14.2 DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

In this section we will show how to evaluate double integrals over regions other than rectangles.

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx \tag{1}$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy \tag{2}$$

We begin with an example that illustrates how to evaluate such integrals.

Evaluate

(a) $\displaystyle\int_0^1 \int_{-x}^{x^2} y^2 x \, dy \, dx$ (b) $\displaystyle\int_0^{\pi/3} \int_0^{\cos y} x \sin y \, dx \, dy$

Solution (a).

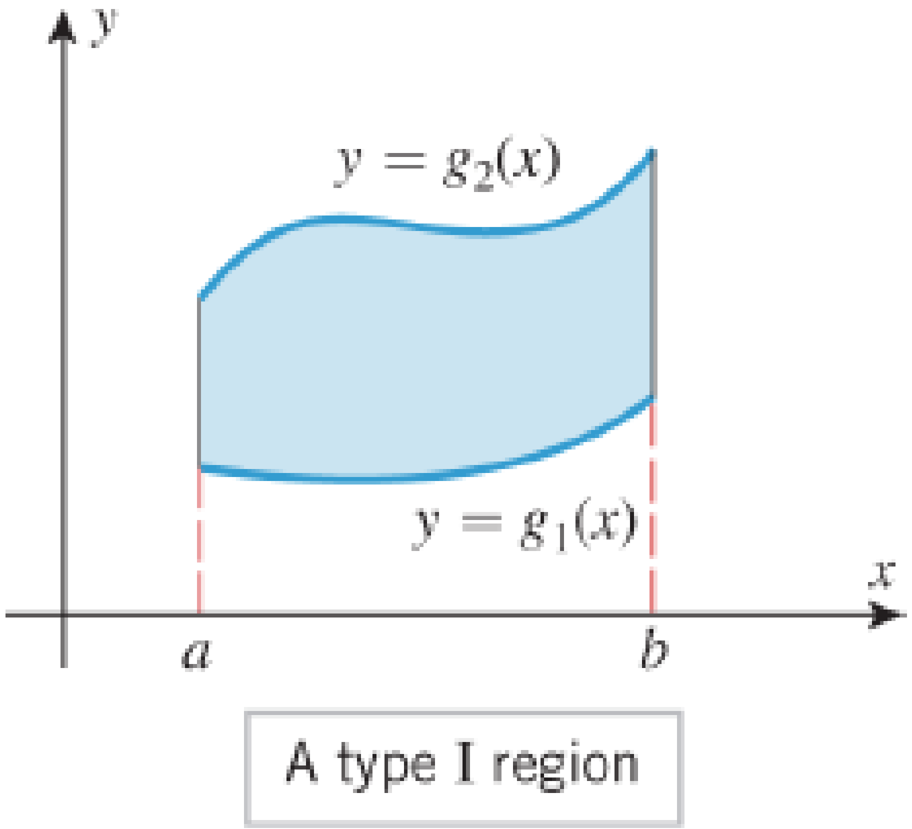
$$\int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x\, dy\right] dx = \int_0^1 \frac{y^3 x}{3}\Bigg]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3}\right] dx = \left(\frac{x^8}{24} + \frac{x^5}{15}\right)\Bigg]_0^1 = \frac{13}{120}$$

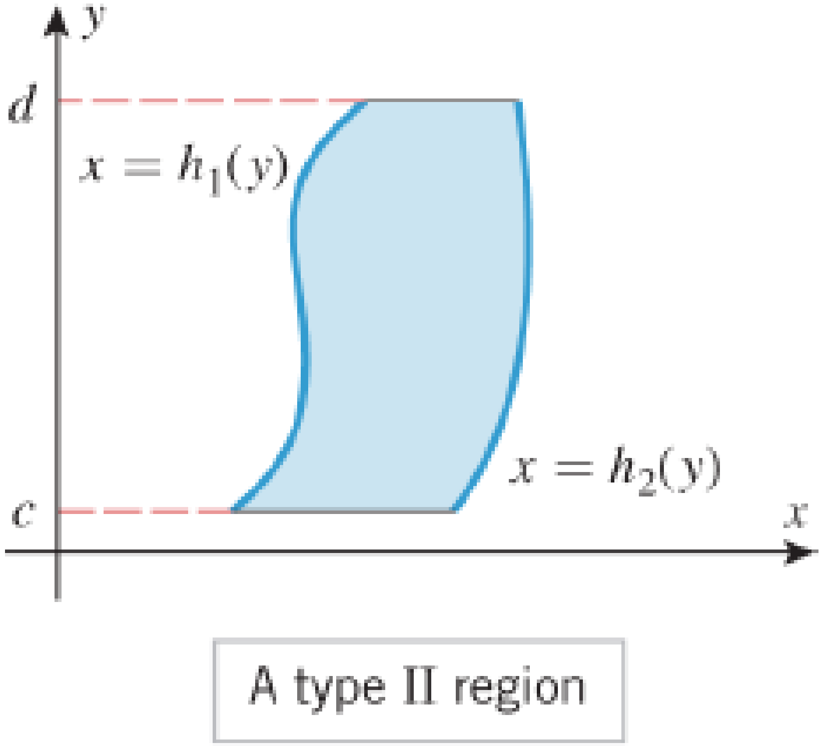
Solution (b).

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y\, dx\, dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y\, dx\right] dy = \int_0^{\pi/3} \frac{x^2}{2} \sin y\Bigg]_{x=0}^{\cos y} dy$$

$$= \int_0^{\pi/3} \left[\frac{1}{2}\cos^2 y \sin y\right] dy = -\frac{1}{6}\cos^3 y\Bigg]_0^{\pi/3} = \frac{7}{48} \blacktriangleleft$$



(a)



(b)

14.2.1 DEFINITION

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$ (Figure 14.2.1a).

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$ (Figure 14.2.1b).

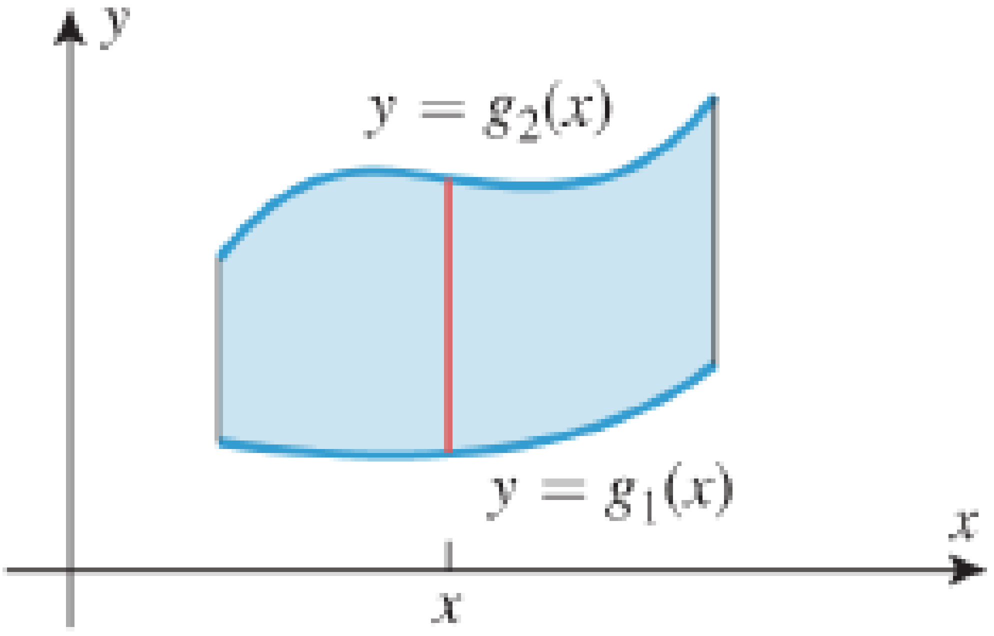
SETTING UP LIMITS OF INTEGRATION FOR EVALUATING DOUBLE INTEGRALS

To apply Theorem 14.2.2, it is helpful to start with a two-dimensional sketch of the region R. [It is not necessary to graph $f(x, y)$.] For a type I region, the limits of integration in Formula (3) can be obtained as follows:

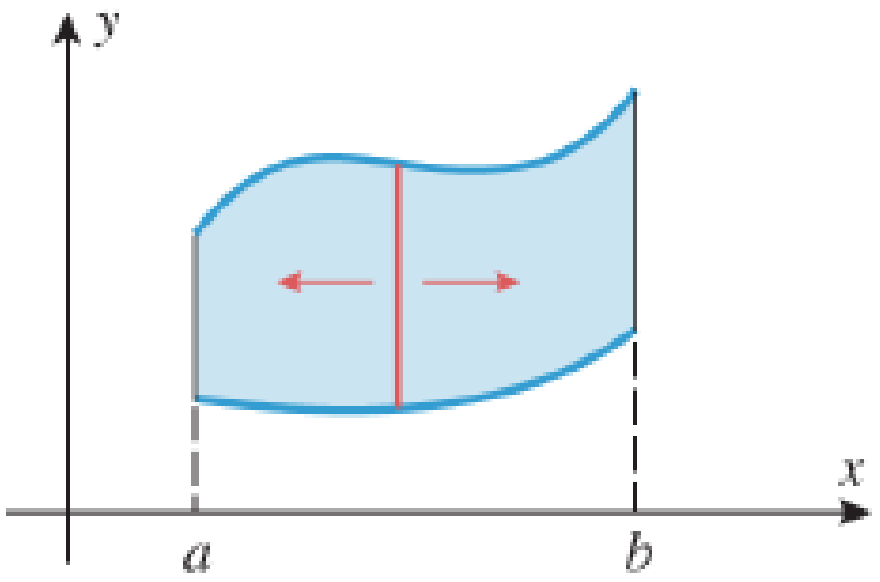
Determining Limits of Integration: Type I Region

Step 1. Since x is held fixed for the first integration, we draw a vertical line through the region R at an arbitrary fixed value x (Figure 14.2.5). This line crosses the boundary of R twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration in Formula (3).

Step 2. Imagine moving the line drawn in Step 1 first to the left and then to the right (Figure 14.2.5). The leftmost position where the line intersects the region R is $x = a$, and the rightmost position where the line intersects the region R is $x = b$. This yields the limits for the x-integration in Formula (3).



y
$y = g_2(x)$
$y = g_1(x)$
x
x



y
x
a
b

Evaluate

$$\iint_R xy\,dA$$

over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Solution. We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure 14.2.6. This line meets the region R at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

$$\iint_R xy\,dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\,dy\,dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

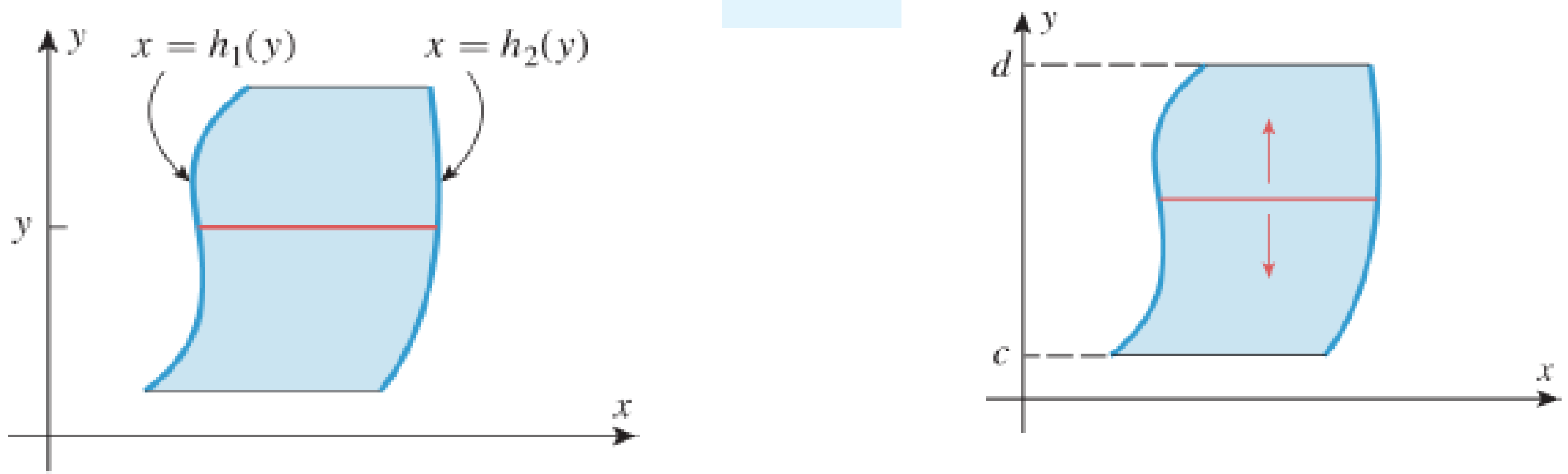
$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}$$

If R is a type II region, then the limits of integration in Formula (4) can be obtained as follows:

Determining Limits of Integration: Type II Region

Step 1. Since y is held fixed for the first integration, we draw a horizontal line through the region R at a fixed value y (Figure 14.2.7). This line crosses the boundary of R twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration in (4).

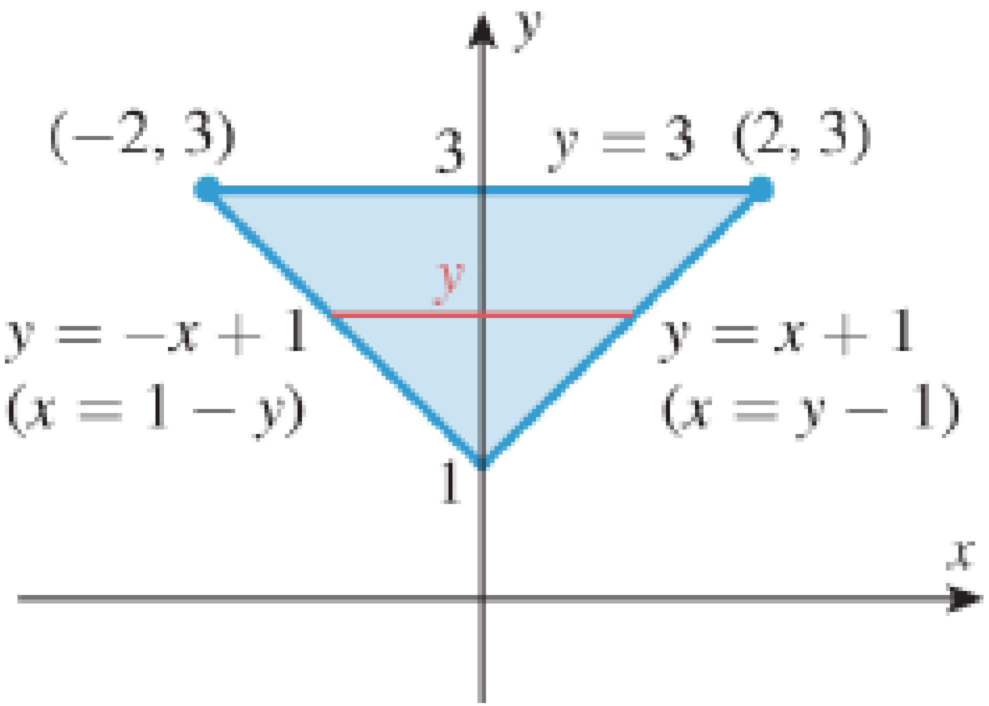
Step 2. Imagine moving the line drawn in Step 1 first down and then up (Figure 14.2.7). The lowest position where the line intersects the region R is $y = c$, and the highest position where the line intersects the region R is $y = d$. This yields the y-limits of integration in (4).



Evaluate

$$\iint_R (2x - y^2)\, dA$$

over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.



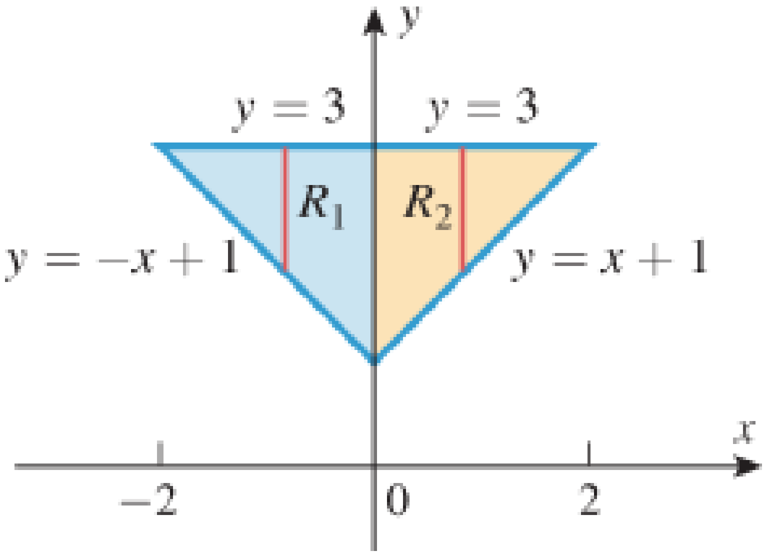
▲ **Figure 14.2.8**

Solution. We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in Figure 14.2.8. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\iint_R (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy$$

$$= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3} \blacktriangleleft$$



$$\iint_R (2x - y^2)\, dA = \iint_{R_1} (2x - y^2)\, dA + \iint_{R_2} (2x - y^2)\, dA$$

$$= \int_{-2}^{0} \int_{-x+1}^{3} (2x - y^2)\, dy\, dx + \int_{0}^{2} \int_{x+1}^{3} (2x - y^2)\, dy\, dx$$

This will yield the same result that was obtained in Example 4. (Verify.)

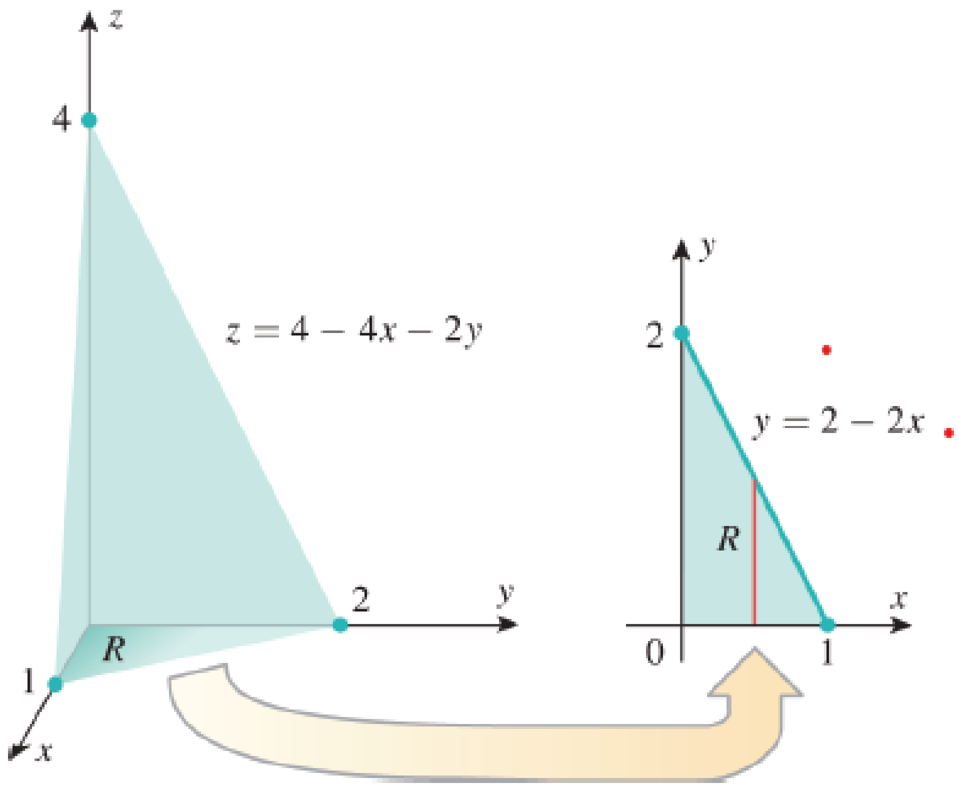
Example 5 Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane

$$z = 4 - 4x - 2y \tag{5}$$

and below by the triangular region R shown in Figure 14.2.10. Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y)\, dA$$



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The region R is bounded by the x-axis, the y-axis, and the line $y = 2 - 2x$ [set $z = 0$ in (5)], so that treating R as a type I region yields

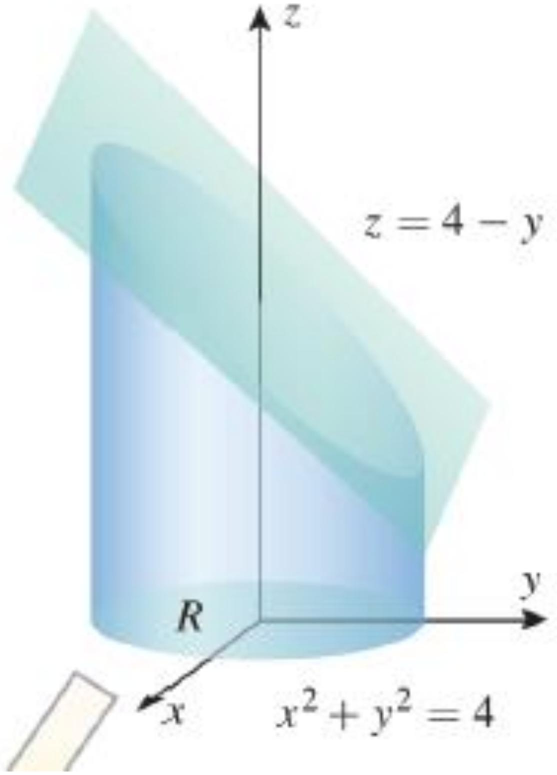
$$V = \iint_R (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_{y=0}^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\, dx = \frac{4}{3}$$

▶ Example 6 Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution. The solid shown in Figure 14.2.11 is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$



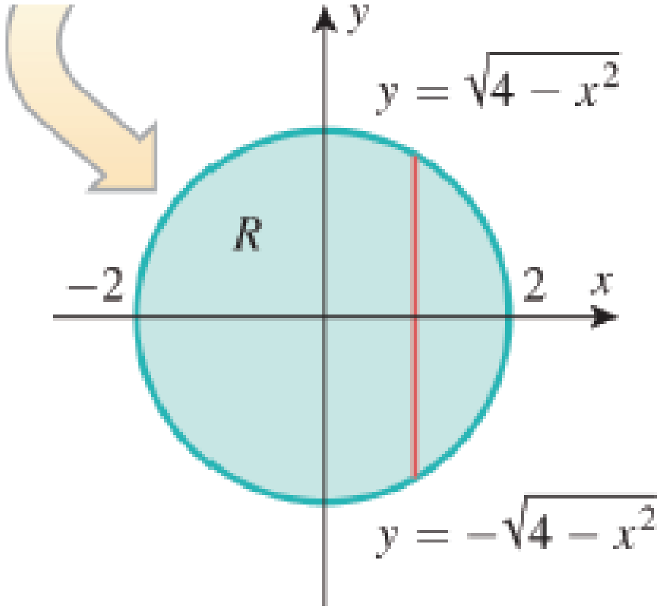


Figure 14.2.11

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\,dx = 8(2\pi) = 16\pi$$

See Formula (3) of Section 7.4.

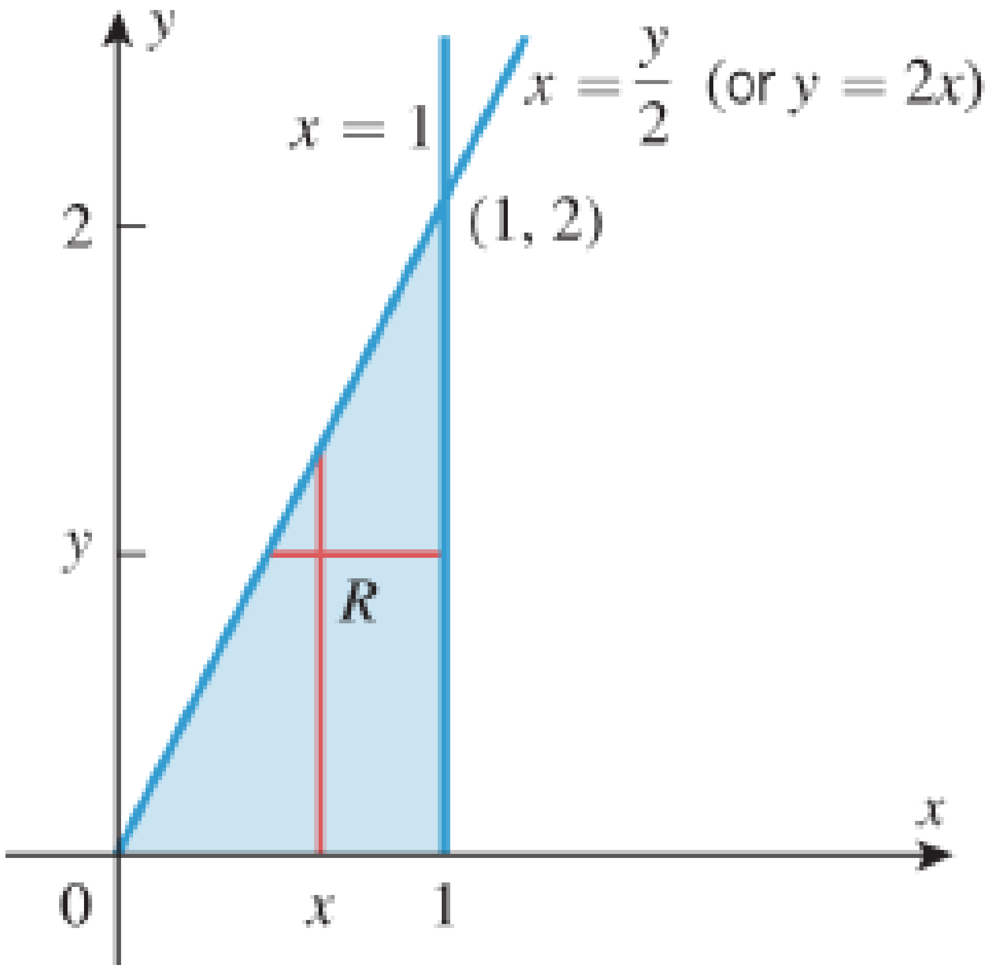
REVERSING THE ORDER OF INTEGRATION

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration. The next example illustrates how this is done.

▶ Example 7 Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$$

cannot be evaluated by performing the x-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.



▲ **Figure 14.2.12**

Solution. For the inside integration, y is fixed and x varies from the line $x = y/2$ to the line $x = 1$ (Figure 14.2.12). For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R in Figure 14.2.12.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \iint_R e^{x^2}\,dA = \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1 \left[e^{x^2} y\right]_{y=0}^{2x} dx$$

$$= \int_0^1 2xe^{x^2}\,dx = e^{x^2}\Big]_0^1 = e - 1 \blacktriangleleft$$

AREA CALCULATED AS A DOUBLE INTEGRAL

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. To see why this is so, recall that a *right cylinder* is a solid that is generated when a plane region is translated along a line that is perpendicular to the region. In Formula (2) of Section 5.2 we stated that the volume V of a right cylinder with cross-sectional area A and height h is

$$V = A \cdot h \tag{6}$$

Now suppose that we are interested in finding the area A of a region R in the xy-plane. If we translate the region R upward 1 unit, then the resulting solid will be a right cylinder that has cross-sectional area A, base R, and the plane $z = 1$ as its top (Figure 14.2.13). Thus, it follows from (6) that

$$\iint_R 1\, dA = (\text{area of } R) \cdot 1$$

which we can rewrite as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA \tag{7}$$

► Example 8 Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution. The region R may be treated equally well as type I (Figure 14.2.14*a*) or type II (Figure 14.2.14*b*). Treating R as type I yields

$$\text{area of } R = \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 [y]_{y=x^2/2}^{2x}\,dx$$
$$= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

Treating R as type II yields

$$\text{area of } R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\,dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}$$

EXERCISE SET 14.2 Graphing Utility C CAS

1–8 Evaluate the iterated integral.

1. $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2\,dy\,dx$

2. $\displaystyle\int_1^{3/2} \int_y^{3-y} y\,dx\,dy$

3. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} y\,dx\,dy$

4. $\displaystyle\int_{1/4}^1 \int_{x^2}^{x} \sqrt{\frac{x}{y}}\,dy\,dx$

5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin\frac{y}{x}\,dy\,dx$

6. $\displaystyle\int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y)\,dy\,dx$

7. $\displaystyle\int_0^1 \int_0^{x} y\sqrt{x^2 - y^2}\,dy\,dx$

8. $\displaystyle\int_1^2 \int_0^{y^2} e^{x/y^2}\,dx\,dy$

Exercise Set 14.2

Sol:

1. $\int_0^1 \int_{x^2}^{x} xy^2\, dy\, dx = \int_0^1 \frac{1}{3}(x^4 - x^7)\, dx = \frac{1}{40}.$

2. $\int_1^{3/2} \int_y^{3-y} y\, dx\, dy = \int_1^{3/2} (3y - 2y^2) dy = \frac{7}{24}.$

3. $\int_0^3 \int_0^{\sqrt{9-y^2}} y\, dx\, dy = \int_0^3 y\sqrt{9-y^2}\, dy = 9.$

4. $\int_{1/4}^1 \int_{x^2}^{x} \sqrt{x/y}\, dy\, dx = \int_{1/4}^1 \int_{x^2}^{x} x^{1/2}y^{-1/2}\, dy\, dx = \int_{1/4}^1 2(x - x^{3/2})\, dx = \frac{13}{80}.$

5. $\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin(y/x)\, dy\, dx = \int_{\sqrt{\pi}}^{\sqrt{2\pi}} [-x\cos(x^2) + x]\, dx = \frac{\pi}{2}.$

6. $\int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y)\, dy\, dx = \int_{-1}^1 2x^4\, dx = \frac{4}{5}.$

7. $\int_0^1 \int_0^x y\sqrt{x^2 - y^2}\, dy\, dx = \int_0^1 \frac{1}{3}x^3\, dx = \frac{1}{12}.$

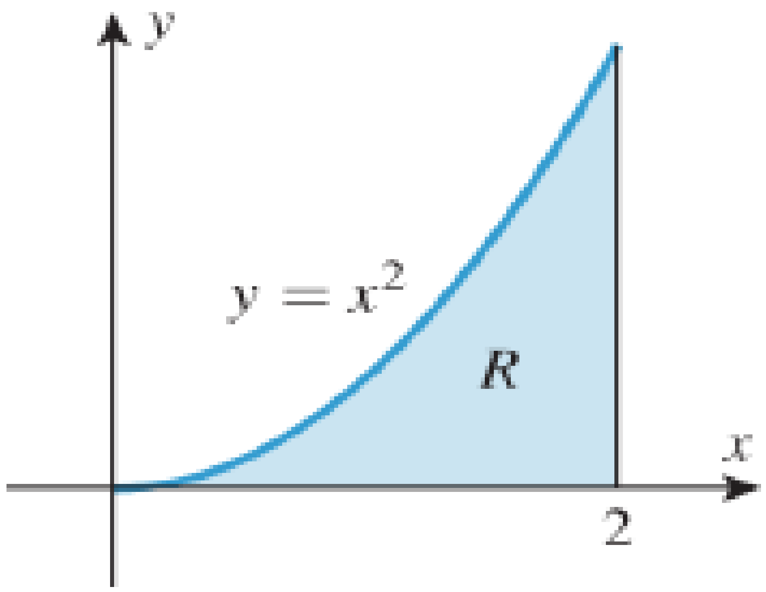
8. $\int_1^2 \int_0^{y^2} e^{x/y^2}\, dx\, dy = \int_1^2 (e-1)y^2\, dy = \frac{7(e-1)}{3}.$

FOCUS ON CONCEPTS

9. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dy\, dx$

(b) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$



▲ Figure Ex-9

Sol:

9. **(a)** $\displaystyle\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx.$

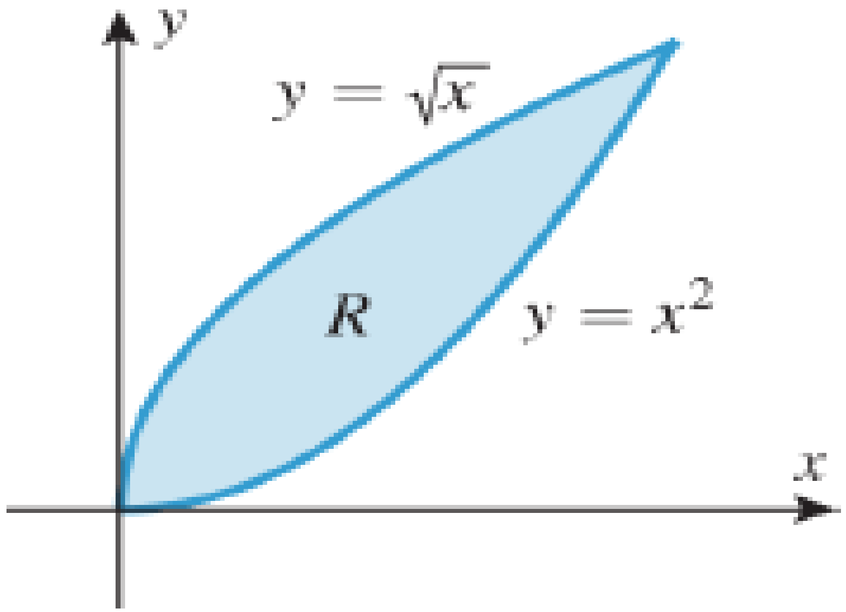
(b) $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 f(x,y)\,dx\,dy.$

R

10. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(x,y)\,dy\,dx$

(b) $\displaystyle\iint_R f(x,y)\,dA = \int_{\square}^{\square}\int_{\square}^{\square} f(x,y)\,dx\,dy$

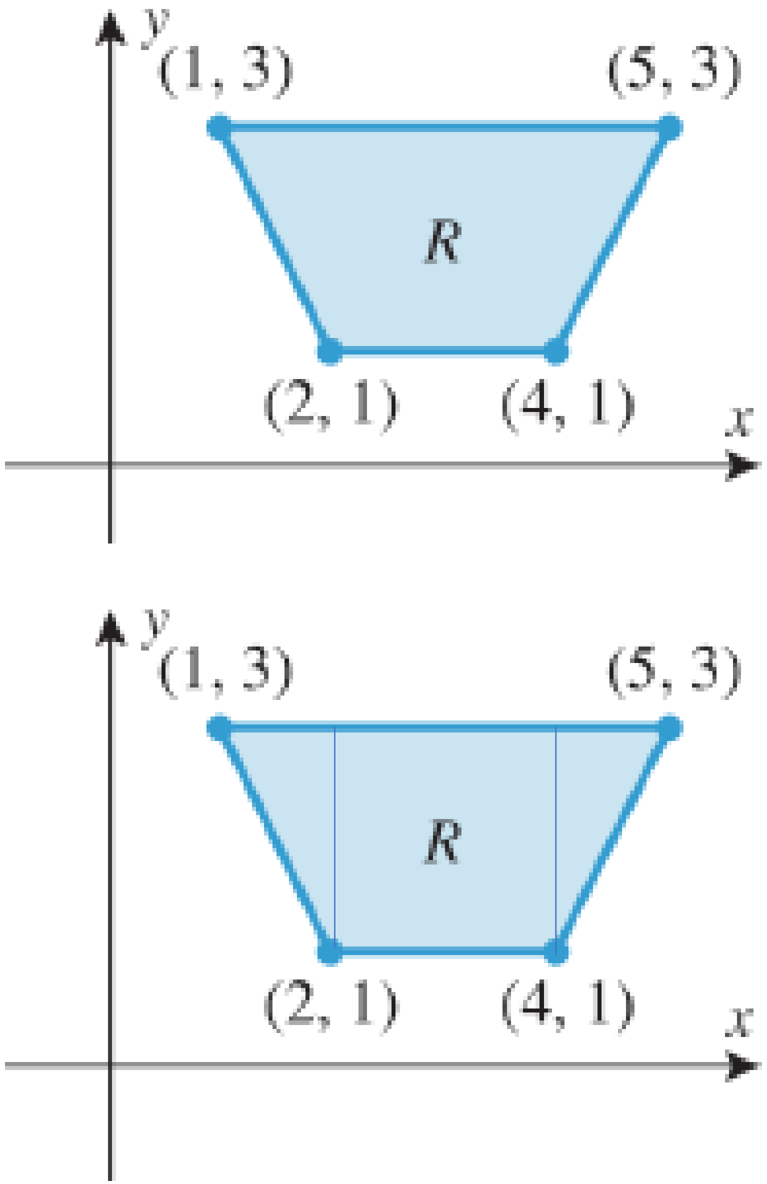


▲ **Figure Ex-10**

11. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $$\iint_R f(x, y)\, dA = \int_1^2 \int_{\square}^{\square} f(x, y)\, dy\, dx + \int_2^4 \int_{\square}^{\square} f(x, y)\, dy\, dx + \int_4^5 \int_{\square}^{\square} f(x, y)\, dy\, dx$$

(b) $$\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$$



▲ **Figure Ex-11**

Sol:

10. (a) $\displaystyle\int_0^1 \int_{x^2}^{\sqrt{x}} f(x,y)\,dy\,dx.$ (b) $\displaystyle\int_0^1 \int_{y^2}^{\sqrt{y}} f(x,y)\,dx\,dy.$

11. (a) $\displaystyle\int_1^2 \int_{-2x+5}^{3} f(x,y)\,dy\,dx + \int_2^4 \int_1^3 f(x,y)\,dy\,dx + \int_4^5 \int_{2x-7}^{3} f(x,y)\,dy\,dx.$

(b) $\displaystyle\int_1^3 \int_{(5-y)/2}^{(y+7)/2} f(x,y)\,dx\,dy.$

12. (a) $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$ (b) $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$

15–18 Evaluate the double integral in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region.

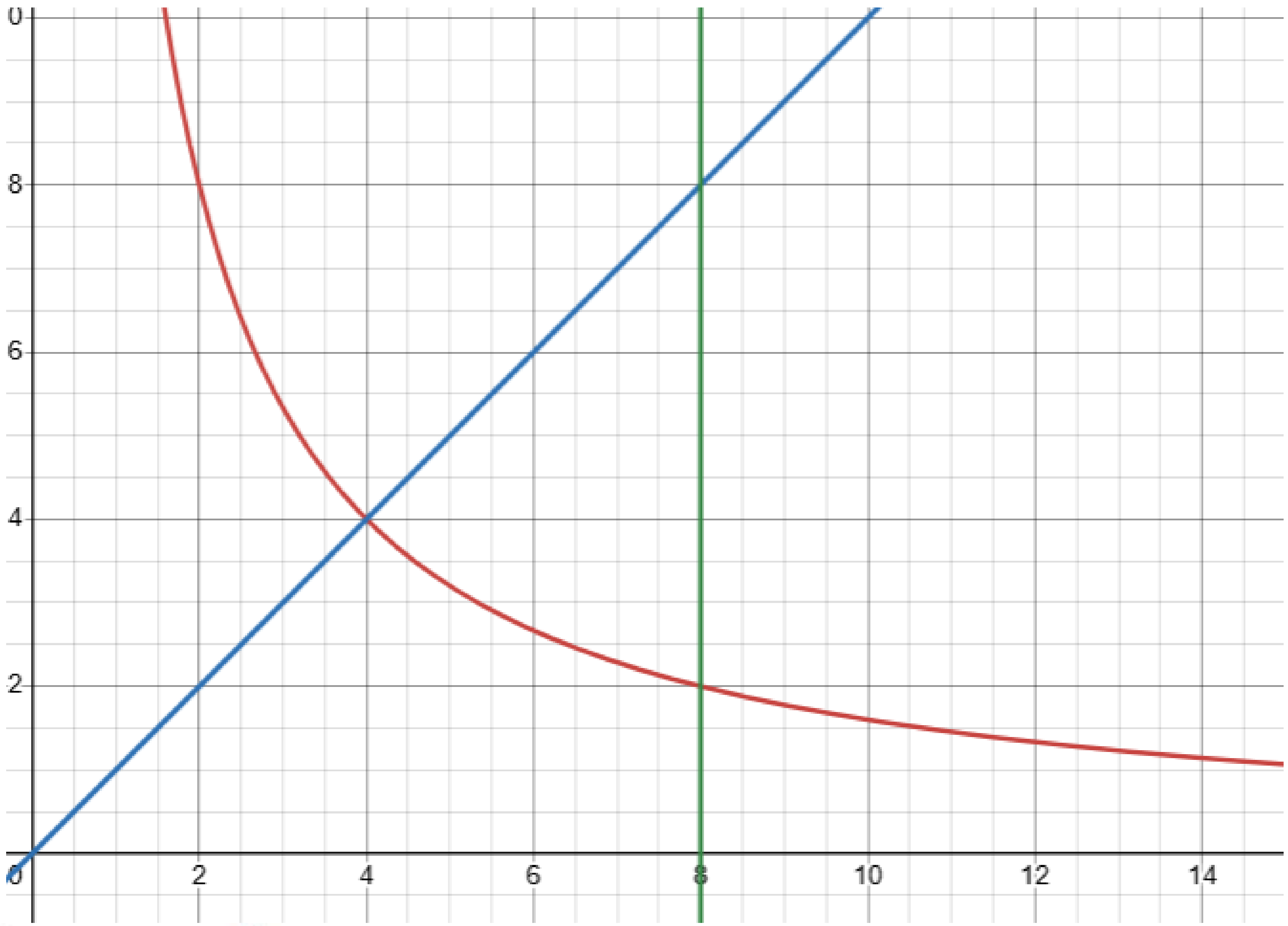
15. $\iint_R x^2\,dA$; R is the region bounded by $y = 16/x$, $y = x$, and $x = 8$.

16. $\iint_R xy^2\,dA$; R is the region enclosed by $y = 1$, $y = 2$, $x = 0$, and $y = x$.

17. $\iint_R (3x - 2y)\,dA$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

18. $\iint_R y\,dA$; R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

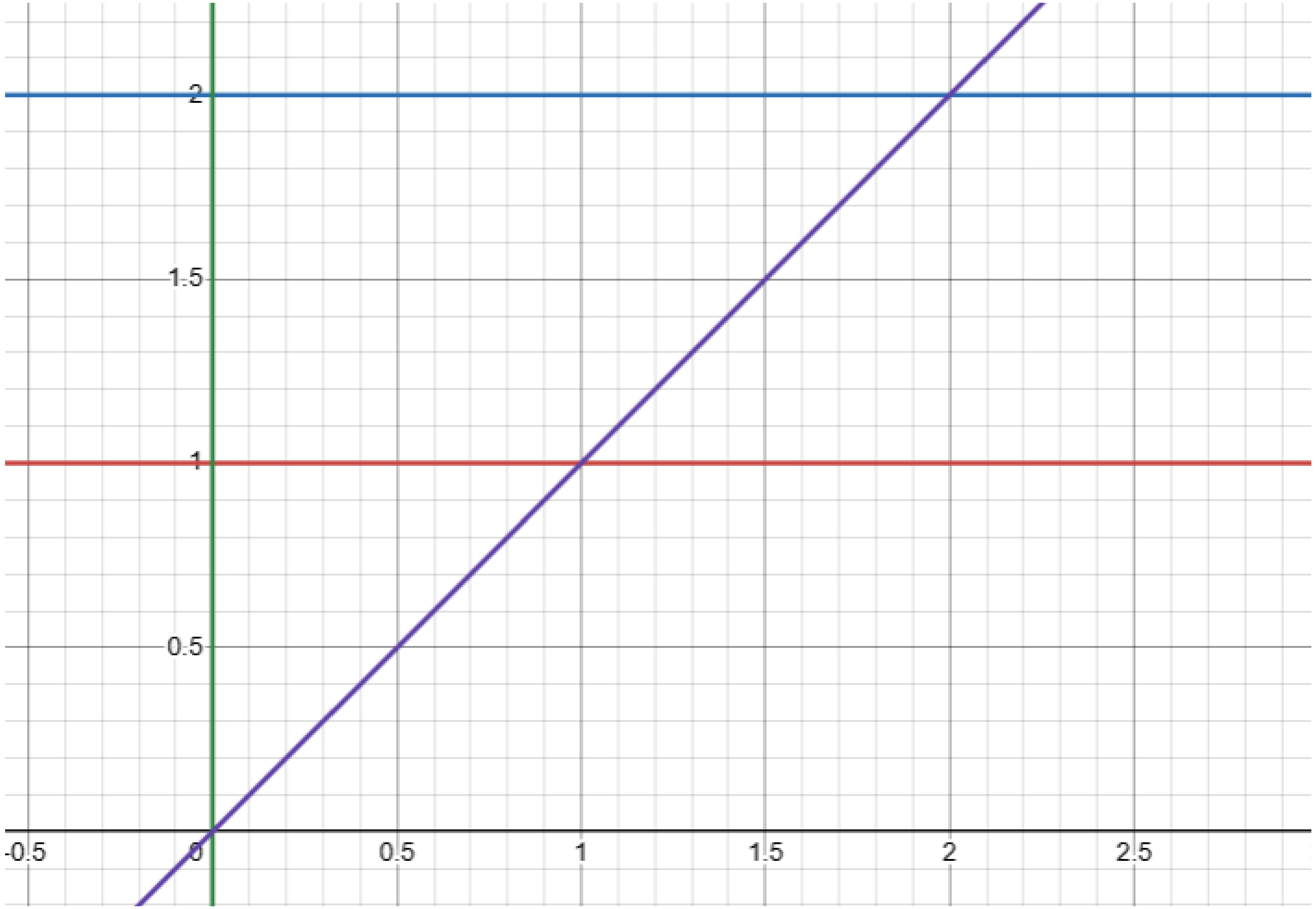
Q#15 graph



15. **(a)** $\int_4^8 \int_{16/x}^{x} x^2 dy\, dx = \int_4^8 (x^3 - 16x)\, dx = 576.$

(b) $\int_2^4 \int_{16/y}^{8} x^2\, dx\, dy + \int_4^8 \int_y^8 x^2\, dx\, dy = \int_4^8 \left[\frac{512}{3} - \frac{4096}{3y^3}\right] dy + \int_4^8 \frac{512 - y^3}{3}\, dy = \frac{640}{3} + \frac{1088}{3} = 576.$

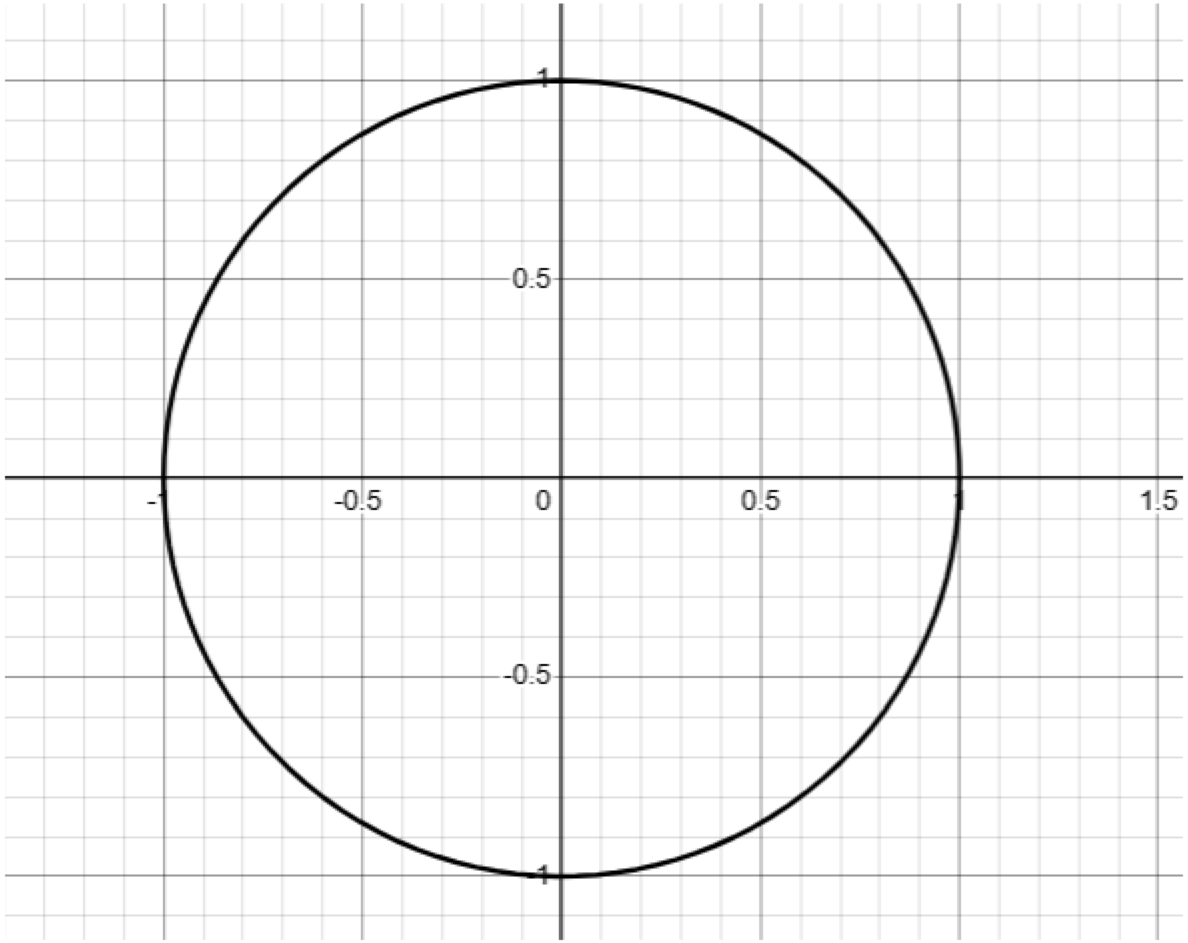
Q16:



16. (a) $\displaystyle\int_0^1\int_1^2 xy^2\,dy\,dx + \int_1^2\int_x^2 xy^2\,dy\,dx = \int_0^1 7x/3\,dx + \int_1^2 \frac{8x - x^4}{3}\,dx = \frac{7}{6} + \frac{29}{15} = \frac{31}{10}.$

(b) $\displaystyle\int_1^2\int_0^y xy^2\,dx\,dy = \int_1^2 \frac{1}{2}y^4\,dy = \frac{31}{10}.$

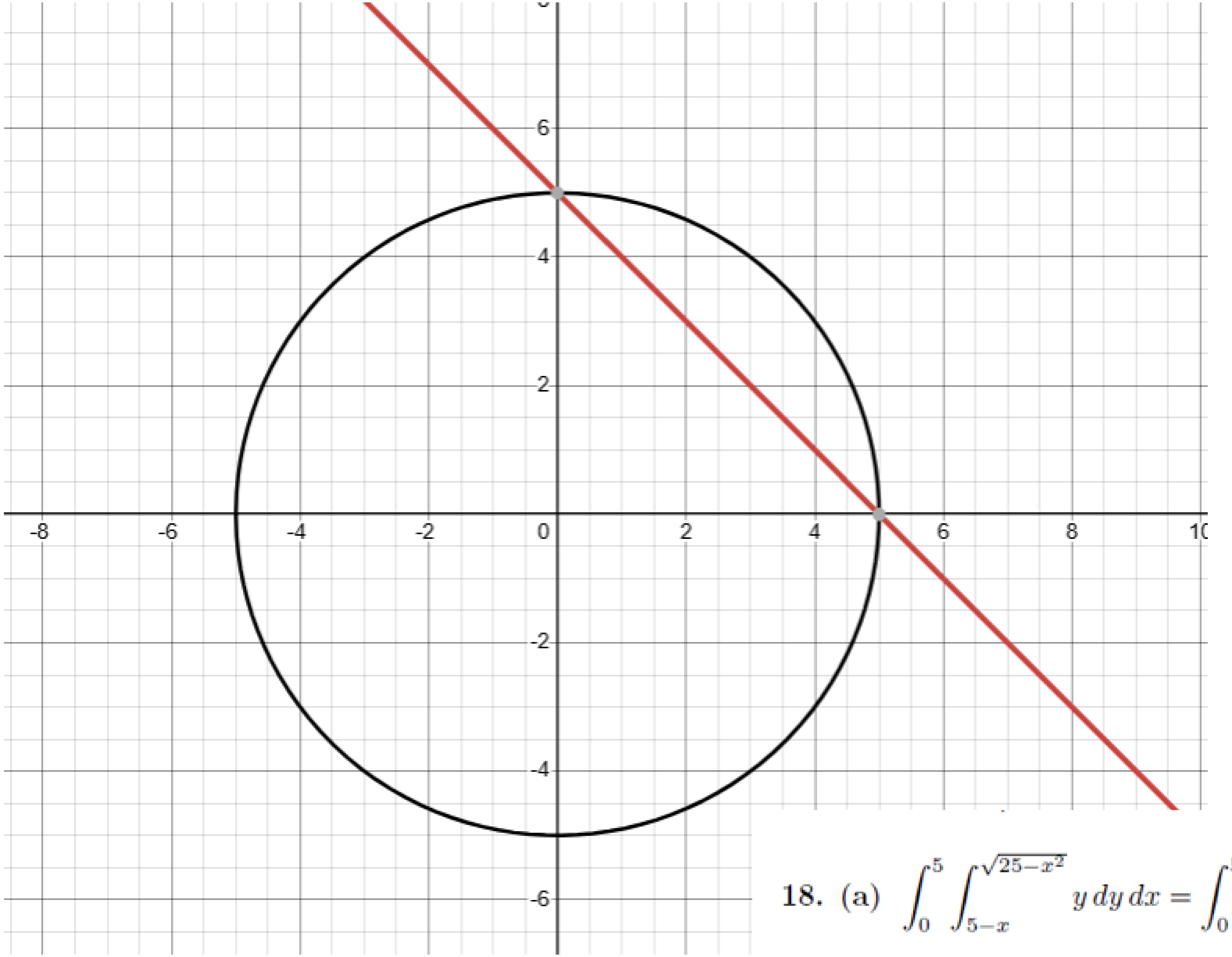
Q17:



17. (a) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(3x-2y)\,dy\,dx=\int_{-1}^{1}6x\sqrt{1-x^2}\,dx=0.$

(b) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(3x-2y)\,dx\,dy=\int_{-1}^{1}-4y\sqrt{1-y^2}\,dy=0.$

Q18:

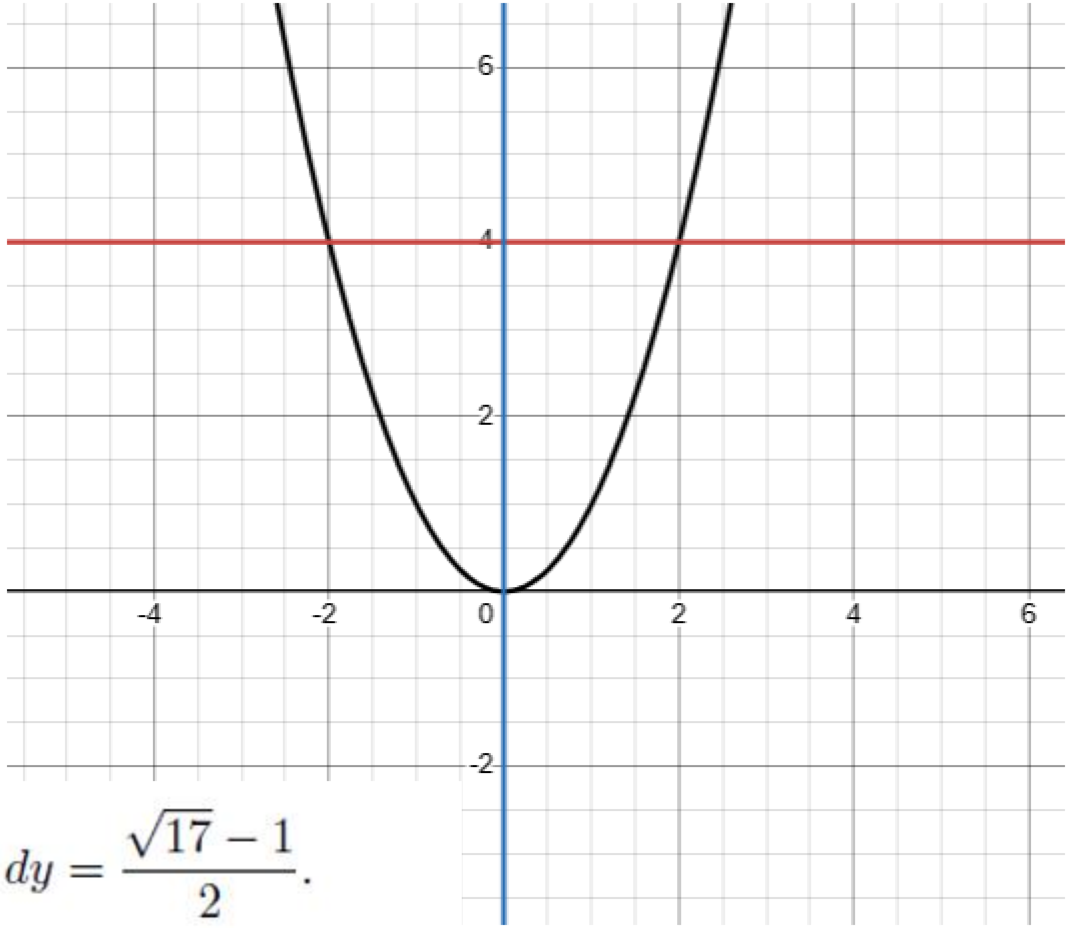


18. (a) $\displaystyle\int_0^5 \int_{5-x}^{\sqrt{25-x^2}} y\,dy\,dx = \int_0^5 (5x - x^2)\,dx = \frac{125}{6}.$

(b) $\displaystyle\int_0^5 \int_{5-y}^{\sqrt{25-y^2}} y\,dx\,dy = \int_0^5 y\left(\sqrt{25-y^2} - 5 + y\right)\,dy = \frac{125}{6}.$

19–24 Evaluate the double integral.

19. $\iint_R x(1+y^2)^{-1/2}\, dA$; R is the region in the first quadrant enclosed by $y = x^2$, $y = 4$, and $x = 0$.



19. $$\int_0^4 \int_0^{\sqrt{y}} x(1+y^2)^{-1/2}\, dx\, dy = \int_0^4 \frac{1}{2} y(1+y^2)^{-1/2}\, dy = \frac{\sqrt{17}-1}{2}.$$

20. $\iint\limits_R x\cos y\,dA$; R is the triangular region bounded by the lines $y = x$, $y = 0$, and $x = \pi$.

21. $\iint\limits_R xy\,dA$; R is the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$.

22. $\iint\limits_R x\,dA$; R is the region enclosed by $y = \sin^{-1} x$, $x = 1/\sqrt{2}$, and $y = 0$.

23. $\iint\limits_R (x - 1)\,dA$; R is the region in the first quadrant enclosed between $y = x$ and $y = x^3$.

24. $\iint\limits_R x^2\,dA$; R is the region in the first quadrant enclosed by $xy = 1$, $y = x$, and $y = 2x$.

25. Evaluate $\iint\limits_R \sin(y^3)\,dA$, where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$. [*Hint:* Choose the order of integration carefully.]

solution

19. $\displaystyle\int_0^4\int_0^{\sqrt{y}} x(1+y^2)^{-1/2}\,dx\,dy = \int_0^4 \frac{1}{2}y(1+y^2)^{-1/2}\,dy = \frac{\sqrt{17}-1}{2}.$

20. $\displaystyle\int_0^{\pi}\int_0^{x} x\cos y\,dy\,dx = \int_0^{\pi} x\sin x\,dx = \pi.$

21. $\displaystyle\int_0^2\int_{y^2}^{6-y} xy\,dx\,dy = \int_0^2 \frac{1}{2}(36y-12y^2+y^3-y^5)\,dy = \frac{50}{3}.$

22. $\displaystyle\int_0^{\pi/4}\int_{\sin y}^{1/\sqrt{2}} x\,dx\,dy = \int_0^{\pi/4} \frac{1}{4}\cos 2y\,dy = \frac{1}{8}.$

23. $\displaystyle\int_0^1\int_{x^3}^{x} (x-1)\,dy\,dx = \int_0^1 (-x^4+x^3+x^2-x)\,dx = -\frac{7}{60}.$

24. $\displaystyle\int_0^{1/\sqrt{2}}\int_x^{2x} x^2\,dy\,dx + \int_{1/\sqrt{2}}^1\int_x^{1/x} x^2\,dy\,dx = \int_0^{1/\sqrt{2}} x^3\,dx + \int_{1/\sqrt{2}}^1 (x-x^3)dx = \frac{1}{8}.$

25. $\displaystyle\int_0^2\int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 y^2\sin(y^3)\,dy = \frac{1-\cos 8}{3}.$

26. $\displaystyle\int_0^1\int_{e^x}^{e} x\,dy\,dx = \int_0^1 (ex-xe^x)\,dx = \frac{e}{2}-1.$

47–52 Express the integral as an equivalent integral with the order of integration reversed.

47. $\displaystyle\int_0^2 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$

48. $\displaystyle\int_0^4 \int_{2y}^{8} f(x, y)\, dx\, dy$

49. $\displaystyle\int_0^2 \int_1^{e^y} f(x, y)\, dx\, dy$

50. $\displaystyle\int_1^e \int_0^{\ln x} f(x, y)\, dy\, dx$

51. $\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\pi/2} f(x, y)\, dx\, dy$

52. $\displaystyle\int_0^1 \int_{y^2}^{\sqrt{y}} f(x, y)\, dx\, dy$

solution

47. $\int_0^{\sqrt{2}} \int_{y^2}^{2} f(x, y)\, dx\, dy.$

48. $\int_0^{8} \int_0^{x/2} f(x, y)\, dy\, dx.$

49. $\int_1^{e^2} \int_{\ln x}^{2} f(x, y)\, dy\, dx.$

50. $\int_0^{1} \int_{e^y}^{e} f(x, y)\, dx\, dy.$

51. $\int_0^{\pi/2} \int_0^{\sin x} f(x, y)\, dy\, dx.$

52. $\int_0^{1} \int_{x^2}^{\sqrt{x}} f(x, y)\, dy\, dx.$

53–56 Evaluate the integral by first reversing the order of integration.

53. $\displaystyle\int_0^1 \int_{4x}^4 e^{-y^2}\,dy\,dx$

54. $\displaystyle\int_0^2 \int_{y/2}^1 \cos(x^2)\,dx\,dy$

55. $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$

56. $\displaystyle\int_1^3 \int_0^{\ln x} x\,dy\,dx$

Solution

53. $\displaystyle\int_0^4 \int_0^{y/4} e^{-y^2}\,dx\,dy = \int_0^4 \frac{1}{4} y e^{-y^2}\,dy = \frac{1-e^{-16}}{8}.$

54. $\displaystyle\int_0^1 \int_0^{2x} \cos(x^2)\,dy\,dx = \int_0^1 2x\cos(x^2)\,dx = \sin 1.$

55. $\displaystyle\int_0^2 \int_0^{x^2} e^{x^3}\,dy\,dx = \int_0^2 x^2 e^{x^3}\,dx = \frac{e^8-1}{3}.$

56. $\displaystyle\int_0^{\ln 3} \int_{e^y}^{3} x\,dx\,dy = \frac{1}{2}\int_0^{\ln 3} (9-e^{2y})\,dy = \frac{9\ln 3 - 4}{2}.$