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$$Rot_{l,\theta}(P) = \cos(\theta)\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} + \sin(\theta)\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ x & y & z\end{vmatrix} + (1-\cos\theta)\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$$

$$= M\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Sem 10: Chapter 6 ex: 1, 3, 4, 5, 6, 7, 8, 9, 10

Quadratic curves in $\mathbb{E}^2$ (ellipses)

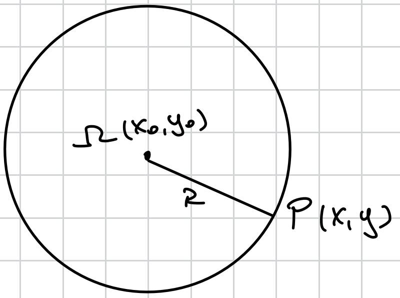
Q quadratic :

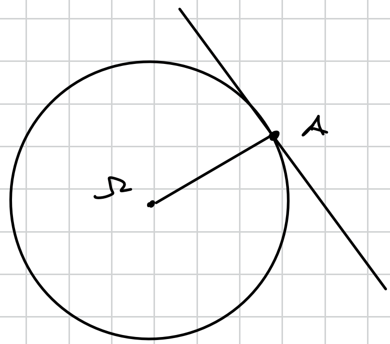
$$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$$

Circle: locus of points in the plane whose distance to a fixed point $\Omega(x_0, y_0)$ is a constant $R > 0$

$$\mathcal{C}(\Omega, R): (x-x_0)^2 + (y-y_0)^2 = R^2$$

parametric: $\mathcal{C}(\Omega, R): \begin{cases} x = x_0 + R\cos t \\ y = y_0 + R\sin t \end{cases}$





$T_{\mathcal{C},A} \perp \Omega A$

6.1. Find the equation of the circle

a) of diameter $[AB]$; $A(-1,2)$ $B(-3,-1)$

$AB = \sqrt{4^2+3^2} = \sqrt{25} = 5$ $\Rightarrow R = \frac{5}{2}$

M middle of $[AB]$ $\Rightarrow$ M center $\Rightarrow \begin{cases} x_M = -2 \\ y_M = \frac{1}{2} \end{cases}$

$\mathcal{C}(M,5): (x+1)^2 + (y-\frac{1}{2})^2 = \left(\frac{5}{2}\right)^2$

b) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $l: 3x-y-2=0$

$\Rightarrow \begin{cases} 3x_0 - y_0 - 2 = 0 \\ (3-x_0)^2 + (1-y_0)^2 = R^2 \\ (-1-x_0)^2 + (3-y_0)^2 = R^2 \end{cases} \Rightarrow \begin{cases} 3x_0 - y_0 - 2 = 0 \\ 9 - 6x_0 + x_0^2 + 1 - 2y_0 + y_0^2 = R^2 \\ 1 + 2x_0 + x_0^2 + 9 - 6y_0 + y_0^2 = R^2 \end{cases} (-)$

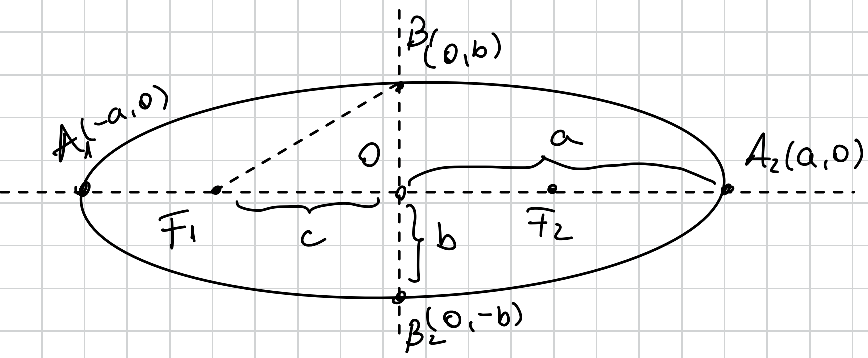
$\Rightarrow \begin{cases} 3x_0 - y_0 - 2 = 0 \\ 8x_0 - 4y_0 = 0 \end{cases} \Rightarrow y_0 = 2x_0 \quad \Rightarrow \begin{cases} x_0 = 2 \\ y_0 = 4 \end{cases}$

$\Rightarrow R^2 = (3-2)^2 + (1-4)^2$
$= 1 + 9 = 10$

$\mathcal{C}: (x-2)^2 + (y-4)^2 = 10$

sau: căutăm ediatoarea segmentului

<u>Ellipse.</u> locus of points in the plane whose sum of distances to 2 distinct points (the focal points, the foci) is a constant $2a$



If we fix a reference system $K = (O, \vec{i}, \vec{j})$ where O is the midpoint of F_1F_2 and $\vec{i} \in D(F_1F_2)$ then we have the canonical eq:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \text{ where } a,b>0, \; a \geq b \qquad c = \sqrt{a^2-b^2}$$

Proof. $B_1F_1 + B_2F_2 = 2a$

$B_1F_1 = B_2F_2 = \sqrt{c^2+b^2}$

$\sqrt{c^2+b^2} = a \Rightarrow c^2 = a^2 - b^2$

$$\boxed{e = \frac{c}{a} = \text{eccentricity}}$$

$e = 0 \Rightarrow$ circle

6.3. Set the foci of the ellipse $9x^2+25y^2=225$

$9x^2+25y^2 = 225 \quad |:225$

$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1 \Rightarrow c = \sqrt{5^2-3^2} = 4 \Rightarrow F_1 = (-4, 0)$
$F_2 = (4, 0)$

6.4. Set the intersection of the line $l: x+2y-7=0$ and the ellipse $\mathcal{E}: x^2+3y^2-25=0$

$\begin{cases} x+2y-7=0 \Rightarrow x = 7-2y \\ x^2+3y^2-25=0 \end{cases} \Rightarrow (7-2y)^2+3y^2-25=0$

$7y^2 - 28y + 24 = 0$

$\Delta = 784 - 672$
$= 112 \Rightarrow y_{1,2} = \dots$

$l: y = kx+m$

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow l \cap \mathcal{E} = \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = kx+m \end{cases}$

$\Rightarrow \begin{cases} \frac{x^2}{a^2} + \frac{(kx+m)^2}{b^2} = 1 \\ y = kx+m \end{cases} \Rightarrow \begin{cases} x^2\left(\frac{1}{a^2}+\frac{k^2}{b^2}\right) + \frac{2km}{b^2}\cdot x + \frac{m^2}{b^2} - 1 = 0 \\ y = kx+m \end{cases}$

$\Delta = \frac{4k^2m^2}{b^4} - 4 \cdot \frac{b^2+a^2k^2}{a^2b^2} \cdot \frac{m^2-b^2}{b^2}$

$$= \frac{4}{b^4}\left(k^2m^2 - \frac{(b^2+a^2k^2)(m^2-b^2)}{a^2}\right)$$

$$= \frac{4}{a^2b^4}\left(\cancel{a^2k^2m^2} - b^2m^2 + b^4 - \cancel{a^2k^2m^2} - b^2a^2k^2\right)$$

$$\Delta = \frac{4}{a^2b^2}\left(b^2 - m^2 + a^2k^2\right)$$

$b^2 - m^2 + a^2k^2$	intersection
< 0	none
$= 0$	1 intersection point (tangent)
> 0	2 intersection points (secant)

If l tangent to $\mathcal{E}$ $\Rightarrow b^2 = m^2 - a^2k^2$

$$\Rightarrow m = \pm\sqrt{b^2+a^2k^2}$$

The eq of the tangent line with the slope k is: $y = kx \pm \sqrt{b^2+a^2k^2}$

$$T_{(x_0,y_0)}\mathcal{E}_{a,b} = \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

$$\Leftrightarrow \quad \frac{x_0}{a^2}(x-x_0) + \frac{y_0}{b^2}(y-y_0) =$$

the tangent to $\mathcal{E}_{a,b}$ in the point $(x_0,y_0) \in \mathcal{E}_{a,b}$

6.6. Det the eq. of a line that is orthogonal to $l: 2x-2y-13=0$ and tangent to the ellipse $\mathcal{E}: x^2+4y^2-20=0$

$m_l = 1 \Rightarrow m_t = -1$

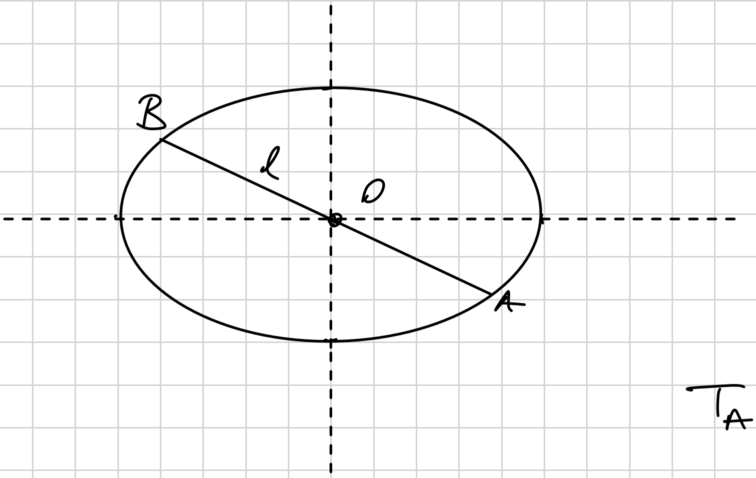
$$\mathcal{E}: \frac{x^2}{20} + \frac{y^2}{5} = 1$$

$$m = \pm\sqrt{5+20} = \pm 5$$

$$t: y = -x \pm 5$$

6.7. A diameter of an ellipse is the line segment det. by the intersection points of the ellipse with the line passing through the center of the ellipse.

Show that the tangent lines to an ellipse at the endpoints of the diameter are parallel



Show that $T_{A}\mathcal{E} \parallel T_{B},\mathcal{E}$

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$l: y = kx$

$T_{A,\mathcal{E}}: \frac{x x_A}{a^2} + \frac{y y_A}{b^2} = 1$

$T_{A,\mathcal{E}}: \frac{x x_A}{a^2} + y\frac{k x_A}{b^2} = 1$

$\Rightarrow m_{T_{A,\mathcal{E}}} = -\frac{\cancel{x_A}}{a^2} \cdot \frac{b^2}{k\cancel{x_A}}$

$= -\frac{b^2}{ka^2}$

$T_{B,\mathcal{E}}: \frac{x x_B}{a^2} + \frac{y k x_B}{b^2} = 1$

$\Rightarrow m_{T_{B,\mathcal{E}}} = -\frac{b^2}{ka^2}$

$\Rightarrow T_{A,\mathcal{E}} \parallel T_{B,\mathcal{E}}$

6.* Consider the ellipse $\mathcal{E}: x^2+4y^2=1$. Find the lines that contain $P(2,3)$ and are tangents to $\mathcal{E}$.

$\mathcal{E}: \frac{x^2}{1^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$

$T_{A,\mathcal{E}}: \frac{x x_A}{1^2} + \frac{y y_A}{\left(\frac{1}{2}\right)^2} = 1$

$P \in T_{A,\mathcal{E}}$

$\Rightarrow 2x_A + 12y_A = 1$

$$A: \begin{cases} 2x_A + 12y_A = 1 \\ x_A^2 + 4y_A^2 = 1 \end{cases} \Rightarrow x_A = \frac{1}{2} - 6y_A$$

$$\Rightarrow \left(\frac{1}{2} - 6y_A\right)^2 + 4y_A^2 = 1$$

$$40y_A^2 - 6y_A + \frac{1}{4} = 1$$

<u>SAU</u>:

$l_P : y - 3 = m(x-2)$

or

$l_P : x = 2$

If $l_P : x = 2 \Rightarrow l_P \cap \mathcal{E} : \begin{cases} x = 2 \\ 4 + 4y^2 = 1 \end{cases} \Rightarrow l_P \cap \mathcal{E} = \emptyset$

If $l_P : y - 3 = m(x-2)$

$$l_P \cap \mathcal{E} : \begin{cases} y = m(x-2) + 3 \\ x^2 + 4y^2 = 1 \end{cases} \Rightarrow \begin{cases} y = 3 + m(x-2) \\ x^2 + 4\left(9 + m^2(x-2)^2 + 6m(x-2)\right) = 1 \end{cases}$$

$$\Rightarrow x^2(1 + 4m^2) + x(-16m^2 + 24m) + 36 - 4m^2 - 12m = 0$$

$$\Rightarrow \Delta = 64(2m^2 - 3m)^2 - 4(1 + 4m^2)(-4m^2 - 12m + 36)$$

next we solve $\Delta = 0$ and get m

Sem 10: Cap 6 ex: $1 \to 10$ $- \{2\}$

6.1. Find the eq of a circle of diameter [AB] with $A(1,2)$ and $B(-3,-1)$

a) $Q(-1, \frac{1}{2})$ mid point of AB

$AB = \sqrt{4^2+3^2} = 5$

$\Rightarrow \mathcal{C}(M, \frac{5}{2}) : (x+1)^2 + (x-\frac{1}{2})^2 = (\frac{5}{2})^2$

b) $\mathcal{C}(i, 7) : (x+1)^2 + (y-2)^2 = 49$

$i(-1,2)$

c) $i(-1,2)$ center and passing through $A(2,6)$

$R = iA = \sqrt{25} = 5$

$\mathcal{C}(i,5) : (x+1)^2 + (y-2)^2 = 25$

d) center $\to$ origin and tangent to $l: 3x-4y+20=0$

$d(0,l) = \frac{20}{5} = 4$

$\mathcal{C}(0,4) = x^2+y^2 = 16$

e) $A(3,1)$, $B(-1,3)$ center on the line $l: 3x-y-2=0$

$$\Rightarrow \begin{cases} 3x_0 - y_0 - 2 = 0 \\ (3-x_0)^2 + (1-y_0)^2 = R^2 \\ (-1-x_0)^2 + (3-y_0)^2 = R^2 \end{cases} \Rightarrow \begin{cases} y_0 = 3x_0 - 2 \\ x_0^2 + 9 - 6x_0 + y_0^2 + 1 - 2y_0 = R^2 \\ x_0^2 + 1 + 2x_0 + y_0^2 + 9 - 6y_0 = R^2 \end{cases} (-)$$

$$\Rightarrow \begin{cases} y_0 = 3x_0 - 2 \\ 8x_0 - 4y_0 = 0 \end{cases} \Rightarrow 8x_0 - 12x_0 + 8 = 0$$

$-4x_0 = -8$

$x_0 = 2$

$\Rightarrow y_0 = 4$

$\Rightarrow O(2,4)$

$R^2 = 10$

$\Rightarrow \mathcal{C}: (x-2)^2 + (y-4)^2 = 10$

f) $A(1,1)$ $B(1,-1)$ $C(2,0)$

$$\mathcal{C}: \begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow \mathcal{C}: \begin{vmatrix} x^2+y^2 & x & y & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 4 & 2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x^2+y^2 & x & y & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -1 & 1 \\ 4 & 2 & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow -2 \cdot \begin{vmatrix} x^2+y^2 & x & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 0$

$-2\left(x^2+y^2+4+4x-4-2x^2-2y^2-2x \right) = 0$

$\Rightarrow -2\left(-x^2-y^2+2x \right) = 0$

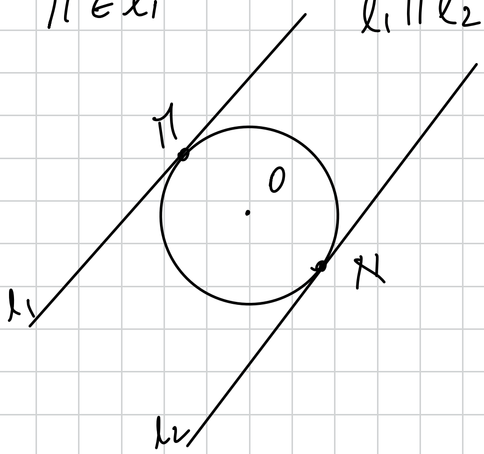
$x^2+y^2-2x = 0$

$(x-1)^2+y^2 = 1$

g) tangent to $l_1: 2x+y-5=0$ $\quad$ $M(3,-1)$ tangency point

$l_2: 2x+y+15=0$

$M \in l_1$ $\quad$ $l_1 \parallel l_2$



$m_{l_1} = -2$

let $d \perp l_2$ $\Rightarrow m_d = \frac{1}{2}$

$M \in d$

$\Rightarrow d: y - y_M = \frac{1}{2}(x - x_M)$

$y+1 = \frac{1}{2}(x-3)$

$2y+2 = x-3$

$N \in l_2 \cap \mathcal{C} \Rightarrow \left.\begin{matrix} N \in d \\ N \in \mathcal{C} \end{matrix}\right\} \Rightarrow \begin{cases} x-2y-5=0 \\ 2x+y+15=0 \quad | \cdot 2 \end{cases}$

$\Rightarrow 5x+25=0$

$\Rightarrow x=-5 \Rightarrow y=-5$

$\Rightarrow N(-5,-5)$ $\quad$ O midpoint of MN $\Rightarrow O(-1,-3)$ $\quad \Rightarrow R^2 = 20$

$\Rightarrow \mathcal{C}: (x+1)^2+(y+3)^2 = 20$

6.3. Det the foci of the ellipse $\mathcal{E}: 9x^2+25y^2-225=0$

$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad c=\sqrt{a^2-b^2}$

$9x^2+25y^2=225 \mid :225$

$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1 \quad \Rightarrow c=\sqrt{25-9}=\sqrt{16}=4$

$\Rightarrow F_1(-4,0)$
$F_2(4,0)$

6.4. Det $l \cap \mathcal{E}$ $\qquad l: x+2y-7=0$
$\mathcal{E}: x^2+3y^2-25=0$

$\Rightarrow \begin{cases} x=7-2y \\ x^2+3y^2-25=0 \end{cases} \Rightarrow (7-2y)^2+3y^2-25=0$

$7y^2-28y+24=0$

$\Delta=784-672=112$

$y_{1,2}=\frac{28\pm\sqrt{112}}{14}=2\pm\frac{4\sqrt{7}}{14}=2\pm\frac{2\sqrt{7}}{7}$

$y=2+\frac{2}{7}\sqrt{7} \Rightarrow x=3-\frac{4\sqrt{7}}{7}$

$y=2-\frac{2}{7}\sqrt{7} \Rightarrow x=3+\frac{4\sqrt{7}}{7}$

$P_1\left(3-\frac{4\sqrt{7}}{7}, 2+\frac{2\sqrt{7}}{7}\right) \quad P_2\left(3+\frac{4\sqrt{7}}{7}, 2-\frac{2\sqrt{7}}{7}\right)$

6.5. Relative position of the line $l: 2x+y-10=0$ to the ellipse $\mathcal{E}: \frac{x^2}{9}+\frac{y^2}{4}=1$

$y=10-2x$

$\Rightarrow \mathcal{E}: 4x^2+9y^2=36$

$4x^2+9(10-2x)^2-36=0$

$40x^2-360x+864=0$

$\Delta<0 \Rightarrow l \parallel \mathcal{E}$

6.6. Find the eq. of the tangents to $l: 2x-2y-13=0$
and to $\mathcal{E}: x^2+4y^2-20=0$

$m_l=1 \Rightarrow m_t=-1 \qquad t: y=m_t x+n$
t - tangent

$\mathcal{E}: x^2+4y^2-20=0 \quad |:20$

$$\frac{x^2}{20}+\frac{y^2}{5}=1 \qquad n=\pm\sqrt{a^2m^2+b^2} = \pm\sqrt{20+5}=\pm 5$$

$$\Rightarrow t: y=-x\pm 5$$

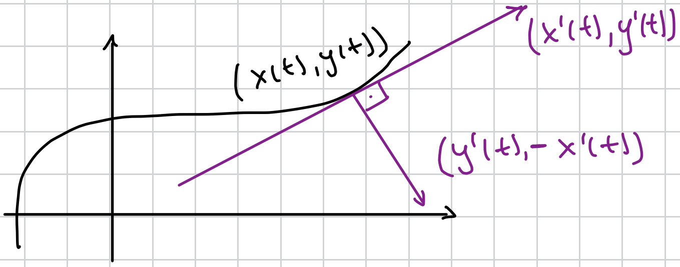
<u>Sem 11</u>: ex: 2, 8, 9, 10, 18, 20, 26, 27, 28 Cap 6

6.2. For a circle $\mathcal{C}$ of radius R.

a) use the parametrization $x \mapsto (x, \pm\sqrt{R^2-x^2})$ to deduce a parametrization of tangent lines to $\mathcal{C}$

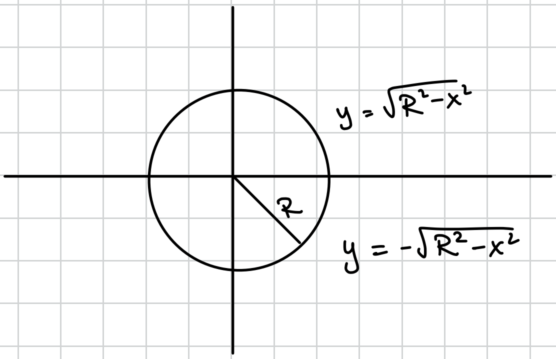
b) use the parametrization $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ to deduce a parametrization of tangent lines to $\mathcal{C}$

c) compare these to the eq. of the tangent line $xx_0+yy_0=R^2$ where $(x_0,y_0)\in\mathcal{C}$



director vector of the tangent line

$$T_{(x(t),y(t))}\mathcal{C}: \frac{x-x(t)}{x'(t)} = \frac{y-y(t)}{y'(t)}$$



Sem 12: Cap 7 ex: 1, 2, 4, 6, 7, 8

$Q: \; a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$

quadratic eq.

$$Q \xrightarrow[\text{(rotation \& translation)}]{\text{isometries}} \lambda_1 x^2 + \lambda_2 y^2 = k$$

OR

$$\lambda_1 x^2 + v_2 y = k$$

$\downarrow$ rescaling

$k, \lambda_1, \lambda_2 = \pm 1$

nr of pos eigenvalues

$r = \text{rank } M_Q$	$(p, i-p)$	equation	name
2	$(0,2)$ or $(2,0)$	$x^2+y^2+1=0$	imaginary ellipse
2	$(0,2)$ or $(2,0)$	$x^2+y^2-1=0$	circle (ellipse)
2	$(1,1)$	$x^2-y^2-1=0$	hyperbola
2	$(0,2)$ or $(2,0)$	$x^2+y^2=0$	two complex lines
2	$(1,1)$	$x^2-y^2=0$	two real lines
1	$(0,1)$ or $(1,0)$	$x^2+1=0$	two complex lines
1	$(1,0)$	$x^2-1=0$	two real lines
1	$(1,0)$	$x^2=0$	a real double line
1	$(0,1)$ or $(1,0)$	$x^2-y=0$	parabola

$$M_Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

7.2. Write down a quadratic eq. with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$

$$P_A(x) = \det(A - x I_2)$$

$$= \begin{vmatrix} 6-x & 2 \\ 2 & 9-x \end{vmatrix} = x^2 - 15x + 54 - 4$$

$$= x^2 - 15x + 50$$

$$= (x-5)(x-10) \Rightarrow \lambda_1 = 10, \; \lambda_2 = 5$$

$$S(\lambda) = \{(x,y) \mid A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}\}$$

$$= \{(x,y) \mid (A - \lambda I_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$S(\lambda_1) = \left\{ (x,y) \mid \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{cases} -4x + 2y = 0 \\ 2x - y = 0 \end{cases} \Leftrightarrow y = 2x \quad \Rightarrow S(\lambda_1) = \{ (x, 2x) \mid x \in \mathbb{R} \}$$

$$= \langle (1,2) \rangle$$

Choose $v_1 = \frac{1}{\sqrt{5}} (1,2)$ $\to$ normalize the vector

$$S(\lambda_2) = \left\{ (x,y) \mid \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases} \Leftrightarrow x = -2y \quad \Rightarrow S(\lambda_2) = \{ (-2y, y) \mid y \in \mathbb{R} \}$$

$$= \langle (-2,1) \rangle$$

Choose $v_2 = \frac{1}{\sqrt{5}} (2,-1)$

$$M \in SO(2) \Leftrightarrow \begin{cases} \det(M) = 1 \\ M \cdot M^T = I_2 \end{cases}$$

The base-change matrix is $M = M_{B'B} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$

$\det(M) = -\frac{1}{5} - \frac{4}{5} = -1$ but we want $\det(M) = 1$

so we choose $v_2 = \frac{1}{\sqrt{5}} (-2,1)$

$$\Rightarrow M = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\underbrace{M^{-1} \cdot A \cdot M = M^T}_{M \in SO(2)} \cdot A \cdot M = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}$$

$$= \frac{1}{5} \begin{pmatrix} 10 & 20 \\ -10 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$Q: \quad (x \ y) \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + (x \ y) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$$

$$(x \ y) \cdot \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + x + 2y - 1 = 0$$

$$\begin{pmatrix} 6x + 2y & 2x + 9y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + x + 2y - 1 = 0$$

$$6x^2+2xy+2xy+9y^2+x+2y-1=0$$

$$Q:\ 6x^2+4xy+9y^2+x+2y-1=0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M\cdot\begin{pmatrix} x' \\ y' \end{pmatrix}$$
$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$(x\ \ y) = \begin{pmatrix} x \\ y \end{pmatrix}^T$$
$$= \left(M\cdot\begin{pmatrix} x' \\ y' \end{pmatrix}\right)^T$$
$$= (x'\ \ y')\cdot M^T$$

$$(A\cdot B)^T = B^T\cdot A^T$$

$$Q:\ (x'\ \ y')\cdot M^T\cdot A\cdot M\begin{pmatrix} x' \\ y' \end{pmatrix} + (x'\ \ y')M^T\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$$

$$Q:\ (x'\ \ y')\begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + (x'\ \ y')\cdot\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$$

$$Q:\ 10x'^2+5y'^2 + (x'\ \ y')\cdot\frac{1}{\sqrt{5}}\begin{pmatrix} 5 \\ 0 \end{pmatrix} - 1 = 0$$

$$Q:\ 10x'^2+5y'^2+\sqrt{5}x'-1=0$$

$$Q:\ (10x'^2+\sqrt{5}x')+5y'^2=0$$

$$Q:\ 10\left(x'^2+2\cdot\frac{\sqrt{5}}{20}x'+\frac{5}{400}\right)+5y'^2-1-\frac{1}{8}=0$$

$$Q:\ 10\left(x'+\frac{\sqrt{5}}{20}\right)^2+5y'^2-\frac{9}{8}=0$$

$$\begin{cases} x'' = x'+\frac{\sqrt{5}}{20} \\ y'' = y' \end{cases}$$

$$Q:\ 10x''^2+5y''^2=\frac{9}{8}$$

$$Q:\ \frac{x''^2}{\frac{9}{80}}+\frac{y''^2}{\frac{9}{40}}=1$$

$$x''' = \frac{x''}{\frac{3}{\sqrt{80}}} \qquad y''' = \frac{y''}{\frac{3}{\sqrt{40}}}$$

$$Q:\ x'''^2+y'''^2=1$$

SAU: Lagrange method

$Q: \ 6x^2+4xy+9y^2+x+2y-1=0$

$Q: \ (6x^2+4xy)+9y^2+x+2y-1=0$

$Q: \ \left(6x^2+2\cdot\sqrt{6}\,x\cdot\frac{2}{\sqrt{6}}\cdot y+\frac{2}{3}y^2\right)+9y^2-\frac{2}{3}y^2+x+2y-1=0$

$Q: \ \left(\sqrt{6}\,x+\frac{2}{\sqrt{6}}y\right)^2+\frac{25}{3}y^2+x+2y-1=0$

$x' = \sqrt{6}\,x+\frac{2}{\sqrt{6}}y$

$y' = y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \sqrt{6} & \frac{2}{\sqrt{6}} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = \frac{x'-\frac{2}{\sqrt{6}}y}{\sqrt{6}}$$

$Q: \ x'^2+\frac{25}{3}y^2+\frac{1}{6}x'-\frac{1}{3}y+2y-1=0$

$Q: \ \left(x'^2+\frac{1}{6}x'\right)+\left(\frac{25}{3}y^2+\frac{5}{3}y\right)-1=0$

$Q: \ \left(x'^2+2\cdot\frac{1}{12}x'+\frac{1}{144}\right)+\left(\frac{25}{3}y^2+2\cdot\frac{5}{\sqrt{3}}y\cdot\frac{\sqrt{3}}{6}+\frac{1}{12}\right)-\frac{1}{144}-\frac{1}{12}-1=0$

$Q: \ \left(x'+\frac{1}{12}\right)^2+\left(\frac{5}{\sqrt{3}}y+\frac{\sqrt{3}}{6}\right)^2=\frac{157}{144}$

$x'' = x'+\frac{1}{12} \qquad y'' = \frac{5}{\sqrt{3}}y+\frac{\sqrt{3}}{6}$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \frac{1}{12} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$

$Q: \ x''^2+y''^2=\frac{157}{144}$

$x''' = x''\cdot\sqrt{\frac{144}{157}}$

$y''' = y''\cdot\sqrt{\frac{144}{157}}$

$Q: \ x'''^2+y'''^2=1$

7.4. $Q: -x^2 + xy - y^2 = 0$

Bring Q to the canonical form

$Q: x^2 - xy + y^2 = 0$

$Q: \left(x^2 - 2 \cdot x \cdot \frac{1}{2}y + \frac{1}{4}y^2\right) + y^2 - \frac{1}{4}y^2 = 0$

$Q: \left(x - \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 = 0$

$x' = x - \frac{1}{2}y$

$y' = \frac{\sqrt{3}}{2}y$

$\Rightarrow Q: x'^2 + y'^2 = 0 \quad \to$ 2 complex imaginary lines

---

Sem 12: ex 1, 2, 4, 6, 7, 8 Cap7

ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

7.2 For A write the quadratic eq. with the associated matrix A and find $M \in SO(2)$ which diagonalizes A

a) $A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$

$Q: (x \ \ y) \cdot A \begin{pmatrix} x \\ y \end{pmatrix} + (x \ \ y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$

$Q: (x \ \ y) \cdot \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + x + 2y - 1 = 0$

$Q: (6x + 2y \quad 2x + 9y)\begin{pmatrix} x \\ y \end{pmatrix} + x + 2y - 1 = 0$

$Q: 6x^2 + 4xy + 9y^2 + x + 2y - 1 = 0$

$\left(6x^2 + 2 \cdot \sqrt{6}x \cdot y \cdot \frac{2}{\sqrt{6}} + \frac{2}{3}y^2\right) + 9y^2 - \frac{2}{3}y^2 + x + 2y - 1 = 0$

$\left(\sqrt{6}x + \frac{2}{\sqrt{6}}y\right)^2 + \frac{25}{3}y^2 + x + 2y - 1 = 0$

let $x' = \sqrt{6}x + \frac{2}{\sqrt{6}}y \quad \Rightarrow x = \dfrac{x' - \frac{2}{\sqrt{6}}y}{\sqrt{6}}$

$y' = y$

$$x'^2 + \frac{25}{3}y'^2 + \frac{1}{\sqrt{6}}x' - \frac{1}{3}y' + 2y' - 1 = 0$$

$$\left(x'^2 + \frac{1}{\sqrt{6}}x'\right) + \left(\frac{25}{3}y'^2 + \frac{2}{3}y'\right) - 1 = 0 \quad \Big| \cdot 6$$

$$(6x'^2 + \sqrt{6}x') + (50y'^2 + 4y') - 6 = 0$$

$$\left(6x'^2 + 2\cdot\sqrt{6}x'\cdot\frac{1}{2} + \frac{1}{4}\right) + \left(50y'^2 + 2\cdot\sqrt{50}y\cdot\frac{2}{\sqrt{50}} + \frac{4}{50}\right) - 6 = 0$$

$$(\underbrace{\sqrt{6}x' + \tfrac{1}{2}}_{x''})^2 + (\underbrace{5\sqrt{2}y' + \tfrac{\sqrt{2}}{5}}_{y''})^2 - 6 - \frac{1}{4} - \frac{\sqrt{2}}{5} = 0$$

$$Q: \; x''^2 + y''^2 - \frac{25}{4} - \frac{\sqrt{2}}{5} = 0$$

Find eigen values

$$\det(A - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(9-\lambda) - 4 = 0$$

$$\lambda^2 - 15\lambda + 50 = 0$$

$$\lambda_1 = 10$$
$$\lambda_2 = 5$$

$$S(\lambda_1) = \{(x,y) \mid (A - \lambda_1 I_2)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\}$$

$$= \langle (1,2) \rangle \Rightarrow \text{We choose } v_1 = \frac{1}{\sqrt{5}}(1,2)$$

$$S(\lambda_2) = \{(x,y) \mid (A - \lambda_2 I_2)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\}$$

$$= \langle (-2,1) \rangle \Rightarrow \text{We choose } v_2 = \frac{1}{\sqrt{5}}(-2,1)$$

$$\Rightarrow M = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$M \cdot M^T = I_2$$

$$M^T \cdot A \cdot M = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Sem 13:

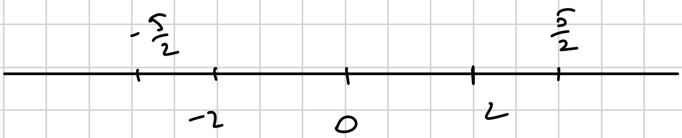
7.9. Discuss the type of the curve $x^2 + \lambda xy + y^2 - 6x - 16 = 0$ in terms of $\lambda \in \mathbb{R}$

$$Q = \begin{bmatrix} 1 & \frac{\lambda}{2} \\ \frac{\lambda}{2} & 1 \end{bmatrix} \qquad \delta = \det(Q) = 1 - \frac{\lambda^2}{4} \qquad T = tr(Q) = 2$$

$$\hat{Q} = \begin{bmatrix} 1 & \frac{\lambda}{2} & -3 \\ \frac{\lambda}{2} & 1 & 0 \\ -3 & 0 & -16 \end{bmatrix} \qquad \hat{\Delta} = \det(\hat{Q}) = -16 - 9 + 4\lambda^2 = 4\lambda^2 - 25$$

$\hat{\Delta} = 0 \iff 4\lambda^2 - 25 = 0 \iff \lambda_{1,2} = \pm \frac{5}{2}$

$\delta = 0 \iff 1 - \frac{\lambda^2}{4} = 0 \iff \lambda_{1,2} = \pm 2$



λ	$-\infty$		$-\frac{5}{2}$		-2		2		$\frac{5}{2}$		$+\infty$
$\hat{\Delta} = 4\lambda^2 - 25$		+	0	−	−	−	−	−	0	+	
$\delta = 1 - \frac{\lambda^2}{4}$		−	−	−	0	+	0	−	−	−	

λ	$\hat{\Delta}$	δ	T	δT	classification
$\lambda \in (-\infty, -\frac{5}{2})$	+	−	+	−	hyperbola
$\lambda = -\frac{5}{2}$	0	−	+	−	2 lines
$\lambda \in (-\frac{5}{2}, -2)$	−	−	+	−	hyperbola
$\lambda = -2$	−	0	+	0	parabola
$\lambda \in (-2, 2)$	−	+	+	+	empty set
$\lambda = 2$	−	0	+	0	parabola
$\lambda \in (2, \frac{5}{2})$	−	−	+	−	hyperbola
$\lambda = \frac{5}{2}$	0	−	+	−	2 lines
$\lambda \in (\frac{5}{2}, \infty)$	+	−	+	−	hyperbola

8.3. Det the intersection of the ellipsoid $\mathcal{E}_{2,\sqrt{3},3}: \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1$ with the line $\ell: x = y = z$

$$\frac{x^2}{4}+\frac{y^2}{3}+\frac{z^2}{9}=1 \quad \Big| \cdot 36$$

$$9x^2+12y^2+4z^2=36$$
$$l: x=y=z$$

$\Rightarrow 25x^2=36$

$x^2=\frac{36}{25}$ $\Rightarrow P_1\left(\frac{6}{5},\frac{6}{5},\frac{6}{5}\right)$

$P_2\left(-\frac{6}{5},-\frac{6}{5},-\frac{6}{5}\right)$

eq. tg: $\frac{x\cdot x_0}{4}+\frac{y\cdot y_0}{9}+\frac{z\cdot z_0}{16}=1$

$$\frac{x\cdot 6}{20}+\frac{y\cdot 6}{45}+\frac{z\cdot 6}{80}=1$$

$\Rightarrow tg_1: \frac{3x}{10}+\frac{2y}{15}+\frac{3z}{40}=1$

$tg_2: -\frac{3x}{10}-\frac{2y}{15}-\frac{3z}{40}=1$

8.4. Det the tangent planes to the ellipsoid $\mathcal{E}_{2,3,2\sqrt{2}}: \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{8}=1$ which are parallel to the plane $\pi: 3x-2y+5z+1=0$

$T_{\mathcal{E}} \parallel \pi$ $\quad \vec{v}(3,-2,5)$ - normal vector

let $P(x_0,y_0,z_0)\in\mathcal{E}_{2,3,2\sqrt{2}} \Rightarrow T_{p}\mathcal{E}: \frac{xx_0}{4}+\frac{yy_0}{9}+\frac{z\cdot z_0}{8}=1$

$\Rightarrow \vec{w}\left(\frac{x_0}{4},\frac{y_0}{9},\frac{z_0}{8}\right)$ normal vector

$\vec{v}\parallel\vec{w} \Leftrightarrow \frac{3}{\frac{x_0}{4}}=\frac{-2}{\frac{y_0}{9}}=\frac{5}{\frac{z_0}{8}}=k$

$x_0=\frac{k}{12}$

$y_0=\frac{-k}{18}$ $\Rightarrow P\left(\frac{k}{12},-\frac{k}{18},\frac{k}{40}\right)$

$z_0=\frac{k}{40}$

$P\in\mathcal{E}_{2,3,2\sqrt{2}}$

$\Rightarrow \frac{k^2}{144\cdot 4}+\frac{k^2}{324\cdot 9}+\frac{k^2}{40^2\cdot 8}=1$

$k=\ldots$

7.10. a) Using the classification and decide what surface is described by the following eq: $x^2+2y^2+z^2+xy+yz+zx=1$

se simetrizează

$$\begin{array}{cccc} & x^2 & xy & xz \\ Q = & \left[\begin{array}{ccc} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \end{array}$$

$$\det(Q-\lambda I_3)=0$$

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2(2-\lambda) + 2 \cdot \frac{1}{8} - \frac{1}{4}(2-\lambda) - \frac{1}{2}(1-\lambda) = 0$$

$$(2-\lambda)\left[(1-\lambda)^2 - \frac{1}{4}\right] + \frac{1}{4} - \frac{1}{2}(1-\lambda) = 0$$

$$(2-\lambda)\left(1-\lambda-\frac{1}{2}\right)\left(1-\lambda+\frac{1}{2}\right) - \frac{1}{2}\left(1-\lambda-\frac{1}{2}\right) = 0$$

$$\left(\frac{1}{2}-\lambda\right)\left[(2-\lambda)\left(\frac{3}{2}-\lambda\right) - \frac{1}{2}\right] = 0$$

$$\lambda_1 = \frac{1}{2}$$

$$\lambda^2 - \frac{7}{2}\lambda + \frac{5}{2} = 0$$

$$2\lambda^2 - 7\lambda + 5 = 0$$

$$\Delta = 49 - 40 = 9$$

$$\lambda_{2,3} = \frac{7 \pm 3}{4} \quad \lambda_2 = \frac{5}{2}, \quad \lambda_3 = 1$$

$\Rightarrow$ ellipsoid