10.11.2023

1) a) $(\sin x)' = \cos x$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$(\sin x)' = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}\right)' = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}\right)' =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\underset{(2n)!}{\cancel{(2n+1)!}}} \cdot \cancel{(2n+1)} \cdot x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = \cos x.$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}$$

$$(\cos x)' = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}\right)' = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(2n)!} \cdot x^{2n}\right)' =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\underset{(2n-1)!}{\cancel{(2n)!}}} \cdot \cancel{2n} \cdot x^{2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)!} \cdot x^{2n-1}$$

b) $\sin x = x - \underbrace{\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}_{<0} < x$

$\Rightarrow \sin x < x$

$\sin x = x - \frac{x^3}{3!} + \underbrace{\frac{x^5}{5!} - \dots}_{>0} > x - \frac{x^3}{3!}$

$\Rightarrow \sin x > x - \frac{x^3}{3!}$

$$\boxed{x - \frac{x^3}{3!} < \sin x < x}$$

$\cos x = 1 - \frac{x^2}{2} + \underbrace{\frac{x^4}{4!} - \dots}_{>0}$

$$\boxed{\cos x > 1 - \frac{x^2}{2}}$$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} \underbrace{- \frac{x^6}{6!} \dots}_{<0}$

$$\boxed{\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{4!}}$$

$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$

c) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \forall x \in \mathbb{R}$

$e^{ix} = 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3}{3!} \cdot x^3 + \frac{i^4}{4!} \cdot x^4 + \frac{i^5}{5!} x^5$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!}$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right)} + i\underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)}$$

$$e^{ix} = \cos x + i \sin x$$

2) $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n$$

if $x_0 = 0$: $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$$

we can calc. this.

$\alpha \in \mathbb{R}$ and $|x| < 1$, $x_0 = 0$

$$f(x) = (1+x)^{\alpha}$$

$$f'(x) = \alpha(1+x)^{\alpha-1}, \quad \underline{f'(0) = \alpha}$$

$$f''(x) = \alpha(\alpha-1)\cdot(1+x)^{\alpha-2}, \quad f''(0) = \alpha(\alpha-1)$$

$$\vdots$$

$$f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)\cdot(1+x)^{\alpha-n},$$

$$f^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)}{n!} \cdot x^n =$$

$$= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

b) $$\sqrt{1+x} = (1+x)^{\frac{1}{2}}; \quad T_3(x) = \sum_{n=0}^{3} \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{n!} \cdot x^n$$

$f_{(0)}^{(0)} = \sqrt{1+0} = \sqrt{1} = 1$

$$= 1 + \alpha \cdot x + \frac{\alpha(\alpha-1)\cdot x^2}{2} + \frac{\alpha(\alpha-1)(\alpha-2)\cdot x^3}{6} =$$

$$= T_3(x) \overset{\alpha=\frac{1}{2}}{=\!=} 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{\cancel{3}\,x^3}{\cancel{48}\,16} = 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16}$$

$$\sqrt{1+x} \approx 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{16} \quad \text{for } x \approx 0$$

$$g(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$$

$$T_3(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} \cdot x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} x^3$$

$$\overset{\alpha=-\frac{1}{2}}{=\!=} 1 - \frac{x}{2} + \frac{3}{8} x^2 - \frac{\cancel{15}\,5}{\cancel{48}\,16} x^3$$

3) a) $f(x) = a^x, \; a > 0$ $\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n \to$ MacLaurin series.

$$f'(x) = a^x \cdot \ln a, \quad f'(0) = \ln a$$

$$f''(x) = a^x \cdot (\ln a)^2, \quad f''(0) = (\ln a)^2$$

:

$$f^{(n)}(x) = a^x \cdot (\ln a)^n \quad , \quad f^{(n)}(0) = (\ln a)^n$$

$$a^x = \sum_{n=0}^{\infty} \underbrace{\frac{(\ln a)^n}{n!}}_{a_n} \cdot x^n$$

For which x, will this converge? (radius of conv.)

$$\boxed{\begin{gathered} \sum a_n (x-x_0)^n \text{ is abs conv for } x \in (x_0 - R, x_0 + R) \\ R = \frac{1}{L}, \quad L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \sqrt[n]{|a_n|} \end{gathered}}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\ln a}{n+1} \longrightarrow 0, \text{ when } n \to \infty \Rightarrow$$

$$\Rightarrow L = 0 \Rightarrow R = \infty \Rightarrow \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} \cdot x^n \text{ conv. } \forall x \in \mathbb{R}$$

b) $f(x) = \underbrace{(1+x)}_{\text{pol.}} \cdot \underbrace{\ln(1+x)}_{g(x)}$ Mc. m.: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$

$$f'(x) = \ln(1+x) + 1 \quad , \quad f'(0) = 1$$

$$f''(x) = \frac{1}{1+x} \quad , \quad f''(0) = 1$$

$$f'''(x) = -\frac{1}{(1+x)^2} \quad , \quad f'''(0) = -1$$

$$f^{(4)}(x) = \frac{2}{(1+x)^3}, \quad f^{(4)}(0) = 2$$

$\vdots$

$$f^{(n)}(x) = \frac{(-1)^n \cdot (n-2)!}{(1+x)^{n-1}}, \quad f^{(n)}(0) = (-1)^n (n-2)!, \; n \geq 2$$

$$f(x) = \underset{\downarrow \atop 1}{f(0)} + \underbrace{f'(0)}_{1} \cdot x + \sum_{n=2}^{\infty} \frac{(-1)^n \overset{1}{\cancel{(n-2)!}}}{\underset{(n-1)n}{\cancel{n!}}} x^n \iff$$

$$\iff f(x) = 1 + x + \sum_{n=2}^{\infty} \frac{(-1)^n \cdot x^n}{(n-1)n}$$

$$g(x) = \ln(1+x), \quad g(0) = 0$$

$$g'(x) = \frac{1}{1+x}, \quad g'(0) = 1$$

Varianta II

$$g''(x) = -\frac{1}{1+x}, \quad g''(0) = -1$$

$$g^{(n)}(x) = \frac{(-1)^{n-1} \cdot (n-1)!}{(1+x)^n}, \quad g^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (n-1)!}{n!} \cdot x^n$$

$$f(x) = (1+x) \cdot \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,(n-1)!}{n!} x^n$$

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\cancel{n}(n-1)}{(n+1)\cancel{n}} = 1 \Rightarrow R = 1$$

$$\left. \begin{array}{l} x_0 = 0 \\ R = 1 \end{array} \right\} \Rightarrow \text{series is abs. conv. } x \in (-1, 1)$$

c) $f(x) = \sin^2(x)$, $f(0) = 0$

$f'(x) = 2 \sin x \cdot \cos x$, $f'(0) = 0$

$f''(x) = (\sin 2x)' = 2 \cdot \cos 2x$, $f''(0) = 2$

$f^{(3)}(x) = -4 \cdot \sin 2x$, $f(0) = 0$

$f^{(4)}(x) = -4 \cdot (\cos 2x) \cdot 2 = -8 \cdot \cos 2x$, $f^{(4)}(0) = -8$

$\vdots$

$f^{(2n+1)}(x) = \ldots$(sth. with $\sin(2x)$) $f^{(2n+1)}(0) = 0$

$f^{(2n)}(x) = (-1)^{n+1} \cdot 2^{2n-1} \cdot \cos(2x)$, $f^{(2n)}(0) = (-1)^{n+1} \cdot 2^{2n-1}$

$$f(x) = \overset{0}{\overset{\shortparallel}{f(0)}} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^{2n-1}}{(2n)!} \cdot x^{2n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{4}{(2n+1)(2n+2)} = 0 \rightarrow R = \infty$$

$\Rightarrow$ series conv. $\forall\, x \in \mathbb{R}$

$\boxed{\text{The smarter way}}$

$$f(x) = \sin^2 x = \frac{1}{2}\left(1 - 1 + \frac{(2x)^2}{2} - \frac{(2x)^4}{4!} + \dots\right)$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin^2 x = \frac{1}{2}\left(\cancel{1} - \underbrace{\cancel{1} + \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \dots}_{-\cos 2x}\right)$$

d) $f(x) = \arctan x = ?$

$$f'(x) = \frac{1}{1+x^2}, \quad f'(0) = 1$$

Write $f'(x)$ in Taylor / power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x^n) = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

$$f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot (x^2)^n = \sum_{n=0} (-1)^n \cdot x^{2n} \quad \Big| \int$$

<u>Integrate:</u> $f(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = \arctan x$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2n+1}{2n+3} = 1 \Rightarrow R = 1$$