LINEAR MOTION

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MOTION IN ONE DIMENSION

- Here we’ll discuss velocity, displacement and acceleration in terms of Cartesian components as they all are vector quantities.
- Considering motion only in the direction of a single component, for example, the x direction, that is, motion in a straight line.
- If we start $(x_0 = 0)$ an object moving in the x direction when it starts from $x_o = 0$ point, we may write

$$\overrightarrow{v_x} = \frac{\vec{x} - \overrightarrow{x_0}}{t - t_0} \; ; (x_0 = 0)$$

$$\overline{v}_x = \frac{x - 0}{t - 0}$$

$$\boxed{x = \overline{v}_x t} \longrightarrow \textbf{Eq. 1.1}$$

Equation 1.1 results from the definition of average velocity; **thus, it holds in all cases whether the acceleration is constant.**

DERIVATION OF THREE EQUATIONS OF MOTION

$$v = v_0 + at$$

$$v^2 - v_0^2 = 2ax$$

$$x = v_0 t + \frac{1}{2} a t^2$$

1^{ST} EQUATION OF MOTION

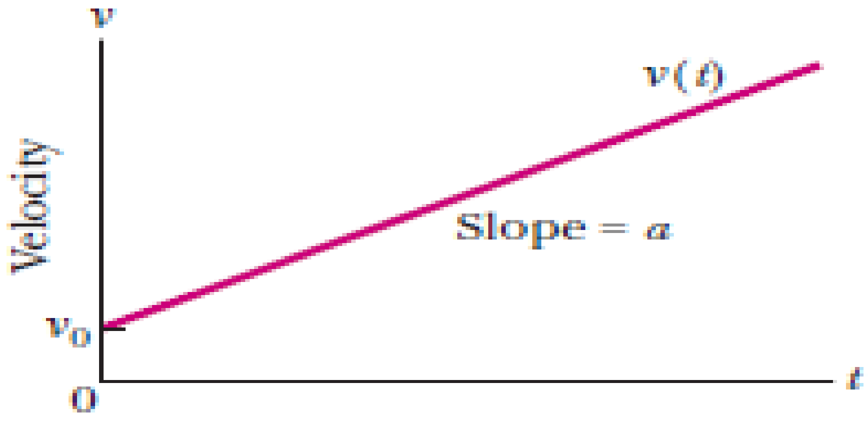
- The acceleration is defined as the rate of change of the velocity.

- If the acceleration is constant, the change in the velocity during the first, second, third, and all succeeding seconds of the motion will be the same and equal to the acceleration $\vec{a}$
- if the motion lasts t seconds, the change in the velocity $\Delta v = v - v_0 = at,$ where v is the final velocity and v_0 is the initial velocity.
- We can rewrite this result as

$$\boxed{v = v_0 + at} \longrightarrow \textbf{Eq.1.2}$$

VELOCITY – TIME GRAPH

- If we plot equation 1.2 on graph we will obtain a straight line, as indicated in the Figure below.
- The slope of this line is the constant acceleration *a.*



2^{ND} EQUATION OF MOTION

- Another important relation that we can have when

the velocity increases at a constant rate the average velocity is one half the sum of the initial velocity v_o and the final velocity namely, v

$$\overline{v} = \frac{v + v_0}{2} \longrightarrow \textbf{Eq. 1.3}$$

- Previously we have,

$$x = \overline{v}\ t \longrightarrow \textbf{Eq. 1.1}$$

- On substituting value of $\overline{v}$ in eq 1.1, we get

$$x = \frac{v + v_0}{2} t$$

2^{ND} EQUATION OF MOTION

$$x = \frac{v + v_0}{2} t \longrightarrow \textbf{Eq. 1.4}$$

If we substitute the value of "v" from 1st equation of motion in eq.1.4 then we will have

$$x = \frac{v_0 + at + v_0}{2} t$$

Or we can have ,

$$\boxed{x = v_0 t + \frac{1}{2} a t^2}$$

3RD EQUATION OF MOTION

- Considering equation 1.4,

$$x = \frac{v + v_0}{2} t \longrightarrow \textbf{Eq. 1.4}$$

- We can find out the value of "t" from 1st equation of motion that is,

$$t = \frac{v - v_0}{a}$$

- On substituting the above value of "t" in eq. 1.4 we will have the following results

$$x = \frac{(v + v_0)}{2} \frac{(v - v_0)}{a}$$

$$\boxed{v^2 - v_0^2 = 2ax}$$

DERIVATIONS BY INTEGRATION

We may derive these equations more formally by integration.

By definition

$$a = \frac{dv}{dt}$$

Rearranging terms and integrating, we write

$$\int_{v_0}^{v} dv = \int_0^t a\, dt$$

acceleration is taken as constant, so a can be taken out of the integral and we write

DERIVATIONS BY INTEGRATION

$$\int_{v_0}^{v} dv = a \int_0^t dt$$

This integrates to

$$v - v_0 = at$$

and

$$v = v_0 + at$$

1st equation of motion

DERIVATIONS BY INTEGRATION

- From definition we know, $v = \frac{dx}{dt}$
- Rearranging terms $$\int_{x_0}^{x} dx = \int_0^t v\, dt$$

$$\int_{x_0}^{x} dx = \int_0^t (v_0 + at) dt$$

$$\int_{x_0}^{x} dx = v_0 \int_0^t dt + a \int_0^t t\, dt$$

DERIVATIONS BY INTEGRATION

- After applying limits, we will have the following result,

$$\boxed{x - x_0 = v_0 t + \frac{1}{2} a t^2}$$

- Above is the 2^{nd} Equation of motion

Note that in this formulation we have not required that $x = 0$ at $t = 0$ as in the previous algebraic derivations.

DERIVATIONS BY INTEGRATION

For 3rd equation of motion

We may use the chain rule to write

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\int_{v_0}^{v} v\,dv = a\int_{x_0}^{x} dx$$

$$\frac{v^2}{2} - \frac{v_0^2}{2} = a(x - x_0)$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

RESULTS BY INTEGRATION

- Fol egration,

$$v = v_0 + at$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

- All the equations that we have derived for motion are in the x-direction. Similar equations can simply be written for motion in the y and z directions when the components of the acceleration in these directions are also constant.

CONSTANT ACCELERATION

$$v = v_0 + at,$$

$$x - x_0 = v_0 t + \tfrac{1}{2}at^2,$$

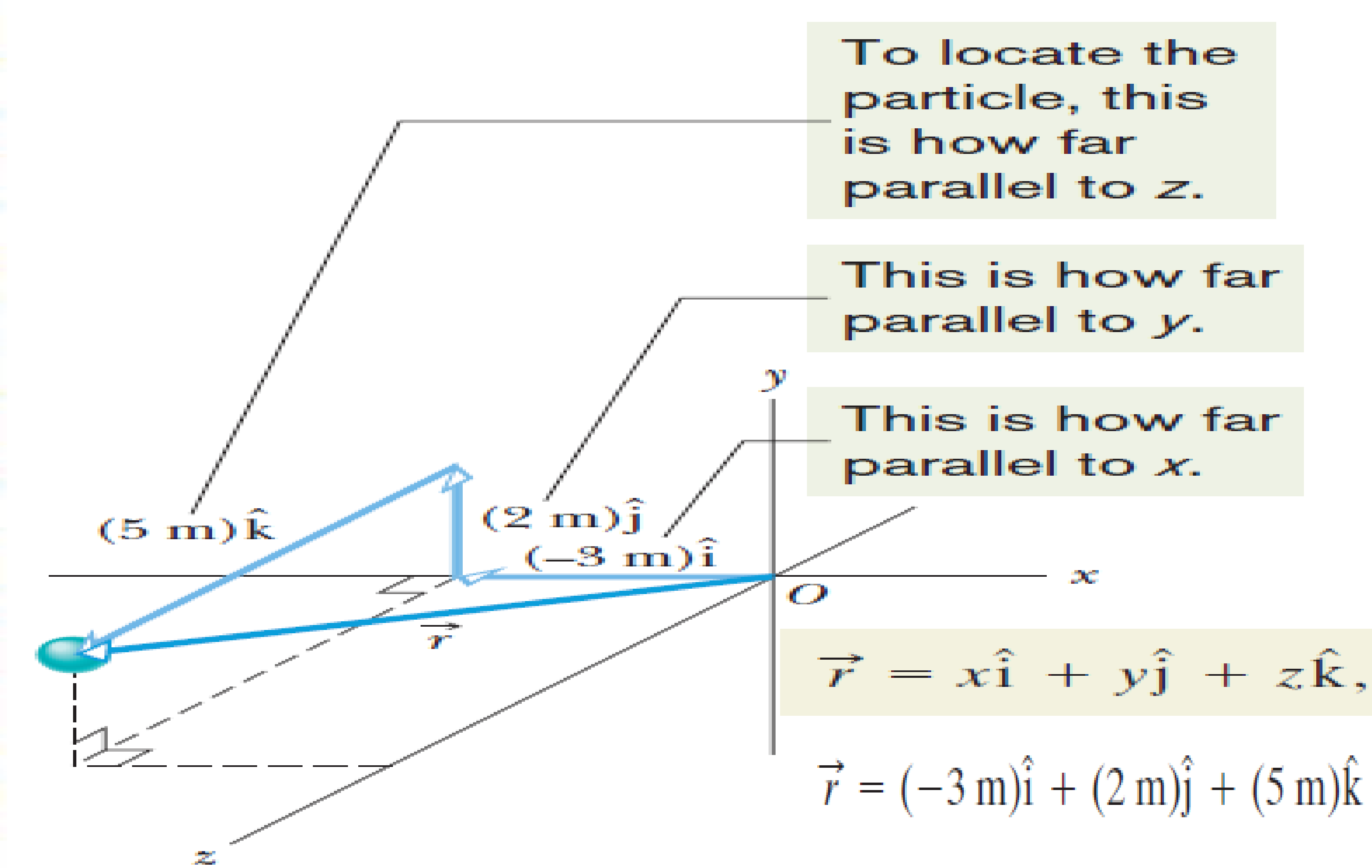
$$v^2 = v_0^2 + 2a(x - x_0),$$

$$x - x_0 = \tfrac{1}{2}(v_0 + v)t,$$

$$x - x_0 = vt - \tfrac{1}{2}at^2.$$

These are *not* valid when the acceleration is not constant.

POSITION OF A POINT IN SPACE



DISPLACEMENT VECTOR

As a particle moves, its position vector changes in such a way that the vector always extends to the particle from the reference point (the origin). If the position vector changes—say, from $\vec{r}_1$ to $\vec{r}_2$ during a certain time interval—then the particle's **displacement** $\Delta\vec{r}$ during that time interval is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$

Using the unit-vector notation we can rewrite this displacement as

$$\Delta\vec{r} = (x_2\hat{\mathrm{i}} + y_2\hat{\mathrm{j}} + z_2\hat{\mathrm{k}}) - (x_1\hat{\mathrm{i}} + y_1\hat{\mathrm{j}} + z_1\hat{\mathrm{k}})$$

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}},$$

$$\Delta\vec{r} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}.$$

AVG. VELOCITY IN 3-DIMENSIONS

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t}.$$

$$\Delta \vec{r} = \Delta x\hat{\text{i}} + \Delta y\hat{\text{j}} + \Delta z\hat{\text{k}}.$$

Put in the formula of average velocity we'll get

$$\vec{v}_{\text{avg}} = \frac{\Delta x\hat{\text{i}} + \Delta y\hat{\text{j}} + \Delta z\hat{\text{k}}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\text{i}} + \frac{\Delta y}{\Delta t}\hat{\text{j}} + \frac{\Delta z}{\Delta t}\hat{\text{k}}.$$

INSTANTANEOUS VELOCITY IN 3-D

When we speak of the **velocity** of a particle, we usually mean the particle's **instantaneous velocity** $\vec{v}$ at some instant. This $\vec{v}$ is the value that $\vec{v}_{avg}$ approaches in the limit as we shrink the time interval Δt to 0 about that instant. Using the language of calculus, we may write $\vec{v}$ as the derivative

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

- Substitute the value of unit vector

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}},$$

$$\vec{v} = \frac{d}{dt}(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}) = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}}.$$

INSTANTANEOUS VELOCITY IN 3-D

- Simply $$\vec{v} = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}}.$$

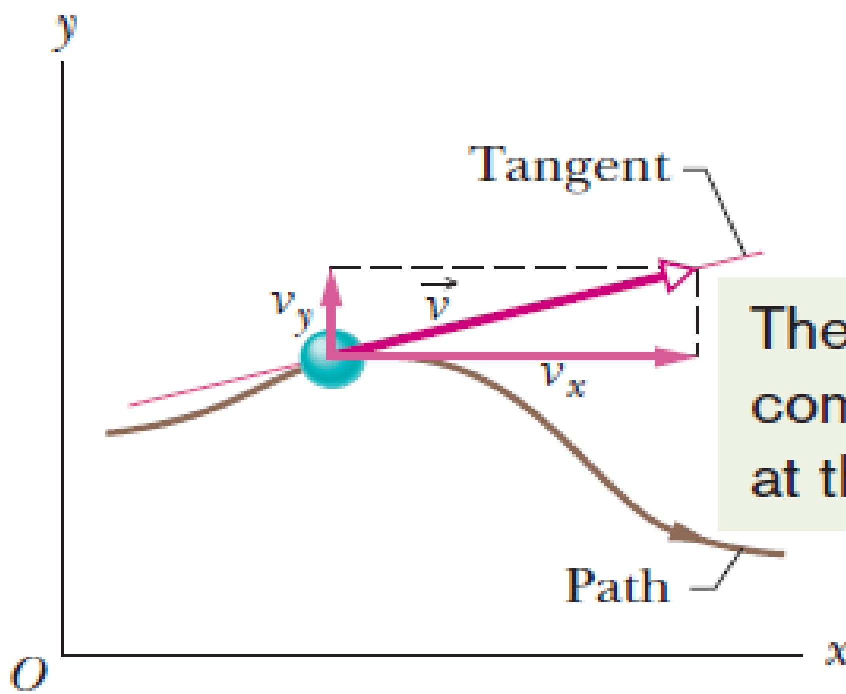
$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}},$$

where the scalar components of $\vec{v}$ are

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad \text{and} \quad v_z = \frac{dz}{dt}.$$

INSTANTANEOUS VELOCITY IN 3-D

The velocity vector is always tangent to the path.



These are the *x* and *y* components of the vector at this instant.

SOLVE FOR INSTANTANEOUS ACCELERATION IN 3-D

- Class task: Find out the scalar components of acceleration.

instantaneous acceleration

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}},$$

$$\vec{a} = \frac{d}{dt}(v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}})$$

$$= \frac{dv_x}{dt}\hat{\mathrm{i}} + \frac{dv_y}{dt}\hat{\mathrm{j}} + \frac{dv_z}{dt}\hat{\mathrm{k}}$$

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}},$$

where the scalar components of $\vec{a}$ are

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad \text{and} \quad a_z = \frac{dv_z}{dt}.$$