Ex#15.5
SURFACE INTEGRALS
Q#1 to 8

DEFINITION OF A SURFACE INTEGRAL

In this section we will define what it means to integrate a function $f(x, y, z)$ over a smooth parametric surface σ. To motivate the definition we will consider the problem of finding the mass of a curved lamina whose density function (mass per unit area) is known. Recall that in Section 5.7 we defined a *lamina* to be an idealized flat object that is thin enough to be viewed as a plane region. Analogously, a ***curved lamina*** is an idealized object that is thin enough to be viewed as a surface in 3-space. A curved lamina may look like a bent plate, as in Figure 15.5.1, or it may enclose a region in 3-space, like the shell of an egg. We will model the lamina by a smooth parametric surface σ. Given any point (x, y, z) on σ, we let $f(x, y, z)$ denote the corresponding value of the density function. To compute the mass of the lamina, we proceed as follows:

The thickness of a curved lamina is negligible.

▲ **Figure 15.5.1**

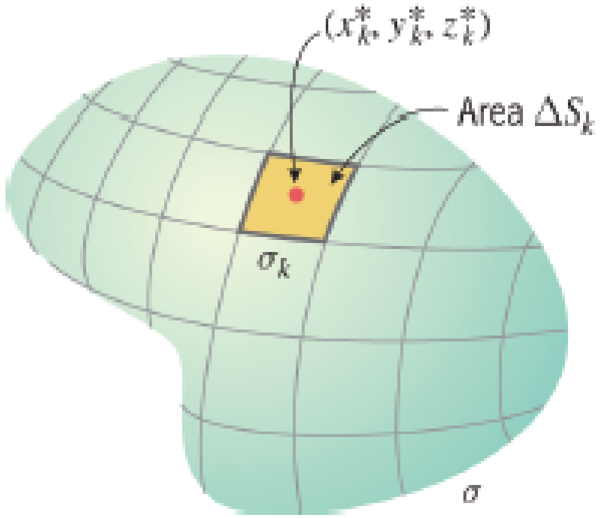


Figure 15.5.2

- As shown in Figure 15.5.2, we divide σ into n very small patches $\sigma_1, \sigma_2, \ldots, \sigma_n$ with areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$, respectively. Let (x_k^*, y_k^*, z_k^*) be a sample point in the kth patch with ΔM_k the mass of the corresponding section.
- If the dimensions of σ_k are very small, the value of f will not vary much along the kth section and we can approximate f along this section by the value $f(x_k^*, y_k^*, z_k^*)$. It follows that the mass of the kth section can be approximated by

$$\Delta M_k \approx f(x_k^*, y_k^*, z_k^*)\Delta S_k$$

- The mass M of the entire lamina can then be approximated by

$$M = \sum_{k=1}^{n} \Delta M_k \approx \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{1}$$

- We will use the expression $n \to \infty$ to indicate the process of increasing n in such a way that the maximum dimension of each patch approaches 0. It is plausible that the error in (1) will approach 0 as $n \to \infty$ and the exact value of M will be given by

$$M = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{2}$$

The limit in (2) is very similar to the limit used to find the mass of a thin wire [Formula (2) in Section 15.2]. By analogy to Definition 15.2.1, we make the following definition.

15.5.1 DEFINITION If σ is a smooth parametric surface, then the ***surface integral*** of $f(x, y, z)$ over σ is

$$\iint_{\sigma} f(x, y, z)\, dS = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*) \Delta S_k \tag{3}$$

provided this limit exists and does not depend on the way the subdivisions of σ are made or how the sample points (x_k^*, y_k^*, z_k^*) are chosen.

EVALUATING SURFACE INTEGRALS

There are various procedures for evaluating surface integrals that depend on how the surface σ is represented. The following theorem provides a method for evaluating a surface integral when σ is represented parametrically.

15.5.2 THEOREM *Let σ be a smooth parametric surface whose vector equation is*

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies over a region R in the uv-plane. If $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \qquad (6)$$

► **Example 1** Evaluate the surface integral $\iint_{\sigma} x^2\, dS$ over the sphere $x^2+y^2+z^2=1$.

15.5.2 THEOREM *Let σ be a smooth parametric surface whose vector equation is*

$$\mathbf{r} = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

where (u,v) varies over a region R in the uv-plane. If $f(x,y,z)$ is continuous on σ, then

$$\iint_{\sigma} f(x,y,z)\, dS = \iint_{R} f(x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \tag{6}$$

Solution. As in Example 11 of Section 14.4 (with $a=1$), the sphere is the graph of the vector-valued function

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k} \quad (0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi) \tag{7}$$

$x = sin\emptyset cos\theta$, $y = sin\emptyset sin\theta$ and $z = cos\emptyset$

$\frac{\partial r}{\partial \emptyset} = \cos\emptyset cos\theta i + cos\emptyset sin\theta \mathrm{j} - \sin\emptyset k$ and $\frac{\partial r}{\partial \theta} = -sin\emptyset sin\theta i + sin\emptyset cos\theta \mathrm{j} + 0k$

$$\frac{\partial r}{\partial \emptyset} \times \frac{\partial r}{\partial \theta} = \begin{vmatrix} i & j & k \\ \cos\emptyset cos\theta & \cos\emptyset sin\theta & -sin\emptyset \\ -sin\emptyset sin\theta & sin\emptyset cos\theta & 0 \end{vmatrix}$$

$$= i(0 + \sin^2 \emptyset\, cos\theta) - j(0 - \sin^2 \emptyset\, sin\theta) +$$
$$\mathrm{k}(\cos\emptyset sin\emptyset \cos^2 \theta + \cos\emptyset sin\emptyset \sin^2 \theta)$$

$$\left\| \frac{\partial r}{\partial \emptyset} \times \frac{\partial r}{\partial \theta} \right\|$$
$$= \sqrt{\sin^4 \emptyset \cos^2 \theta + \sin^4 \emptyset \sin^2 \theta + cos^2\emptyset \sin^2 \emptyset \cos^4 \theta + cos^2\emptyset \sin^2 \emptyset \cos^4 \theta + 2cos^2\emptyset \sin^2 \emptyset \sin^2 \theta \cos^2 \theta}$$
$$=\sqrt{\sin^4 \emptyset (\cos^2 \theta + \sin^2 \theta) + cos^2\emptyset \sin^2 \emptyset(\cos^4 \theta + \cos^4 \theta + 2 \sin^2 \theta \cos^2 \theta)}$$
$$= \sqrt{\sin^4 \emptyset + cos^2\emptyset \sin^2 \emptyset} \quad = \sqrt{\sin^2 \emptyset (\sin^2 \emptyset + cos^2\emptyset)} = sin\emptyset$$

$$\iint_{\sigma} x^2\,dS = \iint_{R} (\sin^2\phi\cos^2\theta)\left\|\frac{\partial \mathbf{r}}{\partial \phi}\times\frac{\partial \mathbf{r}}{\partial \theta}\right\| dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\pi} \sin^3\phi\,d\phi\right]\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi}\cos^2\theta\,d\theta$$

$$= \frac{4}{3} \int_0^{2\pi} \cos^2 \theta \, d\theta$$

$$= \frac{4}{3} \left[\frac{1}{2}\theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{4\pi}{3}$$

1–8 Evaluate the surface integral

$$\iint_\sigma f(x, y, z)\, dS$$

8. $f(x, y, z) = x^2 + y^2$; σ is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Same as Example #1

8. Let $\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\, \mathbf{i} + a \sin\phi \sin\theta\, \mathbf{j} + a\cos\phi\, \mathbf{k}, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$; $\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = a^2 \sin\phi$, $x^2 + y^2 = a^2 \sin^2\phi$,

$$\iint_\sigma f(x, y, z) = \int_0^{2\pi} \int_0^{\pi} a^4 \sin^3\phi\, d\phi\, d\theta = \frac{8}{3}\pi a^4.$$

SURFACE INTEGRALS OVER $z = g(x, y)$, $y = g(x, z)$, AND $x = g(y, z)$

In the case where σ is a surface of the form $z = g(x, y)$, we can take $x = u$ and $y = v$ as parameters and express the equation of the surface as

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + g(u, v)\mathbf{k}$$

in which case we obtain

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

(verify). Thus, it follows from (6) that

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

15.5.3 THEOREM

(*a*) *Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \qquad (8)$$

(*b*) *Let σ be a surface with equation $y = g(x, z)$ and let R be its projection on the xz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, g(x, z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA \qquad (9)$$

(*c*) *Let σ be a surface with equation $x = g(y, z)$ and let R be its projection on the yz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(g(y, z), y, z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA \qquad (10)$$

▸ Example 2 Evaluate the surface integral

$$\iint_{\sigma} xz\, dS$$

where σ is the part of the plane $x + y + z = 1$ that lies in the first octant.

(a) *Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \qquad (8)$$

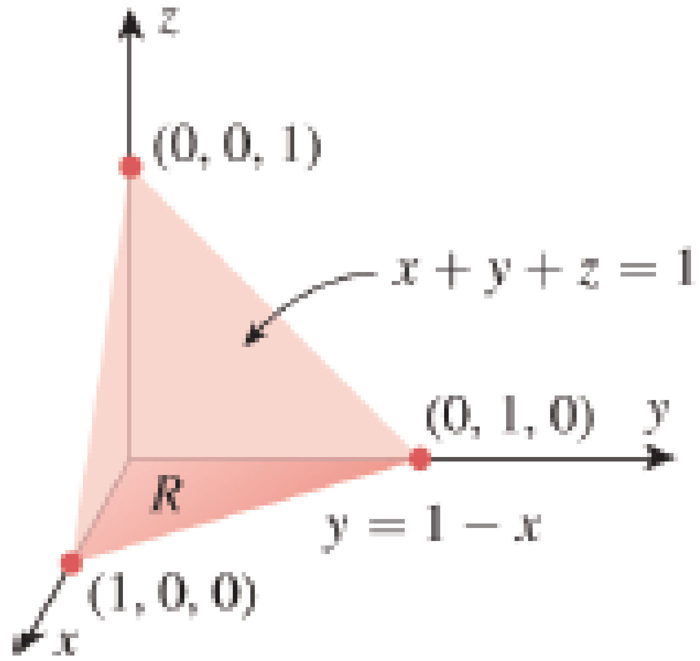
Solution. The equation of the plane can be written as

$$z = 1 - x - y$$

Consequently, we can apply Formula (8) with $z = g(x, y) = 1 - x - y$ and $f(x, y, z) = xz$. We have

$$\frac{\partial z}{\partial x} = -1 \quad \text{and} \quad \frac{\partial z}{\partial y} = -1$$

$$\iint_{\sigma} xz\,dS = \iint_{R} x(1 - x - y)\sqrt{(-1)^2 + (-1)^2 + 1}$$



▲ **Figure 15.5.3**

where R is the projection of σ on the xy-plane (Figure 15.5.3). Rewriting the double integral in (11) as an iterated integral yields

$$\begin{aligned}
\iint_{\sigma} xz\,dS &= \sqrt{3}\int_0^1 \int_0^{1-x} (x - x^2 - xy)\,dy\,dx \\
&= \sqrt{3}\int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_{y=0}^{1-x} dx \\
&= \sqrt{3}\int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx \\
&= \sqrt{3}\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}
\end{aligned}$$

1–8 Evaluate the surface integral

$$\iint_{\sigma} f(x, y, z)\, dS$$

2. $f(x, y, z) = xy$; σ is the portion of the plane $x + y + z = 1$ lying in the first octant.

Same as Example # 2

2. $z = 1 - x - y$, R is the triangular region enclosed by $x + y = 1$, $x = 0$ and $y = 0$;

$$\iint_{\sigma} xy\, dS = \iint_{R} xy\sqrt{3}\, dA = \sqrt{3}\int_0^1 \int_0^{1-x} xy\, dy\, dx = \frac{\sqrt{3}}{24}.$$

4. $f(x, y, z) = (x^2 + y^2)z$; σ is the portion of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$.

Above the plane $z = 1$ so $x^2 + y^2 + 1 = 3$, $x^2 + y^2 = 3$ $(domain)$

$$Z = \sqrt{4 - x^2 - y^2}$$

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

4. $z = \sqrt{4 - x^2 - y^2}$, R is the circular region enclosed by $x^2 + y^2 = 3$; $\iint_{\sigma} (x^2 + y^2) z\, dS =$

$$= \iint_{R} (x^2 + y^2)\sqrt{4 - x^2 - y^2} \sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1}\, dA = \iint_{R} 2(x^2 + y^2)\, dA = 2\int_0^{2\pi} \int_0^{\sqrt{3}} r^3\, dr\, d\theta = 9\pi.$$

6. $f(x, y, z) = x + y$; σ is the portion of the plane $z = 6 - 2x - 3y$ in the first octant.

(a) *Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \qquad (8)$$

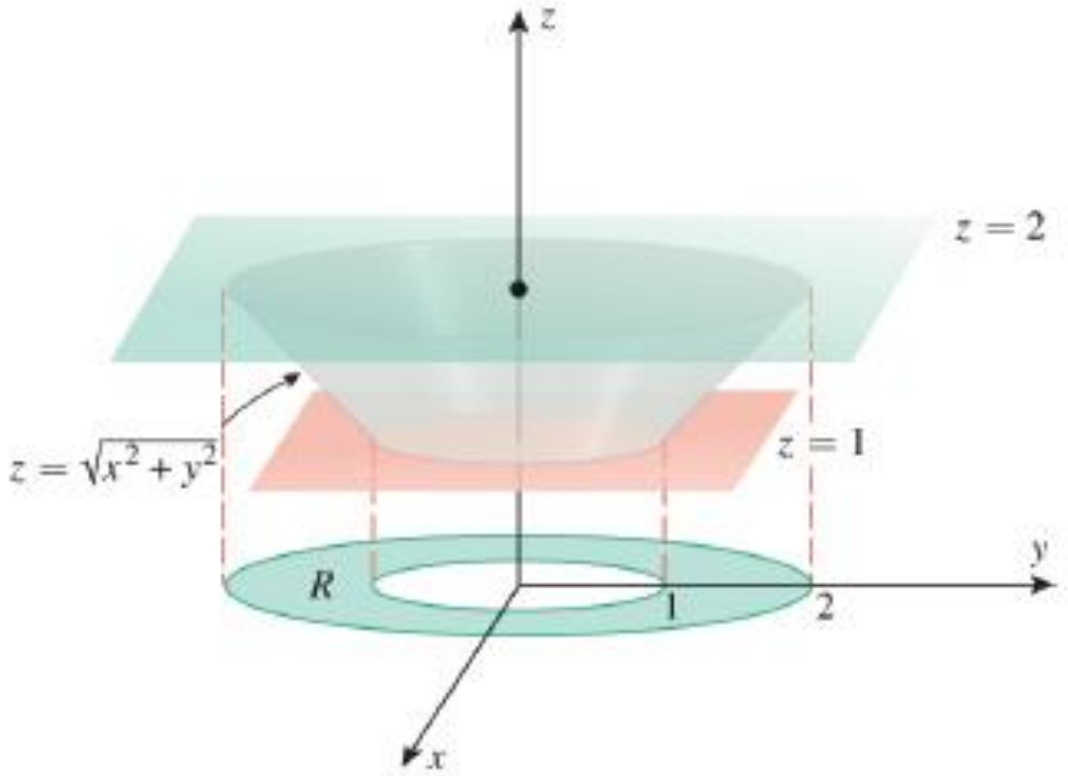
6. R is the triangular region enclosed by $2x + 3y = 6$, $x = 0$, and $y = 0$; $\displaystyle\iint_{\sigma}(x + y)dS = \iint_{R}(x + y)\sqrt{14}\, dA =$

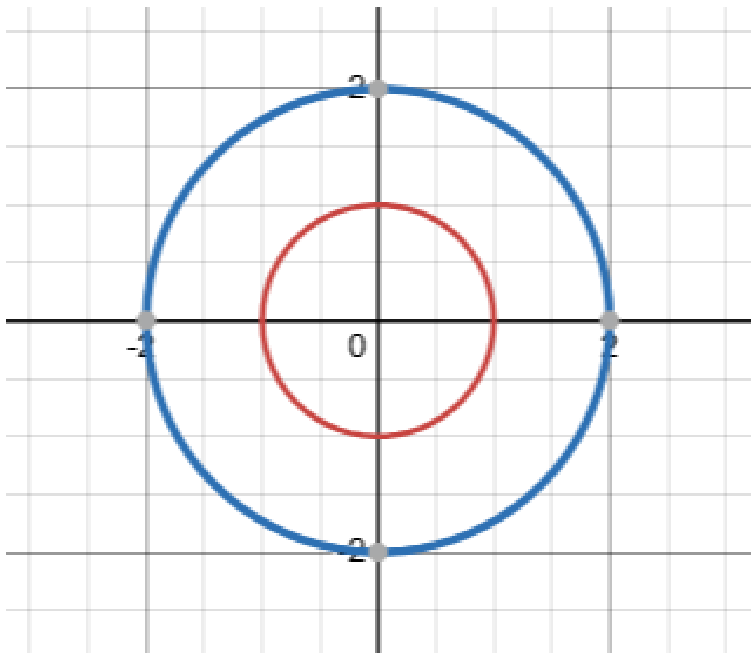
$$\sqrt{14}\int_0^3 \int_0^{(6-2x)/3}(x + y)dy\, dx = 5\sqrt{14}.$$

▶ Example 3 Evaluate the surface integral

$$\iint_{\sigma} y^2 z^2 \, dS$$

where σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$ (Figure 15.5.4).





▶ Figure 15.5.4

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \qquad (8)$$

Solution. We will apply Formula (8) with

$$z = g(x, y) = \sqrt{x^2 + y^2} \quad \text{and} \quad f(x, y, z) = y^2 z^2$$

Thus,

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

so

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2}$$

(verify), and (8) yields

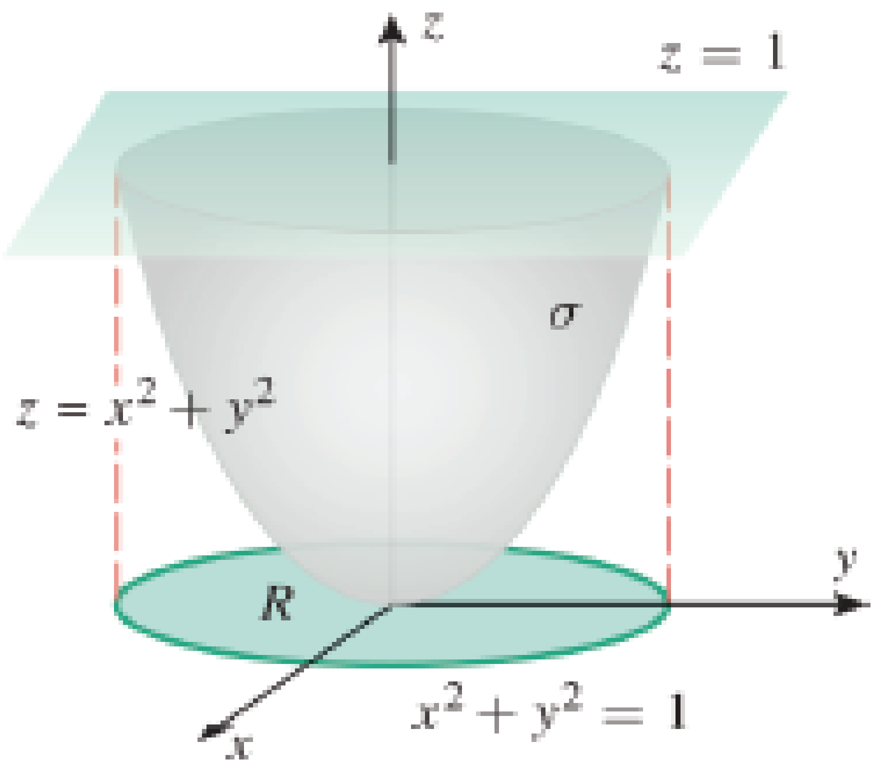
$$\iint_{\sigma} y^2 z^2\, dS = \iint_{R} y^2 \left(\sqrt{x^2 + y^2}\right)^2 \sqrt{2}\, dA = \sqrt{2} \iint_{R} y^2 (x^2 + y^2)\, dA$$

where R is the annulus enclosed between $x^2+y^2=1$ and $x^2+y^2=4$ (Figure 15.5.4). Using polar coordinates to evaluate this double integral over the annulus R yields

$$\begin{aligned}
\iint_{\sigma} y^2z^2\,dS &= \sqrt{2}\int_0^{2\pi}\int_1^2 (r\sin\theta)^2(r^2)r\,dr\,d\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_1^2 r^5\sin^2\theta\,dr\,d\theta \\
&= \sqrt{2}\int_0^{2\pi}\left[\frac{r^6}{6}\sin^2\theta\right]_{r=1}^{2} d\theta = \frac{21}{\sqrt{2}}\int_0^{2\pi}\sin^2\theta\,d\theta \\
&= \frac{21}{\sqrt{2}}\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{21\pi}{\sqrt{2}}
\end{aligned}$$

Formula (7), Section 7.3

► Example 4 Suppose that a curved lamina σ with constant density $\delta(x, y, z) = \delta_0$ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ (Figure 15.5.5). Find the mass of the lamina.



▲ **Figure 15.5.5**

Solution. Since $z = g(x, y) = x^2 + y^2$, it follows that

$$\frac{\partial z}{\partial x} = 2x \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

Therefore,

$$M = \iint_{\sigma} \delta_0 \, dS = \iint_{R} \delta_0 \sqrt{(2x)^2 + (2y)^2 + 1} \, dA = \delta_0 \iint_{R} \sqrt{4x^2 + 4y^2 + 1} \, dA \qquad (12)$$

where R is the circular region enclosed by $x^2 + y^2 = 1$. To evaluate (12) we use polar coordinates:

$$M = \delta_0 \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1} \, r \, dr \, d\theta = \frac{\delta_0}{12} \int_0^{2\pi} (4r^2 + 1)^{3/2} \Big]_{r=0}^{1} \, d\theta$$

$$= \frac{\delta_0}{12} \int_0^{2\pi} (5^{3/2} - 1) \, d\theta = \frac{\pi \delta_0}{6} (5\sqrt{5} - 1) \blacktriangleleft$$

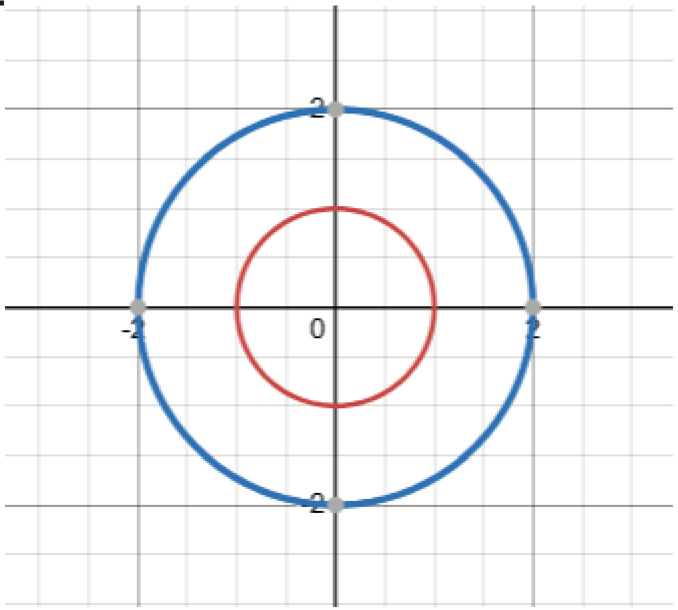
EXERCISE SET 15.5 C CAS

1–8 Evaluate the surface integral

$$\iint_{\sigma} f(x, y, z)\, dS$$

Converts into polar form

1. $f(x, y, z) = z^2$; σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.



Same as Example # 3

1. R is the annular region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$;

$$\iint_{\sigma} z^2 dS = \iint_{R} (x^2 + y^2)\sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1}\, dA = \sqrt{2}\iint_{R} (x^2+y^2) dA = \sqrt{2}\int_0^{2\pi}\int_1^2 r^3 dr\, d\theta = \frac{15}{2}\pi\sqrt{2}.$$

3. $f(x, y, z) = x^2y$; σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$, $y = 1$, and above the xy-plane.

(a) *Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \qquad (8)$$

$$x^2 + z^2 = 1$$

Then $z = \sqrt{1 - x^2}$ above the xy-plane z is +ve

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1-x^2}} \quad , \frac{\partial z}{\partial y} = 0$$

So $\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 + 1} = \sqrt{\frac{x^2}{1-x^2} + 0 + 1} = \frac{1}{\sqrt{1-x^2}}$

$$f(x,y,z) = x^2 y = f\big(x, y, g(x,y)\big) = \mathrm{x}^2\mathrm{y}$$

limit $-1 \le x \le 1$ and $0 \le y \le 1$ $where$ $dA = dydx$

$$\int_{-1}^{1}\int_{0}^{1} x^2 y \frac{1}{\sqrt{1-x^2}} \mathrm{dydx}$$

$$\int_{-1}^{1} x^2 \frac{1}{\sqrt{1-x^2}} \left[\int_0^1 y\,\mathrm{dy}\right]\mathrm{dx} = \int_{-1}^{1} x^2 \frac{1}{\sqrt{1-x^2}} \left[\frac{\mathrm{y}^2}{2}\right]_0^1 \mathrm{dx}$$

$\frac{1}{2}\int_{-1}^{1} x^2 \frac{1}{\sqrt{1-x^2}}$ dx use trigionometruic sub or other method

$$\frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

5. $f(x, y, z) = x - y - z$; σ is the portion of the plane $x + y = 1$ in the first octant between $z = 0$ and $z = 1$.

(b) *Let σ be a surface with equation $y = g(x, z)$ and let R be its projection on the xz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x, g(x, z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA \qquad (9)$$

5. If we use the projection of σ onto the xz-plane then $y = 1-x$ and R is the rectangular region in the xz-plane enclosed by $x = 0$, $x = 1$, $z = 0$ and $z = 1$; $\displaystyle\iint_{\sigma} (x-y-z)dS = \iint_{R} (2x-1-z)\sqrt{2}dA = \sqrt{2}\int_0^1\int_0^1 (2x-1-z)dz\, dx = -\sqrt{2}/2.$

7. $f(x, y, z) = x + y + z$; σ is the surface of the cube defined by the inequalities $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$. [*Hint:* Integrate over each face separately.]

7. There are six surfaces, parametrized by projecting onto planes:

$\sigma_1 : z = 0;\ 0 \le x \le 1,\ 0 \le y \le 1$ (onto xy-plane), $\sigma_2 : x = 0;\ 0 \le y \le 1,\ 0 \le z \le 1$ (onto yz-plane),
$\sigma_3 : y = 0;\ 0 \le x \le 1,\ 0 \le z \le 1$ (onto xz-plane), $\sigma_4 : z = 1;\ 0 \le x \le 1,\ 0 \le y \le 1$ (onto xy-plane),
$\sigma_5 : x = 1;\ 0 \le y \le 1,\ 0 \le z \le 1$ (onto yz-plane), $\sigma_6 : y = 1;\ 0 \le x \le 1,\ 0 \le z \le 1$ (onto xz-plane).

By symmetry the integrals over σ_1, σ_2 and σ_3 are equal, as are those over σ_4, σ_5 and σ_6, and $\iint\limits_{\sigma_1}(x+y+z)dS =$
$\int_0^1\int_0^1 (x+y)dx\,dy = 1$; $\iint\limits_{\sigma_4}(x+y+z)dS = \int_0^1\int_0^1 (x+y+1)dx\,dy = 2$, thus, $\iint\limits_{\sigma}(x+y+z)dS = 3\cdot 1 + 3\cdot 2 = 9$.

How is a surface integral used in determining the total light intensity falling on a curved 3D surface in computer graphics?

Let a surface S be defined as the portion of the paraboloid $z = 4 - x^2 - y^2$ lying above the xy-plane (i.e., where $z \geq 0$). If the light intensity per unit area arriving at each point on the surface is given by the scalar function $f(x, y, z) = z$, compute the **total light intensity** falling on this surface using a surface integral.

Step 1: Understand the physical context

In computer graphics, the **surface integral** allows us to calculate **aggregate quantities** like total light, heat, or flux across a curved surface.

The light intensity at a point on the surface is $f(x, y, z) = z$. So we want to compute:

$$\iint_S f(x, y, z)\, dS = \iint_S z\, dS$$

Step 2: Parametrize the surface

The surface S is given as:

$$z = 4 - x^2 - y^2 \quad \text{(a paraboloid)}$$

Above the xy-plane ⇒ $z \geq 0$

So,

$$4 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 4$$

This is a disk of radius 2 in the xy-plane.

Step 3: Use surface integral formula

For a surface $z = g(x, y)$, the surface area element is:

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Here,

$$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y \Rightarrow dS = \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

So, the surface integral becomes:

$$\iint_S z\, dS = \iint_R (4 - x^2 - y^2)\, \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy$$

Where R is the disk $x^2 + y^2 \leq 4$

Let:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $dx\,dy = r\,dr\,d\theta$

Then:

$$z = 4 - r^2 \quad \text{and} \quad \sqrt{1 + 4x^2 + 4y^2} = \sqrt{1 + 4r^2}$$

The integral becomes:

$$\int_0^{2\pi} \int_0^2 (4 - r^2)\sqrt{1 + 4r^2} \cdot r\,dr\,d\theta$$

First, integrate with respect to r:

Let's compute:

$$I = \int_0^2 (4 - r^2)\, r\, \sqrt{1 + 4r^2}\, dr$$

Let's do substitution:

Let $u = 1 + 4r^2 \Rightarrow du = 8r\, dr \Rightarrow r\, dr = \frac{du}{8}$

When $r = 0$, $u = 1$; when $r = 2$, $u = 1 + 4(4) = 17$

Also:

$$4 - r^2 = 4 - \frac{u - 1}{4} = \frac{17 - u}{4}$$

So:

$$I = \int_{u=1}^{17} \left(\frac{17 - u}{4} \right) \cdot \sqrt{u} \cdot \frac{1}{8}\, du = \frac{1}{32} \int_1^{17} (17 - u) \cdot \sqrt{u}\, du$$

Split:

$$= \frac{1}{32} \left[17 \int_1^{17} \sqrt{u}\, du - \int_1^{17} u^{3/2}\, du \right]$$

Now integrate:

$$\int \sqrt{u}\, du = \frac{2}{3}u^{3/2}, \quad \int u^{3/2}\, du = \frac{2}{5}u^{5/2}$$

Evaluate:

$$I = \frac{1}{32}\left[17 \cdot \frac{2}{3}(17^{3/2} - 1^{3/2}) - \frac{2}{5}(17^{5/2} - 1^{5/2})\right]$$

This can be approximated numerically:

- $\sqrt{17} \approx 4.123$
- $17^{3/2} \approx 17 \cdot 4.123 \approx 70.1$
- $17^{5/2} \approx 289 \cdot 4.123 \approx 1191$

So:

$$I \approx \frac{1}{32}\left[17 \cdot \frac{2}{3}(70.1 - 1) - \frac{2}{5}(1191 - 1)\right] = \frac{1}{32}\left[\frac{34}{3}(69.1) - \frac{2}{5}(1190)\right] \approx \frac{1}{32}[783.5 - 476] = \frac{1}{32}(307.5) \approx 9.61$$

Now integrate with respect to θ:

$$\int_0^{2\pi} d\theta = 2\pi \quad \Rightarrow \quad \text{Total light intensity} = 2\pi \cdot 9.61 \approx 60.4$$

🔶 Question 1: Heat Flow Across a Surface (Robotics & AI)

Problem:

A robotic sensor panel is shaped like the part of the plane $z = 2x + y$ over the rectangle $0 \le x \le 1, 0 \le y \le 2$. The temperature distribution is given by $T(x, y, z) = z$.
Find the total **heat flux** through the surface if heat flows in the direction of the vector field $\vec{F} = T \cdot \vec{n}$, where $\vec{n}$ is the unit normal to the surface.

Hint: Use surface integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R T(x, y, z(x, y)) \cdot |\vec{n}|\; dx\, dy$$

Given:

- Surface: $z = 2x + y$, over $0 \leq x \leq 1,\ 0 \leq y \leq 2$
- Temperature $T(x, y, z) = z = 2x + y$
- Heat vector field: $\vec{F} = T \cdot \vec{n}$, where $\vec{n}$ is the unit normal.

✅ Solution:

The heat flux is:

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S T\, dS = \iint_R T(x, y) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Step 1: Compute partial derivatives

$$\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 1 \Rightarrow \sqrt{1 + 4 + 1} = \sqrt{6}$$

Step 2: Plug in and integrate

$$T(x, y) = 2x + y \Rightarrow \iint_R (2x + y)\sqrt{6}\, dx\, dy = \sqrt{6} \int_0^1 \int_0^2 (2x + y)\, dy\, dx$$

Step 3: Inner integral over y

$$\int_0^2 (2x + y)\, dy = 2x \cdot 2 + \frac{1}{2} \cdot 2^2 = 4x + 2$$

Step 4: Outer integral over x

$$\int_0^1 (4x+2)\,dx = 2x^2 + 2x\Big|_0^1 = 2+2 = 4$$

Final Answer:

$$\text{Heat flux} = \sqrt{6}\cdot 4 \approx \boxed{9.80}$$

🔷 Question 2: Computing Surface Area for Texture Mapping (Computer Graphics)

Problem:

In texture mapping, determining how much texture area is needed depends on the **surface area** of a model. Find the surface area of the hemisphere $z = \sqrt{4 - x^2 - y^2}$, above the xy-plane (i.e., $z \geq 0$).

Hint:

Use:

$$\text{Area} = \iint_S dS = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

🔷 Question 2: Surface Area for Texture Mapping

Given:

- Hemisphere: $z = \sqrt{4 - x^2 - y^2}$, with $x^2 + y^2 \leq 4$

✅ Solution:

Use formula:

$$\text{Area} = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Step 1: Compute partial derivatives

Let $z = (4 - x^2 - y^2)^{1/2}$, then:

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4 - x^2 - y^2}}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{x^2 + y^2}{4 - x^2 - y^2}$$

$$\Rightarrow \sqrt{1 + \frac{x^2 + y^2}{4 - x^2 - y^2}} = \frac{2}{\sqrt{4 - x^2 - y^2}}$$

So:

$$\text{Area} = \iint_R \frac{2}{\sqrt{4 - x^2 - y^2}}\, dx\, dy$$

Switch to **polar coordinates:**

- $x = r\cos\theta,\ y = r\sin\theta, r \in [0, 2],\ \theta \in [0, 2\pi]$
- $dx\, dy = r\, dr\, d\theta$

$$\text{Area} = \int_0^{2\pi} \int_0^2 \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta$$

Substitute: $u = 4 - r^2 \Rightarrow du = -2r\,dr$

When $r = 0,\ u = 4$, and when $r = 2,\ u = 0$

$$\int_0^2 \frac{2r}{\sqrt{4 - r^2}}\,dr = \int_4^0 \frac{-1}{\sqrt{u}}\,du = \int_0^4 u^{-1/2}\,du = 2u^{1/2}\Big|_0^4 = 4$$

$$\Rightarrow \text{Area} = \int_0^{2\pi} 4\,d\theta = 4 \cdot 2\pi = \boxed{8\pi}$$

🟢 Question 3: Light Reflection on Curved Screens (Computer Vision / AR)

Problem:

A concave screen is modeled as the surface $z = x^2 + y^2$ over the disk $x^2 + y^2 \leq 1$.

The incident light intensity per unit area is given by $f(x, y, z) = e^{-z}$.

Calculate the **total reflected light** off the surface.

Hint:

Use the surface integral:

$$\iint_S f(x, y, z)\, dS = \iint_R e^{-z(x,y)} \cdot \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Given:

- Surface: $z = x^2 + y^2$, over disk $x^2 + y^2 \leq 1$
- Light intensity: $f(x, y, z) = e^{-z}$

✅ Solution:

$$\text{Total Light} = \iint_R e^{-z(x,y)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Step 1: Derivatives

$$z = x^2 + y^2, \quad \frac{\partial z}{\partial x} = 2x,\ \frac{\partial z}{\partial y} = 2y \Rightarrow \sqrt{1 + 4x^2 + 4y^2}$$

So the integral becomes:

$$\iint_{x^2+y^2\leq 1} e^{-(x^2+y^2)} \cdot \sqrt{1+4x^2+4y^2}\, dx\, dy$$

Switch to polar:

- $x = r\cos\theta,\ y = r\sin\theta \Rightarrow x^2+y^2 = r^2$
- $dx\, dy = r\, dr\, d\theta$

$$= \int_0^{2\pi} \int_0^1 e^{-r^2} \cdot \sqrt{1+4r^2} \cdot r\, dr\, d\theta$$

Let's compute the inner integral numerically:

Let:

$$I = \int_0^1 r\, e^{-r^2} \cdot \sqrt{1+4r^2}\, dr$$

This is done via substitution or numerically. Approximate using a midpoint or trapezoidal rule (or software):

$$I \approx 0.400 \quad \Rightarrow \text{Total light} = 2\pi \cdot 0.400 = \boxed{0.8\pi \approx 2.51}$$