Week 11: The approximation of a solution of a differential equation

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The geometry behind the differential equations - the direction field

We consider a planar differential system of the form

$$\dot{X} = f(X) \quad \Leftrightarrow \quad \begin{array}{l} \dot{x} = f_1(x, y) \\ \dot{y} = f_2(x, y) \end{array} \tag{1}$$

where $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a given C^1 function.

The direction field is a collection of vectors in the plane such that the vector based in a given point $(x_0, y_0) \in \mathbb{R}^2$ is parallel with the vector $(f_1(x_0, y_0), f_2(x_0, y_0))$.

An example

Consider the system

$$\dot{x} = -y, \quad \dot{y} = x.$$

Show that the direction field is orthogonal to the position vector.

Here $f(x, y) = (-y, x)$.
Fix $(x_0, y_0) \in \mathbb{R}^2$.
The coordinates of the position vector are (x_0, y_0).
The direction field is parallel with the vector $(-y_0, x_0)$.
Since the scalar product of these two vectors is null, we have that they are orthogonal, indeed.

The definition of the direction field

In addition, we consider the scalar differential equation

$$y' = g(x, y) \tag{2}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is a given C^1 function.

The direction field is a collection of vectors in the plane such that the vector based in a given point $(x_0, y_0) \in \mathbb{R}^2$ has the slope $g(x_0, y_0)$.

Remark: Since the slope of the vector $(f_1(x_0, y_0), f_2(x_0, y_0))$ is $\frac{f_2(x_0,y_0)}{f_1(x_0,y_0)}$, we deduce that the planar system (1) and the scalar differential equation of its orbits $\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$ have the same direction field.

Direction fields - Examples

Example 1: $y' = 1 - \frac{x}{y^2}$. Plot the vectors of the direction field corresponding to the points $(1,1)$, $(0,1)$ and $(1,0)$.

Let $g(x,y) = 1 - \frac{x}{y^2}$, the right-hand side of the equation.

The vector based in $(1,1)$ has the slope $m_1 = g(1,1) = 0$, i.e. it is horizontal.

The vector based in $(0,1)$ has the slope $m_2 = g(0,1) = 1$, i.e. it is parallel with the first bisectrix.

The vector based in $(1,0)$ has the slope $m_3 = g(1,0) = \infty$, i.e. it is vertical.

Direction fields - Examples

Example 2: $\frac{dy}{dx} = -\frac{x}{y}$.
Show that the direction field is orthogonal to the position vector.

The direction field based in $(x_0, y_0) \in \mathbb{R}^2$ has the slope $-\frac{x_0}{y_0}$.

The position vector (x_0, y_0) has the slope $\frac{y_0}{x_0}$.

Since the product of the two slopes is equal to -1, we deduce that the two vectors are orthogonal.

Remark: $\frac{dy}{dx} = -\frac{x}{y}$ is the equation of the orbits of the system $\dot{x} = -y, \quad \dot{y} = x$. It is not a surprise that they have the same direction fields.

Why we need the direction field?

The starting point of the important applications of this notion is the fact that the direction field is tangent to the orbits of the system, or, respectively, to the graph of the solutions of the scalar differential eqaution.

For showing this, we start with the scalar differential equation $y' = g(x, y)$.
Let us fix a point $(x_0, y_0) \in \mathbb{R}^2$ and consider a solution ψ whose graph passes through it, i.e. $\psi'(x) = g(x, \psi(x))$ and $\psi(x_0) = y_0$.

The slope of this graph is $\psi'(x_0) = g(x_0, y_0)$, i.e. it is equal to the slope of the direction field.

The direction field is tangent to the orbits

We prove this for the system $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y)$.

Let us fix a point $(x_0, y_0) \in \mathbb{R}^2$ and consider a solution (φ_1, φ_2) whose orbit passes through it at some moment t_0.
Thus, we have $\varphi_1'(t) = f_1(\varphi_1(t), \varphi_2(t)$, $\varphi_2'(t) = f_2(\varphi_1(t), \varphi_2(t)$ and

The direction field is tangent to the orbits

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Thus, we have $\varphi_1'(t) = f_1(\varphi_1(t), \varphi_2(t)$, $\varphi_2'(t) = f_2(\varphi_1(t), \varphi_2(t)$ and $\varphi_1(t_0) = x_0$, $\varphi_2(t_0) = y_0$.

The tangent to the orbit is the vector $(\varphi_1'(t_0), \varphi_2'(t_0)) = (f_1(x_0, y_0), f_2(x_0, y_0))$. That is, the direction field is tangent to the orbit.

An example reloaded

What is the shape of the orbits of the system $\dot{x} = -y, \quad \dot{y} = x$?

We know: the direction field is orthogonal to the position vector.
Thus, an orbit is a planar curve that is orthogonal to the position vector of each point of it.
The only planar curve with this property is a circle centered in the origin of coordinates.

Remark: This is just a simple example. In general, the shape of an orbit is more complicated and it can not be determined exactly.
BUT, Leonard Euler (1707–1783) had the brilliant idea to determine an approximate shape. This was the birth of a new and extremely important field:
Numerical analysis of differential equations.

Introduction to numerical methods

Consider the IVP $y' = g(x, y)$, $y(x_0) = y_0$.

Since g is C^1, it is known that it has a unique solution, denoted $\psi : [x_0, x^*] \to \mathbb{R}$.

In general, one can not find the exact expression of ψ.

But, as we saw, we easily can find the direction field and "imagine" the shape of the graph of ψ. Euler made an algorithm based on this "imagination".

Numerical methods and interpolation methods

Fix $n \in \mathbb{N}^*$. Let $\{x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = x^*\}$, called *partition* of the interval $[x_0, x^*]$. The number n is called the *number of steps*.

The aim is to find $y_1, y_2, \ldots, y_n \in \mathbb{R}$ as "good" approximations of $\psi(x_1), \psi(x_2), \ldots, \psi(x_n)$. Thus, the points $(x_1, y_1), \ldots, (x_n, y_n)$ will be "quite" close to the graph of the solution ψ.

There are methods, called interpolation methods, to represent a "nice" curve that fits the points $(x_1, y_1), \ldots, (x_n, y_n)$. This curve is a "good" approximation of the graph of the solution ψ.

Euler's numerical formula

Let $k \in \{0, 1, \ldots, n-1\}$. The value y_{k+1} is found such that the point (x_{k+1}, y_{k+1}) is on the line that contains (x_k, y_k) and has the slope $g(x_k, y_k)$ (the slope of the direction field). Thus

$$\frac{y_{k+1} - y_k}{x_{k+1} - x_k} = g(x_k, y_k).$$

Then

$$y_{k+1} = y_k + g(x_k, y_k)(x_{k+1} - x_k).$$

A particular case: constant step-size $h > 0$: let $x_{k+1} = x_k + h$ for all $k \in \{0, 1, \ldots, n-1\}$. Then the Euler's numerical formula with constant step-size writes as

$$y_{k+1} = y_k + h\, g(x_k, y_k), \quad k \in \{0, 1, \ldots, n-1\}.$$

Numerical formulas with constant step-size

The Euler's numerical formula: $y_{k+1} = y_k + h\, g(x_k, y_k)$.

The improved Euler's numerical formula:

$$y_{k+1} = y_k + h\frac{g(x_k, y_k) + g(x_{k+1}, y_k + h\, g(x_k, y_k))}{2}.$$

Runge-Kutta formulas ...

There are many formulas, some of them are adapted to the specific of the given differential equation. For the exam we need to know just the Euler's numerical formula with constant step-size.

Errors

It is necessary to estimate the error of the approximation and to know how the minimize the error.

Note that the starting point y_0 is the one that appears in the initial condition. Hence $y_0 = \psi(x_0)$ is an exact value. In fact, only in theory y_0 is an exact value, since, practically, when y_0 has too many decimals (for example it is an irrational number) a human or even a computer use only a truncation of it.

When applying a numerical method, the errors are due to the formula itself and to the truncation made. Moreover, the errors accumulate at each step, thus, in general, the errors are small when the interval $[x_0, x^*]$ is small.
When the step-size h is smaller, the partition of the interval $[x_0, x^*]$ is finer. In general the errors are small when the step-size h is small.

Exercise 1

We consider the IVP $y' = 1 + xy^2, \ y(0) = 0$ whose unique solution is denoted by φ. Write the Euler numerical formula with constant step size $h > 0$ for this IVP. Now take $h = 0.1$. Find the number of steps to reach $x^* = 1$. Compute approximate values for $\varphi(0.1)$, $\varphi(0.2)$ and $\varphi(0.3)$. $\diamond$

Here $f(x, y) = 1 + xy^2$, $x_0 = 0$ and $y_0 = 0$. Let n be the number of steps. Then the Euler numerical formula with constant step-size is: $x_{k+1} = x_k + h$ and $y_{k+1} = y_k + h(1 + x_k y_k^2)$ for $0 = 1, ..., n - 1$.

If we start with $x_0 = 0$ and the step-size is $h = 0.1$ then we need $n = 10$ steps to reach $x^* = 1$ and we have $x_k = k/10$.

Since $x_k = k/10$ we have that y_k is an approximate value for $\varphi(k/10)$.

Exercise 1

$x_k = k/10$, $y_0 = 0$ and $y_{k+1} = y_k + 0.1(1 + x_k y_k^2)$ for $k = 0, ..., 9$.

Using the Euler's numerical formula we have

$\varphi(0.1) \approx y_1 = y_0 + 0.1(1 + x_0 y_0^2) = 0.1;$

$\varphi(0.2) \approx y_2 = y_1 + 0.1(1 + x_1 y_1^2) = 0.1 + 0.1(1 + 0.1^3) = 0.2001.$

$\varphi(0.3) \approx y_3 = y_2 + 0.1(1 + x_2 y_2^2) = 0.2001 + 0.1(1 + 0.2 * 0.2001^2) = 0.3001 + 0.1 * 0.04004001 = 0.3005004001.$

... and we do not have any truncation until now :-)!!! But the number of decimals it seems that will increase with each step :-(and at some moment we will not be able to handle all of them ...

Exercise 2

We consider the IVP $y' = y, \; y(0) = 1$ whose solution we know that it is $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = e^x$. Apply the Euler numerical method with a constant step size $h > 0$ on the interval $[0, x^*]$ where $x^* > 0$ is fixed. Prove that

$$y_k = (1+h)^k, \quad k = 0, ..., n \quad \text{where } h = \frac{x^*}{n}.$$

Prove that $y_n \to \varphi(x^*) = e^{x^*}$ as $n \to \infty$. $\diamond$

The Euler's numerical formula for this IVP is
$x_0 = 0$, $y_0 = 1$, $x_k = kh$, $y_{k+1} = y_k + h * y_k$

Indeed, since the number of steps is n, and we want to cover the interval $[0, x^*]$, we have $h = \frac{x^*}{n}$.

Exercise 2

From $y_0 = 1$ and $y_{k+1} = (1+h)y_k$ we get that the sequence (y_k) is a geometric progression and $y_k = (1+h)^k$.

We have $x_n = nh = n\frac{x^*}{n} = x^*$ and $\varphi(x^*) \approx y_n = \left(1 + \frac{x^*}{n}\right)^n$. Indeed we know that its limit is $e^{x^*} = \varphi(x^*)$.

We deduce that, indeed, the approximation is better when the step-size is smaller.

We proved that, for this IVP, the Euler's method is convergent.