22.11.2023

$$\underbrace{0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < \frac{n}{n}}_{\text{Partition}} < 1$$

$$\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \to \int_0^1 f(x)\, dx$$

1) a) $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \right)$

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right),$$

$$f\left(\frac{k}{n}\right) = \frac{1}{1+\frac{k}{n}} \Leftrightarrow f(x) = \frac{1}{1+x}$$

$$\sum \frac{1}{n} f\left(\frac{k}{n}\right) \to \int_0^1 f(x)\, dx = \int_0^1 \frac{1}{1+x} dx =$$

$$= \ln(1+x) \Big|_0^1 = \ln 2$$

b) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$

$$a_n = \sqrt[n]{\frac{n!}{n^n}} = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = e^{\frac{1}{n} \cdot \ln \frac{n!}{n^n}} \to e^{-1} = \frac{1}{e}$$

$$\frac{1}{n}\ln\frac{n!}{n^n} = \frac{1}{n}\ln\left(\frac{1\cdot 2\cdot\ldots\cdot n}{n\cdot n\cdot\ldots\cdot n}\right) = \frac{1}{n}\cdot\sum_{k=1}^{n}\underbrace{\ln\frac{k}{n}}_{f\left(\frac{k}{n}\right)} \xrightarrow{f(x)=\ln x}$$

$$\rightarrow \int_0^1 \ln(x)\,dx = x\ln x\Big|_0^1 - 1 = 0$$

$\downarrow$

integrate by parts

$$\lim_{x\searrow 0} x\ln x = \lim_{x\searrow 0}\frac{\ln x}{\frac{1}{x}} \overset{l'H}{=} \lim_{x\searrow 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim(-x) = 0$$

Or, let $b_n = \ln a_n = \ln\frac{\sqrt[n]{n!}}{n} = (\ldots$ compute in a similar way $)$

d) $$f(x) = \begin{cases} 1, & x\in\mathbb{Q} \\ 0, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

Let $a = x_0 < x_1 < \ldots < x_n = b$ a partition on $[0,1]$

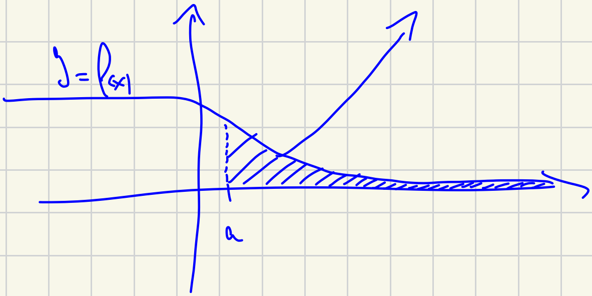
Let $\overline{c_k}\in\mathbb{Q}$, $\underline{c_k}\in\mathbb{R}\setminus\mathbb{Q}$, $\overline{\underline{c_k}}\in[x_{k-1}, x_k]$

$$\overline{\sigma}(f, P, \overline{c_k}) = \sum_{k=1}^{n} 1\cdot(x_k - x_{k-1}) = 1$$

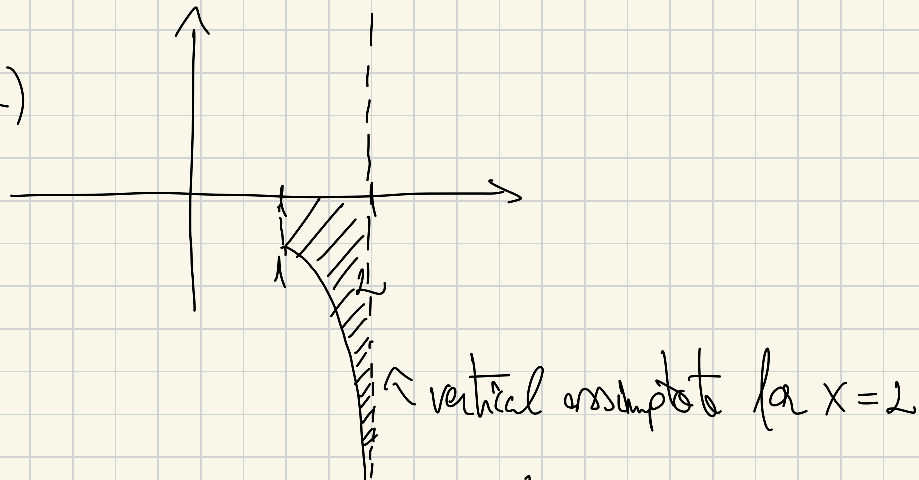
$$\# \Rightarrow \nexists \int_0^1 f(x)\,dx$$

$$\underline{\sigma}(f, P, \underline{c_k}) = \sum_{k=1}^{n} 0\,(x_k - x_{k-1}) = 0$$

$$\int_a^\infty f(x)\,dx = \lim_{t\nearrow\infty}\int_a^t f(x)\,dx$$



3) a)



$$\int_1^2 \frac{1}{x(x-2)}\,dx = \lim_{t\nearrow 2}\int_1^t \frac{1}{x(x-2)}\,dx =$$

$$= \lim_{t\nearrow 2}\left(-\frac{1}{2}\ln t + \frac{1}{2}\ln(2-t)\right) = -\frac{1}{2}\cdot\ln 2 - \infty = -\infty.$$

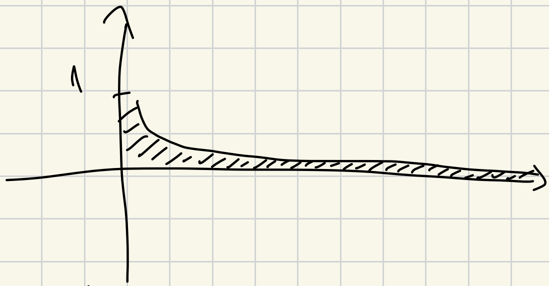
$$\int_1^t \frac{1}{x(x-2)}\,dx = \left(-\frac{1}{2}\right)\cdot\int \frac{1}{x} - \frac{1}{x-2}\,dx =$$

$$= -\frac{1}{2}\cdot\ln x\Big|_1^t + \frac{1}{2}\ln|x-2|\Big|_1^t$$

$$= -\frac{1}{2}\ln t + \frac{1}{2}\ln(2-t)$$

b) $t = x^2$, $\frac{dt}{dx} = 2x$, $dt = 2x\,dx$

$$\int_0^\infty x \cdot e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2} \cdot 2x\,dx = \frac{1}{2}\int_0^\infty e^{-t}\,dt$$



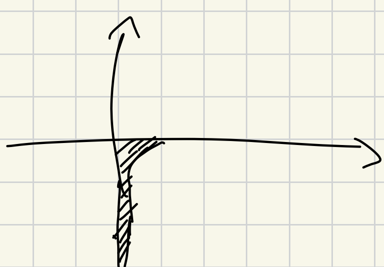
$$\frac{1}{2}\cdot\lim_{u\to\infty}\int_0^u e^{-t}\,dt = \frac{1}{2}\left(-e^{-t}\right)\Big|_0^u = \lim_{u\to\infty}\Big(\underbrace{-\frac{1}{2}e^{-u}}_{\to 0} + \frac{1}{2}\Big) =$$

$$= \frac{1}{2}$$

we can just write: $\frac{1}{2}\int_0^\infty e^{-t}\,dt = -\frac{1}{2}e^{-t}\Big|_0^\infty =$

$$= -\frac{1}{2}e^{-\infty} + \frac{1}{2} = \frac{1}{2}$$

c) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$



$$\lim_{x\to 0} \frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0_+} = -\infty \cdot \infty = -\infty$$

$$\lim_{t \searrow 0} \int_t^1 \frac{\ln x}{\sqrt{x}} \, dx =$$

$$\int_t^1 \frac{\ln x}{\sqrt{x}} \, dx = 2\int_t^1 \ln (\sqrt{x})' \, dx =$$

$$= 2 \ln x \sqrt{x} \Big|_t^1 - 2\int_t^1 \frac{\sqrt{x}}{x} \, dx$$

$$= -2 \ln t \cdot \sqrt{t} - 4\sqrt{x} \Big|_t^1$$

$$= -2 \ln t \cdot \sqrt{t} - 4 + 4t$$

$$\lim_{t \searrow 0} (\underbrace{-2\ln t \sqrt{t}}_{=0} - 4 + \underbrace{4t}_{=0}) = -4$$

$$\lim_{t \searrow 0} \sqrt{t} \cdot \ln t = \lim_{t \searrow 0} \frac{\ln t}{\frac{1}{\sqrt{t}}} \overset{l'H}{=} \lim_{t \searrow 0} \frac{\frac{1}{\cancel{t}}}{-\frac{\frac{1}{2\sqrt{t}}}{\cancel{t}}} =$$

$$= \lim_{t \searrow 0} -2\sqrt{t} = 0$$

i) a) $\int_1^\infty \frac{1}{x\sqrt{1+x^2}}\,dx$

$\sqrt{1+x^2} \approx x$, $\frac{1}{x\sqrt{1+x^2}} \approx \frac{1}{x^2}$,

$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty}+1 = 1$

$\int_1^\infty \frac{1}{x\sqrt{1+x^2}}\,dx$ "like" $\int_1^\infty \frac{1}{x^2}\,dx < \infty \Rightarrow$ Conv.

c) $\int_1^\infty \frac{\ln x}{x\sqrt{x^2-1}}$

$\frac{\ln x}{x\sqrt{x^2-1}} \approx \frac{\ln x}{x^2} < \frac{x^\alpha}{x^2} = \frac{1}{x^{2-\alpha}}$, $\alpha \in (0,1)$

$2-\alpha > 1 \Rightarrow \int_1^\infty \frac{1}{x^{2-\alpha}}\,dx$ Converges $\Rightarrow$

$\Rightarrow \int_1^\infty \frac{\ln x}{x\sqrt{x^2-1}} < \int_1^\infty \frac{1}{x^{2-\alpha}}\,dx < \infty$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx$

$$\frac{1}{\cos x} \underset{\text{as } x\nearrow\frac{\pi}{2}}{\simeq} \frac{1}{\left(\frac{\pi}{2}-x\right)^p} \quad \text{for some } p.=1$$

problem happens in $\frac{\pi}{2}$

$$\frac{1}{\cos x} = \frac{1}{\sin\left(\frac{\pi}{2}-x\right)} \simeq \frac{1}{\frac{\pi}{2}-x} \quad \text{as } x\nearrow\frac{\pi}{2}$$

$$\lim_{x\nearrow\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x} = \lim_{t\searrow 0} \frac{\sin t}{t} = 1$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx \quad \text{"like"} \quad \int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\,dx = +\infty \Rightarrow$$

$\Rightarrow$ divergence.

Integral test → Could be done instead of Cauchy cond. test or ratio test!

$f : [1,\infty) \to [0,\infty)$ decreasing

$\displaystyle\int_1^\infty f(x)\,dx$ has the same nature as $\displaystyle\sum_{n=1}^{\infty} f(n)$

proof in lecture notes

5) a) $\sum \frac{1}{n^p}$, $p>0$

$$\sum_{n\geq 1} \frac{1}{n^p} \text{ „like" } \int_1^\infty \frac{1}{x^p}\,dx$$

conv for $p>1$

div. for $p\leq 1$

b) $$\sum_{n\geq 2} \frac{1}{n \ln^2 n} \text{ „like" } \int_2^\infty \frac{1}{x \ln^2 x}\,dx \overset{t=\ln x}{=} \int_{\ln 2}^\infty \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_{\ln 2}^\infty = \frac{1}{\ln 2} \Rightarrow$$

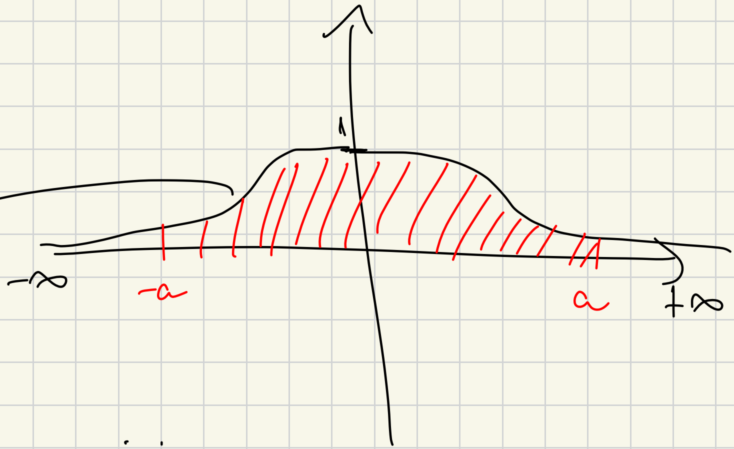
$$\boxed{\frac{dt}{dx} = \frac{1}{x} \Rightarrow dt = \frac{1}{x}\,dx}$$

$\Rightarrow$ conv.

c) $$\sum_{n\geq 2} \frac{\ln n}{n^2}$$

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = \int_2^\infty \ln x \left(-\frac{1}{x}\right)'\,dx = \underbrace{\ln x\left(-\frac{1}{x}\right)\Big|_2^\infty}_{\frac{\ln 2}{2}} + \underbrace{\int_2^\infty \frac{1}{x^2}\,dx}_{<\infty} < \infty \Rightarrow \text{conv.}$$

6) Indications:



CURBA lui Gauss.

$\int e^{-x^2} dx$ is impossible to compute with elementary functions.

Try trapezium rule for the best approximation

or. riemann sum for