13.5 THE CHAIN RULE

In this section we will derive versions of the chain rule for functions of two or three variables. These new versions will allow us to generate useful relationships among the derivatives and partial derivatives of various functions.

CHAIN RULES FOR DERIVATIVES

If y is a differentiable function of x and x is a differentiable function of t, then the chain rule for functions of one variable states that, under composition, y becomes a differentiable function of t with

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

We will now derive a version of the chain rule for functions of two variables.

Assume that $z = f(x, y)$ is a function of x and y, and suppose that x and y are in turn functions of a single variable t, say

$$x = x(t), \quad y = y(t)$$

13.5.1 THEOREM (Chain Rules for Derivatives) *If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and*

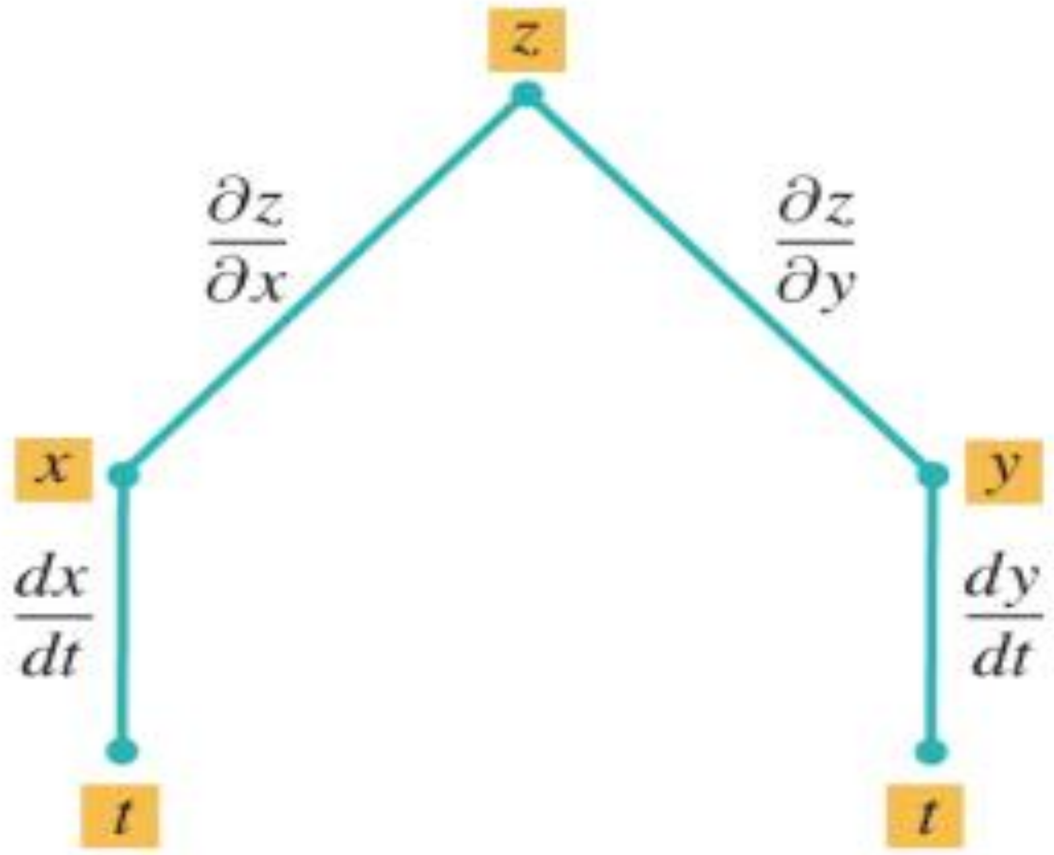
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{5}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then the function $w = f(x(t), y(t), z(t))$ is differentiable at t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \tag{6}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).



$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Figure 13.5.1

Example 1 Suppose that

$$z = x^2 y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find dz/dt, and check the result by expressing z as a function of t and differentiating directly.

Solution. By the chain rule [Formula (5)],

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2)$$
$$= (2t^5)(2t) + (t^4)(3t^2) = 7t^6$$

Alternatively, we can express z directly as a function of t,

$$z = x^2y = (t^2)^2(t^3) = t^7$$

and then differentiate to obtain $dz/dt = 7t^6$. However, this procedure may not always be convenient. ◀

Example 2 Suppose that

$$w = \sqrt{x^2 + y^2 + z^2}, \quad x = \cos\theta, \quad y = \sin\theta, \quad z = \tan\theta$$

Use the chain rule to find $dw/d\theta$ when $\theta = \pi/4$.

Solution. From Formula (6) with θ in the place of t, we obtain

$$\begin{aligned}\frac{dw}{d\theta} &= \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta} \\ &= \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(-\sin\theta) + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(\cos\theta) \\ &\quad + \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(\sec^2\theta)\end{aligned}$$

When $\theta = \pi/4$, we have

$$x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad z = \tan\frac{\pi}{4} = 1$$

Substituting $x = 1/\sqrt{2}, y = 1/\sqrt{2}, z = 1, \theta = \pi/4$ in the formula for $dw/d\theta$ yields

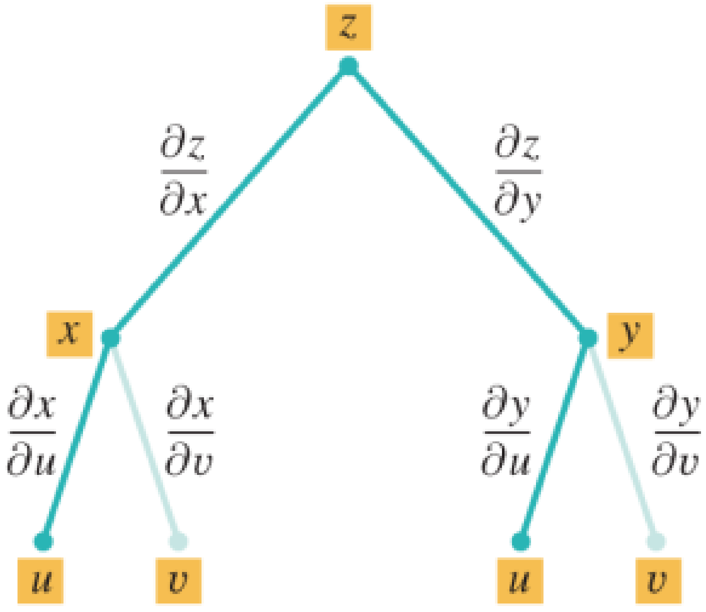
$$\begin{aligned}\left.\frac{dw}{d\theta}\right|_{\theta=\pi/4} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)(2)(2) \\ &= \sqrt{2} \blacktriangleleft\end{aligned}$$

13.5.2 THEOREM ***(Chain Rules for Partial Derivatives)*** *If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v), and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first-order partial derivatives at the point (u, v) given by*

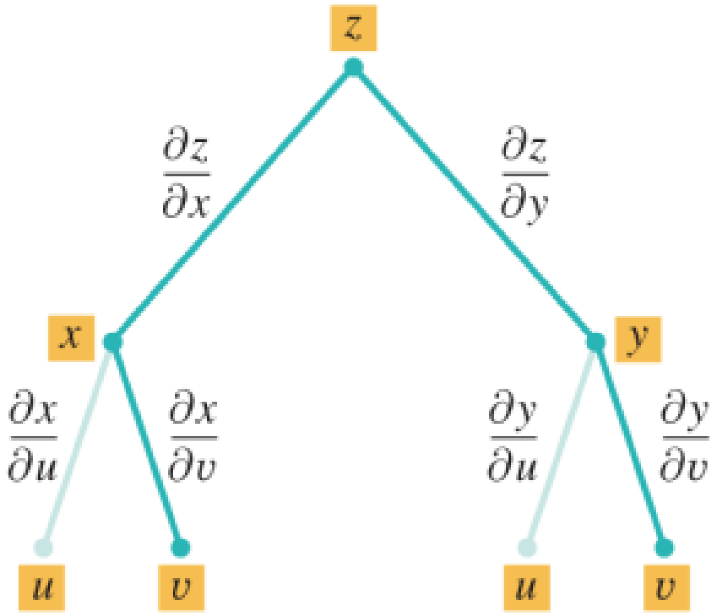
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \tag{7–8}$$

If each function $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ has first-order partial derivatives at the point (u, v), and if the function $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(u, v), y(u, v), z(u, v))$, then $w = f(x(u, v), y(u, v), z(u, v))$ has first-order partial derivatives at the point (u, v) given by

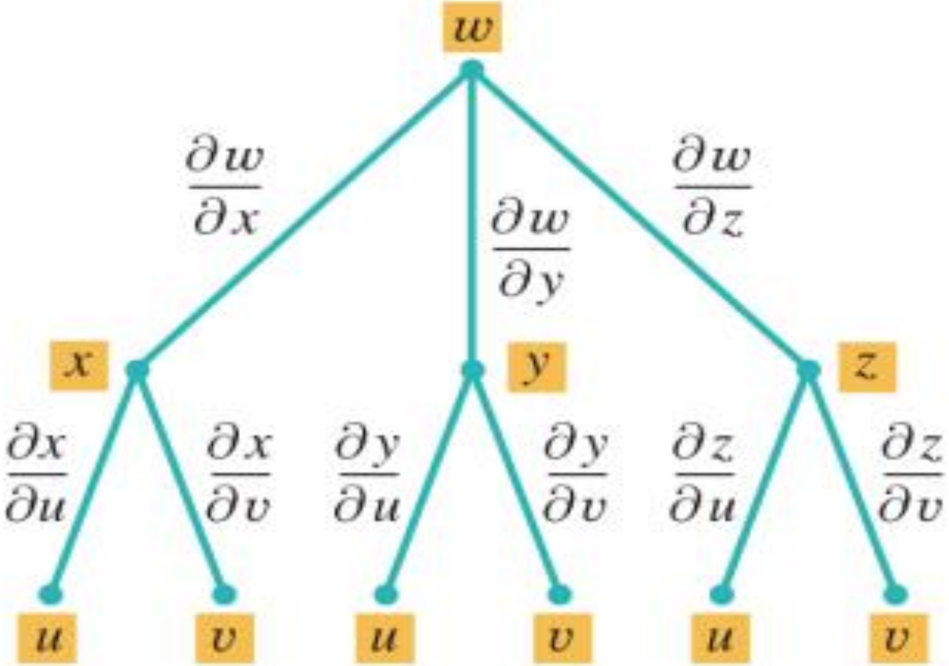
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \tag{9–10}$$



$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}$$



$$\boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$



$$\boxed{\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \end{aligned}}$$

▲ **Figure 13.5.3**

Example 3 Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u/v$$

find $\partial z/\partial u$ and $\partial z/\partial v$ using the chain rule.

Solution.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right]e^{xy}$$

$$= \left[\frac{2u}{v} + \frac{2u+v}{v}\right]e^{(2u+v)(u/v)} = \left[\frac{4u}{v} + 1\right]e^{(2u+v)(u/v)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right)$$

$$= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)(u/v)}$$

$$= -\frac{2u^2}{v^2}e^{(2u+v)(u/v)} \blacktriangleleft$$

▶ Example 4 Suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

Solution. From the tree diagram and corresponding formulas in Figure 13.5.3 we obtain

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

and

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

If desired, we can express $\partial w/\partial u$ and $\partial w/\partial v$ in terms of u and v alone by replacing x, y, and z by their expressions in terms of u and v. ◀

OTHER VERSIONS OF THE CHAIN RULE

Although we will not prove it, the chain rule extends to functions $w = f(v_1, v_2, \ldots, v_n)$ of n variables. For example, if each v_i is a function of t, $i = 1, 2, \ldots, n$, the relevant formula is

$$\frac{dw}{dt} = \frac{\partial w}{\partial v_1}\frac{dv_1}{dt} + \frac{\partial w}{\partial v_2}\frac{dv_2}{dt} + \cdots + \frac{\partial w}{\partial v_n}\frac{dv_n}{dt} \tag{11}$$

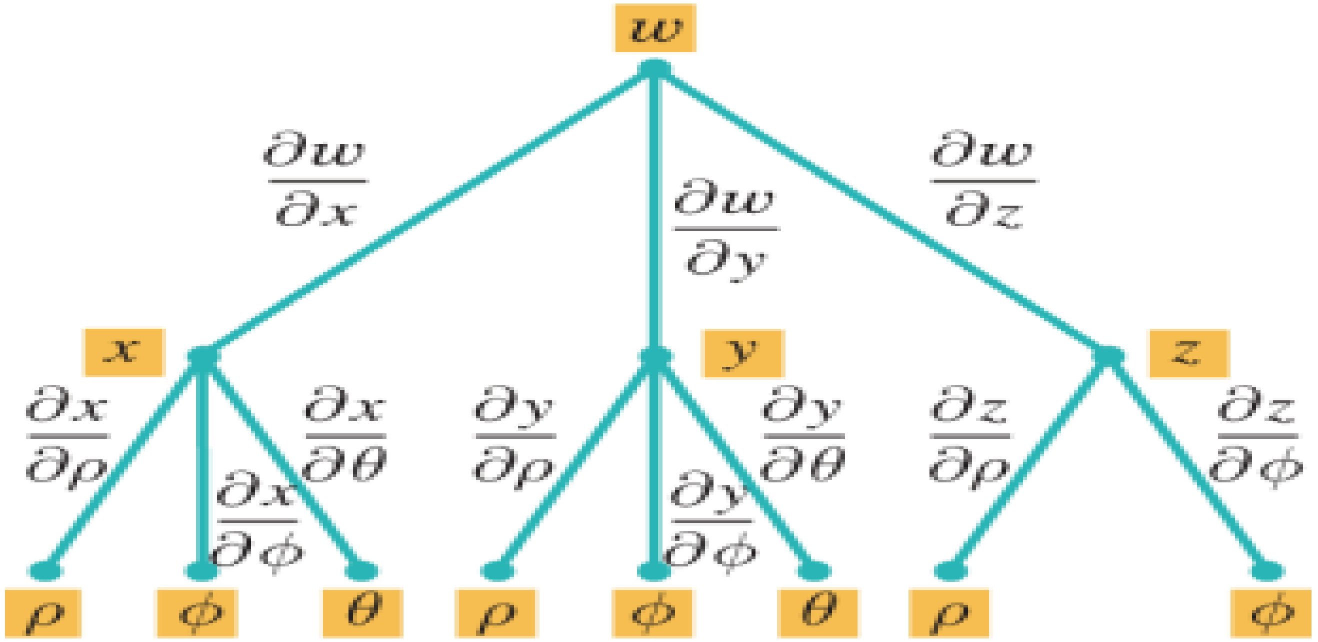
Note that (11) is a natural extension of Formulas (5) and (6) in Theorem 13.5.1.

There are infinitely many variations of the chain rule, depending on the number of variables and the choice of independent and dependent variables. A good working procedure is to use tree diagrams to derive new versions of the chain rule as needed.

▶ **Example 5** Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

Use appropriate forms of the chain rule to find $\partial w/\partial \rho$ and $\partial w/\partial \theta$.



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

Figure 13.5.4

Solution. From the tree diagram and corresponding formulas in Figure 13.5.4 we obtain

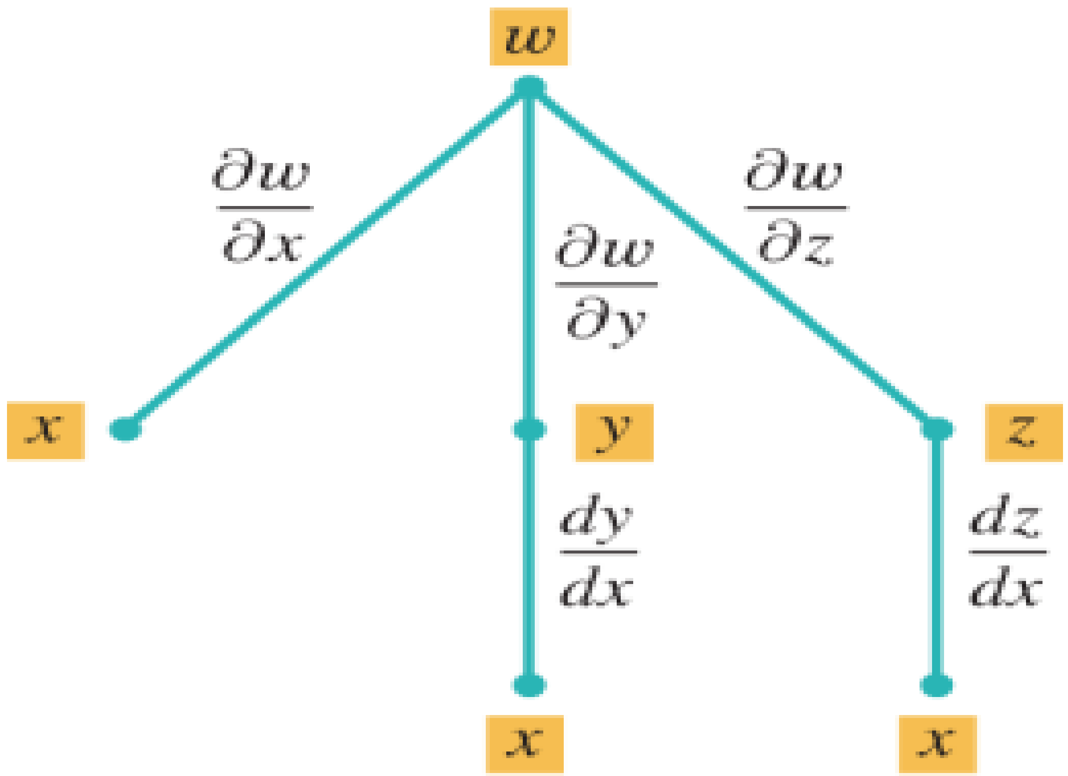
$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= 2x\sin\phi\cos\theta + 2y\sin\phi\sin\theta - 2z\cos\phi \\
&= 2\rho\sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\rho\cos^2\phi \\
&= 2\rho\sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\rho\cos^2\phi \\
&= 2\rho(\sin^2\phi - \cos^2\phi) \\
&= -2\rho\cos 2\phi \\
\frac{\partial w}{\partial \theta} &= (2x)(-\rho\sin\phi\sin\theta) + (2y)\rho\sin\phi\cos\theta \\
&= -2\rho^2\sin^2\phi\sin\theta\cos\theta + 2\rho^2\sin^2\phi\sin\theta\cos\theta \\
&= 0
\end{aligned}$$

This result is explained by the fact that w does not vary with θ. You can see this directly by expressing the variables x, y, and z in terms of ρ, ϕ, and θ in the formula for w. (Verify that $w = -\rho^2\cos 2\phi$.) ◀

► Example 6 Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find dw/dx.



$$\boxed{\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx}}$$

Figure 13.5.5

Solution. From the tree diagram and corresponding formulas in Figure 13.5.5 we obtain

$$\begin{aligned}\frac{dw}{dx} &= y + (x+z)\cos x + ye^x \\ &= \sin x + (x + e^x)\cos x + e^x \sin x\end{aligned}$$

This result can also be obtained by first expressing w explicitly in terms of x as

$$w = x\sin x + e^x \sin x$$

and then differentiating with respect to x; however, such direct substitution is not always possible. ◀

1–6 Use an appropriate form of the chain rule to find dz/dt.

1. $z = 3x^2y^3;\ x = t^4, y = t^2$
2. $z = \ln(2x^2 + y);\ x = \sqrt{t}, y = t^{2/3}$
3. $z = 3\cos x - \sin xy;\ x = 1/t, y = 3t$
4. $z = \sqrt{1 + x - 2xy^4};\ x = \ln t, y = t$
5. $z = e^{1-xy};\ x = t^{1/3}, y = t^3$
6. $z = \cosh^2 xy;\ x = t/2, y = e^t$

7–10 Use an appropriate form of the chain rule to find dw/dt.

7. $w = 5x^2y^3z^4$; $x = t^2, y = t^3, z = t^5$

8. $w = \ln(3x^2 - 2y + 4z^3)$; $x = t^{1/2}, y = t^{2/3}, z = t^{-2}$

9. $w = 5\cos xy - \sin xz$; $x = 1/t, y = t, z = t^3$

10. $w = \sqrt{1 + x - 2yz^4x}$; $x = \ln t, y = t, z = 4t$

17–22 Use appropriate forms of the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

17. $z = 8x^2y - 2x + 3y;\ x = uv,\ y = u - v$

18. $z = x^2 - y\tan x;\ x = u/v,\ y = u^2v^2$

19. $z = x/y;\ x = 2\cos u,\ y = 3\sin v$

20. $z = 3x - 2y;\ x = u + v\ln u,\ y = u^2 - v\ln v$

21. $z = e^{x^2y};\ x = \sqrt{uv},\ y = 1/v$

22. $z = \cos x \sin y;\ x = u - v,\ y = u^2 + v^2$

23–30 Use appropriate forms of the chain rule to find the derivatives.

23. Let $T = x^2y - xy^3 + 2$; $x = r\cos\theta$, $y = r\sin\theta$. Find $\partial T/\partial r$ and $\partial T/\partial\theta$.

24. Let $R = e^{2s-t^2}$; $s = 3\phi$, $t = \phi^{1/2}$. Find $dR/d\phi$.

25. Let $t = u/v$; $u = x^2 - y^2$, $v = 4xy^3$. Find $\partial t/\partial x$ and $\partial t/\partial y$.

26. Let $w = rs/(r^2 + s^2)$; $r = uv$, $s = u - 2v$. Find $\partial w/\partial u$ and $\partial w/\partial v$.

27. Let $z = \ln(x^2 + 1)$, where $x = r\cos\theta$. Find $\partial z/\partial r$ and $\partial z/\partial\theta$.

28. Let $u = rs^2 \ln t$, $r = x^2$, $s = 4y + 1$, $t = xy^3$. Find $\partial u/\partial x$ and $\partial u/\partial y$.

29. Let $w = 4x^2 + 4y^2 + z^2$, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Find $\partial w/\partial\rho$, $\partial w/\partial\phi$, and $\partial w/\partial\theta$.

30. Let $w = 3xy^2z^3$, $y = 3x^2 + 2$, $z = \sqrt{x - 1}$. Find dw/dx.

31. Use a chain rule to find the value of $\left.\dfrac{dw}{ds}\right|_{s=1/4}$ if $w = r^2 - r\tan\theta;\ r = \sqrt{s},\ \theta = \pi s.$

32. Use a chain rule to find the values of

$$\left.\frac{\partial f}{\partial u}\right|_{u=1,v=-2} \quad \text{and} \quad \left.\frac{\partial f}{\partial v}\right|_{u=1,v=-2}$$

if $f(x, y) = x^2y^2 - x + 2y;\ x = \sqrt{u},\ y = uv^3.$

33. Use a chain rule to find the values of

$$\left.\frac{\partial z}{\partial r}\right|_{r=2,\theta=\pi/6} \quad \text{and} \quad \left.\frac{\partial z}{\partial \theta}\right|_{r=2,\theta=\pi/6}$$

if $z = xye^{x/y};\ \ x = r\cos\theta,\ y = r\sin\theta.$

34. Use a chain rule to find $\left.\dfrac{dz}{dt}\right|_{t=3}$ if $z = x^2y;\ x = t^2, y = t + 7.$

35. Let a and b denote two sides of a triangle and let θ denote the included angle. Suppose that a, b, and θ vary with time in such a way that the area of the triangle remains constant. At a certain instant $a = 5$ cm, $b = 4$ cm, and $\theta = \pi/6$ radians, and at that instant both a and b are increasing at a rate of 3 cm/s. Estimate the rate at which θ is changing at that instant.

instant.

36. The voltage, V (in volts), across a circuit is given by Ohm's law: $V = IR$, where I is the current (in amperes) flowing through the circuit and R is the resistance (in ohms). If two circuits with resistances R_1 and R_2 are connected in parallel, then their combined resistance, R, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that the current is 3 amperes and is increasing at 10^{-2} ampere/s, R_1 is 2 ohms and is increasing at 0.4 ohm/s, and R_2 is 5 ohms and is decreasing at 0.7 ohm/s. Estimate the rate at which the voltage is changing.

IMPLICIT DIFFERENTIATION

Consider the special case where $z = f(x, y)$ is a function of x and y and y is a differentiable function of x. Equation (5) then becomes

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{12}$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \tag{13}$$

defines y implicitly as a differentiable function of x and we are interested in finding dy/dx. Differentiating both sides of (13) with respect to x and applying (12) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Thus, if $\partial f/\partial y \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

In summary, we have the following result.

13.5.3 THEOREM *If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\partial f/\partial y \neq 0$, then*

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{14}$$

Example 7 Given that

$$x^3 + y^2x - 3 = 0$$

find dy/dx using (14), and check the result using implicit differentiation.

Solution. By (14) with $f(x, y) = x^3 + y^2x - 3$,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Alternatively, differentiating implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

which agrees with the result obtained by (14). ◀

13.5.4 THEOREM *If the equation $f(x, y, z) = c$ defines z implicitly as a differentiable function of x and y, and if $\partial f/\partial z \neq 0$, then*

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

▶ Example 8 Consider the sphere $x^2+y^2+z^2=1$. Find $\partial z/\partial x$ and $\partial z/\partial y$ at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Solution. By Theorem 13.5.4 with $f(x, y, z)=x^2+y^2+z^2$,

$$\frac{\partial z}{\partial x}=-\frac{\partial f/\partial x}{\partial f/\partial z}=-\frac{2x}{2z}=-\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y}=-\frac{\partial f/\partial y}{\partial f/\partial z}=-\frac{2y}{2z}=-\frac{y}{z}$$

At the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$, evaluating these derivatives gives $\partial z/\partial x=-1$ and $\partial z/\partial y=-\frac{1}{2}$. ◀

41–44 Use Theorem 13.5.3 to find dy/dx and check your result using implicit differentiation.

41. $x^2y^3 + \cos y = 0$

42. $x^3 - 3xy^2 + y^3 = 5$

43. $e^{xy} + ye^y = 1$

44. $x - \sqrt{xy} + 3y = 4$

45–48 Find $\partial z/\partial x$ and $\partial z/\partial y$ by implicit differentiation, and confirm that the results obtained agree with those predicted by the formulas in Theorem 13.5.4.

45. $x^2 - 3yz^2 + xyz - 2 = 0$

46. $\ln(1 + z) + xy^2 + z = 1$

47. $ye^x - 5\sin 3z = 3z$

48. $e^{xy}\cos yz - e^{yz}\sin xz + 2 = 0$