$$T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0) \cdot (x-x_0, y-y_0) +$$
$$+ \frac{1}{2}(x-x_0, y-y_0) \cdot H_f(x_0,y_0) \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

1) $f(x,y) = \sin(x+2y)$ in $(0,0)$

$f(0,0) = 0$

$\nabla f(x_0,y_0) = (1, 2)$

$$\frac{\partial f}{\partial x} = \cos(x+2y) \cdot x' = \cos(x+2y)$$

$$\frac{\partial f}{\partial y} = \cos(x+2y) \cdot (2y)' = 2 \cdot \cos(x+2y)$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+2y)$$

$$\frac{\partial^2 f}{\partial y^2} = (-4)\sin(x+2y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = (-2) \cdot \sin(x+2y)$$

$$H(x_0,y_0) = \begin{pmatrix} -\sin(x+2y) & (-2)\sin(x+2y) \\ (-2)\cdot\sin(x+2y) & (-4)\cdot\sin(x+2y) \end{pmatrix}$$

$$H(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T_2(0,0) = 0 + (1,2)(x,y) = \underline{x+2y}.$$

$$t = \underline{x+2y}, \quad \sin t = t - \frac{t^3}{3!} + \dots, \quad T_2(t) = \underline{t}$$

5) $f(x,y) = e^{x+y}$ at $(0,0)$ and $(1,-1)$

<u>Fastway:</u>

$$t = x+y \Rightarrow f(t) = e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots$$

$$T_2(t) = 1+t+\frac{t^2}{2} \Rightarrow$$

$$\Rightarrow T_2(x,y) = 1+\underbrace{(x+y)}_{t}+\frac{\overbrace{(x+y)}^{t^2}}{2}$$

Other way;

$$f(0,0) = 1$$
$$f(1,-1) = 1$$

$\nabla f(x,y)$

$$\frac{\partial f}{\partial x} = e^{x+y} = \frac{\partial f}{\partial y} = \frac{\partial f^2}{\partial x \partial y} = \frac{\partial f^2}{\partial y \partial x}$$

$$H(x,y) = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix}$$

$$H(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = H(1,-1)$$

$$T_2(0,0) = 1 + (1,1) \cdot (x,y) + \frac{1}{2} \cdot (x,y) \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + x + y + \frac{1}{2} (x,y) \cdot \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

$$= 1 + x + y + \frac{1}{2}(x, y) \cdot (x+y,\ x+y)$$

$$= 1 + x + y + \frac{1}{2}\big(x(x+y) + y(x+y)\big)$$

$$= 1 + x + y + \frac{(x+y)^2}{2}$$

$$a \cdot b = a^T \cdot b = [\quad]\,[\quad]$$

$$T_2(1,-1) = 1 + (1,1)(x-1,\ y+1) + \frac{1}{2} \cdot (x-1, y+1) \cdot$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y+1 \end{pmatrix}$$

$$= 1 + x - \cancel{1} + y + \cancel{1} + \underbrace{\frac{1}{2}(x-1, y+1)\begin{pmatrix} x - \cancel{1} + y + \cancel{1} \\ x - \cancel{1} + y + \cancel{1} \end{pmatrix}}_{\frac{(x+y)^2}{2}}$$

$$= T_2(0,0)$$

c) $f(x,y) = \sin(x) \cdot \sin(y) \cdot \text{at } \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$f(x,y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^3}{3!} - \dots \quad , \quad \forall t \in \mathbb{R}.$$

$$t = x-y; \quad \cos(x-y) = 1 - \frac{(x-y)^2}{2} + \dots$$

$$t = x+y; \quad \cos(x+y) = 1 - \frac{(x+y)^2}{2} + \dots$$

$$T_2(x,y) = \frac{1}{2}\left(\cancel{1} - \frac{(x-y)^2}{2} - \cancel{1} + \frac{(x+y)^2}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\cancel{-x^2} + 2xy \cancel{- y^2} \cancel{+x^2} + 2xy \cancel{+y^2}}{2}\right)$$

$$= xy$$

$$\left(x - \frac{x^3}{3!} + \dots\right)\left(y - \frac{y^3}{3!} + \dots\right) = xy + \dots$$

d) $f(x,y) = e^{-(x^2+y^2)}$ at $(0,0)$

$$f(0,0) = 1$$

$$\nabla f(0,0) = (0,0)$$

$$\frac{\partial f}{\partial x} = e^{-(x^2+y^2)}(-2x)$$

$$\frac{\partial f}{\partial y} = e^{-(x^2+y^2)} \cdot (-2y)$$

$$\frac{\partial^2 f}{\partial x^2} = e^{-(x^2+y^2)} \cdot (-2x) \cdot (-2x) - 2e^{-(x^2+y^2)}$$

$$= e^{-(x^2+y^2)}(4x^2 - 2)$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-(x^2+y^2)}(4y^2 - 2)$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{-(x^2+y^2)} \cdot 4xy = \frac{\partial^2 f}{\partial x \partial y}$$

$$H_{(0,0)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$T_2(x,y) = 1 + 0 + \frac{1}{2} \cdot (x,y) \cdot \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= 1 + \frac{1}{2} \cdot (x,y) \cdot \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

$$= 1 + \frac{1}{2}\left(-2x^2 - 2y^2\right) = 1 - x^2 - y^2$$

<u>Faster way</u>:

$$e^t = 1 + t + \frac{t^2}{2!} + \dots$$

$$t = -(x^2 + y^2)$$

$$e^{-(x^2+y^2)} = 1 - (x^2+y^2) + \frac{(x^2+y^2)^2}{2!}$$

second order terms end here, unlike here

2) a) $f(x,y) = (y-1)\, e^{x} + (x-1) \cdot e^{y}$ at $(0,0)$

$$\frac{\partial f}{\partial x} = (y-1) \cdot e^{x} + e^{y}$$

$$\frac{\partial f}{\partial y} = e^{x} + (x-1)\, e^{y}$$

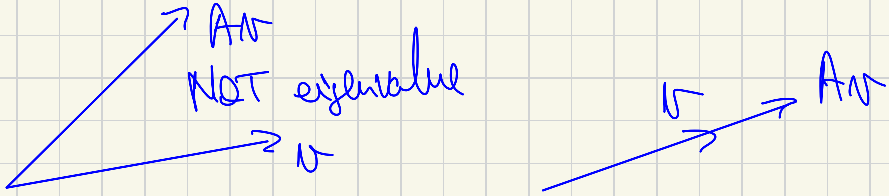
$$\frac{\partial f^{2}}{\partial x^{2}} = (y-1)\, e^{x}$$

$$\frac{\partial f^{2}}{\partial y^{2}} = (x-1)\, e^{y}$$

$$\frac{\partial f^{2}}{\partial y \partial x} = e^{x} + e^{y} = \frac{\partial f^{2}}{\partial x \partial y}$$

$$H(0,0) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

eigenvalues : $A v = \lambda v$



$$(A - \lambda I)\, v = 0 \quad , \quad v \neq 0$$

$$\det(A - \lambda I) = 0$$

$$H(0,0) - \lambda I = \begin{pmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$\det(H(0,0) - \lambda I) = 0 \Rightarrow$$

$$\Rightarrow (-1-\lambda)^2 - 4 = 0$$

$$(-1-\lambda)^2 = 4$$

$$-1-\lambda = \pm 2 \Rightarrow \begin{cases} \lambda_1 = -3 \\ \lambda_2 = 1 \end{cases} \text{, eigenvalues.}$$

3) a) $f(x,y) = x^3 - 3x + y^2$

$$\frac{\partial f}{\partial x} = 3x^2 - 3$$

$$\frac{\partial f}{\partial y} = 2y$$

$3x^2 - 3 = 0 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$2y = 0 \Rightarrow y = 0$

$\Rightarrow (1,0)$ and $(-1,0)$ are critical points

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 6x \\ \frac{\partial^2 f}{\partial y^2} &= 2 \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = 0 \end{aligned}\right\} \quad \begin{aligned} H_{(1,0)} &= \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \\ H_{(-1,0)} &= \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

$\Rightarrow \quad \lambda_1 = 6 > 0, \ \lambda_2 = 2 > 0 \Rightarrow (1,0)$ local min

$\Rightarrow \quad \lambda_3 = -6 < 0, \ \lambda_2 = 2 > 0 \Rightarrow (-1,0)$ saddle point

d) $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$

$$\frac{\partial f}{\partial x} = 2x - y + 1 = 0$$

$$\frac{\partial f}{\partial y} = 2y - x = 0$$

$$\frac{\partial f}{\partial z} = 2z - 2 = 0 \Rightarrow z = 1$$

$$\begin{cases} 2x - y + 1 = 0 \\ 2y - x = 0 \quad | \cdot 2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x - y + 1 = 0 \\ 4y - 2x = 0 \end{cases}$$

$$3y+1=0 \Rightarrow y=-\frac{1}{3} \Rightarrow x=+\frac{2}{3}$$

$$\frac{\partial^2 f}{\partial x^2}=2 \quad \Bigg| \quad \frac{\partial^2 f}{\partial z \partial x}=\frac{\partial^2 f}{\partial x \partial z}=0 \quad \Bigg| \quad \frac{\partial^2 f}{\partial y \partial x}=\frac{\partial^2 f}{\partial x \partial y}=-1$$

$$\frac{\partial^2 f}{\partial y^2}=2 \quad ; \quad \frac{\partial^2 f}{\partial z \partial y}=\frac{\partial^2 f}{\partial y \partial z}=0$$

$$\frac{\partial^2 f}{\partial z^2}=2$$

$$H\left(-\frac{2}{3}, \frac{+1}{3}, 0\right)=\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}=0 \Leftrightarrow$$

$$\Leftrightarrow \quad (2-\lambda)(-1)^{3+3} \cdot \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow \quad (2-\lambda) \cdot \left((2-\lambda)^2 - 1\right) = 0$$

$$\Leftrightarrow \quad (2-\lambda)(2-\lambda-1)(2-\lambda+1) = 0$$

$$\Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases} \Rightarrow H \text{ is positive definite} \Rightarrow$$

$$\Rightarrow \left(-\frac{2}{3}, -\frac{1}{3}, 0\right) \rightarrow \text{loc min.}$$

4) $f(x+h) = f(x) + \nabla f(x) \cdot h + \frac{1}{2} \cdot h^T \cdot H(x) \cdot h + \dots$

$$f(x) = \frac{1}{2} \cdot x^T \cdot A \cdot x.$$

A symmetric matrix

x vector.

Method 1: $\dfrac{\partial f}{\partial x_1} = ?$ (A lot of work $\ddot{\frown}$)

Method 2: $f(x+h) = \frac{1}{2}(x+h)^T \cdot A \cdot (x+h)$

$$= \frac{1}{2} \cdot x^T A x + \frac{1}{2} x^T A h + \frac{1}{2} h^T A x + \frac{1}{2} h^T A h$$

$$x^T A h = \langle x, Ah \rangle = x \cdot (Ah)$$

$\langle a, b \rangle = a \cdot b = a^T b$

$(AB)^T = B^T \cdot A^T$

$$= \langle Ah, x \rangle$$

$$= (Ah)^T \cdot x = h^T \cdot A^T \cdot x$$

$A^T \to A$ symmetric

$$= h^T A x = \langle h, Ax \rangle =$$

$$= \langle Ax, h \rangle$$

$$f(x+h) = f(x) + \underbrace{\boxed{Ax \cdot h}}_{= \nabla f(x)} + \frac{1}{2} h^T \underbrace{\boxed{A}}_{= H(x)} h \Rightarrow$$

$\Rightarrow \nabla f(x) = Ax$ and $H(x) = A$.

5) $f(x) = \|Ax - b\|^2 = \langle Ax - b, Ax - b \rangle$

$f(x) \to \min$, $\nabla f(x) = 0$

$$f(x) = \langle Ax, Ax \rangle - 2\langle Ax, b \rangle + \langle b, b \rangle =$$

$$\langle Ax, Ax \rangle = (Ax)^T \cdot Ax =$$

$$= x^T A^T \cdot Ax = x^T (A^T A) x$$

$$= x^T (A^T A) x - 2 \underbrace{b^T A x}_{= \langle x, A^T b \rangle = x (A^T b)} + \|b\|^2$$

$$\nabla f = 2 A^T A x - 2 A^T b = 0 \quad \Big| \cdot \frac{1}{2}$$

$$= A^T A x - A^T b = 0, \quad \underline{A^T A x = A^T b.}$$