Week 7: Planar dynamical systems

April 8, 2024

Planar dynamical systems (n=2)

We consider differential equations of the form

$$\dot{X} = f(X) \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a given C^1 function.

In the previous lecture it was given the existence and uniqueness theorem that mainly assures that, for any initial state $\eta \in \mathbb{R}^2$ there exists a unique solution of (1) with $X(0) = \eta$. This solution is denoted by $t \mapsto \varphi(t, \eta)$ and it describes the motion of (1) that initiates at η. The orbit is $\gamma_\eta = \{\varphi(t, \eta) \; : t \in I_\eta\}$.

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The function φ of both variables (t, η) is called the flow of (1).

Also, the following notions have been presented:

equilibrium point (constant solution, stationary point, steady state)

orbit (all the states of the system during one motion)

attractor equilibrium point (when starting nearby, the system tend to it in the future)

repeller equilibrium point (the same as "attractor" but reversing the arrow of time)

phase portrait (graphical representation of the orbits)

The case $n = 1$ was discussed.

The main goal of the theory is to represent the phase portrait.

Why? Because from the phase portrait one can reed (deduce) the main properties of the solutions of the differential system, in other words, the long-term behaviour of the dynamical system.

How? If it is possible, without knowing the expressions of the solutions (since, in general, it is impossible to find them).

In the case $n = 1$ we saw in the previous lecture that it is possible to represent the phase without solving the differential equation. We need only to study the sign of the function f from $\dot{x} = f(x)$.

The phase portrait can be more complex in higher dimensions and the study is more complicated, of course. Today we consider $n = 2$. We start with presenting a new notion.

Definition Let $U \subset \mathbb{R}^2$ be open and $H : U \to \mathbb{R}$ be a continuous function.
For some $c \in \mathbb{R}$, the c-level curve of H is the planar curve

$$\Gamma_c = \{X \in U \,:\, H(X) = c\}.$$

We say that H is a first integral in U of (1) if

H is not locally constant and

$$H(\varphi(t,\eta)) = H(\eta), \forall\ \eta \in U, \forall\ t \in I_\eta \ \ \varphi(t,\eta) \in U.$$

We say that U is an invariant set of (1) if $\gamma_\eta \subset U$ for any $\eta \in U$.

Remark
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In other words,

The orbits are contained in the level curves of a first integral.

When H is a first integral in $U = \mathbb{R}^2$, we say that
H is a global first integral.

Recall that an attractor is said to be a global attractor when its basin of attraction is $\mathbb{R}^2$.

Phase portraits of linear planar systems

Let $A \in \mathcal{M}_2(\mathbb{R})$ and

$$\dot{X} = AX \quad \text{or} \quad \left\{ \begin{array}{l} \dot{x} = a_{11}x + a_{12}y, \\ \dot{y} = a_{21}x + a_{22}y \end{array} \right. \tag{2}$$

Remark. We have that $\eta^* = 0 \in \mathbb{R}^2$ is the unique equilibrium point of (2) if and only if $\det(A) \neq 0$.

In this lecture we intend to study the following (simple) linear planar systems.

$$a) \left\{ \begin{array}{l} \dot{x} = -y, \\ \dot{y} = x \end{array} \right. \quad b) \left\{ \begin{array}{l} \dot{x} = -x, \\ \dot{y} = -y \end{array} \right. \quad c) \left\{ \begin{array}{l} \dot{x} = -x, \\ \dot{y} = y \end{array} \right. \quad d) \left\{ \begin{array}{l} \dot{x} = x - y, \\ \dot{y} = x + y \end{array} \right.$$

A center

a) $\dot{x} = -y,\ \dot{y} = x$. Find the flow. Check that $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x^2 + y^2$ is a global first integral. What is the shape of the level curves of H? Represent the phase portrait. Is the equilibrium point at the origin a global attractor/repeller?

In order to find the flow we have to consider the IVP

$$\dot{x} = -y, \quad \dot{y} = x, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 \cos t - \eta_2 \sin t,\ \eta_1 \sin t + \eta_2 \cos t).$$

A center

In order to check that H is a global first integral we first notice that H is continuous and not locally constant on $\mathbb{R}^2$. It remains just to check that

$$H(\varphi(t,\eta)) = H(\eta),\ \forall\, \eta \in \mathbb{R}^2,\ \forall\, t \in \mathbb{R}.$$

As we know, the orbits of the system lie on the level curves of H. The level curves of H are the planar curves of implicit equations $x^2 + y^2 = c$, $c \in \mathbb{R}$, i.e. they are the circles centered in the origin. So, we found the orbits.

In order to insert an arrow on each orbit we note that when $y > 0$ we have $\dot{x} = -y < 0$. So, in the upper half-plane, x decreases along an orbit. Thus, the arrow points to the left in the upper half-plane.

A node

b) $\dot{x} = -x$, $\dot{y} = -y$. Find the flow. Check that the origin is a global attractor. Note that $\mathbb{R} \times (0, \infty)$ is an invariant set. Find the expression of a first integral in $\mathbb{R} \times (0, \infty)$.

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$$\dot{x} = -x, \quad \dot{y} = -y, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{-t},\, \eta_2 e^{-t}).$$

The origin is a global attractor since $\lim_{t\to\infty} \varphi(t, \eta_1, \eta_2) = (0, 0)$ for all $(\eta_1, \eta_2) \in \mathbb{R}^2$.

A node

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{-t}, \, \eta_2 e^{-t}).$$

$\mathbb{R} \times (0, \infty)$ is an invariant set since $(\eta_1, \eta_2) \in \mathbb{R} \times (0, \infty) \implies \eta_2 > 0 \implies \eta_2 e^{-t} > 0, \ \forall \, t \in \mathbb{R} \implies \gamma_\eta \subset \mathbb{R} \times (0, \infty)$.

In order to find the expression of a first integral in $\mathbb{R} \times (0, \infty)$ we have to recall that the time t must disappear after we replace $x = \eta_1 e^{-t}$ and $y = \eta_2 e^{-t}$.

Check that $H : \mathbb{R} \times (0, \infty) \to \mathbb{R}$, $H(x, y) = \frac{x}{y}$ is a first integral in $\mathbb{R} \times (0, \infty)$. Note that the same expression also defines a first integral in $\mathbb{R} \times (-\infty, 0)$.

A node

In order to find the shape of the orbits, note that the level curves of H are $y = \frac{1}{c}x$, $c \in \mathbb{R}$. There are the lines that contain the origin.

On each orbit, the arrow must point to the origin since it is a global attractor.

A saddle

c) $\dot{x} = -x, \ \dot{y} = y$. Find the flow. Find the expression of a global first integral. Find the shape of the orbits.

In order to find the flow we have to consider the IVP

$$\dot{x} = -x, \quad \dot{y} = y, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{-t}, \, \eta_2 e^{t}).$$

In order to find the expression of a first integral we have to recall that the time t must disappear after we replace $x = \eta_1 e^{-t}$ and $y = \eta_2 e^{t}$.

A saddle

Check that $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = xy$ is a global first integral. Indeed, H is continuous on $\mathbb{R}^2$, it is not locally constant, and $H(\varphi(t, \eta_1, \eta_2)) = H(\eta_1 e^{-t}, \eta_2 e^t) = \eta_1\eta_2 = H(\eta_1, \eta_2)$ for all $(\eta_1, \eta_2) \in \mathbb{R}^2$.

The shape of the orbits. The level curves of H are $xy = c$, $c \in \mathbb{R}$. For $c = 0$ we have $x = 0$ or $y = 0$. For $c = 1$ we have $y = \frac{1}{x}$. The level curves are hyperbolas tangent to the coordinate axes.

The arrow must point to the left on each orbit in the right-hand plane (since $\dot{x} = -x < 0$), while the arrow must point to the right on each orbit in the left-hand plane (since $\dot{x} = -x > 0$).

A focus

d) $\dot{x} = x - y, \ \dot{y} = x + y$. Transform the system to polar coordinates. Find the flow in polar coordinates. Find the shape of the orbits. Represent the phase portrait.

Recall that, for each $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ there exists a unique $(\rho, \theta) \in (0, \infty) \times [0, 2\pi)$ such that $x = \rho \cos\theta$, $y = \rho \sin\theta$ (or, equivalently, $x + iy = \rho e^{i\theta}$). Note that we have the reversed relations $\rho^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$.

We consider now the unknown functions $(\rho(t), \theta(t))$ such that, for each t, these are the polar coordinates of $(x(t), y(t))$, that is $\rho^2(t) = x^2(t) + y^2(t)$, $\tan\theta(t) = \frac{y(t)}{x(t)}$. After taking the derivative with respect to t, we get

A focus

$\rho\dot\rho = x\dot x + y\dot y$, $\frac{\dot\theta}{\cos^2\theta} = \frac{x\dot y - y\dot x}{x^2}$.

Now we replace $\dot x = x - y$, $\dot y = x + y$. We get

$\rho\dot\rho = x\dot x + y\dot y = x(x-y) + y(x+y) = x^2 + y^2 = \rho^2$

$\frac{\dot\theta}{\cos^2\theta} = \frac{x\dot y - y\dot x}{x^2} = \frac{x(x+y) - y(x-y)}{x^2} = \frac{\rho^2}{\rho^2\cos^2\theta} = \frac{1}{\cos^2\theta}$.

We arrive to a simple system in polar coordinates

$$\dot\rho = \rho, \quad \dot\theta = 1.$$

If we impose the initial conditions $\rho(0) = \rho_0$, $\theta(0) = \theta_0$, we immediately see that $\rho(t) = \rho_0 e^t$, $\theta(t) = t + \theta_0$. This is the expression of the flow in polar coordinates.

A focus

From the expression of the flow we see that, along an orbit, θ increases (linearly), while ρ increases (exponentially) with respect to the time. Having in mind the geometrical interpretation of the polar coordinates, we see that each orbit is a (logarithmic) spiral that rotates in the trigonometric sense around the origin while departing from it.

In general, we have the following rules.
$\dot{\theta} > 0$ means that the orbit rotates in the trigonometric sense around the origin, while $\dot{\theta} < 0$ means that the orbit rotates clockwise around the origin.
$\dot{\rho} > 0$ means that the state goes further from the origin (along the orbit), while $\dot{\rho} > 0$ means that the state approaches the origin (along the orbit).