Week 12: Discrete scalar dynamical systems

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First order linear difference equations

We consider the first order linear homogeneous difference equation with constant coefficient

$$x_{k+1} - a x_k = 0, \quad k \in \mathbb{N} \tag{1}$$

where $a \in \mathbb{R}$ is given, and the unknown is the sequence of real numbers $(x_k)_{k \geq 0}$.

Its characteristic equation is $r - a = 0$ with the real root $r = a$. We associate the sequence a^k and write the general solution

$$x_k = c\, a^k, \quad k \in \mathbb{N},$$

with $c \in \mathbb{R}$ an arbitrary constant.

First order linear difference equations

Now we consider the first order linear non-homogeneous difference equation

$$x_{k+1} - ax_k = b_k, \quad k \in \mathbb{N} \tag{2}$$

where $(b_k)_{k \geq 0}$ is a given sequence of real numbers.
It can be proved that

its general solution is the sum between a particular solution of it and the general solution of the linear homogeneous difference equation associated $x_{k+1} - ax_k = 0$.

Second order linear homogeneous difference equations

We consider the second order linear homogeneous difference equation with constant coefficients

$$x_{k+2} + a_1 x_{k+1} + a_2 x_k = 0, \quad k \in \mathbb{N} \tag{3}$$

where $a_1, a_2 \in \mathbb{R}$ are given, and the unknown is the sequence of real numbers $(x_k)_{k \geq 0}$. We present the characteristic equation method to find its general solution.

First note that, if we look for solutions of the form $x_k = r^k$ we get $r^{k+2} + a_1 r^{k+1} + a_2 r^k = 0$ for all $k \in \mathbb{N}$. Thus, we must have $r^2 + a_1 r + a_2 = 0$.

The characteristic equation method

$$x_{k+2} + a_1 x_{k+1} + a_2 x_k = 0, \quad k \in \mathbb{N} \tag{4}$$

Step 1. Write the characteristic equation $r^2 + a_1 r + a_2 = 0$ and find its roots $r_1, r_2 \in \mathbb{C}$.

Step 2. According to the nature of the roots we associate two sequences following the rules:

If $r_{1,2} \in \mathbb{R}$ and $r_1 \neq r_2$ then we associate r_1^k and r_2^k.

If $r_1 = r_2 \in \mathbb{R}$ then we associate r_1^k and kr_1^k.

If $r_1 = \overline{r_2} \in \mathbb{C} \setminus \mathbb{R}$ then we associate the real part and, respectively, the imaginary part of r_1^k.

Step 3. We write the general solution as a linear combination with arbitrary coefficients of the two sequences found at Step 2.

Scalar discrete dynamical systems

For a given map $f : \mathbb{R} \to \mathbb{R}$, starting with an initial value $x_0 = \eta \in \mathbb{R}$, we construct a sequence $(x_k)_{k \geq 0}$ such that

$$x_{k+1} = f(x_k).$$

This is said to be the positive orbit of the initial state η; or the sequence of iterates of f that starts with η.

Notation: $f^2 = f \circ f$, $f^3 = f \circ f \circ f$ and $f^k =$ the k times composition of f with itself. Then $x_k = f^k(\eta)$.

We are interested in studying the long-term behavior of each orbit.
We will always assume at least that f is continuous.

The simplest sequence is a constant one and it is obtained when the initial value is a fixed point of f, i.e. $f(\eta) = \eta$.

Fixed points - an exercise

We consider the logistic map $f_\lambda : \mathbb{R} \to \mathbb{R}$, $f_\lambda(x) = \lambda x(1-x)$, where $\lambda \in [1,4]$ is a fixed parameter. Our aim is to find the fixed points of f. $\diamond$

We have to solve the equation $x = f_\lambda(x) \iff x = \lambda x(1-x) \iff \lambda x^2 - (\lambda - 1)x = 0$. This equation has two roots, which means that f_λ has two fixed points $\eta_1^* = 0$ and $\eta_2^* = \frac{\lambda-1}{\lambda}$. $\square$

In particular, this implies that the unique solution of the IVP
(a) $x_{k+1} = \lambda x_k(1-x_k)$, $x_0 = 0$ is $x_k = 0$ for all $k \geq 0$.
(b) $x_{k+1} = 2x_k(1-x_k)$, $x_0 = 0.5$ is $x_k = 0.5$ for all $k \geq 0$.

Graphically, the fixed points of f are found at the intersection between the graph of f and the first bisectrix (i.e. the line $y = x$).

Scalar discrete dynamical systems

Another simple behavior is when the sequences is convergent, i.e. there exists $\eta^* \in \mathbb{R}$ such that $x_k \to \eta^*$ when $k \to \infty$.
In this case, using also the continuity of f, we have $x_{k+1} \to \eta^*$ and $f(x_k) \to f(\eta^*)$ when $k \to \infty$.
Since $x_{k+1} = f(x_k)$ we deduce that $\eta^* = f(\eta^*)$. Thus, The limit of a convergent sequence of iterates of f is a fixed point of f.

We say that a fixed point η^* is an attractor of f when there exists $\rho > 0$ such that for each η with $|\eta - \eta^*| \leq \rho$ we have $f^k(\eta) \to \eta^*$ when $k \to \infty$.
The basin of attraction of an attractor η^* is defined as

$$A_{\eta^*} = \{\eta \in \mathbb{R} \quad : \quad f^k(\eta) \to \eta^* \text{ when } k \to \infty\}.$$

When $A_{\eta^*} = \mathbb{R}$ we say that η^* is a global attractor of f.

A linear map

We consider $f(x) = a\,x$, where $a \in \mathbb{R} \setminus \{-1, 0, 1\}$ and the associated linear difference equation

$$x_{k+1} = ax_k.$$

It is easy to see that the only fixed point of f is $\eta^* = 0$ and $x_k = f^k(\eta) = a^k\eta$ for any $\eta \in \mathbb{R}$, $k \in \mathbb{N}$.

If $|a| < 1$ then $\eta^* = 0$ is a global attractor of f.

If $|a| > 1$ then $f^k(\eta)$ is unbounded for any $\eta \neq 0$.

The linearization method

Theorem

Assume that f is a C^1 function and let η^* be a fixed point of f.
If $|f'(\eta^*)| < 1$ then η^* is an attractor.
If $|f'(\eta^*)| > 1$ then η^* is not an attractor.

Exercise: Apply the linearization method to the fixed points of the logistic map $f_\lambda(x) = \lambda x(1-x)$ where $\lambda \in [1,4]$ is a parameter. $\diamond$

We found that f_λ has two fixed points $\eta_1^* = 0$ and $\eta_2^* = \frac{\lambda-1}{\lambda}$.
We compute the derivative of $f_\lambda(x) = \lambda x - \lambda x^2$ and obtain
$f'_\lambda(x) = \lambda - 2\lambda x$.
Then $|f'_\lambda(\eta_1^*)| = \lambda$ and $|f'_\lambda(\eta_2^*)| = |\lambda - 2(\lambda-1)| = |2-\lambda|$.
The above theorem only assures that:
If $\lambda \in (1,3)$ then $\eta_1^* = 0$ is not an attractor, but η_2^* is an attractor.
If $\lambda \in (3,4]$ then neither of the two fixed points is an attractor.

The cobweb or stair-step diagram

The cobweb diagram is an intuitive geometric way of showing the behavior of the orbits $x_k = f^k(\eta)$ of a discrete dynamical system.

The algorithm works as follows for $k = 0, 1, 2, \ldots$:
Beginning with x_k, move vertically to find the point on the graph of the map f that corresponds to it, i.e. the point of coordinates $(x_k, f(x_k))$.
Then move horizontally to a point on the line $y = x$, i.e the point of coordinates $(f(x_k), f(x_k))$.
Repeat as many times as needed.

The cobweb diagram for the logistic map can be better visualized here:
http://sites.saintmarys.edu/~sbroad/example-logistic-cobweb.html
https://www.geogebra.org/m/gHYqKMSJ

We proved that, when $\lambda \in (1, 3)$ the fixed point $\eta_2^* = \frac{\lambda - 1}{\lambda}$ is an attractor.
It is important to find its basin of attraction A_2.
After we have seen, it seems that $A_2 = (0, 1)$.
In fact, this is mathematically proved. (see Elaydi: Discrete chaos, 2008)

Periodic points

We will employ again the notation $f^k = f \circ f \circ \ldots \circ f$, the k times composition of the map f with itself. We start by noting that a fixed point of f is also a fixed point of f^k for any $k \in \mathbb{N}$.

Definition

Let $\eta^* \in \mathbb{R}$ and $p \in \mathbb{N}$, $p \geq 2$. We say that η^* is a p-periodic point of f when η^* is a fixed point of f^p but η^* is not a fixed point of f^{p-1}, ..., f^2, f.

Remark. If η^* is a p-periodic point of f then the unique solution of the IVP $x_{k+1} = f(x_k)$, $x_0 = \eta^*$ is a sequence whose first p terms are η^*, $f(\eta^*)$, ..., $f^{p-1}(\eta^*)$ and then they are repeated. Such an orbit is also called a p-cycle.

Exercise: Check that $\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$ is a 2-cycle of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$. $\diamond$

We compute $f\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1-\sqrt{5}}{2}$ and $f^2\left(\frac{1+\sqrt{5}}{2}\right) = f\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1+\sqrt{5}}{2}$. $\square$

Periodic points - an exercise

Exercise: Find the fixed points and the 2-periodic points of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$.

To find the fixed points we solve the equation $x = f(x) \Leftrightarrow$ $2x^2 + x - 1 = 0 \Leftrightarrow (2x-1)(x+1) = 0$.
Thus, f has two fixed points: $\eta_1^* = 1/2$ and $\eta_2^* = 1$.

To find the fixed points we first compute $f^2 = f \circ f$, then find the fixed points of f^2, i.e. solve the equation $x = f^2(x)$. We will use that the fixed points of f are also fixed points of f^2.

We have
$f^2(x) = f(f(x)) = 1 - 2f(x)^2 = 1 - 2(1-2x^2)^2 = 1 - 2(1 - 4x^2 + 4x^4)$.
Then $f^2(x) = -8x^4 + 8x^2 - 1$ for all $x \in \mathbb{R}$.

The equation $x = f^2(x) \Leftrightarrow 8x^4 - 8x^2 + x + 1 = 0$. This is a polynomial equation of degree 4. We have that the fixed points of f ($1/2$ and 1) are roots of this equation. Since
$8x^4 - 8x^2 + x + 1 = (2x^2 + x - 1)(4x^2 - 2x - 1)$, the 2-periodic points are the roots of $4x^2 - 2x - 1 = 0$. Thus, they are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

Attracting cycle

Definition

Let η^* be a p-periodic point of f. We say that the corresponding p-cycle is an attracting cycle of f when η^* is an attracting fixed point of f^p. The basin of attraction of the fixed point η^* of f^p is said to be the basin of attraction of the p-cycle.

Remark 1. Let η^* be a p-periodic point of f such that the corresponding p-cycle is an attracting cycle of f. Let η be in the basin of attraction of the fixed point η^* of f^p. Then the unique solution of the IVP $x_{k+1} = f(x_k)$, $x_0 = \eta$ is a sequence that can be split into p convergent sub-sequences. The limits are the terms of the attracting p-cycle, i.e. η^*, $f(\eta^*)$, ..., $f^{p-1}(\eta^*)$.

Remark 2. What we see in the cobweb-diagram for the logistic map with $\lambda = 3.15$ (for example) suggests us that there exists an attracting 2-cycle having the interval $(0, 1)$ as its basin of attraction.
Note that it is very difficult (impossible) to detect (using the cobweb diagram) a cycle which is not attracting.

The linearization method for a 2-cycle

Theorem

Let $\{\eta^*, f(\eta^*)\}$ be a 2-cycle of f.
If $|f'(\eta^*)f'(f(\eta^*))| < 1$ then the cycle is an attractor.
If $|f'(\eta^*)f'(f(\eta^*))| > 1$ then the cycle is not an attractor.

Proof. Using the definition of an attracting cycle and the linearization method for fixed points, the following statements are valid.
If $|(f^2)'(\eta^*)| < 1$ then the cycle is an attractor.
If $|(f^2)'(\eta^*)| > 1$ then the cycle is not an attractor.
The proof is done after we notice that
$(f^2)'(x) = \frac{d}{dx} f(f(x)) = f'(f(x))f'(x)$ for all $x \in \mathbb{R}$. $\square$

Exercise. Prove that neither the fixed points, nor the 2-cycle of $f(x) = 1 - 2x^2$ are attractors. $\diamond$

Exercises

1. We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50} x(100 - x).$$

(a) Find its fixed points and study their stability.
(b) Using the stair-step diagram estimate the basin of attraction of the attractor fixed point.
(c) If $(x_k)_{k \geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50} x_k (100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$. $\diamond$

Exercises

2. Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x} \ .$$

All the exercises have been solved during the lecture.