Seminar 3

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The dot product & orthogonality

$v, w \in V$

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\widehat{\vec{v}, \vec{w}})$

$\Rightarrow \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

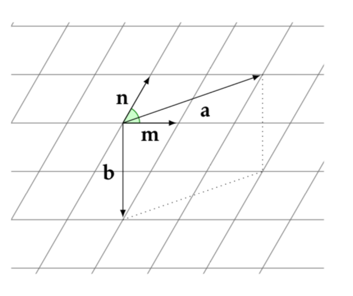
If we have an orthonormal basis $(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ on V, then we use the following simplified formulas:

If $\vec{v}(a_1, \dots, a_n)$, $\vec{w}(b_1, b_2, \dots, b_n)$

$\vec{v} \cdot \vec{w} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

1. Let **m** and **n** be two unit vectors such that $\measuredangle(\mathbf{m}, \mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$ and $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$.



$m(\widehat{\vec{m}, \vec{n}}) = \frac{\pi}{3}$

We need to find $\|\vec{a} + \vec{b}\|$ and $\|\vec{a} - \vec{b}\|$

$\vec{a} + \vec{b} = 2\vec{m} + \vec{n} + \vec{m} - 2\vec{n} = 3\vec{m} - \vec{n}$

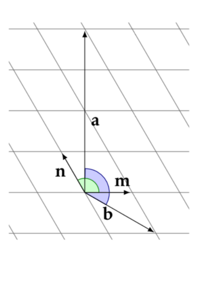
$\vec{a} - \vec{b} = 2\vec{m} + \vec{n} - \vec{m} + 2\vec{n} = \vec{m} + 3\vec{n}$

$\|\vec{a} + \vec{b}\|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (3\vec{m} - \vec{n}) \cdot (3\vec{m} - \vec{n}) = 9\vec{m}^2 - 3\vec{m}\vec{n} - 3\vec{m}\vec{n} + \vec{n}^2$

$\|\vec{a} + \vec{b}\|^2 = 9 - 6\cos\frac{\pi}{3} + 1 = 9 - 3 + 1 = 7 \Rightarrow \|\vec{a} + \vec{b}\| = \sqrt{7}$

$\|\vec{a} - \vec{b}\|^2 = (\vec{m} + 3\vec{n})(\vec{m} + 3\vec{n}) = \vec{m}^2 + 6\vec{m}\vec{n} + 9\vec{n}^2 = 13 \Rightarrow \|\vec{a} - \vec{b}\| = \sqrt{13}$

2. Let **m** and **n** be two unit vectors such that $\measuredangle(\mathbf{m}, \mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$.



$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(a, b) \Rightarrow \cos(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} = \frac{-3}{\sqrt{12}} = \frac{-3}{2\sqrt{3}} \Rightarrow m(\vec{a}, \vec{b}) = \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

$\vec{a} \cdot \vec{b} = (2\vec{m} + 4\vec{n})(\vec{m} - \vec{n}) = 2\vec{m}^2 + 4\vec{m}\vec{n} - 2\vec{m}\vec{n} - 4\vec{n}^2 = -3$

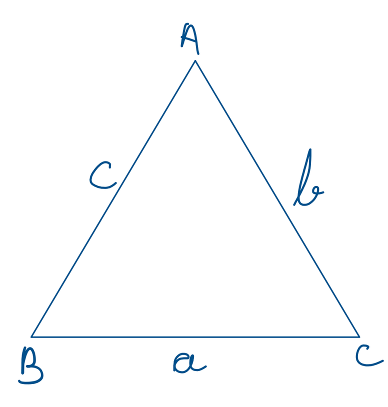
$\|\vec{a}\|^2 = \vec{a} \cdot \vec{a} = (2\vec{m} + 4\vec{n})(2\vec{m} + 4\vec{n}) = 4\vec{m}^2 + 16\vec{m}\vec{n} + 16\vec{n}^2 = 12 \Rightarrow \|\vec{a}\| = \sqrt{12}$

$\|\vec{b}\|^2 = \vec{b} \cdot \vec{b} = (\vec{m} - \vec{n})^2 = \vec{m}^2 - 2\vec{m}\vec{n} + \vec{n}^2 = 1 \Rightarrow \|\vec{b}\| = 1$

6. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = 2\overrightarrow{AB} \cdot \overrightarrow{AC}$$

and deduce the law of cosines in a triangle.



$\overrightarrow{AB}^2 = \|\vec{c}\|^2 = c^2$

$\overrightarrow{AC}^2 = \|\vec{b}\|^2 = b^2$

$\overrightarrow{BC}^2 = (\overrightarrow{BA} + \overrightarrow{AC})^2 = \overrightarrow{AB}^2 - 2\overrightarrow{AB}\,\overrightarrow{AC} + \overrightarrow{AC}^2$

$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = \overrightarrow{AB}^2 + \overrightarrow{AC} + 2\overrightarrow{AB}\,\overrightarrow{AC} - \overrightarrow{AB}^2 + \overrightarrow{AC}^2 = 2\overrightarrow{AB}\,\overrightarrow{AC}$

$\|\overrightarrow{BC}\|^2 = \|\vec{a}\|^2 = a^2$

$2\overrightarrow{AB}\,\overrightarrow{AC} = 2bc\cos A$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

9. Consider the vector **v** which is perpendicular on $\mathbf{a}(4, -2, -3)$ and on $\mathbf{b}(0, 1, 3)$. If **v** describes an acute angle with Ox and $\|\mathbf{v}\| = 26$ determine the components of **v**.

$\vec{v} \cdot \vec{a} = 4x - 2y - 3z = 0 \Leftrightarrow 4x + 6z - 3z = 0 \Leftrightarrow z = -\frac{4x}{3}$

$\vec{v} \cdot \vec{b} = y + 3z = 0 \Rightarrow y = -3z$

$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2} = 26 \Leftrightarrow x^2 + y^2 + z^2 = 676$

$x^2 + 9z^2 + z^2 = 676$

$x^2 + 10z^2 = 676$

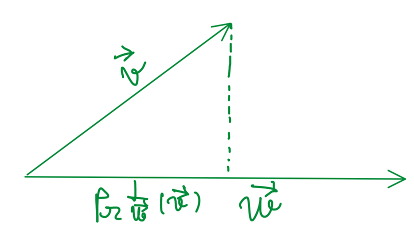
$x^2 + 10 \cdot \frac{16x^2}{9} = 676$

$169x^2 = 9 \cdot 676$

$x^2 = 9$

$\begin{cases} x = 3 \\ y = 12 \\ z = -4 \end{cases}$

10. Show that the Gram-Schmidt orthogonalization process yields an orthonormal basis.



$Pr^{\perp}_{\vec{w}}(\vec{v}) = \frac{1}{\|\vec{w}\|}\vec{w} \cdot \|Pr^{\perp}_{\vec{w}}(\vec{v})\|$

$\|Pr^{\perp}_{w}(\vec{v})\| = pr^{1}_{\vec{w}}(\vec{v}) = \|\vec{v}\| \cdot \cos(\vec{u}, \vec{w})$

$Pr^{\perp}_{\vec{w}}(\vec{v}) = \frac{1}{\|\vec{w}\|} \cdot \|\vec{v}\|\cos(\vec{v}, \vec{w}) \cdot \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \cdot \vec{w}$

$\vec{u} = \vec{v} \; Pr^{\perp}_{\vec{w}}(\vec{v})$

Gram Schmidt orthogonalization

$(v_1, v_2, \dots, v_n)$ basis of V

① $v_1' = v_1$

② $v_2' = v_2 - Pr^{\perp}_{v_1'}(v_2)$

③ $v_3' = v_3 - Pr^{\perp}_{v_1}(v_3) - Pr^{\perp}_{v_2}(v_3) = v_3 - Pr^{\perp}_{\langle v_1, v_2 \rangle}(v_3)$

⋮

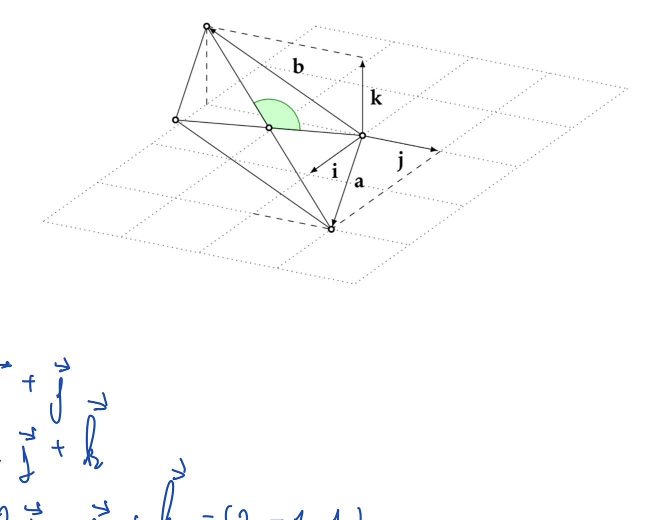
ⓝ $v_n' = v_n - Pr^{\perp}_{\langle v_1, v_2, \dots, v_n \rangle}(v_n)$

This gives us an orthogonal basis $(v_1', v_2', \dots, v_n')$

To make it orthonormal

$v_i'' = \frac{1}{\|v_i'\|} v_i'$

3. You are given two vectors $\mathbf{a}(2, 1, 0)$ and $\mathbf{b}(0, -2, 1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by **a** and **b**.



$\vec{a} = 2\vec{i} + \vec{j}$

$\vec{b} = -2\vec{j} + \vec{k}$

$\vec{a} + \vec{b} = 2\vec{i} - \vec{j} + \vec{k} = (2, -1, 1)$

$\vec{a} - \vec{b} = 2\vec{i} + 3\vec{j} - \vec{k} = (2, 3, -1)$

$(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 4 - 3 - 1 = 0 \Rightarrow \vec{a} + \vec{b} \perp \vec{a} - \vec{b}$

4. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p}, \mathbf{q})$ is $5/12$.

$\vec{q} = 3\vec{i} + \vec{j}$

$\vec{p} = \vec{i} + 2\vec{j} + \lambda\vec{k}$

$\vec{p} \cdot \vec{q} = 3 + 2 = 5$

$\vec{p} \cdot \vec{q} = \|\vec{p}\| \cdot \|\vec{q}\| \cdot \cos(p, q)$

$\|\vec{p}\| = \sqrt{10}$

$\|\vec{q}\| = \sqrt{5 + \lambda^2}$

$\cos(p, q) = \frac{5}{12}$

$\Rightarrow \sqrt{10(5 + \lambda^2)} = 12$

$10(5 + \lambda^2) = 144$

$\lambda = \pm\sqrt{\frac{47}{5}}$

$\lambda = \pm\frac{\sqrt{235}}{5}$

5. Using the scalar product, prove the Cauchy-Bunyakovsky-Schwarz inequality, i.e. show that for any $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ we have

$$(a_1 b_1 + \dots + a_n b_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2).$$

$(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ orthonormal basis

$\vec{a} = (a_1, a_2, \dots, a_n)$

$\vec{b} = (b_1, b_2, \dots, b_n)$

$\|\vec{a}\|^2 = a_1^2 + a_2^2 + \dots + a_n^2$

$\|\vec{b}\|^2 = b_1^2 + b_2^2 + \dots + b_n^2$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(a, b) \quad |()^2$

$(\vec{a} \cdot \vec{b})^2 = \|\vec{a}\|^2 \cdot \|\vec{b}\|^2 \cdot \underbrace{(\cos(a, b))^2}_{\leq 1} \leq \|\vec{a}\|^2 \cdot \|\vec{b}\|^2$

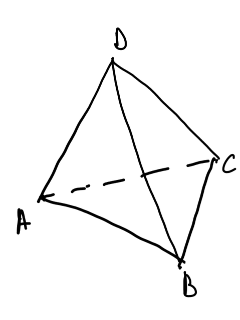
$(\vec{a} \cdot \vec{b})^2 = (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2$

$\Rightarrow (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$

7. Let $ABCD$ be a tetrahedron. Show that

$$\cos(\measuredangle(\overrightarrow{AB}, \overrightarrow{CD})) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2 \cdot AB \cdot CD}.$$

This is a $3D$-version of the law of cosine.



$\cos(\measuredangle(\overrightarrow{AB}, \overrightarrow{CD})) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2AB \cdot CD}$

$\cos(\measuredangle(\overrightarrow{AB}, \overrightarrow{CD})) = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{AB}| \cdot |\overrightarrow{CD}|}$

$\Rightarrow AD^2 + BC^2 - AC^2 - BD^2 = 2\overrightarrow{AB} \cdot \overrightarrow{CD}$

$AD^2 = (x_D - x_A)^2 + (y_D - y_A)^2 + (z_D - z_A)^2 = x_A^2 - 2x_A x_D + x_D^2 + y_A^2 - 2y_A y_D + y_D^2 + z_A^2 - 2z_A z_D + z_D^2$

$BC^2 = (x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2 = x_B^2 - 2x_B x_C + x_C^2 + y_B^2 - 2y_B y_C + y_C^2 + z_B^2 - 2z_B z_C + z_C^2$

$AC^2 = (x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2 = x_C^2 - 2x_C x_A + x_A^2 + y_C^2 - 2y_C y_A + y_A^2 + z_C^2 - 2z_C z_A + z_A^2$

$BD^2 = (x_D - x_B)^2 + (y_D - y_B)^2 + (z_D - z_B)^2 = x_B^2 - 2x_B x_D + x_D^2 + y_B^2 - 2y_B y_D + y_D^2 + z_D^2 - 2z_D z_B + z_B^2$

$2\overrightarrow{AB} \cdot \overrightarrow{CD} = 2(x_B - x_A)(x_D - x_C) + 2(y_B - y_A)(y_D - y_C) + 2(z_B - z_A)(z_D - z_C)$

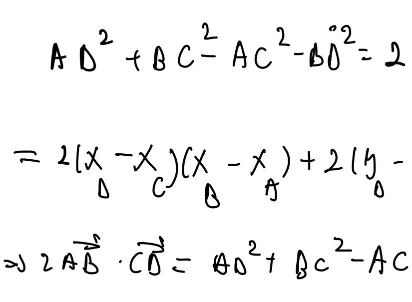
$AD^2 + BC^2 - AC^2 - BD^2 = 2(x_A x_C + x_B x_D - x_A x_D - x_B x_C) + 2(y_A y_C + y_B y_D - y_A y_D - y_B y_C) + 2(z_A z_C + z_B z_D - z_A z_D - z_B z_C)$

$= 2(x_D - x_C)(x_B - x_A) + 2(y_D - y_C)(y_B - y_A) + 2(z_D - z_C)(z_B - z_A) =$

$\Rightarrow 2\overrightarrow{AB} \cdot \overrightarrow{CD} = AD^2 + BC^2 - AC^2 - BD^2$

8. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA} \cdot \overrightarrow{OC} = \overrightarrow{OB} \cdot \overrightarrow{OD} \quad \text{and} \quad \overrightarrow{OA}^2 + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \overrightarrow{OD}^2.$$



$\overrightarrow{OA} = (\overrightarrow{OB} + \overrightarrow{BA})$

$\overrightarrow{OC} = (\overrightarrow{OD} + \overrightarrow{DC}) = (\overrightarrow{OD} + \overrightarrow{AB})$

$\overrightarrow{OA} \cdot \overrightarrow{OC} = \overrightarrow{OB} \cdot \overrightarrow{OD} + \overrightarrow{BA} \cdot \overrightarrow{OD} + \overrightarrow{AB} \cdot \overrightarrow{OB} + \overrightarrow{BA} \cdot \overrightarrow{AB}$

$= \overrightarrow{OB} \cdot \overrightarrow{OD} + \overrightarrow{BA} \cdot (\overrightarrow{AB} + \overrightarrow{BO} + \overrightarrow{OD}) = \overrightarrow{OB} \cdot \overrightarrow{OD} + \underbrace{\overrightarrow{BA} \cdot \overrightarrow{AD}}_{=0} = \overrightarrow{OB} \cdot \overrightarrow{OD}$

14. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{R}$ be the map $\phi(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to an orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \cdot \mathbf{v}$?

$\phi(\vec{w}) = \vec{v} \cdot \vec{w}$, $v = (v_1, v_2, v_3)$

ϕ linear map $\Leftrightarrow \phi(\lambda_1 \vec{w}_1 + \lambda_2 \vec{w}_2) = \lambda_1 \phi(\vec{w}_1) + \lambda_2 \phi(\vec{w}_2)$

$\phi(\lambda_1 \vec{w}_1 + \lambda_2 \vec{w}_2) = \vec{v} \cdot (\lambda_1 \vec{w}_1 + \lambda_2 \vec{w}_2) = \lambda_1 \vec{v}\vec{w}_1 + \lambda_2 \vec{v}\vec{w}_2$

$\lambda_1 \phi(\vec{w}_1) + \lambda_2 \phi(\vec{w}_2) = \lambda_1 \vec{v}\vec{w}_1 + \lambda_2 \vec{v}\vec{w}_2$

$\Rightarrow \phi$ is a linear map

$\phi(w) = v_1 w_1 + v_2 w_2 + v_3 w_3$

$M(\phi) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

11. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0, 1, 0)$, $\mathbf{v}_2(2, 1, 0)$ and $\mathbf{v}_3(-1, 0, 1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

$Pr^{\perp}_{u} v = \frac{\langle v, u \rangle}{\langle u, u \rangle} u$

① $v_1' = v_1$ $\quad e_1' = (0, 1, 0)$

② $v_2' = v_2 - Pr^{\perp}_{v_1'}(v_2) = v_2 - \frac{\langle v_2, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1' = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2 \cdot 0 + 1 \cdot 1 + 0 \cdot 0}{1} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\quad e_2' = (1, 0, 0)$

③ $v_3' = v_3 - Pr^{\perp}_{v_1'}(v_3) - Pr^{\perp}_{v_2'}(v_3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\langle v_3, v_1' \rangle}{\langle v_1', v_1' \rangle} v_1' - \frac{\langle v_3, v_2' \rangle}{\langle v_2', v_2' \rangle} v_2' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 0 + \frac{2}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad e_3' = (0, 0, 1)$