8.11.2023

1) $f: \mathbb{R}^2 \to \mathbb{R}, \; f(x,y) = x^2 + xy$

$$\nabla f(x) = \left(\frac{\partial f}{\partial x}(x), \frac{\partial f}{\partial y}(x) \right)$$

$$\frac{\partial f}{\partial x}(x,y) = 2x + y$$

$$\frac{\partial f}{\partial y}(x,y) = x$$

$$\nabla f(1,0) = (2,1)$$

steepest descent is: $-\nabla f(x,y) = (-2,-1)$

b) $$D_v f(x,y) = \nabla f(x,y) \cdot v$$

$$= (2,1) \cdot (1,1)$$

$$= 2 + 1 = 3$$

c) $z = f(x,y) \Rightarrow z = x^2 + xy \Rightarrow \underbrace{x^2 + xy - z}_{g(x,y,z)} = 0$

Gradient ⊥ Level set

$$\frac{\partial g}{\partial x} = 2x + y$$

$$\frac{\partial g}{\partial y} = x$$

$$\frac{\partial g}{\partial z} = -1$$

$$\nabla g \left(\underbrace{\frac{\partial g}{\partial x}}_{= \frac{\partial f}{\partial x}}, \underbrace{\frac{\partial g}{\partial y}}_{= \frac{\partial f}{\partial y}}, \frac{\partial g}{\partial z} \right) =$$

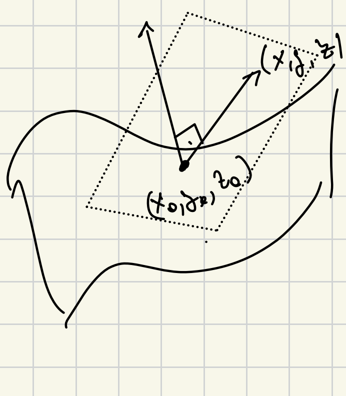
$$= (2x + y, \; x, \; -1)$$

$$\nabla g \left(\boxed{1, 0, 1} \right) = (2, 1, -1)$$

$$(2, 1, -1) \cdot (x - 1, y, z - 1) = 0$$

$$2x - 1 + y - z + 1 = 0 \Rightarrow 2x + y - z - 1 = 0 \Rightarrow$$

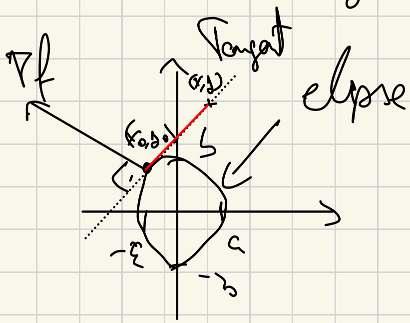
$$\Rightarrow \boxed{z = 2x + y - 1}$$



2) $$\underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2}}_{f(x,y)} = 1$$

$\Rightarrow$ elipse = the level set $= \{(x,y) \mid f(x,y) = 1\}$

$\nabla f(x,y) \perp$ Level set.



$$\nabla f(x_0,y_0) \cdot (x - x_0, y - y_0) = 0$$

$$\Leftrightarrow \frac{\partial f}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0) \cdot (y - y_0) = 0$$

$$\frac{\partial f}{\partial x} = \frac{2x}{a^2} \quad ; \quad \frac{\partial f}{\partial y} = \frac{2y}{b^2}$$

$$\Leftrightarrow \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2} \cdot (y - y_0) = 0 \quad \Big| \cdot \frac{1}{2}$$

$$\Leftrightarrow \frac{x_0 \cdot x - x_0^2}{a^2} + \frac{y_0 \cdot y - y_0^2}{b^2} = 0$$

$$\Leftrightarrow \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = f(x_0,y_0) = 1$$

$\Rightarrow y = \ldots\ldots$ (in terms of x)

3) $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = \frac{1}{2}\|x\|^2 = \frac{1}{2} \cdot x \cdot x =$

$= \frac{1}{2} \cdot (x_1^2 + x_2^2 + \ldots + x_n^2)$

$\nabla f(x) = x$

$\frac{\partial f}{\partial x_1} = x_1$; $\frac{\partial f}{\partial x_2} = x_2$; ..., $\frac{\partial f}{\partial x_n} = x_n$

$D_v f(x) = \nabla f(x) \cdot v = x \cdot v$

$D_v f(x) \overset{\text{Def.}}{=} \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h}$

$f(x+hv) = \frac{1}{2}\|x+hv\|^2 = \frac{1}{2}(x+hv)(x+hv) =$

$= \frac{1}{2}(\underbrace{x \cdot x}_{f(x)?} + 2h(x \cdot v) + h^2 \cdot v \cdot v)$

$f(x+hv) - f(x) = \frac{1}{2}(2h(x \cdot v) + h^2 \cdot v \cdot v)$

$= h(x \cdot v) + \frac{h^2}{2} \cdot (v \cdot v)$

$$\lim_{h \to 0} \frac{\cancel{h}(x\cdot v) + \frac{h^{\cancel{2}}}{2}\cdot(v\cdot v)}{\cancel{h}} = x\cdot v$$

$x \cdot (D_x)$

DOT PROD

$$D = \operatorname{diag}(d_1, \dots, d_n)$$

$$f: \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \frac{1}{2} x^T \cdot D \cdot x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad D_x = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$$

$$= \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}$$

$$f(x) = \frac{1}{2} x^T \cdot D_x = \frac{1}{2} \underbrace{\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}}_{1\times n} \underbrace{\begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}}_{n\times 1} =$$

$$= \frac{d_1 x_1^2 + d_2 x_2^2 + \dots + d_n x_n^2}{2} =$$

$$x \cdot y = x^T y = \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{x^T} \cdot \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{y}$$

$$\frac{\partial f}{\partial x_1} = \frac{2 d_1 x_1}{2} = d_1 x_1 \quad \dots \quad \Rightarrow \frac{\partial f}{\partial x_i} = d_i x_i \Rightarrow$$

$$\Rightarrow \nabla f = D x$$

Hessian matrix: $H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 x_n} & \frac{\partial^2 f}{\partial x_2 x_n} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 x_n} & \frac{\partial^2 f}{\partial x_2 x_n} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ 0 & 0 & d_3 \dots & 0 \\ \vdots & & & \\ 0 & 0 \dots & 0 & d_n \end{bmatrix}$$

$$= D \qquad \boxed{\text{q.e.d.}}$$

$$\frac{\partial f}{\partial x_1^2} = \frac{\partial f}{\partial x_1}\underbrace{(d_1 x_1)}_{\frac{\partial f}{\partial x_1}} = d_1$$

$$\frac{\partial f}{\partial x_i \partial x_j} = 0 \,, \quad i \neq j \quad \text{and} \quad \frac{\partial f}{\partial x_i^2} = d_i \quad (i=j)$$

$$\Rightarrow \quad f(x,y) = \ln(x^2+y^2) \;;\; l(t) = \ln(t^2+t^4)$$

$$x = t \,,\quad y = t^2 \qquad \frac{dl}{dt} = \frac{2t+4t^3}{t^2+t^4} = \frac{2+4t^2}{t+t^3}$$

$$\frac{dl}{dt} = \frac{\partial l}{\partial x}\cdot\frac{dx}{dt}^{=1} + \frac{\partial l}{\partial y}\cdot\frac{dy}{dt}^{=2t} =$$

$$\frac{\partial l}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}$$

$$= \frac{2t}{t^2+t^4}\cdot 1 + \frac{2t^2}{t^2+t^4}\cdot 2t =$$

$$= \frac{2t+4t^3}{t^2+t^4} = \frac{2+4t^2}{t^2+t^4}$$

b) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$x = \cos t \; ; \; y = \sin t \; ; \; z = t > 0.$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \underbrace{\frac{dx}{dt}}_{= -\sin t} + \frac{\partial f}{\partial y} \cdot \underbrace{\frac{dy}{dt}}_{= \cos t} + \frac{\partial f}{\partial z} + \underbrace{\frac{dz}{dt}}_{= 1} =$$

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \; ; \; \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= \cancel{\frac{\cos t \cdot (-\sin t)}{\sqrt{\cos^2 t + \sin^2 t + t^2}}} + \cancel{\frac{\sin t \cdot \cos t}{\sqrt{\cos^2 t + \sin^2 t + t^2}}} + \frac{t}{\sqrt{\cos^2 t + \sin^2 t + t^2}} =$$

$$= \frac{t}{\sqrt{1+t^2}}.$$

$$f(t) = \sqrt{\cos^2 t + \sin^2 t + t^2} = \sqrt{1+t^2}$$

$$\frac{df}{dt} = \frac{\cancel{2}t}{\cancel{2}\sqrt{1+t^2}} = \frac{t}{\sqrt{1+t^2}}.$$

b) $(x,y) = (g_1(u,v), g_2(u,v)) = g(u,v)$

$g: \mathbb{R}^2 \to \mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

$$(f \circ g)(u,v) = f(g_1(u,v), g_2(u,v))$$

$(u,v) \to f(g(u,v))$

Chain Rule $\quad D(f\circ g)(\cdot) = Df(g(\cdot)) \cdot Dg(\cdot)$

$$Df(u,v) = \nabla f(u,v) = \left(\frac{\partial f}{\partial x}(u,v), \frac{\partial f}{\partial y}(u,v)\right)_{1\times 2}$$

$$Dg(u,v) = \begin{bmatrix} \nabla g_1(u,v) \\ \nabla g_2(u,v) \end{bmatrix} = \begin{bmatrix} \underbrace{\frac{\partial g_1}{\partial u}}_{=\frac{\partial x}{\partial u}}(u,v) & \frac{\partial g_1}{\partial v}(u,v) \\ \frac{\partial g_2}{\partial u}(u,v) & \frac{\partial g_2}{\partial v}(u,v) \end{bmatrix}_{2\times 2}$$

$$D(f \circ g)(u,v) = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} ,\right.$$

$$\left. \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \right) =$$

$$= \left(\frac{\partial f}{\partial u} , \frac{\partial f}{\partial v} \right)$$