Lecture 5

Wednesday, March 23, 2022 7:29 AM

Linear differential systems

Let $n \in \mathbb{N}^*$, $A \in C(I, \mathcal{M}_n(\mathbb{R}))$ where $I \subset \mathbb{R}$ open interval, $F \in C(I, \mathbb{R}^n)$.
We consider the system

(1) $\quad X' = A(t)X + F(t)$

this is a linear non-hom (when $F \not\equiv 0$) differential system in $\mathbb{R}^n$.
Here $A(t)$ is the coefficients matrix, $F(t)$ is the non-hom. part of (1),
while $X' = A(t)X$ is the homogeneous part of (1).

Def 1 A solution of (1) is a function $\varphi : I \to \mathbb{R}^n$ of class C^1 such that

$$\varphi'(t) = A(t)\varphi(t) + F(t), \quad \forall t \in I.$$

Now we write the system by components

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & & & \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix} \qquad F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \dots \\ f_n(t) \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$(1) \iff \begin{cases} x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + f_2(t) \\ \dots \\ x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t) \end{cases}$$

Def 2. A matrix solution of $X' = A(t)X$ is a function $U : I \to \mathcal{M}_n(\mathbb{R})$ of class C^1 s.t.

$$U'(t) = A(t)U(t), \quad \forall t \in I.$$

Property A matrix function $U : I \to \mathcal{M}_n(\mathbb{R})$, of class C^1, is a matrix solution of $X' = A(t)X$ if and only if each column of $U(t)$ is a solution of $X' = A(t)X$.

Theorem 1. (The existence and uniqueness theorem for the IVP).
Let $t_0 \in I$ and $\eta \in \mathbb{R}^n$. The IVP $\begin{cases} X' = A(t)X + F(t) \\ X(t_0) = \eta \end{cases}$ has a unique solution.

Definition The matrix function $E : \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ which is the unique

... the exponential of

Definition The matrix function $E: \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ which is the unique matrix function such that $\begin{cases} E'(t) = A\,E(t) \\ E(0) = I_n \end{cases}$ is said to be the exponential of the matrix (tA) and is denoted by $E(t) = e^{tA}$ or $E(t) = exp(tA)$.

Here $A \in \mathcal{M}_n(\mathbb{R})$ and $I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix}$ is the identity matrix.

Property Let $A \in \mathcal{M}_n(\mathbb{R})$. The series

$$I_n + A + \frac{1}{2!}A^2 + \dots + \frac{1}{k!}A^k + \dots$$

is convergent to e^A.

Thus, $\boxed{e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k}$ (an equivalent definition of the matrix exponential)

Examples. Compute $e^{t\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$ in two ways where $\lambda_1, \lambda_2 \in \mathbb{R}$.

Sol $E(t) = e^{tA}$, where $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

First we use the definition of e^{tA}.

Step 1 we write the LHS with matrix A

$$X' = AX \iff \begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \end{cases}$$

this is an uncoupled system, so it is easy to solve it.

$x_1' = \lambda_1 x_1 \iff x_1' - \lambda_1 x_1 = 0$

$r - \lambda_1 = 0$

$r = \lambda_1 \mapsto e^{\lambda_1 t}$

$\Rightarrow \begin{cases} x_1 = c_1 e^{\lambda_1 t} & c_1 \in \mathbb{R} \\ x_2 = c_2 e^{\lambda_2 t} & c_2 \in \mathbb{R} \end{cases}$

Step 2. we formulate 2 ivp's

$$(2) \begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \\ x_1(0) = 1 \\ x_2(0) = 0 \end{cases}$$

has the unique solution $\varphi(t) = \begin{pmatrix} e^{\lambda_1 t} \\ 0 \end{pmatrix}$

$$(3) \begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \lambda_2 x_2 \\ x_1(0) = 0 \\ x_2(0) = 1 \end{cases}$$

has the unique solution $\psi(t) = \begin{pmatrix} 0 \\ e^{\lambda_2 t} \end{pmatrix}$

Step 3. $E(t) = \begin{pmatrix} \text{sol. of } (2) & \text{sol. of } (3) \end{pmatrix} = \begin{pmatrix} \varphi(t) & \psi(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

Conclusion $e^{t\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}, \ \forall t \in \mathbb{R}.$

Conclusion $$e^{t\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{pmatrix}, \quad \forall t\in\mathbb{R}.$$

Now we use the equivalent definition with Taylor series.

$$e^{tA} = I_2 + tA + \frac{1}{2!}(tA)^2 + \dots + \frac{1}{k!}(tA)^k + \dots = I_2 + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^k}{k!}A^k + \dots$$

$$A = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \qquad A^2 = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = \begin{pmatrix}\lambda_1^2 & 0\\ 0 & \lambda_2^2\end{pmatrix} \dots$$

By induction one can prove that $A^k = \begin{pmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{pmatrix}$ $\forall k\in\mathbb{N}$

$$e^{tA} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\begin{pmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{pmatrix} = \begin{pmatrix}\sum_{k=0}^{\infty}\frac{1}{k!}(\lambda_1 t)^k & 0\\ 0 & \sum_{k=0}^{\infty}\frac{1}{k!}(\lambda_2 t)^k\end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{pmatrix}$$

Linear homogeneous systems of dimension 2 with constant coefficients solved using the reduction method.

$$(4)\ \begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y\end{cases} \qquad A = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

We assume that (4) is a coupled system, i.e. either $a_{12}\neq 0$ or $a_{21}\neq 0$.

To fix, consider that $a_{12}\neq 0$. Our aim is to find a second order LDE for x, i.e. a relation between x, x' and x''. (Comment: if $a_{12}=0$ but $a_{21}\neq 0$ we find a relation between y, y', y'')

$$x' = a_{11}x + a_{12}y \Rightarrow y = \frac{1}{a_{12}}(x' - a_{11}x) \quad (5)$$

$$x' = a_{11}x + a_{12}y \Rightarrow x'' = a_{11}x' + a_{12}y' \underset{y' = a_{21}x + a_{22}y}{\Rightarrow} x'' = a_{11}x' + a_{12}\cdot(a_{21}x + a_{22}y) \overset{(5)}{\Rightarrow}$$

$$\Rightarrow x'' = a_{11}x' + a_{12}a_{21}x + a_{22}(x' - a_{11}x) \Rightarrow$$

$$\Rightarrow x'' - (a_{11}+a_{22})x' + (a_{11}a_{22} - a_{12}a_{21})x = 0 \quad (6)$$ this is a second

$$\Rightarrow \quad x'' - (a_{11}+a_{22})x' + (a_{11}a_{22} - a_{12}a_{21})x = 0 \quad (6)$$ this is a second order LHDE with constant coefficients that can be solved using the characteristic equation method.

Remarks. The characteristic polynomial of (6) is $\ell(r) = r^2 - \underbrace{(a_{11}+a_{22})}_{\mathrm{tr}\,A} r + \underbrace{(a_{11}a_{22}-a_{12}a_{21})}_{\det A}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Note that $\ell(r)$ is the characteristic polynomial of A.

Recall that the roots of $\ell(r)$ are the eigenvalues of A.

Recall the definition of an eigenvalue and an eigenvector of a matrix $A \in \mathcal{M}_n(\mathbb{R})$:

we say that $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix A when $\exists\, u \in \mathbb{R}^n$, $u \neq 0$ such that $Au = \lambda u$. This u is said to be an eigenvector of A corresp. to the eigenvalue λ.

Example. Compute in two ways $e^{t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$, $t \in \mathbb{R}$.

Method I. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the LHDS having A as the coeff. matrix is

Step 1

$\begin{cases} x' = y \\ y' = -x \end{cases}$ this is a coupled system. We use the reduction method to find its general sol.

$x'' = y' \Rightarrow x'' = -x \Rightarrow x'' + x = 0$, $y = x'$

$r^2 + 1 = 0$ $r^2 = -1$ $r_{1,2} = \pm i \mapsto \cos t, \sin t$

$(\alpha \pm i\beta \mapsto e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t)$

$\begin{cases} x = c_1\cos t + c_2 \sin t \\ y = -c_1 \sin t + c_2\cos t \end{cases}$, $c_1, c_2 \in \mathbb{R}$

Step 2 $\begin{cases} x' = y \\ y' = -x \\ x(0) = 1 \\ y(0) = 0 \end{cases}$

$x = c_1\cos t + c_2\sin t$ $\quad x(0) = c_1$

$y = -c_1\sin t + c_2\cos t$ $\quad y(0) = c_2$ $\quad \Rightarrow c_1 = 1$ and $c_2 = 0$

the unique sol. of this IVP is $\varphi(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

$\begin{cases} x' = y \\ y' = -x \end{cases}$ $c_1 = 0$ the unique sol. of this IVP is $\psi(t) = \begin{pmatrix} \sin t \\ \ldots \end{pmatrix}$

$$\begin{cases} x' = y \\ y' = -x \\ x(0) = 0 \\ y(0) = 1 \end{cases} \qquad \begin{array}{l} c_1 = 0 \\ c_2 = 1 \end{array}$$

the unique sol. of this IVP is $\varphi(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

Step 3

$$e^{t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \forall t \in \mathbb{R}$$

Method II HW