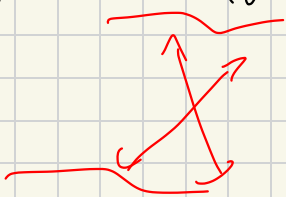
22.11.2023

1) a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\|x+y\|^2 = \langle x+y, x+y \rangle =$ $\|x\|^2 = \langle x, x \rangle$

$= \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle$

$\|x-y\|^2 = \langle x-y, x-y \rangle =$

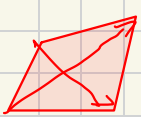


$= \langle x, x \rangle + \langle y, y \rangle - 2\langle x, y \rangle$ (+)

(+) $\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\langle x, x \rangle + 2\langle y, y \rangle =$

$= 2(\langle x, x \rangle + \langle y, y \rangle)$

$= 2(\|x\|^2 + \|y\|^2)$



$d_1^2 + d_2^2 =$

b) $\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$

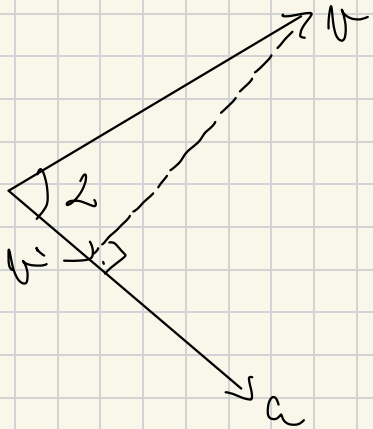
$\|x+y\|^2 = \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle$

$\|x-y\|^2 = \langle x, x \rangle + \langle y, y \rangle - 2\langle x, y \rangle$ (−)

$\Leftarrow$ $\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$ $(\div 4)$

$\Leftrightarrow$ $\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$

3. Find the orthogonal projection of a vector $v \in \mathbb{R}^2$ onto a vector $a \in \mathbb{R}^2$.



$v' = C \cdot a$, $C = ?$

$\cos(\alpha) = \frac{\|v'\|}{\|v\|}$

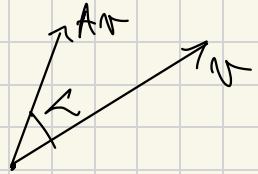
$\cos(\alpha) = \frac{\|C \cdot a\|}{\|v\|} = \frac{C \cdot \|a\|}{\|v\|}$

$\langle v, a\rangle = v \cdot a = \|v\| \cdot \|a\| \cdot \cos(\alpha) \Rightarrow$

$\Rightarrow \cos(\alpha) = \frac{\langle v, a\rangle}{\|v\| \cdot \|a\|}$

$C = \frac{\cos(\alpha) \cdot \|v\|}{\|a\|} = \frac{\frac{\langle v, a\rangle}{\|v\| \cdot \|a\|} \cdot \|v\|}{\|a\|} = \frac{\langle v, a\rangle}{\|a\|^2}$

4) $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

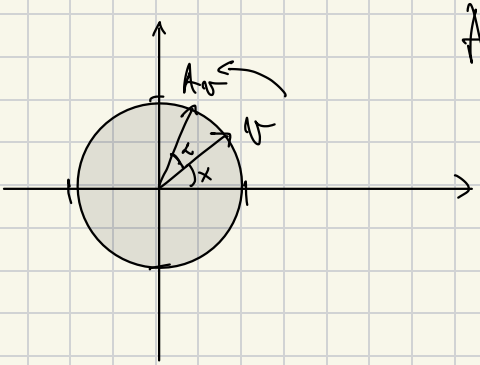


$Av = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$v = \begin{bmatrix} r\cos x \\ r\sin x \end{bmatrix}$

$$Av = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} r\cos x \\ r\sin x \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\alpha\cdot\cos x - r\sin\alpha\sin x \\ r\sin\alpha\cos x + r\cos\alpha\sin x \end{bmatrix} = \begin{bmatrix} r\cos(x+\alpha) \\ r\sin(\alpha+x) \end{bmatrix}$$

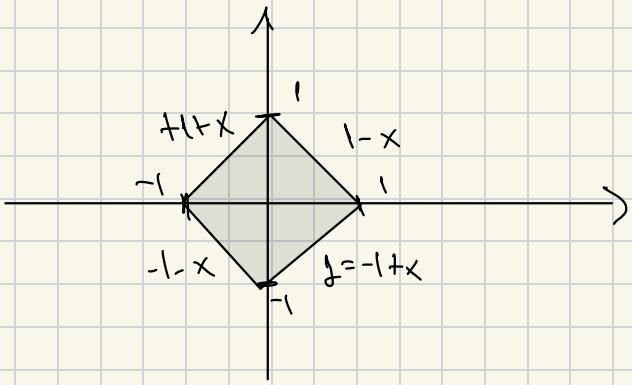


A^{-1} → will rotate with an angle of $(-\alpha)$.

5) $\|x\|_p := \left(|x_1|^p + \dots + |x_m|^p\right)^{\frac{1}{p}}, \quad p \geq 1$

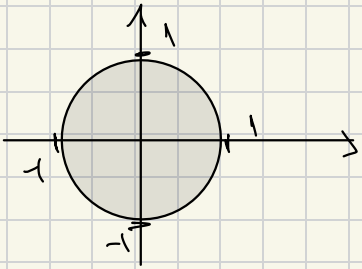
$\|x\|_\infty := \max\{|x_1|, \dots, |x_m|\}$

- $p=1 \quad \|x\| = |x_1| + \dots + |x_m|$



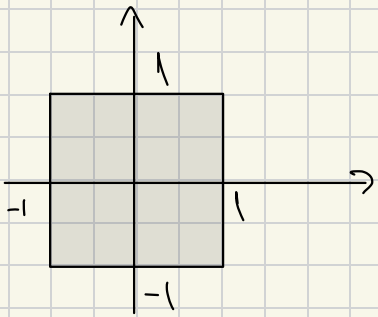
$$\|x\|_1 = |x_1| + |x_2| = 1$$

- $p=2 \Rightarrow \|x\|_2 = \sqrt{x_1^2 + x_2^2}$



$x_1^2 + x_2^2 = 1 \Rightarrow$ cerchio

- $p=\infty \Rightarrow \|x\|_\infty = \max\{|x_1|, |x_2|\}$



$\|x\|_\infty = 1$

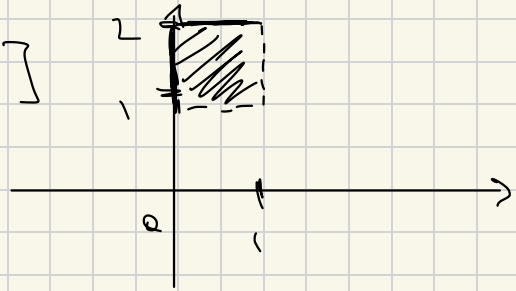
$x_1 = 1 \;,\; x_2 \in [-1, 1]$

$x_2 = 1 \;,\; x_1 \in [-1, 1]$

$x_1 = -1 \;,\; x_2 \in [-1, 1]$

$x_2 = -1 \;,\; x_1 \in [-1, 1]$

7) a) $[0,1) \times (1,2]$



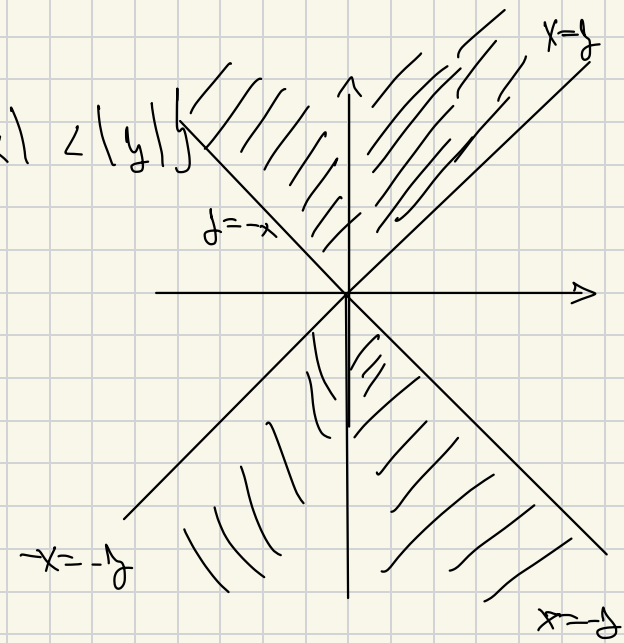
$\text{Int}(A) = (0,1) \times (1,2)$

$\text{cl}(A) = [0,1] \times [1,2]$

$\text{bd}(A) = [0,1] \times \{1\} \cup [0,1] \times \{2\} \cup \{1\} \times [1,2] \cup$

$\cup \{0\} [1,2]$

b) $B = \{(x,y) \in \mathbb{R}^2 \mid |x| < |y|\}$



$x, y \geq 0$, $x < y$

$x < 0, y > 0$, $-x < y$

$x < 0, y < 0$, $-x < -y$

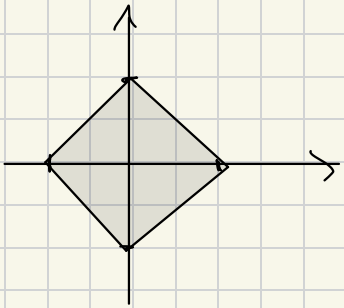
$x > 0, y < 0$ $x < -y$

$\text{Int}(B) = \{(x,y) \in \mathbb{R}^2 \mid |x| < |y|\} = B$

$\text{cl}(B) = \{(x,y) \in \mathbb{R}^2 \mid |x| \leq |y|\}$

$\text{bd}(B) = \{(x,y) \in \mathbb{R}^2 \mid |x| = |y|\}$

c) $C = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\}$

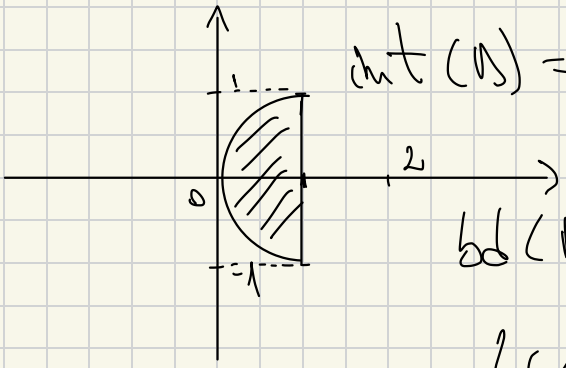


$\text{int}(C) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| < 1\} = C$

$\text{cl}(C) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$

$\text{bd}(C) = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$

d) $D = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1, \ x \leq 1\}$



$\text{int}(D) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1, \ x < 1\}$

$\text{cl}(D) = D$

$\text{bd}(D) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\} \cup$

$\{(1,y) \in \mathbb{R}^2 \mid y^2 \leq 1\}$

$\{1\} \times [-1,1]$

g) the Geogebra