Drift modelling of gravitational capture by PBHs

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Evolution eqaution for Drift modelling

- gravitational capture of monopoles by PBHs is simmilar to the $M\bar{M}$ annihilation.
- monopole will attain a drift velocity $u_{\rm D}(\bar{R})$ when the drag force from plasma is balanced by the gravitational force, where $\bar{R}$ is monopole-PBH distance

$$F_{\rm drag} = -CT^2 u_{\rm D}(\bar{R}) = \frac{m m_{\rm bh}}{M_{\rm Pl}^2 \bar{R}^2}$$

The typical separation between PBHs is $n_{\rm bh}^{-1/3}$

$$u_{\rm D}(n_{\rm bh}^{-1/3}) = \frac{m m_{\rm bh} n_{\rm bh}^{2/3}}{M_{\rm Pl}^2 C T^2}$$

the typical capture time

$$\tau_{\rm gc} = \frac{n_{\rm bh}^{-1/3}}{u_{\rm D}(n_{\rm bh}^{-1/3})} = \frac{M_{\rm Pl}^2 C T^2}{n_{\rm bh} m_{\rm bh} m}$$

Evolution eqation for Drift modelling

the capture frequency

$$F \equiv \tau_{\rm gc}^{-1} = \frac{n_{\rm bh} m_{\rm bh} m}{M_{\rm Pl}^2 C T^2}$$

the evolution eqation of the monopole number density

$$\dot{n}_M = -D n_M^2 - F n_M - 3\frac{\dot{a}}{a} n_M$$

Capture requirement for T

monopole mean free path ℓ must be smaller than the capture radius, i.e. thermal kinetic energy of monopoles is comparable to the gravitational potential energy of PBH

$$\ell < r_c^{\mathrm{gc}} \quad \text{or} \quad T = \frac{m m_{\mathrm{bh}}}{M_{\mathrm{Pl}}^2 r_c^{\mathrm{gc}}}$$

by solving Newton's equation of motion for monopoles that $m\dot{\mathrm{v}} = -CT^2\mathrm{v}$, the mean free path is $\ell \simeq \dfrac{1}{CT}\left(\dfrac{m}{T}\right)^{1/2}$

$$\ell \simeq \frac{1}{CT}\left(\frac{m}{T}\right)^{1/2} < r_c^{\mathrm{gc}} = \frac{m m_{\mathrm{bh}}}{M_{\mathrm{Pl}}^2 T} \quad \Rightarrow \quad T > T_{\mathrm{gc}} \equiv \frac{M_{\mathrm{Pl}}^4}{C^2 m_{\mathrm{bh}}^2 m}$$

Energy density fraction

energy density fraction of PBHs at temperature T

$$\beta_T \equiv \frac{n_{\rm bh} m_{\rm bh}}{K_1 T^4} \propto T^{-1} \quad \Rightarrow \quad n_{\rm bh} m_{\rm bh} = \beta_T K_1 T^4 \propto T^3$$

energy density of PBHs when they form, with T_b the temperature of universe and β the energy density fraction at formation

$$(n_{\rm bh} m_{\rm bh})\big|_{\rm formation} = \beta K_1 T_b^4$$

for any temperature

$$n_{\rm bh} m_{\rm bh} = \beta K_1 T_b T^3$$

Mass fraction

total mass within a particle horizon

$$m_H \equiv \frac{4}{3}\pi\rho H^{-3} = \frac{4\pi K_1}{3K^2} \cdot M_{\rm Pl}^2 H^{-1} = 0.5H^{-1}M_{\rm Pl}^2$$

mass fraction of PBHs at formation

$$\gamma \equiv \frac{m_{\rm bh}}{0.5H_{\rm form}^{-1}M_{\rm Pl}^2} \simeq 0.2$$

Hubble parameter at formation in terms of T_b: $H_{\rm form} = \frac{KT_b^2}{M_{\rm Pl}}$

$$T_b^2 = \frac{H_{\rm form}M_{\rm Pl}}{K} = \frac{0.5\gamma M_{\rm Pl}^2}{m_{\rm bh}}\frac{M_{\rm Pl}}{K} = \frac{\gamma M_{\rm Pl}^3}{2m_{\rm bh}K}$$

Solve F

$$T_b^2 = \frac{\gamma M_{\rm Pl}^3}{2 m_{\rm bh} K} \quad \Rightarrow \quad T_b = \left(\frac{\gamma}{2K}\right)^{1/2} \left(\frac{m_{\rm bh}}{M_{\rm Pl}}\right)^{-1/2} M_{\rm Pl}$$

F when $n_{\rm bh} m_{\rm bh} = \beta_b K_1 T_b T^3$

$$F = \frac{n_{\rm bh} m_{\rm bh} m}{M_{\rm Pl}^2 C T^2} = \frac{\beta K_1 T_b m}{M_{\rm Pl}^2 C} T \quad \propto T$$

reduced variable $\delta = \dfrac{m}{T_c}$

$$F = \frac{\beta K_1 T_b \delta T_c}{M_{\rm Pl}^2 C} T = K_1 \left(\frac{\gamma}{2K}\right)^{1/2} C^{-1} \delta \times \beta \left(\frac{m_{\rm bh}}{M_{\rm Pl}}\right)^{-1/2} \left(\frac{T_c}{M_{\rm Pl}}\right) T$$

Yield evolution in an adiabatic expansion

evolution eqaution of monopole number density in Drift modelling

$$\dot{n}_M = -Dn_M^2 - Fn_M - 3\frac{\dot{a}}{a}n_M$$

To separate the effects of annihilation and expansion, transform it into the form for monopole yield $r = n_M/s$, s being entropy density.

$$\dot{r}s + r\dot{s} = -Dr^2s^2 - Frs - 3\frac{\dot{a}}{a}rs$$

In an adiabatic expansion, entropy within a comoving volume is conserved, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3 s\right) = 0$. Thus $\dot{s} = -3\frac{\dot{a}}{a}s$

$$\dot{r} = -Dsr^2 - Fr$$

Yield evolution in an adiabatic expansion

entropy density s at temperature T is given by $s = K_2 T^3$, with K_2 constant in a radiation-dominated regime.

$$\frac{\mathrm{d}}{\mathrm{d}t}(a^3 s) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(aT) = 0 \quad \Rightarrow \quad \frac{\dot{T}}{T} = -\frac{\dot{a}}{a} = -H = -\frac{KT^2}{M_{\mathrm{Pl}}}$$

Transform the previous equation to the form with T as variable

$$\frac{\mathrm{d}r}{\mathrm{d}T} = \frac{\dot{r}}{\dot{T}} = \frac{-Dsr^2 - Fr}{-KT^3 M_{\mathrm{Pl}}^{-1}} = \frac{\Delta}{T^2} r^2 + \frac{\Phi}{T^2} r$$

Where Δ and Φ are reduced forms of D and F respectively, both independent of T

$$\Delta \equiv K_2 K^{-1} M_{\mathrm{Pl}} D T^2 = \frac{K_2 \chi^2 g^2 M_{\mathrm{Pl}}}{KC} \quad \Phi \equiv \frac{M_{\mathrm{Pl}} F}{KT}$$

Solution to evolution equation $r(T)$

$$\frac{\mathrm{d}r}{\mathrm{d}T} = \frac{\Delta}{T^2} r^2 + \frac{\Phi}{T^2} r$$

Integrate both sides

$$\int \frac{\mathrm{d}r}{\Delta r^2 + \Phi r} = \int \frac{\mathrm{d}T}{T^2} \quad \Rightarrow \quad \frac{1}{\Phi} \log \frac{r}{\frac{\Delta}{\Phi} r + 1} = -\frac{1}{T} + \text{constant}$$

introduce $r_{\mathrm{cr}} \equiv \frac{\Phi}{\Delta}$, $\bar{\Phi} \equiv \frac{\Phi}{T_c}$, replace T with reduced temperature $z(T) = \frac{T}{T_c}$

$$\frac{1}{\Phi} \log \frac{r r_{\mathrm{cr}}}{r + r_{\mathrm{cr}}} = -\frac{1}{\Phi} \log \left(\frac{1}{r} + \frac{1}{r_{\mathrm{cr}}} \right) = -\frac{1}{T} + \text{constant}$$

$$\log \left(\frac{1}{r} + \frac{1}{r_{\mathrm{cr}}} \right) = \frac{\bar{\Phi}}{z} + \text{constant}$$

Solution to evolution equation $r(T)$

The solution to the evolution equation is

$$\log\left(\frac{1}{r}+\frac{1}{r_{\rm cr}}\right)=\frac{\bar{\Phi}}{z}+\text{constant}$$

Suppose r_1, r_2 the corresponding yield at temperature z_1, z_2

$$\log\left(\frac{1}{r_2}+\frac{1}{r_{\rm cr}}\right)-\log\left(\frac{1}{r_1}+\frac{1}{r_{\rm cr}}\right)=\frac{\bar{\Phi}}{z_2}-\frac{\bar{\Phi}}{z_1}$$

$$r_2=\left\{\left(\frac{1}{r_{\rm cr}}+\frac{1}{r_1}\right)\exp\left[\bar{\Phi}\left(\frac{1}{z_2}-\frac{1}{z_1}\right)\right]-\frac{1}{r_{\rm cr}}\right\}^{-1}$$

to separate the contribution of gravitational capture from that of annihilation, let $\Delta=0$, thus $r_{\rm cr}\to\infty$

$$r_2=r_1\exp\left[-\bar{\Phi}\left(\frac{1}{z_2}-\frac{1}{z_1}\right)\right]$$

Calculate captured monopole number from evolution equation of $r(T)$

$$\frac{\mathrm{d}r}{\mathrm{d}T} = \frac{\Delta}{T^2}r^2 + \frac{\Phi}{T^2}r$$

integrate the second term on the right-hand side from T_s to T_t, gravitational capture is effective in this interval and $T_s \equiv \max\{T_{\mathrm{ev}}, T_{\mathrm{gc}}\},\ T_t \equiv \min\{T_c, T_b\}$

$$\kappa \equiv \Phi \int_{T_s}^{T_t} \frac{r(T)}{T^2}\mathrm{d}T$$

1. T_{c}, temperature when monopoles are produced, approximately equal to critical temperature of symmetry breaking phase transition.
2. T_b, temperature of universe when PBHs are formed. When $T < T_{\mathrm{b}}$, PBH have formed.
3. T_{ev}, temperature of universe when PBH have evaporated completely, i.e. when $T > T_{\mathrm{ev}}$, PBH have not evaporated yet.
4. T_{gc}, when $T > T_{\mathrm{gc}}$ gravitational capture of monopoles by PBHs remains effective

Calculate captured monopole number from evolution equation of $r(z)$

κ is Δr caused by gravitational capture, can be expressed in terms of reduced variables $z(T)$ and $\bar{\Phi}$

$$\kappa = \bar{\Phi} \int_{z_s}^{z_t} r(z) z^{-2} \mathrm{d}z$$

The average number of monopoles or antimonopoles captured by each PBH is then

$$n_2 = \frac{\Delta n_M}{n_{\mathrm{bh}}} = \kappa \frac{s(T_t)}{n_{\mathrm{bh}}(T_t)} = \frac{\bar{\Phi} s(T_t)}{n_{\mathrm{bh}}(T_t)} \int_{z_s}^{z_t} r(z) z^{-2} \mathrm{d}z$$

let $y \equiv m_{\mathrm{bh}}/M_{\mathrm{Pl}}$ and $\delta \equiv m/T_c$

$$\begin{cases} F = \dfrac{n_{\mathrm{bh}} m_{\mathrm{bh}} m}{M_{\mathrm{Pl}}^2 C T^2} \\ \bar{\Phi} = \dfrac{M_{\mathrm{Pl}} F}{K T T_c} = \dfrac{M_{\mathrm{Pl}} F_t}{K T_t T_c} \end{cases} \Rightarrow \quad \frac{\bar{\Phi} s(T_t)}{n_{\mathrm{bh}}(T_t)} = \frac{M_{\mathrm{Pl}} \cdot m_{\mathrm{bh}} m \cdot K_2 T_t^3}{K T_t T_c \cdot M_{\mathrm{Pl}}^2 C T_t^2} = \frac{K_2}{K} \delta y C^{-1}$$

Calculate captured monopole number from evolution equation of $r(z)$

$K_2 = \frac{2\pi^2}{45}\mathcal{N}$ and $K = \left(\frac{4\pi^3\mathcal{N}}{45}\right)^{1/2}$, then $\frac{K_2}{K} = \frac{1}{3}\left(\frac{\pi\mathcal{N}}{5}\right)^{1/2}$

$$n_2 = \frac{1}{3}\left(\frac{\pi\mathcal{N}}{5}\right)^{1/2} \delta y C^{-1} \int_{z_s}^{z_t} r(z) z^{-2} \mathrm{d}z$$

before magnetic charge of PBH is big enough, monopoles and antimonopoles are captured at the same rate. So the residual magnetic charge of PBHs can be treated as a one-dimensional random walk.

$$\chi_{\rm gc} = \chi\sqrt{n_2}$$

Initial magnetic charge of PBH

At $T = T_b < T_c$ (at which PBHs are forming and monopoles have been produced), the expected number of monopoles (or antimonopoles) per horizon volume is

$$\langle N_{\rm col}\rangle \simeq \frac{4\pi}{3} n_M(T_b) H_{\rm form}^{-3}$$

$$\begin{cases} n_M(T_b) = r(T_b)s(T_b) \\ H_{\rm form} = K\dfrac{T_b^2}{M_{\rm Pl}} \\ T_b = \left(\dfrac{\gamma}{2K}\right)^{1/2}(y)^{-1/2} M_{\rm Pl} \end{cases} \quad \Rightarrow \quad \langle N_{\rm col}\rangle \simeq \frac{4\pi}{3} r(z_b) K_2 K^{-3/2}\left(\frac{\gamma}{2}\right)^{-3/2} y^{3/2}$$

Initial magnetic charge of PBH

For a monochromatic PBH mass function, between T_c and T_b only monopole annihilation is effective, then the yield follows

$$r_2 = \left[\frac{1}{r_1} + \frac{\Delta}{T_c}\left(\frac{1}{z_2} - \frac{1}{z_1}\right)\right]^{-1}$$

let r_i the initial yield at T_c. $\dfrac{\Delta}{T_c} = \dfrac{\bar{\Phi}}{r_{\rm cr}}$

$$r(z_b) = \left[\frac{1}{r_i} + \frac{\bar{\Phi}}{r_{\rm cr}}\left(\frac{1}{z_b} - 1\right)\right]^{-1}$$

the initial magnetic charge of PBHs

$$\chi_{\rm col} = \chi\sqrt{2\left\langle N_{\rm col}\right\rangle}$$

Thank you!