13.9 LAGRANGE MULTIPLIERS

In this section we will study a powerful new method for maximizing or minimizing a function subject to constraints on the variables. This method will help us to solve certain optimization problems that are difficult or impossible to solve using the methods studied in the last section.

▶ Example 6 Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution. Let

$$x = \text{length of the box (in feet)}$$
$$y = \text{width of the box (in feet)}$$
$$z = \text{height of the box (in feet)}$$
$$S = \text{surface area of the box (in square feet)}$$

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

$$S = xy + 2xz + 2yz \tag{5}$$

(Figure 13.8.10) subject to the volume requirement

$$xyz = 32 \tag{6}$$

EXTREMUM PROBLEMS WITH CONSTRAINTS

In Example 6 of the last section, we solved the problem of minimizing

$$S = xy + 2xz + 2yz \tag{1}$$

subject to the constraint

$$xyz - 32 = 0 \tag{2}$$

This is a special case of the following general problem:

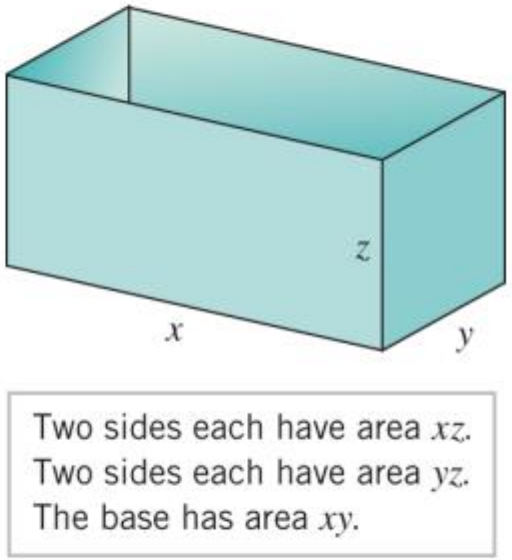


Figure 13.8.10

13.9.1 ***Three-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

We will also be interested in the following two-variable version of this problem:

13.9.2 ***Two-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

13.9.3 THEOREM ***(Constrained-Extremum Principle for Two Variables and One Constraint)*** *Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

If the set of points satisfying the constraint is closed and bounded, such as a circle or line segment, then there must be a global maximum and minimum of f subject to the constraint. If the constraint is not closed and bounded, such as a line or hyperbola, then there may or may not be a global maximum and minimum.

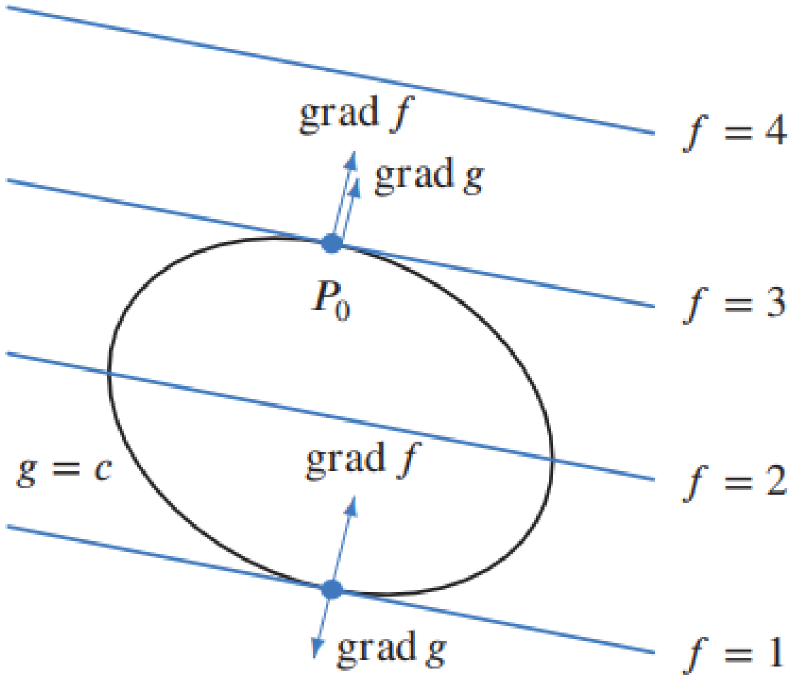


Figure 15.28: Maximum and minimum values of $f(x, y)$ on $g(x, y) = c$ are at points where grad f is parallel to grad g

Example 1 Find the maximum and minimum values of $x + y$ on the circle $x^2 + y^2 = 4$.

Solution The objective function is

$$f(x, y) = x + y,$$

and the constraint is

$$g(x, y) = x^2 + y^2 = 4.$$

Since $\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} = \vec{i} + \vec{j}$ and $\operatorname{grad} g = g_x\vec{i} + g_y\vec{j} = 2x\vec{i} + 2y\vec{j}$, the condition $\operatorname{grad} f = \lambda \operatorname{grad} g$ gives

$$1 = 2\lambda x \quad \text{and} \quad 1 = 2\lambda y,$$

so

$$x = y.$$

We also know that

$$x^2 + y^2 = 4,$$

We also know that

$$x^2 + y^2 = 4,$$

giving $x = y = \sqrt{2}$ or $x = y = -\sqrt{2}$. The constraint has no endpoints (it's a circle) and $\operatorname{grad} g \neq \vec{0}$ on the circle, so we compare values of f at $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Since $f(x, y) = x + y$, the maximum value of f is $f(\sqrt{2}, \sqrt{2}) = 2\sqrt{2}$; the minimum value is $f(-\sqrt{2}, -\sqrt{2}) = -2\sqrt{2}$. (See Figure 15.29.)

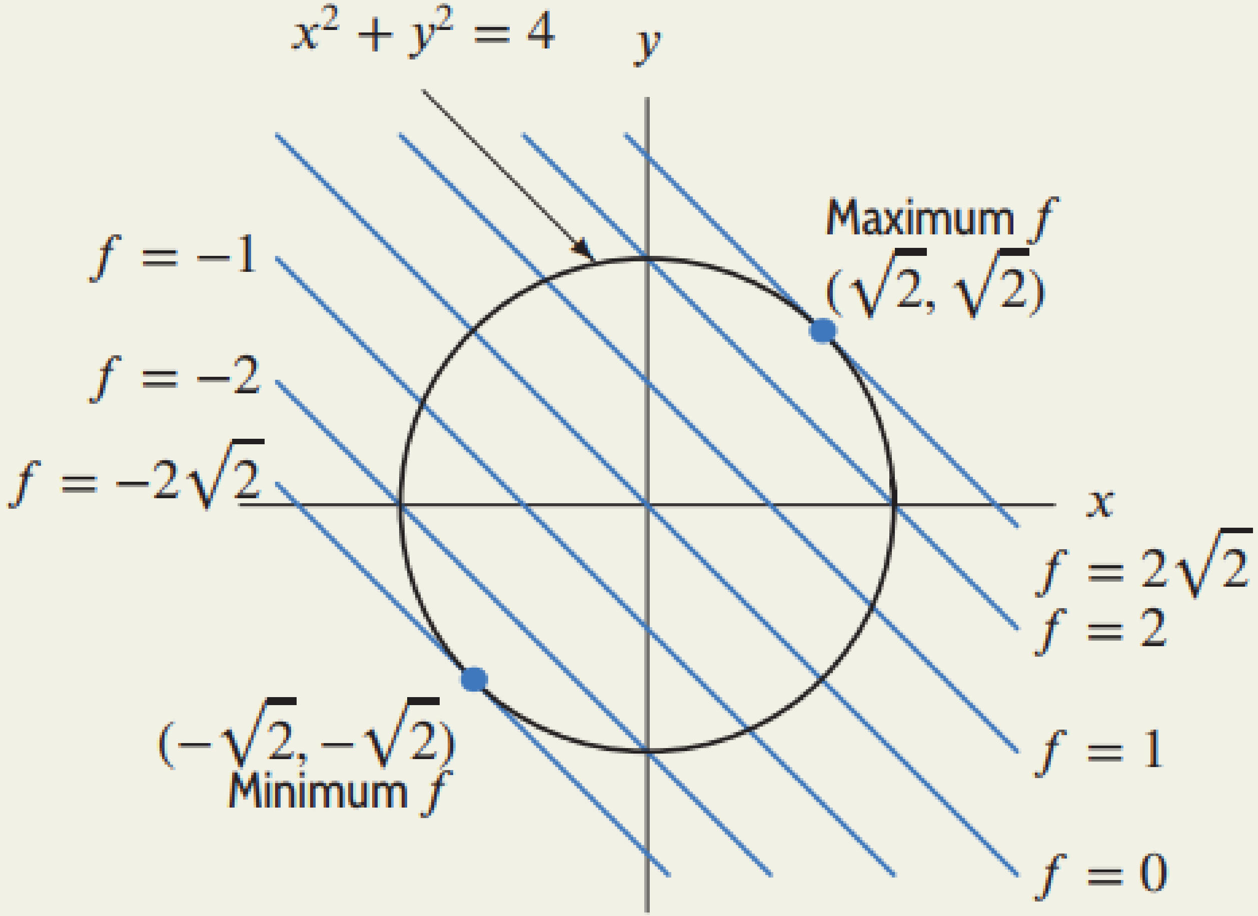


Figure 15.29: Maximum and minimum values of $f(x, y) = x + y$ on the circle $x^2 + y^2 = 4$ are at points where contours of f are tangent to the circle

▶ **Example 1** At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution. The circle $x^2 + y^2 = 1$ is a closed and bounded set and $f(x, y) = xy$ is a continuous function, so it follows from the Extreme-Value Theorem (Theorem 13.8.3) that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.

We want to maximize $f(x, y) = xy$ subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0 \tag{5}$$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad \text{or} \quad y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

which is equivalent to the pair of equations

$$y = 2x\lambda \quad \text{and} \quad x = 2y\lambda$$

It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

$$\lambda = \frac{y}{2x} \quad \text{and} \quad \lambda = \frac{x}{2y}$$

from which we obtain

$$\frac{y}{2x} = \frac{x}{2y}$$

or

$$y^2 = x^2 \tag{6}$$

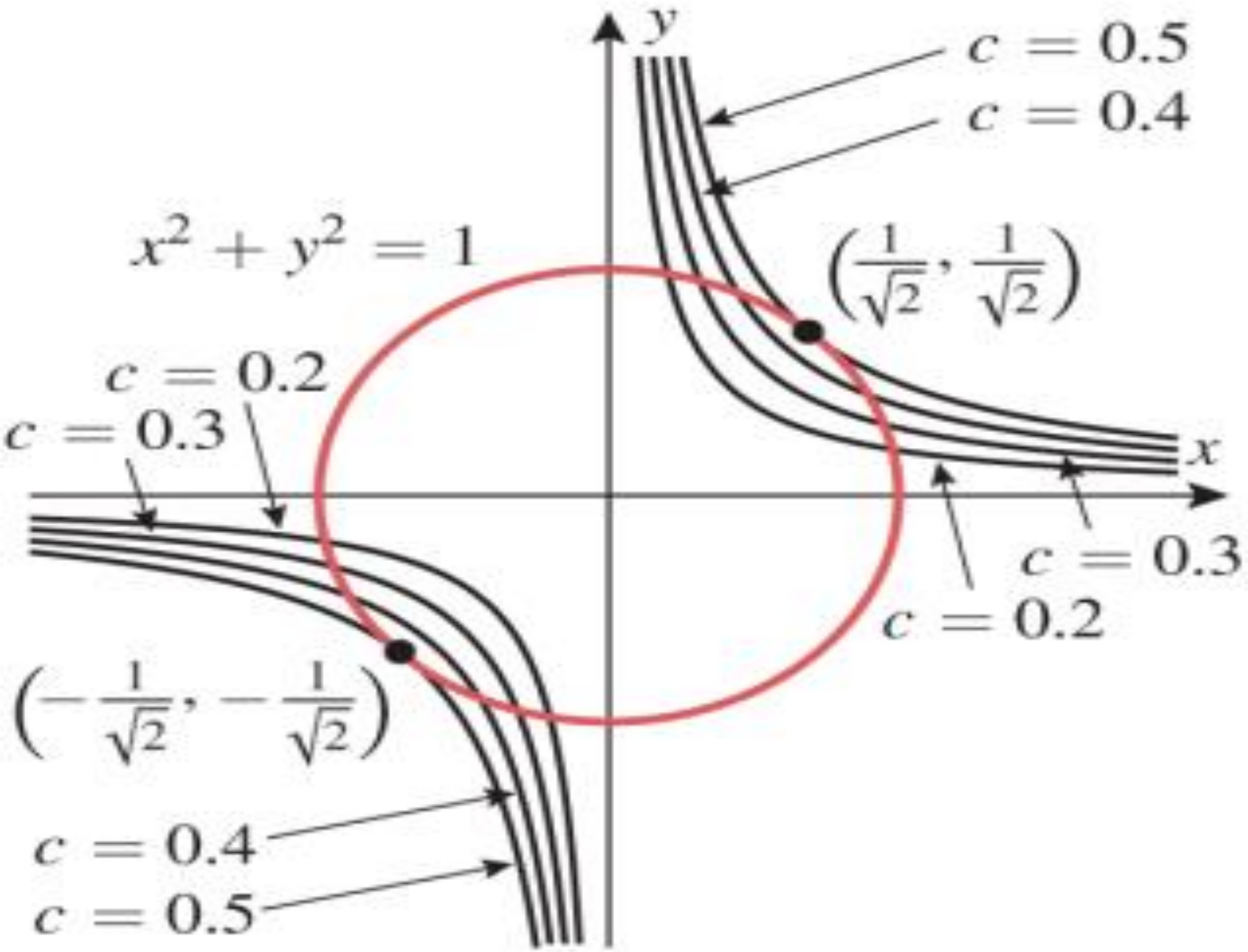
Substituting this in (5) yields

$$2x^2 - 1 = 0$$

from which we obtain $x = \pm 1/\sqrt{2}$. Each of these values, when substituted in Equation (6), produces y-values of $y = \pm 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	1/2	–1/2	–1/2	1/2

Thus, the function $f(x, y) = xy$ has an absolute maximum of $\frac{1}{2}$ occurring at the two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Although it was not asked for, we can also see that f has an absolute minimum of $-\frac{1}{2}$ occurring at the points $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$. Figure 13.9.3 shows some level curves $xy = c$ and the constraint curve in the vicinity of the maxima. A similar figure for the minima can be obtained using negative values of c for the level curves $xy = c$. ◀



y
x
c = 0.5
c = 0.4
$x^2 + y^2 = 1$
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
c = 0.2
c = 0.3
c = 0.3
c = 0.2
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
c = 0.4
c = 0.5

Figure 13.9.3

5–12 Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint. Also, find the points at which these extreme values occur.

5. $f(x, y) = xy;\; 4x^2 + 8y^2 = 16$

6. $f(x, y) = x^2 - y^2;\; x^2 + y^2 = 25$

7. $f(x, y) = 4x^3 + y^2;\; 2x^2 + y^2 = 1$

8. $f(x, y) = x - 3y - 1;\; x^2 + 3y^2 = 16$

9. $f(x, y, z) = 2x + y - 2z;\; x^2 + y^2 + z^2 = 4$

10. $f(x, y, z) = 3x + 6y + 2z;\; 2x^2 + 4y^2 + z^2 = 70$

11. $f(x, y, z) = xyz;\; x^2 + y^2 + z^2 = 1$

12. $f(x, y, z) = x^4 + y^4 + z^4;\; x^2 + y^2 + z^2 = 1$

Q1 sol:

To find the maximum and minimum values of $f(x, y) = xy$ subject to the constraint $4x^2 + 8y^2 = 16$ using **Lagrange multipliers**, follow these steps:

Step 1: Define the Constraint Function

The given constraint is:

$$g(x, y) = 4x^2 + 8y^2 - 16 = 0$$

Step 2: Compute the Gradient

The **gradients** of $f(x, y)$ and $g(x, y)$ are:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y, x)$$

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = (8x, 16y)$$

Step 3: Solve the Lagrange System

By the **Lagrange multiplier** method, we set:

$$\nabla f = \lambda \nabla g$$

This gives the system:

1. $y = \lambda(8x)$
2. $x = \lambda(16y)$
3. $4x^2 + 8y^2 = 16$ (original constraint)

From equation (1):

$$\lambda = \frac{y}{8x}$$

From equation (2):

$$\lambda = \frac{x}{16y}$$

Equating both expressions for λ:

$$\frac{y}{8x} = \frac{x}{16y}$$

Cross multiply:

$$y \cdot 16y = x \cdot 8x$$

$$16y^2 = 8x^2$$

$$2y^2 = x^2$$

$$x^2 = 2y^2$$

Step 5: Solve for x, y Using the Constraint

Substituting $x^2 = 2y^2$ into $4x^2 + 8y^2 = 16$:

$$4(2y^2) + 8y^2 = 16$$

$$8y^2 + 8y^2 = 16$$

$$16y^2 = 16$$

$$y^2 = 1 \Rightarrow y = \pm 1$$

Using $x^2 = 2y^2$:

$$x^2 = 2(1) = 2$$

$$x = \pm\sqrt{2}$$

Step 6: Compute the Function Values

Evaluating $f(x, y) = xy$ at the critical points:

1. $(\sqrt{2}, 1) \rightarrow f(\sqrt{2}, 1) = \sqrt{2} \cdot 1 = \sqrt{2}$
2. $(\sqrt{2}, -1) \rightarrow f(\sqrt{2}, -1) = \sqrt{2} \cdot (-1) = -\sqrt{2}$
3. $(-\sqrt{2}, 1) \rightarrow f(-\sqrt{2}, 1) = -\sqrt{2}$
4. $(-\sqrt{2}, -1) \rightarrow f(-\sqrt{2}, -1) = \sqrt{2}$

Step 7: Identify Maximum and Minimum Values

- Maximum value: $\sqrt{2}$ at points $(\sqrt{2}, 1)$ and $(-\sqrt{2}, -1)$.
- Minimum value: $-\sqrt{2}$ at points $(\sqrt{2}, -1)$ and $(-\sqrt{2}, 1)$.

Final Answer

- Maximum value: $\sqrt{2}$ at $(\sqrt{2}, 1)$ and $(-\sqrt{2}, -1)$.
- Minimum value: $-\sqrt{2}$ at $(\sqrt{2}, -1)$ and $(-\sqrt{2}, 1)$.

Q9 sol:

We will use **Lagrange multipliers** to find the maximum and minimum values of $f(x,y,z) = 2x + y - 2z$ subject to the constraint:

$$g(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$$

Step 1: Compute the Gradient

The gradients of $f(x,y,z)$ and $g(x,y,z)$ are:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2, 1, -2)$$

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = (2x, 2y, 2z)$$

Step 2: Solve the Lagrange System

By the **Lagrange multiplier** method, we set:

$$\nabla f = \lambda \nabla g$$

This gives the system:

1. $2 = \lambda(2x) \to \lambda = \frac{2}{2x} = \frac{1}{x}$
2. $1 = \lambda(2y) \to \lambda = \frac{1}{2y}$
3. $-2 = \lambda(2z) \to \lambda = \frac{-2}{2z} = \frac{-1}{z}$

Step 3: Equate the Expressions for λ

From (1) and (2):

$$\frac{1}{x} = \frac{1}{2y}$$

Cross multiply:

$$2y = x$$

From (1) and (3):

$$\frac{1}{x} = \frac{-1}{z}$$

Cross multiply:

$$z = -x$$

Step 4: Solve for x, y, z Using the Constraint

The constraint equation is:

$$x^2 + y^2 + z^2 = 4$$

Substituting $x = 2y$ and $z = -x$:

$$(2y)^2 + y^2 + (-2y)^2 = 4$$

$$4y^2 + y^2 + 4y^2 = 4$$

$$9y^2 = 4$$

$$y^2 = \frac{4}{9}$$

$$y = \pm\frac{2}{3}$$

Now, solve for x and z:

$$x = 2y = \pm\frac{4}{3}$$

$$z = -x = \mp\frac{4}{3}$$

Step 5: Compute $f(x, y, z)$

$$f\left(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right) = 2\left(\frac{4}{3}\right) + \frac{2}{3} - 2\left(-\frac{4}{3}\right)$$

$$= \frac{8}{3} + \frac{2}{3} + \frac{8}{3} = \frac{18}{3} = 6$$

$$f\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right) = 2\left(-\frac{4}{3}\right) + \left(-\frac{2}{3}\right) - 2\left(\frac{4}{3}\right)$$

$$= -\frac{8}{3} - \frac{2}{3} - \frac{8}{3} = -\frac{18}{3} = -6$$

Step 6: Identify Maximum and Minimum Values

- Maximum value: 6 at $\left(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right)$.
- Minimum value: -6 at $\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)$.

Final Answer

- Maximum value: 6 at $\left(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right)$.
- Minimum value: -6 at $\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right)$.

$$f(x, y, z) = x^4 + y^4 + z^4$$

subject to the constraint:

$$g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0.$$

Step 1: Compute the Gradients

The gradients of $f(x, y, z)$ and $g(x, y, z)$ are:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (4x^3, 4y^3, 4z^3)$$

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) = (2x, 2y, 2z)$$

Step 2: Solve the Lagrange System

Using the Lagrange multiplier λ:

$$\nabla f = \lambda \nabla g$$

This gives the system:

1. $4x^3 = \lambda(2x) \rightarrow \lambda = \frac{4x^3}{2x} = 2x^2$ (if $x \neq 0$)
2. $4y^3 = \lambda(2y) \rightarrow \lambda = \frac{4y^3}{2y} = 2y^2$ (if $y \neq 0$)
3. $4z^3 = \lambda(2z) \rightarrow \lambda = \frac{4z^3}{2z} = 2z^2$ (if $z \neq 0$)

Since all three expressions equal λ, we equate:

$$2x^2 = 2y^2 = 2z^2$$

$$x^2 = y^2 = z^2$$

Thus, x^2, y^2, and z^2 must be equal.

Step 3: Solve for x, y, z Using the Constraint

Since $x^2 = y^2 = z^2$, let $x^2 = y^2 = z^2 = k$.

From the constraint:

$$x^2 + y^2 + z^2 = 1$$

$$3k = 1$$

$$k = \frac{1}{3}$$

Thus:

$$x^2 = y^2 = z^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}, \quad y = \pm\frac{1}{\sqrt{3}}, \quad z = \pm\frac{1}{\sqrt{3}}$$

Step 4: Compute $f(x, y, z)$ at These Points

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4$$

$$= 3 \times \left(\frac{1}{3^2}\right) = 3 \times \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

Thus, $f = \frac{1}{3}$ at these points.

Step 5: Check Boundary Cases

Another possible case is when one variable is **1 or -1** and the others are **0** (since $x^2 + y^2 + z^2 = 1$).

For example, consider $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, etc.

$$f(1, 0, 0) = 1^4 + 0^4 + 0^4 = 1.$$

Thus, $f = 1$ at these points.

Step 6: Identify Maximum and Minimum Values

- Maximum value: $f_{\max} = 1$ at $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)$.
- Minimum value: $f_{\min} = \frac{1}{3}$ at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and permutations.

Final Answer

- Maximum value: 1 at $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)$.
- Minimum value: $\frac{1}{3}$ at $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.