B+ tree index

CSC 443

1. Overview of B+ tree

In the previous article (./04.basic-index.html), we discussed several important things about indexing:

1. Indexes are disk based data structures to speed up record search (and / or scan)
2. Organizing page pointers into sorted index tree structures speeds up record search. (ISAM (./04.basic-index.html#isam))
3. Maintaining sortedness of the index tree is expensive, so we may create overflow pages in ISAM, but the overflow pages degrade search performance from log to linear scan.

We will introduce a highly robust and popular data structure B+ tree, which is an extension of ISAM.

Generally speaking, a B+ tree is an efficient disk based data structure that stores (key / value) pairs. It supports very efficient lookup by key, and iteration of entries in the range of two specified key values.

B+ tree offers:

- fast record search

- fast record traversal
- maintaining sorted tree structure without overflow pages

The key idea behind B+ tree is that it utilizes balanced sorted tree of page pointers, as opposed to just sorted tree in the case of ISAM.

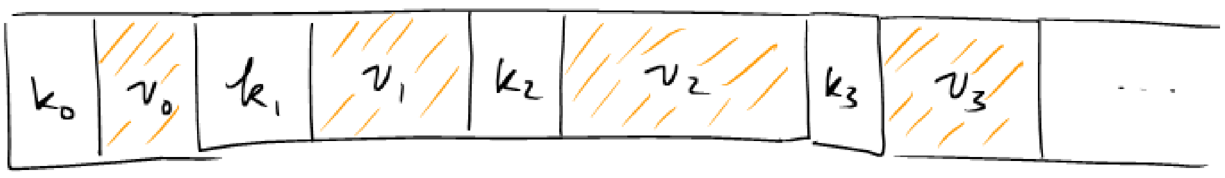
2. Definition of B+ tree

A B+ tree is a tree whose nodes are pages on disk. We will distinguish the leaf nodes and the interior nodes of a B+ tree.

Since each node is exactly one disk page, in the context of B+ tree, we use the term page and node interchangably.

2.1. Leaf nodes

The leaf nodes store the data entries in the form of (key, value). All leaf nodes are also organized into an linked list of pages. Leaf nodes of a B+ tree can be visualized as follows.



A leaf node

The following data structure abstracts the leaf nodes.

```
struct LeafNode {
    vector<Key> keys;
    vector<Value> values;
    PagePointer next_page;
}
```

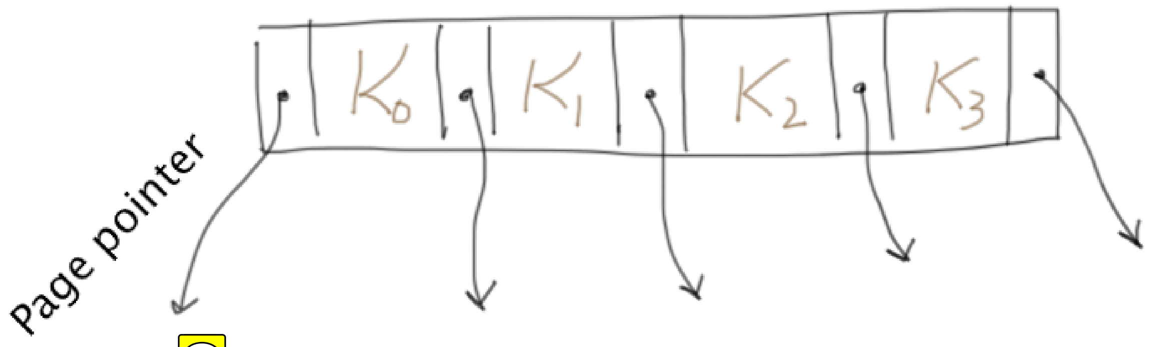
We can always assert that

`p.keys.size() == p.values.size()` for all nodes `p`.

2.2. Interior nodes

The interior nodes form a tree structure, starting from a root node, to speed up lookup of the leaf node that contains a key of interest.

The structure of an interior node can be visualized as a sequence of alternating page pointers and keys.



An interior node

The following data structure abstracts the interior nodes:

```
struct InteriorNode {
    vector<Key> keys;
    vector<PagePointer> pointers;
}
```

We can always assert that `p.keys.size() +1 == p.pointers.size()`.

Definition: Neighbouring pointers

Give a key k_i, we define before(k_i) to be the page pointer before k_i, and after(k_i) the page pointer after k_i.

Namely,

- `p.before(`k_i`) = p.pointers[i]`
- `p.after(`k_i`) = p.pointers[i+1]`

2.3. Constraints and properties of B+ tree

Keys are sorted in the nodes.

Let `p` be a node in B+ tree. We need to maintain that `p.keys` is sorted by the key values.

<u>Nodes are sorted by their keys</u>

B+ tree is a sorted tree in the following sense:

- Leaf nodes are sorted:

  $$\forall p \in \text{LeafNode},\ \forall k \in p.\text{keys},\ \forall k' \in p.\mathtt{next\_page.keys},\ k < k'$$

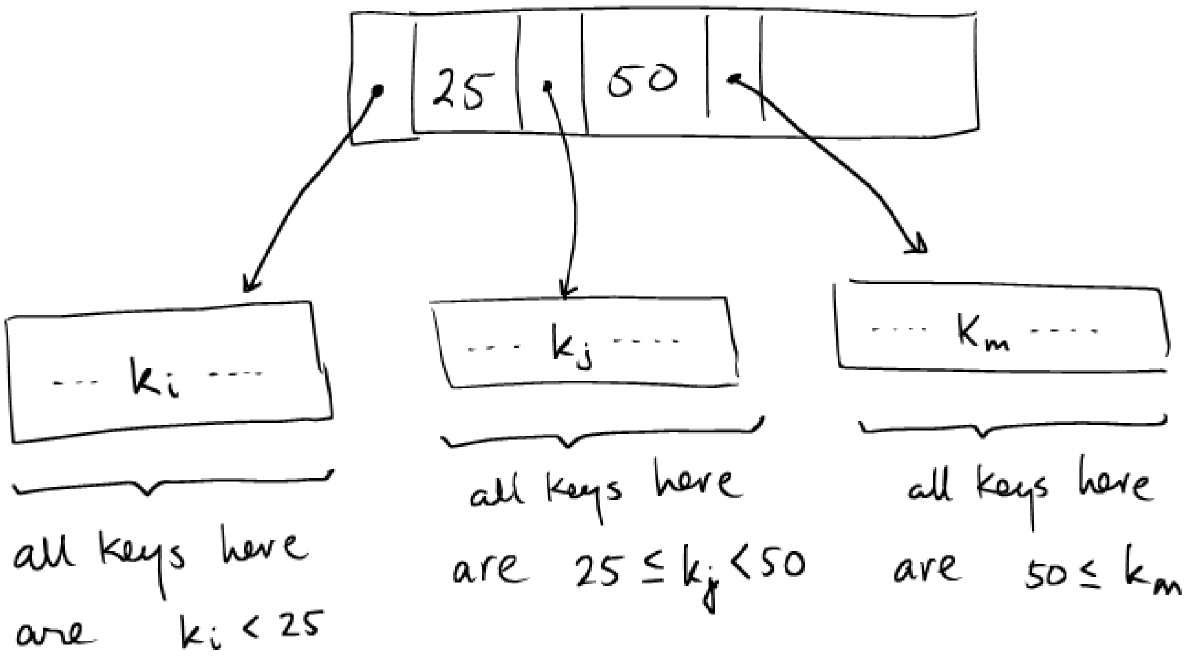
  The sorted linked list of leaf nodes allows very efficient traveral of (key, value) pairs between two leaf nodes.

- Interior nodes are sorted:

  B+ tree asserts that for all key k and the adjacent pointers, $\mathtt{after}(k)$ and $\mathtt{after}(k)$, it must be that:

  - $k > \max(\text{keys}(\text{before}(k)))$
  - $k \leq \min(\text{keys}(\mathtt{after}(k)))$

  In other words, k is the key that divides the keys in pages $\text{before}(k)$ and $\mathtt{after}(k)$.



25
50
... k_i ...
... k_j ...
... K_m ...
all keys here are $k_i < 25$
all keys here are $25 \leq k_j < 50$
all keys here are $50 \leq k_m$

<u>The tree is balanced</u>

B+ tree is a balanced tree, so all paths from the root to the leaf nodes must be of equal length.

The nodes are sufficiently filled.

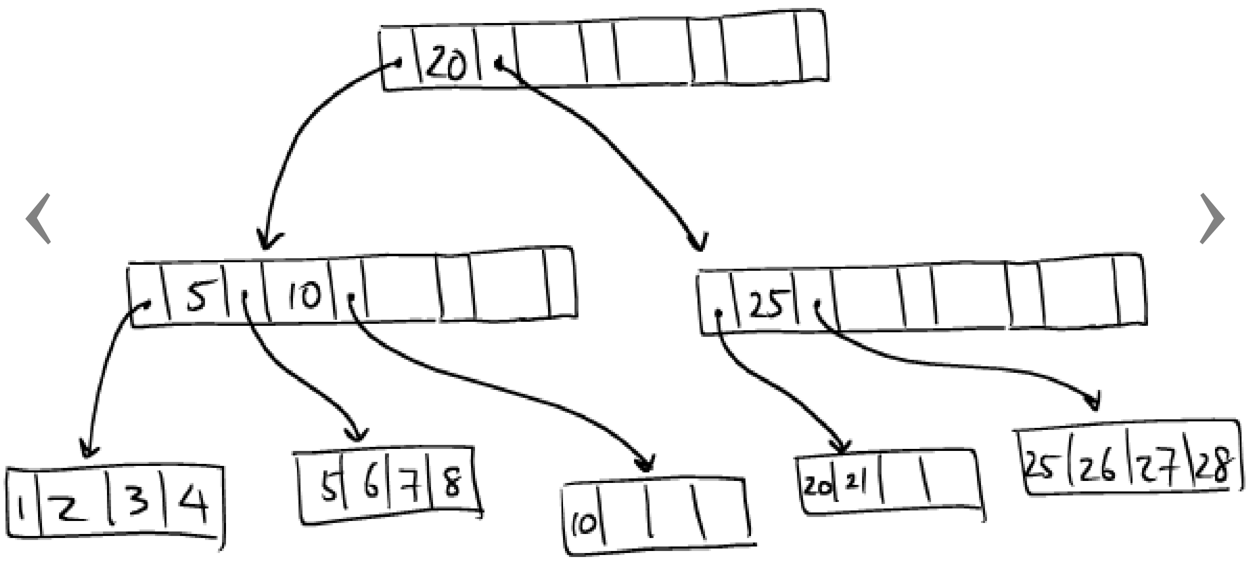
B+ tree permits nodes to be partially filled. As a design parameter, B+ tree specifies a percentage, known as the fill factor of the tree, that controls the minimal occupacity of all the non-root nodes.

If a non-root node is too empty, we say it has an underflow. Only the root node can have an underflow.

Here is an example of a problematic B+ tree. Suppose we have the following parameters:

```
Capacity of each node: 4 keys
Fill factor: 50%
```

While the tree is balanced and sorted, it does not satisfy the fill-factor.



3. B+ Tree Search and Insertion

The main operations on B+ tree are:

```
/**
 * finds the leaf node that _should_ contain the entry with the specified
key
 */
LeafNode search(Node root, Key key)

/**
 * Inserts a key/val pair into the tree.
 * Returns the root of the new tree which _may_ be different
 * from the old root node.
 */
InteriorNode insert_into_tree(InteriorNode root, Key newkey, Value val
)
```

The algorithm for `insert` must ensure that the properties of the B+ tree are preserved after the respective operation.

3.1. Searching

The B+ tree search algorithm is a straight-forward tree traveral algorithm.

```
LeafNode search(Node p, Key key) {
    if(p is LeafNode)
        return root;
    else {
        if(key < p.keys[0])
            return search(before(p.keys[0]), key);
        else if(key > p.keys[-1])
            return search(after(p.keys[-1]), key);
        else {
            let i be p.keys[i] <= key < p.keys[i+1]
            return search(after(p.keys[i]), key)
        }
    }
}
```

3.2. Inserting

Insertion into a B+ tree is quite tricky. Unlike the algorithm (http://en.wikipedia.org/wiki/AVL_tree#Insertion) for insertion into a balanced sorted binary tree, the B+ tree insertion needs to deal with node overflows and underflows.

The insertion algorithm starts with:

- look for the right leaf for insertion
- try to insert into the leaf node

```
InteriorNode insert_into_tree(InteriorNode root, Key newkey, Value val
) {
    LeafNode leaf = search(root, newkey);
    return insert_into_node(leaf, newkey, val);
}
```

The insertion algorithm, `insert_into_node`, looks something like this:

```
/**
 * Tries to inserts the (newkey/val) pair into
 * the node.
 *
 * If `target` is an interior node, then `val` must be a page pointer.
 */
InteriorNode insert_into_node(Node target, newkey, val)
{
    if( ... CASE 1 ... ) {
        /* handle CASE 1 */
    } else if( ... CASE 2 ... ) {
        /* handle CASE 2 */
    } else if( ... CASE 3 ... ) {
        /* handle CASE 2 */
    }
}
```

The tree distinct cases are:

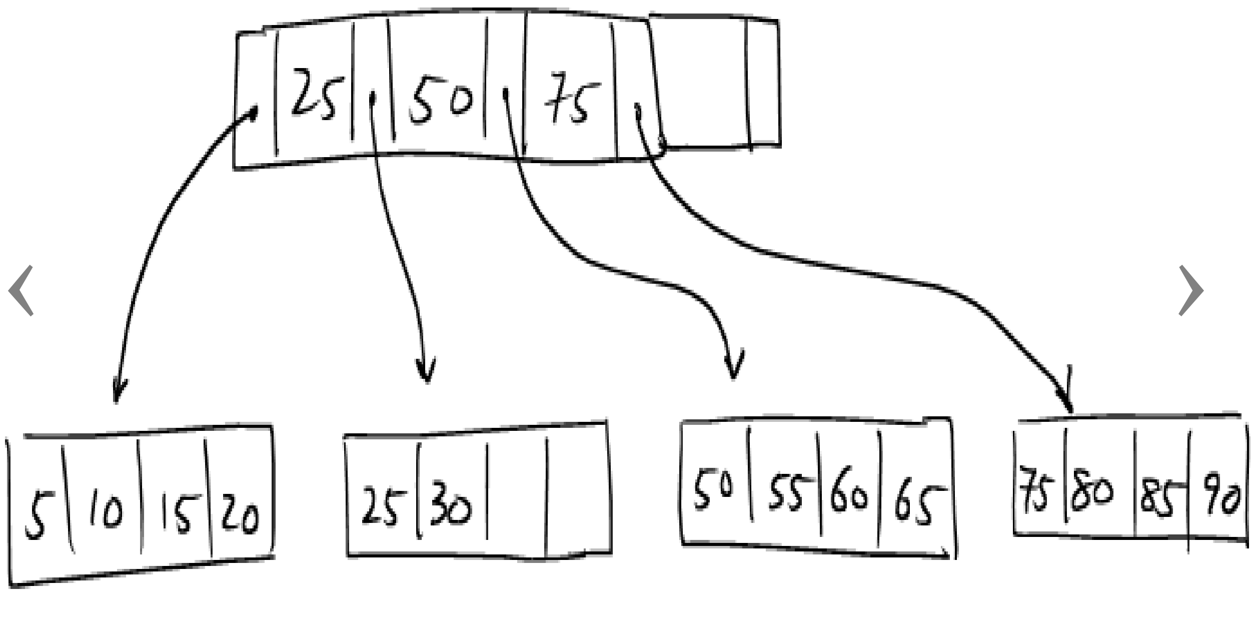
1. the target node has available space for one more key.
2. the target node is full, but its parent has space for one more key.
3. the target node and its parent are both full.

3.2.1. Case 1

This is the easiest case. Simply insert the entry (newkey, val) into `target`.

- The root of the B+ tree does not change.
- We have not discussed the disk I/O operations involved. Since all the node operations operate on one page at a time, the buffer manager (02.memory-hierarchy.html#buffermanager) can be used to provide the transparent virtual memory, as if all the nodes are stored in memory.

Example



(1): Suppose this is the initial tree. We are to insert (28, val) into this tree.

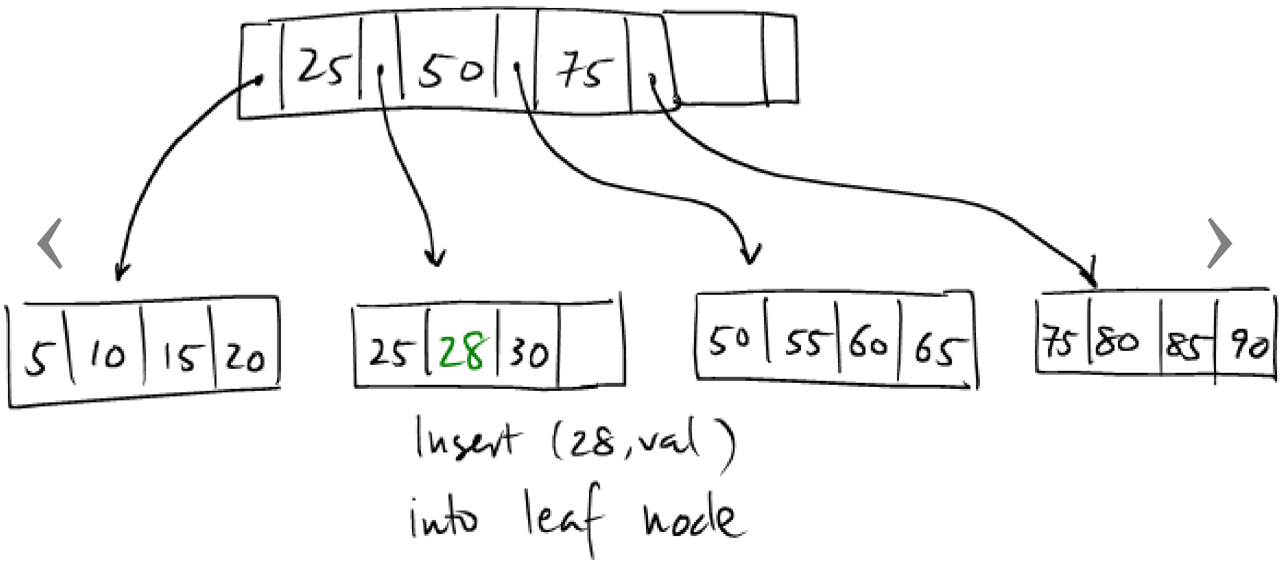
3.2.2. Case 2

In this case, the target node is full, but its parent node has available space for at least one more key.

- Create a new sibling node to target, call it `new_target`, and insert it <u>after</u> `target`.
- Distribute the data entries between `target` and `new_target`. From the assumption that `target` is full, we can safely assert that after distribution, there will be no underflow.
- `new_target` must be pointed to by `(k,p) = (leaf2.keys[0], ADDRESS[leaf2])` in `PARENT[target]`. So `(k,p)` are to be inserted into `PARENT[leaf]`. By assumption, we know that `PARENT[leaf]` will not overflow.

Example

Let's continue with the previous example, and try to insert (70, val).



(1): We now insert an entry with key = 70.

3.2.3. Case 3

This is the case that both `target` and `PARENT[target]` are full. We have to recursively attempt to insert the new key into the ancestors of `target`. It is possible that even the root does not have sufficient space for the new key, in which case we must split the root, and create a new root node of the B+ tree.

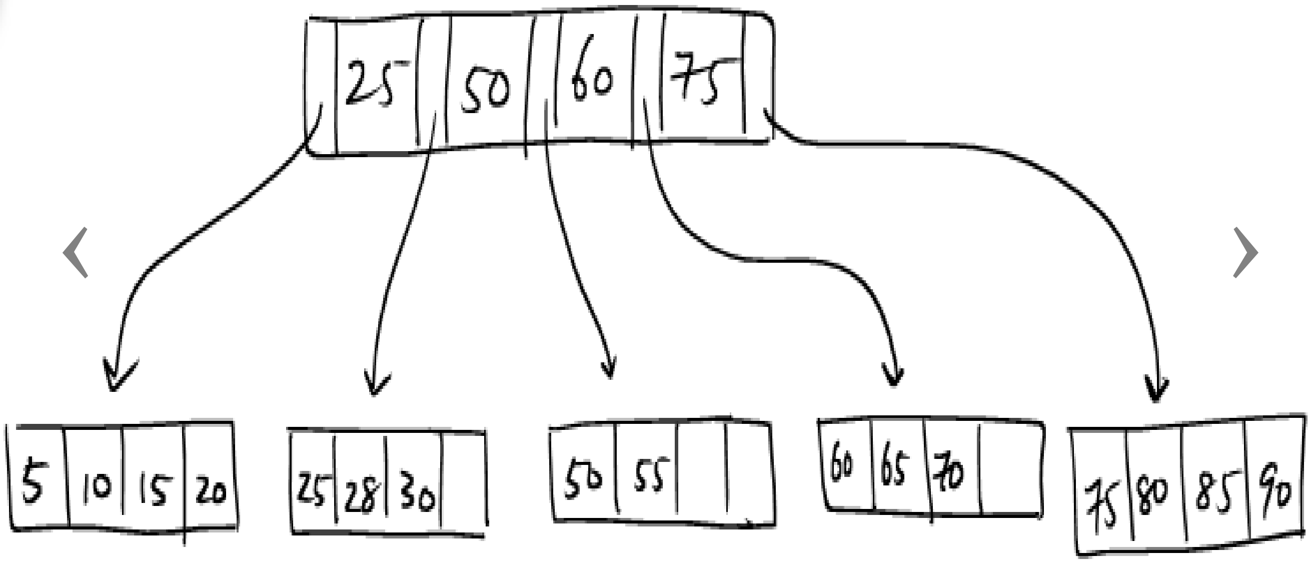
The details are as follows:

1. Create `new_target` node, and insert it after `target`.
2. Distribute data entries among `target` and `new_target`.

Now, let `k = leaf2.keys[0]` and `p = ADDRESS[leaf2]`. We need to insert `(k,p)` into `PARENT[leaf]`, which is full.

1. Let `target_parent = PARENT[target]`
2. Let `all_keys = sorted(target_parent.keys ∪ {k})`
3. Allocate new node `new_interior`
4. Let `i = floor(all_keys.size() / 2)`.
   Let `middle_key = all_keys[i]`
5. Distribute `all_keys[0 .. i-1]` to `target_parent`, and
   Distribute `all_keys[i+1 .. n]` to `new_interior`.
6. If `target_parent` is the root, then let `grandparent` be a newly allocated node.
   Otherwise, `grandparent = PARENT[target_parent]`.
7. Recursively call:
   `insert_into_node(grandparent, middle_key, ADDRESS[new_interior])`

Example



(1): We try to insert (95, ...) into this tree.

4. Other things about B+ Tree

- B+ tree also offers efficient deletion. The deletion algorithm is a mirror reversal of the insertion algorithm. During deletion, the algorithm trys to merge nodes to avoid underflow. If a merge took place, deletion recursively deletes the `(Key, PagePointer)` pair from the parent node.
- If the data entries are stored in a sequential file, sorted by a key, then one can very efficiently bulk load the sequential file into a B+ tree.
- B+ tree can be used as a sorting algorithm for disk based sorting.