National University of Computer & Emerging Sciences – FAST

School of Computer Science

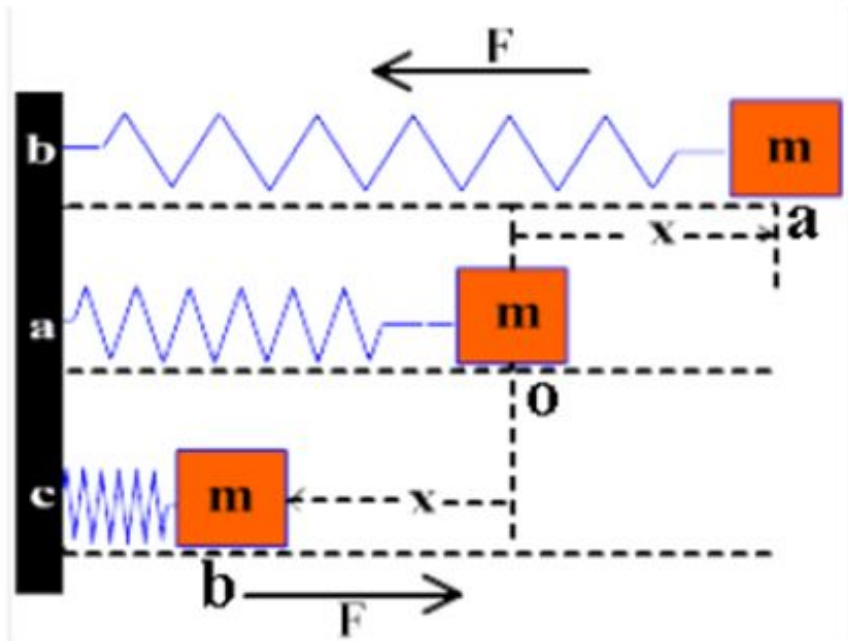
Course Code : NS 1001

Course Title : Applied Physics

Simple Harmonic Motion

Definition:

- Such a motion in which acceleration is directly proportional to the displacement and is directed towards the mean position is called simple harmonic motion(SHM).



Condition FOR SHM:

- The system should haves restoring force.
- The system should have inertia.
- The system should be frictionless.

Hooke's Law

- force that is applied to spring is **directly proportional** to the displacement.
- If the spring is un stretched, there is no net force on the mass or the system is in equilibrium.
- if the mass is displaced from equilibrium, the spring will exert a restoring force, which is a force that tends to restore it to the equilibrium position.

$$F \propto x$$

where,
F → Elastic force
k → Spring constant
x → Displacement

$$\boxed{F = kx}$$

Expression for acceleration of the body executing SHM:

- Consider a mass ‘m’ attached to one end of elastic spring which can move freely on a frictionless horizontal surface.

- When the mass is released, it begins to vibrate about its mean or equilibrium position.

- But due to elasticity, spring opposes the applied force which produces the displacement. This opposing force is called restoring force.

Expression for acceleration of the body executing SHM:

The restoring force is written by;

$$F_r = -kx \text{.......................(i)}$$

If 'a' is the acceleration produced by force 'F' in mass-spring system at any instant, then according to Newton's law of motion.

$$F = ma$$

$$F = m\ddot{x} \text{ ....(ii)}$$

Comparing (i) and (ii)

$$m\ddot{x} = -kx \qquad \Rightarrow \ddot{x} = \left(\frac{-k}{m}\right)x$$

$$\because \frac{k}{m} = constant$$

a α -x

From above equation we can write it as

$$\ddot{x} + \left(\frac{k}{m}\right)x = 0$$

Solution of the form

$$x = A\cos(\omega t + \phi).$$

$$\therefore \omega = \sqrt{\frac{k}{m}}$$

- Velocity can found by differentiating displacement

$$v = -A\omega \sin(\omega t + \emptyset)$$

- Acceleration can found by differentiating velocity

$$a = -A\omega^2 \cos(\omega t + \emptyset)$$

Simplifying acceleration in terms of displacement:

$$a = \frac{d^2x}{dt^2} = -\omega^2 x$$

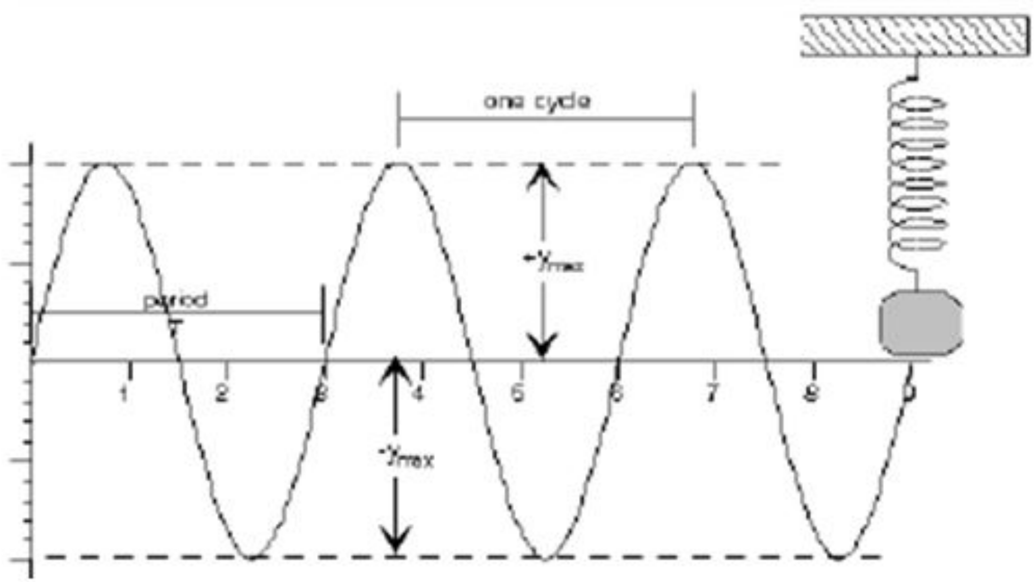
$$a = \frac{d^2x}{dt^2} = -\omega^2 x$$

Acceleration can also be expressed as:

$$a(t) = -(2\pi f)^2 x(t)$$

Characteristic of Mass-spring system executing SHM:

When a body is vibrating, its displacement from the mean position changes with time. The value of its distance from the mean position at any time is known as its instantaneous displacement.



- The maximum value of displacement is known as its amplitude.

- A vibration means one complete round trip of the body in motion.

- The time required to complete one vibration is called time period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- The number of cycles per second. A cycle is a complete round trip is called Frequency

$$f = \frac{1}{T} \qquad f = \frac{1}{2\pi\sqrt{\frac{m}{k}}} \qquad f = 2\pi\sqrt{\frac{k}{m}}$$

- If T is time period of a body executing SHM, its angular frequency can be written as;

$$\omega = \frac{2\pi}{T} = 2\pi f$$

Phase angle:

The angle $\theta = \omega t$ which specifies the displacement as well as the direction of the point executing SHM is known as phase angle.

General Equation

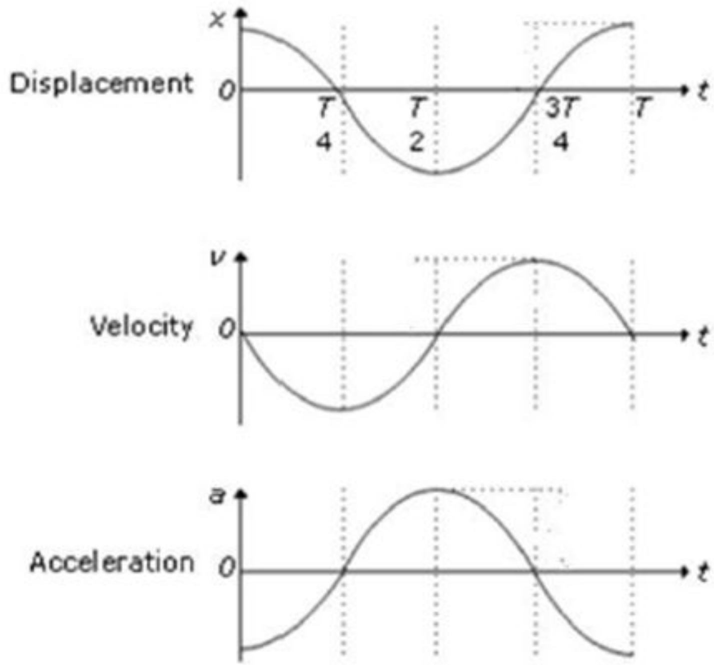
$$x(t) = A\cos(2\pi ft + \phi)$$

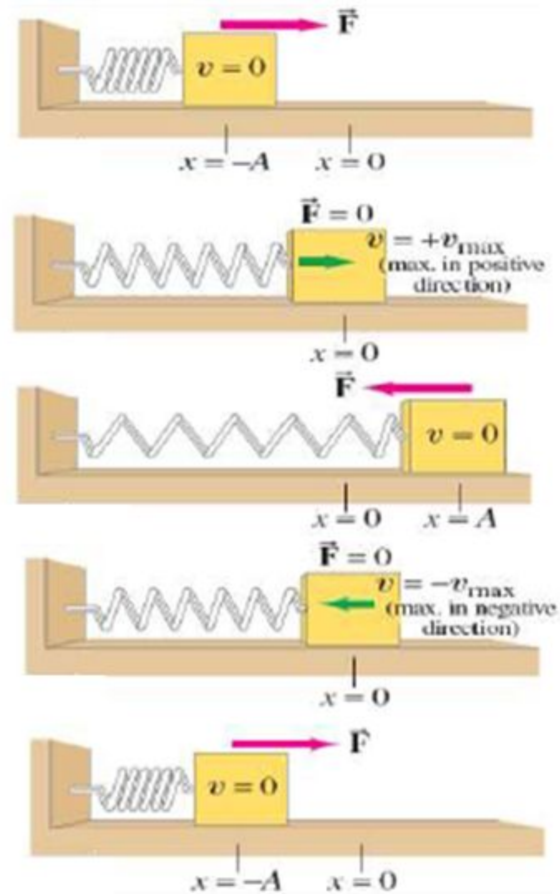
where,

- x → Displacement
- A → Amplitude of the oscillation
- f → Frequency
- t → time
- ϕ → Phase of oscillation

If there is no displacement at time t = 0, the phase is $\phi = \pi/2$

Graphical representation





A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

KEY IDEA The block–spring system forms a linear simple harmonic oscillator, with the block undergoing SHM.

Calculations: The angular frequency is given by Eq. 15-12:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\text{ N/m}}{0.68\text{ kg}}} = 9.78\text{ rad/s}$$

$$\approx 9.8\text{ rad/s.} \qquad \text{(Answer)}$$

The frequency follows from Eq. 15-5, which yields

$$f = \frac{\omega}{2\pi} = \frac{9.78\text{ rad/s}}{2\pi\text{ rad}} = 1.56\text{ Hz} \approx 1.6\text{ Hz.} \qquad \text{(Answer)}$$

The period follows from Eq. 15-2, which yields

$$T = \frac{1}{f} = \frac{1}{1.56\text{ Hz}} = 0.64\text{ s} = 640\text{ ms.} \qquad \text{(Answer)}$$

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(b) What is the amplitude of the oscillation?

KEY IDEA With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11 \text{ cm}. \qquad \text{(Answer)}$$

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

KEY IDEA The maximum speed v_m is the velocity amplitude ωx_m in Eq. 15-6.

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m}) = 1.1 \text{ m/s.} \qquad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin; compare Figs. 15-4*a* and 15-4*b*, where you can see that the speed is a maximum whenever $x = 0$.

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(d) What is the magnitude a_m of the maximum acceleration of the block?

KEY IDEA The magnitude a_m of the maximum acceleration is the acceleration amplitude $\omega^2 x_m$ in Eq. 15-7.

Calculation: So, we have

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2(0.11 \text{ m})$$
$$= 11 \text{ m/s}^2. \qquad \text{(Answer)}$$

This maximum acceleration occurs when the block is at the ends of its path. At those points, the force acting on the block has its maximum magnitude; compare Figs. 15-4*a* and 15-4*c*, where you can see that the magnitudes of the displacement and acceleration are maximum at the same times.

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(e) What is the phase constant ϕ for the motion?

Calculations: Equation 15-3 gives the displacement of the block as a function of time. We know that at time $t = 0$, the block is located at $x = x_m$. Substituting these *initial conditions*, as they are called, into Eq. 15-3 and canceling x_m give us

$$1 = \cos \phi. \tag{15-14}$$

Taking the inverse cosine then yields

$$\phi = 0 \text{ rad}. \qquad \text{(Answer)}$$

(Any angle that is an integer multiple of 2π rad also satisfies Eq. 15-14; we chose the smallest angle.)

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(f) What is the displacement function $x(t)$ for the spring–block system?

Calculation: The function $x(t)$ is given in general form by Eq. 15-3. Substituting known quantities into that equation gives us

$$\begin{aligned} x(t) &= x_m \cos(\omega t + \phi) \\ &= (0.11\ \text{m}) \cos[(9.8\ \text{rad/s})t + 0] \\ &= 0.11 \cos(9.8t), \qquad \text{(Answer)} \end{aligned}$$

where x is in meters and t is in seconds.

Energy conservation in Simple harmonic motion:

- if the friction effect are neglected, total mechanical energy of vibrating mass spring system remains constant
- The velocity and position of the vibrating body are continually changing
- The kinetic and potential energies also change, but their sum must have the same values at any instant.

- By hook's law
- F=-kx
- $W = \int f\,dx$
- U=-W
- U=$-\int f\,dx$

- $U = -\int -kx\,dx$
- $U = k\int x\,dx$
- $U = k\frac{x^2}{2}$

- Putting value of displacement

$$U(t) = \tfrac{1}{2}k\left(x_m \cos(\omega t + \phi)\right)^2$$
$$U(t) = \tfrac{1}{2}k\, x_m^2 \cos^2(\omega t + \phi)$$

- Kinetic energy is given by $K.E = \frac{1}{2}mv^2$

- Kinetic energy is maximum if x=0 when the mass is at equilibrium position.

$$K(t) = \tfrac{1}{2}m\left(-\omega\, x_m \sin(\omega t + \phi)\right)^2$$

$$k = m\omega^2 \qquad K(t) = \tfrac{1}{2}k\, x_m^2 \sin^2(\omega t + \phi)$$

- The energy is partly P.E and partly K.E.

$$E = P.E + K.E$$

$$\therefore E = \tfrac{1}{2}k\, x_m^2 = constant$$

Many tall building have mass dampers, which are anti-sway devices to prevent them from oscillating in a wind. The device might be a block oscillating at the end of a spring and on a lubricated track. If the building sways, say eastward, the block also moves eastward but delayed enough so that when it finally moves, the building is then moving back westward. Thus, the motion of the oscillator is out of step with the motion of the building.

Suppose that the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring-block system?

KEY IDEA The mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position $x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-12 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$k = m\omega^2 = m(2\pi f)^2$$
$$= (2.72 \times 10^5 \text{ kg})(2\pi)^2(10.0 \text{ Hz})^2$$
$$= 1.073 \times 10^9 \text{ N/m}.$$

We can now evaluate E as

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$
$$= 0 + \tfrac{1}{2}(1.073 \times 10^9 \text{ N/m})(0.20 \text{ m})^2$$
$$= 2.147 \times 10^7 \text{ J} \approx 2.1 \times 10^7 \text{ J}. \qquad \text{(Answer)}$$

(b) What is the block's speed as it passes through the equilibrium point?

Calculations: We want the speed at $x = 0$, where the potential energy is $U = \frac{1}{2}kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$2.147 \times 10^7\,\text{J} = \tfrac{1}{2}(2.72 \times 10^5\,\text{kg})v^2 + 0,$$

or

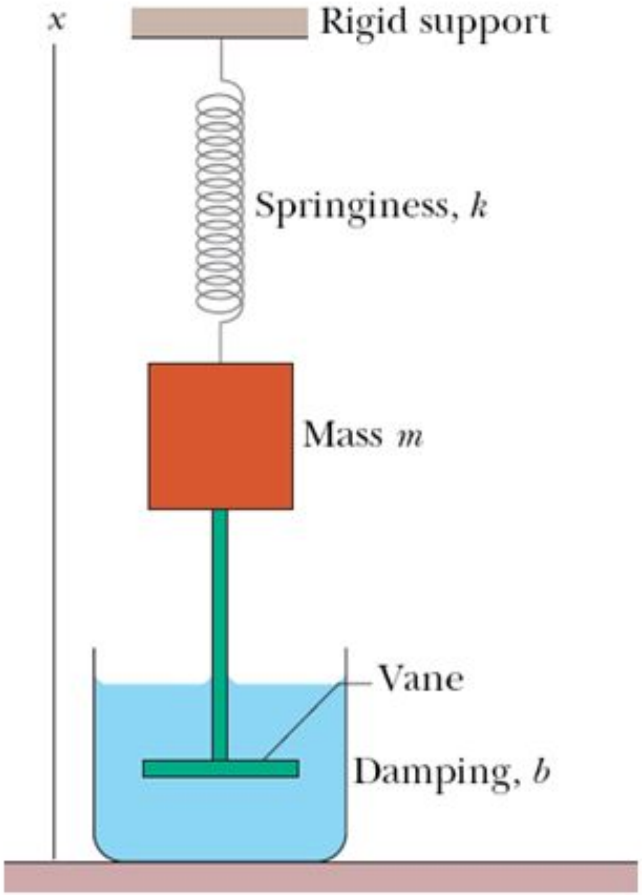
$$v = 12.6\,\text{m/s}. \qquad \text{(Answer)}$$

Because E is entirely kinetic energy, this is the maximum speed v_m.

Damped Oscillations

In a damped oscillation, the motion of the oscillator is reduced by an external force.

Example: A block of mass m oscillates vertically on a spring with spring constant k.
From the block a rod extends to a vane which is submerged in a liquid.
The liquid provides the external damping force, F_d.



Let us assume the liquid exerts a **damping force** F_d that is proportional to the velocity $\vec{v}$ of the vane and block (an assumption that is accurate if the vane moves slowly). Then, for force and velocity components along the x axis in Fig. 15-16, we have

$$F_d = -bv, \tag{15-39}$$

where b is a **damping constant** that depends on the characteristics of both the vane and the liquid and has the SI unit of kilogram per second. The minus sign indicates that $\vec{F}_d$ opposes the motion.

Damped Oscillations. The force on the block from the spring is $F_s = -kx$. Let us assume that the gravitational force on the block is negligible relative to F_d and F_s. Then we can write Newton's second law for components along the x axis ($F_{\text{net},x} = ma_x$) as

$$-bv - kx = ma. \tag{15-40}$$

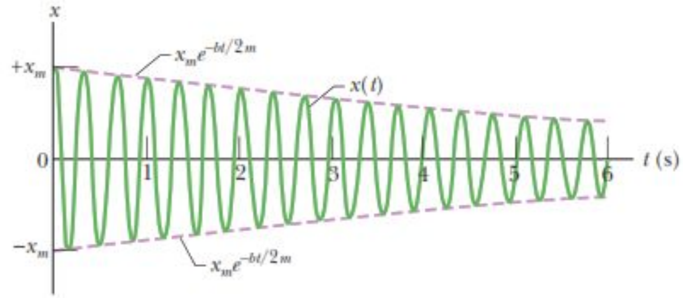


Figure 15-17 The displacement function $x(t)$ for the damped oscillator of Fig. 15-16. The amplitude, which is $x_m e^{-bt/2m}$, decreases exponentially with time.

Substituting dx/dt for v and d^2x/dt^2 for a and rearranging give us the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{15-41}$$

The solution of this equation is

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi), \tag{15-42}$$

where x_m is the amplitude and ω' is the angular frequency of the damped oscillator. This angular frequency is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \tag{15-43}$$

If $b = 0$ (there is no damping), then Eq. 15-43 reduces to Eq. 15-12 ($\omega = \sqrt{k/m}$) for the angular frequency of an undamped oscillator, and Eq. 15-42 reduces to Eq. 15-3 for the displacement of an undamped oscillator. If the damping constant is small but not zero (so that $b \ll \sqrt{km}$), then $\omega' \approx \omega$.

Damped Energy. We can regard Eq. 15-42 as a cosine function whose amplitude, which is $x_m e^{-bt/2m}$, gradually decreases with time, as Fig. 15-17 suggests. For an undamped oscillator, the mechanical energy is constant and is given by Eq. 15-21 ($E = \frac{1}{2}kx_m^2$). If the oscillator is damped, the mechanical energy is not constant but decreases with time. If the damping is small, we can find $E(t)$ by replacing x_m in Eq. 15-21 with $x_m e^{-bt/2m}$, the amplitude of the damped oscillations. By doing so, we find that

$$E(t) \approx \tfrac{1}{2}kx_m^2 e^{-bt/m}, \qquad (15\text{-}44)$$

which tells us that, like the amplitude, the mechanical energy decreases exponentially with time.

Sample Problem 15.06 Damped harmonic oscillator, time to decay, energy

For the damped oscillator of Fig. 15-16, $m = 250$ g, $k = 85$ N/m, and $b = 70$ g/s.

(a) What is the period of the motion?

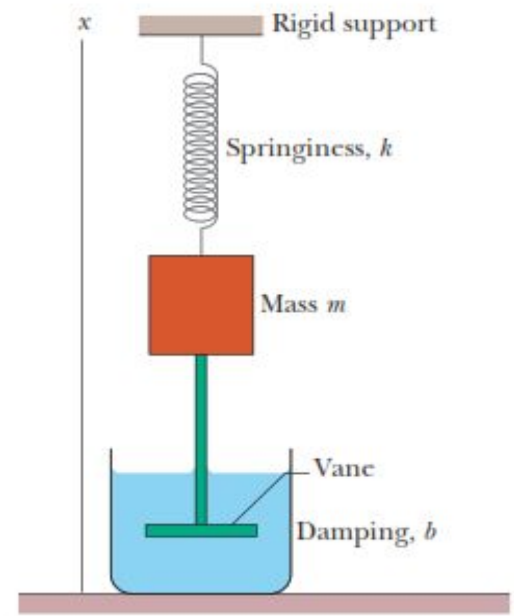


Figure 15-16 An idealized damped simple harmonic oscillator. A vane immersed in a liquid exerts a damping force on the block as the block oscillates parallel to the x axis.

KEY IDEA

Because $b \ll \sqrt{km} = 4.6$ kg/s, the period is approximately that of the undamped oscillator.

Calculation: From Eq. 15-13, we then have

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25\text{ kg}}{85\text{ N/m}}} = 0.34\text{ s.} \qquad \text{(Answer)}$$

For the damped oscillator of Fig. 15-16, $m = 250$ g, $k = 85$ N/m, and $b = 70$ g/s.

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

The amplitude at time t is displayed in Eq. 15-42 as $x_m e^{-bt/2m}$.

Calculations: The amplitude has the value x_m at $t = 0$. Thus, we must find the value of t for which

$$x_m e^{-bt/2m} = \tfrac{1}{2}x_m.$$

Canceling x_m and taking the natural logarithm of the equation that remains, we have $\ln \frac{1}{2}$ on the right side and

$$\ln(e^{-bt/2m}) = -bt/2m$$

on the left side. Thus,

$$t = \frac{-2m \ln \frac{1}{2}}{b} = \frac{-(2)(0.25\text{ kg})(\ln \frac{1}{2})}{0.070\text{ kg/s}}$$

$$= 5.0\text{ s.} \qquad \text{(Answer)}$$

For the damped oscillator of Fig. 15-16, $m = 250$ g, $k = 85$ N/m, and $b = 70$ g/s.

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

KEY IDEA

From Eq. 15-44, the mechanical energy at time t is $\frac{1}{2}kx_m^2\, e^{-bt/m}$.

Calculations: The mechanical energy has the value $\frac{1}{2}kx_m^2$ at $t = 0$. Thus, we must find the value of t for which

$$\tfrac{1}{2}kx_m^2\, e^{-bt/m} = \tfrac{1}{2}\left(\tfrac{1}{2}kx_m^2\right).$$

If we divide both sides of this equation by $\frac{1}{2}kx_m^2$ and solve for t as we did above, we find

$$t = \frac{-m \ln \frac{1}{2}}{b} = \frac{-(0.25\text{ kg})(\ln \frac{1}{2})}{0.070\text{ kg/s}} = 2.5\text{ s.} \qquad \text{(Answer)}$$

This is exactly half the time we calculated in (b), or about 7.5 periods of oscillation. Figure 15-17 was drawn to illustrate this sample problem.