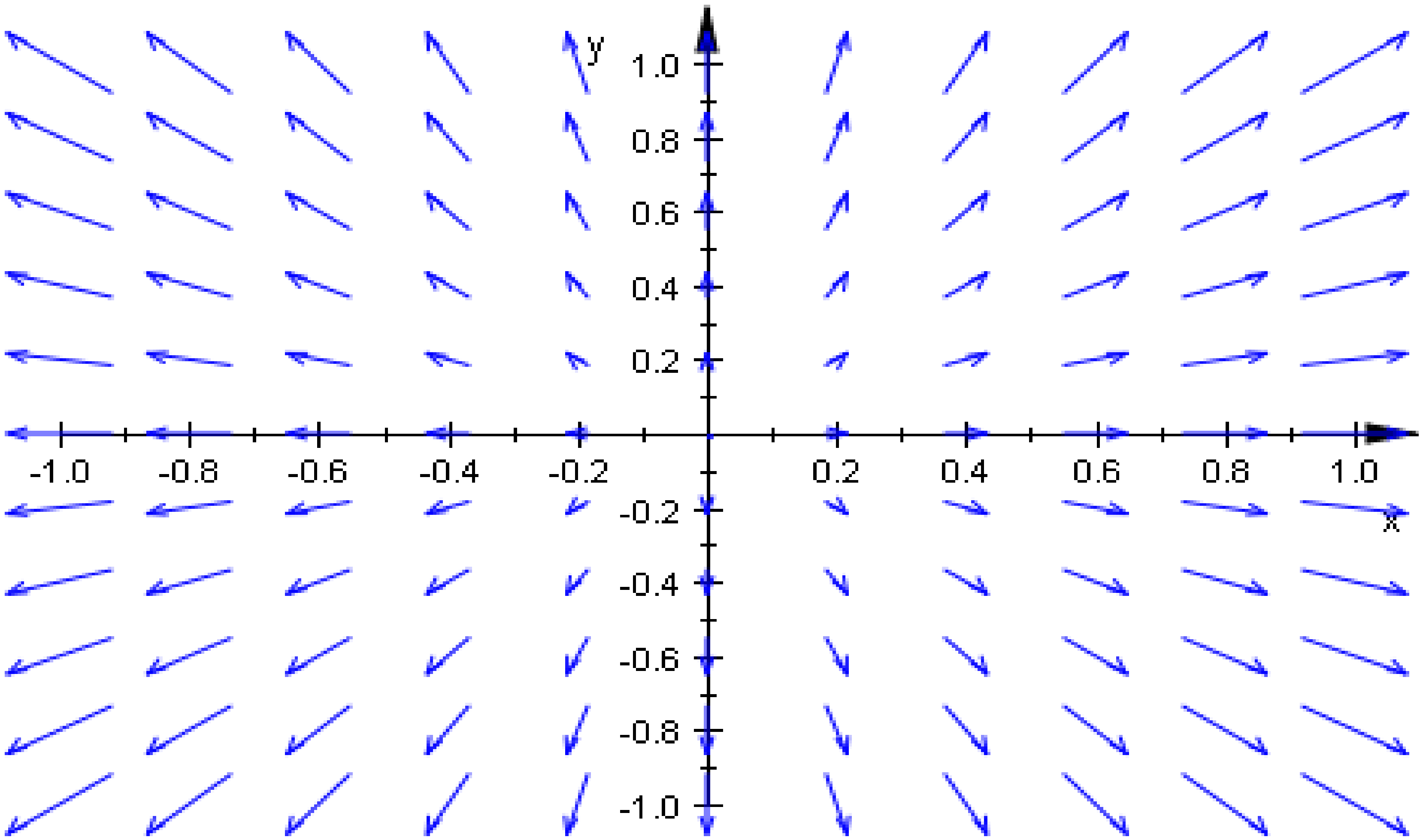
Vector Fields, gradient, divergence, and curl

15.1

VECTOR FIELD



VECTOR FIELDS

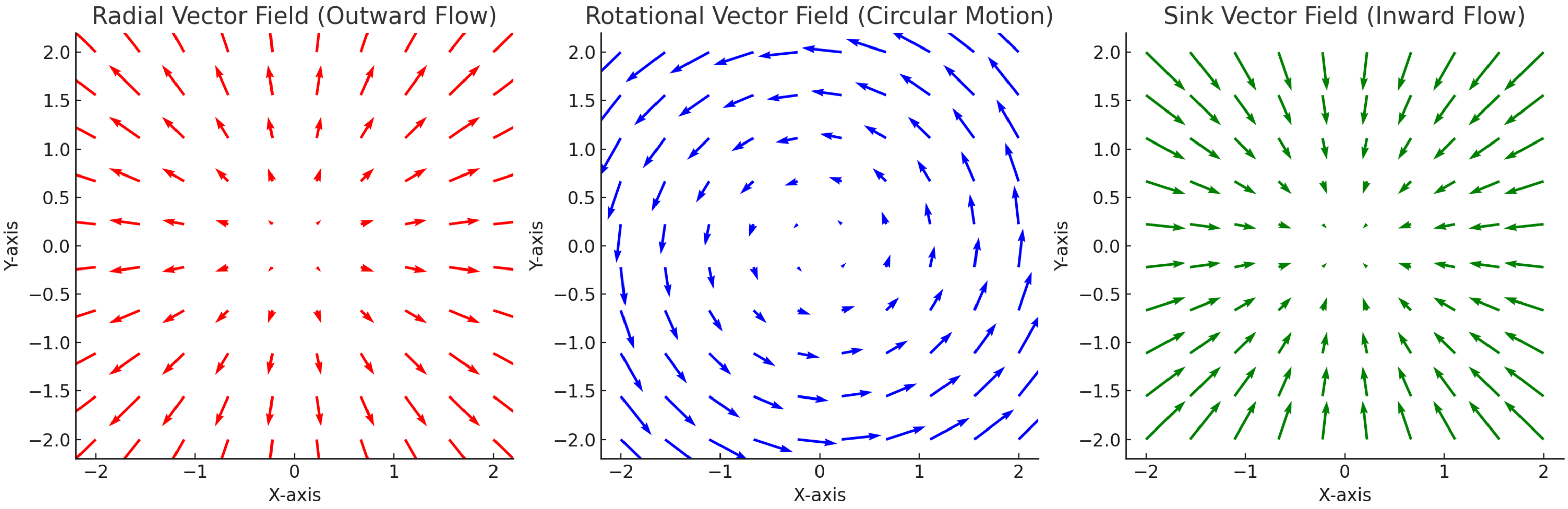
Consider a *unit* point-mass located at any point in the Universe. According to Newton's Law of Universal Gravitation, the Earth exerts an attractive force on the mass that is directed toward the center of the Earth and has a magnitude that is inversely proportional to the square of the distance from the mass to the Earth's center (Figure 15.1.1). This association of force vectors with points in space is called the Earth's *gravitational field*. A similar association occurs in fluid flow. Imagine a stream in which the water flows horizontally at every level, and consider the layer of water at a specific depth. At each point of the layer, the water has a certain velocity, which we can represent by a vector at that point (Figure 15.1.2). This association of velocity vectors with points in the two-dimensional layer is called the *velocity field* at that layer. These ideas are captured in the following definition.

▲ **Figure 15.1.1**

15.1.1 DEFINITION A *vector field* in a plane is a function that associates with each point P in the plane a unique vector $\mathbf{F}(P)$ parallel to the plane. Similarly, a vector field in 3-space is a function that associates with each point P in 3-space a unique vector $\mathbf{F}(P)$ in 3-space.

Vector Fields in Computer Science

- • A vector field assigns a vector to every point in space.
- • Used in fluid simulations, robotics, and game physics.
- • Example: Wind flow simulation in a game engine.



Equation: F(x, y) = (x, y)

Equation: F(x, y) = (-y, x)

Equation: F(x, y) = (-x, -y)

GRADIENT : (NORMAL TO FUNCTION)

The vector differential operator ∇ , called "del" or "nabla", is defined in three dimensions to be:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} .$$

Note that these are *partial derivatives*!
This vector operator may be applied to (differentiable) scalar functions (scalar fields) and the result is a special case of a vector field, called a gradient vector field.

2. Gradient (Grad)

The **gradient** of a function, $f(x, y)$, in two dimensions is defined as:

$$\boxed{\operatorname{grad} f(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}\,.}$$

The gradient of a function is a vector field. It is obtained by applying the vector operator ∇ to the scalar function $f(x, y)$. Such a vector field is called a gradient (or conservative) vector field.

Example 1 The gradient of the function $f(x, y) = x + y^2$ is given by:

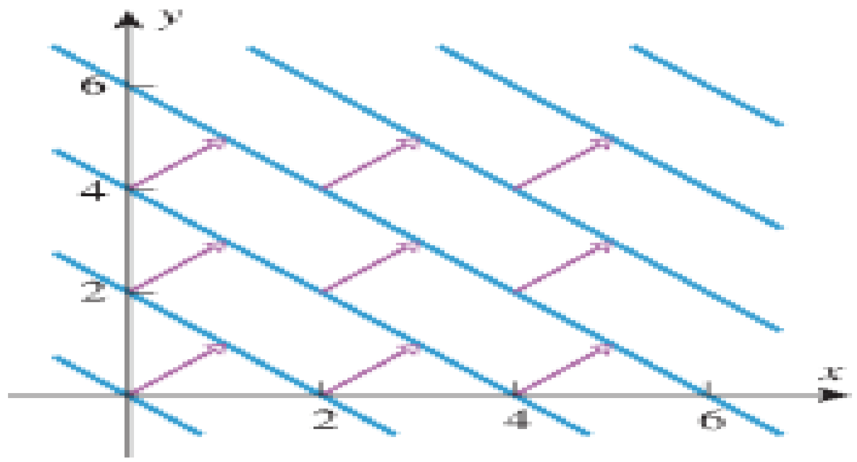
$$\begin{aligned}
\nabla f(x, y) &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} \\
&= \frac{\partial}{\partial x}(x + y^2)\boldsymbol{i} + \frac{\partial}{\partial y}(x + y^2)\boldsymbol{j} \\
&= (1 + 0)\boldsymbol{i} + (0 + 2y)\boldsymbol{j} \\
&= \boldsymbol{i} + 2y\boldsymbol{j}\,.
\end{aligned}$$

▶ **Example 2** Sketch the gradient field of $\phi(x, y) = x + y$.

Solution. The gradient of ϕ is

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = \mathbf{i} + \mathbf{j}$$

which is constant [i.e., is the same vector at each point (x, y)]. A portion of the vector field is sketched in Figure 15.1.4 together with some level curves of ϕ. Note that at each point, $\nabla\phi$ is normal to the level curve of ϕ through the point (Theorem 13.6.6). ◀



▲ **Figure 15.1.4**

CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

If $\mathbf{F}(\mathbf{r})$ is an arbitrary vector field in 2-space or 3-space, we can ask whether it is the gradient field of some function ϕ, and if so, how we can find ϕ. This is an important problem and we will study it in more detail later. However, there is some terminology for such fields that we will introduce now.

15.1.3 DEFINITION A vector field $\mathbf{F}$ in 2-space or 3-space is said to be *conservative* in a region if it is the gradient field for some function ϕ in that region, that is, if

$$\mathbf{F} = \nabla\phi$$

The function ϕ is called a *potential function* for $\mathbf{F}$ in the region.

Conservative(independent of path) and Non-Conservative Vector Fields in Computer Science

In **computer science**, conservative and non-conservative fields appear in **robotics, physics-based simulations, machine learning, and computer graphics**. Here are some relevant questions with solutions

Q1: A robotic arm moves in a 2D plane under a **force field** given by: $F = (2xy, x^2)$ Check if the field is **conservative**,

1. Compute the **partial derivatives:**
   - $P = 2xy$, so $\frac{\partial P}{\partial y} = 2x$
   - $Q = x^2$, so $\frac{\partial Q}{\partial x} = 2x$
2. Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the field is **conservative.**

Q#2 In a game physics engine, a particle moves in a velocity field:
$F = (-y, x)$ Determine if the field is **conservative**.

1. Compute the partial derivatives:
   - $P = -y$, so $\frac{\partial P}{\partial y} = -1$
   - $Q = x$, so $\frac{\partial Q}{\partial x} = 1$
2. Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, the field is non-conservative.

Gradient and Its Role in Optimization

- Gradient represents the steepest ascent direction.
- Used in Machine Learning (Gradient Descent) to optimize neural networks.
- Applications: Image processing (edge detection), terrain mapping in robotics.

Curl and Divergence

Curl

- If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of **F** is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}}$$

- Let’s rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ (“del”) as

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

Curl

- It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field **F** as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Curl

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \text{curl}\,\mathbf{F}$$

- So the easiest way to remember Definition 1 is by means of the symbolic expression

$$\boxed{2} \qquad \boxed{\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}}$$

Example 1

- If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

- Solution:
- Using Equation 2, we have

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

Example 1 – *Solution*

cont'd

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

Ex#2 A robotic system detects an **electromagnetic field** given by:

$$F = (x^2y, -yx^2, z)$$

Find the **curl** $\nabla \times F$.

1. **Given field:**

$$\mathbf{F} = (P, Q, R) = (x^2y, -yx^2, z)$$

2. Compute **curl** in 3D:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -yx^2 & z \end{vmatrix}$$

Expanding along the first row:

$$\mathbf{i}\left(\frac{\partial z}{\partial y} - \frac{\partial(-yx^2)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial z}{\partial x} - \frac{\partial(x^2y)}{\partial z}\right) + \mathbf{k}\left(\frac{\partial(-yx^2)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\right)$$

3. Compute **partial derivatives:**

- $\frac{\partial z}{\partial y} = 0$, $\frac{\partial(-yx^2)}{\partial z} = 0$
- $\frac{\partial z}{\partial x} = 0$, $\frac{\partial(x^2y)}{\partial z} = 0$
- $\frac{\partial(-yx^2)}{\partial x} = -2yx$, $\frac{\partial(x^2y)}{\partial y} = x^2$

4. **Final curl:**

$$\nabla \times \mathbf{F} = (0, 0, -2yx - x^2)$$

$$= (0, 0, -x^2 - 2yx)$$

✅ **Conclusion:** Since the curl is **nonzero**, this magnetic field has a **rotational component**, affecting robotic sensors and electromagnetism simulations.

Curl

- Recall that the gradient of a function *f* of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl.
- The following theorem says that the curl of a gradient vector field is **0**.

$\boxed{3}$ **Theorem** If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

Curl

- Since a conservative vector field is one for which **F** = ∇f, Theorem 3 can be rephrased as follows:

If **F** is conservative, then curl **F** = **0**.

This gives us a way of verifying that a vector field is not conservative.

Curl

- The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if **F** is defined everywhere. (More generally it is true if the domain is simply-connected, that is, “has no hole.”)

$\boxed{4}$ **Theorem** If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field.

Curl

- The reason for the name *curl* is that the curl vector is associated with rotations.

- Another occurs when **F** represents the velocity field in fluid flow. Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1).

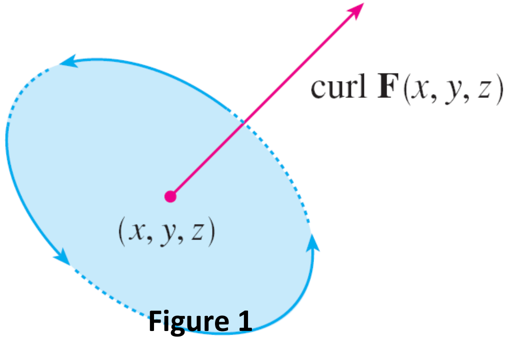


Figure 1

Curl

- If curl $\mathbf{F} = \mathbf{0}$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called **irrotational** at P. In other words, there is no whirlpool or eddy at P. If curl $\mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If curl $\mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis.

Divergence

Divergence

- If **F** = P **i** + Q **j** + R **k** is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

$$\boxed{9} \qquad \boxed{\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

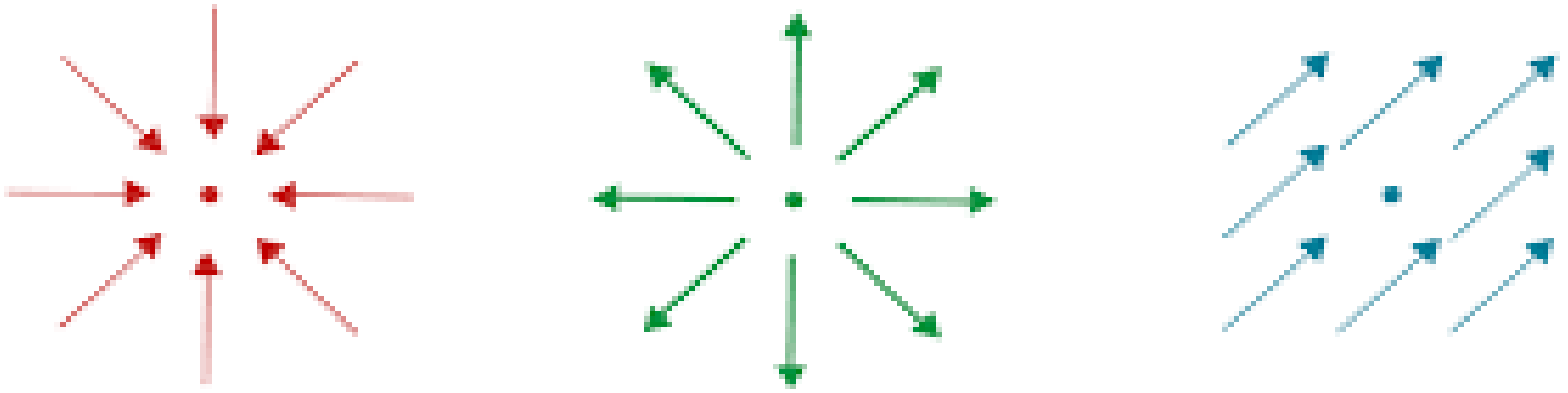
- Observe that curl **F** is a vector field but div **F** is a scalar field.

Divergence

- In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

$$\boxed{10} \qquad \boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}}$$

$\nabla \cdot \vec{\mathbf{v}} < 0$ $\qquad$ $\nabla \cdot \vec{\mathbf{v}} > 0$ $\qquad$ $\nabla \cdot \vec{\mathbf{v}} = 0$



Example 4

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} + y^2\,\mathbf{k}$, find div $\mathbf{F}$.

Solution:

By the definition of divergence (Equation 9 or 10) we have

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

Divergence

- If **F** is a vector field on $\mathbb{R}^3$ then curl **F** is also a vector field on $\mathbb{R}^3$ As such, we can compute its divergence.

The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Again, the reason for the name *divergence* can be understood in the context of fluid flow.

Divergence

- If **F**(x, y, z) is the velocity of a fluid (or gas), then div **F**(x, y, z) represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

- In other words, div **F**(x, y, z) measures the tendency of the fluid to diverge from the point (x, y, z).

- If div **F** = 0, then **F** is said to be **incompressible** OR Solenoidal.

- Another differential operator occurs when we compute the divergence of a gradient vector field ∇f.

Divergence

- If f is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator
- $$\nabla^2 = \nabla \cdot \nabla$$
- is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Divergence

- We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$

The **divergence** of a vector field measures how much a vector field spreads out (sources) or contracts (sinks). In **computer science**, divergence is used in **fluid dynamics, computer graphics, game physics, and machine learning (vector-based optimization problems).**

Q1:A **fluid simulation** in a video game uses the velocity field: $F = (x^2, y^2)$ Find the **divergence** of F and interpret its meaning.

Solution:

1. **Given field:**

$$\mathbf{F} = (P, Q) = (x^2, y^2)$$

2. Compute **divergence:**

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

   - $\frac{\partial P}{\partial x} = \frac{\partial (x^2)}{\partial x} = 2x$
   - $\frac{\partial Q}{\partial y} = \frac{\partial (y^2)}{\partial y} = 2y$

3. **Final divergence:**

$$\nabla \cdot \mathbf{F} = 2x + 2y$$

✅ **Conclusion:** The divergence **increases as x and y increase**, meaning that the field behaves like a **source** (fluid is expanding outward). This can be used to simulate **explosions or expanding smoke** in a game engine.

Q#2: In a **machine learning gradient descent problem**, a cost function is influenced by the force field:

$F = (e^x, -y)$ Determine if the vector field represents a **source** or a **sink**.

Solution:

1. **Given field:**

$$\mathbf{F} = (P, Q) = (e^x, -y)$$

2. Compute **divergence:**

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

- $\frac{\partial P}{\partial x} = \frac{\partial (e^x)}{\partial x} = e^x$
- $\frac{\partial Q}{\partial y} = \frac{\partial (-y)}{\partial y} = -1$

3. **Final divergence:**

$$\nabla \cdot \mathbf{F} = e^x - 1$$

✅ **Conclusion:**

- If $x > 0$, then $e^x - 1 > 0$, meaning the field acts as a **source** (expanding forces).
- If $x < 0$, then $e^x - 1 < 0$, meaning the field acts as a **sink** (contracting forces).

This is useful in **AI for gradient-based optimization**, where we analyze how gradients **expand or contract** in training models.

Q3: In **computer vision**, optical flow is modeled as:
$F = (x + y, y - x, -z)$Compute the **divergence** and explain its meaning.

Solution:

1. Given field:

$$\mathbf{F} = (P, Q, R) = (x + y, y - x, -z)$$

2. Compute divergence:

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- $\frac{\partial P}{\partial x} = \frac{\partial(x+y)}{\partial x} = 1$
- $\frac{\partial Q}{\partial y} = \frac{\partial(y-x)}{\partial y} = 1$
- $\frac{\partial R}{\partial z} = \frac{\partial(-z)}{\partial z} = -1$

3. **Final divergence:**

$$\nabla \cdot \mathbf{F} = 1 + 1 - 1 = 1$$

✅ **Conclusion:** Since the divergence is **positive**, this optical flow field represents an **expansion** of pixels (e.g., an object moving away from the camera). This is useful in **motion tracking** and **image segmentation.**

Summary: Applications in Computer Science

Concept	Formula	Applications
Vector Field	$F(x, y, z)$	Fluid simulation, robotics, physics engines
Gradient	∇f	Machine learning, terrain mapping, edge detection
Divergence	$\nabla \cdot F$	Game physics, network data flow analysis
Curl	$\nabla \times F$	Motion detection, self-driving cars, image processing

17–22 Find div $\mathbf{F}$ and curl $\mathbf{F}$.

17. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2\mathbf{j} + yz\mathbf{k}$

18. $\mathbf{F}(x, y, z) = xz^3\mathbf{i} + 2y^4x^2\mathbf{j} + 5z^2y\mathbf{k}$

19. $\mathbf{F}(x, y, z) = 7y^3z^2\mathbf{i} - 8x^2z^5\mathbf{j} - 3xy^4\mathbf{k}$

20. $\mathbf{F}(x, y, z) = e^{xy}\mathbf{i} - \cos y\,\mathbf{j} + \sin^2 z\,\mathbf{k}$

21. $\mathbf{F}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

22. $\mathbf{F}(x, y, z) = \ln x\,\mathbf{i} + e^{xyz}\mathbf{j} + \tan^{-1}(z/x)\mathbf{k}$

23–24 Find $\nabla \cdot (\mathbf{F} \times \mathbf{G})$.

23. $\mathbf{F}(x, y, z) = 2x\mathbf{i} + \mathbf{j} + 4y\mathbf{k}$
$\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$

24. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
$\mathbf{G}(x, y, z) = xy\mathbf{j} + xyz\mathbf{k}$

25–26 Find $\nabla \cdot (\nabla \times \mathbf{F})$.

25. $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos(x - y)\mathbf{j} + z\mathbf{k}$

26. $\mathbf{F}(x, y, z) = e^{xz}\mathbf{i} + 3xe^{y}\mathbf{j} - e^{yz}\mathbf{k}$

27–28 Find $\nabla \times (\nabla \times \mathbf{F})$.

27. $\mathbf{F}(x, y, z) = xy\mathbf{j} + xyz\mathbf{k}$

28. $\mathbf{F}(x, y, z) = y^2x\mathbf{i} - 3yz\mathbf{j} + xy\mathbf{k}$