

DATA STRUCTURES AND ALGORITHMS

LECTURE 11

Lect. PhD. Oneț-Marian Zsuzsanna

Babeș - Bolyai University
Computer Science and Mathematics Faculty

2023 - 2024

- Collision resolution through coalesced chaining
- Collision resolution through open addressing

Today

- Cuckoo hashing
- Perfect hashing
- Linked hash table
- Trees
- Binary trees

Cuckoo hashing

Cuckoo hashing

- In cuckoo hashing we have two hash tables of the same size, each of them more than half empty and each hash table has its own hash function (so we have two different hash functions).
- For each element to be added we can compute two positions: one from the first hash table and one from the second. In case of cuckoo hashing, it is guaranteed that an element will be on one of these positions.
- Search is simple, because we only have to look at these two positions.
- Delete is simple, because we only have to look at these two positions and set to empty the one where we find the element.

- When we want to insert a new element we will compute its position in the first hash table. If the position is empty, we will place the element there.
- If the position in the first hash table is not empty, we will kick out the element that is currently there, and place the new element into the first hash table.
- The element that was kicked off, will be placed at its position in the second hash table. If that position is occupied, we will kick off the element from there and place it into its position in the first hash table.
- We repeat the above process until we will get an empty position for an element.
- If we get back to the same location with the same key we have a cycle and we cannot add this element \Rightarrow resize, rehash

Cuckoo hashing - example

- Assume that we have two hash tables, with $m = 11$ positions and the following hash functions:
 - $h1(k) = k \% 11$
 - $h2(k) = (k \text{ div } 11) \% 11$

Position	0	1	2	3	4	5	6	7	8	9	10
T											

Position	0	1	2	3	4	5	6	7	8	9	10
T											

Cuckoo hashing - example

- Insert key 20

Cuckoo hashing - example

- Insert key 20
 - $h_1(20) = 9$ - empty position, element added in the first table
- Insert key 50

Cuckoo hashing - example

- Insert key 20
 - $h_1(20) = 9$ - empty position, element added in the first table
- Insert key 50
 - $h_1(50) = 6$ - empty position, element added in the first table

Position	0	1	2	3	4	5	6	7	8	9	10
T							50			20	

Position	0	1	2	3	4	5	6	7	8	9	10
T											

Cuckoo hashing - example

- Insert key 53

Cuckoo hashing - example

- Insert key 53
 - $h_1(53) = 9$ - occupied
 - 53 goes in the first hash table, and it sends 20 in the second to position $h_2(20) = 1$

Position	0	1	2	3	4	5	6	7	8	9	10
T							50			53	

Position	0	1	2	3	4	5	6	7	8	9	10
T		20									

Cuckoo hashing - example

- Insert key 75

Cuckoo hashing - example

- Insert key 75
 - $h_1(75) = 9$ - occupied
 - 75 goes in the first hash table, and it sends 53 in the second to position $h_2(53) = 4$

Position	0	1	2	3	4	5	6	7	8	9	10
T							50			75	

Position	0	1	2	3	4	5	6	7	8	9	10
T		20			53						

Cuckoo hashing example

- Insert key 100

Cuckoo hashing example

- Insert key 100
 - $h_1(100) = 1$ - empty position
- Insert key 67

Cuckoo hashing example

- Insert key 100
 - $h_1(100) = 1$ - empty position
- Insert key 67
 - $h_1(67) = 1$ - occupied
 - 67 goes in the first hash table, and it sends 100 in the second to position $h_2(100) = 9$

Position	0	1	2	3	4	5	6	7	8	9	10
T		67					50			75	

Position	0	1	2	3	4	5	6	7	8	9	10
T		20			53					100	

Cuckoo hashing example

- Insert key 105

Cuckoo hashing example

- Insert key 105
 - $h_1(105) = 6$ - occupied
 - 105 goes in the first hash table, and it sends 50 in the second to position $h_2(50) = 4$
 - 50 goes in the second hash table, and it sends 53 to the first one, to position $h_1(53) = 9$
 - 53 goes in the first hash table, and it sends 75 to the second one, to position $h_2(75) = 6$

Position	0	1	2	3	4	5	6	7	8	9	10
T		67					105			53	

Position	0	1	2	3	4	5	6	7	8	9	10
T		20			50		75			100	

Cuckoo hashing example

- Insert key 3

Cuckoo hashing example

- Insert key 3
 - $h_1(3) = 3$ - empty position
- Insert key 36

Cuckoo hashing example

- Insert key 3
 - $h_1(3) = 3$ - empty position
- Insert key 36
 - $h_1(36) = 3$ - occupied
 - 36 goes in the first hash table, and it sends 3 in the second to position $h_2(3) = 0$

Position	0	1	2	3	4	5	6	7	8	9	10
T		67		36			105			53	

Position	0	1	2	3	4	5	6	7	8	9	10
T	3	20			50		75			100	

Cuckoo hashing example

- Insert key 39

Cuckoo hashing example

- Insert key 39
 - $h1(39) = 6$ - occupied
 - 39 goes in the first hash table and it sends 105 in the second to position $h2(105) = 9$
 - 105 goes to the second hash table and it sends 100 in the first to position $h1(100) = 1$
 - 100 goes in the first hash table and it sends 67 in the second to position $h2(67) = 6$
 - 67 goes in the second hash table and it sends 75 in the first to position $h1(75) = 9$
 - 75 goes in the first hash table and it sends 53 in the second to position $h2(53) = 4$
 - 53 goes in the second hash table and it sends 50 in the first to position $h1(50) = 6$
 - 50 goes in the first hash table and it sends 39 in the second to position $h2(39) = 3$

Cuckoo hashing example

Position	0	1	2	3	4	5	6	7	8	9	10
T		100		36			50			75	

Position	0	1	2	3	4	5	6	7	8	9	10
T	3	20		39	53		67			105	

- It can happen that we cannot insert a key because we get in a cycle. In these situation we have to increase the size of the tables and rehash the elements.
- While in some situation insert moves a lot of elements, it can be shown that if the load factor of the tables is below 0.5, the probability of a cycles is low and it is very unlikely that more than $O(\log_2 n)$ elements will be moved.

- If we use two tables and each position from a table holds one element at most, the tables have to have load factor below 0.5 to work well.
- If we use three tables, the tables can have load factor of 0.91 and for 4 tables we have 0.97

Perfect hashing

Perfect hashing

- Assume that we know all the keys in advance and we use *separate chaining* for collision resolution \Rightarrow the more lists we make, the shorter the lists will be (reduced number of collisions) \Rightarrow if we could make a large number of list, each would have one element only (no collision).
- How large should we make the hash table to make sure that there are no collisions?
- If $M = N^2$, it can be shown that the table is collision free with probability at least $1/2$.
- Start building the hash table. If you detect a collision, just choose a new hash function and start over (expected number of trials is at most 2).

Perfect hashing

- Having a table of size N^2 is impractical.
- Solution instead:
 - Use a hash table of size N (*primary* hash table).
 - Instead of using linked list for collision resolution (as in separate chaining) each element of the hash table is another hash table (*secondary hash table*)
 - Make the secondary hash table of size n_j^2 , where n_j is the number of elements from this hash table.
 - Each secondary hash table will be constructed with a different hash function, and will be reconstructed until it is collision free.
- This is called **perfect hashing**.
- It can be shown that the total space needed for the secondary hash tables is at most $2N$.

Perfect hashing

- Perfect hashing requires multiple hash functions, this is why we use *universal hashing*.

Perfect hashing

- Perfect hashing requires multiple hash functions, this is why we use *universal hashing*.
- Let p be a prime number, larger than the largest possible key.
- The universal hash function family \mathcal{H} can be defined as:

$$\mathcal{H} = \{H_{a,b}(x) = ((a * x + b) \% p) \% m\}$$

$$\text{where } 1 \leq a \leq p - 1, 0 \leq b \leq p - 1$$

- a and b are chosen randomly when the hash function is initialized.

Perfect hashing - example

- Insert into a hash table with perfect hashing the letters from "PERFECT HASHING EXAMPLE". Since we want no collisions at all, we are going to consider only the unique letters: "PERFCTHASINGXML"
- Since we are inserting $N = 15$ elements, we will take $m = 15$.
- For each letter, the *hashCode* is the index of the letter in the alphabet.

Letter	P	E	R	F	C	T	H	A	S	I	N	G	X	M	L
HashCode	16	5	18	6	3	20	8	1	19	9	14	7	24	13	12

Perfect hashing - example

- p has to be a prime number larger than the maximum key \Rightarrow 29
- The hash function will be:

$$H_{a,b}(x) = ((a * x + b) \% p) \% m$$

- where a will be 3 and b will be 2 (chosen randomly).

Letter	P	E	R	F	C	T	H	A	S	I	N	G	X	M	L
HashCode	16	5	18	6	3	20	8	1	19	9	14	7	24	13	12

Perfect hashing - example

- p has to be a prime number larger than the maximum key \Rightarrow 29
- The hash function will be:

$$H_{a,b}(x) = ((a * x + b) \% p) \% m$$

- where a will be 3 and b will be 2 (chosen randomly).

Letter	P	E	R	F	C	T	H	A	S	I	N	G	X	M	L
HashCode	16	5	18	6	3	20	8	1	19	9	14	7	24	13	12
H(HashCode)	6	2	12	5	11	4	11	5	1	0	0	8	1	12	9

Perfect hashing - example

- Collisions:

- position 0 - I, N
- position 1 - S, X
- position 2 - E
- position 4 - T
- position 5 - F, A
- position 6 - P
- position 8 - G
- position 9 - L
- position 11 - C, H
- position 12 - R, M

Perfect hashing - example

- For the positions where we have no collision (only one element hashed to it) we will have a secondary hash table with only one element, and hash function $h(x) = 0$
- For the positions where we have two elements, we will have a secondary hash table with 4 positions and different hash functions, taken from the same universe, with different random values for a and b .
- For example for position 0, we can define $a = 4$ and $b = 11$ and we will have:
$$h(I) = h(9) = 2$$
$$h(N) = h(14) = 1$$

Perfect hashing - example

- Assume that for the secondary hash table from position 1 we will choose $a = 5$ and $b = 2$.
- Positions for the elements will be:
$$h(S) = h(19) = ((5 * 19 + 2) \% 29) \% 4 = 2$$
$$h(X) = h(24) = ((5 * 24 + 2) \% 29) \% 4 = 2$$
- In perfect hashing we should not have collisions, so we will simply chose another hash function: another random values for a and b . Choosing for example $a = 2$ and $b = 13$, we will have $h(S) = 2$ and $h(X) = 3$.

Perfect hashing

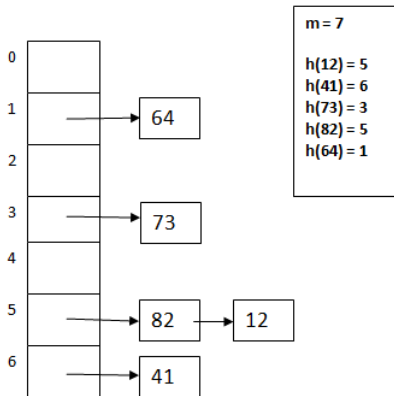
- When perfect hashing is used and we search for an element we will have to check at most 2 positions (position in the primary and in the secondary table).
- This means that worst case performance of the table is $\Theta(1)$.
- But in order to use perfect hashing, we need to have static keys: once the table is built, no new elements can be added.

Linked Hash Table

Linked Hash Table

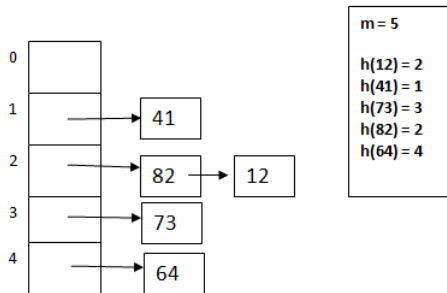
- Assume we build a hash table using separate chaining as a collision resolution method.
- We have discussed how an iterator can be defined for such a hash table.
- When iterating through the elements of a hash table, the order in which the elements are visited is *undefined*
- For example:
 - Assume an initially empty hash table (we do not know its implementation)
 - Insert one-by-one the following elements: 12, 41, 73, 82, 64
 - Use an iterator to display the content of the hash table
 - In what order will the elements be displayed?

Linked Hash Table



- Iteration order: 64, 73, 82, 12, 41

Linked Hash Table



- Iteration order: 41, 82, 12, 73, 64

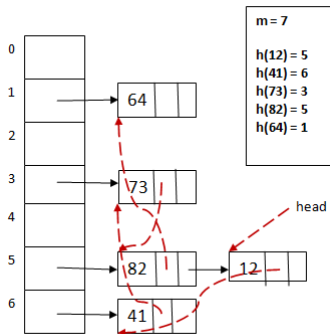
Linked Hash Table

- A *linked hash table* is a data structure which has a *predictable* iteration order. This order is the order in which elements were inserted.
- So if we insert the elements 12, 41, 73, 82, 64 (in this order) in a linked hash table and iterate over the hash table, the iteration order is guaranteed to be: 12, 41, 73, 82, 64.
- How could we implement a linked hash table which provides this iteration order?

Linked Hash Table

- A linked hash table is a combination of a hash table and a linked list. Besides being stored in the hash table, each element is part of a linked list, in which the elements are added in the order in which they are inserted in the table.
- Since it is still a hash table, we want to have, on average, $\Theta(1)$ for insert, remove and search, these are done in the same way as before, the *extra* linked list is used only for iteration.

Linked Hash Table



- Red arrows show how the elements are linked in insertion order, starting from a *head* - the first element that was inserted, 12.

Linked Hash Table

- Do we need a doubly linked list for the order of elements or is a singly linked list sufficient? (think about the operations that we usually have for a hash table).

Linked Hash Table

- Do we need a doubly linked list for the order of elements or is a singly linked list sufficient? (think about the operations that we usually have for a hash table).
- The only operation that cannot be efficiently implemented if we have a singly linked list is the *remove* operation. When we remove an element from a singly linked list we need the element before it, but finding this in our linked hash table takes $O(n)$ time.

Linked Hash Table - Implementation

- What structures do we need to implement a Linked Hash Table?

Node:

info: TKey

nextH: \uparrow Node // *pointer to next node from the collision*

nextL: \uparrow Node // *pointer to next node from the insertion-order list*

prevL: \uparrow Node // *pointer to prev node from the insertion-order list*

LinkedHT:

m: Integer

T: (\uparrow Node)[]

h: TFunction

head: \uparrow Node

tail: \uparrow Node

Linked Hash Table - Insert

- How can we implement the *insert* operation?

```
subalgorithm insert(lht, k) is:  
//pre: lht is a LinkedHT, k is a key  
//post: k is added into lht  
  allocate(newNode)  
  [newNode].info  $\leftarrow$  k  
  @set all pointers of newNode to NIL  
  pos  $\leftarrow$  lht.h(k)  
  //first insert newNode into the hash table  
  if lht.T[pos] = NIL then  
    lht.T[pos]  $\leftarrow$  newNode  
  else  
    [newNode].nextH  $\leftarrow$  lht.T[pos]  
    lht.T[pos]  $\leftarrow$  newNode  
  end-if  
//continued on the next slide...
```

Linked Hash Table - Insert

```
//now insert newNode to the end of the insertion-order list  
if lht.head = NIL then  
    lht.head  $\leftarrow$  newNode  
    lht.tail  $\leftarrow$  newNode  
else  
    [newNode].prevL  $\leftarrow$  lht.tail  
    [lht.tail].nextL  $\leftarrow$  newNode  
    lht.tail  $\leftarrow$  newNode  
end-if  
end-subalgorithm
```

Linked Hash Table - Remove

- How can we implement the *remove* operation?

subalgorithm remove(lht, k) **is:**

//pre: lht is a LinkedHT, k is a key

//post: k was removed from lht

pos \leftarrow lht.h(k)

current \leftarrow lht.T[pos]

nodeToBeRemoved \leftarrow NIL

//first search for k in the collision list and remove it if found

if current \neq NIL **and** [current].info = k **then**

nodeToBeRemoved \leftarrow current

lht.T[pos] \leftarrow [current].nextH

else

prevNode \leftarrow NIL

while current \neq NIL **and** [current].info \neq k **execute**

prevNode \leftarrow current

current \leftarrow [current].nextH

end-while

//continued on the next slide...

```
if current  $\neq$  NIL then  
    nodeToBeRemoved  $\leftarrow$  current  
    [prevNode].nextH  $\leftarrow$  [current].nextH  
else  
    @k is not in lht  
end-if  
end-if
```

```
//if k was in lht then nodeToBeRemoved is the address of the node containing  
//it and the node was already removed from the collision list - we need to  
//remove it from the insertion-order list as well
```

```
if nodeToBeRemoved  $\neq$  NIL then  
    if nodeToBeRemoved = lht.head then  
        if nodeToBeRemoved = lht.tail then  
            lht.head  $\leftarrow$  NIL  
            lht.tail  $\leftarrow$  NIL  
        else  
            lht.head  $\leftarrow$  [lht.head].nextL  
            [lht.head].prev  $\leftarrow$  NIL  
        end-if
```

```
//continued on the next slide...
```

```
else if nodeToBeRemoved = lht.tail then  
    lht.tail  $\leftarrow$  [lht.tail].prev  
    [lht.tail].next  $\leftarrow$  NIL  
else  
    [[nodeToBeRemoved].next].prev  $\leftarrow$  [nodeToBeRemoved].prev  
    [[nodeToBeRemoved].prev].next  $\leftarrow$  [nodeToBeRemoved].next  
end-if  
    deallocate(nodeToBeRemoved)  
end-if  
end-subalgorithm
```

- Trees are one of the most commonly used data structures because they offer an efficient way of storing data and working with the data.
- In graph theory a *tree* is a connected, acyclic graph (usually undirected).
- When talking about trees as a data structure, we actually mean *rooted trees*, trees in which one node is designated to be the *root* of the tree.

Tree - Definition

- A tree is a finite set \mathcal{T} of 0 or more elements, called *nodes*, with the following properties:
 - If \mathcal{T} is empty, then the tree is empty
 - If \mathcal{T} is not empty then:
 - There is a special node, R , called the *root* of the tree
 - The rest of the nodes are divided into k ($k \geq 0$) disjunct *trees*, T_1, T_2, \dots, T_k , the root node R being linked by an edge to the root of each of these trees. The trees T_1, T_2, \dots, T_k are called the *subtrees* (*children*) of R , and R is called the *parent* of the subtrees.

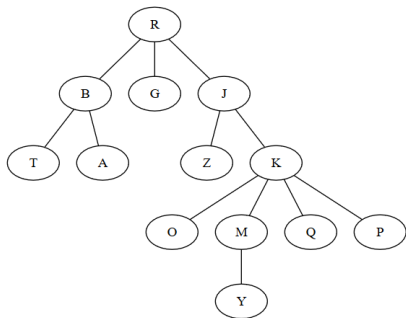
Tree - Terminology I

- An *ordered tree* is a tree in which the order of the children is well defined and relevant (instead of having a set of children, each node has a list of children).
- The *degree* of a node is defined as the number of children of the node.
- The nodes with the degree 0 (nodes without children) are called *leaf nodes*.
- The nodes that are not leaf nodes are called *internal nodes*.

Tree - Terminology II

- The *depth* or *level* of a node is the length of the path (measured as the number of edges traversed) from the root to the node. This path is unique. The root of the tree is at level 0 (and has depth 0).
- The *height* of a node is the length of the longest path from the node to a leaf node.
- The *height of the tree* is defined as the height of the root node, i.e., the length of the longest path from the root to a leaf.

Tree - Terminology Example



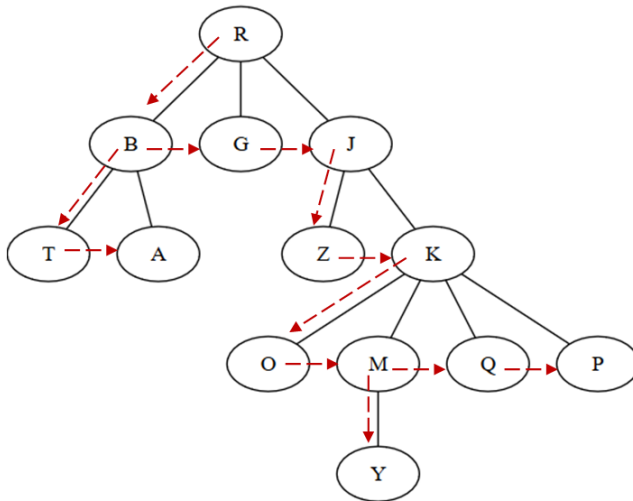
- Root of the tree: R
- Children of R : B, G, J
- Parent of M : K
- Leaf nodes: T, A, G, Z, O, Y, Q, P
- Internal nodes: R, B, J, K, M
- Depth of node K : 2 (path $R-J-K$)
- Height of node K : 2 (path $K-M-Y$)
- Height of the tree (height of node R): 4
- Nodes on level 2: T, A, Z, K

- How can we represent a tree in which every node has at most k children?
- One option is to have a structure for a *node* that contains the following:
 - information from the node
 - address of the parent node (not mandatory)
 - k fields, one for each child
- Obs: this is doable if k is not too large

- Another option is to have a structure for a *node* that contains the following:
 - information from the node
 - address of the parent node (not mandatory)
 - an array of dimension k , in which each element is the address of a child
 - number of children (number of occupied positions from the above array)
- Disadvantage of these approaches is that we occupy space for k children even if most nodes have less children.

- A third option is the so-called *left-child right-sibling* representation in which we have a structure for a node which contains the following:
 - information from the node
 - address of the parent node (not mandatory)
 - address of the leftmost child of the node
 - address of the right sibling of the node (next node on the same level from the same parent).

Left-child right sibling representation example



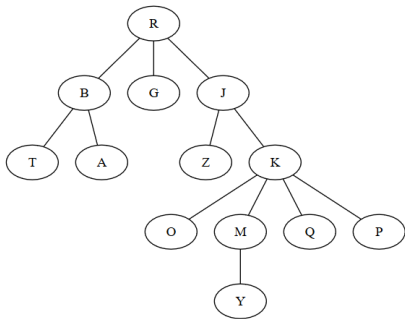
Tree traversals

- A node of a tree is said to be *visited* when the program control arrives at the node, usually with the purpose of performing some operation on the node (printing it, checking the value from the node, etc.).
- *Traversing* a tree means visiting all of its nodes.
- For a k-ary tree there are 2 possible traversals:
 - Depth-first traversal
 - Level order (breadth first) traversal

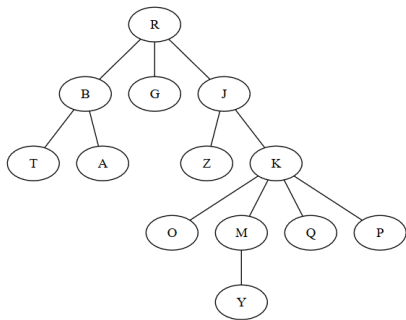
Depth first traversal

- Traversal starts from root
- From root we visit one of the children, then one child of that child, and so on. We go down (in depth) as much as possible, and continue with other children of a node only after all descendants of the "first" child were visited.
- For depth first traversal we use a stack to remember the nodes that have to be visited.

Depth first traversal example



Depth first traversal example

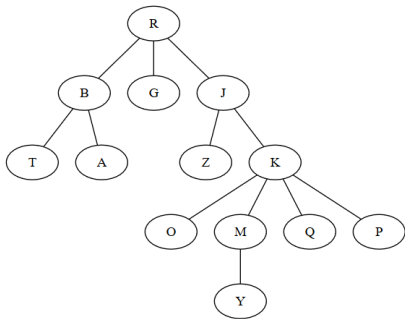


- Stack s with the root: R
- Visit R (pop from stack) and push its children: $s = [B\ G\ J]$
- Visit B and push its children: $s = [T\ A\ G\ J]$
- Visit T and push nothing: $s = [A\ G\ J]$
- Visit A and push nothing: $s = [G\ J]$
- Visit G and push nothing: $s = [J]$
- Visit J and push its children: $s = [Z\ K]$
- etc...

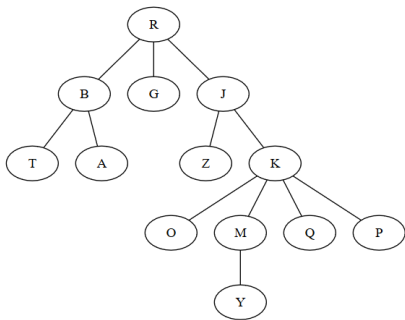
Level order traversal

- Traversal starts from root
- We visit all children of the root (one by one) and once all of them were visited we go to their children and so on. We go down one level, only when all nodes from a level were visited.
- For level order traversal we use a queue to remember the nodes that have to be visited.

Level order traversal example



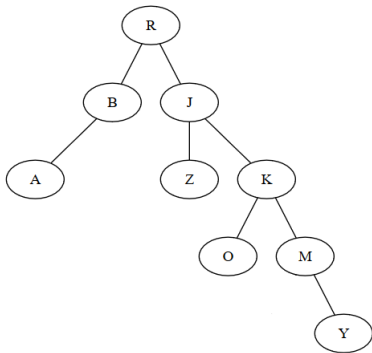
Level order traversal example



- Queue q with the root: R
- Visit R (pop from queue) and push its children: $q = [B\ G\ J]$
- Visit B and push its children: $q = [G\ J\ T\ A]$
- Visit G and push nothing: $q = [J\ T\ A]$
- Visit J and push its children: $q = [T\ A\ Z\ K]$
- Visit T and push nothing: $q = [A\ Z\ K]$
- Visit A and push nothing: $q = [Z\ K]$
- etc...

- An ordered tree in which each node has at most two children is called *binary tree*.
- In a binary tree we call the children of a node the *left child* and *right child*.
- Even if a node has only one child, we still have to know whether that is the left or the right one.

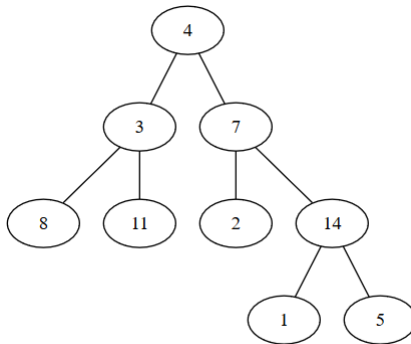
Binary tree - example



- *A* is the left child of *B*
- *Y* is the right child of *M*

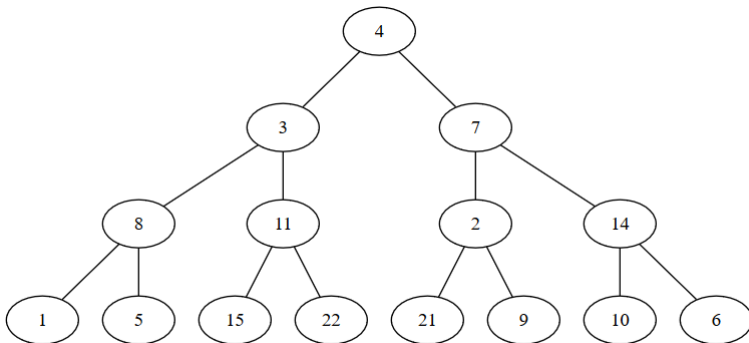
Binary tree - Terminology I

- A binary tree is called *full* if every internal node has exactly two children.



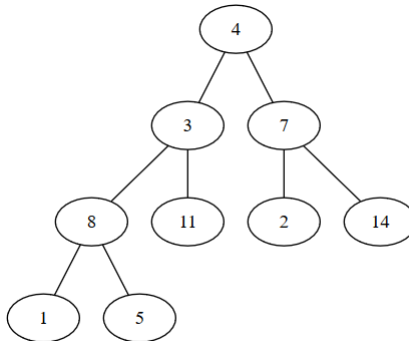
Binary tree - Terminology II

- A binary tree is called *complete* if all leaves are on the same level and all internal nodes have exactly 2 children.



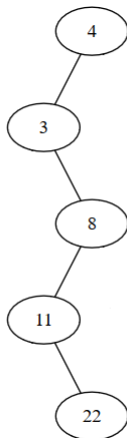
Binary tree - Terminology III

- A binary tree is called *almost complete* if it is a *complete* binary tree except for the last level, where nodes are completed from left to right (binary heap - structure).



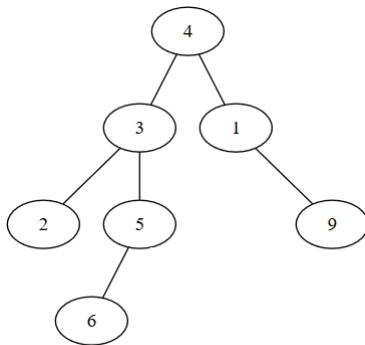
Binary tree - Terminology IV

- A binary tree is called *degenerate* if every internal node has exactly one child (it is actually a chain of nodes).



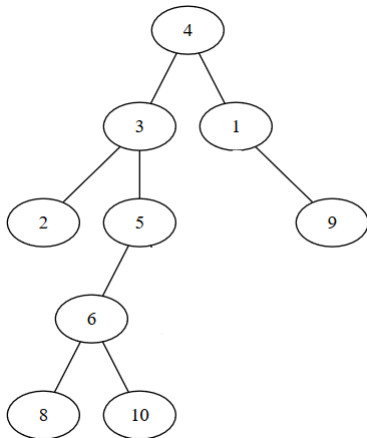
Binary tree - Terminology V

- A binary tree is called *balanced* if the difference between the height of the left and right subtrees is at most 1 (for every node from the tree).



Binary tree - Terminology VI

- Obviously, there are many binary trees that are none of the above categories, for example:



Binary tree - properties

- A binary tree with n nodes has exactly $n - 1$ edges (this is true for every tree, not just binary trees)
- The number of nodes in a complete binary tree of height N is $2^{N+1} - 1$ (it is $1 + 2 + 4 + 8 + \dots + 2^N$)
- The maximum number of nodes in a binary tree of height N is $2^{N+1} - 1$ - if the tree is complete.
- The minimum number of nodes in a binary tree of height N is $N + 1$ - if the tree is degenerate.
- A binary tree with N nodes has a height between $\log_2 N$ and $N - 1$.

- Domain of ADT Binary Tree:

$$\mathcal{BT} = \{bt \mid bt \text{ binary tree with nodes containing information of type TElem}\}$$

- `init(bt)`
 - **descr:** creates a new, empty binary tree
 - **pre:** true
 - **post:** $bt \in \mathcal{BT}$, *bt* is an empty binary tree

- **initLeaf(bt, e)**
 - **descr:** creates a new binary tree, having only the root with a given value
 - **pre:** $e \in TElem$
 - **post:** $bt \in \mathcal{BT}$, bt is a binary tree with only one node (its root) which contains the value e

- `initTree(bt, left, e, right)`
 - **descr:** creates a new binary tree, having a given information in the root and two given binary trees as children
 - **pre:** $left, right \in \mathcal{BT}$, $e \in TElem$
 - **post:** $bt \in \mathcal{BT}$, bt is a binary tree with left child equal to $left$, right child equal to $right$ and the information from the root is e

- `insertLeftSubtree(bt, left)`
 - **descr:** sets the left subtree of a binary tree to a given value (if the tree had a left subtree, it will be changed)
 - **pre:** $bt, left \in \mathcal{BT}$
 - **post:** $bt' \in \mathcal{BT}$, the left subtree of bt' is equal to $left$

- `insertRightSubtree(bt, right)`
 - **descr:** sets the right subtree of a binary tree to a given value (if the tree had a right subtree, it will be changed)
 - **pre:** $bt, right \in \mathcal{BT}$
 - **post:** $bt' \in \mathcal{BT}$, the right subtree of bt' is equal to $right$

- **root(bt)**
 - **descr:** returns the information from the root of a binary tree
 - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
 - **post:** $root \leftarrow e$, $e \in TElem$, e is the information from the root of bt
 - **throws:** an exception if bt is empty

- **left(bt)**
 - **descr:** returns the left subtree of a binary tree
 - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
 - **post:** $left \leftarrow l$, $l \in \mathcal{BT}$, l is the left subtree of bt
 - **throws:** an exception if bt is empty

- **right(*bt*)**
 - **descr:** returns the right subtree of a binary tree
 - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
 - **post:** $right \leftarrow r$, $r \in \mathcal{BT}$, r is the right subtree of bt
 - **throws:** an exception if bt is empty

- $\text{isEmpty}(bt)$
 - **descr:** checks if a binary tree is empty
 - **pre:** $bt \in \mathcal{BT}$
 - **post:**

$$empty \leftarrow \begin{cases} True, & \text{if } bt = \Phi \\ False, & \text{otherwise} \end{cases}$$

- **iterator** (bt , $traversal$, i)
 - **descr:** returns an iterator for a binary tree
 - **pre:** $bt \in \mathcal{BT}$, $traversal$ represents the order in which the tree has to be traversed
 - **post:** $i \in \mathcal{I}$, i is an iterator over bt that iterates in the order given by $traversal$

- **destroy(bt)**
 - **descr:** destorys a binary tree
 - **pre:** $bt \in \mathcal{BT}$
 - **post:** bt was destroyed

- Other possible operations:
 - change the information from the root of a binary tree
 - remove a subtree (left or right) of a binary tree
 - search for an element in a binary tree
 - return the number of elements from a binary tree

- Today we have talked about:
 - Cuckoo hashing
 - Perfect hashing
 - Linked hash table
 - Trees
 - Binary trees (will be continued)