SF modelling of gravitainoal capture by PBHs

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Modelling Objects

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Promodial Black Holes

- mass: m_{bh}
- Schwarzschild radius: $R_{bh} = \dfrac{2m_{bh}}{M_{Pl}^2}$
- number density: n_{bh}

Magnetic monopoles

- velocity: v_M
- number density: n_M

Assumptions

- **random walk model**:

$$v_M = \frac{v_T}{\sqrt{N}}$$

  N is the number of steps in the random walk and v_T is the thermal velocity

- **non-relativistic approximation** $(v_M \ll 1)$ giving the gravitainoal capture cross section:

$$\sigma_g = \frac{4\pi R_{bh}^2}{v_M^2}$$

  Justification: "Monopoles move with non-relativistic velocities $v_M \ll c$ in the early universe, since they are magnetically charged and their velocity is damped due to interaction with surrounding plasma."

Evolution Equation of n_M in SF Modelling

Evolution Equation of n_M in SF Modelling

- Evolution equation of n_M without considering the gravitational capture by PBHs:

$$\dot{n}_M = -Dn_M^2 - 3\frac{\dot{a}}{a}n_M$$

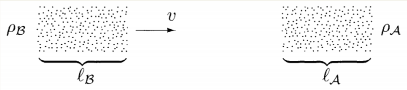
- Adding a new term for gravitational capture

$$\dot{n}_M = -Dn_M^2 - F_{\mathrm{SF}}n_M - 3\frac{\dot{a}}{a}n_M$$

$$F_{SF} \equiv n_{bh}\sigma_g v_M$$

Cross section σ_g and number density

- 反应截面的另一种定义方式（参考M. Peskin & D. Schroeder 量子场论）



$$\sigma \equiv \frac{\text{Number of scattering events}}{\rho_{\mathcal{A}}\, \ell_{\mathcal{A}}\, \rho_{\mathcal{B}}\, \ell_{\mathcal{B}}\, A}.$$

Number of particles captured within $\mathrm{d}t$: $-\mathrm{d}n_{gc} \cdot d_H^3$

$$\sigma_g = \frac{-\mathrm{d}n_{gc} \cdot d_H^3}{n_{bh} d_H n_M (v_M \cdot \mathrm{d}t)\, d_H^2} \quad \Rightarrow \quad \dot{n}_{gc} = \frac{\mathrm{d}n_{gc}}{\mathrm{d}t} = -n_{bh}\sigma_g v_M n_M = -F_{\mathrm{SF}} n_M$$

Cross section σ_g and Monopoles Velocity v_M

Equation of monopole motion

- Schwarzschild metric: spherically symmetric vaccum solution of the Einstein field equations under Schwarzschild coordinates (t, r, θ, ϕ)
- All the orbits are planar
- consider the equatorial plane $\theta = \frac{\pi}{2}$, the geodesics of the particle and light exterior to a black hole are described by

$$\mathrm{d}s^2 = -g\mathrm{d}t^2 + g^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2, \quad g = 1 - \frac{R_{bh}}{r}$$

Particle motion equation around a black hole

$$ds^2 = -g\mathrm{d}t^2 + g^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2, \quad g = 1 - \frac{R_{bh}}{r}$$

- Divide $\mathrm{d}\lambda$ $(= \frac{\mathrm{d}\tau}{m}$ for massive particles) on both sides of the above equation
- Use on-shell condition $p^\mu p_\mu = p^2 = -m^2$ with signature convention $(-, +, +, +)$

$$-g\dot{t}^2 + g^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \frac{\mathrm{d}s^2}{\mathrm{d}\lambda^2} = p^2 = -m^2$$

Killing vectors and conserved quantities

$$-g\dot{t}^2 + g^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -m^2$$

- The spacetime have two Killing vectors $\boldsymbol{\xi}_{(t)} = \partial_t, \boldsymbol{\xi}_{(\phi)} = \partial_\phi$, correspondingly two conserved quantities: energy E and angular momentum L

$$E \equiv -\xi^\mu_{(t)} p_\mu = g\dot{t}$$

$$L \equiv \xi^\mu_{(\phi)} p_\mu = r^2\dot{\phi}$$

$$-E^2 + \dot{r}^2 + g\frac{L^2}{r^2} = -gm^2$$

Effective potential

$$-E^2 + \dot{r}^2 + g\frac{L^2}{r^2} = -gm^2$$

- Introducing three dimensionless quantities:

$$\zeta = \frac{R_{bh}}{r}, \quad \ell = \frac{L}{mR_{bh}}, \quad \mathcal{E} = \frac{E}{m}$$

$$\mathcal{E}^2 = \frac{\dot{r}^2}{m^2} + g(1 + \ell^2\zeta^2)$$

- Effective potential energy U is following

$$U = g(1 + \ell^2\zeta^2) = (1 - \zeta)(1 + \ell^2\zeta^2)$$

Effective potential for a given ℓ

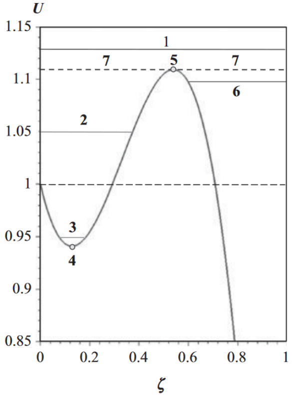
$$U = (1-\zeta)(1+\ell^2\zeta^2)$$

- For a given ℓ, U has a maximum value U_+ and a minimum value U_-

$$U_+(\ell) = U(\zeta_+, \ell): \quad \frac{\partial U}{\partial \zeta} = 0, \quad \frac{\partial^2 U}{\partial \zeta^2} < 0$$

$$\Rightarrow \quad U_+(\ell) = \frac{2[\ell(\ell^2+9)+(\ell^2-3)^{3/2}]}{27\ell}$$

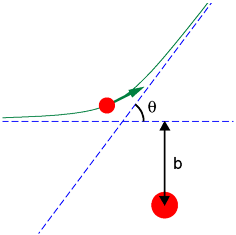
- When $\mathcal{E}^2 > U_+$, monopoles captured by gravity.



Effective potential U for $\ell = 2.2$

Impact parameter b

$$\sigma = \pi b^2$$



$$\begin{cases} U_+(\ell) = \dfrac{2[\ell(\ell^2+9)+(\ell^2-3)^{3/2}]}{27\ell} = \mathcal{E}^2 \\ L = bp = mR_{bh}\ell \\ E^2 = (m\mathcal{E})^2 = p^2 + m^2 \end{cases}$$

$$\Downarrow$$

- Non-relativistically approximation: $p \ll m \to \mathcal{E}^2 = 1$, $p = mv$

$$U_+(\ell) = \frac{2[\ell(\ell^2+9)+(\ell^2-3)^{3/2}]}{27\ell} = 1 \quad \Rightarrow \quad \ell = 2$$

$$b = \frac{2R_{bh}}{v} \quad \sigma_g = \pi b^2 = \frac{4\pi R_{bh}^2}{v_M^2}$$

Monopole Velocity v_M and Thermal Velocity v_T

Thank you!