13.2 LIMITS AND CONTINUITY

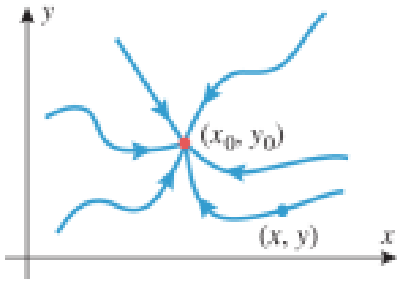
In this section we will introduce the notions of limit and continuity for functions of two or more variables. We will not go into great detail—our objective is to develop the basic concepts accurately and to obtain results needed in later sections. A more extensive study of these topics is usually given in advanced calculus.

LIMITS ALONG CURVES

For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left. For functions of two or three variables there are infinitely many different curves along which one point can approach another (Figure 13.2.1). Our first objective in this section is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).



▲ **Figure 13.2.1**

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y) \to (x_0, y_0) \\ (\text{along } C)}} f(x, y) \quad \text{and} \quad \lim_{\substack{(x,y,z) \to (x_0, y_0, z_0) \\ (\text{along } C)}} f(x, y, z)$$

are defined by

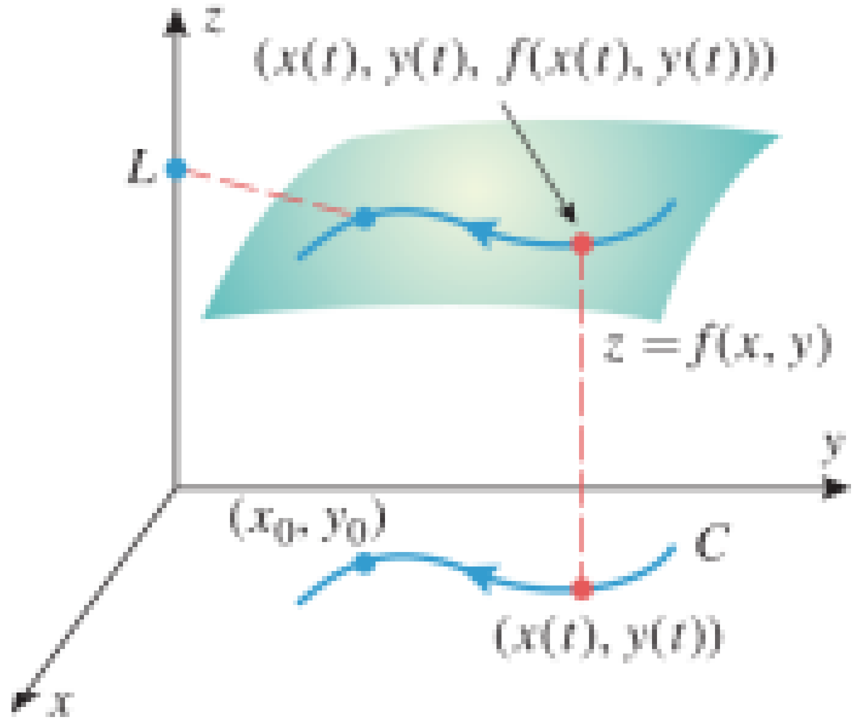
$$\lim_{\substack{(x,y) \to (x_0, y_0) \\ (\text{along } C)}} f(x, y) = \lim_{t \to t_0} f(x(t), y(t)) \tag{1}$$

$$\lim_{\substack{(x,y,z) \to (x_0, y_0, z_0) \\ (\text{along } C)}} f(x, y, z) = \lim_{t \to t_0} f(x(t), y(t), z(t)) \tag{2}$$

In words, Formulas (1) and (2) state that a limit of a function f along a parametric curve can be obtained by substituting the parametric equations for the curve into the formula for the function and then computing the limit of the resulting function of one variable at the appropriate point.

In these formulas the limit of the function of t must be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

A geometric interpretation of the limit along a curve for a function of two variables is shown in Figure 13.2.2: As the point $(x(t), y(t))$ moves along the curve C in the xy-plane toward (x_0, y_0), the point $(x(t), y(t), f(x(t), y(t)))$ moves directly above it along the graph of $z = f(x, y)$ with $f(x(t), y(t))$ approaching the limiting value L. In the figure we followed a common practice of omitting the zero z-coordinate for points in the xy-plane.



$$\lim_{\substack{(x, y) \to (x_0, y_0) \\ (\text{along } C)}} f(x, y) = L$$

▲ **Figure 13.2.2**

LIMITS AND CONTINUITY

- Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \quad \text{and} \quad g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as *x* and *y* both approach 0
(and thus the point (*x*, *y*) approaches
the origin).

LIMITS AND CONTINUITY

- The following tables show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin.

LIMITS AND CONTINUITY

Table 1

- This table shows values of $f(x, y)$.

TABLE 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

LIMITS AND CONTINUITY

Table 2

- This table shows values of $g(x, y)$.

TABLE 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

LIMITS AND CONTINUITY

- Notice that neither function is defined at the origin.

  - It appears that, as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1, whereas the values of $g(x, y)$ aren't approaching any number.

LIMITS AND CONTINUITY

- It turns out that these guesses based on numerical evidence are correct.
- Thus, we write:
  - $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$
  - $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$ does not exsit

LIMITS AND CONTINUITY

- In general, we use the notation

to indicate that: $$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

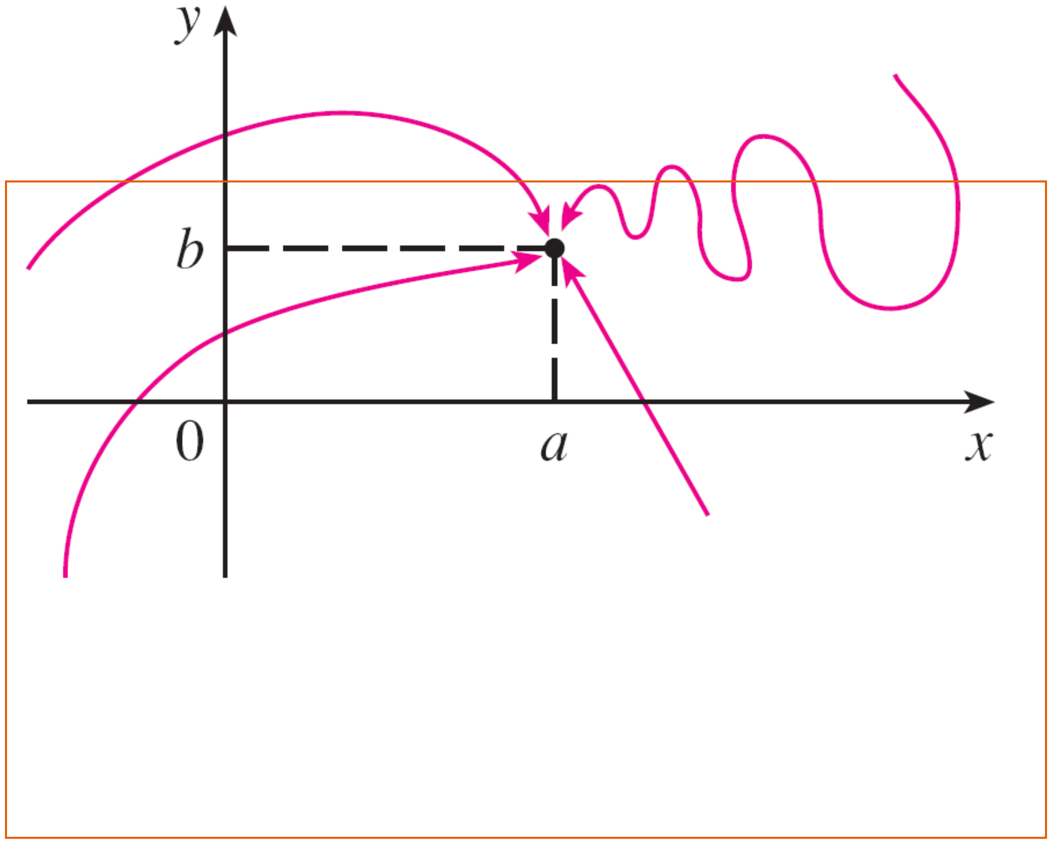
  - The values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.

DOUBLE VARIABLE FUNCTIONS

- For functions of two variables, the situation is not as simple.

DOUBLE VARIABLE FUNCTIONS

- This is because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever as long as (x, y) stays within the domain of f.



LIMIT OF A FUNCTION

- Definition 1 refers only to the distance between (x, y) and (a, b).
  - It does not refer to the direction of approach.

LIMIT OF A FUNCTION

- Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).

LIMIT OF A FUNCTION

- Thus, if we can find two different paths of approach along which the function *f*(*x*, *y*) has different limits, then it follows that

$$\lim_{(x,y)\to(a,b)} f(x, y)$$ does not exist.

LIMIT OF A FUNCTION

- If
  $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and
  $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$,
  then $$\lim_{(x,y)\to(a,b)} f(x, y)$$
  does not exist.

LIMIT OF A FUNCTION

Example 1

- Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

  - Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.

Example 1

LIMIT OF A FUNCTION

- First, let's approach (0, 0) along the x-axis.

  - Then, $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$.

  - So, $f(x, y) \to 1$ as $(x, y) \to (0, 0)$ along the x-axis.

Example 1

LIMIT OF A FUNCTION

- We now approach along the y-axis by putting $x = 0$.

  - Then, $f(0, y) = -y^2/y^2 = -1$ for all $y \neq 0$.

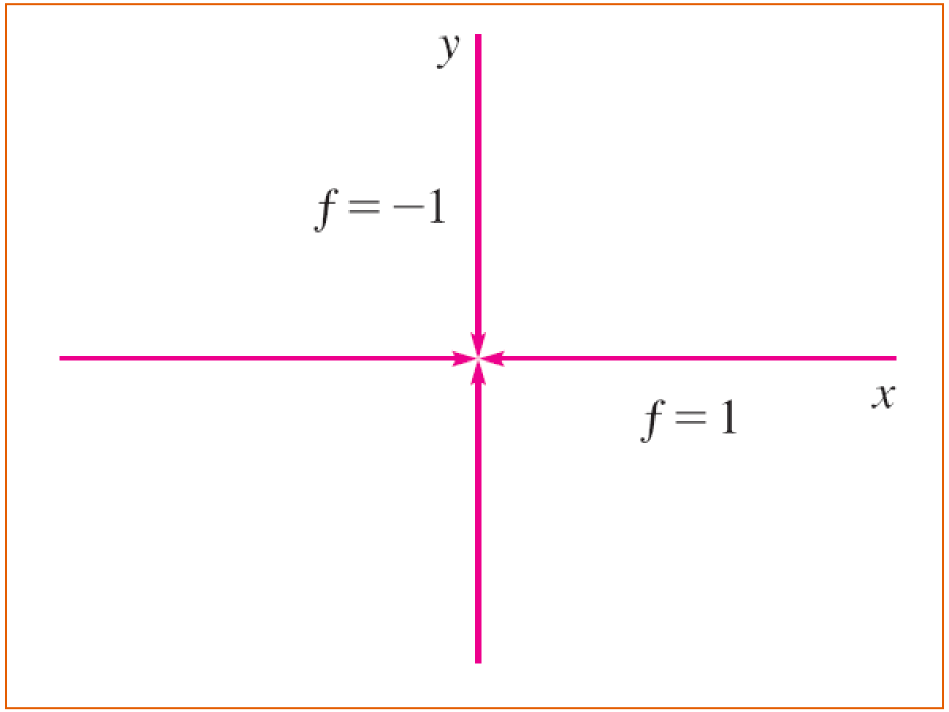
  - So, $f(x, y) \to -1$ as $(x, y) \to (0, 0)$ along the y-axis.

LIMIT OF A FUNCTION

Example 1

- Since f has two different limits along two different lines, the given limit does not exist.

  - This confirms the conjecture we made on the basis of numerical evidence at the beginning of the section.



LIMIT OF A FUNCTION

Example 2

- If $$f(x,y)=\frac{xy}{x^2+y^2}$$

  does

  $$\lim_{(x,y)\to(0,0)} f(x,y)$$

  exist?

LIMIT OF A FUNCTION

Example 2

- If $y = 0$, then $f(x, 0) = 0/x^2 = 0$.

- Therefore,

  $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the x-axis.

LIMIT OF A FUNCTION

Example 2

- If $x = 0$, then $f(0, y) = 0/y^2 = 0$.

- So,

  $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the y-axis.

Example 2

LIMIT OF A FUNCTION

- Although we have obtained identical limits along the axes, that does not show that the given limit is 0.

Example 2

LIMIT OF A FUNCTION

- Let's now approach (0, 0) along another line, say $y = x$.
  - For all $x \neq 0$,

$$f(x,x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

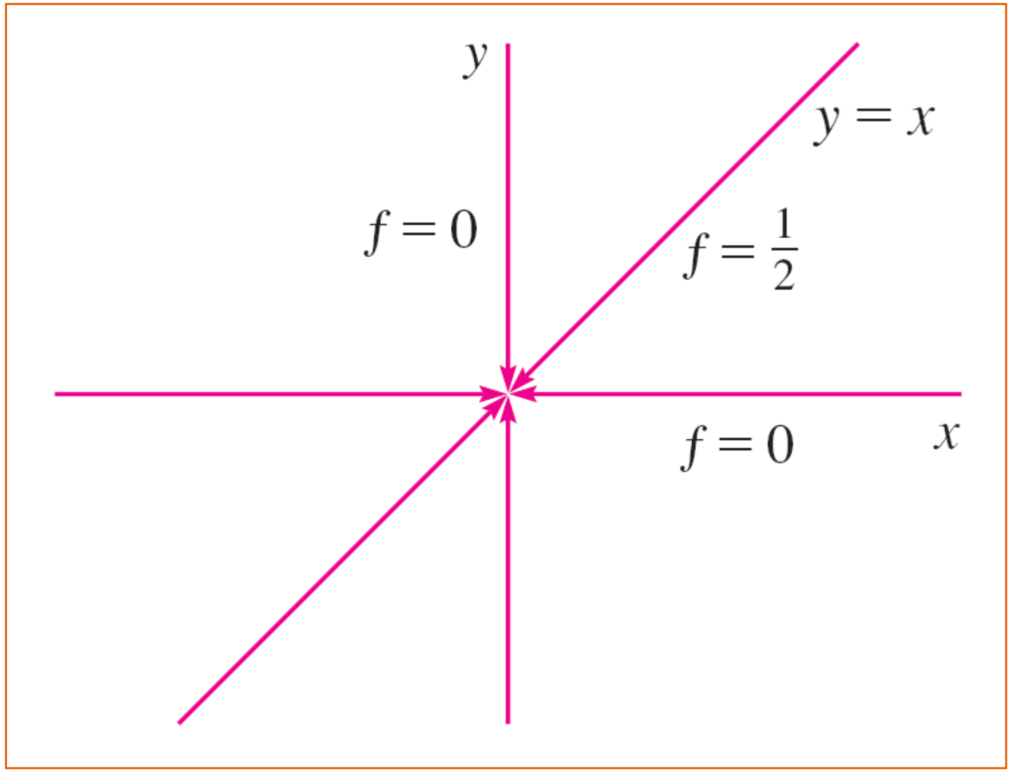
  - Therefore,

$$f(x,y) \to \tfrac{1}{2} \text{ as } (x,y) \to (0,0) \text{ along } y = x$$

LIMIT OF A FUNCTION

Example 2

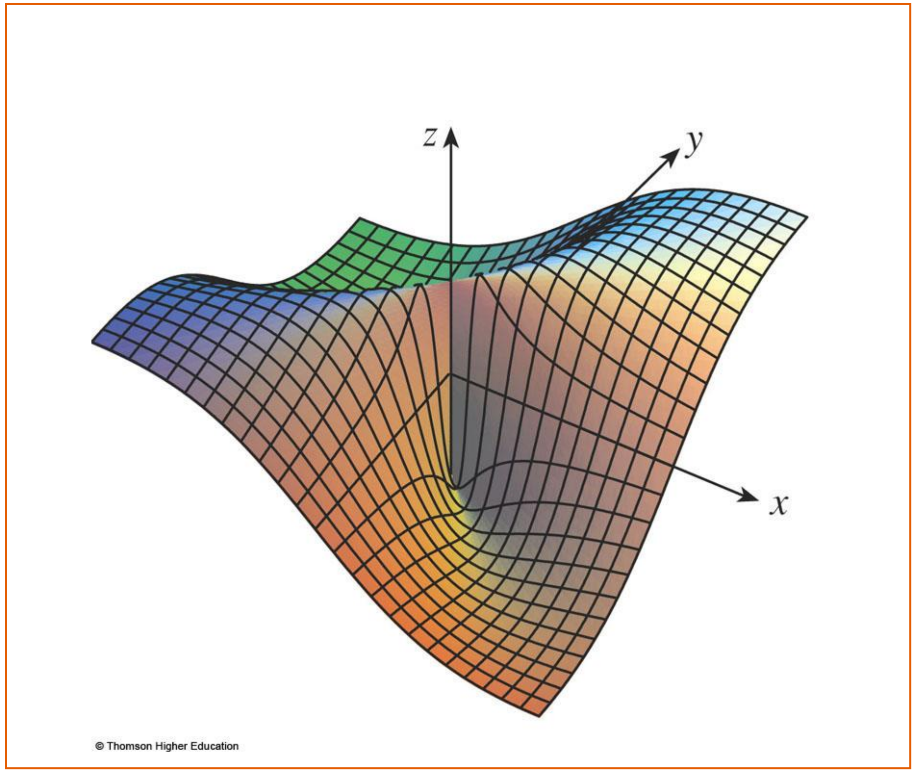
- Since we have obtained different limits along different paths, the given limit does not exist.



LIMIT OF A FUNCTION

- This figure sheds some light on Example 2.

  - The ridge that occurs above the line $y = x$ corresponds to the fact that $f(x, y) = ½$ for all points (x, y) on that line except the origin.



LIMIT OF A FUNCTION

Example 3

- If

$$f(x,y)=\frac{xy^2}{x^2+y^4}$$

does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

Example 3

LIMIT OF A FUNCTION

- With the solution of Example 2 in mind,
  let's try to save time by letting $(x, y) \rightarrow (0, 0)$ along any nonvertical line through the origin.

LIMIT OF A FUNCTION

Example 3

- Then, $y = mx$, where m is the slope, and

$$f(x,y) = f(x,mx)$$

$$= \frac{x(mx)^2}{x^2+(mx)^4}$$

$$= \frac{m^2x^3}{x^2+m^4x^4}$$

$$= \frac{m^2x}{1+m^4x^2}$$

LIMIT OF A FUNCTION

Example 3

- Therefore,

$$f(x, y) \to 0 \text{ as } (x, y) \to (0, 0) \text{ along } y = mx$$

- Thus, f has the same limiting value along every nonvertical line through the origin.

LIMIT OF A FUNCTION

Example 3

- However, that does not show that the given limit is 0.
  - This is because, if we now let
    $(x, y) \to (0, 0)$ along the parabola $x = y^2$
    we have:

$$f(x,y) = f(y^2,y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

  - So,
    $f(x, y) \to ½$ as $(x, y) \to (0, 0)$ along $x = y^2$

Example 3

LIMIT OF A FUNCTION

- Since different paths lead to different limiting values, the given limit does not exist.

LIMIT OF A FUNCTION

- Now, let’s look at limits that do exist.

LIMIT OF A FUNCTION

- Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

LIMIT OF A FUNCTION

- The Limit Laws listed in Section 2.3 can be extended to functions of two variables.

- For instance,

  - The limit of a sum is the sum of the limits.

  - The limit of a product is the product of the limits.

Equations 2

LIMIT OF A FUNCTION

- In particular, the following equations are true.

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

$$\lim_{(x,y)\to(a,b)} c = c$$

▶ Example 1 Figure 13.2.3*a* shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$

(d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a). The x-axis has parametric equations $x = t, y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=0)}} f(x, y) = \lim_{t\to 0} f(t, 0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 13.2.3b.

Solution (b). The y-axis has parametric equations $x = 0, y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} f(x, y) = \lim_{t\to 0} f(0, t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 13.2.3b.

Solution (c). The line $y = x$ has parametric equations $x = t, y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} f(x, y) = \lim_{t\to 0} f(t, t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure 13.2.3b.

Solution (d). The line $y = -x$ has parametric equations $x = t, y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=-x)}} f(x, y) = \lim_{t\to 0} f(t, -t) = \lim_{t\to 0} \frac{t^2}{2t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

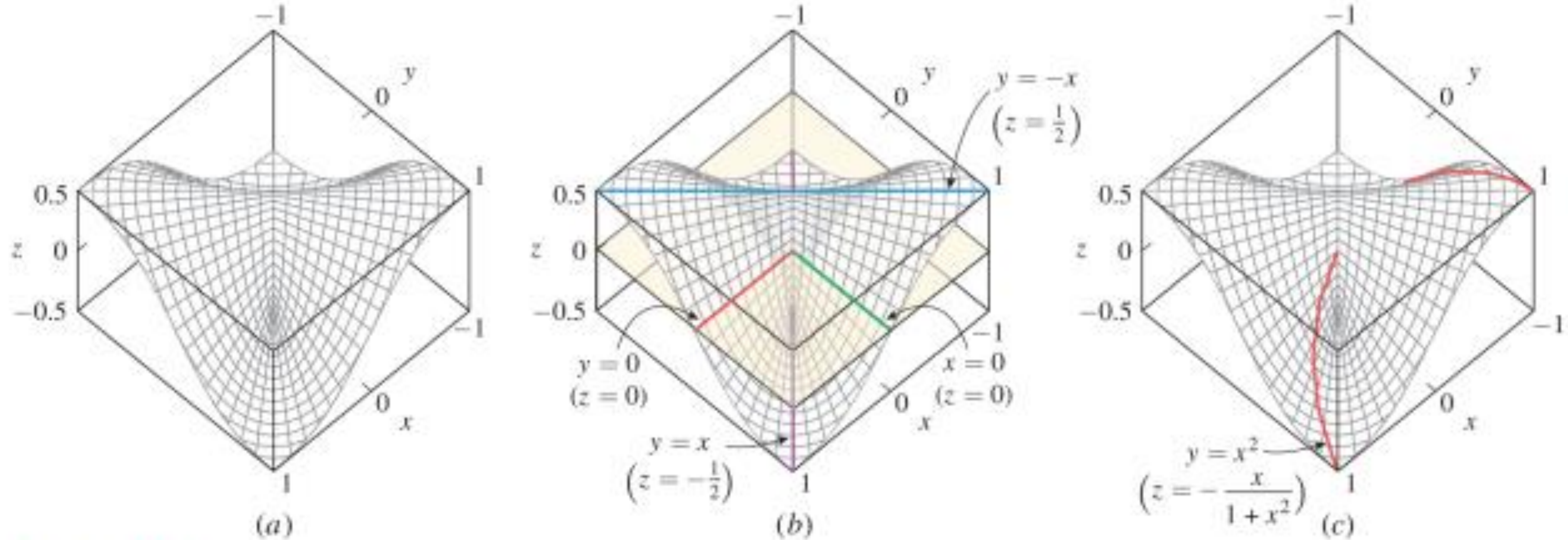
Solution (e). The parabola $y = x^2$ has parametric equations $x = t, y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x^2)}} f(x, y) = \lim_{t\to 0} f(t, t^2) = \lim_{t\to 0} \left(-\frac{t^3}{t^2 + t^4}\right) = \lim_{t\to 0} \left(-\frac{t}{1 + t^2}\right) = 0$$

This is consistent with Figure 13.2.3*c*, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1 + t^2}$$

superimposed on the surface. ◄



▲ **Figure 13.2.3**

OPEN AND CLOSED SETS

Although limits along specific curves are useful for many purposes, they do not tell the complete story about the limiting behavior of a function at a point; what is required is a limit concept that accounts for the behavior of the function in an *entire vicinity* of a point, not just along smooth curves passing through the point. For this purpose, we start by introducing some terminology.

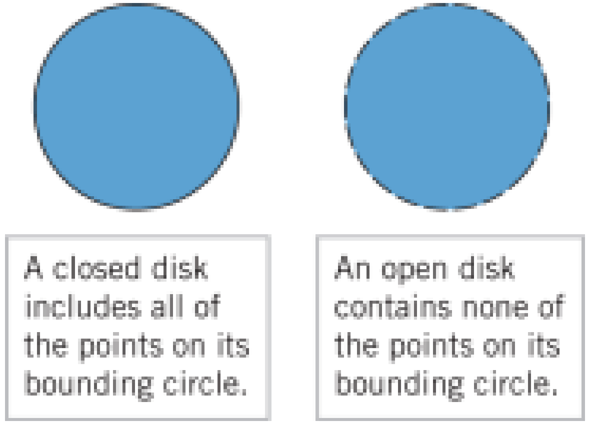


Figure 13.2.4

Let C be a circle in 2-space that is centered at (x_0, y_0) and has positive radius δ. The set of points that are enclosed by the circle, but do not lie on the circle, is called the ***open disk*** of radius δ centered at (x_0, y_0), and the set of points that lie on the circle together with those enclosed by the circle is called the ***closed disk*** of radius δ centered at (x_0, y_0) (Figure 13.2.4). Analogously, if S is a sphere in 3-space that is centered at (x_0, y_0, z_0) and has positive radius δ, then the set of points that are enclosed by the sphere, but do not lie on the sphere, is called the ***open ball*** of radius δ centered at (x_0, y_0, z_0), and the set of points that lie on the sphere together with those enclosed by the sphere is called the ***closed ball*** of radius δ centered at (x_0, y_0, z_0). Disks and balls are the two-dimensional and three-dimensional analogs of intervals on a line.

The notions of "open" and "closed" can be extended to more general sets in 2-space and 3-space. If D is a set of points in 2-space, then a point (x_0, y_0) is called an ***interior point*** of D if there is *some* open disk centered at (x_0, y_0) that contains only points of D, and (x_0, y_0) is called a ***boundary point*** of D if *every* open disk centered at (x_0, y_0) contains both points in D and points not in D. The same terminology applies to sets in 3-space, but in that case the definitions use balls rather than disks (Figure 13.2.5).

For a set D in either 2-space or 3-space, the set of all interior points is called the ***interior*** of D and the set of all boundary points is called the ***boundary*** of D. Moreover, just as for disks, we say that D is ***closed*** if it contains all of its boundary points and ***open*** if it contains *none* of its boundary points. The set of all points in 2-space and the set of all points in 3-space have no boundary points (why?), so by agreement they are regarded to be both open and closed.

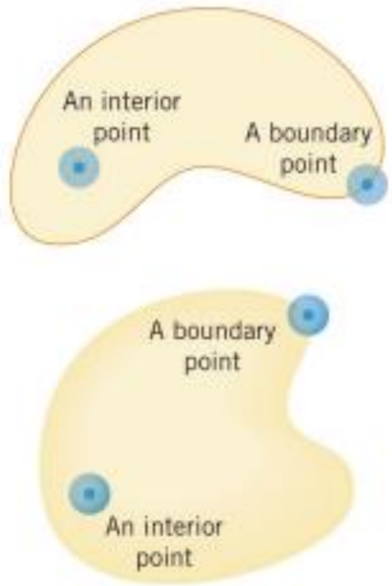


Figure 13.2.5

open and closed.

GENERAL LIMITS OF FUNCTIONS OF TWO VARIABLES

The statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

is intended to convey the idea that the value of $f(x, y)$ can be made as close as we like to the number L by restricting the point (x, y) to be sufficiently close to (but different from) the point (x_0, y_0). This idea has a formal expression in the following definition and is illustrated in Figure 13.2.6.

13.2.1 DEFINITION Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

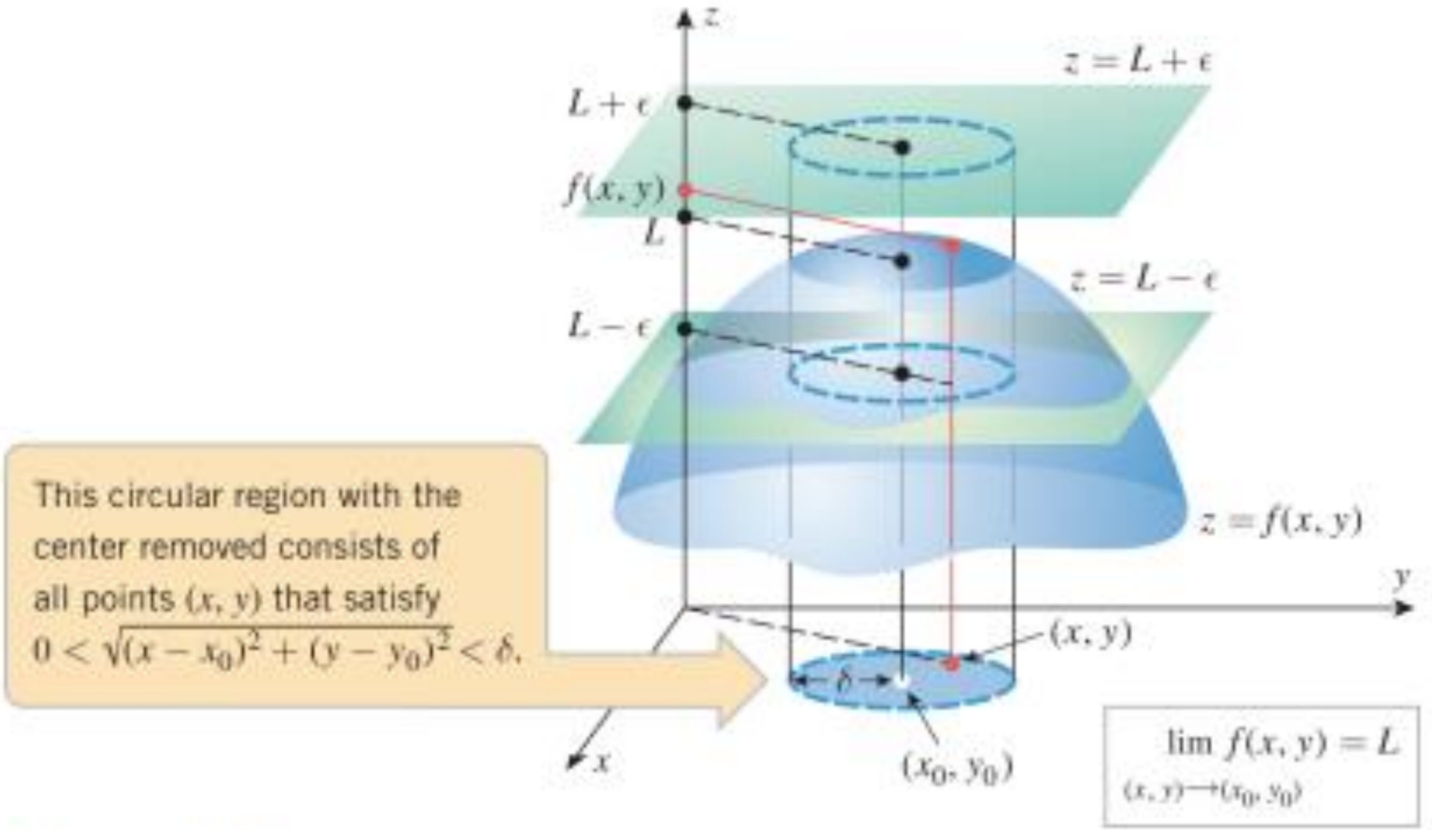
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \tag{3}$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x,y) - L| < \epsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$



z
$z = L + \epsilon$
$L + \epsilon$
$f(x, y)$
L
$z = L - \epsilon$
$L - \epsilon$
This circular region with the center removed consists of all points (x, y) that satisfy $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.
$z = f(x, y)$
y
(x, y)
δ
x
(x_0, y_0)
$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L$

▲ **Figure 13.2.6**

▶ Example 2

$$\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] = \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9$$

$$= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3 \left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9$$

$$= 5(1)^3(4)^2 - 9 = 71 \blacktriangleleft$$

RELATIONSHIPS BETWEEN GENERAL LIMITS AND LIMITS ALONG SMOOTH CURVES

Stated informally, if $f(x, y)$ has limit L as (x, y) approaches (x_0, y_0), then the value of $f(x, y)$ gets closer and closer to L as the distance between (x, y) and (x_0, y_0) approaches zero. Since this statement imposes no restrictions on the direction in which (x, y) approaches (x_0, y_0), it is plausible that the function $f(x, y)$ will also have the limit L as (x, y) approaches (x_0, y_0) along *any* smooth curve C. This is the implication of the following theorem, which we state without proof.

13.2.2 THEOREM

(*a*) *If* $f(x, y) \to L$ *as* $(x, y) \to (x_0, y_0)$, *then* $f(x, y) \to L$ *as* $(x, y) \to (x_0, y_0)$ *along any smooth curve.*

(*b*) *If the limit of* $f(x, y)$ *fails to exist as* $(x, y) \to (x_0, y_0)$ *along some smooth curve, or if* $f(x, y)$ *has different limits as* $(x, y) \to (x_0, y_0)$ *along two different smooth curves, then the limit of* $f(x, y)$ *does not exist as* $(x, y) \to (x_0, y_0)$.

Example 3 The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist because in Example 1 we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} -\frac{xy}{x^2+y^2} = -\frac{1}{2}$$

CONTINUITY

Stated informally, a function of one variable is continuous if its graph is an unbroken curve without jumps or holes. To extend this idea to functions of two variables, imagine that the graph of $z = f(x, y)$ is formed from a thin sheet of clay that has been molded into peaks and valleys. We will regard f as being continuous if the clay surface has no jumps, tears, or holes. For example, the function graphed in Figure 13.2.8 fails to be continuous because its graph exhibits a vertical jump at the origin.

The precise definition of continuity at a point for functions of two variables is similar to that for functions of one variable—we require the limit of the function and the value of the function to be the same at the point.

13.2.3 DEFINITION A function $f(x, y)$ is said to be ***continuous at*** (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

In addition, if f is continuous at every point in an open set D, then we say that f is ***continuous on D***, and if f is continuous at every point in the xy-plane, then we say that f is ***continuous everywhere***.

The following theorem, which we state without proof, illustrates some of the ways in which continuous functions can be combined to produce new continuous functions.

13.2.4 THEOREM

(*a*) *If* $g(x)$ *is continuous at* x_0 *and* $h(y)$ *is continuous at* y_0, *then* $f(x, y) = g(x)h(y)$ *is continuous at* (x_0, y_0).

(*b*) *If* $h(x, y)$ *is continuous at* (x_0, y_0) *and* $g(u)$ *is continuous at* $u = h(x_0, y_0)$, *then the composition* $f(x, y) = g(h(x, y))$ *is continuous at* (x_0, y_0).

(*c*) *If* $f(x, y)$ *is continuous at* (x_0, y_0), *and if* $x(t)$ *and* $y(t)$ *are continuous at* t_0 *with* $x(t_0) = x_0$ *and* $y(t_0) = y_0$, *then the composition* $f(x(t), y(t))$ *is continuous at* t_0.

▶ **Example 4** Use Theorem 13.2.4 to show that the functions $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution. The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by part (*a*) of Theorem 13.2.4, the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane. Since $3x^2y^5$ is continuous at every point in the xy-plane and $\sin u$ is continuous at every real number u, it follows from part (*b*) of Theorem 13.2.4 that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous everywhere. ◀

Example 5 Evaluate $\lim_{(x,y)\to(-1,2)} \dfrac{xy}{x^2+y^2}$.

Solution. Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+(2)^2} = -\frac{2}{5}$$

Example 6 Since the function

$$f(x, y) = \frac{x^3y^2}{1-xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$.

LIMITS AT DISCONTINUITIES

Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y)\to(0,0)$ along any smooth curve (Figure 13.2.9). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0\cdot\infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.

▶ Example 7 Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution. Let (r, θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2+y^2}$, so that $r\to 0^+$ if and only if $(x, y)\to(0,0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2\ln r^2 \\
&= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \text{This converts the limit to an indeterminate form of type } \infty/\infty. \\
&= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \text{L'Hôpital's rule} \\
&= \lim_{r\to 0^+} (-r^2) = 0 \ ◀
\end{aligned}$$

CONTINUITY

Example 7

- Let

$$g(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

  - Here, g is defined at (0, 0).
  - However, it is still discontinuous there because

$$\lim_{(x,y)\to(0,0)} g(x,y)$$

    does not exist (see Example 1).

CONTINUITY

Example 8

- Let

$$f(x,y)=\begin{cases}\dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)\neq(0,0)\end{cases}$$

Example 8

CONTINUITY

- We know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there.

- Also, from Example 4, we have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2}$$

$$= 0 = f(0,0)$$

Example 8

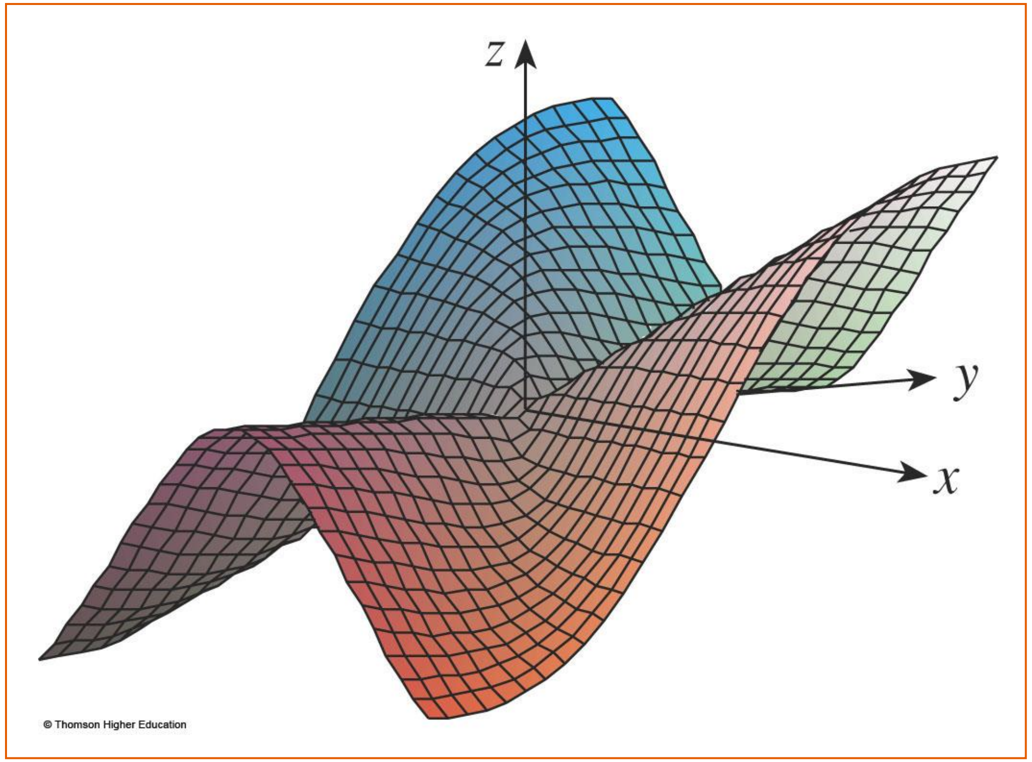
CONTINUITY

- Thus, f is continuous at (0, 0).

  - So, it is continuous on R^2.

CONTINUITY

- This figure shows the graph of the continuous function in Example 8.



Example 9

COMPOSITE FUNCTIONS

- Where is the function $h(x, y) = \arctan(y/x)$ continuous?

    - The function $f(x, y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$.

    - The function $g(t) = \arctan t$ is continuous everywhere.

Example 9

COMPOSITE FUNCTIONS

- So, the composite function

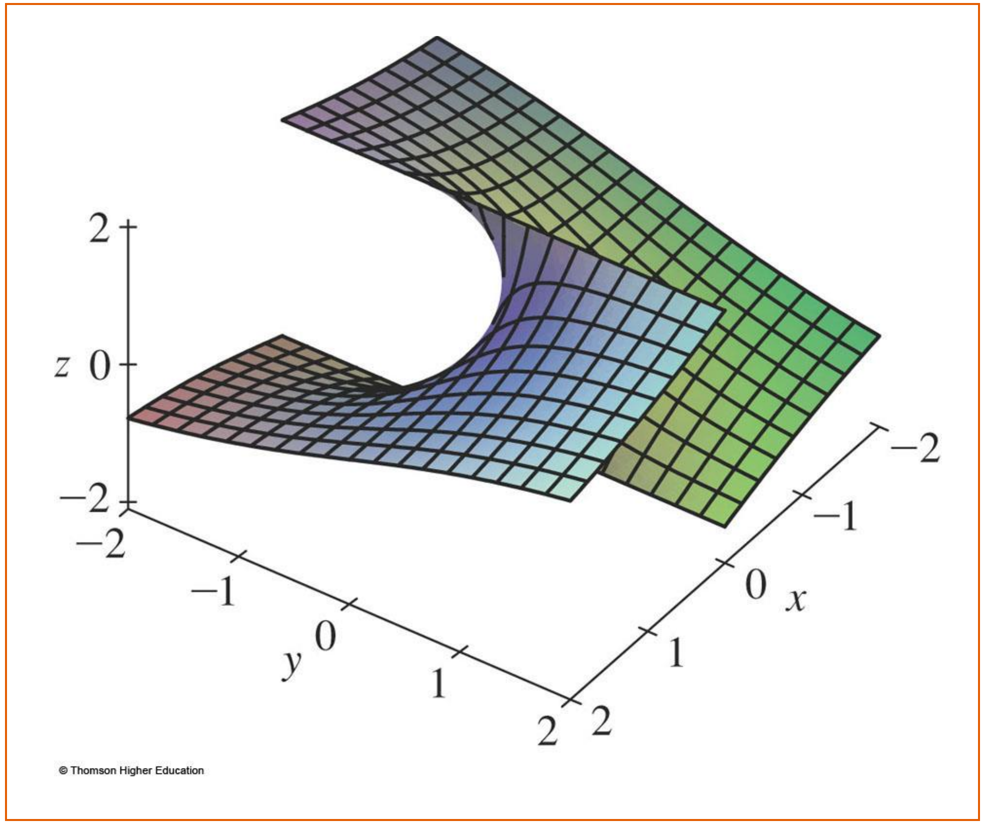
$$g(f(x, y)) = \arctan(y, x) = h(x, y)$$

is continuous except where $x = 0$.

COMPOSITE FUNCTIONS

Example 9

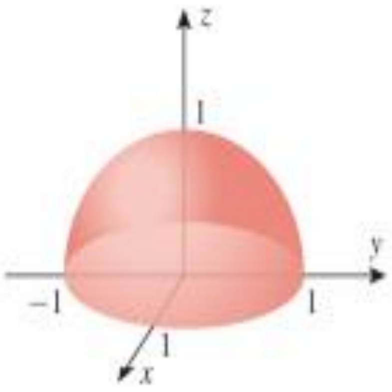
- The figure shows the break in the graph
of h above the y-axis.



CONTINUITY AT BOUNDARY POINTS

Recall that in our study of continuity for functions of one variable, we first defined continuity at a point, then continuity on an open interval, and then, by using one-sided limits, we extended the notion of continuity to include endpoints of the interval. Similarly, for functions of two variables one can extend the notion of continuity of $f(x, y)$ to the boundary

of its domain by modifying Definition 13.2.1 appropriately so that (x, y) is restricted to approach (x_0, y_0) through points lying wholly in the domain of f. We will omit the details.



$z = \sqrt{1 - x^2 - y^2}$

▲ **Figure 13.2.11**

▶ **Example 8** The graph of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ is the upper hemisphere shown in Figure 13.2.11, and the natural domain of f is the closed unit disk

$$x^2 + y^2 \leq 1$$

The graph of f has no jumps, tears or holes, so it passes our "intuitive test" of continuity. In this case the continuity at a point (x_0, y_0) on the boundary reflects the fact that

$$\lim_{(x,y)\to(x_0,y_0)} \sqrt{1 - x^2 - y^2} = \sqrt{1 - x_0^2 - y_0^2} = 0$$

when (x, y) is restricted to points on the closed unit disk $x^2 + y^2 \leq 1$. It follows that f is continuous on its domain. ◀

EXERCISE SET 13.2

1–6 Use limit laws and continuity properties to evaluate the limit. ■

1. $\lim_{(x,y)\to(1,3)} (4xy^2 - x)$

2. $\lim_{(x,y)\to(0,0)} \dfrac{4x - y}{\sin y - 1}$

3. $\lim_{(x,y)\to(-1,2)} \dfrac{xy^3}{x + y}$

4. $\lim_{(x,y)\to(1,-3)} e^{2x-y^2}$

5. $\lim_{(x,y)\to(0,0)} \ln(1 + x^2y^3)$

6. $\lim_{(x,y)\to(4,-2)} x\sqrt[3]{y^3 + 2x}$

7–8 Show that the limit does not exist by considering the limits as $(x, y) \to (0, 0)$ along the coordinate axes.

7. (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{3}{x^2 + 2y^2}$ (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x + y}{2x^2 + y^2}$

8. (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x - y}{x^2 + y^2}$ (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\cos xy}{x^2 + y^2}$

9–12 Evaluate the limit using the substitution $z = x^2 + y^2$ and observing that $z \to 0^+$ if and only if $(x, y) \to (0, 0)$.

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

11. $\displaystyle\lim_{(x,y)\to(0,0)} e^{-1/(x^2+y^2)}$

12. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{-1/\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

13–22 Determine whether the limit exists. If so, find its value.

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 + y^2}$

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4 - 16y^4}{x^2 + 4y^2}$

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{3x^2 + 2y^2}$

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1 - x^2 - y^2}{x^2 + y^2}$

17. $\lim\limits_{(x,y,z)\to(2,-1,2)} \dfrac{xz^2}{\sqrt{x^2 + y^2 + z^2}}$

18. $\lim\limits_{(x,y,z)\to(2,0,-1)} \ln(2x + y - z)$

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin(x^2 + y^2 + z^2)}{\sqrt{x^2 + y^2 + z^2}}$

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin\sqrt{x^2 + y^2 + z^2}}{x^2 + y^2 + z^2}$

21. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2 + y^2 + z^2}}$

22. $\lim\limits_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\dfrac{1}{x^2 + y^2 + z^2}\right]$

23–26 Evaluate the limits by converting to polar coordinates, as in Example 7.

23. $\displaystyle\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\,\ln(x^2+y^2)$

24. $\displaystyle\lim_{(x,y)\to(0,0)} y\ln(x^2+y^2)$

25. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}}$

26. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+2y^2}}$

34. (a) Show that as $(x, y) \to (0, 0)$ along any straight line $y = mx$, or along any parabola $y = kx^2$, the value of

$$\frac{x^3y}{2x^6 + y^2}$$

approaches 0.

(b) Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{2x^6 + y^2}$$

does not exist by letting $(x, y) \to (0, 0)$ along the curve $y = x^3$.

35. (a) Show that the value of

$$\frac{xyz}{x^2 + y^4 + z^4}$$

approaches 0 as $(x, y, z) \to (0, 0, 0)$ along any line $x = at$, $y = bt$, $z = ct$.

(b) Show that the limit

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^4 + z^4}$$

does not exist by letting $(x, y, z) \to (0, 0, 0)$ along the curve $x = t^2$, $y = t$, $z = t$.

36. Find $\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\dfrac{x^2+1}{x^2+(y-1)^2}\right]$.

37. Find $\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\dfrac{x^2-1}{x^2+(y-1)^2}\right]$.

38. Let $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0). \end{cases}$

Show that f is continuous at $(0,0)$.

39–40 A function $f(x, y)$ is said to have a ***removable discontinuity*** at (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists but f is not continuous at (x_0, y_0), either because f is not defined at (x_0, y_0) or because $f(x_0, y_0)$ differs from the value of the limit. Determine whether $f(x, y)$ has a removable discontinuity at $(0, 0)$.

39. $f(x, y) = \dfrac{x^2}{x^2 + y^2}$

40. $f(x) = \begin{cases} x^2 + 7y^2, & \text{if } (x, y) \neq (0, 0) \\ -4, & \text{if } (x, y) = (0, 0) \end{cases}$

▶ Exercise Set 13.2 (Page 823)

1. 35 **3.** −8 **5.** 0

7. **(a)** along $x = 0$ limit does not exist
(b) along $x = 0$ limit does not exist

9. 1 **11.** 0 **13.** 0 **15.** limit does not exist **17.** $\frac{8}{3}$ **19.** 0

21. limit does not exist **23.** 0 **25.** 0 **27.** 0

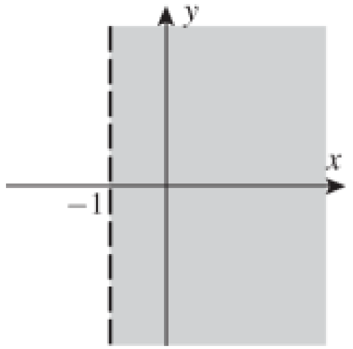
Responses to True–False questions may be abridged to save space.

29. True; by the definition of open set.

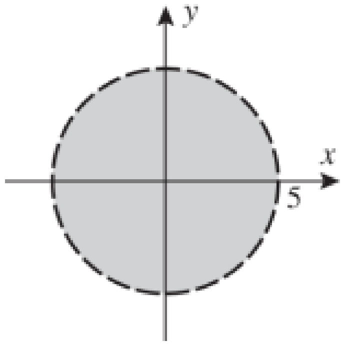
31. False; let $f(x, y) = \begin{cases} 1, & x \le 0 \\ -1, & x > 0 \end{cases}$ and let $g(x, y) = -f(x, y)$.

33. **(a)** no **(d)** no; yes **37.** $-\pi/2$ **39.** no

41.



43.



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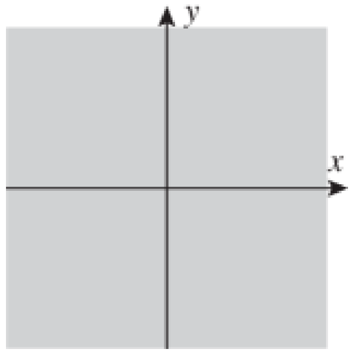
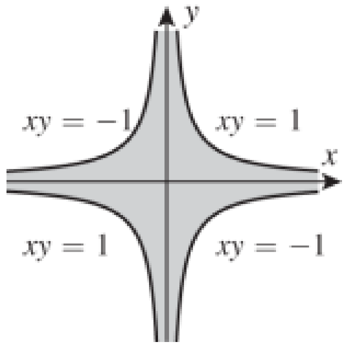


47.



49. all of 3-space **51.** all points not on the cylinder $x^2 + z^2 = 1$