SIMPLE HARMONIC MOTION (CONTD..)

Dr Muhammad Adeel

Loudspeaker.

The cone of a loudspeaker oscillates in SHM at a frequency of 262 Hz. The amplitude at the center of the cone is $A = 1.5 \times 10^{-4}$ m, and at $t = 0$, $x = A$. (a) What equation describes the motion of the center of the cone? (b) What are the velocity and acceleration as a function of time? (c) What is the position of the cone at $t = 1.00$ min $(= 1.00 \times 10^{-3}$ s$)$?

Energy conservation in Simple harmonic motion:

- if the friction effect are neglected, total mechanical energy of vibrating mass spring system remains constant
- The velocity and position of the vibrating body are continually changing
- The kinetic and potential energies also change, but their sum must have the same values at any instant.
- By hook's law
- F=-kx
- $W = \int f\, dx$
- U=-W
- U=$- \int f\, dx$
- $U = - \int -kx\, dx$
- $U = k \int x\, dx$
- $U = k\frac{x^2}{2}$

- Putting value of displacement

$$U(t) = \tfrac{1}{2}k\left(x_m \cos(\omega t + \phi)\right)^2$$
$$U(t) = \tfrac{1}{2}k\, x_m^2 \cos^2(\omega t + \phi)$$

- Kinetic energy is given by $K.E = \frac{1}{2}mv^2$

- Kinetic energy is maximum if x=0 when the mass is at equilibrium position.

$$K(t) = \tfrac{1}{2}m\left(-\omega\, x_m \sin(\omega t + \phi)\right)^2$$

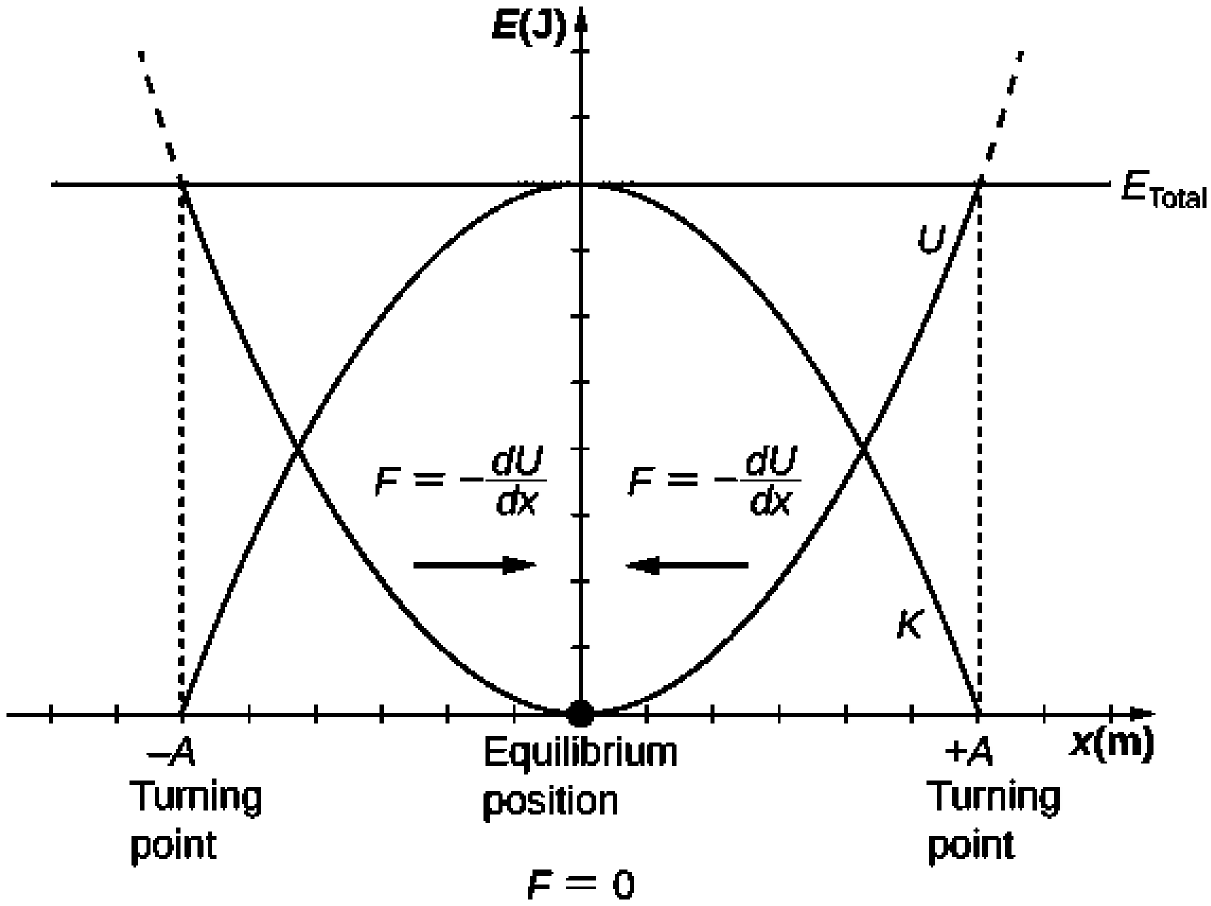
$$k = m\omega^2 \qquad\qquad K(t) = \tfrac{1}{2}k\, x_m^2 \sin^2(\omega t + \phi)$$

- The energy is partly P.E and partly K.E.

$$E = P.E + K.E$$

$$\therefore\ E = \tfrac{1}{2}k\, x_m^2 = constant$$

In a SHM, discuss how kinetic energy “K” and potential energy “U” change with respect to position “x”, only with help of graph between energy and position. Does the total energy change if the mass is doubled but the amplitude is not changed?



Many tall building have mass dampers, which are anti-sway devices to prevent them from oscillating in a wind. The device might be a block oscillating at the end of a spring and on a lubricated track. If the building sways, say eastward, the block also moves eastward but delayed enough so that when it finally moves, the building is then moving back westward. Thus, the motion of the oscillator is out of step with the motion of the building.

Suppose that the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring-block system?

KEY IDEA The mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position $x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-12 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$\begin{aligned} k &= m\omega^2 = m(2\pi f)^2 \\ &= (2.72 \times 10^5\ \text{kg})(2\pi)^2(10.0\ \text{Hz})^2 \\ &= 1.073 \times 10^9\ \text{N/m}. \end{aligned}$$

We can now evaluate E as

$$\begin{aligned} E &= K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \\ &= 0 + \tfrac{1}{2}(1.073 \times 10^9\ \text{N/m})(0.20\ \text{m})^2 \\ &= 2.147 \times 10^7\ \text{J} \approx 2.1 \times 10^7\ \text{J}. \qquad \text{(Answer)} \end{aligned}$$

(b) What is the block's speed as it passes through the equilibrium point?

Calculations: We want the speed at $x = 0$, where the potential energy is $U = \frac{1}{2}kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$2.147 \times 10^7\ \text{J} = \tfrac{1}{2}(2.72 \times 10^5\ \text{kg})v^2 + 0,$$

or

$$v = 12.6\ \text{m/s}. \qquad \text{(Answer)}$$

Because E is entirely kinetic energy, this is the maximum speed v_m.

Example:

In an engine, a piston oscillates with simple harmonic motion so that its position varies according to the expression:

$$x = (5.00\ cm)Cos(2t + \pi/6)$$

Where, x is in centimeters and t is in seconds. At t =0, find

(a) the position of the piston,

(b) its velocity

(a) The equation for the piston's position is given as

$$x = (5.00\ \mathrm{cm})\cos\left(2t + \frac{\pi}{6}\right)$$

At $t = 0$,

$$x = (5.00\ \mathrm{cm})\cos\left(\frac{\pi}{6}\right) = \boxed{4.33\ \mathrm{cm}}$$

(b) Differentiating the equation for position with respect to time gives us the piston's velocity:

$$v = \frac{dx}{dt} = -(10.0\ \mathrm{cm/s})\sin\left(2t + \frac{\pi}{6}\right)$$

At $t = 0, v = \boxed{-5.00\ \mathrm{cm/s}}$

$$x = (5.00\ cm)Cos(2t + \pi/6)$$

(iii) its acceleration.

(iv) the period and amplitude of the motion.

(c) Differentiating again gives its acceleration:

$$a = \frac{dv}{dt} = -(20.0\ \mathrm{cm/s^2})\cos\left(2t + \frac{\pi}{6}\right)$$

At $t = 0, a = \boxed{-17.3\ \mathrm{cm/s^2}}$

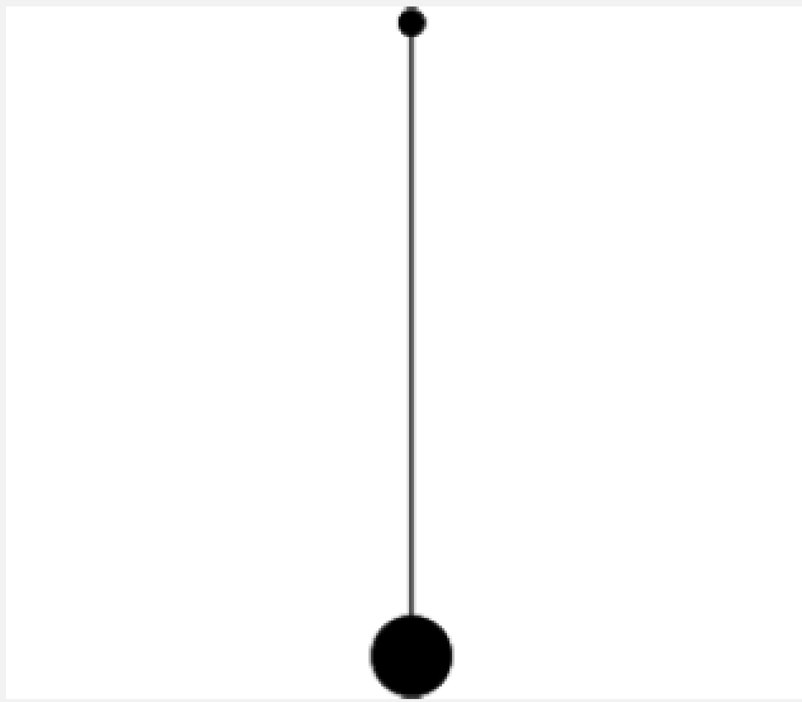
(d) The period of motion is

$$T = \frac{2\pi}{\omega} - \frac{2\pi}{2} = \boxed{3.14\ \mathrm{s}}$$

(e) We read the amplitude directly from the equation for x:

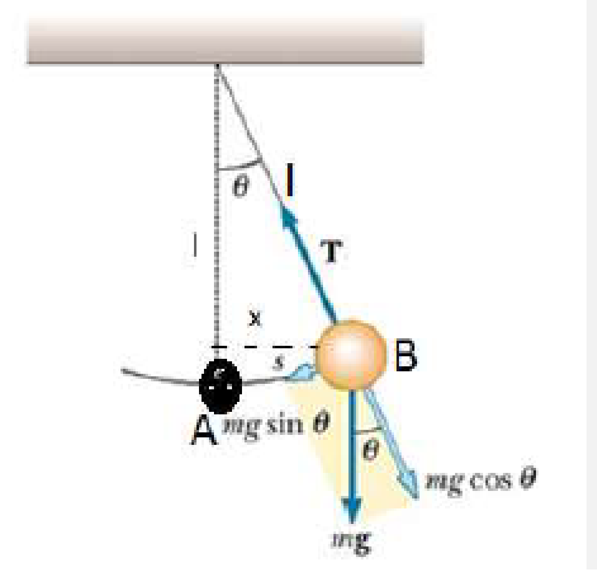
$$A = \boxed{5.00\ \mathrm{cm}}$$

Simple pendulum:



Simple pendulum:

- A simple pendulum is idealized model consist of a point mass suspended by an inextensible string of length *l* fixed
- When pulled to one side of its equilibrium position A to the position B through a small angle θ and released, it starts oscillating to and fro over the small.



Simple pendulum performs SHM:

- Condition for SHM is that restoring force F should be directly proportional to the displacement oppositely directed. The path of the bob is not straight line, but the arc of the circle of radius l

- Let T is the tension in the spring. When the particle is at point B two forces are acting on it:

1. mg, the weight of the point bob acting vertically downward.
2. T, the tension along the string.

- The weight mg can be resolved into two rectangular components.

1. Component of weight mg along the spring=$mg\cos\theta$
2. Component of weight mg perpendicular to the string=$mg\sin\theta$

- Since there is no motion of bob along the string, so the component mg cosθ must be equal to tension in the string T

$$T=mg\cos\theta$$

- Component mg sinθ is responsible for the motion of the bob towards the mean position. Thus, the restoring force F is;

$$F=-mg\sin\theta \qquad ....(iv)$$

$$\text{So } \sin\theta=\frac{x}{l}$$

$$F=-mg\frac{x}{l}$$

- By 2nd law of motion

$F=ma$ ....(v)

- Comparing (iv) and (v)

$$ma=-\,mg\frac{x}{l}$$

$$a=-\,\frac{gx}{l}$$

$$a=-(\text{constant})x$$

$$a\propto -x$$

Expression for time period:

We know that

$a = -x\omega^2 \qquad \Rightarrow \quad a = -\frac{gx}{l}$

$-x\omega^2 = -\frac{gx}{l} \qquad \Rightarrow \omega = \sqrt{g/l}$

$T = \frac{2\pi}{\omega} \qquad \Rightarrow T = 2\pi\sqrt{\frac{l}{g}}$

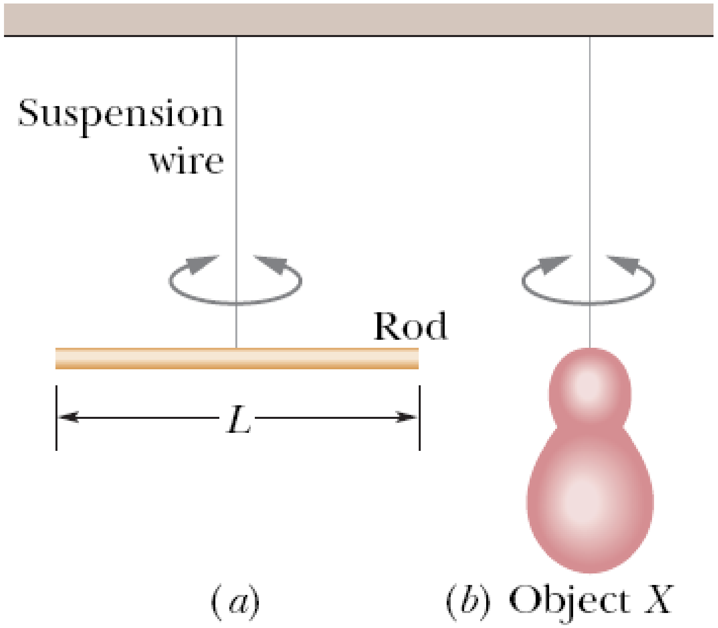
Frequency:

$F = {}^{1}/_{T}$

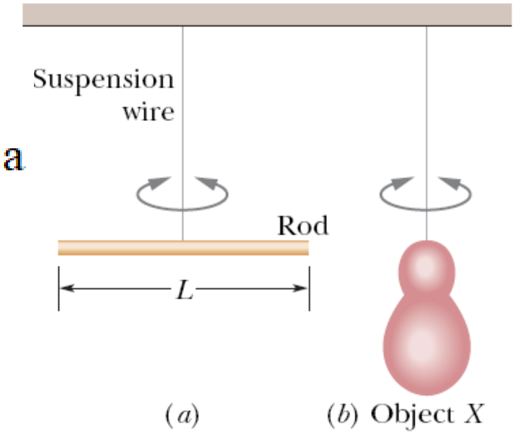
$F = {}^{1}/_{2\pi} \times \sqrt{g/l}$

Angular SHM

Figure *a shows a thin rod whose length L is 12.4 cm* and whose mass *m is 135 g, suspended at its midpoint* from a long wire. Its period T_a *of angular SHM is measured* to be 2.53 s. An irregularly shaped object, which we call object *X, is then hung from the same wire, as in Fig. b, and its period* T_b *is found to be 4.76 s. What is the* rotational inertia of object *X about its suspension axis?*



Angular SHM:



Answer: The rotational inertia of either the rod or object X *is related to the measured period.* The rotational inertia of a thin rod about a perpendicular axis through its midpoint is given as 1/12 mL^2. *Thus, we have, for the rod in Fig. a,*

$$I_a = \tfrac{1}{12}mL^2 = (\tfrac{1}{12})(0.135\text{ kg})(0.124\text{ m})^2$$
$$= 1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2.$$

Now let us write the periods, once for the rod and once for object *X:*

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}.$$

The constant κ, which is a property of the wire, is the same for both figures; only the periods and the rotational inertias differ.

Let us square each of these equations, divide the second by the first, and solve the resulting equation for I_b. *The result is*

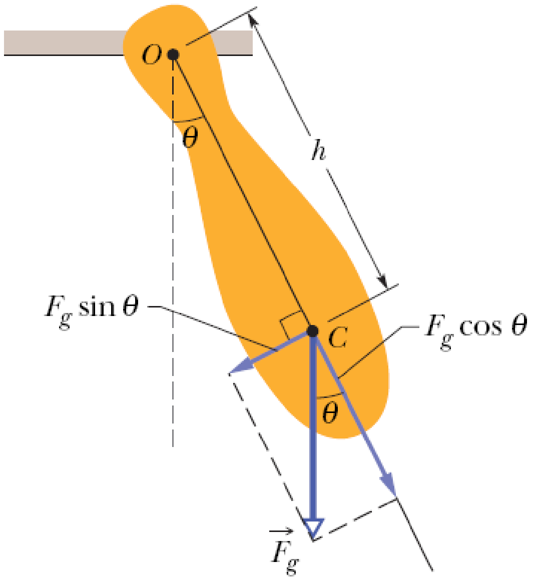
$$I_b = I_a\frac{T_b^2}{T_a^2} = (1.73 \times 10^{-4}\text{ kg}\cdot\text{m}^2)\,\frac{(4.76\text{ s})^2}{(2.53\text{ s})^2}$$
$$= 6.12 \times 10^{-4}\text{ kg}\cdot\text{m}^2. \qquad \text{(Answer)}$$

Pendulums

A physical pendulum can have a complicated distribution of mass. If the center of mass, C, is at h from the pivot point (figure), then for *small angular amplitudes*, the motion is simple harmonic.

The period, T, is:

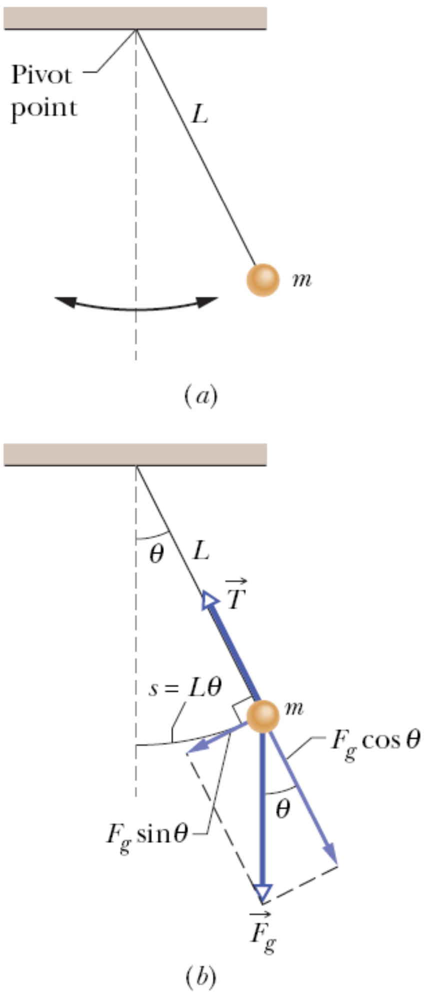
$$T = 2\pi\sqrt{\frac{I}{mgh}}$$



Here, I is the rotational inertia of the pendulum about O.

Pendulums

In a *simple pendulum*, a particle of mass m is suspended from one end of an un-stretchable massless string of length L that is fixed at the other end.



The restoring torque acting on the mass when its angular displacement is θ, is:

$$\tau = -L(F_g \sin\theta) = I\alpha$$

α is the angular acceleration of the mass. Finally,

$$\alpha = -\frac{mgL}{I}\theta, and$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

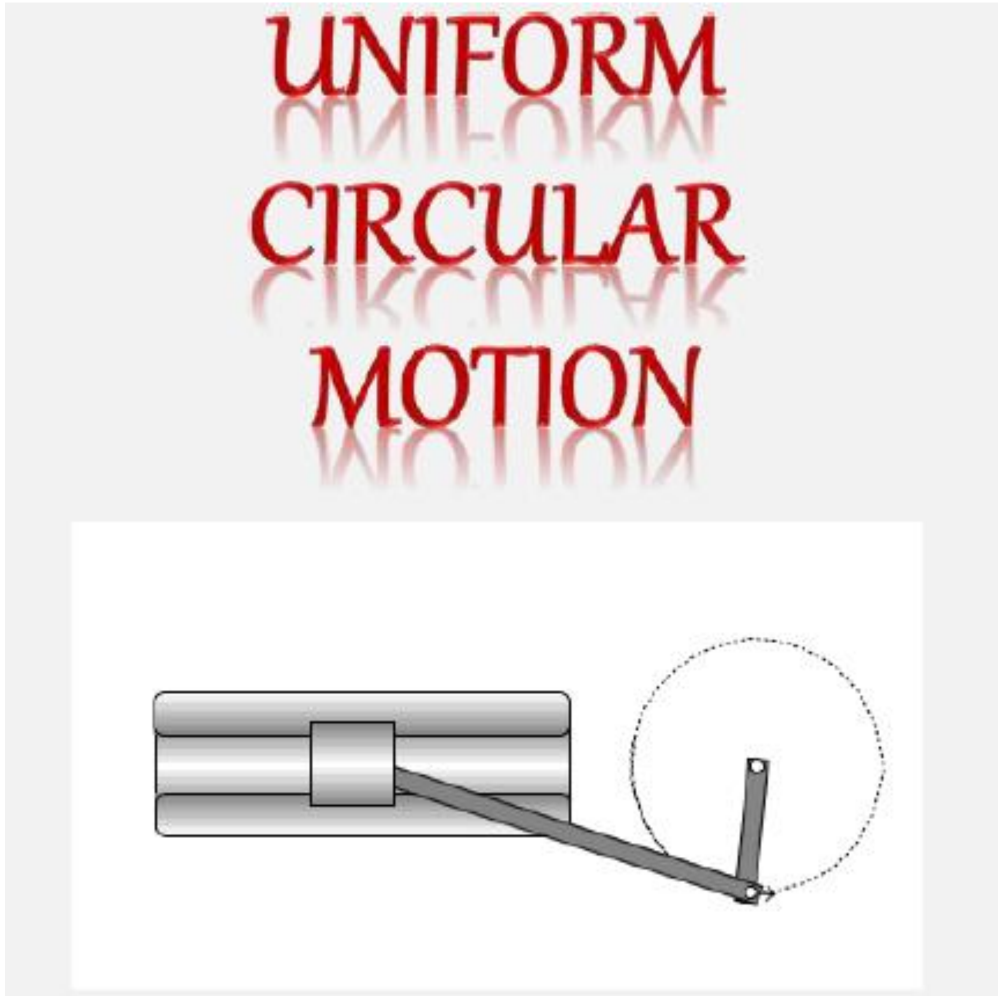
This is true for *small angular displacements*, θ.

Pendulums

In the **small-angle approximation** we can assume that $\theta << 1$ and use the approximation $\sin\theta \cong \theta$. Let us investigate up to what angle θ is the approximation reasonably accurate?

θ (degrees)	θ (radians)	$\sin\theta$
5	0.087	0.087
10	0.174	0.174
15	0.262	0.259 (1% off)
20	0.349	0.342 (2% off)

Conclusion: If we keep $\theta < 10°$ we make less than 1 % error.


UNIFORM
CIRCULAR
MOTION

Relation between uniform circular motion and SHM

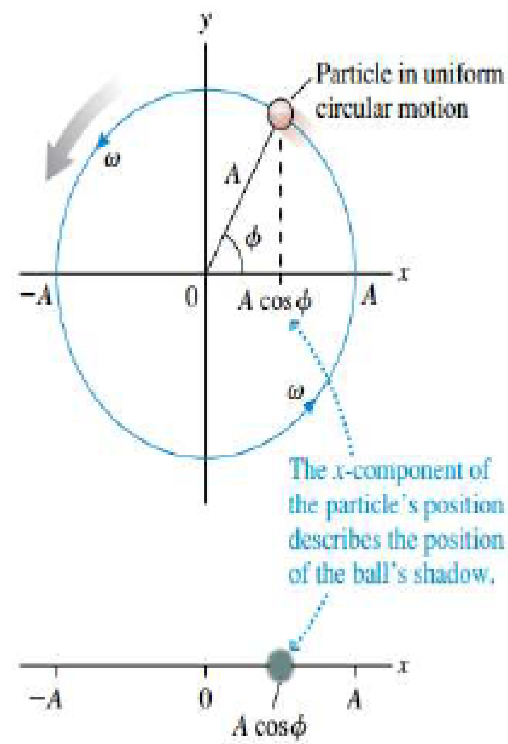
An object in simple harmonic motion has the same motion as of an object in uniform circular motion:

- Consider the particle in uniform circular motion with radius A and angle ϕ
- x= A cos ϕ
- Particle's angular velocity, in rad/s, is
- $\frac{d\varphi}{dt}$=ω
- This is the rate at which the angle ϕ is increasing.

If the particle starts from $\emptyset_0$= 0 at t = 0, its angle at a later time t is simply

- φ =ωt

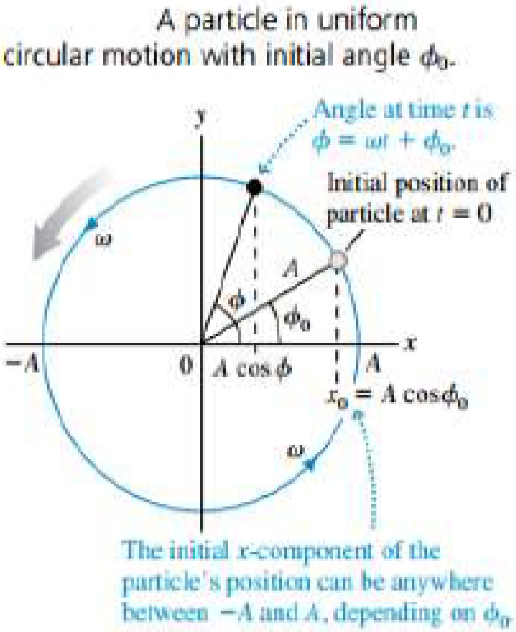
A particle in uniform circular motion with radius A and angular velocity ω.



- As φ increases, the particle's x-component is $x(t) = A\cos\omega t$
- The particle in started at $\emptyset_0 = 0$. fig shows a more general situation in which the initial angle $\emptyset_0$ can have any value. The angle at a later time t is then

$$\varphi = \omega t + \emptyset_0$$

- $v(t) = -\omega A\sin(\omega t + \emptyset_0) =$
$v(t) = -v_{max}\sin(\omega t + \emptyset_0)$



Example

A particle rotates counterclockwise in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. As the particle has an *x* coordinate of 2.00 m and is moving to the right. (a) Determine the *x* coordinate as a function of time.

Solution Because the amplitude of the particle's motion equals the radius of the circle and $\omega = 8.00$ rad/s, we have

$$x = A\cos(\omega t + \phi) = (3.00\text{ m})\cos(8.00t + \phi)$$

We can evaluate ϕ by using the initial condition that $x = 2.00$ m at $t = 0$:

$$2.00\text{ m} = (3.00\text{ m})\cos(0 + \phi)$$

$$\phi = \cos^{-1}\left(\frac{2.00\text{ m}}{3.00\text{ m}}\right)$$

If we were to take our answer as $\phi = 48.2°$, then the coordinate $x = (3.00\text{ m})\cos(8.00t + 48.2°)$ would be decreasing at time $t = 0$ (that is, moving to the left). Because our particle is first moving to the right, we must choose $\phi = -48.2° = -0.841$ rad. The *x* coordinate as a function of time is then

$$x = \boxed{(3.00\text{ m})\cos(8.00t - 0.841)}$$

Note that ϕ in the cosine function must be in radians.

(b) Find the *x* components of the particle's velocity and acceleration at any time *t*.

Solution

$$v_x = \frac{dx}{dt} = (-3.00\text{ m})(8.00\text{ rad/s})\sin(8.00t - 0.841)$$

$$= \boxed{-(24.0\text{ m/s})\sin(8.00t - 0.841)}$$

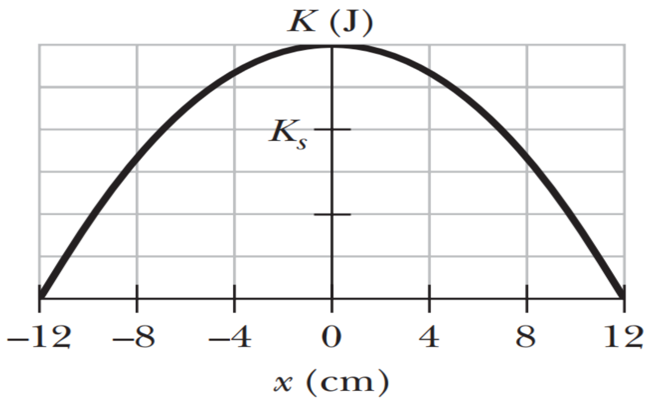
$$a_x = \frac{dv_x}{dt} = (-24.0\text{ m/s})(8.00\text{ rad/s})\cos(8.00t - 0.841)$$

$$= \boxed{-(192\text{ m/s}^2)\cos(8.00t - 0.841)}$$

From these results, we conclude that $v_{max} = 24.0$ m/s and that $a_{max} = 192$ m/s^2. Note that these values also equal the tangential speed ωA and the centripetal acceleration $\omega^2 A$.

Example:

Figure shows the kinetic energy K of a simple harmonic oscillator versus its position x. The vertical axis scale is set by $K_s = 4.0$ J. What is the spring constant?



The total energy in the system is 6.0J

The amplitude is apparently x_m=12cm=0.12m.

The maximum potential energy is equal to 6.0J

Solve for the spring constant

$1/2kx_m^2 = 6.0J$

$\Rightarrow$ $k=8.3\times10^2 N/m$.