20.12.2023

Lagrange multipliers : min / max $f(x)$ subject to $g(x)=c$

$$L(x,\lambda) = f(x) + \lambda(g(x)-c)$$

$$\nabla L = 0 \ , \ \underbrace{\frac{\partial L}{\partial x}}_{= \nabla_x L} = 0 = \frac{\partial L}{\partial \lambda}$$

(all the partial derivatives should be zero)

(x could be a vector)

1) a) $f(x,y) = x^2 + y^2 \ , \ g(x,y) = x - y + 1$

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$$

$$= x^2 + y^2 + \lambda \cdot (x - y + 1)$$: Lagrange function

$$\frac{\partial L}{\partial x} = 2x + \lambda$$

$$\frac{\partial L}{\partial y} = 2y - \lambda$$

$$\frac{\partial L}{\partial \lambda} = \underbrace{x - y + 1}_{g(x,y)}$$

$$\begin{cases} 2x+\lambda = 0 \\ 2y-\lambda = 0 \\ x-y+1 = 0 \end{cases}$$

$(+) \Leftrightarrow x+y=0,\ y=-x$

$\Rightarrow 2x+1=0 \Rightarrow x=-\frac{1}{2},\ y=\frac{1}{2}$

$\lambda = 1$

$f(-\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2} \Rightarrow$ the extremum of the function f (either the minimum or the maximum of the func.)

Another solution: $x-y+1=0,\ y=x+1;\ f(x,y)=x^2+y^2=$

$=x^2+(x+1)^2 = 2x^2+2x+1,$ Vertex $: x=-\frac{1}{2}$ (min)

b) $f(x,y)=(x+y)^2,\ g(x,y)=x^2+y^2-1=0$

$=x^2+2xy+y^2 = 1+2xy$

$L(x,y,\lambda) = f(x,y)+\lambda\cdot g(x,y)$

$= x^2+2xy+y^2+\lambda(x^2+y^2-1)$

$$\frac{\partial L}{\partial x} = 2x + 2y + 2\lambda x = 0$$

$$\frac{\partial L}{\partial y} = 2x + 2y + 2\lambda y = 0$$

$- \iff \lambda(x-y) = 0 \iff$

$\iff \lambda = 0$ or $x = y$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

(i) $\lambda = 0$, $x + y = 0$, $y = -x$, $x^2 = y^2$,

$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}}$; $y = \mp\sqrt{\frac{1}{2}}$

(ii) $x = y$, $(2+\lambda)x = 0 \Rightarrow \lambda = -2$

$2x^2 = 1$, $x = \pm\sqrt{\frac{1}{2}} = y$

obs: we will have 4 extremum points with these constraints.

(i) $f(x,y) = (x+y)^2 = 0 \rightarrow$ MIN points

(ii) $f(x,y) = (x+y)^2 = 4x^2 = 2 \rightarrow$ MAX points.

$\hookrightarrow$ can be skipped but it is natural to do

d) $f(x,y,z) = x + 2y + 3z$

$g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda \cdot g(x,y,z)$$

$$= x + 2y + 3z + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial L}{\partial x} = 1 + 2\lambda x = 0$$

$$\frac{\partial L}{\partial y} = 2 + 2\lambda y = 0$$

$$\frac{\partial L}{\partial z} = 3 + 2\lambda z = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0$$

$$\begin{cases} 1 + 2\lambda x = 0 \quad , \quad x = -\dfrac{1}{2\lambda} \\ 2 + 2\lambda y = 0 \quad , \quad y = -\dfrac{1}{\lambda} \\ 3 + 2\lambda z = 0 \quad , \quad z = -\dfrac{3}{2\lambda} \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases}$$

$$\hookrightarrow \quad \frac{1}{4\lambda^2} + \frac{\overset{4}{\cancel{1}}}{\lambda^2} + \frac{9}{4\lambda^2} = 1 \iff \frac{14}{4\lambda^2} = 1$$

$$14 = 4\lambda^2 \quad ; \quad 2\lambda^2 = 7 \iff \lambda = \pm\sqrt{\frac{7}{2}} \ \ldots$$

$$f(x,y,z) = x + 2y + 3z = -\frac{1}{2\lambda} - \frac{\overset{3}{\cancel{2}}}{\lambda} - \frac{9}{2\lambda} = -\frac{14}{2\lambda} = -\frac{7}{\lambda}$$

$\lambda = \sqrt{\frac{7}{2}}$, $f(x,y,z) \longrightarrow$ MIN

$\lambda = -\sqrt{\frac{7}{2}}$, $f(x,y,z) \rightarrow$ MAX

e) $f(x,y,z) = 2x^2 + y^2 + 3z^2$

$g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda \cdot g(x,y,z)$$

$$= 2x^2 + y^2 + 3z^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial L}{\partial x} = 4x + 2\lambda x = 0 \quad , \quad (2+\lambda)x = 0$$

$$\frac{\partial L}{\partial y} = 2y + 2\lambda y = 0 \quad , \quad (1+\lambda)y = 0$$

$$\frac{\partial L}{\partial z} = 6z + 2\lambda z = 0 \quad , \quad (3+\lambda)\cdot z = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0$$, Note that we cannot have that $x = y = 0$

i) $\lambda = -2$, $y = 0 = z$, $x^2 = 1 \Leftrightarrow x = \pm 1$

$f(x,y,z) = 2x^2 = 2$

ii) $\lambda = -1$, $x = 0 = z$, $y^2 = 1 \Leftrightarrow y = \pm 1$

$f(x,y,z) = y^2 = 1 \rightarrow$ MIN

iii) $\lambda = -3$, $x = 0 = y$, $z^2 = 1 \Leftrightarrow z = \pm 1$

$f(x,y,z) = 3z^2 = 3 \rightarrow$ MAX
↳ highest value of the function.

6 solutions total. (all of them are <u>extremum</u> points)

2) a) $x_1+x_2+x_3=3$

$$f(x_1,x_2,x_3)=\frac{1}{2}(x_1^2+x_2^2+x_3^2)\ ,\ g(x_1,x_2,x_3)=$$

$$=x_1+x_2+x_3-3=0$$

$$L(x_1,x_2,x_3,\lambda)=\frac{1}{2}(x_1^2+x_2^2+x_3^2)+\lambda(x_1+x_2+x_3-3)$$

$$\left.\begin{aligned}
\frac{\partial L}{\partial x_1}&=x_1+\lambda=0\\
\frac{\partial L}{\partial x_2}&=x_2+\lambda=0\\
\frac{\partial L}{\partial x_3}&=x_3+\lambda=0
\end{aligned}\right\}\quad x_1=x_2=x_3=-\lambda=1$$

$$\frac{\partial L}{\partial \lambda}=x_1+x_2+x_3-3=0\ \Rightarrow 3(-\lambda)-3=0\Rightarrow \lambda=-1$$

$$f(x_1,x_2,x_3)=\frac{3}{2}\to \text{MiN}.$$

b) we use 2 Lagrange multipliers.

$$L(x_1, x_2, x_3, \lambda, \mu) = f(x_1, x_2, x_3) + \lambda \cdot \underbrace{g(1)}_{\text{first constraint}} + \mu \cdot \underbrace{g(2)}_{\text{second constraint}}$$

$$= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda(x_1 + x_2 + x_3 - 3) + \mu(x_1 + 2x_2 + 3x_3 - 12)$$

$$\frac{\partial L}{\partial x_1} = x_1 + \lambda x_1 + \mu x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = x_2 + \lambda x_2 + 2\mu x_2 = 0$$

$$\frac{\partial L}{\partial x_3} = x_3 + \lambda x_3 + 3\mu x_3 = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 3 = 0$$

$$\frac{\partial L}{\partial \mu} = x_1 + 2x_2 + 3x_3 - 12 = 0$$

$$(1)+(2)+(3): \quad 3+3\lambda+6\mu = 0 \iff$$
$$\iff \lambda + 2\mu = -1$$
$$\iff 2\mu = -1-\lambda$$

$$(1)+2(2)+3(3): \quad 12+6\lambda+14\mu = 0$$
$$12+6\lambda-7-7\lambda = 0; \quad \lambda = 5; \quad \mu = -3$$

$$x_1 = -2$$
$$x_2 = 1$$
$$x_3 = 4$$

$$f(-2,1,4) = \frac{1}{2}(4+1+16) = \frac{21}{2}$$

3) a) $\iint_R \cos x \cdot \sin y \, dx\, dy$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

(Seperable function)

$$\int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \cos x \cdot \underbrace{\sin y}_{\text{const.}} \, dx \right) dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y \underbrace{\left(\int_0^{\frac{\pi}{2}} \cos x \, dx \right)}_{\text{const.}} dy =$$

$$= \left(\int_0^{\frac{\pi}{2}} \sin y \, dy\right) \cdot \left(\int_0^{\frac{\pi}{2}} \cos x \, dx\right)$$

$$= 1 \cdot 1 = 1$$

b) $R = [1,2] \times [0,1]$

1) $$\int_0^1 \left(\int_1^2 \frac{1}{(x+y)^2} \, dx\right) dy = \int_0^1 \left(-\frac{1}{y+2} + \frac{1}{y+1}\right) dy =$$

$$\int_1^2 \frac{1}{(x+y)^2} dx = -\frac{1}{x+y}\Big|_1^2 = -\frac{1}{y+2} + \frac{1}{y+1}$$

$$= -\ln(y+2)\Big|_0^1 + \ln(y+1)\Big|_0^1 = -\ln 3 + \ln 2 + \ln 2 - \ln \frac{4}{3}$$

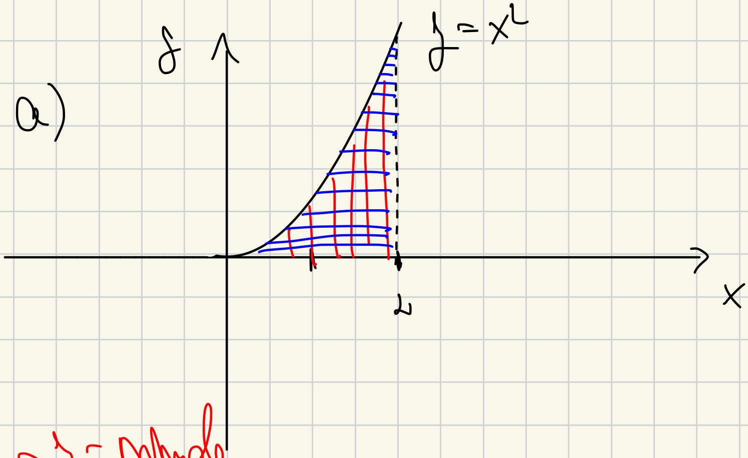
2) $$\int_0^1 \int_1^2 y \cdot e^{xy} \, dx \, dy = \int_0^1 \left(e^{xy}\right)\Big|_{x=1}^{x=2} dy =$$

$$= \int_0^1 \left(e^{2y} - e^{y}\right) dy = \frac{e^{2y}}{2}\Big|_0^1 - e^{y}\Big|_0^1 =$$

$$= \frac{e^2}{2} - \frac{1}{2} - e + 1 = \frac{e^2}{2} - e + \frac{1}{2}$$

$$\int y \cdot e^{xy} dx = e^{xy} \quad , \quad \int 2e^{2x} dx = e^{2x}$$

4) a)



y - simple

• $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$ $\to$ $\sqrt{y} \le x$

OR

x - simple

• $D = \{(x,y) \mid 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}$

b) $\iint_D x \cdot y \, dx\, dy =$

$$\overset{(1)}{=} \int_0^2 \left(\int_0^{x^2} xy \, dy \right) dx = \int_0^2 x \cdot \left. \frac{y^2}{2} \right|_{y=0}^{y=x^2} dx =$$

$$= \int_0^2 \frac{x^5}{2} \, dx = \left. \frac{x^6}{12} \right|_0^2 = \frac{64}{12} = \frac{16}{3}$$

(2)

$$= \int_0^4 \left(\int_{\sqrt{y}}^{2} x \cdot y \, dx \right) dy = \int_0^4 y \cdot \frac{x^2}{2} \Big|_{x=\sqrt{y}}^{x=2} dy =$$

$$= \int_0^4 2y - \frac{y^2}{2} \, dy = y^2 \Big|_0^4 - \frac{y^3}{6} \Big|_0^4$$

$$= 16 - \frac{64}{6} = 16 - \frac{32}{3} = \frac{16}{3}$$