Week 8: Planar dynamical systems

April 15, 2024

Planar dynamical systems (continuation)

We consider differential equations of the form

$$\dot{X} = f(X) \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a given C^1 function. As usual, denote by $\varphi(t, \eta)$ its flow. Recall that we are interested (generally speaking) in representing the phase portrait (the significant orbits). The simplest orbit is the one corresponding to an equilibrium point. In the first part of this lecture we will study the behaviour of the orbits in a neighborhood of an equilibrium point. More precisely, we study the stability of the equilibrium points. In addition, we classify the linear systems using names that reflect the geometry of the orbits.

Let $\eta^* \in \mathbb{R}^2$ be an equilibrium point, i.e. $f(\eta^*) = 0$. In the previous lecture we have seen a system with

an attractor equilibrium point (if the system initiates nearby, the future states are closer and closer to the equilibrium point)

and, also, a system whose orbits were circles centered in the equilibrium point. In this case, if the system initiates nearby, the future states remain close to the equilibrium point. This situation is also important and is described, in general, by the notion of *stability*.

Definition

We say that the equilibrium point η^* of (1) is stable if $\forall\ \varepsilon > 0\ \exists$ $\delta > 0$ such that whenever $||\eta - \eta^*|| \leq \delta$ we have $||\varphi(t, \eta) - \eta^*|| \leq \varepsilon$, $\forall\ t \in [0, \infty)$.
We say that an equilibrium point is unstable when it is not stable.

Remark. The formulation *study the stability of an equilibrium point* means to decide whether the equilibrium point is attractor, repeller, stable or unstable.

Definition

We say that γ_η is a periodic orbit (or closed orbit) when the corresponding solution $\varphi(\cdot, \eta)$ is a periodic function.

The type and stability of linear planar systems

$$\dot{X} = AX \tag{2}$$

where $A \in \mathcal{M}_2(\mathbb{R})$ with $\det(A) \neq 0$.
Denote by $\lambda_1, \lambda_2 \in \mathbb{C}$ the eigenvalues of A. We know that $\det(A) = \lambda_1 \lambda_2$. Note that, since $\det(A) \neq 0$, the only equilibrium point of (2) is the origin, and, in addition, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

We will classify the linear planar systems using words (node, saddle, center, focus) that reflect the geometry of the orbits, but using pure algebraic criteria (the nature of the eigenvalues).

The type of linear planar systems

Definition

We say that the equilibrium point at the origin of (2) is a
node when $\lambda_1, \lambda_2 \in \mathbb{R}$ and they have the same sign.
saddle when $\lambda_1, \lambda_2 \in \mathbb{R}$ and they have opposite signs.
center when $\lambda_{1,2} = \pm i\beta$, with $\beta > 0$, $\beta \in \mathbb{R}$.
focus when $\lambda_{1,2} = \alpha \pm i\beta$, with $\alpha \neq 0$, $\beta > 0$, $\alpha, \beta \in \mathbb{R}$.

In the previous lecture we represented phase portraits for linear systems of each type.

The stability of linear planar systems

Theorem

(i) If Re$(\lambda_1) < 0$ *and Re*$(\lambda_2) < 0$ *then the equilibrium point at the origin of* (2) *is a global attractor.*
(ii) If Re$(\lambda_1) > 0$ *and Re*$(\lambda_2) > 0$ *then the equilibrium point at the origin of* (2) *is a global repeller.*
(iii) Any center is stable.
(iv) Any saddle is unstable.

The *proof* uses that $\varphi(t, \eta) = e^{tA}\eta$ for all $t \in \mathbb{R}$, $\eta \in \mathbb{R}^2$ and is based on the analysis of e^{tA} in each situation.

(i) If $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$ then the equilibrium point at the origin of (2) is a global attractor.

Proof of (i) in the case that A is diagonalizable over $\mathbb{C}$.
It is sufficient if we find that $\lim_{t\to\infty} e^{tA}$ is the null matrix.
There exists an invertible matrix P such that $A = PJP^{-1}$,
where J is a diagonal matrix with (λ_1, λ_2) on the main diagonal.
Then $e^{tA} = Pe^{tJ}P^{-1}$ for all $t \in \mathbb{R}$. Also, e^{tJ} is a diagonal matrix with $(e^{t\lambda_1}, e^{t\lambda_2})$ on the main diagonal.
The proof is done if we justify that $\lim_{t\to\infty} e^{t\lambda_k} = 0$ for $k = 1, 2$.
When λ_k is not real we write $\lambda_k = \alpha + i\beta$. Then $e^{t\lambda_k} = e^{t\alpha}e^{it\beta}$.
We have $\lim_{t\to\infty} e^{t\alpha} = 0$ since $\alpha < 0$ and
$e^{it\beta} = (\cos(t\beta) + i\sin(t\beta))$ is bounded. The proof is finished.

The linearization method to study the stability of equilibria of nonlinear planar systems

We consider the nonlinear planar system $\dot{X} = f(X)$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a given C^1 function. Let $\eta^* \in \mathbb{R}^2$ be an equilibrium point. Let $Jf(\eta^*)$ be the Jacobian matrix of f computed in η^*. The linear system

$$\dot{X} = Jf(\eta^*)X$$

is called the linearization of $\dot{X} = f(X)$ around the equilibrium point η^*.

The linearization method

Denote by $\lambda_1, \lambda_2 \in \mathbb{C}$ the eigenvalues of $Jf(\eta^*)$.

Definition

We say that the equlibrium point η^* is hyperbolic if $Re(\lambda_1) \neq 0$ and $Re(\lambda_2) \neq 0$.

Theorem

Let η^* be a hyperbolic equilibrium point of $\dot{X} = f(X)$.
If the linear system $\dot{X} = Jf(\eta^*)X$ is an attractor/repeller, then the equilibrium point η^* of $\dot{X} = f(X)$ is also an attractor/repeller.
If the linear system $\dot{X} = Jf(\eta^*)X$ is a saddle, then the equilibrium point η^* of $\dot{X} = f(X)$ is unstable.

Remark

In order to study the stability of the equilibrium points of the second order scalar differential equation

$$\ddot{x} = f(x, \dot{x}),$$

we transform it to a planar system with unknowns x and $y = \dot{x}$. So, the planar system is

$$\dot{x} = y, \quad \dot{y} = f(x, y).$$

Stability and First integrals

Theorem

Let η^* be an equilibrium point of $\dot{X} = f(X)$.
(i) If η^* is an attractor/repeller, then there is no first integral in a neighborhood of η^*.
(ii) In particular, if the origin is an attractor/repeller of $\dot{X} = AX$, then this linear system does not have a global first integral.

Proof. (ii) Let η^* be an attractor of $\dot{X} = AX$. Then it is a global attractor, i.e. $\lim_{t\to\infty} \varphi(t,\eta) = \eta^*$ for all $\eta \in \mathbb{R}^2$. Assume by contradiction that there exists $H : \mathbb{R}^2 \to \mathbb{R}$ a global first integral. Then, for all $\eta \in \mathbb{R}^2$ and all $t \in \mathbb{R}$, we have $H(\varphi(t,\eta)) = H(\eta)$. Then $\lim_{t\to\infty} H(\varphi(t,\eta)) = H(\eta)$ for all $\eta \in \mathbb{R}^2$. Since H is continuous, we get $H(\eta^*) = H(\eta)$ for all $\eta \in \mathbb{R}^2$. This implies that H is constant, which contradicts the definition of the first integral.

Stability and First integrals

Proof. (i) Let η^* be an attractor. Then, by definition, $\exists\ \rho > 0$ such that, whenever $||\eta - \eta^*|| \leq \rho$ we have $\lim_{t\to\infty} ||\varphi(t,\eta) - \eta^*|| = 0$ (equivalently $\lim_{t\to\infty} \varphi(t,\eta) = \eta^*$). Assume by contradiction that there exists V a neighborhood of η^* and $H : V \to \mathbb{R}$ a first integral. Then, for all $\eta \in V$ and all t with $\varphi(t,\eta) \in V$, we have

$$H(\varphi(t,\eta)) = H(\eta).$$

Since η^* is an attractor, we can assume that $\varphi(t,\eta) \in V$ whenever $||\eta - \eta^*|| \leq \rho$ and for sufficiently large t. Then $\lim_{t\to\infty} H(\varphi(t,\eta)) = H(\eta)$ whenever $||\eta - \eta^*|| \leq \rho$. Since H is continuous, we get $H(\eta^*) = H(\eta)$ whenever $||\eta - \eta^*|| \leq \rho$. This implies that H is locally constant in a neighborhood of η^*, which contradicts the definition of the first integral.

Stability and First integrals

Theorem

Let η^* be an equilibrium point of the planar system $\dot{X} = f(X)$. Assume that the eigenvalues of $Jf(\eta^*)$ are $\lambda_{1,2} \pm i\beta$, with $\beta > 0$ (thus, η^* is not hyperbolic). If $\dot{X} = f(X)$ has a first integral well-defined in a neighborhood of η^*, then η^* is a stable equilibrium point.

First integrals

In the previous lecture we considered few simple examples of linear planar systems for which we first found the flow. Having the flow, using the definition of the first integral, we were able to check that a given function is a first integral of a given system.
Moreover, in some example, you were able to guess the expression of a first integral just looking at the flow.

But, in general, for nonlinear systems, one can not find the flow!
In this situation,

How to check that a given function is a first integral?

How to find a first integral?

How to check that a given function is a first integral

Theorem

A nonconstant C^1 function $H : U \to \mathbb{R}$ is a first integral in U of

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y)$$

if and only if it satisfies the first order linear partial differential equation

$$f_1(x, y)\frac{\partial H}{\partial x}(x, y) + f_2(x, y)\frac{\partial H}{\partial y}(x, y) = 0, \text{ for any } (x, y) \in U.$$

How to check that a given function is a first integral

Proof. Let $H : U \to \mathbb{R}$ be a nonconstant C^1 function. We have

H is a first integral of the given system $\iff$
$H(\varphi(t,\eta)) = H(\eta)$ for all $\eta \in U$ and all t with $\varphi(t,\eta) \in U \iff$
$\frac{d}{dt} H(\varphi_1(t,\eta), \varphi_2(t,\eta)) = 0 \iff$
$\frac{\partial H}{\partial x}(\varphi(t,\eta))\dot{\varphi}_1(t,\eta) + \frac{\partial H}{\partial y}(\varphi(t,\eta))\dot{\varphi}_2(t,\eta) = 0 \iff$
$\frac{\partial H}{\partial x}(\varphi(t,\eta)) f_1(\varphi(t,\eta)) + \frac{\partial H}{\partial y}(\varphi(t,\eta)) f_2(\varphi(t,\eta)) = 0 \iff$
$f_1(\eta)\frac{\partial H}{\partial x}(\eta) + f_2(\eta)\frac{\partial H}{\partial y}(\eta) = 0,$ for any $\eta \in U$.

How to check that a given function is a first integral

An example. Check that $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x^2 + y^2$ is a first integral of the planar system $\dot{x} = -y$, $\dot{y} = x$.

Recall that we already checked this in the previous lecture, using the definition of the first integral. Now we use this new method.

First we note that H is a C^1 function, not locally constant.
It remains to check that

$$-y\frac{\partial H}{\partial x}(x, y) + x\frac{\partial H}{\partial y}(x, y) = 0, \quad \forall\ (x, y) \in \mathbb{R}^2.$$

Since $\frac{\partial H}{\partial x} = 2x$ and $\frac{\partial H}{\partial y} = 2y$ we immediately see the validity of the relation above.

A method to find a first integral

$$\dot{x} = f_1(x, y), \quad \dot{y} = f_2(x, y)$$

Step 1. Write $\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$.

Step 2. Integrate the above DE and write its general solution as

$$H(x, y) = c, \quad c \in \mathbb{R}.$$

Step 3. Find a domain U for the function H found at Step 2 and check that H is a first integral in U.

How to integrate a separable DE

The DE written at Step 1 is said to be separable if it has the form

$$\frac{dy}{dx} = g_1(x)g_2(y).$$

This DE can be integrated after the "separation of the variables":

$$\frac{dy}{g_2(y)} = g_1(x)dx \quad \text{(here the variables are separated)}$$

$$\int \frac{dy}{g_2(y)} = \int g_1(x)dx \quad \text{(we integrate and obtain)}$$

$$G_2(y) = G_1(x) + c, \quad c \in \mathbb{R}.$$

If it is possible, we simplify the previous relation. If not, take

$$H(x, y) = G_2(y) - G_1(x).$$

Find a first integral of the planar system $\dot{x} = -y$, $\dot{y} = x$.

Step 1. Write $\frac{dy}{dx} = -\frac{x}{y}$.

Step 2. (Integrate the above DE and write its general solution as $H(x, y) = c$, $c \in \mathbb{R}$.) We note that the DE is separable.So,

$$ydy = -xdx.$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c, \quad c \in \mathbb{R}.$$

$$H(x, y) = c, \quad c \in \mathbb{R}, \text{ with } H(x, y) = x^2 + y^2.$$

Step 3. (Find a domain U for the function H found at Step 2 and check that H is a first integral in U.) We can consider $H : \mathbb{R}^2 \to \mathbb{R}$ and check that (we already did it, in fact)

$$-y\frac{\partial H}{\partial x}(x, y) + x\frac{\partial H}{\partial y}(x, y) = 0, \quad \forall\ (x, y) \in \mathbb{R}^2.$$