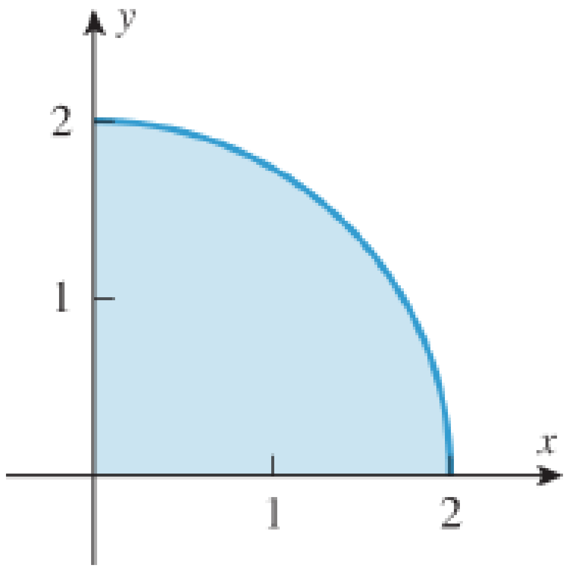
14.3 DOUBLE INTEGRALS IN POLAR COORDINATES

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.

SIMPLE POLAR REGIONS

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. For example, the quarter-disk in Figure 14.3.1 is described in rectangular coordinates by

$$0 \le y \le \sqrt{4 - x^2}, \quad 0 \le x \le 2$$



▲ Figure 14.3.1

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

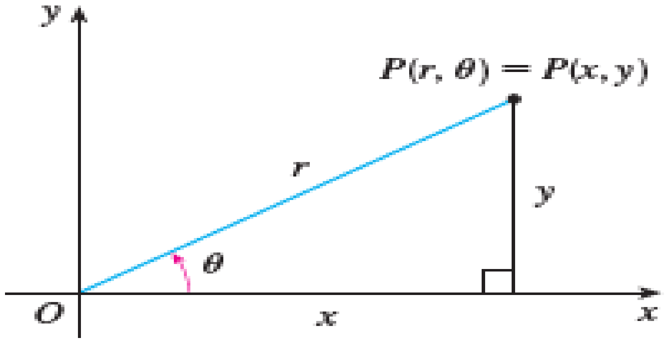


FIGURE 2

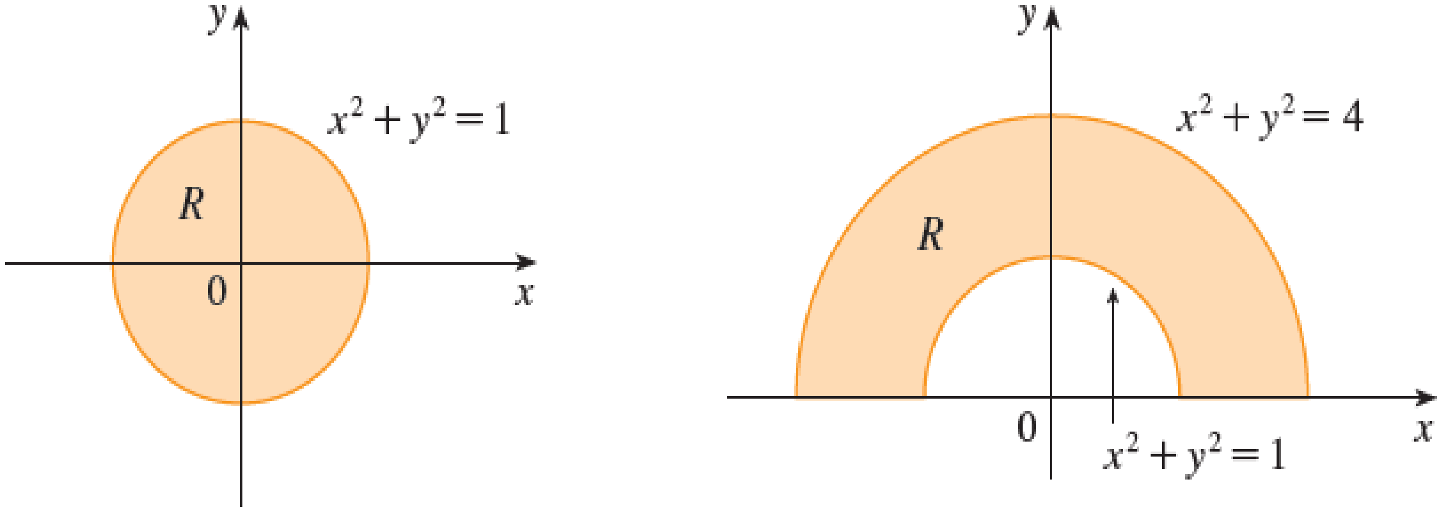
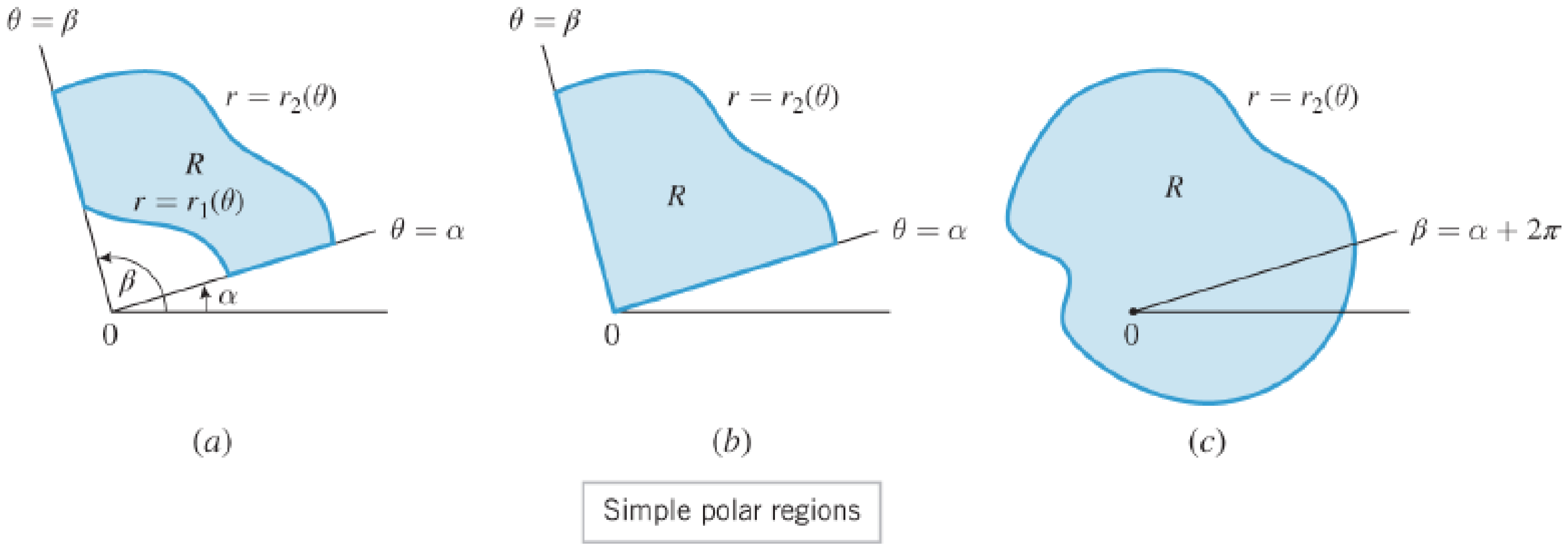


FIGURE 1 (a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

14.3.1 DEFINITION A *simple polar region* in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \leq \beta$ (ii) $\beta - \alpha \leq 2\pi$ (iii) $0 \leq r_1(\theta) \leq r_2(\theta)$



▲ **Figure 14.3.2**

DOUBLE INTEGRALS IN POLAR COORDINATES

Next we will consider the polar version of Problem 14.1.1.

> **14.3.2 THE VOLUME PROBLEM IN POLAR COORDINATES** Given a function $f(r, \theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$ (Figure 14.3.4).

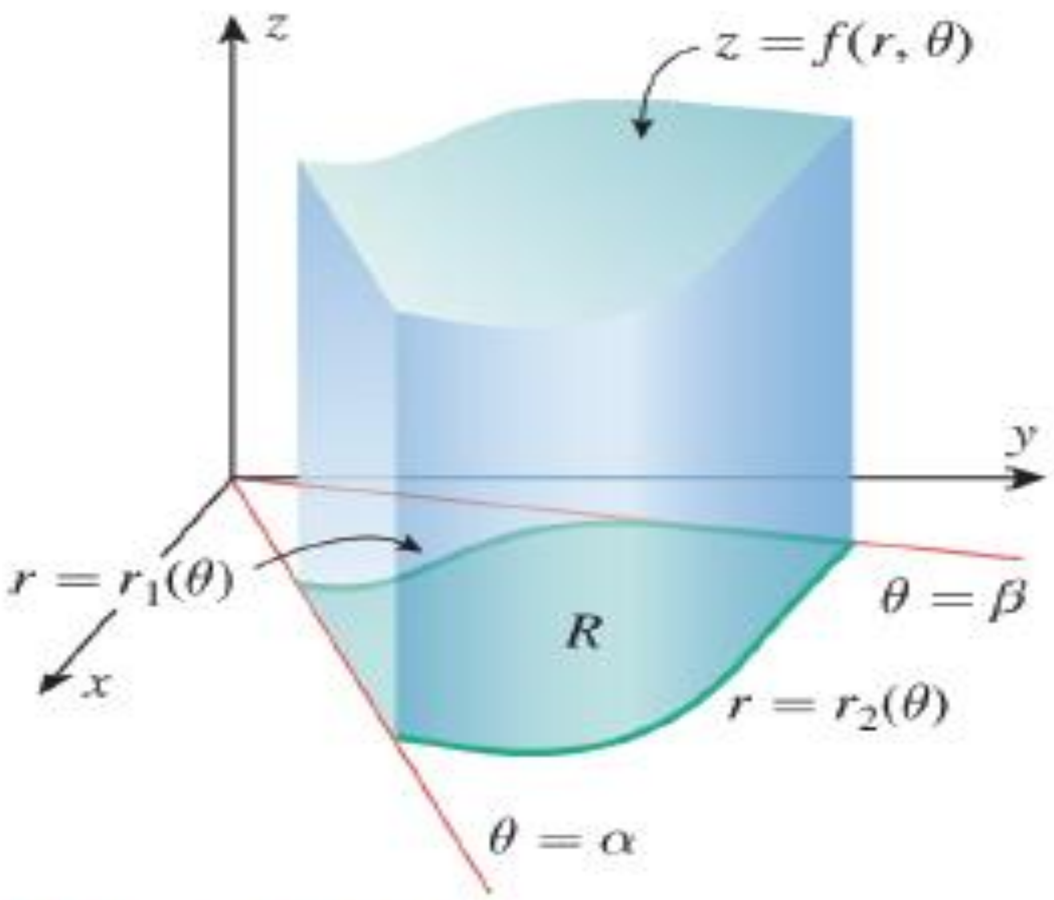


Figure 14.3.4

To motivate a formula for the volume V of the solid in Figure 14.3.4, we will use a limit process similar to that used to obtain Formula (2) of Section 14.1, except that here we will use circular arcs and rays to subdivide the region R into polar rectangles. As shown in Figure 14.3.5, we will exclude from consideration all polar rectangles that contain any points outside of R, leaving only polar rectangles that are subsets of R. Assume that there are n such polar rectangles, and denote the area of the kth polar rectangle by ΔA_k. Let (r_k^*, θ_k^*) be any point in this polar rectangle. As shown in Figure 14.3.6, the product $f(r_k^*, \theta_k^*)\Delta A_k$ is the volume of a solid with base area ΔA_k and height $f(r_k^*, \theta_k^*)$, so the sum

$$\sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

can be viewed as an approximation to the volume V of the entire solid.

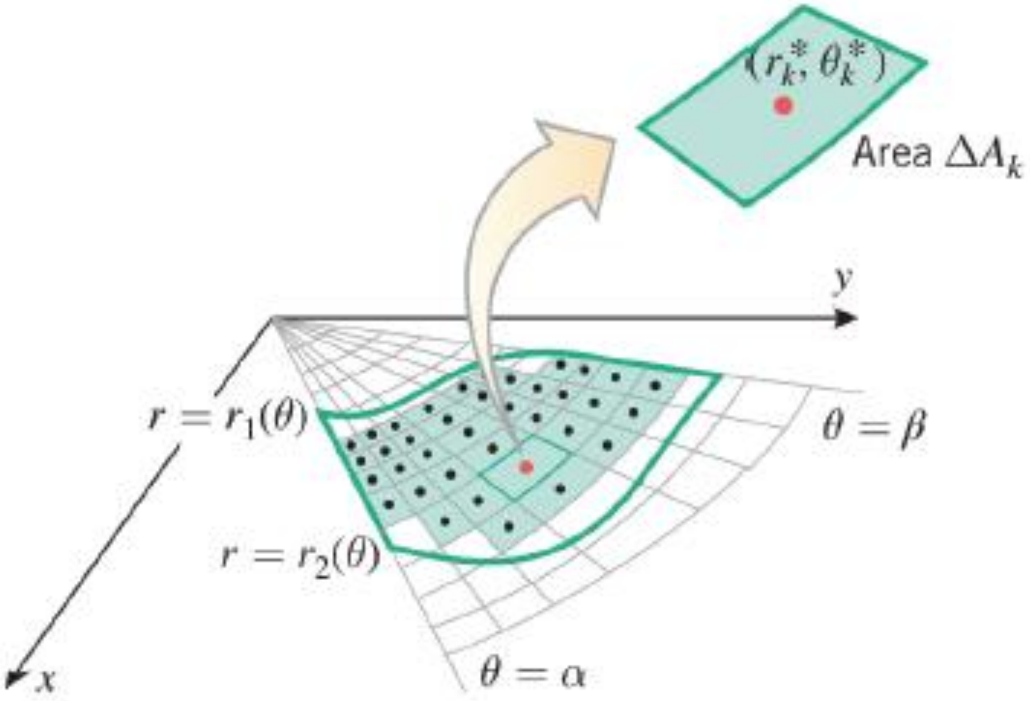


Figure 14.3.5

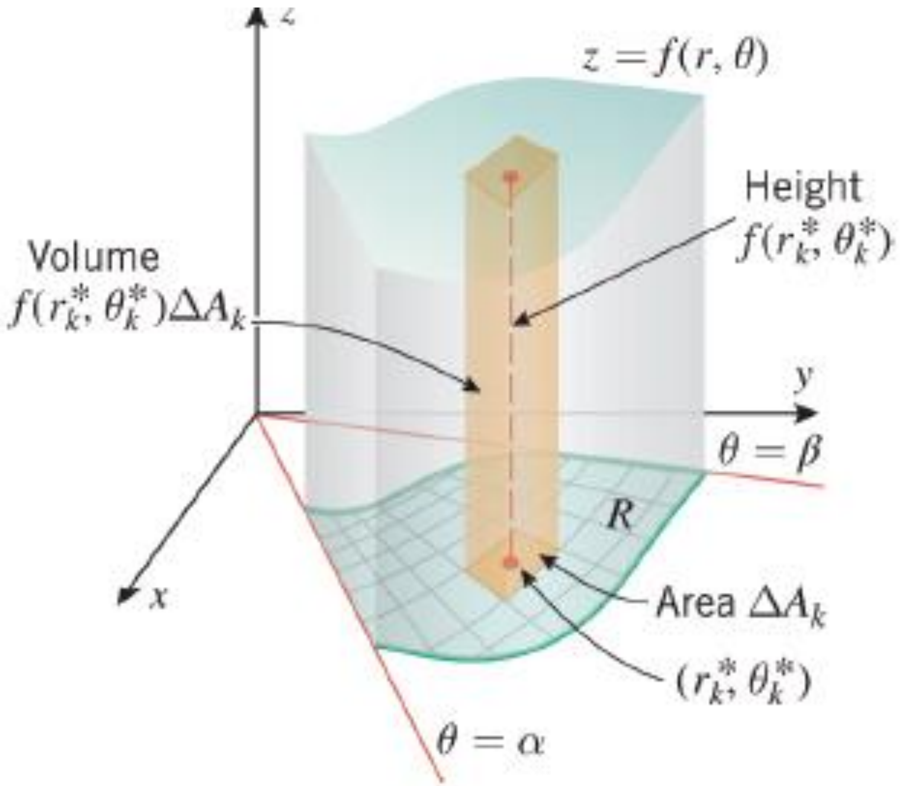


Figure 14.3.6

EXERCISE SET 14.3

1–6 Evaluate the iterated integral.

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\, dr\, d\theta$

2. $\displaystyle\int_0^{\pi}\int_0^{1+\cos\theta} r\, dr\, d\theta$

3. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\, dr\, d\theta$

4. $\displaystyle\int_0^{\pi/6}\int_0^{\cos 3\theta} r\, dr\, d\theta$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$

6. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^3\, dr\, d\theta$

Solution
1 to 6

1. $$\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{2}\sin^2\theta\cos\theta\,d\theta = \frac{1}{6}.$$

2. $$\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi}\frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{3\pi}{4}.$$

3. $$\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\,dr\,d\theta = \int_0^{\pi/2}\frac{a^3}{3}\sin^3\theta\,d\theta = \frac{2}{9}a^3.$$

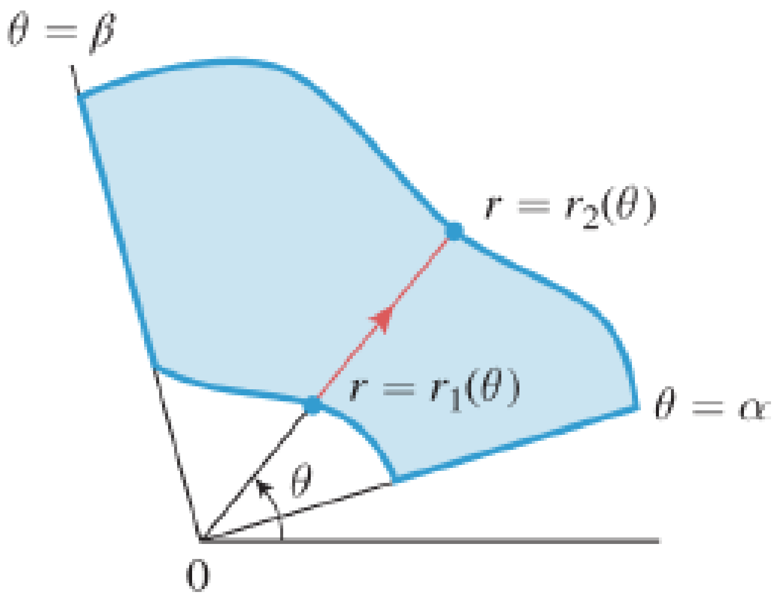
4. $$\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_0^{\pi/6}\frac{1}{2}\cos^2 3\theta\,d\theta = \frac{\pi}{24}.$$

5. $$\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\,dr\,d\theta = \int_0^{\pi}\frac{1}{3}(1-\sin\theta)^3\cos\theta\,d\theta = 0.$$

6. $$\int_0^{\pi/2}\int_0^{\cos\theta} r^3\,dr\,d\theta = \int_0^{\pi/2}\frac{1}{4}\cos^4\theta\,d\theta = \frac{3\pi}{64}.$$

14.3.3 THEOREM *If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in Figure 14.3.8, and if $f(r, \theta)$ is continuous on R, then*

$$\iint_R f(r, \theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r \, dr \, d\theta \tag{7}$$



▲ Figure 14.3.8

Determining Limits of Integration for a Polar Double Integral: Simple Polar Region

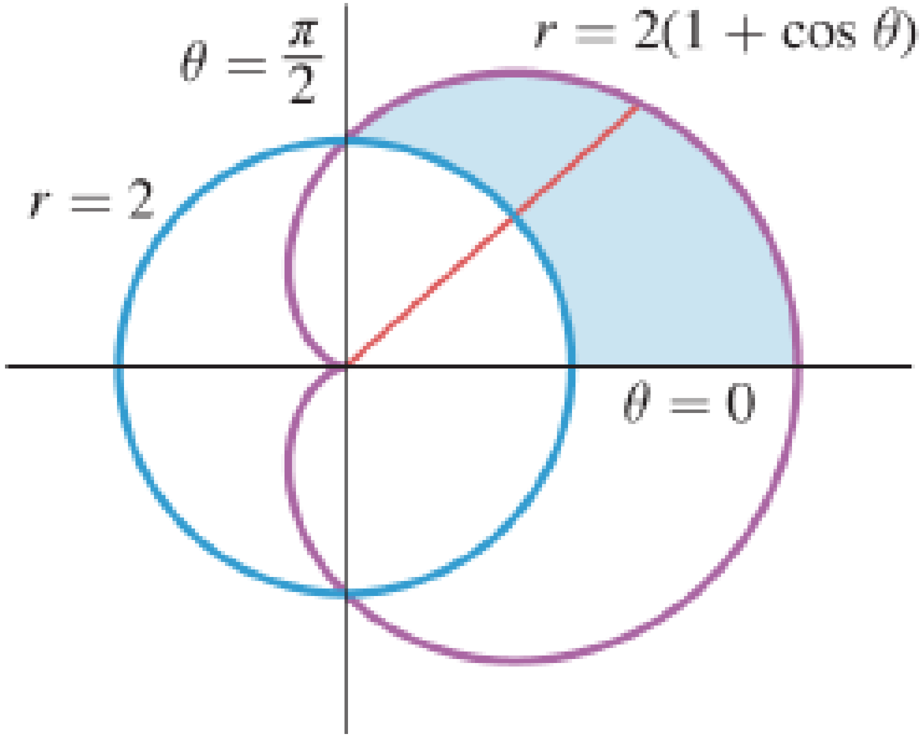
Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ (Figure 14.3.9a). This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration in (7).

Step 2. Imagine rotating the radial line from Step 1 about the origin, thus sweeping out the region R. The least angle at which the radial line intersects the region R is $\theta = \alpha$ and the greatest angle is $\theta = \beta$ (Figure 14.3.9b). This determines the θ-limits of integration.

▶ Example 1 Evaluate

$$\iint\limits_{R} \sin\theta \, dA$$

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.



▲ **Figure 14.3.10**

Solution. The region R is sketched in Figure 14.3.10. Following the two steps outlined above we obtain

$$\iint_R \sin\theta \, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin\theta) r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

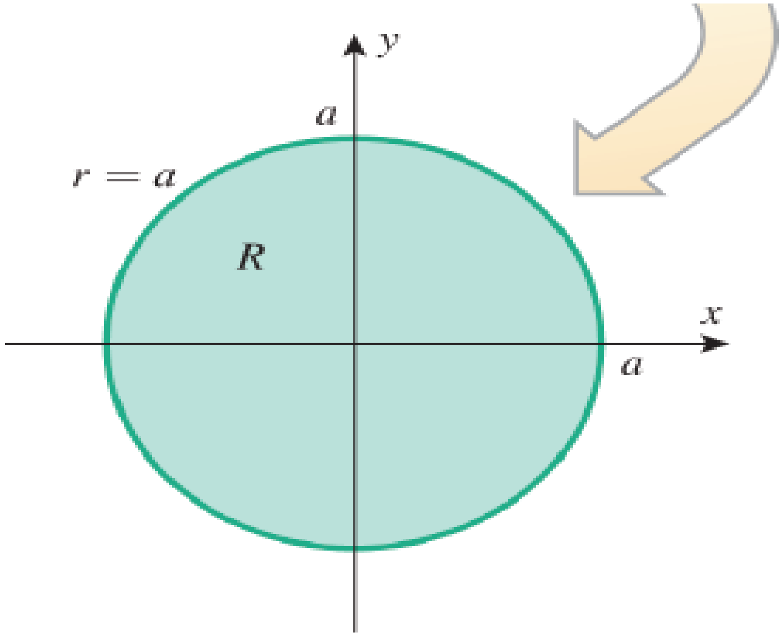
$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta] \, d\theta$$

$$= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\pi/2}$$

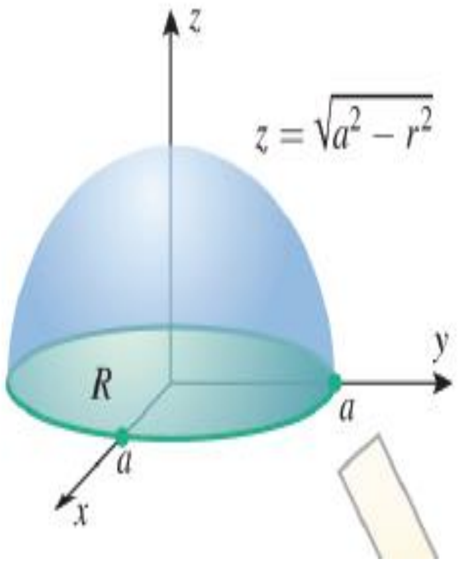
$$= 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$$

▶ Example 2 The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

$hint: x^2 + y^2 = r^2$ so equation is convert $r^2 + z^2 = a^2$
For region on x y axis $z = 0$ so $r^2 = a^2$ or $x^2 + y^2 = a^2$ circle of radius a



▲ Figure 14.3.11



$z = \sqrt{a^2 - r^2}$ for sphere use
$z = 2\sqrt{a^2 - r^2}$

$$z = \sqrt{a^2 - r^2}$$

so the volume enclosed by the entire sphere is

$$V = 2\iint\limits_R \sqrt{a^2 - r^2}\, dA$$

where R is the circular region shown in Figure 14.3.11. Thus,

$$V = 2\iint\limits_R \sqrt{a^2 - r^2}\, dA = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}(2r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(a^2 - r^2)^{3/2}\right]_{r=0}^{a} d\theta = \int_0^{2\pi}\frac{2}{3}a^3\, d\theta$$

$$= \left[\frac{2}{3}a^3\theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3 \blacktriangleleft$$

Solution. In cylindrical coordinates the upper hemisphere is given by the equation

$$z = \sqrt{a^2 - r^2}$$

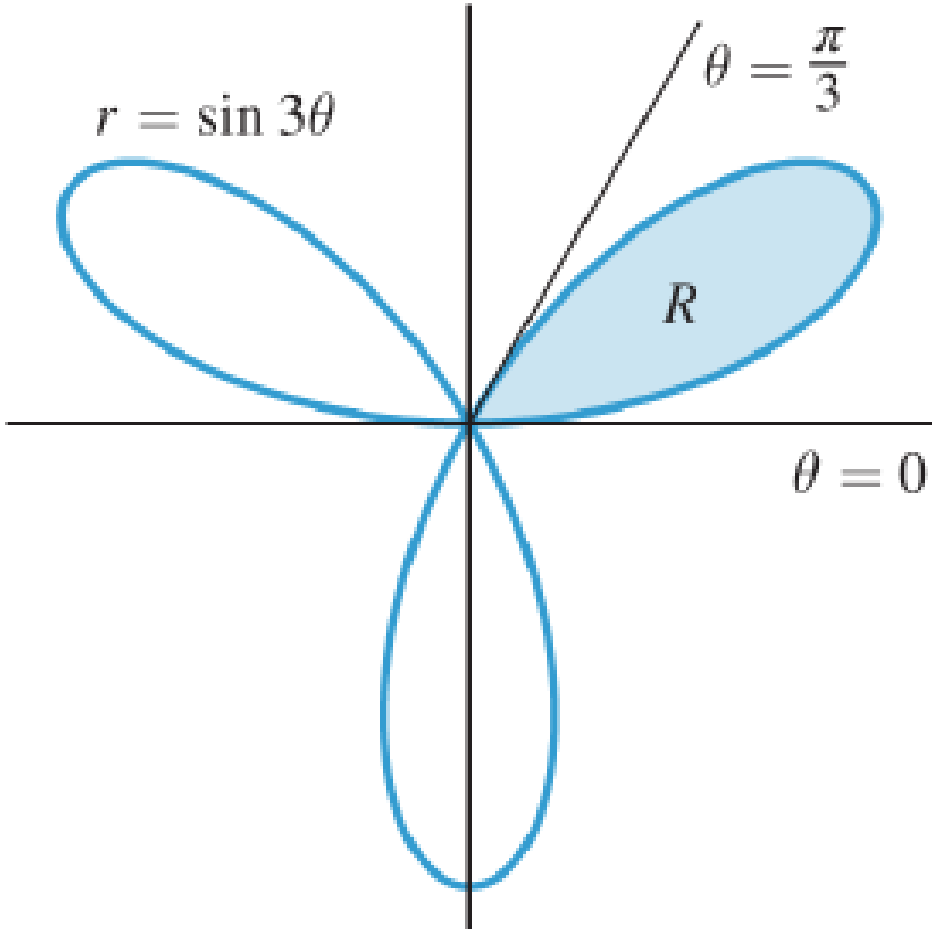
so the volume enclosed by the entire sphere is

$$V = 2 \iint_R \sqrt{a^2 - r^2}\, dA$$

where R is the circular region shown in Figure 14.3.11. Thus,

$$\begin{aligned}
V = 2 \iint_R \sqrt{a^2 - r^2}\, dA &= \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}(2r)\, dr\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{2}{3}(a^2 - r^2)^{3/2}\right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{2}{3} a^3\, d\theta \\
&= \left[\frac{2}{3} a^3 \theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3 \blacktriangleleft
\end{aligned}$$

► **Example 3** Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.



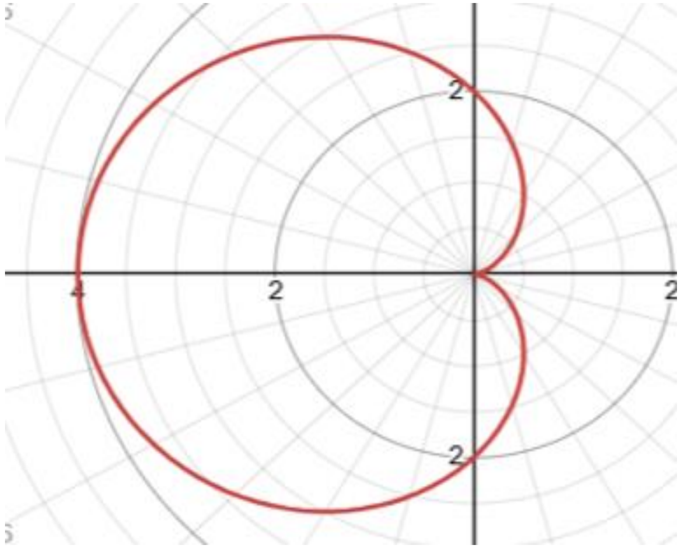
Solution. The rose is sketched in Figure 14.3.12. We will use Formula (8) to calculate the area of the petal R in the first quadrant and multiply by 3.

$$\begin{aligned}
A = 3 \iint_R dA &= 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r\,dr\,d\theta \\
&= \frac{3}{2} \int_0^{\pi/3} \sin^2 3\theta\,d\theta = \frac{3}{4} \int_0^{\pi/3} (1 - \cos 6\theta)\,d\theta \\
&= \frac{3}{4} \left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{1}{4}\pi \blacktriangleleft
\end{aligned}$$

7–10 Use a double integral in polar coordinates to find the area of the region described.

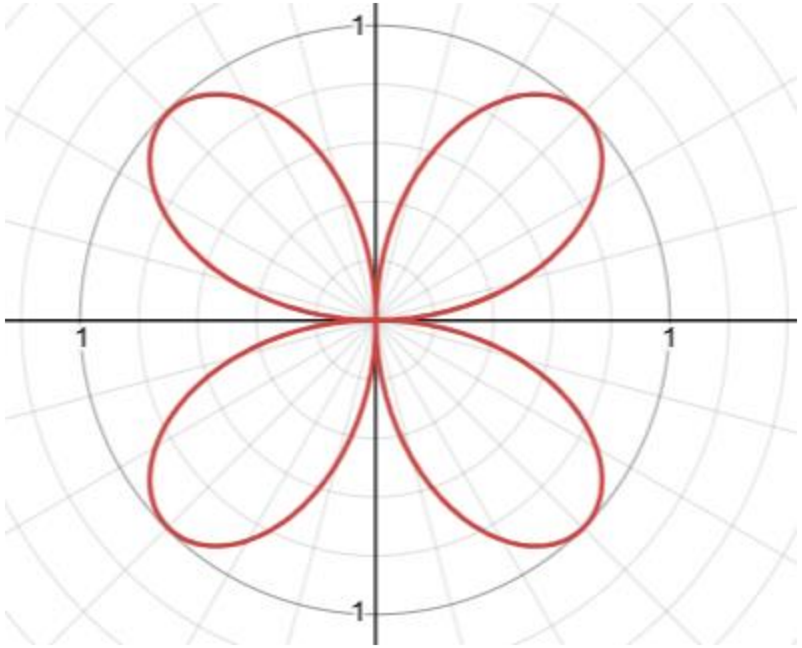
7. The region enclosed by the cardioid $r = 1 - \cos\theta$.

Same as example #1



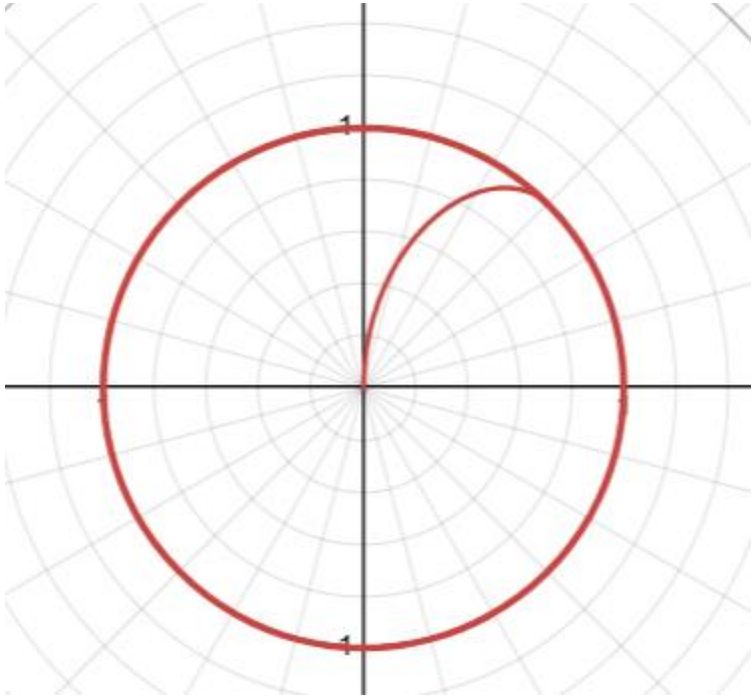
$$\textbf{7. } A = \int_0^{2\pi} \int_0^{1-\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2\,d\theta = \frac{3\pi}{2}.$$

8. The region enclosed by the rose $r = \sin 2\theta$.



8. $A = 4\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = 2\int_0^{\pi/2} \sin^2 2\theta\,d\theta = \frac{\pi}{2}.$

9. The region in the first quadrant bounded by $r = 1$ and $r = \sin 2\theta$, with $\pi/4 \le \theta \le \pi/2$.



$$A = \int_{\pi/4}^{\pi/2} \int_{\sin 2\theta}^{1} r \, dr \, d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{2}(1 - \sin^2 2\theta) \, d\theta = \frac{\pi}{16}.$$

10. The region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.

$x^2 + y^2 = 4 \; means \; r = 2$

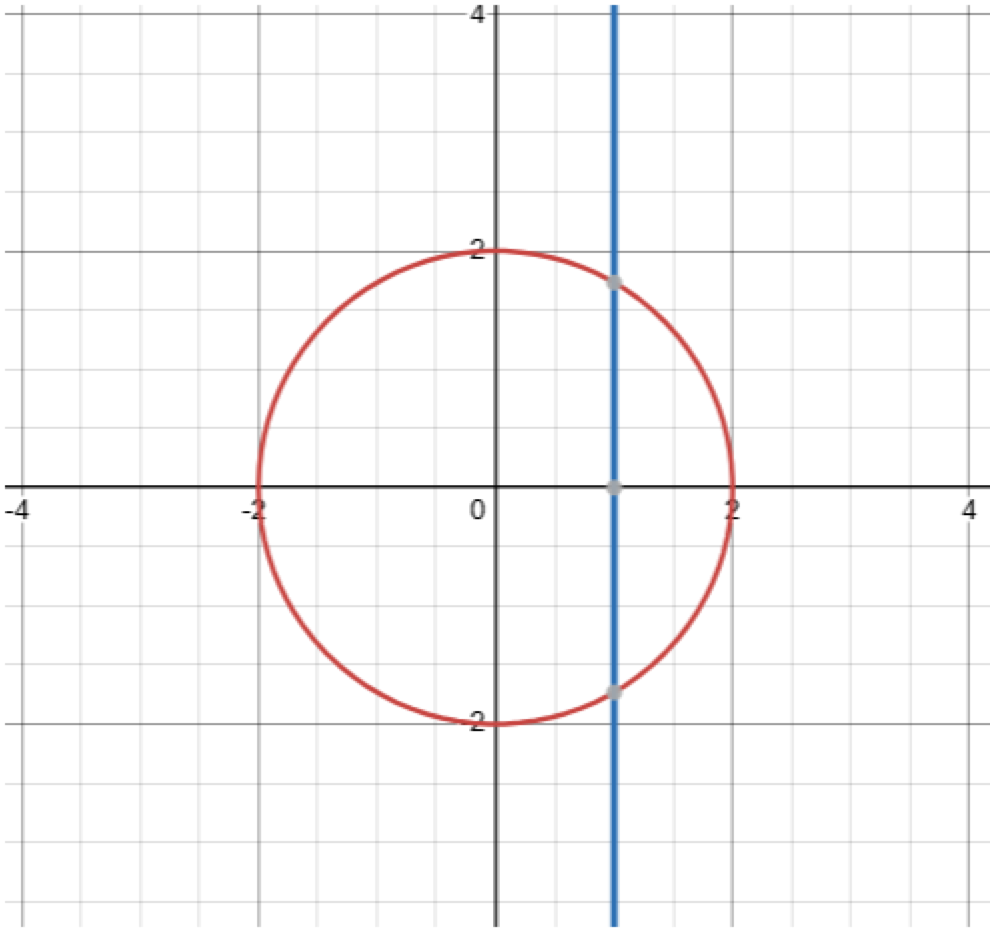
$x = 1 \quad , x = r\cos\theta$ so $r\cos\theta = 1 \, , \; r = \sec\theta$

Area = $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r \, dr d\theta$

For $\theta_1 = 0$ *where* θ_2 *is the* **intersection of r=2 and r** $= \sec\theta$

$$2 = \sec\theta$$

$$\cos\theta = \frac{1}{2} \quad and \; \theta = \frac{\pi}{3}$$



10. $A = 2\int_0^{\pi/3}\int_{\sec\theta}^{2} r\,dr\,d\theta = \int_0^{\pi/3}(4 - \sec^2\theta)\,d\theta = \frac{4\pi}{3} - \sqrt{3}.$

23–26 Use polar coordinates to evaluate the double integral.

23. $\displaystyle\iint_R \sin(x^2+y^2)\,dA$, where R is the region enclosed by the circle $x^2+y^2=9$.

23. $\displaystyle\int_0^{2\pi}\int_0^3 \sin(r^2) r\,dr\,d\theta = \frac{1}{2}(1-\cos 9)\int_0^{2\pi} d\theta = \pi(1-\cos 9).$

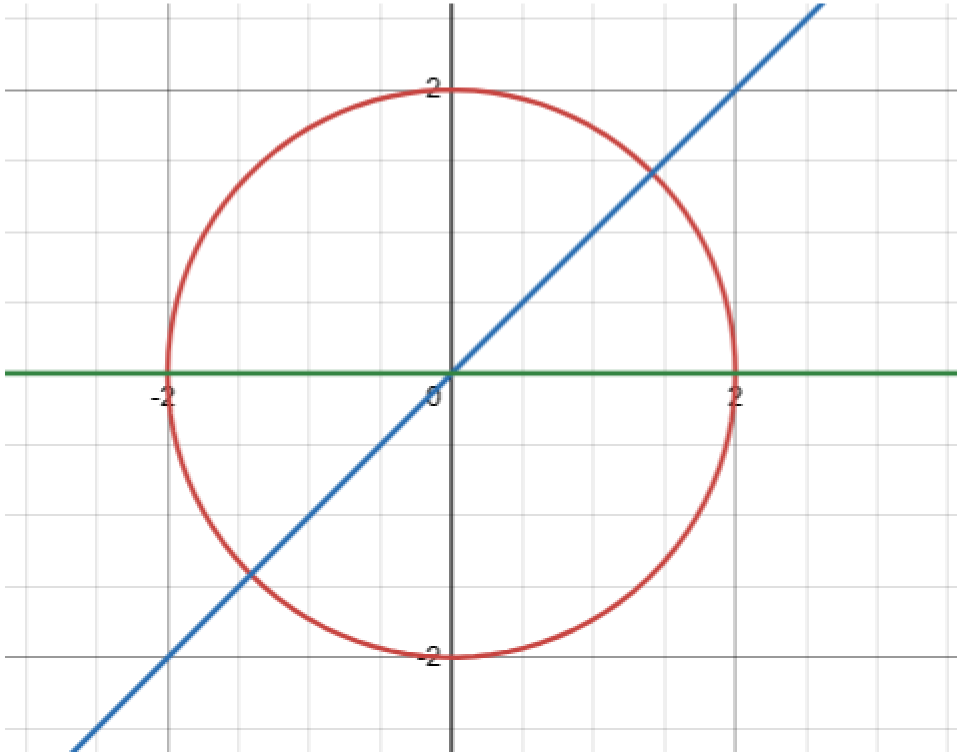
24. $\displaystyle\iint_R \sqrt{9 - x^2 - y^2}\, dA$, where R is the region in the first quadrant within the circle $x^2 + y^2 = 9$.

24. $$\int_0^{\pi/2}\int_0^3 r\sqrt{9 - r^2}\, dr\, d\theta = 9\int_0^{\pi/2} d\theta = \frac{9\pi}{2}.$$

quadrant within the circle $x^2 + y^2 = 9$.

25. $\displaystyle\iint\limits_R \frac{1}{1+x^2+y^2}\, dA$, where R is the sector in the first quadrant bounded by $y = 0$, $y = x$, and $x^2 + y^2 = 4$.

$$\int_0^{\pi/4}\int_0^2 \frac{1}{1+r^2} r\, dr\, d\theta = \frac{1}{2}\ln 5 \int_0^{\pi/4} d\theta = \frac{\pi}{8}\ln 5.$$



26. $\iint_R 2y\,dA$, where R is the region in the first quadrant bounded above by the circle $(x-1)^2+y^2=1$ and below by the line $y=x$.

We convert the given equations to polar form:

- The circle $(x-1)^2+y^2=1$ transforms into:

$$(r\cos\theta-1)^2+(r\sin\theta)^2=1$$

  Expanding:

$$r^2\cos^2\theta-2r\cos\theta+1+r^2\sin^2\theta=1$$

$$r^2(\cos^2\theta+\sin^2\theta)-2r\cos\theta+1=1$$

$$r^2-2r\cos\theta=0$$

$$r(r-2\cos\theta)=0$$

  Since $r\neq 0$, we get:

$$r=2\cos\theta$$

- The line $y=x$ in polar form is:

$$r\sin\theta=r\cos\theta$$

- The line $y = x$ in polar form:

$$\theta = \frac{\pi}{4}.$$

Thus, the region R is bounded by:

- $0 \leq \theta \leq \frac{\pi}{4}$.
- $0 \leq r \leq 2\cos\theta$.

In polar coordinates, $dA = r\,dr\,d\theta$, so the integral becomes:

$$\int_0^{\pi/4} \int_0^{2\cos\theta} 2(r\sin\theta)r\,dr\,d\theta.$$

27–34 Evaluate the iterated integral by converting to polar coordinates.

27. $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$

The limits suggest that:

- x varies from 0 to 1.
- y varies from 0 to $\sqrt{1 - x^2}$.

The upper bound of y, $\sqrt{1 - x^2}$, represents the upper semicircle $x^2 + y^2 = 1$, so the region is a **quarter-circle** in the first quadrant with radius 1.

Using polar transformations:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dA = r\,dr\,d\theta.$$

The region corresponds to:

- $0 \le r \le 1$ (since it's a quarter-circle of radius 1).
- $0 \le \theta \le \frac{\pi}{2}$.

Rewriting the function $x^2 + y^2$ in polar form:

$$x^2 + y^2 = r^2.$$

Thus, the integral becomes:

$$\int_0^{\pi/2} \int_0^1 r^2 \cdot r\,dr\,d\theta.$$

First, evaluate the inner integral:

$$\int_0^1 r^3 \, dr = \frac{r^4}{4}\Big|_0^1 = \frac{1}{4}.$$

Now, evaluate the outer integral:

$$\int_0^{\pi/2} \frac{1}{4} \, d\theta = \frac{1}{4}\theta\Big|_0^{\pi/2} = \frac{\pi}{8}.$$

28. $\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\, dx\, dy$

- The outer integral has limits $y = -2$ to $y = 2$.
- The inner integral runs from $x = -\sqrt{4 - y^2}$ to $x = \sqrt{4 - y^2}$, which suggests that for a given y, x spans symmetrically around zero.

This describes the **disk** $x^2 + y^2 \leq 4$ (a circle of radius 2 centered at the origin).

Using polar transformations:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dA = r\,dr\,d\theta.$$

The region corresponds to:

- $0 \leq r \leq 2$ (radius of the disk).
- $0 \leq \theta \leq 2\pi$ (full circle).

Rewriting the function in polar form:

$$e^{-(x^2+y^2)} = e^{-r^2}.$$

Thus, the integral transforms into:

$$\int_0^{2\pi} \int_0^2 e^{-r^2} r\,dr\,d\theta.$$

We use substitution:

Let $u = r^2$, so $du = 2r\,dr$, which gives:

$$\frac{1}{2}\int_0^4 e^{-u}\,du.$$

Since $\int e^{-u}\,du = -e^{-u}$, we evaluate:

$$\frac{1}{2}\left[-e^{-u}\right]_0^4 = \frac{1}{2}\left(-e^{-4}+1\right) = \frac{1-e^{-4}}{2}.$$

$$\int_0^{2\pi} \frac{1-e^{-4}}{2}\,d\theta.$$

Since $\frac{1-e^{-4}}{2}$ is constant, we get:

$$\frac{1-e^{-4}}{2}\cdot 2\pi = \pi(1-e^{-4}).$$

$$\mathbf{29.}\ \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

X varies 0 to 2

Y varies $0\ to\ \sqrt{2x-x^2}$

$y = \sqrt{2x-x^2}$ squaring on both side

$$y^2 = 2x - x^2$$
$$x^2 - 2x + y^2 = 0$$

Converts into circle :

$$(x-1)^2 + y^2 = 1$$
$$center\ of\ the\ circle\ is\ (1,0)\ and\ radius\ 1.$$
$$x = rcos\theta\ and\ y = r\ sin\theta\ put\ (x-1)^2 + y^2 = 1$$

And get $\mathrm{r(r-2cos\theta)} = 0$

$$r = 0\ to\ r = 2\cos\theta$$

we consider only the **upper half** of the circle in the first quadrant.

$$0 \le \theta \le \frac{\pi}{2}$$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 dr\, d\theta = \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{16}{9}.$$

30. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\,dx\,dy$

31. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{(1+x^2+y^2)^{3/2}} \quad (a>0)$

32. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2+y^2}\,dx\,dy$

33. $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dx\,dy$

34. $\displaystyle\int_{-4}^0 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 3x\,dy\,dx$

27. $\int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{\pi}{8}.$

28. $\int_0^{2\pi}\int_0^2 e^{-r^2} r\,dr\,d\theta = \frac{1}{2}(1-e^{-4})\int_0^{2\pi} d\theta = (1-e^{-4})\pi.$

29. $\int_0^{\pi/2}\int_0^{2\cos\theta} r^2 dr\,d\theta = \frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta = \frac{16}{9}.$

30. $\int_0^{\pi/2}\int_0^1 \cos(r^2) r\,dr\,d\theta = \frac{1}{2}\sin 1\int_0^{\pi/2} d\theta = \frac{\pi}{4}\sin 1.$

31. $\int_0^{\pi/2}\int_0^a \frac{r}{(1+r^2)^{3/2}}\,dr\,d\theta = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{1+a^2}}\right).$

32. $\int_0^{\pi/4}\int_0^{\sec\theta\tan\theta} r^2\,dr\,d\theta = \frac{1}{3}\int_0^{\pi/4}\sec^3\theta\tan^3\theta\,d\theta = \frac{2(\sqrt{2}+1)}{45}.$

33. $\int_0^{\pi/4}\int_0^2 \frac{r}{\sqrt{1+r^2}}\,dr\,d\theta = \frac{\pi}{4}(\sqrt{5}-1).$

34. $\int_{\pi/2}^{3\pi/2}\int_0^4 3r^2\cos\theta\,dr\,d\theta = \int_{\pi/2}^{3\pi/2} 64\cos\theta\,d\theta = -128.$