Seminar 8

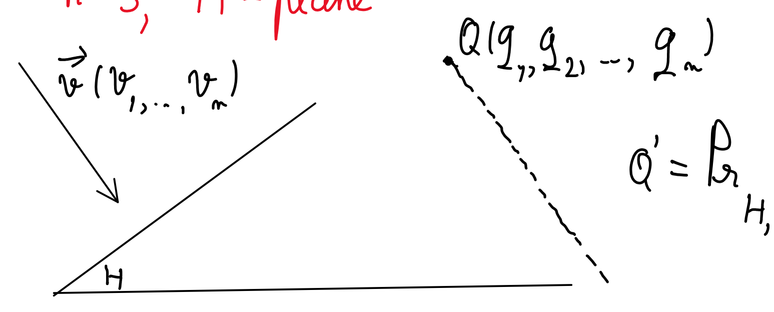
24.04.2023 10:24

Projections and reflections onto a hyperplane

Def: In a n-dimensional euclidean space $\mathbb{E}_n$ a hyperplane is a set of the form $H = \{(x_1, x_2, x_3, \dots, x_n) \mid a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a_{n+1} = 0\}$

For $n = 2$, H = line

$n = 3$, H = plane



$Q' = Pr_{H,B}(Q)$

$Q' = l \cap H$

$$l = Q + t\vec{v} : \begin{cases} x_1 = q_1 + t v_1 \\ x_2 = q_2 + t v_2 \\ \vdots \\ x_n = q_n + t v_n \end{cases}$$

$$Q' : \begin{cases} x_1 = q_1 + t v_1 \\ x_2 = q_2 + t v_2 \\ \vdots \\ a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a_{n+1} = 0 \end{cases}$$

$\vdots$

$$Q' : \begin{cases} \text{—— || ——} \\ a_1(q_1 + t v_1) + a_n(q_n + t v_n) + \dots + a_{n+1} = 0 \end{cases}$$

$$t(a_1 v_1 + \dots + a_n v_n) + a_1 q_1 + a_2 q_2 + \dots + a_n q_n + a_{n+1} = 0$$

$$t = -\frac{a_1 q_1 + \dots + a_n q_n + a_{n+1}}{a_1 v_1 + \dots + a_n v_n}$$

$$Q' = \begin{cases} x_1 = q_1 - \dfrac{a_1 q_1 + \dots + a_n q_n + a_{n+1}}{a_1 v_1 + \dots + a_n v_n} \cdot v_1 \\ \vdots \\ x_n = q_n - \dfrac{a_1 q_1 + \dots + a_n q_n + a_{n+1}}{a_1 v_1 + \dots + a_n v_n} \cdot v_n \end{cases}$$

$P(x_1, \dots, x_n)$, $P'(x_1', \dots, x_n')$, $P' = Pr_{H,\vec{v}}(P)$

$$[P']_R = [P]_R - \frac{a_1 x_1 + \dots + a_n x_n + a_{n+1}}{a_1 v_1 + \dots + a_n v_n} \cdot [\vec{v}]_R \qquad (*)$$

Tensor product:

$\vec{v}(v_1, v_2, \dots, v_n)$, $\vec{w}(w_1, w_2, \dots, w_n) \in V^n$

$\vec{v} \otimes \vec{w} = {}^t\vec{v} \cdot \vec{w}$

$$\vec{v} \otimes \vec{w} = \begin{pmatrix} v_1 w_1 + v_1 w_2 + \dots + v_1 w_n \\ v_2 w_1 + v_2 w_2 + \dots + v_2 w_n \\ \vdots \\ v_n w_1 + v_n w_2 + \dots + v_n w_n \end{pmatrix}$$

$$A, B \in M_{n,m}(\mathbb{R}), \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & -2 \\ 1 & 1 & 2 & 2 \\ 0 & -3 & 0 & -4 \\ 3 & 3 & 4 & 4 \end{pmatrix}$$

$$e(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a_{n+1}$$

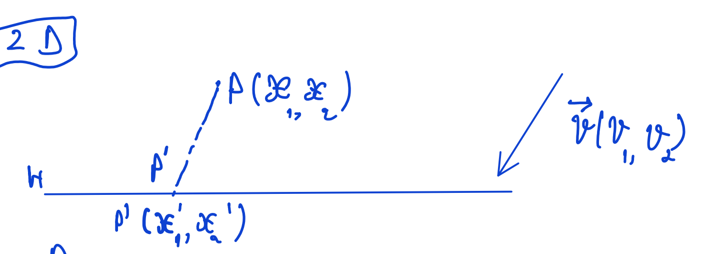
$H : e(x_1, \dots, x_n) = 0$

$(\text{lin } e)(x_1, x_2, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$

$[P']_R = [P]$

! $$[P']_R = \frac{1}{(\text{lin} e)(\vec{v})} \cdot \Big(\underbrace{(\vec{a} \cdot \vec{v}) \cdot I_n - \vec{v} \otimes \vec{a}}_{M_n(R)} \Big) \cdot [P]_R - \frac{a_{n+1}}{(\text{lin} e)(\vec{v})} \cdot [\vec{v}]_R$$

$\boxed{2D}$



$Pr_{H,\vec{v}}(P)$

From $(*)$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{a_1 x_1 + a_2 x_2 + a_3}{a_1 v_1 + a_2 v_2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{a_1 x_1 + a_2 x_2}{(\text{lin } e)(v)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \frac{a_3}{\text{lin } e(v)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{\text{lin } e\, v} \left(\vec{a} \cdot \vec{v} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - a_1 x_1 + a_2 x_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) - \frac{a_3}{\text{lin } e(v)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{\text{lin } e\, v} \left(\vec{a} \cdot \vec{v} - (a_1 \; a_2) - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (a_1 \; a_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) - \frac{}{\text{lin } e(v)} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{\text{lin } e\, v} \left((\vec{a} \cdot \vec{v}) \cdot I_n - \vec{v} \otimes \vec{a} \right) - [P]_R - \frac{a_{n+1}}{\text{lin } e(\vec{v})} [\vec{v}]_R$$

6) $\vec{v}(2, 1, 1) \in V^3$

$Q(3, 2, 2) \in \mathbb{E}^3$

a) Give the matrix form to the parallel proj: $z = 0$ among the line $Q + \langle v \rangle$

$e(x, y, z) = z$

$\text{lin } e(x, y, z) = z$

$\vec{a} = (0, 0, 1)$

$\vec{v} = (2, 1, 1)$

$$[Pr_{H\vec{v}}(P)'] = \frac{1}{1} \left(1 \cdot I_n - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot (0 \;\; 0 \;\; 1) \right) \cdot [P]$$

$$= \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right) \cdot [P] =$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} [P]$$

$$\text{Refl}(P) = 2 \cdot [Pr_{H\vec{v}}(P)] - [P] = 2 \cdot \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} [P] - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [P] = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix} [P]$$