ELECTRIC FIELD OF A CONTINUOUS CHARGE DISTRIBUTION

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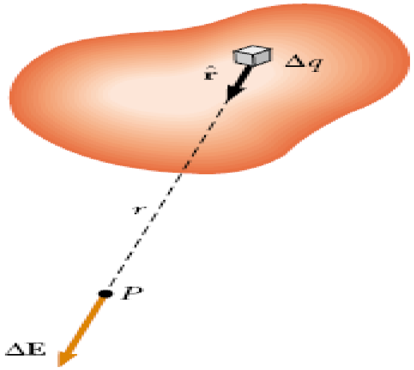
Electric Field of a Continuous Charge Distribution

Electric field is said to exist in a region of space around a charged object, the source charge when another charged object, the test charge enters this electric field, an electric force acts on it.

To evaluate the electric field created by a continuous charge distribution, we use the following procedure:



- we divide the charge distribution into small elements, each of which contains a small charge Δq.

- we use Equation $$\mathbf{E} = k_e \frac{q}{r^2}\,\hat{\mathbf{r}}$$

to calculate the electric field due to one of these elements at a point P.

- Finally, we evaluate the total electric field at P due to the charge distribution by summing the contributions of all the charge elements (that is, by applying the superposition principle).

The total electric field at P due to all elements in the charge distribution is approximately,

$$\mathbf{E} \approx k_e \sum_i \frac{\Delta q_i}{r_i^{\,2}} \hat{\mathbf{r}}_i$$

where the index i refers to the ith element in the distribution.

Because the charge distribution is modeled as continuous, the total field at P in the limit $\Delta q_i \rightarrow 0$ is

$$\mathbf{E} = k_e \lim_{\Delta q_i \rightarrow 0} \sum_i \frac{\Delta q_i}{r_i^{\,2}} \hat{\mathbf{r}}_i = k_e \int \frac{dq}{r^2} \hat{\mathbf{r}}$$

where the integration is over the entire charge distribution. This is a vector operation and must be treated appropriately.

- If a charge Q is uniformly distributed throughout a volume V, the **volume charge density** ρ is defined by

$$\rho \equiv \frac{Q}{V}$$

where ρ has units of coulombs per cubic meter (C/m^3).

- If a charge Q is uniformly distributed on a surface of area A, the **surface charge density** σ (lowercase Greek sigma) is defined by

$$\sigma \equiv \frac{Q}{A}$$

where σ has units of coulombs per square meter (C/m^2).

- If a charge Q is uniformly distributed along a line of length ℓ, the **linear charge density** λ is defined by

$$\lambda \equiv \frac{Q}{\ell}$$

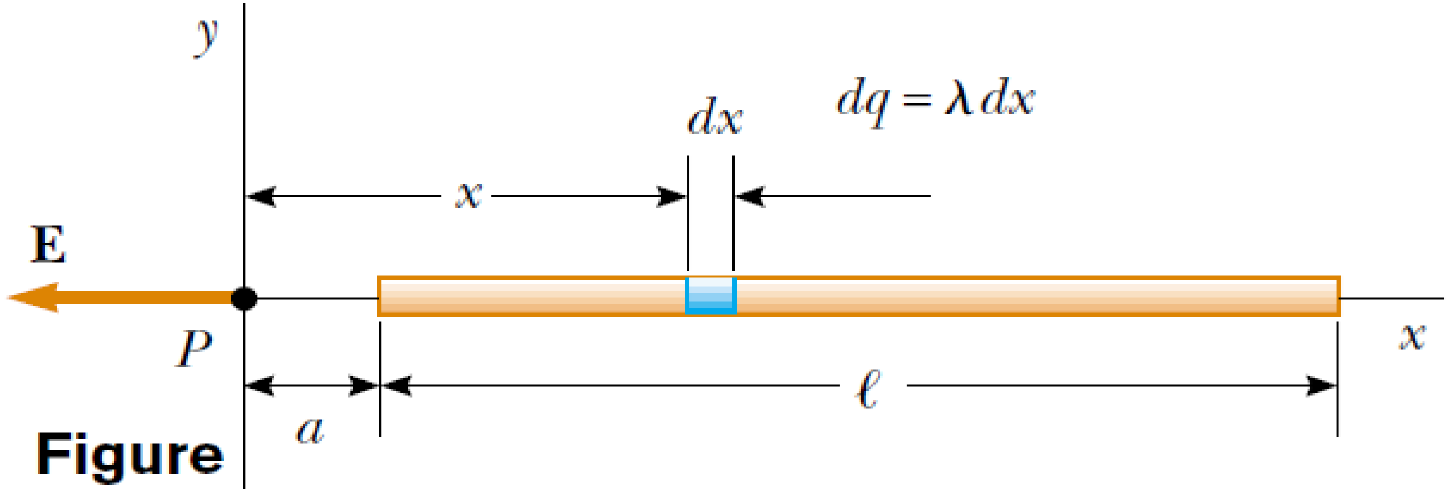
where λ has units of coulombs per meter (C/m).

- If the charge is nonuniformly distributed over a volume, surface, or line, the amounts of charge dq in a small volume, surface, or length element are

$$dq = \rho\, dV \qquad dq = \sigma\, dA \qquad dq = \lambda\, d\ell$$

Electric Field due to Charged Rod

A rod of length ℓ has a uniform positive charge per unit length λ and a total charge Q. Calculate the electric field at a point P that is located along the long axis of the rod and a distance a from one end (Fig.)



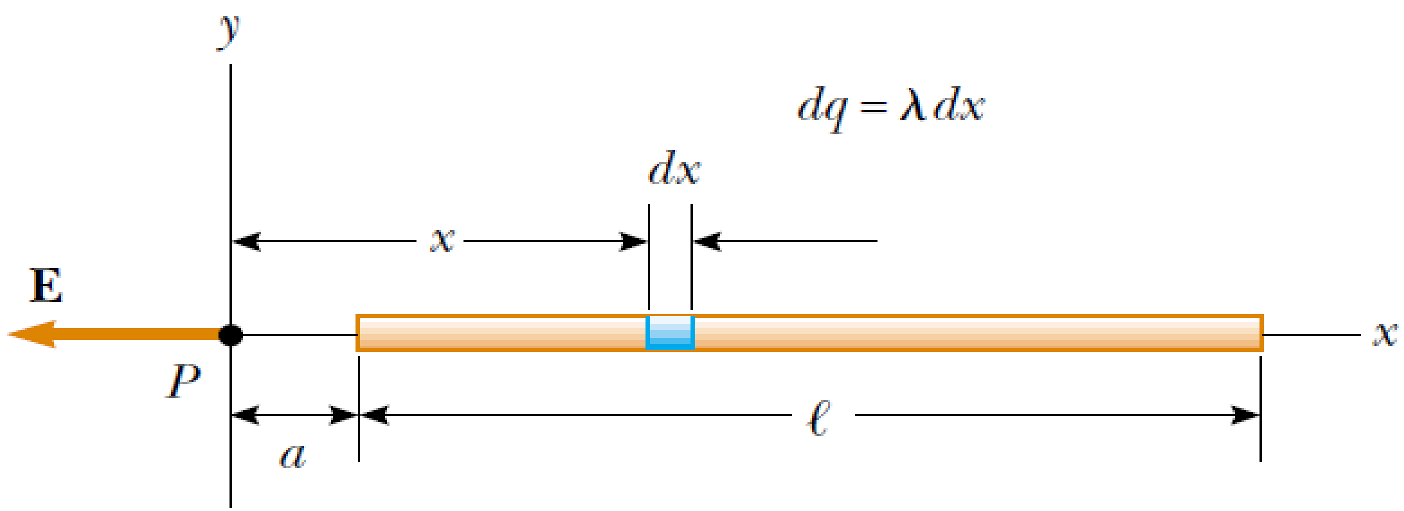
Figure

The electric field at P due to a uniformly charged rod lying along the x axis. The magnitude of the field at P due to the segment of charge dq is $k_e dq/x^2$. The total field at P is the vector sum over all segments of the rod.

Solution Let us assume that the rod is lying along the x axis, that dx is the length of one small segment, and that dq is the charge on that segment. Because the rod has a charge per unit length λ, the charge dq on the small segment is $dq = \lambda\, dx$.

The field $d\mathbf{E}$ at P due to this segment is in the negative x direction (because the source of the field carries a positive charge), and its magnitude is

$$dE = k_e \frac{dq}{x^2} = k_e \frac{\lambda\, dx}{x^2}$$

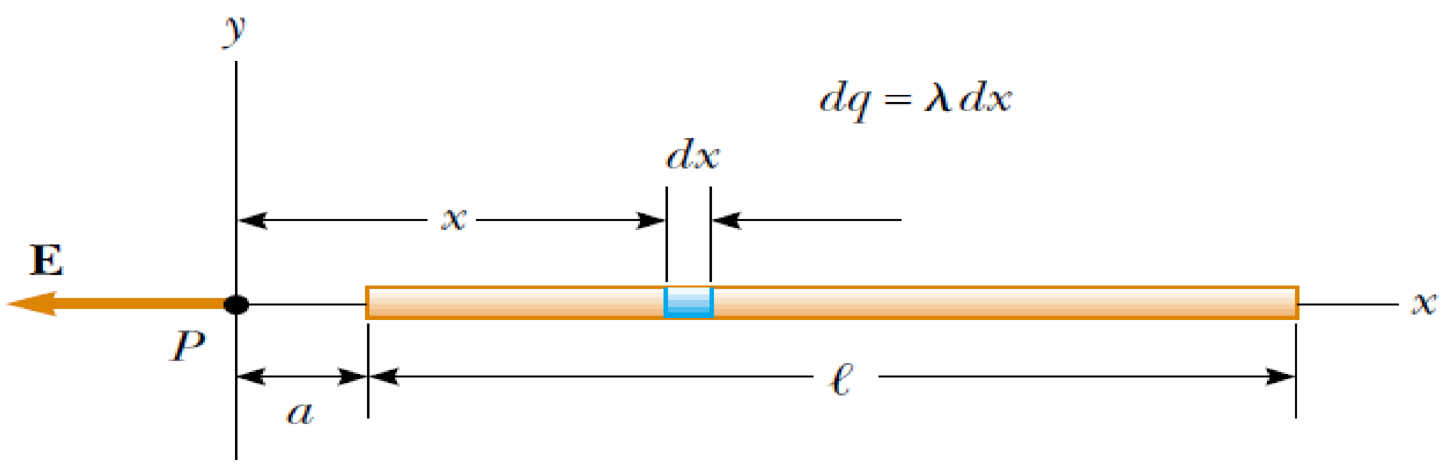


Because every other element also produces a field in the negative x direction, the problem of summing their contributions is particularly simple in this case. The total field at P due to all segments of the rod, which are at different distances from P, is given by Equation

$$= k_e \int \frac{dq}{r^2}\,\hat{\mathbf{r}}$$

which in this case becomes[3]

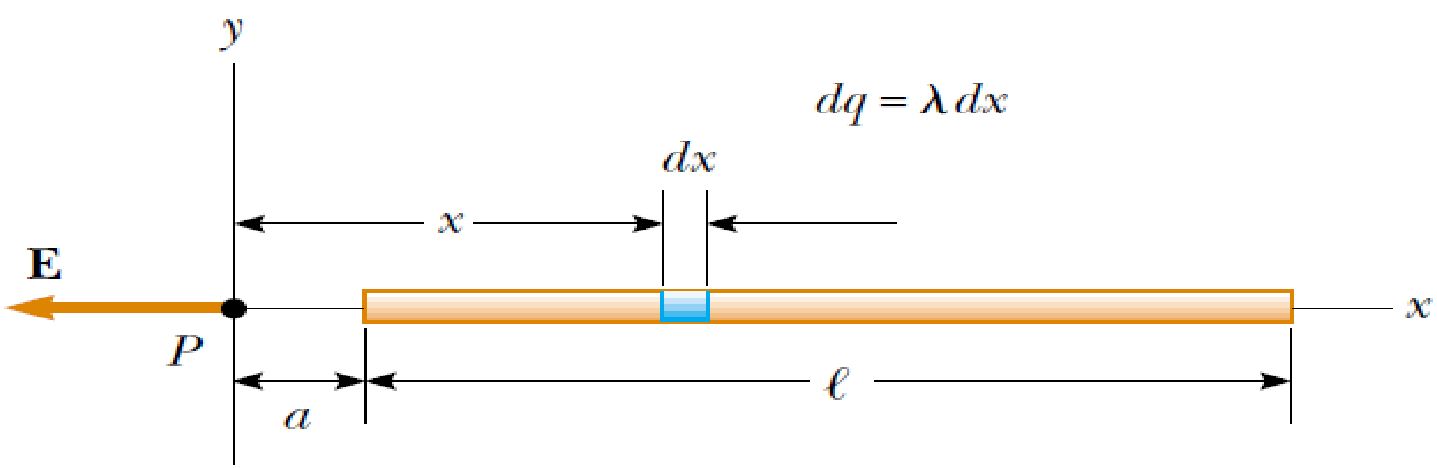
$$E = \int_a^{\ell + a} k_e \boldsymbol{\lambda} \frac{dx}{x^2}$$



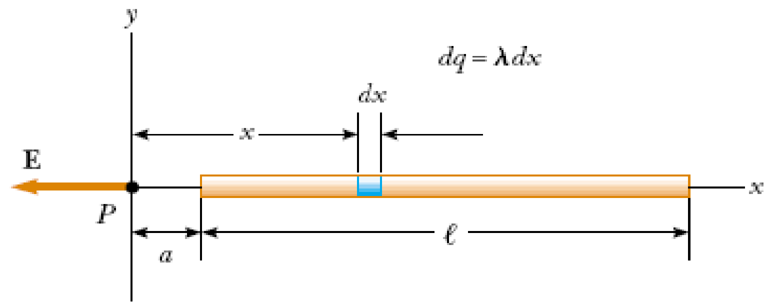
where the limits on the integral extend from one end of the rod ($x = a$) to the other ($x = \ell + a$). The constants k_e and λ can be removed from the integral to yield

$$E = k_e \lambda \int_a^{\ell + a} \frac{dx}{x^2} = k_e \lambda \left[-\frac{1}{x} \right]_a^{\ell + a}$$

$$= k_e \lambda \left(\frac{1}{a} - \frac{1}{\ell + a} \right) = \boxed{\frac{k_e Q}{a(\ell + a)}}$$



where we have used the fact that the total charge $Q = \lambda\ell$.

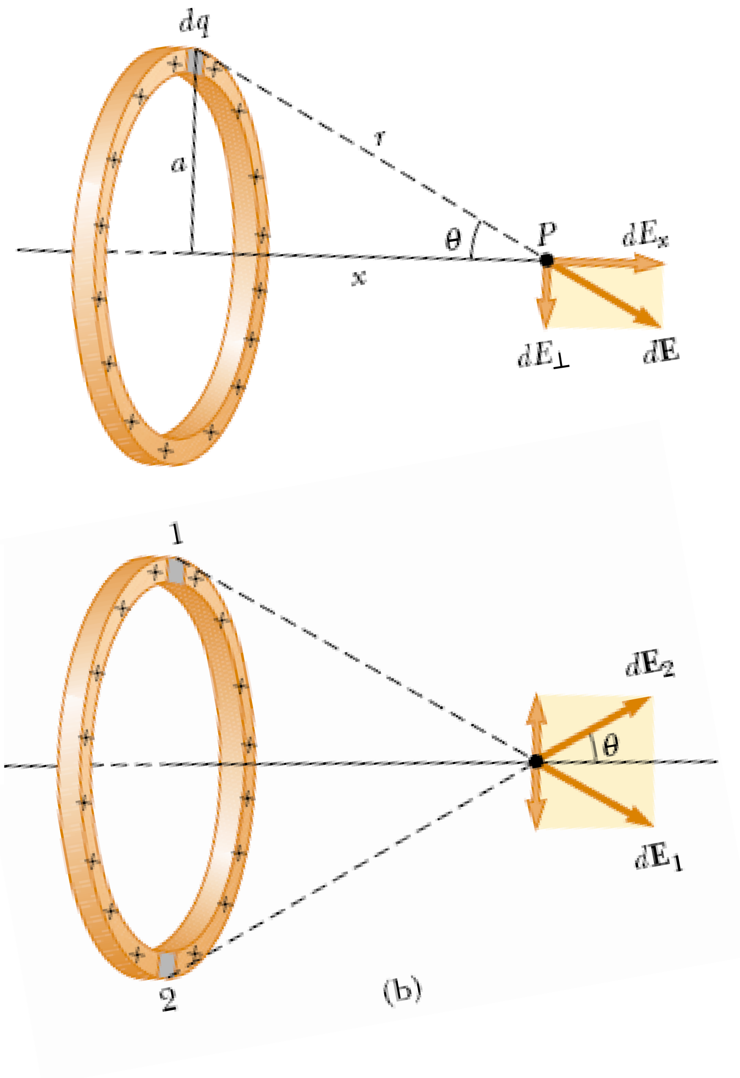


What If? Suppose we move to a point *P* very far away from the rod. What is the nature of the electric field at such a point?

Answer If *P* is far from the rod ($a \gg \ell$), then ℓ in the denominator of the final expression for E can be neglected, and $E \approx k_e Q/a^2$. This is just the form you would expect for a point charge. Therefore, at large values of a/ℓ, the charge distribution appears to be a point charge of magnitude Q—we are so far away from the rod that we cannot distinguish that it has a size. The use of the limiting technique ($a/\ell \to \infty$) often is a good method for checking a mathematical expression.

Electric Field due to Uniform Ring of Charge

A ring of radius a carries a uniformly distributed positive total charge Q. Calculate the electric field due to the ring at a point P lying a distance x from its center along the central axis perpendicular to the plane of the ring (Fig. 23.18a).

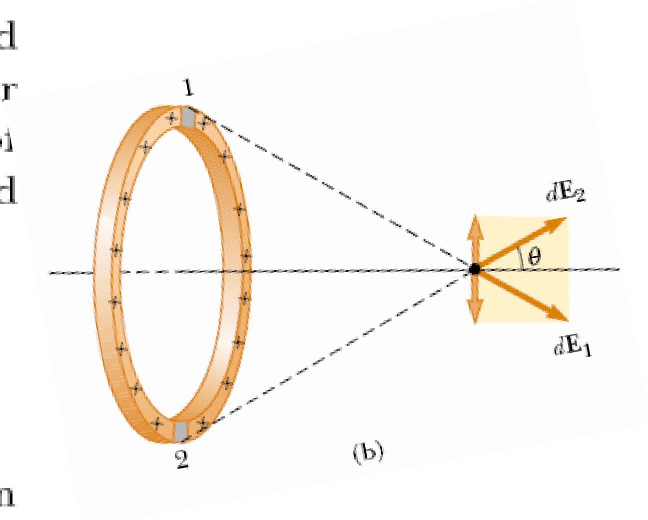


Solution The magnitude of the electric field at P due to the segment of charge dq is

$$dE = k_e \frac{dq}{r^2}$$

This field has an x component $dE_x = dE \cos\theta$ along the x axis and a component $dE_\perp$ perpendicular to the x axis. As we see in Figure 23.18b, however, the resultant field at P must lie along the x axis because the perpendicular components of all the various charge segments sum to zero.

That is, the perpendicular component of the field created by any charge element is canceled by the perpendicular component created by an element on the opposite side of the ring. Because $r = (x^2 + a^2)^{1/2}$ and $\cos\theta = x/r$, we find that



$$dE_x = dE\cos\theta = \left(k_e \frac{dq}{r^2}\right)\frac{x}{r} = \frac{k_e x}{(x^2 + a^2)^{3/2}}\, dq$$

All segments of the ring make the same contribution to the field at P because they are all equidistant from this point. Thus, we can integrate to obtain the total field at P:

$$E_x = \int \frac{k_e x}{(x^2 + a^2)^{3/2}}\, dq = \frac{k_e x}{(x^2 + a^2)^{3/2}} \int dq$$

$$= \boxed{\frac{k_e x}{(x^2 + a^2)^{3/2}}\, Q}$$

This result shows that the field is zero at $x = 0$.

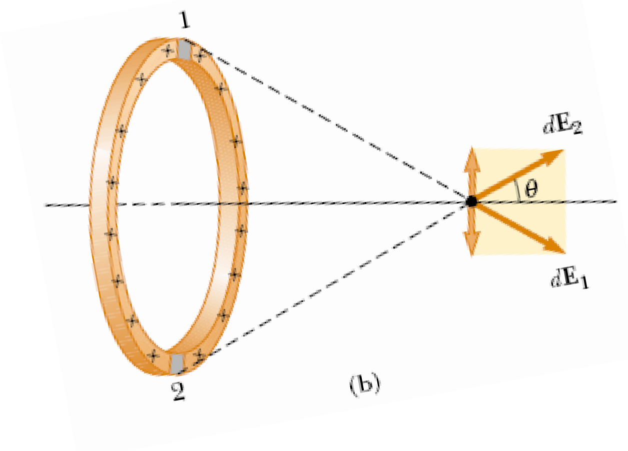
Let us check the equation for a point on the central axis that is so far away that $x >> R$.

For such a point, the expression $x^2 + R^2$ in approximated as x^2

$$E = \frac{KQ}{x^2}$$

This is a reasonable result because from a large distance, the ring "looks like" a point charge.

What If? Suppose a negative charge is placed at the center of the ring in Figure 23.18 and displaced slightly by a distance $x \ll a$ along the x axis. When released, what type of motion does it exhibit?



Answer In the expression for the field due to a ring of charge, we let $x \ll a$, which results in

$$E_x = \frac{k_e Q}{a^3}x$$

Thus, from Equation 23.8, the force on a charge $-q$ placed near the center of the ring is

$$F_x = -\frac{k_e q Q}{a^3}x$$