Seminar 7

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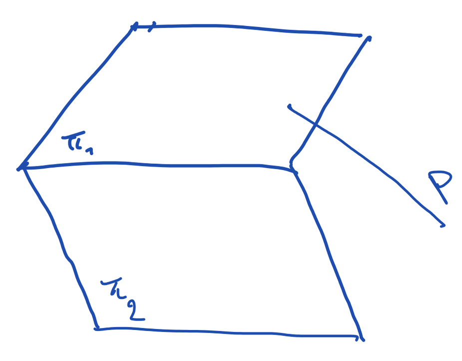
Lines and planes in 3D

1. Determine parametric equations of the line passing through $P(5,0,-2)$ and parallel to the planes $\pi_1 : x-4y+2z=0$ and $2x+3y-z+1=0$.



$\ell \parallel \pi \Leftrightarrow \vec{\ell} \subset \vec{\pi}$

$\vec{n}_{\pi_1}(1,-4,2)$

$\vec{n}_{\pi_2}(2,3,-1)$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = \begin{vmatrix} -4 & 2 \\ 3 & -1 \end{vmatrix} \vec{i} + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & -4 \\ 2 & 3 \end{vmatrix} \vec{k}$$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = -2\vec{i} + 5\vec{j} + 11\vec{k}$$

$\vec{\ell} = \langle(-2,5,11)\rangle$

__Parametric equation__

$$\begin{cases} x = 5 - 2\lambda \\ y = 5\lambda \\ z = -2 + 11\lambda \end{cases}$$

3. For the points $A(2,1,-1)$ and $B(-3,0,2)$, determine

a) an equation of the bundle of planes passing through A and B,

b) the plane π from the bundle, which is orthogonal to Oxy,

c) the plane ρ from the bundle, which is orthogonal to π.

__Bundle of planes__

$$\ell: \begin{cases} \pi_1 : A_1x + B_1y + C_1z + D_1 = 0 \\ \pi_2 : A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

The planes that contain ℓ are of the form

$\pi_{\alpha,\beta} : \alpha(A_1x + B_1y + C_1z + D_1) + \beta(A_2x + B_2y + C_2z + D_2) = 0$

(the bundle of planes corresponding to ℓ)

a) $\vec{AB}(-5,-1,3)$

$$\begin{cases} x = 2 - 5\lambda \\ y = 1 - \lambda \\ z = -1 + 3\lambda \end{cases}$$

$$AB: \frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A}$$

$$\frac{x-2}{-5} = \frac{y-1}{-1} = \frac{z+1}{3}$$

$$AB: \begin{cases} \frac{x-2}{5} - (y-1) = 0 \quad | \cdot 5 \\ \frac{x-2}{-5} - \frac{z+1}{3} = 0 \quad | \cdot (-15) \end{cases}$$

$$AB: \begin{cases} x - 2 - 5y + 5 = 0 \\ 3x - 6 + 5z + 5 = 0 \end{cases}$$

$$AB: \begin{cases} x - 5y - 3 = 0 \\ 3x + 5z - 1 = 0 \end{cases}$$

$\pi_{\alpha,\beta} = \alpha(x - 5y - 3) + \beta(3x + 5z - 1) = 0$

$\pi_{\alpha,\beta} = (\alpha + 3\beta)x - 5\alpha y + 5\beta z + 3\alpha - \beta = 0$

b) $\vec{n}_{\pi_{\alpha,\beta}} = (\alpha + 3\beta, -5\alpha, 5\beta)$

$\vec{n}_{Oxy} = (0,0,1)$

$\vec{n}_{\pi_{\alpha,\beta}} \cdot \vec{n}_{Oxy} = 0$

$5\beta = 0$

$\beta = 0$

$\pi: \alpha x - 5\alpha y + 3\alpha = 0 \quad | : \alpha$

$x - 5y + 3 = 0$

c) $\vec{n}_\pi = (1,-5,0)$

$\rho \perp \pi \Leftrightarrow \vec{n}_\rho \perp \vec{n}_\pi$

$(\alpha + 3\beta, -5\alpha, 5\beta) \cdot (1,-5,0) = 0$

$\alpha + 3\beta + 25\alpha = 0$

$26\alpha = 3\beta$

$\beta = -\frac{26}{3}\alpha$

$\rho: -25\alpha x - 5\alpha y - \frac{130\alpha}{3} z + 3\alpha + \frac{26}{3}\alpha = 0 \quad | : \alpha$

$-25x - 5y - \frac{130}{3}z + \frac{35}{3} = 0$

internal angle bisector of $\measuredangle A$.

6. Determine the angles between the plane $\pi_1 : x - \sqrt{2}y + z - 1 = 0$ and the plane $\pi_2 : x + \sqrt{2}y - z + 3 = 0$.

$\vec{n}_{\pi_1} = (1, -\sqrt{2}, 1)$

$\vec{n}_{\pi_2} = (1, \sqrt{2}, -1)$

$\measuredangle(\pi_1, \pi_2) = \measuredangle(\vec{n}_{\pi_1}, \vec{n}_{\pi_2})$

$$\cos(\measuredangle \vec{n}_{\pi_1}, \vec{n}_{\pi_2}) = \frac{\vec{n}_{\pi_1} \cdot \vec{n}_{\pi_2}}{\|\vec{n}_{\pi_1}\| \cdot \|\vec{n}_{\pi_2}\|} = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \measuredangle(\pi_1, \pi_2) = -\frac{\pi}{3} = \frac{2\pi}{3}$$

9. Determine the values a and c for which the line $3x-2y+z+3 = 0 \cap 4x-3y+4z+1 = 0$ is perpendicular to the plane $ax + 8y + cz + 2 = 0$.

$\vec{n}_{\pi_1}(3,-2,1)$

$\vec{n}_{\pi_2}(4,-3,4)$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = \begin{vmatrix} -2 & 1 \\ -3 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 1 \\ 4 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} \vec{k}$$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = -5\vec{i} - 8\vec{j} - \vec{k}$$

$\vec{\ell} = \langle(5,8,1)\rangle$

$-5a + 64 - c = 0$

$c = 64 - 5a$

$ax + 8y + 64 - 5a + 2 = 0$

$\vec{n}_\pi(a,8,c)$

$\ell \parallel \vec{n}_\pi \Leftrightarrow (-5,-8,-1) = \lambda(a,8,c)$

$\lambda = -1$

$\ell \perp \pi$

10. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x + 4y - 3z + 7 = 0$.

$\pi: x + 4y - 3z + 7 = 0$

$\vec{\ell} = \langle \vec{n}_\pi \rangle = \langle(1,4,-3)\rangle$

$$AA': \begin{cases} x = 2 + t \\ y = 11 + 4t \\ z = -5 - 3t \end{cases}$$

$$A' = \begin{cases} x = 2 + t \\ y = 11 + 4t \\ z = -5 - 3t \end{cases}$$

$x + 4y - 3z + 7 = 0 \Leftrightarrow 2 + t + 44 + 16t + 15 + 9t + 7 = 0 \Leftrightarrow$

$\Leftrightarrow 26t = -68 \Leftrightarrow t = -\frac{34}{3}$

2. Determine an equation of the plane containing $P(2,0,3)$ and the line $\ell : x = -1 + t, y = t, z = -4 + 2t, t \in \mathbb{R}$.

$t = 0$: $A(-1,0,-4)$

$\vec{PA}(-3,0,-7)$

$t = 1$: $B(0,1,-2)$

$\vec{PB}(-2,1,-5)$

$$\vec{PA} \times \vec{PB} = \begin{vmatrix} 0 & -7 \\ 1 & -5 \end{vmatrix} \vec{i} + \begin{vmatrix} -5 & -2 \\ -7 & -3 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 0 \\ -2 & 1 \end{vmatrix} \vec{k}$$

$$\vec{PA} \times \vec{PB} = -7\vec{i} - \vec{j} - 3\vec{k} \Rightarrow v(-7,-1,-3)$$

__Parametric eq__

$$\begin{cases} x = 2 - 7\lambda \\ y = -\lambda \\ z = 3 - 3\lambda \end{cases}$$