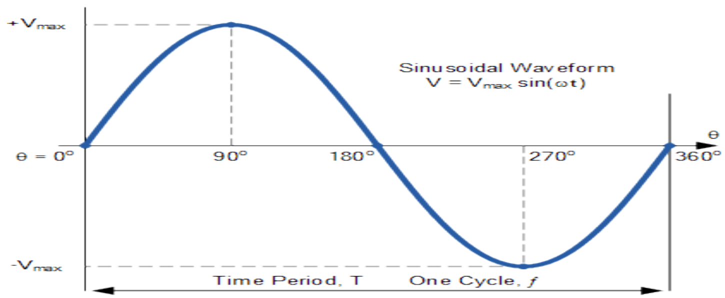
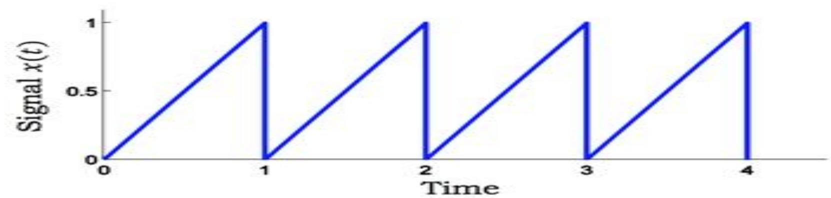
SUPERPOSITION & STANDING WAVES

Dr Muhammad Adeel

CONTENTS

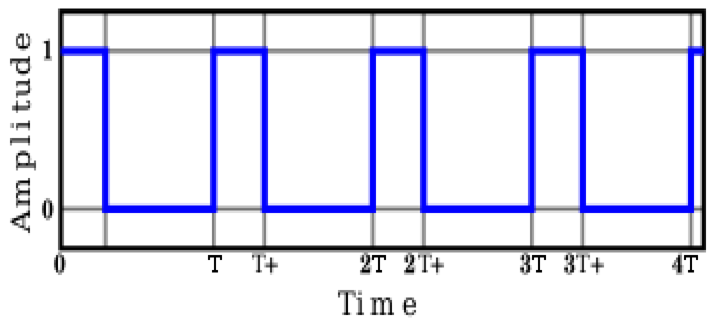
DIFFERENT TYPES OF WAVES/WAVEFORMS

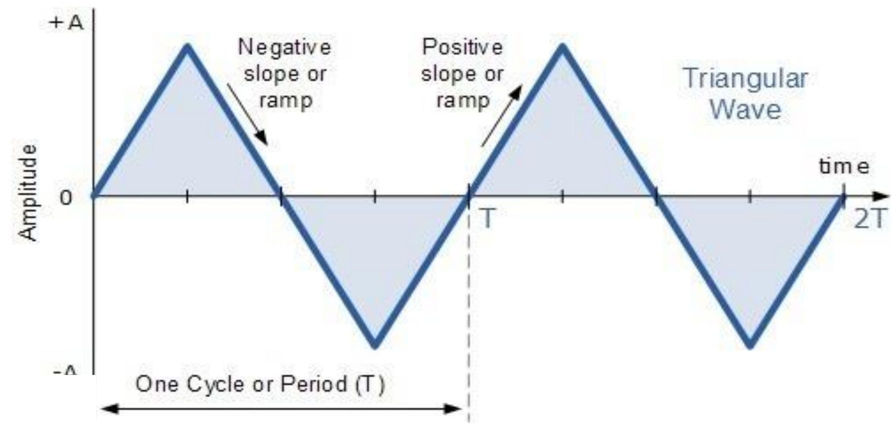




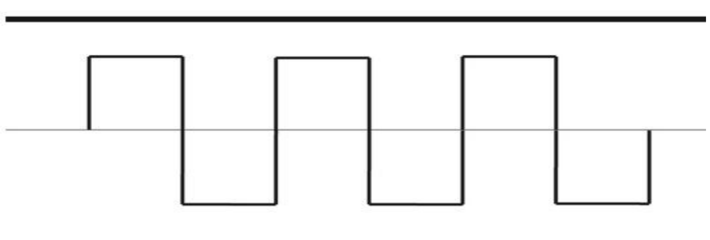
Sawtooth wave

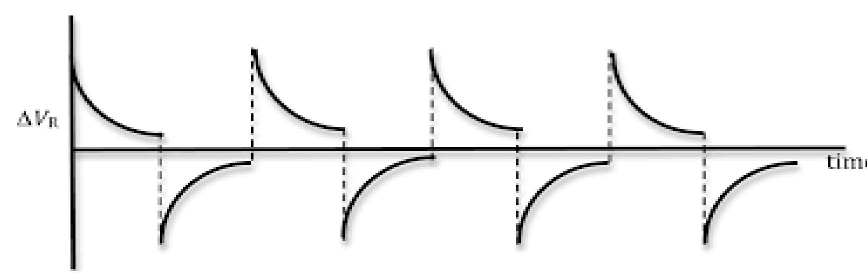
A sawtooth wave of period 1 second is given by the figure.





SQUARE WAVE





SUPERPOSITION

- Many interesting wave phenomenon can not be described by single travelling wave. For which we need to analyze some complex waves.
- One can make use of the Superposition Principle
- The term Superposition means "overlapping or combining effect of something"
- In physics we deal with wave phenomenon therefore this term is related to wave.

SUPERPOSITION PRINCIPLE

- If two or more traveling waves are moving through a medium, the resultant value of the wave function at any point is the algebraic sum of the values of the wave functions of the individual waves.

OR

- Overlapping waves algebraically add to produce a **resultant wave** (or **net wave**).

$$y'(x, t) = y_1(x, t) + y_2(x, t)$$

SUPERPOSITION PRINCIPLE(CONT'D)

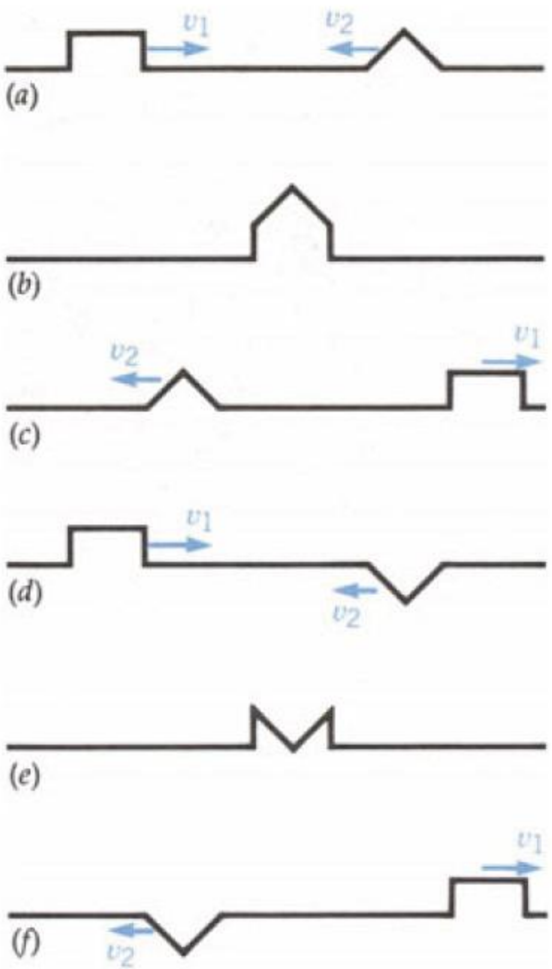
To be more precise….

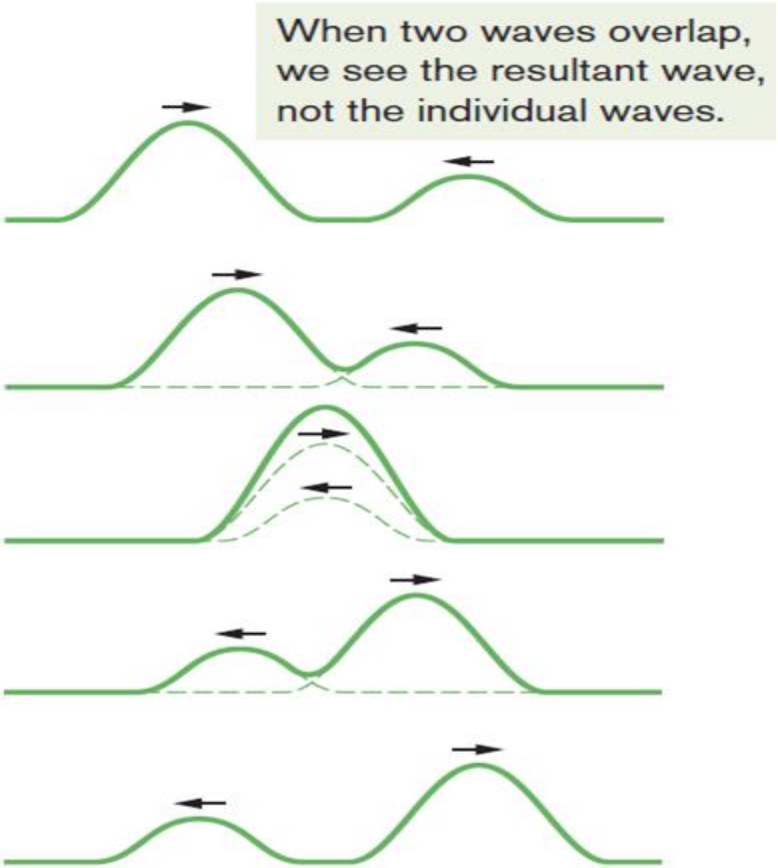
- When particle of one medium is under effect of more than one waves then its displacement at any time is the vector sum of all the displacements which would have been given by individual waves that is,

$$\vec{y} = \overrightarrow{y_1} + \overrightarrow{y_2} + \cdots + \overrightarrow{y_n}$$

- Waves that obey this principle are called ***linear waves***
- Waves that violate the superposition principle are called ***nonlinear waves***

SUPERPOSITION PRINCIPLE(CONT'D)





A series of snapshots that show two pulses traveling in opposite directions along a stretched string. The superposition principle applies as the pulses move through each other.

Superposition Principle(cont'd)

Note: Overlapping waves do not in any way alter the travel of each other.

INTERFERENCE & THEIR TYPES

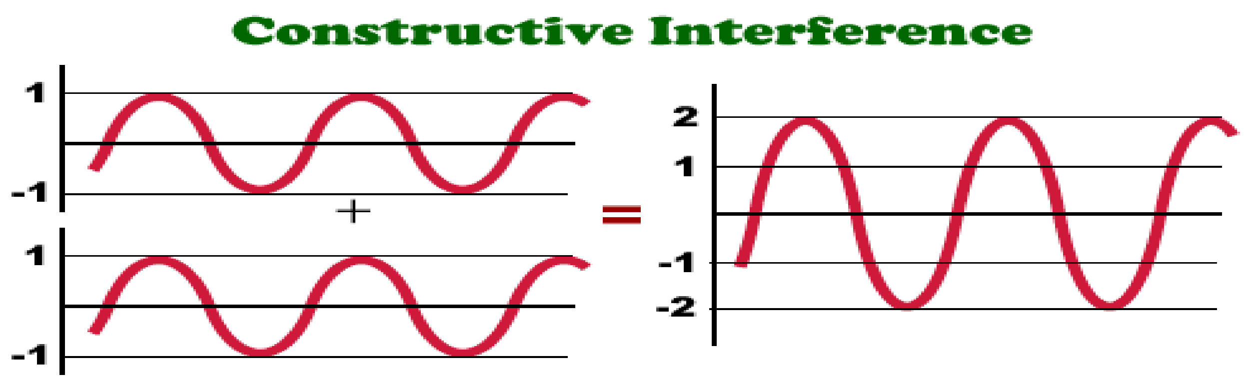
- The superposition principle leads to a wave phenomenon known as ***interference.***
- The combination of separate waves in the same region of space to produce a resultant wave is called ***interference.***
- There are two types of interference that we usually observe in waves,

1. Constructive Interference
2. Destructive Interference

CONSTRUCTIVE INTERFERENCE

- It is the type of interference where two interfering waves have a displacement in the same direction at a time.
- **Fully constructive** interference occurs when the phase

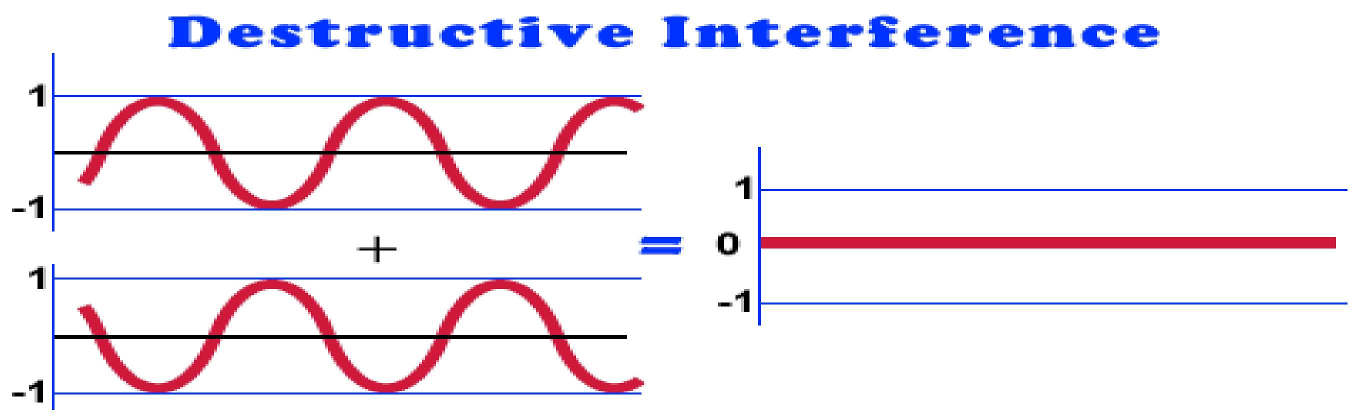
$$\Delta r = \frac{\phi}{2\pi}\lambda \qquad \Delta r = (2n)\frac{\lambda}{2} \qquad \text{for constructive interference}$$



DESTRUCTIVE INTERFERENCE

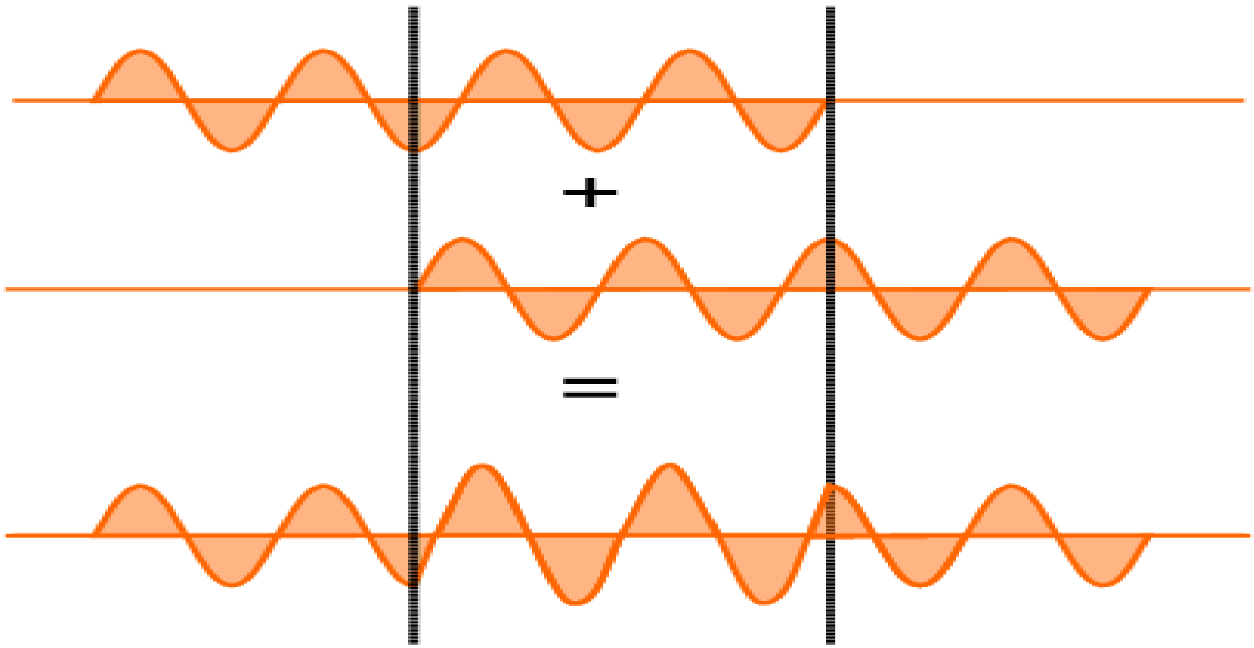
- It is the type of interference where two interfering waves have a displacement in the opposite direction simultaneously.
- **Fully destructive** interference is obtained when the phase difference between two waves is an odd multiple of π

$$\Delta r = \frac{\phi}{2\pi}\lambda \qquad \Delta r = (2n+1)\frac{\lambda}{2} \qquad \text{for destructive interference}$$



INTERMEDIATE PHASE DIFFERENCE

- If the difference between the phases is intermediate between these two extremes (fully constructive or destructive), then the magnitude of displacement of the summed wave lies between the minimum and maximum values.



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SUPERPOSITION OF SINE WAVES

Mathematical Approach

SUPERPOSITION OF SINUSOIDAL WAVES

Let us now apply the principle of superposition to two sinusoidal waves traveling in the same direction in a linear medium. If the two waves are traveling to the right and have the same frequency, wavelength, and amplitude but differ in phase, we can express their individual wave functions as

$$\boxed{y_1 = A\sin(kx - \omega t)} \qquad \boxed{y_2 = A\sin(kx - \omega t + \phi)}$$

where, as usual, $k = 2\pi/\lambda$, $\omega = 2\pi f$, and ϕ is the phase constant, Hence, the resultant wave function y is

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)]$$

To simplify this expression, we use the trigonometric identity

$$\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

If we let $a = kx - \omega t$ and $b = kx - \omega t + \phi$, we find that the resultant wave function y reduces to

$$y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)$$

SUPERPOSITION OF SINUSOIDAL WAVES(RESULT)

$$\boxed{y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)}$$

This result has several important features.

- The resultant wave function **y** also is **sinusoidal.**
- Has the **same frequency** and **wavelength** as the individual waves because the sine function incorporates the same values of k and ω that appear in the original wave functions.
- The amplitude of the resultant wave is **2A cos (Ø/2),** and its phase is Ø/2. If the phase constant Ø = 0, then cos (Ø/2) " cos 0 = 1", and **the amplitude of the resultant wave is 2A**—twice the amplitude of either individual wave.
- Waves are said to be everywhere **in phase** and thus interfere **constructively**

SUPERPOSITION OF SINUSOIDAL WAVES

- <u>Home Work</u>: $$\boxed{y = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t + \frac{\phi}{2}\right)}$$

By using

You all need to find the mathematical conclusions for destructive interference as we have done for constructive interference. Also discuss their important features.

Phase Difference and Resulting Interference Types[a]

Phase Difference, in			Amplitude of Resultant Wave	Type of Interference
Degrees	Radians	Wavelengths		
0	0	0	$2y_m$	Fully constructive
120	$\frac{2}{3}\pi$	0.33	y_m	Intermediate
180	π	0.50	0	Fully destructive
240	$\frac{4}{3}\pi$	0.67	y_m	Intermediate
360	2π	1.00	$2y_m$	Fully constructive
865	15.1	2.40	$0.60y_m$	Intermediate

[a]The phase difference is between two otherwise identical waves, with amplitude y_m, moving in the same direction.

Example:

Two identical sinusoidal waves, moving in the same direction along a stretched string, interfere with each other. The amplitude of each wave is 15mm. Calculate the amplitudes of the resultant waves due to the interference for phase differences 120° and 180°.Give the types of both interferences?

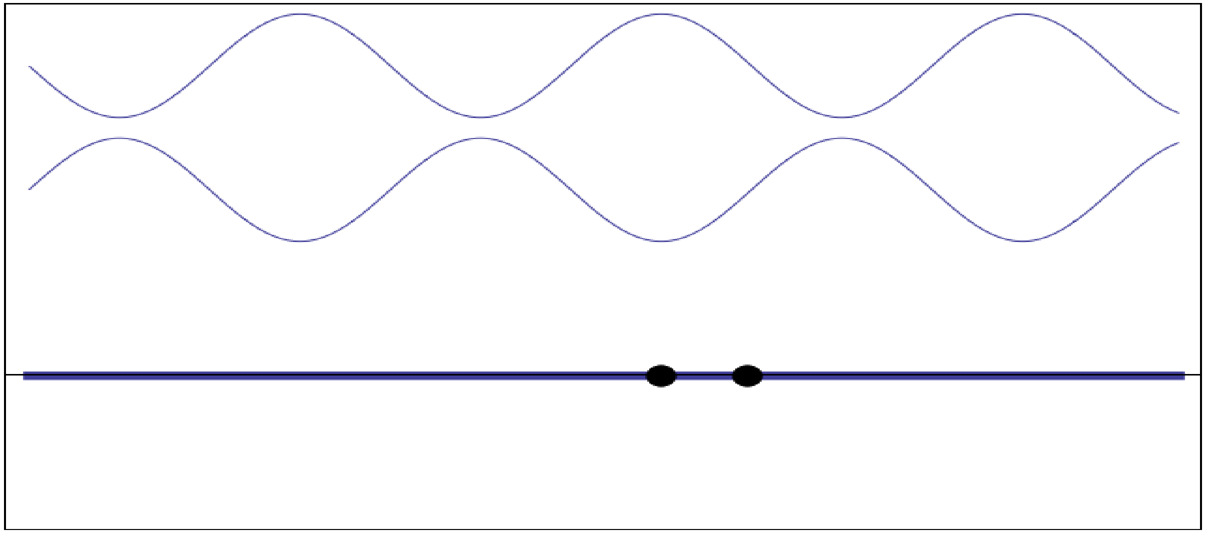
$$Y_m = 15mm$$

$$Y'_m = 2y_m\cos\Phi/2 = 2(15)\cos(120/2) = 15mm$$

$$Y'_m = 2y_m\cos\Phi/2 = 2(15)\cos(180/2) = 0$$

STANDING WAVES

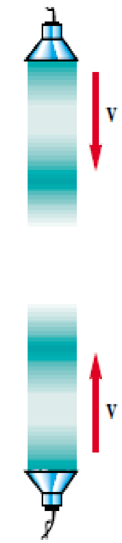
- Another interesting phenomenon resulting from the superposition principle is the formation of ***standing waves.***
- Also known as Stationary wave.

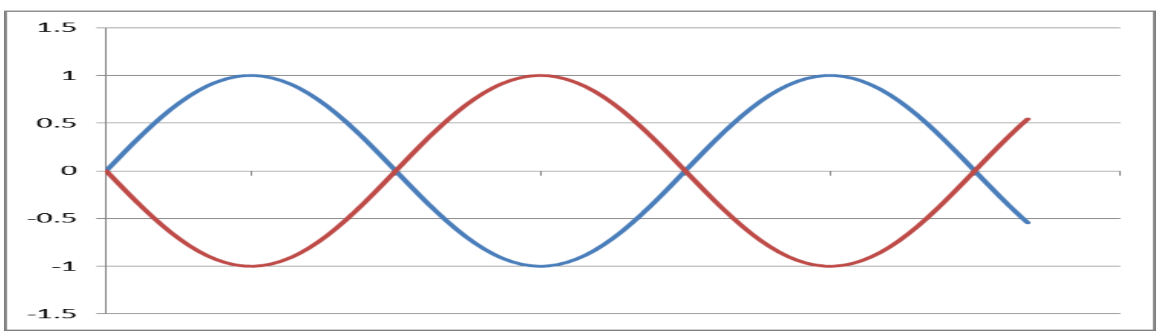


FORMATION OF STANDING WAVE

- Consider that the string is of finite length and the other end is clamped to a rigid support. When the wave disturbances reach the fixed end, they will propagate in the opposite direction. The reflected waves will add to the incident waves according to the superposition principle and, under certain conditions, a standing wave pattern will be formed.
- As the waves move through each other

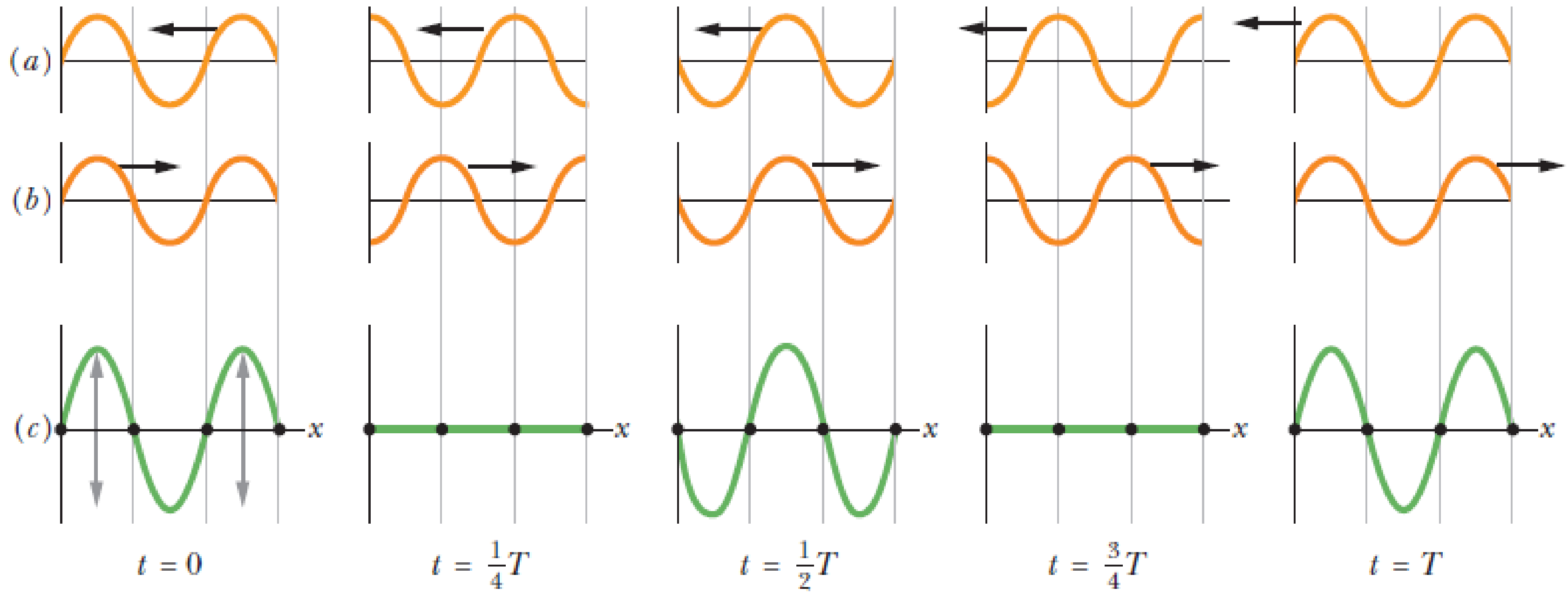
  some points never move and some move the most.





Two speakers emit sound waves toward each other. When they overlap, identical waves traveling in opposite directions will combine to form standing waves.

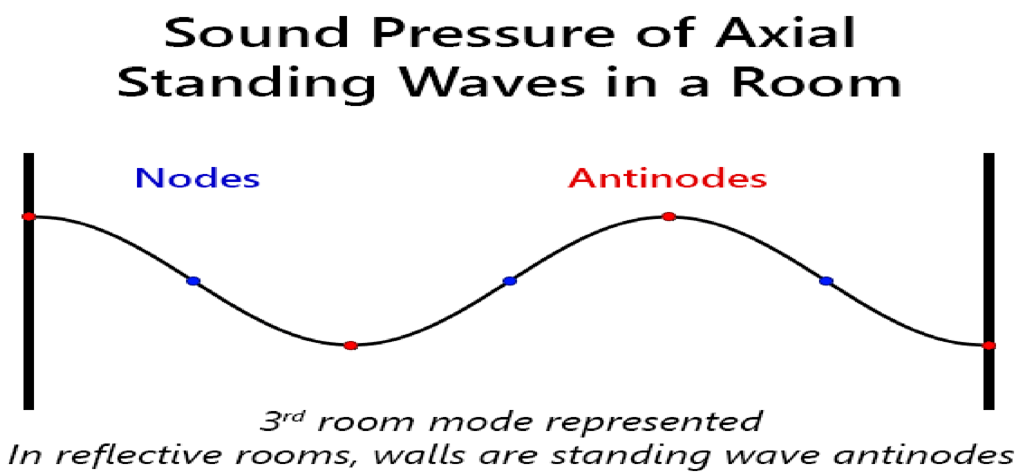
Pictorial View of Standing Wave



(a) Five snapshots of a wave traveling to the left, at the times t indicated below part (c) (T is the period of oscillation). (b) Five snapshots of a wave identical to that in (a) but traveling to the right, at the same times t. (c) Corresponding snapshots for the superposition of the two waves on the same string. At $t = 0, \frac{1}{2}T$, and T, fully constructive interference occurs because of the alignment of peaks with peaks and valleys with valleys. At $t = \frac{1}{4}T$ and $\frac{3}{4}T$, fully destructive interference occurs because of the alignment of peaks with valleys. Some points (the nodes, marked with dots) never oscillate; some points (the antinodes) oscillate the most.

NODES & ANTI-NODES

- If two sinusoidal waves of the same amplitude and wavelength travel in *opposite* directions along a stretched string, their interference with each other produces a standing wave.
- Another interesting feature of this is its Nodes and anti-Nodes.
- The **moving part** on string is known as **Anti-Node.**
- The **stat** **ode.**



STANDING WAVE(MATHEMATICAL ANALYSIS)

To analyze a standing wave, we represent the two combining waves with the equations

$$y_1(x,t) = y_m \sin(kx - \omega t)$$

and

$$y_2(x,t) = y_m \sin(kx + \omega t).$$

The principle of superposition gives, for the combined wave,

$$y'(x,t) = y_1(x,t) + y_2(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t).$$

Applying the trigonometric relation

$$y'(x,t) = [2y_m \sin kx] \cos \omega t.$$

The resultant wave is a standing wave and is due to the interference of two sinusoidal waves of the same amplitude and wavelength that travel in opposite directions.

EQUATIONS FOR NODES & ANTI-NODES

Sin(kx) appears zero at kx = nπ $where\ n = 0,1,2,3 \ldots n$

Substituting $k = 2\pi/\lambda$ in this equation and rearranging, we get

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{[2y_m \sin kx]}_{\substack{\text{Magnitude} \\ \text{gives} \\ \text{amplitude} \\ \text{at position } x}} \underbrace{\cos \omega t}_{\substack{\text{Oscillating} \\ \text{term}}}$$

$$x = n\frac{\lambda}{2}, \qquad \text{for } n = 0, 1, 2, \ldots \qquad \text{(nodes)},$$

Sin(kx) appears 1 at kx = n+(½)π $where\ n = 0,1,2,3 \ldots n$

Substituting $k = 2\pi/\lambda$ in $kx = (n + \frac{1}{2})\pi$, and rearranging, we get

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \qquad \text{for } n = 0, 1, 2, \ldots \qquad \text{(antinodes)},$$

The distance between adjacent antinodes is equal to $\lambda/2$.

The distance between adjacent nodes is equal to $\lambda/2$.

The distance between a node and an adjacent antinode is $\lambda/4$.

EXAMPLE PROBLEM 1(INCLUDES PART A,B)

Two waves traveling in opposite directions produce a standing wave. The individual wave functions are

$$y_1 = (4.0\text{ cm})\sin(3.0x - 2.0t)$$
$$y_2 = (4.0\text{ cm})\sin(3.0x + 2.0t)$$

where x and y are measured in centimeters.

(A) Find the amplitude of the simple harmonic motion of the element of the medium located at $x = 2.3$ cm.

Solution The standing wave is described by standing wave equation in this problem, we have $A = 4.0$ cm, $k = 3.0$ rad/cm, and $\omega = 2.0$ rad/s. Thus,

$$y = (2A\sin kx)\cos\omega t = [(8.0\text{ cm})\sin 3.0x]\cos 2.0t$$

Thus, we obtain the amplitude of the simple harmonic motion of the element at the position $x = 2.3$ cm by evaluating the coefficient of the cosine function at this position:

$$y_{\text{max}} = (8.0\text{ cm})\sin 3.0x\big|_{x=2.3}$$
$$= (8.0\text{ cm})\sin(6.9\text{ rad}) = \boxed{4.6\text{ cm}}$$

EXAMPLE PROBLEM 1(CONT'D)

(B) Find the positions of the nodes and antinodes if one end of the string is at $x = 0$.

Solution With $k = 2\pi/\lambda = 3.0$ rad/cm, we see that the wavelength is $\lambda = (2\pi/3.0)$ cm. Therefore, from Equation 18.4 we find that the nodes are located at

$$x = n\frac{\lambda}{2} = n\left(\frac{\pi}{3}\right) \text{ cm} \qquad n = \boxed{0, 1, 2, 3, \ldots}$$

and from standing wave eq. we find that the antinodes are located at

$$x = n\frac{\lambda}{4} = n\left(\frac{\pi}{6}\right) \text{ cm} \qquad n = \boxed{1, 3, 5, \ldots}$$

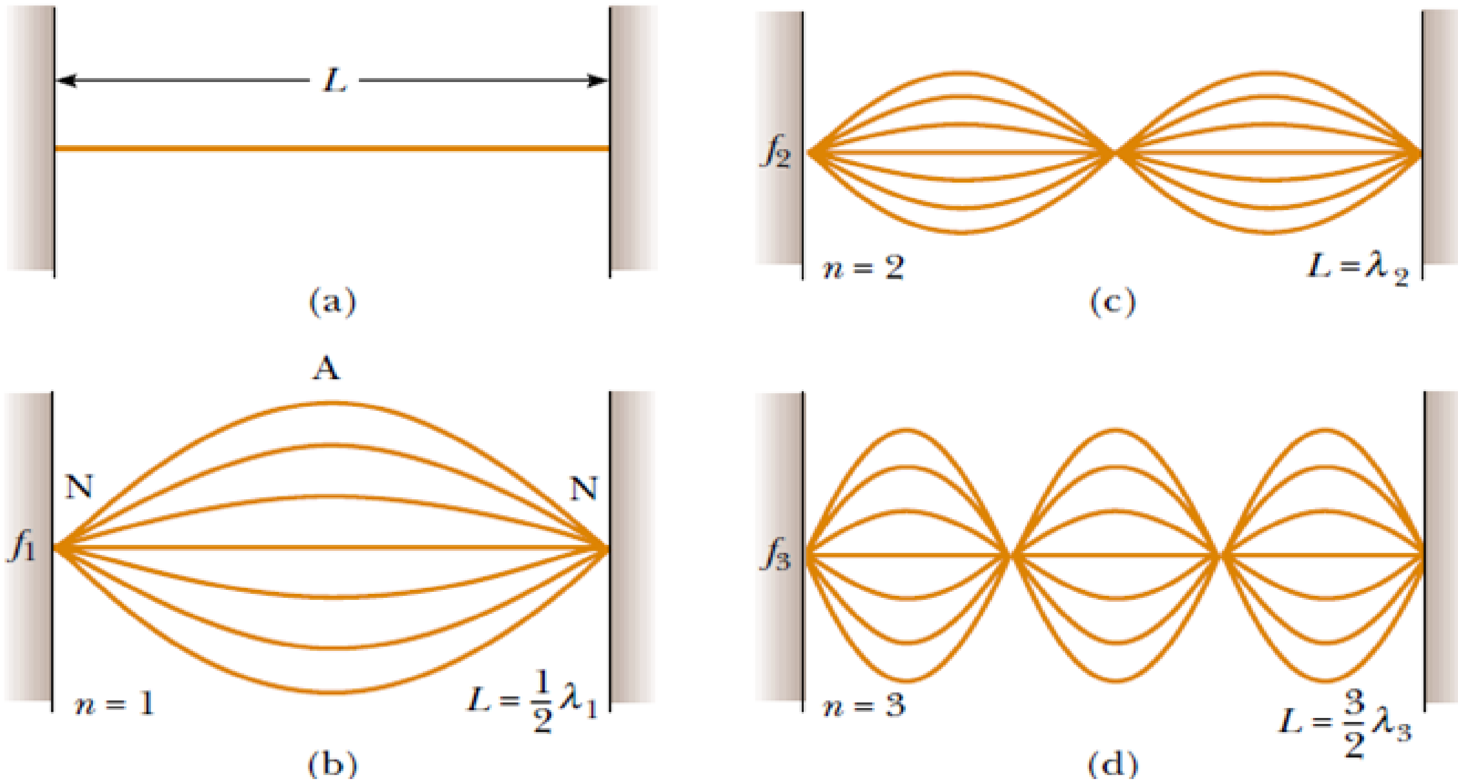
STANDING WAVES IN A STRING FIXED AT BOTH ENDS

- We have two conditions on which standing waves are observed,

1. Boundary Condition
2. With out Boundary

- We cover the boundary condition in this section of study, or standing wave in a string fixed at both ends.

FIXED ENDS STANDING WAVE

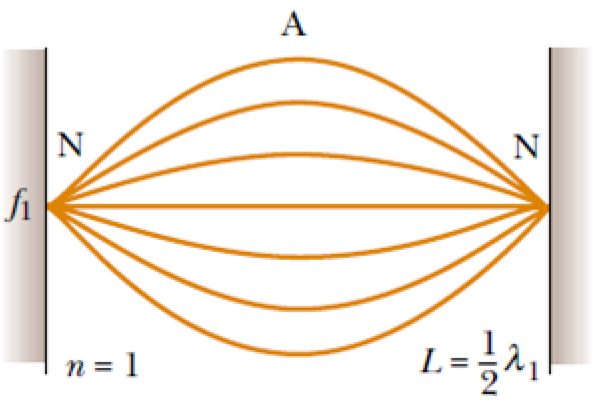


(a) A string of length L fixed at both ends. The normal modes of vibration form a harmonic series: (b) the fundamental, or first harmonic; (c) the second harmonic; (d) the third harmonic.

HARMONICS

- The boundary condition results in the string having a number of natural patterns of oscillation, called **normal modes**, each of which has a characteristic frequency that is easily calculated.
- Normal modes are referred to as **harmonics**
- The first normal mode that is consistent with the boundary conditions, shown in Figure(b) has nodes at its ends and one antinode in the middle
- This is the longest-wavelength mode

  that is consistent with our requirements



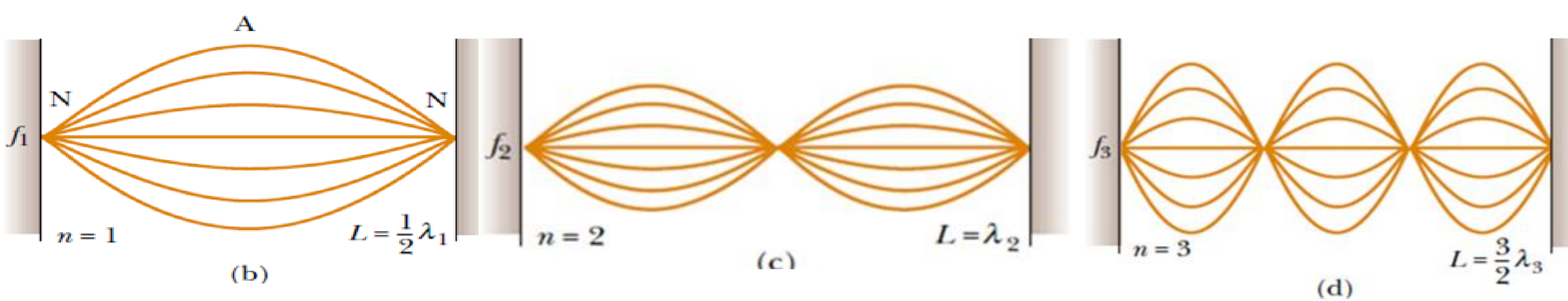
(b)

HARMONICS

- The first normal mode occurs when the length of the string is half the wavelength. λ_1, as indicated in Figure or $\lambda_1 = 2L$.
- The next normal mode of wavelength

  λ_2 occurs when the wavelength equals the length of the string, that is, when $\lambda_2 = L$.
- The third normal mode

  corresponds to the case in which $\lambda_3 = 2L/3$.



HARMONICS:NATURAL FREQUENCY (FN)

- From normal modes $\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ldots$ on i.e,
- These are the *possible* modes of oscillation for the string.
- The natural frequencies associated $f = v/\lambda$ ese modes are obtained from th

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \qquad n = 1, 2, 3, \ldots$$

- These natural frequencies are also called the *quantized frequencies associated with the vibrating* string fixed at both ends.

HARMONICS: FUNDAMENTAL FREQUENCY

$v = \sqrt{T/\mu}$, where T is the tension in the string and μ is its linear mass density, we can also express the natural frequencies of a taut string as

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \qquad n = 1, 2, 3, \ldots$$

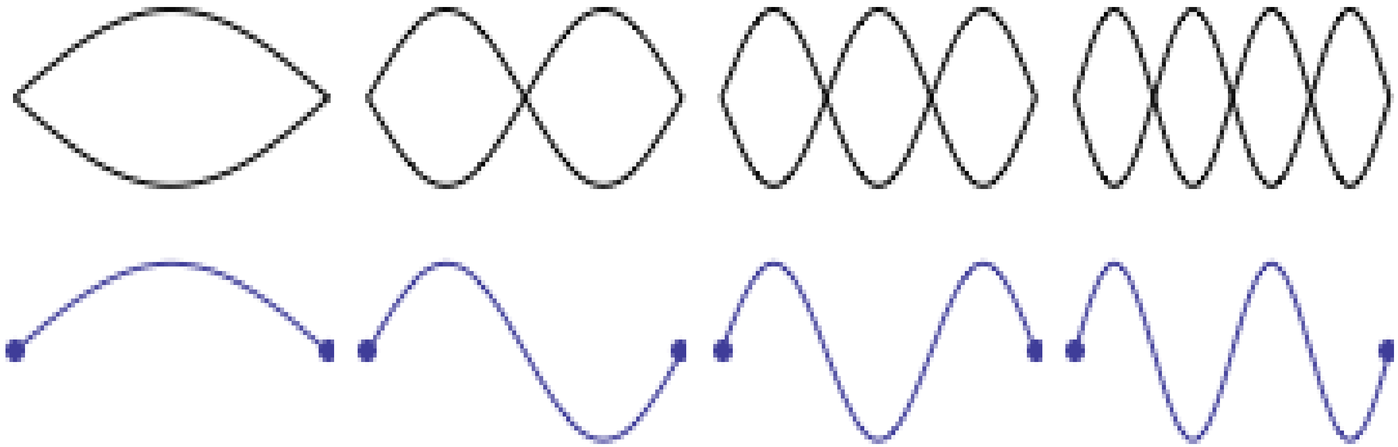
The lowest frequency f_1, which corresponds to $n = 1$, is called either the **fundamental** or the **fundamental frequency** and is given by

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

The frequencies of the remaining normal modes are integer multiples of the fundamental frequency. Frequencies of normal modes that exhibit an integer-multiple relationship such as this form a **harmonic series,** and the normal modes are called **harmonics.**

HARMONICS

harmonics. The fundamental frequency f_1 is the frequency of the first harmonic; the frequency $f_2 = 2f_1$ is the frequency of the second harmonic; and the frequency $f_n = nf_1$ is the frequency of the nth harmonic.



EXAMPLE PROBLEM 2

Middle C on a piano has a fundamental frequency of 262 Hz, and the first A above middle C has a fundamental frequency of 440 Hz.

(A) Calculate the frequencies of the next two harmonics of the C string.

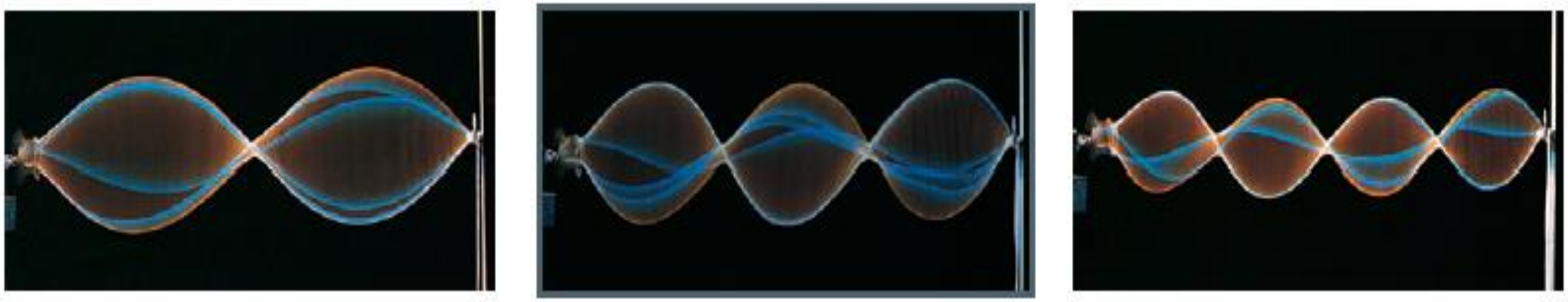
Solution Knowing that the frequencies of higher harmonics are integer multiples of the fundamental frequency $f_1 = 262$ Hz, we find that

$$f_2 = 2f_1 = \boxed{524 \text{ Hz}}$$

$$f_3 = 3f_1 = \boxed{786 \text{ Hz}}$$

RESONANCE & STANDING WAVES

- An oscillating system is in resonance with some driving force whenever the frequency of the driving force matches one of the natural frequencies of the system.
- When the system is resonating, it responds by oscillating with a relatively large amplitude.
- For certain frequencies, the interference produces a standing wave pattern (or **oscillation mode**) with nodes and large antinodes like those in Fig. shown
- Such a standing wave is said to be produced at **resonance,** and the string is said to *resonate* at these certain frequencies, called **resonant frequencies.**



Stroboscopic photographs reveal (imperfect) standing wave patterns on a string being made to oscillate by an oscillator at the left end. The patterns occur at certain frequencies of oscillation.

CONDITION TO HAVE STANDING WAVE

For certain frequencies, the interference produces a standing wave pattern (or **oscillation mode**) with nodes and large antinodes like those in Fig. Such a standing wave is said to be produced at **resonance,** and the string is said to *resonate* at these certain frequencies, called **resonant frequencies.** If the string is oscillated at some frequency other than a resonant frequency, a standing wave is not set up. Then the interference of the right-going and left-going traveling waves results in only small (perhaps imperceptible) oscillations of the string.

PRACTICE PROBLEMS

1. Use the wave equation to find the speed of a wave given by

$$y(x,t) = (2.00\ \text{mm})[(20\ \text{m}^{-1})x - (4.0\ \text{s}^{-1})t]^{0.5}.$$

The wave

$$y(x,t) = (2.00\ \text{mm})[(20\ \text{m}^{-1})x - (4.0\ \text{s}^{-1})t]^{1/2}$$

is of the form $h(kx - \omega t)$ with angular wave number $k = 20\ \text{m}^{-1}$ and angular frequency $\omega = 4.0$ rad/s. Thus, the speed of the wave is

$$v = \omega / k = (4.0\ \text{rad/s})/(20\ \text{m}^{-1}) = 0.20\ \text{m/s}.$$

PRACTICE PROBLEMS

2. Two identical traveling waves, moving in the same direction, are out of phase by $\pi/2$ rad. What is the amplitude of the resultant wave in terms of the common amplitude y_m of the two combining waves?

THINK By superposition principle, the resultant wave is the algebraic sum of the two interfering waves.

EXPRESS The displacement of the string is given by

$$y = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi) = 2y_m \cos\left(\tfrac{1}{2}\phi\right)\sin\left(kx - \omega t + \tfrac{1}{2}\phi\right),$$

where we have used

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta).$$

ANALYZE The two waves are out of phase by $\phi = \pi/2$, so the amplitude is

$$A = 2y_m \cos\left(\tfrac{1}{2}\phi\right) = 2y_m \cos(\pi/4) = 1.41y_m .$$

LEARN The interference between two waves can be constructive or destructive, depending on their phase difference.