1.4 Exercises

1. Let $A_0,\dots,A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1}+\overrightarrow{A_1A_2}+\dots+\overrightarrow{A_{n-1}A_n}+\overrightarrow{A_nA_0}$.

2. In each of the following cases, decide if the indicated vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ can be represented with the vertices of a triangle:

a) $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.

b) $\mathbf{u}(1,2,-1)$, $\mathbf{v}(2,-1,0)$, $\mathbf{w}(1,-3,1)$.

3. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

$$\overrightarrow{MN}+\overrightarrow{PQ}=0.$$

Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.

4. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF}=\frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{CD}\right)=\frac{1}{2}\left(\overrightarrow{AD}+\overrightarrow{CB}\right).$$

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF}=\frac{1}{2}\left(\overrightarrow{AD}+\overrightarrow{BC}\right).$$

6. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'}+\overrightarrow{OB'}+\overrightarrow{OC'}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}.$$

7. Show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.

8. In each of the following cases, decide if the given points are collinear:

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$.

b) $A(11,2)$, $B(1,-3)$, $C(31,13)$.

c) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$.

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$.

9. Let $ABCD$ be a tetrahedron. Determine the sums

a) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$,

b) $\overrightarrow{AD}+\overrightarrow{BC}+\overrightarrow{DB}$,

c) $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{BC}+\overrightarrow{DA}$.

10. Let $ABCD$ be a tetrahedron. Show that

$$\overrightarrow{AD}+\overrightarrow{BC}=\overrightarrow{BD}+\overrightarrow{AC}.$$

11. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.

12. Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x = 1$, $y = 3$ and $z = -2$.

13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

14. Which of the following sets of vectors form a basis?

a) $\mathbf{v}(1,2)$, $\mathbf{w}(3,4)$;

b) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(1,0,-1)$;

c) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(0,5,3)$;

d) $\mathbf{v}_1(-1,2,1,0)$, $\mathbf{v}_2(2,1,1,0)$, $\mathbf{v}_3(1,0-1,1)$, $\mathbf{v}_4(1,0,0,1)$;

15. With respect to the basis $\mathcal{B} = (\mathbf{i},\mathbf{j},\mathbf{k})$ consider the vectors $\mathbf{u} = \mathbf{i}+\mathbf{j}$, $\mathbf{v} = \mathbf{j}+\mathbf{k}$ and $\mathbf{w} = \mathbf{i}+\mathbf{k}$. Check that $\mathcal{B}' = (\mathbf{u},\mathbf{v},\mathbf{w})$ is a basis and give the base change matrix $\mathrm{M}_{\mathcal{B}',\mathcal{B}}$.

16. Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}')$ given in Example 1.20. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix}1\\2\end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix}2\\4\end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix}3\\1\end{bmatrix}$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

17. Consider the tetrahedron $ABCD$ and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

b) the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,

c) the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

18. Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j},\mathbf{k})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}',\mathbf{k}')$ given in Example 1.21. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix}1\\4\\-1\end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix}1\\5\\1\end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix}-3\\7\\1\end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix}0\\3\\1\end{bmatrix}$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

Midterm : - week 7 : 13.04 => G_1
- week 14 : 08.06 => G_2

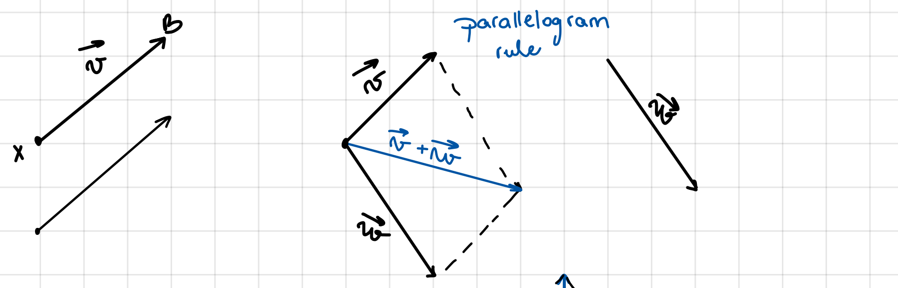
Final : $0.4 * G_1 + 0.6 * G_2 + \underbrace{B}_{\in [0,1]}$

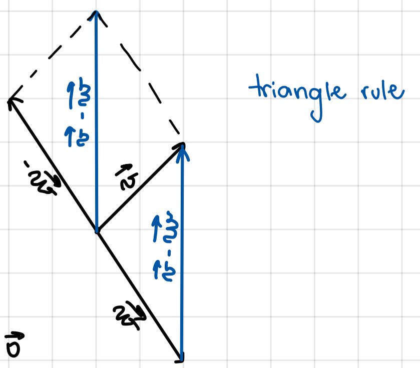
Minimal conditions:
- 75% attendance
- $0.4 * G_1 + 0.6 * G_2 \geq 5$

Retake : One final exam , $\cancel{B}$

E^n n-dimension Euclidean space
(usually n = 2 or 3)
"space of points"

V^n space of vectors in E^n





$A_1, A_2, \ldots, A_k \in E^n$

$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \ldots + \overrightarrow{A_{k-1}A_k} + \overrightarrow{A_kA_1} = \vec{0}$

Fix $O \in E^n$

$\forall A \in E^n$ we define $\vec{r_A} = \overrightarrow{OA}$, the position vector of A (with respect to O)

$A, B, M \in E^n$

M midpoint of (AB) $\Longleftrightarrow$ $\vec{r_M} = \dfrac{\vec{r_A} + \vec{r_B}}{2}$

1.4 ABCD - quadrilateral

E midpoint of [AC]

F midpoint of [BD]

$$\rightarrow \vec{EF} = \frac{1}{2}\left(\vec{AB}+\vec{CD}\right) = \frac{1}{2}\left(\vec{AD}+\vec{CB}\right)$$

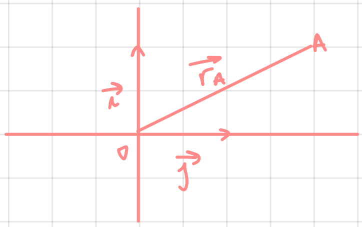
$$\vec{EF} = \vec{EO} + \vec{OF} = \vec{r}_F - \vec{r}_E$$

$$\vec{r}_F = \frac{\vec{r}_A + \vec{r}_C}{2}, \quad \vec{r}_E = \frac{\vec{r}_B + \vec{r}_D}{2}$$

$$\frac{1}{2}\left(\vec{r}_A + \vec{r}_C - \vec{r}_B - \vec{r}_D\right) = \frac{1}{2}\left(\vec{AB}+\vec{CD}\right) = \frac{1}{2}\left(\vec{AD}+\vec{CB}\right)$$

$A, B, C \in E^n$ collinear $\Leftrightarrow \exists\, l$ line, $l \in E^n$ st. $l \ni A, B, C \Leftrightarrow \operatorname{rank}(\vec{AB}, \vec{BC}, \vec{CA}) = 1$
$\Leftrightarrow \exists\, \alpha \in \mathbb{R} : \vec{AB} = \alpha \vec{BC}$

$\mathcal{K} = (O, \mathcal{B})$
reference system
- $O \in E^n$
- $\mathcal{B}$ basis of V^n



$[A]_{\mathcal{K}} = [\vec{r}_A]_{\mathcal{B}}$

A, B, C collinear
$\Leftrightarrow \exists\, \alpha \in \mathbb{R}$

$$[\vec{AB}]_{\mathcal{K}} = \alpha\, [\vec{BC}]_{\mathcal{B}}$$

1.8 Decide if the following points are collinear:

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$

$\vec{v}, \vec{w}$ are collinear if $\vec{v} = k\vec{w}$, $k \in \mathbb{R}$

$$\vec{PQ} = \vec{r}_Q - \vec{r}_P = (x_Q - x_P)\vec{i} + (y_Q - y_P)\vec{j}$$
$$= (-1-3)\vec{i} + (2+5)\vec{j}$$
$$= -4\vec{i} + 7\vec{j}$$

$$\vec{QR} = \vec{r}_R - \vec{r}_Q = (x_R - x_Q)\vec{i} + (y_R - y_Q)\vec{j}$$
$$= (-5+1)\vec{i} + (9-2)\vec{j}$$
$$= -4\vec{i} + 7\vec{j}$$

$\vec{PQ} = \vec{QR} \Rightarrow P, Q, R$ are collinear

b) $A(11,2)$, $B(1,-3)$, $C(31,13)$

$$\vec{AB} = \vec{r_B} - \vec{r_A} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$$
$$= (1-11)\vec{i} + (-3-2)\vec{j}$$
$$= -10\vec{i} - 5\vec{j}$$

$$\vec{BC} = \vec{r_C} - \vec{r_B} = (x_C - x_B)\vec{i} + (y_C - y_B)\vec{j}$$
$$= (31-11)\vec{i} + (13+3)\vec{j}$$
$$= 20\vec{i} + 16\vec{j}$$

A, B, C are not collinear

c) $A(1,0,-1)$, $B(0,-1,2)$, $C(-1,-2,5)$

$$\vec{AB} = \vec{r_B} - \vec{r_A} = -\vec{i} - \vec{j} + 3\vec{k}$$
$$\vec{BC} = \vec{r_C} - \vec{r_B} = -\vec{i} - \vec{j} + 3\vec{k}$$

A, B, C collinear

d) $A(-1,-1,4)$, $B(1,1,0)$, $C(2,2,5)$

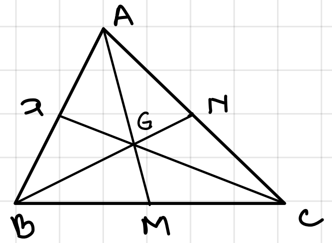
$$\vec{AB} = (0,0,-4)$$
$$\vec{BC} = (1,1,5)$$

$0 \neq 0 \neq \frac{-4}{5}$ => A, B, C are not collinear

$A, B, T \in E^n$

$T \in AB \iff \exists\, \lambda \in \mathbb{R} \,.\, \vec{r_T} = \lambda \vec{r_A} + (1-\lambda)\vec{r_B}$ (vector equation for the line AB)

1.7 Show that the medians in a $\triangle ABC$ intersect in a point, and find the ratio that this point divides the medians in.



M, N, P midpoints of [BC], [AC], [AB]

Let $\{G\} = AM \cap BN$

We will show that $G \in CP$

$G \in AM \Rightarrow \exists\, \lambda \in \mathbb{R}$

$$\vec{r_G} = \lambda \vec{r_A} + (1-\lambda)\vec{r_M} = \lambda \vec{r_A} + \frac{1-\lambda}{2}\vec{r_B} + \frac{1-\lambda}{2}\vec{r_C}$$

$G \in BN \Rightarrow \exists\, \mu \in \mathbb{R}$

$$\vec{r_G} = \mu \vec{r_B} + (1-\mu)\vec{r_N} = \frac{1-\mu}{2}\vec{r_A} + \mu\vec{r_B} + \frac{1-\mu}{2}\vec{r_C}$$

$$\vec{r_A}\left(\lambda - \frac{1-\mu}{2}\right) + \vec{r_B}\left(\frac{1-\lambda}{2} - \mu\right) + \vec{r_C}\left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right) = \vec{0}$$

$\triangle ABC$ non-degenerate

So, $\vec{v} = \vec{AB}$, $\vec{w} = \vec{AC}$ are linearly independent

$$\vec{r_B} = \vec{r_A} + \vec{v}, \quad \vec{r_C} = \vec{r_A} + \vec{w}$$

$$\vec{r_A}\left(\lambda - \frac{1-\mu}{2}\right) + (\vec{r_A} + \vec{v})\left(\frac{1-\lambda}{2} - \mu\right) + (\vec{r_A} + \vec{w})\left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right) = \vec{0}$$

$$\vec{r_A}\left(\lambda - \frac{1-\mu}{2} + \frac{1-\lambda}{2} - \mu + \frac{1-\lambda}{2} - \frac{1-\mu}{2}\right) + \vec{v}\left(\frac{1-\lambda}{2} - \mu\right) + \vec{w}\left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right) = \vec{0}$$

$\vec{v}, \vec{w}$ lin. indep.

$$\Rightarrow \left.\begin{cases} \dfrac{1-\lambda}{2} = \mu \\ \dfrac{1-\lambda}{2} = \dfrac{1-\mu}{2} \Rightarrow \lambda = \mu \end{cases}\right\} \Rightarrow \frac{1-\lambda}{2} = \lambda \Rightarrow \lambda = \frac{1}{3}$$

$$\Rightarrow \mu = \lambda = \frac{1}{3}$$

$$\Rightarrow \vec{r_G} = \frac{1}{3}\vec{r_A} + \frac{1}{3}\vec{r_B} + \frac{1}{3}\vec{r_C}$$

$$\vec{CG} = \vec{r_G} - \vec{r_C} = \frac{1}{3}\vec{r_A} + \frac{1}{3}\vec{r_B} - \frac{2}{3}\vec{r_C}$$

$$\vec{CP} = \vec{r_P} - \vec{r_C} = \frac{1}{2}\vec{r_A} + \frac{1}{2}\vec{r_B} - \vec{r_C}$$

$\Rightarrow \frac{2}{3}\vec{CP} = \vec{CG} \Rightarrow C, G, P$ collinear and $\dfrac{CG}{CP} = \dfrac{2}{3}$

Me practicing :

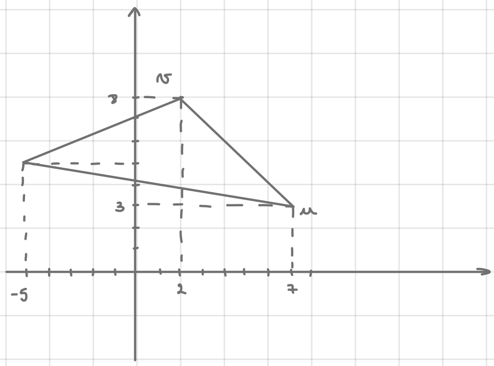
1. $A_0 \dots A_n$ be the vertices of a polygon.

$$\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0}$$

$$= \cancel{\vec{r}_1} - \cancel{\vec{r}_0} + \cancel{\vec{r}_2} - \cancel{\vec{r}_1} + \cancel{\vec{r}_3} - \cancel{\vec{r}_2} + \dots + \cancel{\vec{r}_{n-1}} - \cancel{\vec{r}_{n-2}} + \cancel{\vec{r}_n} - \cancel{\vec{r}_{n-1}} + \cancel{\vec{r}_0} - \cancel{\vec{r}_n}$$

$$= \vec{0}$$

2. a) $u(7,3)$, $v(2,8)$, $w(-5,5)$



if the are not colinear, they can form a triangle

$\overrightarrow{uv} = \vec{r}_v - \vec{r}_u = -5\vec{i} + 5\vec{j}$

$\overrightarrow{vw} = \vec{r}_w - \vec{r}_v = 7\vec{i} - 3\vec{j}$

b) $u(1,2,-1)$, $v(2,-1,0)$, $w(1,-3,1)$

$\overrightarrow{uv} = \vec{r}_v - \vec{r}_u = \vec{i} - 3\vec{j} + \vec{k}$

$\overrightarrow{vw} = \vec{r}_w - \vec{r}_v = -\vec{i} - 2\vec{j} + \vec{k}$

$\Big\}$ they are not collinear, so they can form a Δ

3. ABCD - quadrilateral, M, N, P, Q midpoints of [AB], [BC], [CD] and [DA]

$$\overrightarrow{MN} + \overrightarrow{PQ} \stackrel{?}{=} \vec{0}$$

2.5 Exercises

2.1. Determine parametric equations for the line $\ell \subseteq \mathbb{A}^2$ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2.2. For the lines ℓ in the previous exercise

a) determine a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

2.3. With the assumptions in Example 1.20, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

2.4. Find a Cartesian equation of the line ℓ in $\mathbb{A}^2$ containing the points $P = S \cap S'$ and $Q = T \cap T'$ where

$$S : x+5y-8=0, \quad S' : 3x+6=0, \quad T : 5x-\frac{1}{2}y=1, \quad T' : x-y=5.$$

2.5. Deterimine an equation for the line in $\mathbb{A}^2$ parallel to $\mathbf{v}$ and passing through $S \cap T$ in each of the following cases:

1. $\mathbf{v} = (2,4)$, $S : 3x-2y-7=0$, $T : 2x+3y=0$,
2. $\mathbf{v} = (-5\sqrt{2},7)$, $S : x-y=0$, $T : x+y=1$.

2.6. Let ABC be a triangle in the affine space $\mathbb{A}^n$. Consider the points C' and B' on the sides AB and AC respectively, such that

$$\overrightarrow{AC'} = \lambda \overrightarrow{BC'} \quad \text{and} \quad \overrightarrow{AB'} = \mu \overrightarrow{CB'}.$$

The lines BB' and CC' meet in the point M. For a fixed but arbitrary point $O \in \mathbb{A}^n$, show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda \overrightarrow{OB} - \mu \overrightarrow{OC}}{1-\lambda-\mu}.$$

Deduce a formula for $\overrightarrow{OG}$ where G is the centroid of the triangle.

2.7. In $\mathbb{A}^n$, consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}$$

where $M = AB' \cap A'B$ and $N = AA' \cap BB'$ and where $\overrightarrow{OB} = m\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\overrightarrow{OA'}$.

2.8. Show that the midpoints of the diagonals of a complete quadrilateral are collinear.

2.9. Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$,

c) π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,

d) π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz,

e) π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,

f) π contains the point $A(1,5,7)$ and the coordinate axis Ox.

2.10. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2+3u-4v,\ y = 4-v,\ z = 2+3u$;

b) $\pi : x = u+v,\ y = u-v,\ z = 5+6u-4v$.

2.11. Determine parametric equations for the plane π in the following cases:

a) $3x-6y+z=0$;

b) $2x-y-z-3=0$.

2.12. With the assumptions in Example 1.21, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

2.13. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

2.14. Determine the relative positions of the planes in the following cases

a) $\pi_1 : x+2y+3z-1=0$, $\pi_2 : x+2y-3z-1=0$.

b) $\pi_1 : x+2y+3z-1=0$, $\pi_2 : 2x+y+3z-2=0$, $\pi_3 : x+2y+3z+2=0$.

2.15. Show that the planes

$$\pi_1 : 3x+y+z-1=0, \quad \pi_2 : 2x+y+3z+2=0, \quad \pi_3 : -x+2y+z+4=0$$

have a point in common.

2.16. Show that the pairwise intersection of the planes

$$\pi_1 : 3x+y+z-5=0, \quad \pi_2 : 2x+y+3z+2=0, \quad \pi_3 : 5x+2y+4z+1=0$$

are parallel lines.

2.17. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,

b) ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,

c) ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

2.18. Give Cartesian equations for the lines ℓ in the previous exercise.

2.19. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

2.20. Consider the lines $\ell_1 : x = 1+t, y = 1+2t, z = 3+t, t \in \mathbb{R}$ and $\ell_2 : x = 3+s, y = 2s, z = -2+s, s \in \mathbb{R}$. Show that ℓ_1 and ℓ_2 are parallel and find the equation of the plane determined by the two lines.

2.21. Determine parametric equations of the line passing through $P(5,0,-2)$ and parallel to the planes $\pi_1 : x-4y+2z=0$ and $\pi_2 : 2x+3y-z+1=0$.

2.22. Determine an equation of the plane containing $P(2,0,3)$ and the line $\ell : x = -1+t, y = t, z = -4+2t, t \in \mathbb{R}$.

2.23. For the points $A(2,1,-1)$ and $B(-3,0,2)$, determine an equation of the bundle of planes passing through A and B.

2.24. Determine the relative positions of the lines $x = -3t, y = 2+3t, z = 1, t \in \mathbb{R}$ and $x = 1+5s, y = 1+13s, z = 1+10s, s \in \mathbb{R}$.

2.25. Determine the parameter m for which the line $x = -1+3t, y = 2+mt, z = -3-2t$ doesn't intersect the plane $x+3y+3z-2=0$.

2.26. Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane $ax+y-2z+d=0$.

2.27. In each of the following, find a Cartesian equation of the plane in $\mathbb{A}^3$ passing through Q and parallel to the lines ℓ and ℓ':

a) $Q(1,-1,-2)$, $\ell : x-y=1, x+z=5$, $\ell' : x=1, z=2$

b) $Q(0,1,3)$, $\ell : x+y=-5, x-y+2z=0$, $\ell : 2x-2y=1, x-y+2z=1$

2.28. In each of the following, find the relative positions of the line ℓ and the plane π in $\mathbb{A}^3$, and, if they are incident, determine the point of intersection.

a) $\ell : x = 1+t, y = 2-2t, z = 1-4t$, $\pi : 2x-y+z-1=0$

b) $\ell : x = 2-t, y = 1+2t, z = -1+3t$, $\pi : 2x+2y-z+1=0$

2.29. In each of the following, find a Cartesian equation for the plane in $\mathbb{A}^3$ containing the point Q and the line ℓ.

a) $Q=(3,3,1)$, $\ell : x = 2+3t, y = 5+t, z = 1+7t$

b) $Q=(2,1,0)$, $\ell: x-y+1=0, 3x+5z-7=0$

2.30. In each of the following, find Cartesian equations for the line ℓ in $\mathbb{A}^3$ passing through Q, contained in the plane π and intersecting the line ℓ'

a) $Q=(1,1,0)$, $\pi: 2x-y+z-1=0$, $\ell': x=2-t, y=2+t, z=t$

b) $Q=(-1,-1,-1)$, $\pi: x+y+z+3=0$, $\ell': x-2z+4=0, 2y-z=0$

2.31. In each of the following, find Cartesian equations for the line ℓ in $\mathbb{A}^3$ passing through Q and coplanar to the lines ℓ' and ℓ''. Furthermore, establish whether ℓ meets or is parallel to ℓ' and ℓ''

a) $Q=(1,1,2)$, $\ell': 3x-5y+z=-1, 2x-3z=-9$, $\ell'': x+5y=3, 2x+2y-7z=-7$

b) $Q=(2,0,-2)$, $\ell': -x+3y=2, x+y+z=-1$, $\ell'': x=2-t, y=3+5t, z=-t$

2.32. In each of the following, find the value of the real parameter k for which the lines ℓ and ℓ' are coplanar. Find a Cartesian equation for the plane that contains them, and find the point of intersection whenever they meet

a) $\ell: x=k+t, y=1+2t, z=-1+kt$, $\ell': x=2-2t, y=3+3t, z=1-t$

b) $\ell: x=3-t, y=1+2t, z=k+t$, $\ell': x=1+t, y=1+2t, z=1+3t$

2.33. Find a Cartesian equation for the plane π in $\mathbb{A}^3$ which contains the line of intersection of the two planes

$$x+y=3 \quad \text{and} \quad 2y+3z=4$$

and is parallel to the vector $\mathbf{v}=(3,-1,2)$.

2.34. In the affine space $\mathbb{A}^4$ consider

$$\text{the plane} \quad \alpha=\langle\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\rangle+\begin{bmatrix}2\\4\\1\\2\end{bmatrix} \quad \text{and the line} \quad \beta=\langle\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\rangle+\begin{bmatrix}2\\3\\-1\\1\end{bmatrix}.$$

Determine $\alpha\cap\beta$.

2.35. In $\mathbb{A}^4$ consider the affine subspaces

$$\alpha=\begin{bmatrix}2\\1\\2\\1\end{bmatrix} \quad \beta=\langle\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\rangle+\begin{bmatrix}1\\3\\0\\0\end{bmatrix} \quad \gamma=\langle\begin{bmatrix}2\\1\\3\\-1\end{bmatrix},\begin{bmatrix}1\\0\\2\\-2\end{bmatrix}\rangle+\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \delta=\langle\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\rangle.$$

Which of the following is true?

a) $\alpha \in \beta$

b) $\alpha \in \gamma$

c) $\alpha \in \delta$

d) $\beta \parallel \gamma$

e) $\beta \parallel \delta$

f) $\gamma \parallel \delta$

g) $\beta \subseteq \gamma$

h) $\gamma \subseteq \delta$

i) $\beta \subseteq \delta$

2.36. Consider the following affine subspaces of $\mathbb{A}^4$

$$Y : \begin{cases} x_1 + x_3 - 2 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 &= 0 \end{cases}$$

$$Z : \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 + x_3 - 3x_4 &= -1 \\ x_1 - x_2 + 3x_4 &= 3 \end{cases}$$

a) Determine the dimensions of Y and Z.

b) Find parametric equations for each of the two affine subspaces.

c) Is $Y \parallel Z$?

2.37. In Section 2.2.2 we deduce a linear equation for a plane in $\mathbb{A}^3$ via a determinant. What is the analogue of this description for lines? I.e. deduce Cartesian equations for lines starting from linear dependence of vectors (both in $\mathbb{A}^2$ and $\mathbb{A}^3$).

2.38. Consider the affine space $\mathbb{A}^3$. Show that if a line ℓ doesn't intersect a plane π then $\ell \parallel \pi$ in the sense of the Definition 2.14. Moreover, give an example in $\mathbb{A}^4$ of a line and a plane which do not intersect and which are not parallel.

2.39. Consider the affine space $\mathbb{A}^4$. Describe the relative positions of two planes.

2.40. In $\mathbb{A}^3$ discuss the relative positions of a plane and a line in terms of their Cartesian equations.

2.41. In $\mathbb{A}^3$ discuss the relative positions two lines in terms of their Cartesian equations.

$\mathcal{K} = (O, \mathcal{B})$

reference system for E^n

$O \in E^n$

$\mathcal{B}$ basis of V^n

$\forall P \in E^n$, $\boxed{[P]_{\mathcal{K}} := [\overrightarrow{OP}]_{\mathcal{B}}}$

Notations: V, V' - vector spaces

$\mathcal{B}, \mathcal{B}'$ - bases of V and V'

$\varphi : V \to V'$ - linear map $\qquad M_{\mathcal{B}',\mathcal{B}}^{-1} = M_{\mathcal{B},\mathcal{B}'}$

$$M_{\mathcal{B}',\mathcal{B}}(\varphi) = [\varphi]_{\mathcal{B},\mathcal{B}'} = \left([\varphi(v_1)]_{\mathcal{B}'} \dots [\varphi(v_n)]_{\mathcal{B}'} \right) \text{ where } \mathcal{B} = (v_1, \dots, v_n)$$

V - vector space

$\mathcal{B}, \mathcal{B}'$ bases of V

The base-change matrix from $\mathcal{B}$ to $\mathcal{B}'$ is:

$$M_{\mathcal{B}',\mathcal{B}} := M_{\mathcal{B}',\mathcal{B}}(id) = [id]_{\mathcal{B},\mathcal{B}'} = \left([v_1]_{\mathcal{B}'} [v_2]_{\mathcal{B}'} \dots [v_n]_{\mathcal{B}'} \right)$$

we use this in the following way, $\forall v \in V$: $\boxed{[v]_{\mathcal{B}'} = M_{\mathcal{B}',\mathcal{B}} \cdot [v]_{\mathcal{B}}}$

ex: $M_{\mathcal{B}_1,\mathcal{B}_3}(\varphi \circ \psi) = M_{\mathcal{B}_1,\mathcal{B}_2}(\varphi) \cdot M_{\mathcal{B}_2,\mathcal{B}_3}(\psi)$

$\mathcal{K} = (O, \mathcal{B})$, $\mathcal{K}' = (O', \mathcal{B}')$

$$\begin{aligned}
[P]_{\mathcal{K}'} = [\overrightarrow{O'P}]_{\mathcal{B}'} &= M_{\mathcal{B}',\mathcal{B}} \cdot [\overrightarrow{O'P}]_{\mathcal{B}} \\
&= M_{\mathcal{B}',\mathcal{B}} \cdot [\overrightarrow{OP} - \overrightarrow{OO'}]_{\mathcal{B}} \\
&= M_{\mathcal{B}',\mathcal{B}} \cdot \left([\overrightarrow{OP}]_{\mathcal{B}} - [\overrightarrow{OO'}]_{\mathcal{B}} \right) \\
&= M_{\mathcal{B}',\mathcal{B}} \cdot \left([P]_{\mathcal{K}} - [O']_{\mathcal{K}} \right) \\
&= M_{\mathcal{B}',\mathcal{B}} \cdot [P]_{\mathcal{K}} - M_{\mathcal{B}',\mathcal{B}} [\overrightarrow{OO'}]_{\mathcal{B}} \\
&= M_{\mathcal{B}',\mathcal{B}} [P]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{B}'} \\
&= M_{\mathcal{B}',\mathcal{B}} [P]_{\mathcal{K}} + [\overrightarrow{O'O}]_{\mathcal{B}'} \\
&= M_{\mathcal{B}',\mathcal{B}} [P]_{\mathcal{K}} + [O]_{\mathcal{K}'}
\end{aligned}$$

From now on: $M_{\mathcal{B}',\mathcal{B}} =: M_{\mathcal{K}',\mathcal{K}}$

1.16 $\mathcal{K} = (O, \vec{i}, \vec{j})$

$\mathcal{K}' = (O', \vec{i'}, \vec{j'})$ reference systems for $\mathbb{E}^2$

$[O']_{\mathcal{K}} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$

$$\begin{cases} \vec{i'} = -2\vec{i} + \vec{j} \Rightarrow \vec{i'} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}_{\mathcal{K}} \\ \vec{j'} = \vec{i} + 2\vec{j} \Rightarrow \vec{j'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{K}} \end{cases}$$

$[A]_{\mathcal{K}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, [B]_{\mathcal{K}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, [C]_{\mathcal{K}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Find the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and $[A]_{\mathcal{K}'}, [B]_{\mathcal{K}'}, [C]_{\mathcal{K}'}$

Find the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and check the previous result.

$[A]_{\mathcal{K}'} = [\vec{O'A}]_{\mathcal{K}'} = [\vec{OA}]_{\mathcal{K}} - [\vec{OO'}]_{\mathcal{K}}$

$[\vec{OA}]_{\mathcal{K}'} = M_{\mathcal{K}',\mathcal{K}} [\vec{OA}]_{\mathcal{K}}$

$[\vec{OA}]_{\mathcal{K}'} = M_{\mathcal{K}',\mathcal{K}} ([\vec{OA}]_{K} - [\vec{OO'}]_{K}) = M_{\mathcal{K}',K} [\vec{OA}]_{K} - M_{K',K} [\vec{OO'}]_{K}$

$= M_{K',K} [\vec{OA}]_{K} - [\vec{OO'}]_{K'}$

$[\vec{OA}]_{\mathcal{K}'} = M_{\mathcal{K}'\mathcal{K}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - [\vec{O'O}]_{K'}$

$[A]_{K'} = M_{K',K}([A]_{K} - [O']_{K}) = M_{K'K} [A]_{K} + [O]_{K'}$

$M_{\mathcal{K},\mathcal{K}'} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{\Delta} \cdot A^* = \frac{1}{-5} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$ I did it

$[A]_{\mathcal{K}'} = \begin{bmatrix} \frac{-2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} \frac{-2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$[B]_{\mathcal{K}'} = [\vec{OB}]_{\mathcal{K}'} = [\vec{OB}]_{\mathcal{K}'} - [\vec{OO'}]_{\mathcal{K}'} = M_{\mathcal{K}'\mathcal{K}}([\vec{OB}]_{\mathcal{K}} - [\vec{OO'}]_{\mathcal{K}})$

$= M_{\mathcal{K}',\mathcal{K}}([B]_{\mathcal{K}} - [O]_{\mathcal{K}})$

$= \begin{bmatrix} \frac{-2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$[C]_{\mathcal{K}'} = [\overrightarrow{O'C}]_{\mathcal{K}'} = [\overrightarrow{OC}]_{\mathcal{K}'} - [\overrightarrow{OO'}]_{\mathcal{K}'} = M_{\mathcal{K}'\mathcal{K}}\left([\overrightarrow{OC}]_{\mathcal{K}} - [\overrightarrow{OO'}]_{\mathcal{K}}\right)$$

$$= \begin{bmatrix} -2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Now the other way around, from $\mathcal{K}$ to $\mathcal{K}'$

$$[A]_K = [\overrightarrow{OA}]_K = M_{KK'}\left([\overrightarrow{O'A}]_{K'} - [\overrightarrow{O'O}]_{K'}\right) = M_{KK'} \cdot [\overrightarrow{O'A}]_{K'} - M_{KK'} \cdot [\overrightarrow{O'O}]_{K'}$$

$$= M_{KK'}[A]_{K'} - M_{KK'}[O]_{K'}$$

$$[O]_{K'} = [\overrightarrow{O'O}]_{K'} = M_{K'K}[\overrightarrow{O'O}]_K = -M_{K'K}[\overrightarrow{OO'}]_K = -M_{K'K}[O']_K$$

$$= -\begin{bmatrix} -2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} -14/5 & -1/5 \\ 7/5 & -2/5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} - \begin{bmatrix} -6-1 \\ 3-2 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} - \begin{bmatrix} -7 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$[B]_K = [\overrightarrow{OB}]_K = [\overrightarrow{O'B}]_K - [\overrightarrow{O'O}]_K = M_{KK'}[\overrightarrow{O'B}]_{K'} + [\overrightarrow{OO'}]_K$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} + \begin{pmatrix} 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$[C]_K = [\overrightarrow{O'C}]_K - [\overrightarrow{O'O}]_K = M_{KK'}[\overrightarrow{O'C}]_{K'} + M_{KK'}[\overrightarrow{OO'}]_{K'} = M_{KK'}[\overrightarrow{OC}]_{K'} + [O']_K$$

$$= \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

1.17 ABCD tetrahedron

$K_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})$

$K'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC})$

$K_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD})$

M midpoint of $[BC]$

(a) Write the coordinates of A, B, C, D, M in the 3 reference systems

(b) Write the base change matrix from K_A to K'_A

(c) Write the base change matrix from K_B to K_A

$$[A]_{K_A} = [\vec{AA}]_{K_A} = 0\cdot[\vec{AB}]_{K_A} + 0[\vec{AD}]_{K_A} + 0[\vec{AC}]_{K_A} = \begin{bmatrix}0\\0\\0\end{bmatrix}_A$$

$$[B]_{K_A} = [\vec{AB}]_{K_A} = \begin{bmatrix}1\\0\\0\end{bmatrix}_{K_A}$$

$$[C]_{K_A} = [\vec{AC}]_{K_A} = \begin{bmatrix}0\\1\\0\end{bmatrix}_{K_A}$$

$$[D]_{K_A} = [\vec{AD}]_{K_A} = \begin{bmatrix}0\\0\\1\end{bmatrix}_{K_A}$$

$$[M]_{K_A} = [\vec{AM}]_{K_A} = \frac{1}{2}[\vec{AB}]_{K_A} + [\vec{AC}]_{K_A} = \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}$$

$$[A]_{K'A} = [\vec{AA}]_{K'A} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$[B]_{K'A} = [\vec{AB}]_{K'A} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$[C]_{K'A} = [\vec{AC}]_{K'A} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$r_M = \frac{\vec{r}_B + \vec{r}_C}{2} = \frac{\vec{AB}}{2} + \frac{\vec{AC}}{2} \Rightarrow [M]_{K'A} = [\vec{AM}]_{K'A} = \begin{pmatrix}1/2\\0\\1/2\end{pmatrix}$$

$$[A]_{K_B} = [\vec{BA}]_{K_B} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$[B]_{K_B} = [\vec{BB}]_{K_B} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$[C]_{K_B} = [\vec{BC}]_{K_B} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$[D]_{K_B} = [\vec{BD}]_{K_B} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$[M]_{K_B} = [\vec{BM}]_{K_B} = \frac{1}{2}[\vec{BC}]_{K_B} = \begin{pmatrix}0\\1/2\\0\end{pmatrix}$$

b) $M_{K'_A, K_A} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}$

c) $M_{K_A K_B} = \begin{bmatrix}-1 & -1 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

Seminar 3

Affine variety: $\mathcal{A} = a + U = \{a + \vec{v} \mid \vec{v} \in U\}$

$a \in E^n$, $U \leq_{\mathbb{R}} V^n$

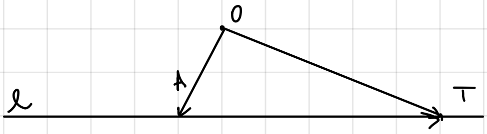
$\dim \mathcal{A} := \dim_{\mathbb{R}} U$

$U =: \mathcal{D}(\mathcal{A})$ = direction subspace of $\mathcal{A}$

If $\dim \mathcal{A} = 1 \Rightarrow \mathcal{A}$ linear

$\dim \mathcal{A} = 2 \Rightarrow \mathcal{A}$ plane

ℓ line, $A \in \ell$, $\vec{v} \in \mathcal{D}(\ell)$ (ie. $\vec{v} \parallel \ell$)



Fix an origin O

$$\vec{r}_T = \vec{r}_A + \vec{AT}$$

$$\vec{AT} \in \mathcal{D}(\ell) \Rightarrow \exists \lambda \in \mathbb{R} : \vec{AT} = \lambda \vec{v}$$

$$\vec{r}_T = \vec{r}_A + \lambda \vec{v}, \lambda \in \mathbb{R}$$

(vector eq of ℓ)

Fix a reference system $\mathcal{K} = (O, \mathcal{B})$, suppose $n = 2$

$$[T]_{\mathcal{K}} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad [A]_{\mathcal{K}} = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$$

$$[\vec{v}]_{\mathcal{K}} = \begin{pmatrix} x_{\vec{v}} \\ y_{\vec{v}} \end{pmatrix}$$

$$\ell : \begin{cases} x = x_A + \lambda x_{\vec{v}} \\ y = y_A + \lambda y_{\vec{v}} \end{cases}$$ parametric eq. of ℓ

If $x_{\vec{v}}, y_{\vec{v}} \neq 0 \Rightarrow \dfrac{x - x_A}{x_{\vec{v}}} = \dfrac{y - y_A}{y_{\vec{v}}}$

If $x_{\vec{v}} = 0 \Rightarrow \ell : x = x_A$

If $y_{\vec{v}} = 0 \Rightarrow \ell : y = y_A$

- implicite form: $y_{\vec{v}}(x - x_A) - x_{\vec{v}}(y - y_A) = 0$

$Ax + By + C = 0$

- explicite form:

If $AB \neq 0 \Rightarrow y = -\frac{A}{B}x - \frac{C}{B}$

If $A = 0,\ B \neq 0 \Rightarrow y = -\frac{C}{B}$

If $A \neq 0,\ B = 0 \Rightarrow x = -\frac{C}{A}$

$D(\ell) = \langle \vec{v} \rangle = \langle v_1, v_2, \dots, v_3 \rangle = \{\alpha_1 v_1 + \dots + \alpha_n v_n \mid \alpha \in K\}$

$\langle v \rangle = \{\alpha v \mid \alpha \in K\}$

2.1 / 2.2 Determine the paramatic and cartesian eq. for the line ℓ and its direction subspace.

a) $\ell \ni A(1,2)$, $\ell \parallel \vec{a}(3,-1)$

b) $\ell \ni O(0,0)$, $\ell \parallel \vec{b}(4,5)$

c) $\ell \ni M(1,7)$, $\ell \parallel O_y$

d) $\ell \ni M(2,4)$, $M(2,-5)$

a) parametric eq. $\ell: \begin{cases} x = x_A + \lambda x_{\vec{v}} \\ y = y_A + \lambda y_{\vec{v}} \end{cases}$

$$\ell: \begin{cases} x = 1 + \lambda 3 \\ y = 2 - \lambda \end{cases}$$

$\Rightarrow \frac{x-1}{3} = 2 - y \Rightarrow x - 1 = 6 - 3y \Rightarrow x + 3y - 7 = 0$ (implicite)

$y = -\frac{x}{3} + \frac{7}{3}$ (explicite)

$D(\ell) = \langle \vec{a} \rangle = \langle (3,-1) \rangle$

b) $\ell \ni O(0,0)$, $\ell \parallel \vec{b}(4,5)$

$\ell: \begin{cases} x = 4\lambda \\ y = 5\lambda \end{cases} \Rightarrow \frac{x}{4} = \frac{y}{5} \Rightarrow 5x - 4y = 0$

$y = \frac{5}{4}x$

$D(\ell) = \langle \vec{b} \rangle = \langle (4,5) \rangle$

c) $\ell \ni M(1,7)$, $\ell \parallel O_y$

$$\ell: \begin{cases} x = 1 \\ y = 7 + \lambda \end{cases}$$

$\ell: x = 1$

$D(\ell) = \langle (0,1) \rangle$

d) $M(2,4), N(2,-5) \in \ell$

$\vec{MN} = (2-2)\vec{i} + (-5-4)\vec{j} = 0 - 9\vec{j}$

$\vec{MN}(0,-9)$

$$\ell: \begin{cases} x = 2 \\ y = 4 - 9\lambda \end{cases}$$

$\ell: x = 2$

$D(\ell) = \langle (0,-9) \rangle$

2.5 Determine the eq. of the line ℓ parallel to $\vec{v}$ and passing through $S \cap T$ if:

$\vec{v} = (2,4)$

$S: 3x - 2y - 7 = 0$ + write $D(\ell)$

$T: 2x + 3y = 0$

$$\begin{cases} 3x - 2y - 7 = 0 \quad |\cdot 3 \\ 2x + 3y = 0 \quad |\cdot 2 \end{cases} \Rightarrow \begin{cases} 9x - 6y - 21 = 0 \\ 4x + 6y = 0 \end{cases} (+)$$

$13x = 21 \Rightarrow x = \frac{21}{13}$

$\frac{42}{13} + 3y = 0 \Rightarrow y = -\frac{42}{39} = -\frac{14}{13}$

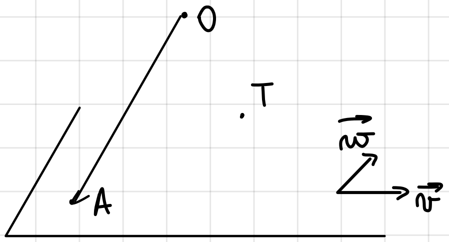
$\xRightarrow{\text{we denote}} A\left(\frac{21}{13}, -\frac{14}{13}\right)$

$$\ell: \begin{cases} x = \frac{21}{13} + 2\lambda \\ y = -\frac{14}{13} + 4\lambda \end{cases} \Rightarrow \frac{x - \frac{21}{13}}{2} = \frac{y + \frac{14}{13}}{4} \Rightarrow 4x - \frac{84}{13} = 2y + \frac{28}{13}$$

$$\Rightarrow 4x - 2y - \frac{112}{13} = 0$$

$\Rightarrow D(\ell) = \langle \vec{v} \rangle = \langle (2,4) \rangle = \langle (1,2) \rangle$

π plane, $A \in \pi$, $\underbrace{\vec{v}, \vec{w}}_{\text{linear indep.}} \in D(\pi)$



$\vec{r}_T = \vec{r}_A + \vec{AT}$

$AT \in D(\pi) \Rightarrow \exists \lambda, \mu \in \mathbb{R}, \ \vec{AT} = \lambda \vec{v} + \mu \vec{w}$

$\pi : \vec{r}_T = \vec{r}_A + \lambda \vec{v} + \mu \vec{w}$ ← vector equation

Fix $\mathcal{K} = (O, \mathcal{B})$ a reference system. So:

$$\pi : \begin{cases} x = x_A + \lambda x_{\vec{v}} + \mu x_{\vec{w}} \\ y = y_A + \lambda y_{\vec{v}} + \mu y_{\vec{w}} \\ z = z_A + \lambda z_{\vec{v}} + \mu z_{\vec{w}} \end{cases}$$ parametric equation

$$\Leftrightarrow \quad \pi : \begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_{\vec{v}} & y_{\vec{v}} & z_{v} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix} = 0$$ (symmetric form)

- implicite form: $Ax + By + Cz + D = 0$
- explicite form: if $c \neq 0$: $z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}$

2.10 Determine Cartesian eq. for the plane π:

a) $\pi : \begin{cases} x = 2 + 3u - 4v \\ y = 4 - v \\ z = 2 + 3u \end{cases}$

b) $\pi : \begin{cases} x = u + v \\ y = u - v \\ z = 5 + 6u - 4v \end{cases}$

a) $$\begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0 - 3(z-2) - 12(y-4) - 0 - 0 + 3(x-2)$$
$$= -3z + 6 - 12y + 48 + 3x - 6 = 3x - 12y - 3z + 48$$

$\Rightarrow x - 4y - z + 16 = 0$

$D(\pi) = \langle (3,0,3), (-4,-1,0) \rangle$

b) $\begin{vmatrix} x & y & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = \begin{vmatrix} \cancel{x} & x+y & z-5+4x \\ \cancel{1} & 2 & 10 \\ \boxed{1} & \cancel{0} & \cancel{0} \end{vmatrix} = 5x+5y-z+5-4x$

$= x+5y-z+5 = 0$

$D(\pi) = \langle (1,1,6), (1,-1,-4) \rangle$

2.11 Determine parametric eq. for π: a) $3x - 6y + z = 0$

b) $2x - y - z - 3 = 0$

a) $\begin{cases} x = 2y - \frac{1}{3}z = 2\lambda - \frac{1}{3}\mu \\ y = \lambda \\ z = \mu \end{cases}$

$D(\pi) = \langle (2,1,0), (-\frac{1}{3},0,1) \rangle = \langle (2,1,0), (1,0,-3) \rangle$

b) $\begin{cases} x = \frac{y}{2} + \frac{z}{2} + 3 = \frac{3}{2} + \alpha\frac{1}{2} + \beta\frac{1}{2} \\ y = \alpha \\ z = \beta \end{cases}$

$D(\pi) = \langle (\frac{1}{2},1,0), (\frac{1}{2},0,1) \rangle = \langle (1,2,0), (1,0,2) \rangle$

Seminar 4

Dot product: (scalar product)

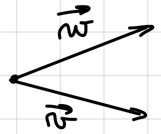
$\vec{v}, \vec{w} \in V$, $\vec{v} \cdot \vec{w} = \begin{cases} 0, \text{ if } \vec{v} = \vec{0} \text{ or } \vec{w} = \vec{0} \\ |\vec{v}| \cdot |\vec{w}| \cdot \cos(\vec{v}, \vec{w}) \text{ otherwise} \end{cases}$

properties:

- bilinearity: $\forall \vec{v}_1, \vec{v}_2, \vec{w} \in V$

  $(\alpha\vec{v}_1 + \beta\vec{v}_2) \cdot \vec{w} = \alpha\vec{v}_1 \cdot \vec{w} + \beta\vec{v}_2 \cdot \vec{w}$

• symmetry: $\forall \vec{v}, \vec{w} \in V$
$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$



• positive definedness: $\forall \vec{v} \in V$
$\vec{v} \cdot \vec{v} \in \mathbb{R} \geq 0$ if $\vec{v} \neq \vec{0}$
then $\vec{v} \cdot \vec{v} > 0$

consequence:
- $\vec{v} \cdot \vec{v} = |\vec{v}|^2$
- $\vec{v}^2 - \vec{w}^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} + \vec{w})$

- if we have an orthonormal basis $B = (\vec{v}_1, \ldots, \vec{v}_n)$
  $\forall i \neq j : \vec{v}_i \cdot \vec{v}_j = 0$
  $\forall i : |\vec{v}_i| = 1$

Let $\vec{w}, \vec{w}' \in V$, $[\vec{w}]_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $[\vec{w}']_B = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}$

$$\vec{w} \cdot \vec{w}' = x_1 x_1' + x_2 x_2' + \ldots + x_n x_n'$$
$$|\vec{w}| = \sqrt{x_1^2 + \ldots + x_n^2}$$

2.1. $\vec{m}, \vec{n}$ unit vectors st. $\measuredangle(\vec{m}, \vec{n}) = 60^\circ$
Determine the length of the diagonals in the palalelogram spanned by:
$\vec{a} = 2\vec{m} + \vec{n}$
$\vec{b} = \vec{m} - 2\vec{n}$

$$\vec{d}_1 = \vec{a} + \vec{b}$$

$$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})(\vec{a} + \vec{b})$$
$$= (2\vec{m} + \vec{n} + \vec{m} - 2\vec{n})(2\vec{m} + \vec{n} + \vec{m} - 2\vec{n})$$
$$= (3\vec{m} - \vec{n})(3\vec{m} - \vec{n})$$
$$= 9\vec{m}^2 + \vec{n}^2 - 6\vec{m}\vec{n}$$
$$\vec{m}^2 = |\vec{m}|^2 = 1$$
$$\vec{n}^2 = |\vec{n}|^2 = 1$$

$$\vec{m} \cdot \vec{n} = |\vec{m}||\vec{n}| \cos(\vec{m}, \vec{n}) = 1 \cdot 1 \cdot \frac{1}{2}$$

$$\Rightarrow |d_1|^2 = 9 - 6\frac{1}{2} + 1 = 7 \Rightarrow d_1 = \sqrt{7}$$

$$\vec{d}_2 = \vec{a} - \vec{b}$$

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})(\vec{a} - \vec{b}) = (2\vec{m} + \vec{n} - \vec{m} + 2\vec{n})(2\vec{m} + \vec{n} - \vec{m} + 2\vec{n})$$
$$= (\vec{m} + 3\vec{n})(\vec{m} + 3\vec{n}) = \vec{m}^2 + 9\vec{n} + 6\vec{m}\vec{n}$$

$\vec{m}\vec{n} = \frac{1}{2}$

$|d_2|^2 = 1 + 9 + 6\cdot\frac{1}{2} = 13 \Rightarrow d_2 = \sqrt{13}$

3.4. Let $(\vec{i}, \vec{j}, \vec{k})$ be an orthonormal basis Consider the vectors:

$\vec{q} = 3\vec{i} + \vec{j}$

$\vec{p} = \vec{i} + 2\vec{j} + \lambda\vec{k}$

Determine λ such that $\cos(\measuredangle(\vec{p}, \vec{q})) = \frac{5}{12}$

$\vec{p}\vec{q} = 3 + 2 = 5$

$\underbrace{\vec{p}\vec{q}}_{5} = |\vec{p}|\cdot|\vec{q}|\cdot\underbrace{\cos(\vec{p}, \vec{q})}_{\frac{5}{12}}$

$\Rightarrow 5 = |\vec{p}|\cdot|\vec{q}|\cdot\frac{5}{12}$

$\Rightarrow |\vec{p}|\cdot|\vec{q}| = 12$

$|\vec{q}| = \sqrt{9+1} = \sqrt{10}$

$|\vec{p}| = \sqrt{1+4+\lambda^2} = \sqrt{5+\lambda^2}$

$|\vec{p}| = \frac{12}{\sqrt{10}}$

$\sqrt{5+\lambda^2} = \frac{12}{\sqrt{10}} \Rightarrow 5 + \lambda^2 = \frac{144}{10} \Rightarrow \lambda^2 = \frac{94}{10}$

$\lambda = \pm\sqrt{\frac{94}{10}}$

3.6 $\triangle ABC$

$\vec{AB}^2 + \vec{AC}^2 - \vec{BC}^2 = 2\vec{AB}\cdot\vec{AC}$ (cosine law)

proof: $\vec{AB}^2 - \vec{BC}^2 + \vec{AC}^2 = (\vec{AB} - \vec{BC})\underbrace{(\vec{AB} + \vec{BC})}_{\vec{AC}} + \vec{AC}^2 = (\vec{AB} - \vec{BC} + \vec{AC})\cdot\vec{AC}$

$= (\vec{r}_B - \vec{r}_A - \vec{r}_C + \vec{r}_B + \vec{r}_C - \vec{r}_A)\cdot\vec{AC} = 2(\vec{r}_B - \vec{r}_A)\vec{AC} = 2\vec{AB}\cdot\vec{AC}$

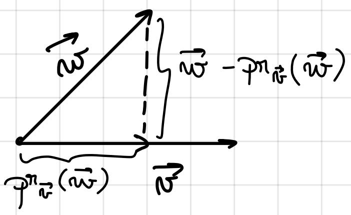
3.7. ABCD tetrahedro

$\cos(\measuredangle(\vec{AB}, \vec{CD})) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2AB\cdot CD}$ (cos. law in 3D)

$$2\vec{AB}\cdot\vec{CD} = AD^2 - AC^2 + BC^2 - BD^2$$

$$(\vec{AD}-\vec{AC})(\vec{AD}+\vec{AC}) + (\vec{BC}-\vec{BD})(\vec{BC}+\vec{BD}) = \vec{CD}(\vec{AD}+\vec{AC}) - \vec{CD}(\vec{BC}+\vec{BD})$$

$$= \vec{CD}(\vec{AD}-\vec{BD}+\vec{AC}-\vec{BC}) = \vec{CD}(\vec{AB}+\vec{AB}) = 2\vec{AB}\cdot\vec{CD}$$



$$|pr_{\vec{v}}(\vec{w})| = \vec{w}\cdot\cos(\vec{v},\vec{w})$$

$$pr_{\vec{v}}(\vec{w}) = \vec{v}\cdot\frac{|pr_{\vec{v}}(\vec{w})|}{|\vec{v}|} = \frac{|\vec{w}|\cdot\cos(\vec{v},\vec{w})}{|\vec{v}|}\cdot\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\cdot\vec{v}$$

<u>Gram-Schmid:</u>

$B = (v_1, v_2, \dots, v_n)$ basis for V

I Orthogonalization

1. $\vec{v}_1{}' = \vec{v}_1$

2. $\vec{v}_2{}' = \vec{v}_2 - pr_{\vec{v}_1{}'}(\vec{v}_2)$

3. $\vec{v}_3{}' = \vec{v}_3 - pr_{span(\vec{v}_1{}', \vec{v}_2{}')}(\vec{v}_3{}') = \vec{v}_3 - pr_{\vec{v}_1{}'}(\vec{v}_3) - pr_{\vec{v}_2{}'}(\vec{v}_3)$

$$= \vec{v}_3 - \frac{\vec{v}_3\cdot\vec{v}_1{}'}{\vec{v}_1\cdot\vec{v}_1{}'}\cdot\vec{v}_1{}' - \frac{\vec{v}_3 - \vec{v}_2{}'}{\vec{v}_2{}'\cdot\vec{v}_2{}'}\cdot\vec{v}_2{}'$$

Repeat this until we get the orthogonal basis $B' = (\vec{v}_1{}', \vec{v}_2{}', \dots, \vec{v}_n{}')$

II Normalization

$$\vec{v}_1{}' = \frac{\vec{v}_1{}'}{|\vec{v}_1{}'|} \Rightarrow B'' \text{– orthonormal basis.}$$

3.10 In an orthonormal basis consider the vectors:
$\vec{v}_1(0,1,0)$, $\vec{v}_2(2,1,0)$, $\vec{v}_3(-1,0,1)$

Use the Gram - Schmid process to find an orthonormal basis containing $\vec{v}_1$.

I
orthogonalization

$$\vec{v}_1' = \vec{v}_1 = (0,1,0)$$

$$\vec{v}_2 = \vec{v}_2 - pr_{\vec{v}_1'}(\vec{v}_2)$$

$$= \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1'}{\vec{v}_1' \cdot \vec{v}_1'} \cdot \vec{v}_1'$$

$$= (2,1,0) - \frac{1}{1} \cdot (0,1,0)$$

$$= (2,0,0)$$

$$\vec{v}_3' = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{v}_1'}{\vec{v}_1' \cdot \vec{v}_1} \cdot \vec{v}_1' - \frac{\vec{v}_3 \cdot \vec{v}_2'}{\vec{v}_2' \cdot \vec{v}_2'} \cdot \vec{v}_2'$$

$$= (-1,0,1) - \frac{0}{1} \cdot (0,1,0) - \frac{(1,0,1) \cdot (2,0,0)}{4} \cdot (2,0,0)$$

$$= (-1,0,1) - 0 - \frac{-2}{4} \cdot (2,0,0)$$

$$= (-1,0,1) + (1,0,0) = (0,0,1)$$

II
normalization

$$\vec{V}_1'' = \frac{\vec{v}_1'}{|\vec{v}_1'|} = \frac{(0,1,0)}{1} = (0,1,0)$$

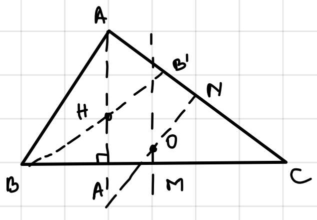
$$\vec{V}_2'' = \frac{\vec{v}_2'}{|\vec{v}_2'|} = \frac{(2,0,0)}{\sqrt{4}} = (1,0,0)$$

$$\vec{v}_3'' = \vec{v}_3' = (0,0,1)$$

3.18. Determine the circumcenter and the orthocenter of ΔABC
$A(1,2)$, $B(3,-2)$, $C(5,6)$ (orthonormal ref. system)

circumcenter $= \cap$ (perpendicular bisectors $\Leftrightarrow$ mediators)
(centrul cercului circumscris)

orthocenter $= \cap$ (heights)
(centrul cercului înscris)



$$m_{AB} = \frac{-2-2}{3-1} = -2$$

$$AA' \perp BC \Rightarrow m_{AA'} \cdot m_{BC} = -1$$

$$m_{BC} = \frac{6+2}{5-3} = 4 \Rightarrow m_{AA'} = -\frac{1}{4}$$

$AA' : y - y_A = m_{AA'} (x - x_A) \Rightarrow AA' : y - y_A = m_{AA'} (x - x_A) \Rightarrow AA' : y - 2 = -\frac{1}{4}(x-1)$

$\Rightarrow AA' : x + 4y - 9 = 0$

$m_{CC'} = \frac{1}{2}$

$CC' : y - 6 = \frac{1}{2}(x-5) \Rightarrow CC' : x - 2y + 7 = 0$

$H : \begin{cases} x + 4y - 9 = 0 \Rightarrow x = 9 - 4y \\ x - 2y + 7 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 9 - 4y & \Rightarrow x = -\frac{5}{3} \\ 16 = 6y \Rightarrow y = \frac{8}{3} \end{cases}$

$m_{NO} = m_{BB'}$ $\quad$ $m_{BB'} \cdot m_{AC} = -1$

$m_{MO} = m_{AA'}$ $\quad$ $m_{AC} = \frac{6-2}{5-1} = 1$

$\Rightarrow m_{BB'} = -1$

$NO : y - y_N = m_{NO}(x - x_N)$ $\quad$, $x_N = \frac{x_A + x_C}{2} = 4$

$NO : y - 3 = -(x-4)$ $\quad$ $y_N = \frac{y_A + y_C}{2} = 3$

$NO : x + y - 7 = 0$

$MO : y - y_M = m_{MO}(x - x_M)$ $\quad$, $x_M = 4$

$MO : y - 2 = -\frac{1}{4}(x-4)$ $\quad$ $y_M = 2$

$MO : x + 4y - 12 = 0$

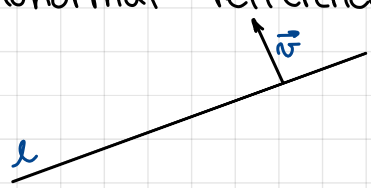
$O : \begin{cases} x + y - 7 = 0 \\ x + 4y - 12 = 0 \end{cases} \Rightarrow \begin{matrix} x = \frac{16}{3} \\ y = \frac{5}{3} \end{matrix}$

Seminar 5 : 26, 29, 30, 31, 32, 33, 35, 36, 38, 39, 42

We will work with respect to the orthonormal refference system.

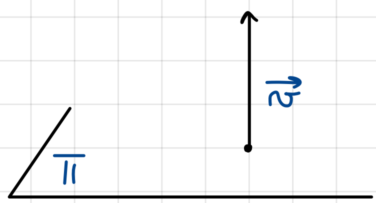
$\ell : ax + by + c = 0$ (in 3D)

$\vec{v}(a,b)$ normal vector for ℓ



$\pi : Ax + By + C \in 3D$

$\Rightarrow \vec{v}(A,B,C)$ normal vector for π

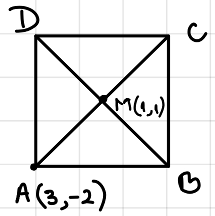


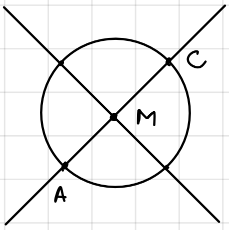
$\Pi : Ax + By + Cz + D = 0$

$P(x_0, y_0, z_0)$

$$\text{dist}(P, \Pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

3.26. The point $A(3,-2)$ is the vertex of a square $ABCD$ and $M(1,1)$ is the intersection point of the diagonals.
Determine cartesian equations for the sides of the square.





$M\left(\frac{x_A + x_C}{2}, \frac{y_A + y_C}{2}\right) = M(1,1)$

$1 = \frac{3 + x_C}{2} \Rightarrow x_C = -1$

$1 = \frac{-2 + y_C}{2} \Rightarrow y_C = 4$

$\Rightarrow C(-1, 4)$

$A(3,-2)$, $C(-1,4)$

$AC: mx + n = y \Leftrightarrow \begin{array}{l} 3m + n = -2 \\ -m + n = 4 \end{array} \quad (-)$

$4m = -6 \Rightarrow m = \frac{-3}{2}$

$n = 4 - \frac{3}{2} = \frac{5}{2}$

$\Rightarrow AC: -\frac{3}{2}x + \frac{5}{2} = y$

$-3x + 5 = 2y$

$AC: -3x + 5 = 2y$

$MB \perp AC \Rightarrow \underbrace{m_{AC}}_{-\frac{3}{2}} \cdot m_{BM} = -1 \Rightarrow m_{BM} = \frac{2}{3}$

$BM: mx + n = y$, $M(1,1)$

$\frac{2}{3}x + n = y$

$2x + 3n = 3y$

$2 + 3n = 3 \Rightarrow 3n = 1 \Rightarrow n = \frac{1}{3}$

$BM: 2x + n = 3y$

The eq. of the circle centered is $\Omega(x_0, y_0)$ with radius R is: $(x-x_0)^2 + (y-y_0)^2 = R^2$

$C(-1,4)$

$AM = \sqrt{(1-3)^2+(1+2)^2} = \sqrt{4+9} = \sqrt{13}$

$m_{AC} = -\frac{3}{2} \Rightarrow m_{BD} = \frac{2}{3}$

$BD: y - y_M = m_{BD}(x - x_M)$

$y - 1 = \frac{2}{3}(x-1) \Leftrightarrow 2x+1 = 3y$

$BD: \begin{cases} 2x - 3y + 1 = 0 \\ (x-1)^2 + (y-1)^2 = 13 \end{cases}$

$BD: \begin{cases} x = \frac{3y-1}{2} \\ \left(\frac{3y-1}{2} - 1\right)^2 + (y-1)^2 = 13 \end{cases}$

$\left(\frac{3y-3}{2}\right)^2 + (y-1)^2 = 13$

$\ldots \quad (y-1)^2 13 = 52 \Rightarrow (y-1)^2 = 4 \Rightarrow y = \{3, -1\}$

For $y = 3 \Rightarrow 2x - 8 = 0 \Rightarrow x = 4$

For $y = -1 \Rightarrow 2x + 4 = 0 \Rightarrow x = -2$

So, $B(4,3)$, $D(-2,-1)$ or $B(-2,-1)$, $D(4,3)$

$AB: \frac{x - x_A}{x - x_B} = \frac{y - y_A}{y - y_B} \ldots$

$AB: y = 5x - 17$

$m_{AB} = 5 \Rightarrow CD: y - y_C = 5(x - x_C)$

$CD: y = 5x + 9$

$m_{BC} = -\frac{1}{5}$

$BC: y - y_B = -\frac{1}{5}(x - x_B)$

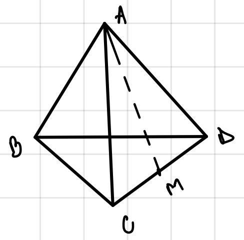
$BC: x + 5y - 19 = 0$

$m_{AD}: -\frac{1}{5}$

$AD: \; y+1 = -\frac{1}{5}(x+2)$

$AD: \; 5y + x + 7 = 0$

3.30 $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$. Vertex of a tetrahedron. Determine a cartesian equation for the plane π that contains $[AB]$ and the midpoint M of $[CD]$.



$x_M = \frac{x_C + x_D}{2}$; $\quad x_M = 3$

$y_M = \frac{y_C + y_A}{2}$; $\quad y_M = -2$

$z_M = \frac{z_C + z_A}{2}$; $\quad z_M = 6$

$\vec{BM} = (x_M - x_B)\vec{i} + (y_B - y_M)\vec{j} + (z_M - z_B)\vec{k}$

$\vec{BM} = 2\vec{i} - 5\vec{j} + \vec{k}$

$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$

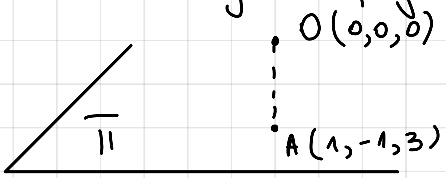
$\vec{AB} = -\vec{i} + 2\vec{j} + 5\vec{k}$

$$\pi: \begin{vmatrix} x - x_M & y - y_M & z - z_M \\ x_{\vec{BM}} & y_{\vec{BM}} & z_{\vec{BM}} \\ x_{\vec{AB}} & y_{\vec{AB}} & z_{\vec{AB}} \end{vmatrix} = 0$$

$$\pi: \begin{vmatrix} x-3 & y+2 & z-6 \\ 2 & -5 & 1 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

$= $ (regula Δ-lui) $-27x - 11y - z + 65 = 0$

3.32 Determine a Cartesian equation for the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π



$\vec{n}\,(1,-1,3)$

$\pi : x - y + 3z + d = 0$

$A \in \pi$

$1 + 1 + 9 + d = 0 \Rightarrow d = -11$

$\Rightarrow \pi : x - y + 3z - 11 = 0$

3.33 Determine the distance between the planes :

$\pi_1 : x - 2y - 2z + 7 = 0$

$\pi_2 : 2x - 4y - 4z + 17 = 0$

$\left.\begin{array}{l} \vec{n_{\pi_1}}\ (1, -2, -2) \\ \vec{n_{\pi_2}}\ (2, -4, -4) = 2\vec{n_{\pi_1}} \end{array}\right\} \Rightarrow \pi_1 \parallel \pi_2$

$d(\pi_1, \pi_2) = d(P, \pi_2)\ ,\ \forall P \in \pi_1$

$P(-7, 0, 0)$

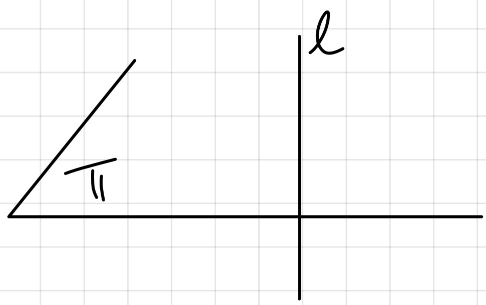
$$d(P, \pi_2) = \frac{|-14 + 17|}{\sqrt{36}} = \frac{3}{6} = \frac{1}{2}$$

3.37 Determine the values a and c for which the line ℓ :

$$\begin{cases} 3x - 2y + z + 3 = 0 \\ 4x - 3y + 4z + 1 = 0 \end{cases}$$

is perpendicular to the plane $\pi : ax + 8y + cz + 2 = 0$

$$\begin{cases} 3x - 2y + z + 3 = 0 \quad | \cdot (-4) \\ 4x - 3y + 4z + 1 = 0 \end{cases} \quad (-)$$



$$\begin{cases} -8x + 5y - 11 = 0 \\ 4x - 3y + 4z + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \dfrac{-5y + 11}{-8} \\ 4x - 3y + 4z + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \dfrac{-5y + 11}{-8} \\ \dfrac{5y - 11}{2} - 3y + 4z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{5y - 11}{8} \\ z = \left(3y - \dfrac{5y - 11}{2} - 1\right) \cdot \dfrac{1}{4} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \dfrac{5y-11}{8} \\ z = \dfrac{y+9}{8} \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{5\lambda-11}{8} \\ z = \dfrac{\lambda+9}{8} \\ y = \lambda \end{cases}$$

$$D(\ell) = \langle (\tfrac{5}{8}, 1, \tfrac{1}{8}) \rangle = \langle \underbrace{(5, 8, 1)}_{\vec{v}} \rangle$$

$$\ell \perp \pi \Leftrightarrow \vec{v} \parallel \vec{n_\pi}$$

$$\vec{n_\pi} = (a, 8, c) \Rightarrow \frac{a}{5} = \frac{8}{8} = \frac{c}{1} \Rightarrow \begin{cases} a = 5 \\ c = 1 \end{cases}$$