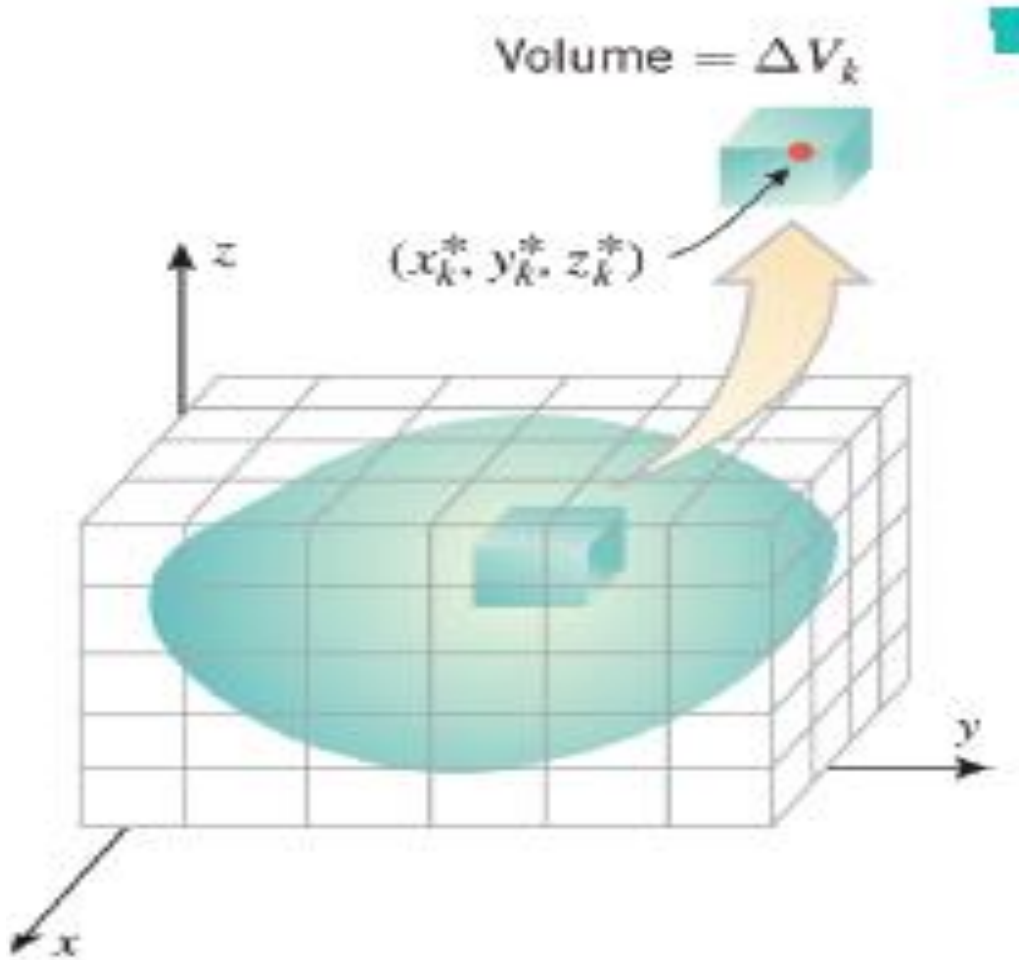
Triple Integrals

Ex # 14.5



▲ **Figure 14.5.1**

DEFINITION OF A TRIPLE INTEGRAL

A single integral of a function $f(x)$ is defined over a finite closed interval on the x-axis, and a double integral of a function $f(x, y)$ is defined over a finite closed region R in the xy-plane. Our first goal in this section is to define what is meant by a *triple integral* of $f(x, y, z)$ over a closed solid region G in an xyz-coordinate system. To ensure that G does not extend indefinitely in some direction, we will assume that it can be enclosed in a suitably large box whose sides are parallel to the coordinate planes (Figure 14.5.1). In this case we say that G is a ***finite solid***.

14.5.1 THEOREM (***Fubini's Theorem****) *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le l$$

If f is continuous on the region G, then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^l f(x, y, z)\, dz\, dy\, dx \tag{2}$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

▶ Example 1 Evaluate the triple integral

$$\iiint_G 12xy^2z^3\, dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Solution. Of the six possible iterated integrals we might use, we will choose the one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned}
\iiint_G 12xy^2z^3\,dV &= \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3\,dz\,dy\,dx \\
&= \int_{-1}^{2}\int_{0}^{3} \left[3xy^2z^4\right]_{z=0}^{2}\,dy\,dx = \int_{-1}^{2}\int_{0}^{3} 48xy^2\,dy\,dx \\
&= \int_{-1}^{2} \left[16xy^3\right]_{y=0}^{3}\,dx = \int_{-1}^{2} 432x\,dx \\
&= 216x^2\Big]_{-1}^{2} = 648
\end{aligned}$$

1–8 Evaluate the iterated integral.

1. $\displaystyle\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz$

2. $\displaystyle\int_{1/3}^{1/2}\int_{0}^{\pi}\int_{0}^{1} zx\sin xy\,dz\,dy\,dx$

3. $\displaystyle\int_{0}^{2}\int_{-1}^{y^2}\int_{-1}^{z} yz\,dx\,dz\,dy$

4. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{1}\int_{0}^{x^2} x\cos y\,dz\,dx\,dy$

Solution Q1 to 4

1. $\displaystyle\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz=\int_{-1}^{1}\int_{0}^{2}(1/3+y^2+z^2)\,dy\,dz=\int_{-1}^{1}(10/3+2z^2)\,dz=8.$

2. $\displaystyle\int_{1/3}^{1/2}\int_{0}^{\pi}\int_{0}^{1}zx\sin xy\,dz\,dy\,dx=\int_{1/3}^{1/2}\int_{0}^{\pi}\frac{1}{2}x\sin xy\,dy\,dx=\int_{1/3}^{1/2}\frac{1}{2}(1-\cos\pi x)\,dx=\frac{1}{12}+\frac{\sqrt{3}-2}{4\pi}.$

3. $\displaystyle\int_{0}^{2}\int_{-1}^{y^2}\int_{-1}^{z}yz\,dx\,dz\,dy=\int_{0}^{2}\int_{-1}^{y^2}(yz^2+yz)\,dz\,dy=\int_{0}^{2}\left(\frac{1}{3}y^7+\frac{1}{2}y^5-\frac{1}{6}y\right)dy=\frac{47}{3}.$

4. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{1}\int_{0}^{x^2}x\cos y\,dz\,dx\,dy=\int_{0}^{\pi/4}\int_{0}^{1}x^3\cos y\,dx\,dy=\int_{0}^{\pi/4}\frac{1}{4}\cos y\,dy=\frac{\sqrt{2}}{8}.$

5. $\displaystyle\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^x xy\,dy\,dx\,dz$

6. $\displaystyle\int_1^3 \int_x^{x^2} \int_0^{\ln z} xe^y\,dy\,dz\,dx$

7. $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx$

8. $\displaystyle\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{x^2+y^2}\,dx\,dy\,dz$

Solution Q5 to 8

5. $$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^x xy\,dy\,dx\,dz = \int_0^3 \int_0^{\sqrt{9-z^2}} \frac{1}{2}x^3 dx\,dz = \int_0^3 \frac{1}{8}(81 - 18z^2 + z^4)\,dz = \frac{81}{5}.$$

6. $$\int_1^3 \int_x^{x^2} \int_0^{\ln z} xe^y\,dy\,dz\,dx = \int_1^3 \int_x^{x^2} (xz - x)\,dz\,dx = \int_1^3 \left(\frac{1}{2}x^5 - \frac{3}{2}x^3 + x^2\right) dx = \frac{118}{3}.$$

7. $$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx = \int_0^2 \int_0^{\sqrt{4-x^2}} [2x(4-x^2) - 2xy^2]\,dy\,dx = \int_0^2 \frac{4}{3}x(4-x^2)^{3/2}\,dx = \frac{128}{15}.$$

8. $$\int_1^2 \int_z^2 \int_0^{\sqrt{3}y} \frac{y}{x^2+y^2} dx\,dy\,dz = \int_1^2 \int_z^2 \frac{\pi}{3} dy\,dz = \int_1^2 \frac{\pi}{3}(2-z)\,dz = \frac{\pi}{6}.$$