3.11.2023

$A \subseteq \mathbb{R}$, $x_0 \in \overline{\mathbb{R}}$ is an accumulation point if
$\forall V \in \mathcal{V}(x_0) : V \cap (A \setminus \{x_0\}) \neq \emptyset$

1) a) $A = [0, 1) \cup \{2\}$

$A' = [0, 1]$

b) $B = \mathbb{Z}$

$B' = \{-\infty, \infty\}$

c) $C = \{0.1, 0.11, \dots\}$

$C' = \{\frac{1}{9}\}$ $0.\dot{1} = 0,(1) = \frac{1}{9}$

2) $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ -1, & \text{if } x \notin \mathbb{Q} \end{cases}$

Let $x \in \mathbb{R}$ $x_n \in \mathbb{Q}$ s.t. $x_n \to x_0$, $f(x_n) = 1$
Let $y_n \in \mathbb{R} \setminus \mathbb{Q}$, $y_n \to x_0$, $f(y_n) = -1$ $\Rightarrow$

$\Rightarrow \nexists \lim\limits_{x \to x_0} f(x) \Rightarrow f$ is discontinuous

$$|f(x)| = 1, \quad \forall x \in \mathbb{R} \text{ continuous}$$

3) $f : [a,b] \to [a,b]$, then it has at least one fixed point x^* s.t. $x^* = f(x^*)$

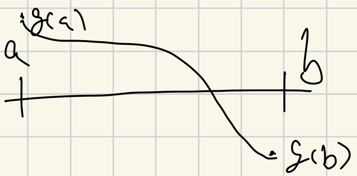
$\exists!\ x^* = f(x^*) \Leftrightarrow f(x^*) - x^* = 0$ Darboux

$\Updownarrow$

I.V.P

$f(a) \geq a \Rightarrow f(a) - a \geq 0$, $g(a) \geq 0$

$f(b) \leq b \Rightarrow f(b) - b \leq 0$, $g(b) \leq 0$ $\Big\}$ $\xrightarrow{\text{I.V.P}}$



$\Rightarrow \exists\, x^* \in [a,b]$ s.t. $g(x^*) = 0$

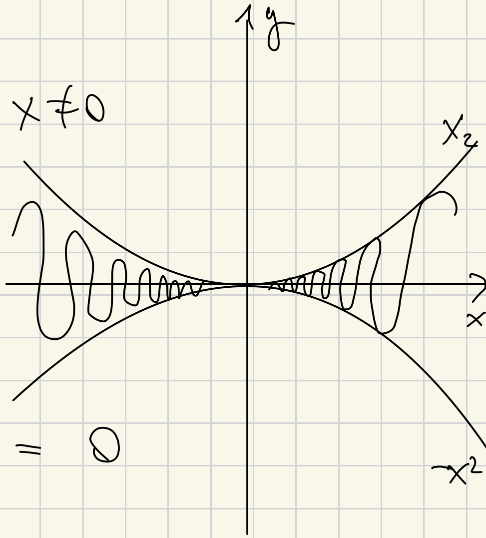
$\Rightarrow \exists\, x^* \in [a,b]$ s.t. $f(x^*) = x^*$

Let $g : [a,b] \to \mathbb{R}$, $g(x) = f(x) - x$, cont. $\Rightarrow$

$\Rightarrow$ has I.V.P (intermediate value property).

4) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



f cont. $\forall x \in \mathbb{R}^*$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \cdot \underbrace{\sin\frac{1}{x}}_{\in[-1,1]} = 0$$

$$-1 \le \sin\frac{1}{x} \le 1 \quad \Big|\cdot x^2$$

$$-x^2 \le f(x) \le x^2 \quad \Big| \lim_{x\to 0}$$

$$0 \le \lim_{x\to 0} f(x) \le 0 \overset{S.T.}{\Longrightarrow} \lim_{x\to 0} f(x) = 0$$

f differentiable $\forall x \in \mathbb{R}^*$

$$f'(x) = \begin{cases} 2x\cdot\sin\frac{1}{x} - \cancel{x^2}\cdot\cos\frac{1}{x}\cdot\frac{1}{\cancel{x^2}}, & \text{if } x \neq 0 \\ \cancel{?}\ \ 0, & \text{if } x = 0 \end{cases}$$

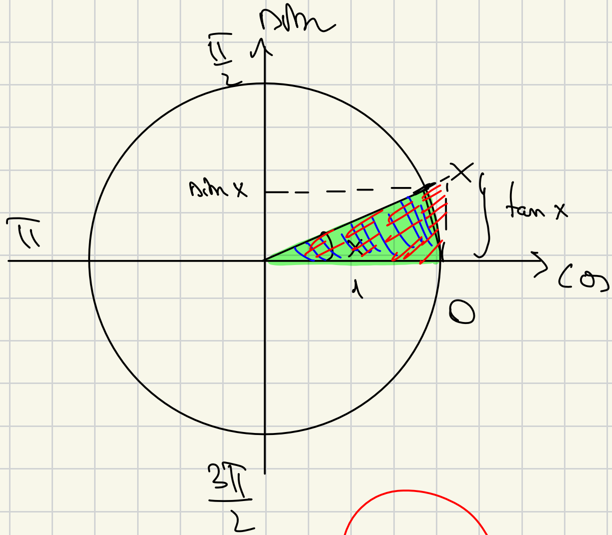
$$\boxed{f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}} = \lim_{x\to 0}\frac{\cancel{x^2}\cdot\sin\frac{1}{x}}{\cancel{x}} = \lim_{x\to 0} x\cdot\underbrace{\sin\frac{1}{x}}_{\in[-1,1]} =$$

$$= 0$$

$$\lim_{x\to 0} f'(x) = \underbrace{2x \cdot \sin\frac{1}{x}}_{0} - \underbrace{\cos\frac{1}{x}}_{\nexists}, \quad \nexists \lim_{x\to 0} f'(x) \Rightarrow$$

$\Rightarrow f'$ not continuous $\Rightarrow f'$ not differential

5)



$$\frac{\sin x}{2} < \frac{x}{2} = \text{Area sector} < \frac{\tan x}{2} \iff$$

x .... Area

2π ...... π

π ...... $\frac{\pi}{2}$

$\frac{\pi}{2}$ ...... $\frac{\pi}{4}$

$$\iff \sin x < x < \tan x = \frac{\sin x}{\cos x}, \quad x > 0$$

$$\cos x < \frac{\sin x}{x} < 1$$

$$x \to 0: \quad \cos x \to 1, \quad \frac{\sin x}{x} \to 1, \quad 1 \to 1$$

• $(\sin x)' \stackrel{?}{=} \cos x$

$$\boxed{\sin a - \sin b = 2 \sin \frac{a-b}{2} \cdot \cos \frac{a+b}{2}}$$

$$\lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0} \stackrel{?}{=} \cos x_0$$

$$\lim_{x \to x_0} \frac{2 \cdot \sin \frac{x-x_0}{2} \cdot \cos \frac{x-x_0}{2}}{x-x_0} = \cos x_0$$

$$\frac{x-x_0}{2} \qquad \to 1$$

$$\boxed{\begin{gathered} \lfloor\ \rfloor = \text{floor} \\ \lceil\ \rceil = \text{ceiling} \\ \text{ex } \lfloor 1.5 \rfloor = 1,\ \lceil 1.5 \rceil = 2 \end{gathered}}$$

6) a) $$\lim_{x \to \infty} \frac{\lfloor x \rfloor}{x} =$$

$$= \lim_{x \to \infty} \frac{x - \{x\}}{x} = 1 - \underbrace{\lim_{x \to \infty} \frac{\{x\}}{x}}_{\to 0} = 1 - 0 = 1$$

b) $$\lim_{x \to \infty} x\left(\ln(x+2) - \ln(x+1)\right) =$$

$$= \lim_{x \to \infty} x \cdot \ln \frac{x+2}{x+1} = \lim_{x \to \infty} \ln \left(\frac{x+2}{x+1}\right)^x =$$

$$= \ln \lim_{x\to\infty} \underbrace{\left(1+\frac{1}{x+1}\right)}_{\to e}{}^{(x+1)\cdot\frac{x}{x+1}} = \ln e^{\lim\limits_{x\to\infty} \frac{x}{x+1}} = \ln e = 1$$

c) $$\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = \lim_{x\to\infty} \left(1+\cos x-1\right)^{\frac{1}{\cos x-1}\cdot\frac{\cos x-1}{x^2}} =$$

$$= e^{\lim\limits_{x\to 0} \frac{\cos x-1}{x^2}} \overset{l'H}{=} e^{\lim\limits_{x\to 0} \frac{-\sin x}{x}} = e^{-1} = \frac{1}{e}$$

d) $$\lim_{\substack{x\to 0\\ x>0}} x^x = \lim_{x\searrow 0} e^{x\ln x} = e^0 = 1$$

$$\lim_{x\searrow 0} x\cdot\ln x = \lim_{x\searrow 0} \frac{\ln x}{\frac{1}{x}} \underset{l'H}{\overset{\frac{\infty}{\infty}}{=}}$$

$$= \lim_{x\searrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\searrow 0} -\frac{x^2}{x} = 0$$

e) $$\lim_{x\searrow 0} (\sin x)^x = \lim_{x\searrow 0} e^{x\ln(\sin x)} = e^0 = 1$$

$$\lim_{x\searrow 0} x\cdot\ln(\sin x) = \lim_{x\searrow 0} \frac{\ln(\sin x)}{\frac{1}{x}} \underset{l'H}{\overset{\frac{\infty}{\infty}}{=}}$$

$$= \lim_{x\searrow 0} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\searrow 0} -\frac{x^2\cdot\cos x}{\sin x} = \lim_{x\searrow 0} -\frac{\overset{\to 0}{x\cos x}}{\underset{\to 1}{\frac{\sin x}{x}}} = 0$$

7) a) $f : (-1, +\infty) \to \mathbb{R}$, $f_{(x)} = \ln(1+x)$

$$f'(x) = \left(\ln(1+x)\right)' = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f^{(3)}(x) = \frac{2\cancel{(1+x)}}{(1+x)^{\cancel{4}}} = \frac{2}{(1+x)^3}$$

$$f^{(4)}(x) = -\frac{2 \cdot 3 \cdot \cancel{(1+x)^2}}{(1+x)^{\cancel{6}\,4}} = -\frac{6}{(1+x)^4}$$

$\vdots$

$$f^{(n)} = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}$$

$$f^{(n+1)} = \left(f^{(n)}\right)' = (-1)^{n+1} \cdot (n-1)! \cdot \left(-\frac{n(1+x)^{n-1}}{(1+x)^{2n}}\right) =$$

$$= (-1)^{n+2} \cdot \frac{n!}{(1+x)^{n+1}}$$