Lecture 3

Wednesday, March 9, 2022 7:29 AM

Linear homogeneous differential equations with constant coefficients (continue)

Given the coefficients $a_1, a_2, \dots, a_n \in \mathbb{R}$, with $n \in \mathbb{N}^*$, we consider the LHDE with cc

(1) $x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = 0$

looking for sol. $x = e^{rt}$, $r \in \mathbb{R}$! we found that

(2) $r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$ (called the characteristic eq. of (1)).

It is evident that for $r \in \mathbb{R}$ a root of (2) we have that e^{rt} is a solution of (1).

Q! We want to justify now why for the roots of (2) $r_{1,2} = \alpha \pm i\beta$ (with $\alpha, \beta \in \mathbb{R}, \beta \neq 0$) we have that $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$ are solutions of (1).

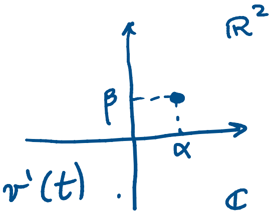
A! The complex exponential: $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$, $\forall \alpha, \beta \in \mathbb{R}$ Euler's formula

A complex function of real variable: $\gamma : I \to \mathbb{C}$, where $I \subset \mathbb{R}$ interval.

$\gamma(t) = u(t) + i v(t)$, $u(t) \overset{not}{=} \mathrm{Re}(\gamma(t))$, $v(t) \overset{not}{=} \mathrm{Im}(\gamma(t))$

we have $u, v : I \to \mathbb{R}$ (real functions)

$\mathbb{C} \approx \mathbb{R}^2$ $\alpha + i\beta \in \mathbb{C} \leftrightarrow (\alpha, \beta) \in \mathbb{R}^2$



When $u, v \in C^1(I)$ we have that $\gamma'(t) = u'(t) + i v'(t)$.

$\gamma' : I \to \mathbb{C}$

Proposition 1 Let $u, v \in C^n(\mathbb{R})$. If $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma = u + iv$, satisfies eq. (1) then u and v are solutions of eq. (1). (this prop. is valid also for LHDE with varying coeff.)

Proof. $\mathcal{L}(x)(t) \overset{not}{=} x^{(n)}(t) + a_1 \cdot x^{(n-1)}(t) + \dots + a_n \cdot x(t)$, $\forall t \in \mathbb{R}$.

$\mathcal{L}(\gamma) = \mathcal{L}(u + iv) \underset{\mathcal{L} \text{ is linear}}{=} \underbrace{\mathcal{L}(u)}_{real} + i \underbrace{\mathcal{L}(v)}_{real} \to \left.\begin{array}{l} \mathcal{L}(u) = \mathrm{Re}(\mathcal{L}(\gamma)) \\ \mathcal{L}(v) = \mathrm{Im}(\mathcal{L}(\gamma)) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \mathcal{L}(u) = 0 \\ \mathcal{L}(v) = 0 \end{array}\right\} \Rightarrow$

ip is that γ satisfies eq (1) $\Rightarrow \mathcal{L}(\gamma) = 0$

$\Rightarrow u$ and v are solutions of (1) (recall that u and v are real functions and $u, v \in C^n(\mathbb{R}, \mathbb{R})$).

Proposition 2 a) $e^{rt} = e^{\alpha t}\cos\beta t + i e^{\alpha t}\sin\beta t$, $\forall t \in \mathbb{R}$

$r = \alpha + i\beta$
$\alpha, \beta \in \mathbb{R}$

b) $(e^{rt})' = r e^{rt}$, $\forall t \in \mathbb{R}$

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Proof. a) use Euler Formula. $u(t) \overset{\text{not}}{=} e^{\alpha t} \cdot \cos \beta t$, $v(t) = e^{\alpha t} \sin \beta t$

Note that $u, v \in C^{\infty}(\mathbb{R})$.

b) $e^{rt} = u(t) + i\, v(t)$, $\forall t \in \mathbb{R}$ $\Rightarrow$ $(e^{rt})' = u'(t) + i\, v'(t)$ $\Big\}$ $\Rightarrow$

$u'(t) = \alpha \underline{e^{\alpha t} \cos \beta t} - \beta e^{\alpha t} \sin \beta t$

$v'(t) = \alpha \underline{e^{\alpha t} \sin \beta t} + \beta e^{\alpha t} \cos \beta t$

$\Rightarrow (e^{rt})' = \alpha \cdot e^{rt} + i\beta \cdot e^{rt} = (\alpha + i\beta) e^{rt} = r e^{rt}$ $\square$.

From here we have:

Let $r \in \mathbb{C}$. e^{rt} satisfies the LHDE (1) iff r is a root of the ch. eq. (2).

1).

Now we apply Prop 1 and Prop 2 a) and obtain that:

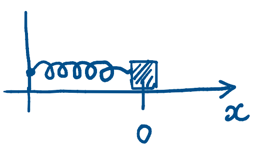
for $r = \alpha + i\beta$, $\beta \neq 0$ root of (2) we get $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$ solutions of (1).

Note that eq (2) has real coefficients, thus if $r_1 = \alpha + i\beta$, $\beta \neq 0$ is a root of (2) then $r_2 = \alpha - i\beta$ is also a root of (2).

$$e^{(\alpha - i\beta)t} = e^{\alpha t} \cdot \cos(-\beta t) + i\, e^{\alpha t} \sin(-\beta t) = \underbrace{e^{\alpha t} \cos \beta t}_{\mathrm{Re}} + i \cdot \underbrace{[-e^{\alpha t} \sin \beta t]}_{\mathrm{Im}}$$

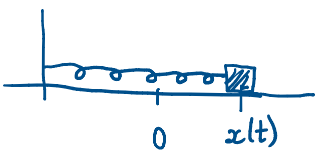
$\Rightarrow e^{\alpha t} \cos \beta t$ and $-e^{\alpha t} \sin(\beta t)$ are solutions. Nothing new!

An application: the spring-mass system



this is the situation when the system is at an equilibrium position (there is no movement)

$x(t)$ measures the displacement of the mass from the equilibrium position $x = 0$.



$x'(t)$ is the velocity, while $x''(t)$ is the acceleration

Newton's second law: $F = m \cdot a$ $\quad m>0 \quad a = x''(t)$

The forces: - the elastic (or restoring) force $F_r = -k\,x(t)$ with $k>0$ (Hooke's law)

- the friction (damping) force $F_d = -\gamma \cdot x'(t)$ with $\gamma>0$

- an external force $F_{ext} = m f(t)$ where f is a given function.

Then $\left.\begin{array}{l} F = F_r + F_d + F_{ext} = -kx - \gamma x' + mf(t) \\ F = ma \end{array}\right\} \Rightarrow -kx - \gamma x' + mf(t) = mx'' \Rightarrow$

$$\Rightarrow \boxed{x'' + \frac{\gamma}{m}x' + \frac{k}{m}x = f(t)}$$

second order L non-HDE with constant coefficients $\frac{\gamma}{m} > 0$, $\frac{k}{m} > 0$.

we consider the following cases:

Case 1. Motion without damping, without external force

$$x'' + \frac{k}{m}x = 0$$

Case 2. Motion with damping, without external force

$$x'' + \frac{\gamma}{m}x' + \frac{k}{m}x = 0$$

Case 3 Motion without damping, with external force

$$x'' + \frac{k}{m}x = A\cos\omega t$$, where $A>0$ and $\omega>0$ parameters.

Case 1. $\omega_0^2 = \frac{k}{m}$ $\quad \omega_0 > 0$

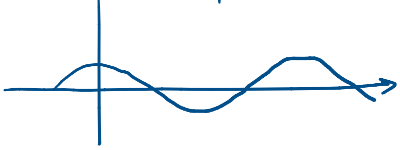
$x'' + \omega_0^2 x = 0$ $\quad r^2 + \omega_0^2 = 0$ $\quad r^2 = -\omega_0^2$ $\quad r_{1,2} = \pm i\omega_0 \mapsto \cos\omega_0 t,\ \sin\omega_0 t$

$x = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$, $\quad c_1, c_2 \in \mathbb{R}$ arbitrary.

Any function like this is periodic with minimal period $\frac{2\pi}{\omega_0}$,

it oscillates around $x=0$ with constant amplitude

is bounded.



perpetuum mobile (no realistic)

perpetuum mobile (no realistic)

Case 2. $x'' + \frac{\gamma}{m} x' + \frac{k}{m} x = 0$

$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0 \qquad \Delta = \frac{\gamma^2}{m^2} - 4\frac{k}{m} = \frac{\gamma^2 - 4km}{m^2}$

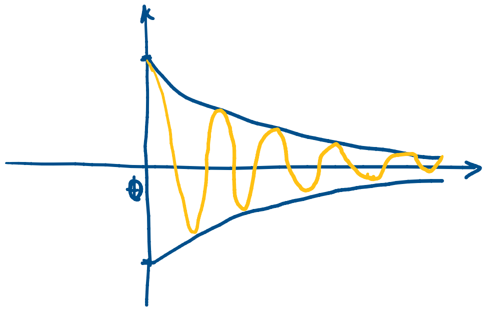
Case 2.1. $\gamma^2 < 4km$ (the damping coeff. is smaller than a critical value $\sqrt{4km}$)

$$r_{1,2} = \frac{-\frac{\gamma}{m} \pm i\frac{\sqrt{4km-\gamma^2}}{m}}{2} = \underbrace{-\frac{\gamma}{2m}}_{\alpha} \pm i \underbrace{\frac{\sqrt{4km-\gamma^2}}{2m}}_{\beta} \mapsto e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t$$

$\boxed{\alpha < 0}$

$x = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t \quad , \quad c_1, c_2 \in \mathbb{R}$

$e^{\alpha t}\cos\beta t$



oscillates with amplitude that decreases to 0 as $t \to \infty$.

Case 2.2. $\gamma^2 > 4km$ (the damping coeff. is bigger than $\sqrt{4km}$)

$r_1, r_2 \in \mathbb{R},\ r_1 \neq r_2 \mapsto e^{r_1 t},\ e^{r_2 t}$

$r_1 + r_2 = -\frac{\gamma}{m} < 0 \quad r_1 \cdot r_2 = \frac{k}{m} > 0 \Rightarrow r_1 < 0 \text{ and } r_2 < 0$

$x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad , \quad c_1, c_2 \in \mathbb{R}$

any such function decreases exponentially to 0 and will not oscillate around 0.

Case 3. $x'' + \omega_0^2 x = A\cos(\omega t)$

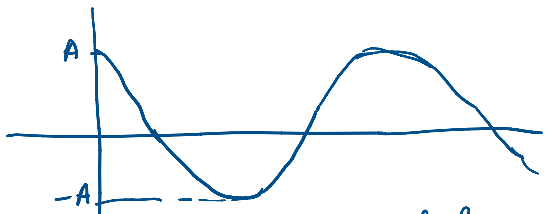
$x'' + \omega_0^2 x = 0 \qquad x_h = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$

$x_p = ? \qquad x = x_h + x_p$

$\omega \neq \omega_0 \qquad x_p = a\cos\omega t + b\sin\omega t$

$a, b \in \mathbb{R} \quad a = ? \quad b = ?$

$\omega = \omega_0 \qquad x_p = t\,(a\cos\omega t + b\sin\omega t)$



the external force has period $\frac{2\pi}{\omega}$ and has frequency ω. is bounded!

Question: $\omega \neq \omega_0$ is any solution periodic ?

is bounded !

Questions: $\omega \neq \omega_0$ is any solution periodic?

$\omega = \omega_0$ any sol. is unbounded resonance.

Tacoma bridge disaster

see report of Alex Pop.

Remark. Resonance is a complex phenomenon. The DE $x'' + \omega_0^2 x = A\cos\omega_0 t$ is just one mathematical model for resonance. It is now known that the mathematical model for the Tacoma bridge disaster is not this simple DE.