REPRESENTATION OF A WAVE

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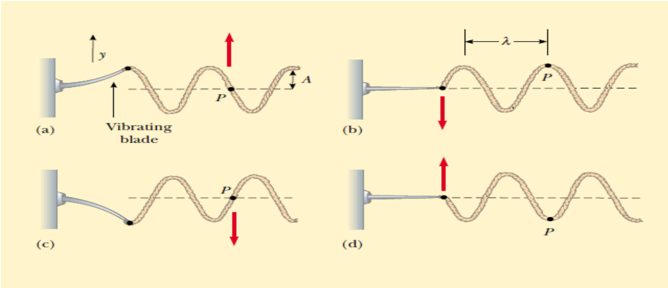
$$y = A\sin(kx - \omega t)$$



WAVE VELOCITY

- Below is the fundamental wave velocity equation where υ is the wave velocity, λ is the wavelength, and ν is the frequency of the wave.

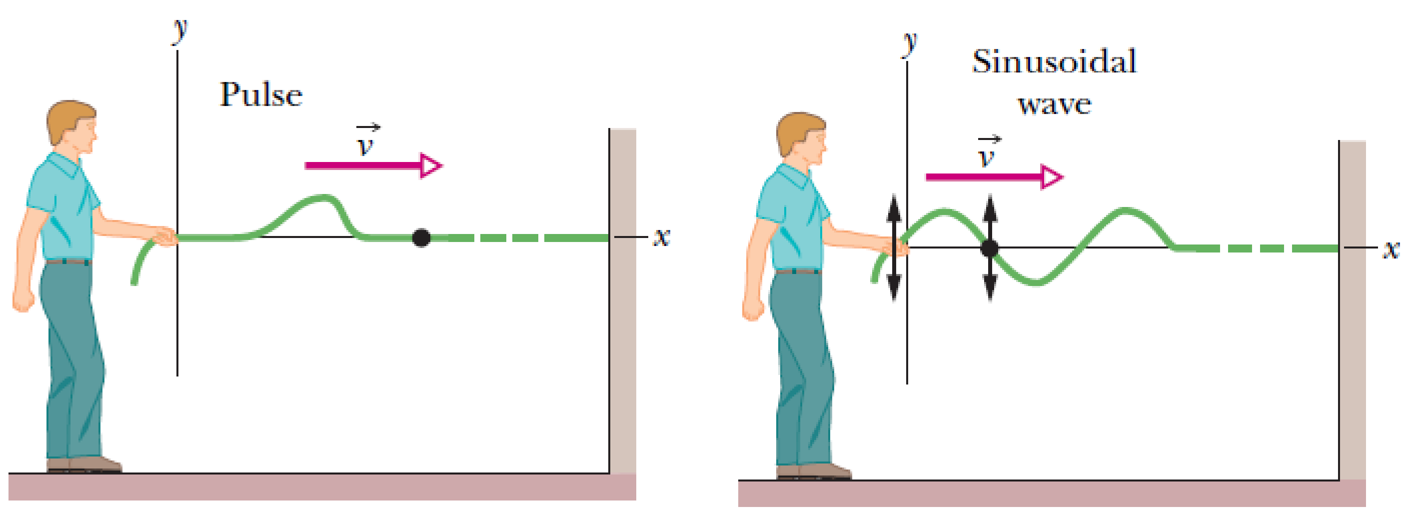
$$\upsilon = \lambda \nu$$

- Frequency is the number of consecutive risings of wave /cycles per second measured in hertz.

- This is a fundamental equation obeyed by all waves.

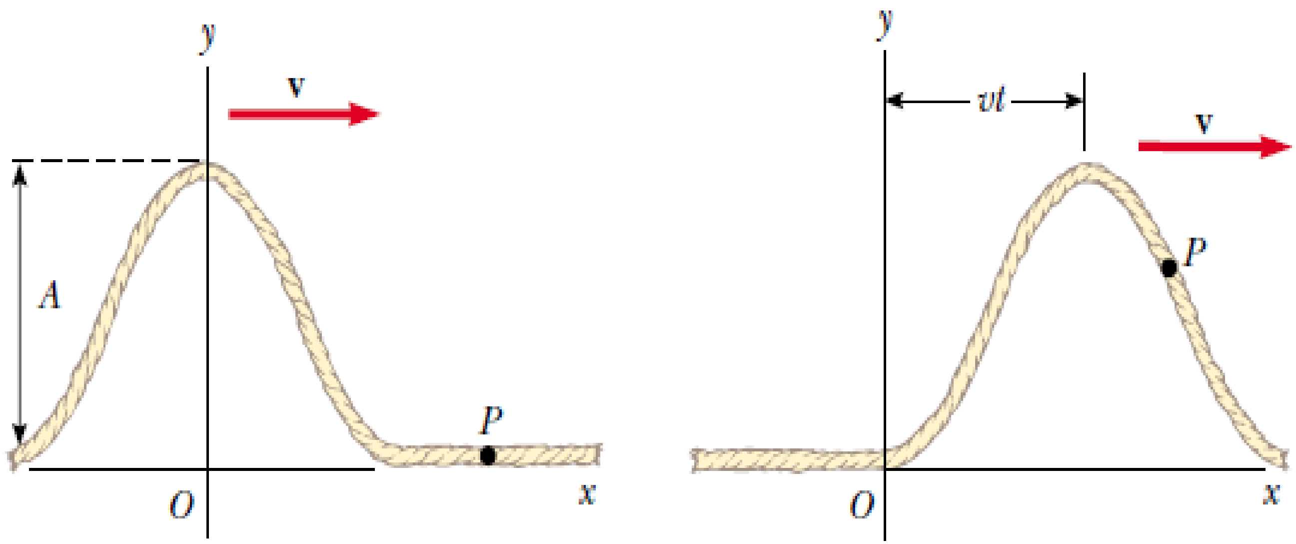
ONE DIMENSIONAL PULSE AND A WAVE FUNCTION

- We need a function that gives the shape of the wave



(a) (b)

One Dimensional Pulse and a Wave Function



A one-dimensional pulse traveling to the right with a speed v. (a) At $t = 0$, the shape of the pulse is given by $y = f(x)$. (b) At some later time t, the shape remains unchanged and the vertical position of an element of the medium any point P is given by $y = f(x - vt)$.

WAVE FUNCTION

- To completely describe a wave on a string (and the motion of any element along its length), we need a function that gives the shape of the wave.
- y is the perpendicular displacement , x is the horizontal distance covered by wave in time t.

Wave Function

Consequently, an element of the string at x at this time has the same y position as an element located at $x - vt$ had at time $t = 0$:

$$y(x, t) = y(x - vt, 0)$$

In general, then, we can represent the transverse position y for all positions and times, measured in a stationary frame with the origin at O, as

$$y(x, t) = f(x - vt)$$ **Pulse traveling to the right**

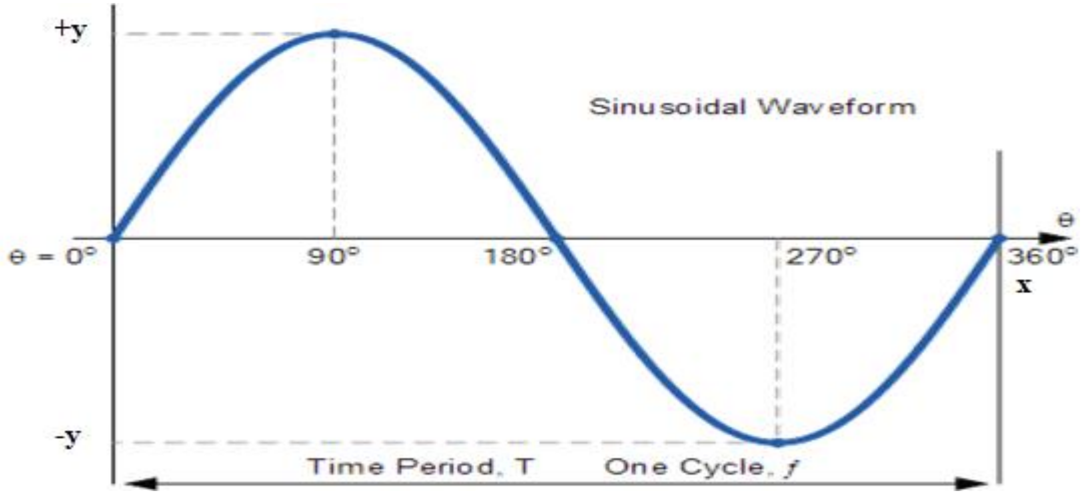
Similarly, if the pulse travels to the left, the transverse positions of elements of the string are described by

$$y(x, t) = f(x + vt)$$ **Pulse traveling to the left**

The function y, sometimes called the **wave function,** depends on the two variables x and t. For this reason, it is often written $y(x, t)$, which is read "y as a function of x and t."

Wave Form

It is important to understand the meaning of y. Consider an element of the string at point P, identified by a particular value of its x coordinate. As the pulse passes through P, the y coordinate of this element increases, reaches a maximum, and then decreases to zero. **The wave function $y(x, t)$ represents the y coordinate—the transverse position—of any element located at position x at any time t.** Furthermore, if t is fixed (as, for example, in the case of taking a snapshot of the pulse), then the wave function $y(x)$, sometimes called the **waveform,** defines a curve representing the actual geometric shape of the pulse at that time.



EXAMPLE PROBLEM 1

- Pulse moving to the Right:

A pulse moving to the right along the x axis is represented by the wave function

$$y(x, t) = \frac{2}{(x - 3.0t)^2 + 1}$$

where x and y are measured in centimeters and t is measured in seconds. Plot the wave function at $t = 0$, $t = 1.0$ s, and $t = 2.0$ s.

Solution First, note that this function is of the form $y = f(x - vt)$. By inspection, we see that the wave speed is $v = 3.0$ cm/s. Furthermore, the maximum value of y is given by $A = 2.0$ cm. (We find the maximum value of the function representing y by letting $x - 3.0t = 0$.) The wave function expressions are

$$y(x, 0) = \frac{2}{x^2 + 1} \qquad \text{at } t = 0$$

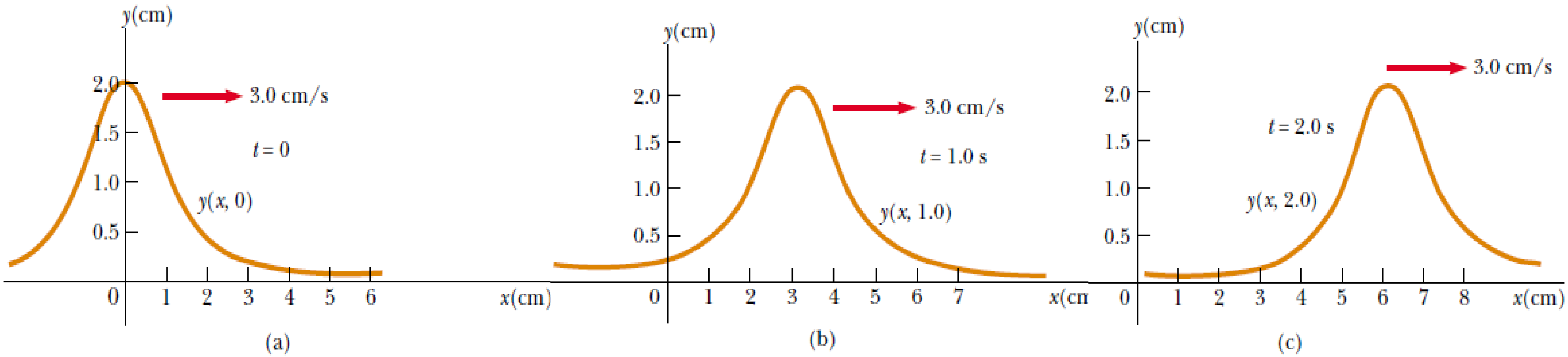
$$y(x, 1.0) = \frac{2}{(x - 3.0)^2 + 1} \qquad \text{at } t = 1.0 \text{ s}$$

$$y(x, 2.0) = \frac{2}{(x - 6.0)^2 + 1} \qquad \text{at } t = 2.0 \text{ s}$$

We now use these expressions to plot the wave function versus x at these times. For example, let us evaluate $y(x, 0)$ at $x = 0.50$ cm:

$$y(0.50, 0) = \frac{2}{(0.50)^2 + 1} = 1.6 \text{ cm}$$

Likewise, at $x = 1.0$ cm, $y(1.0, 0) = 1.0$ cm, and at $\mathbf{x} = 2.0$ cm, $y(2.0, 0) = 0.40$ cm. Continuing this procedure for other values of x yields the wave function shown in Figure **a** . In a similar manner, we obtain the graphs of $y(x, 1.0)$ and $y(x, 2.0)$, shown in Figure b and c respectively. These snapshots show that the pulse moves to the right without changing its shape and that it has a constant speed of 3.0 cm/s.



What If? (A) What if the wave function were

$$y(x, t) = \frac{2}{(x + 3.0t)^2 + 1}$$

How would this change the situation?

Answer (A) The new feature in this expression is the plus sign in the denominator rather than the minus sign. This results in a pulse with the same shape as that in Figure **a,b,c** but moving to the left as time progresses.

(B) What if the wave function were

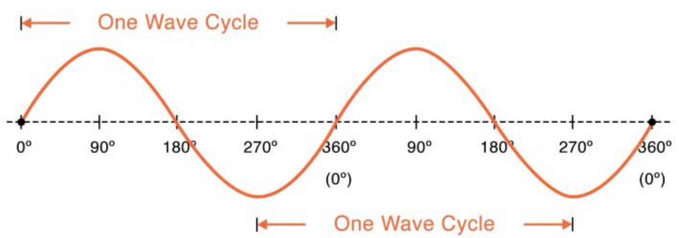
$$y(x, t) = \frac{4}{(x - 3.0t)^2 + 1}$$

How would this change the situation?

(B) The new feature here is the numerator of 4 rather than 2. This results in a pulse moving to the right, but with twice the height of that in Figure **a,b,c**

SINUSOIDAL WAVE

Consider the sinusoidal wave in Figure below, which shows the position of the wave at $t = 0$. Because the wave is sinusoidal, we expect the wave function at this instant to be expressed as $y(x, 0) = A \sin ax$, where A is the amplitude and a is a constant to be determined. At $x = 0$, we see that $y(0, 0) = A \sin a(0) = 0$, consistent with Figure The next value of x for which y is zero is $x = \lambda/2$. Thus,



$$y(x, 0) = A \sin ax,$$

$$y\left(\frac{\lambda}{2}, 0\right) = A \sin a\left(\frac{\lambda}{2}\right)$$

For this to be true, we must have

$$a(\lambda/2) = \pi,$$

$$\text{or } a = 2\pi/\lambda.$$

therefore

$$y(x, 0) = A \sin\left(\frac{2\pi}{\lambda} x\right)$$

$$y(x, 0) = A \sin\left(\frac{2\pi}{\lambda} x\right)$$ at t=0

$$y(x, t) = A \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$ at time "t" .....equation 1

we know $$v = \frac{\lambda}{T}$$

Substituting this expression for v into Equation 1

$$y = A \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

We can express the wave function in a convenient form by defining two other quantities, the **angular wave number** k (usually called simply the **wave number**) and the **angular frequency** ω:

$$k \equiv \frac{2\pi}{\lambda}$$ Angular wave number

$$\omega \equiv \frac{2\pi}{T}$$ Angular frequency

$$k \equiv \frac{2\pi}{\lambda} \qquad \omega \equiv \frac{2\pi}{T}$$

$$y = A \sin(kx - \omega t)$$ Wave function for a sinusoidal wave ...equation 2

we can express the wave speed v originally

in the alternative forms $$v = \frac{\omega}{k}$$

$$v = \lambda f$$

The wave function given by Equation 2 assumes that the vertical position y of an element of the medium is zero at $x = 0$ and $t = 0$. This need not be the case. If it is not, we generally express the wave function in the form

$$y = A \sin(kx - \omega t + \phi)$$ ...Equation 3 General expression for a sinusoidal wave

where ϕ is the **phase constant,**

This constant can be determined from the initial conditions.

Example:

The equation of a transverse wave on a string is

$$y = (2.0\ \text{mm})\ \sin\ [(20\ \text{m}m^{-1})\ x - (600\ \ s^{-1})\ t]$$

Calculate (i) Wavelength (ii) Time period (iii) Wave speed
(iv) Frequency (v) Wave number

y_m=2mm, k=2mm^{-1}, ω=600s^{-1}

(i) λ=2π/K=0.3142 mm

(ii) T=2π/ω=2π/600=0.01047sec

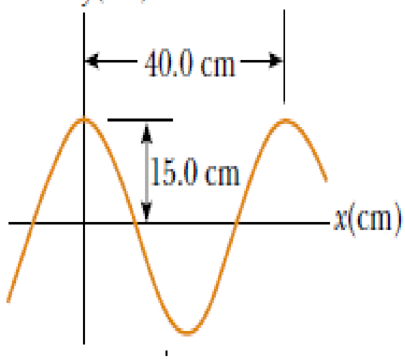
(iii) v=λ/T=30mm/s

(iv) f=1/T=3.1415Hz

(v) K=20mm^{-1}

EXAMPLE PROBLEM 2

A sinusoidal wave traveling in the positive x direction has an amplitude of 15.0 cm, a wavelength of 40.0 cm, and a frequency of 8.00 Hz. The vertical position of an element of the medium at $t = 0$ and $x = 0$ is also 15.0 cm, as shown in Figure below



A sinusoidal wave of wavelength $\lambda = 40.0$ cm and amplitude $A = 15.0$ cm. The wave function can be written in the form $y = A\cos(kx - \omega t)$.

(A) Find the wave number k, period T, angular frequency ω, and speed v of the wave.

(B) Determine the phase constant ϕ, and write a general expression for the wave function.

Solution **A** Using Equations below we find the following:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{40.0 \text{ cm}} = \boxed{0.157 \text{ rad/cm}}$$

$$T = \frac{1}{f} = \frac{1}{8.00 \text{ s}^{-1}} = \boxed{0.125 \text{ s}}$$

$$\omega = 2\pi f = 2\pi(8.00 \text{ s}^{-1}) = \boxed{50.3 \text{ rad/s}}$$

$$v = \lambda f = (40.0 \text{ cm})(8.00 \text{ s}^{-1}) = \boxed{320 \text{ cm/s}}$$

Solution **B** Because $A = 15.0$ cm and because $y = 15.0$ cm at $x = 0$ and $t = 0$, substitution into

Equation $y = A\sin(kx - \omega t + \phi)$

$$15.0 = (15.0)\sin\phi \quad \text{or} \quad \sin\phi = 1$$

We may take the principal value $\phi = \pi/2$ rad (or $90°$). Hence, the wave function is of the form

$$y = A\sin\left(kx - \omega t + \frac{\pi}{2}\right) = A\cos(kx - \omega t)$$

By inspection, we can see that the wave function must have this form, noting that the cosine function has the same shape as the sine function displaced by $90°$. Substituting the values for A, k, and ω into this expression, we obtain

$$y = \boxed{(15.0 \text{ cm})\cos(0.157x - 50.3t)}$$

Example Problem 3

The string shown in Figure 16.10 is driven at a frequency of 5.00 Hz. The amplitude of the motion is 12.0 cm, and the wave speed is 20.0 m/s. Determine the angular frequency ω and wave number k for this wave, and write an expression for the wave function.

Solution Using Equations below we find that

$$\omega = \frac{2\pi}{T} = 2\pi f = 2\pi(5.00\ \text{Hz}) = \boxed{31.4\ \text{rad/s}}$$

$$k = \frac{\omega}{v} = \frac{31.4\ \text{rad/s}}{20.0\ \text{m/s}} = \boxed{1.57\ \text{rad/m}}$$

Because $A = 12.0\ \text{cm} = 0.120\ \text{m}$, we have

$$y = A\sin(kx - \omega t)$$

$$= \boxed{(0.120\ \text{m})\sin(1.57x - 31.4t)}$$

PRACTICE PROBLEMS

1. At $t = 0$, a transverse pulse in a wire is described by the function

$$y = \frac{6}{x^2 + 3}$$

where x and y are in meters. Write the function $y(x, t)$ that describes this pulse if it is traveling in the positive x direction with a speed of 4.50 m/s.

2. Ocean waves with a crest-to-crest distance of 10.0 m can be described by the wave function

$$y(x, t) = (0.800 \text{ m}) \sin[0.628(x - vt)]$$

where $v = 1.20$ m/s. (a) Sketch $y(x, t)$ at $t = 0$. (b) Sketch $y(x, t)$ at $t = 2.00$ s. Note that the entire wave form has shifted 2.40 m in the positive x direction in this time interval.

3. A pulse moving along the x axis is described by

$$y(x, t) = 5.00e^{-(x+5.00t)^2}$$

where x is in meters and t is in seconds. Determine (a) the direction of the wave motion, and (b) the speed of the pulse.

4. For a certain transverse wave, the distance between two successive crests is 1.20 m, and eight crests pass a given point along the direction of travel every 12.0 s. Calculate the wave speed

5 A sinusoidal wave on a string is described by

$$y = (0.51 \text{ cm}) \sin(kx - \omega t)$$

where $k = 3.10$ rad/cm and $\omega = 9.30$ rad/s. How far does a wave crest move in 10.0 s? Does it move in the positive or negative x direction?

SINUSOIDAL WAVE ON STRING

We can use this expression to describe the motion of any element of the string. An element at point P (or any other element of the string) moves only vertically, and so its x coordinate remains constant. Therefore, the **transverse speed** v_y (not to be confused with the wave speed v) and the **transverse acceleration** a_y of elements of the string are

$$v_y = \left.\frac{dy}{dt}\right]_{x = \text{constant}} = \frac{\partial y}{\partial t} = -\omega A \cos(kx - \omega t)$$

$$a_y = \left.\frac{dv_y}{dt}\right]_{x = \text{constant}} = \frac{\partial v_y}{\partial t} = -\omega^2 A \sin(kx - \omega t)$$

In these expressions, we must use partial derivatives because y depends on both x and t. In the operation $\partial y/\partial t$, for example, we take a derivative with respect to t while holding x constant. The maximum values of the transverse speed and transverse acceleration are simply the absolute values of the coefficients of the cosine and sine functions:

$$v_{y,\,\max} = \omega A$$

$$a_{y,\,\max} = \omega^2 A$$

The transverse speed and transverse acceleration of elements of the string do not reach their maximum values simultaneously. The transverse speed reaches its maximum value (ωA) when $y = 0$, whereas the magnitude of the transverse acceleration reaches its maximum value ($\omega^2 A$) when $y = \pm A$.