MULTIPLE INTEGRALS

In this chapter we will extend the concept of a definite integral to functions of two and three variables. Whereas functions of one variable are usually integrated over intervals, functions of two variables are usually integrated over regions in 2-space and functions of three variables over regions in 3-space. Calculating such integrals will require some new techniques that will be a central focus in this chapter. Once we have developed the basic methods for integrating functions of two and three variables, we will show how such integrals can be used to calculate surface areas and volumes of solids; and we will also show how they can be used to find masses and centers of gravity of flat plates and three-dimensional solids. In addition to our study of integration, we will generalize the concept of a parametric curve in 2-space to a parametric surface in 3-space. This will allow us to work with a wider variety of surfaces than previously possible and will provide a powerful tool for generating surfaces using computers and other graphing utilities.

14.1 DOUBLE INTEGRALS

The notion of a definite integral can be extended to functions of two or more variables. In this section we will discuss the double integral, which is the extension to functions of two variables.

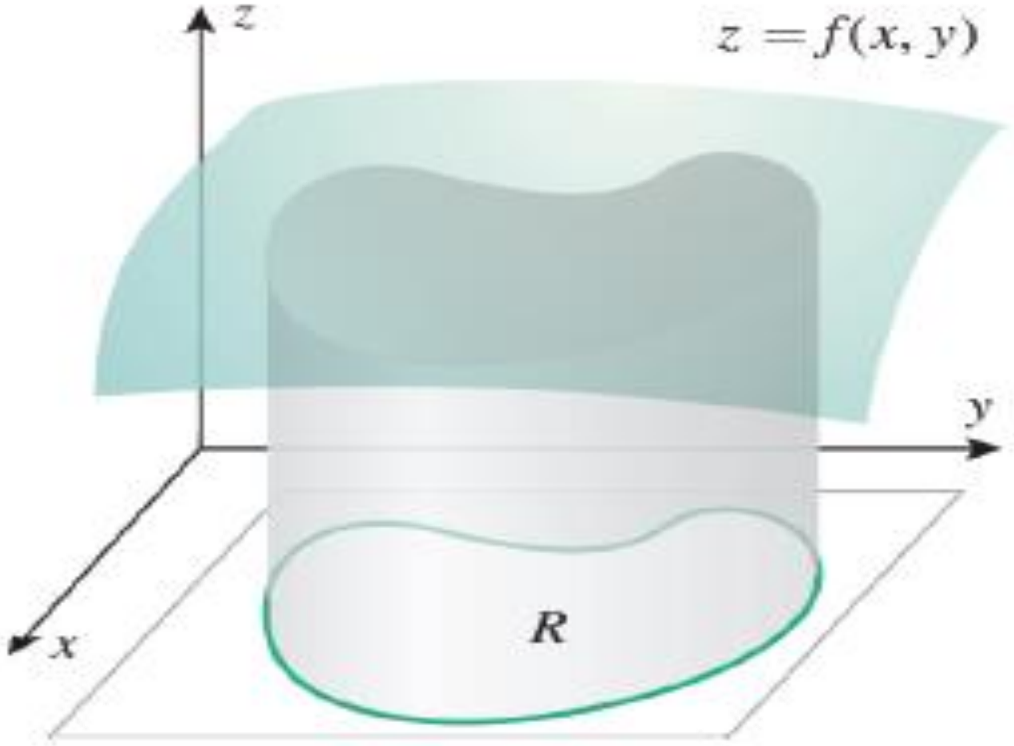
VOLUME

Recall that the definite integral of a function of one variable

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{1}$$

arose from the problem of finding areas under curves. [In the rightmost expression in (1), we use the "limit as $n \to +\infty$" to encapsulate the process by which we increase the number of subintervals of $[a, b]$ in such a way that the lengths of the subintervals approach zero.] Integrals of functions of two variables arise from the problem of finding volumes under surfaces.

14.1.1 THE VOLUME PROBLEM Given a function f of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R (Figure 14.1.1).



▲ **Figure 14.1.1**

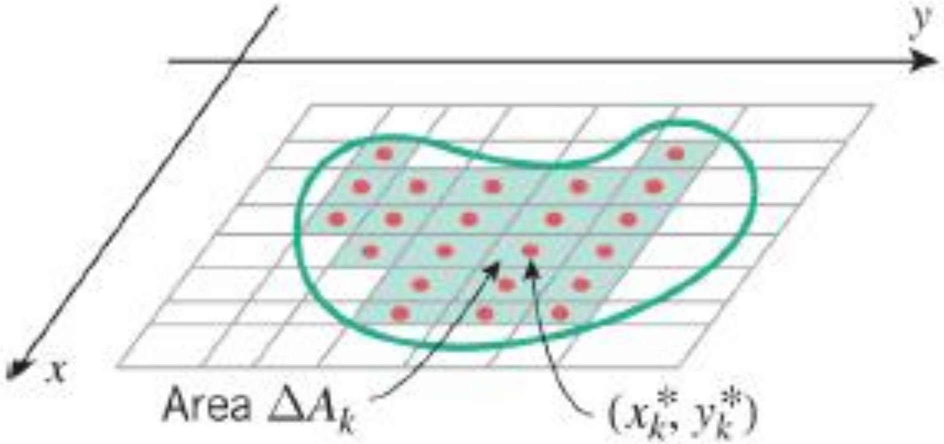
- Using lines parallel to the coordinate axes, divide the rectangle enclosing the region R into subrectangles, and exclude from consideration all those subrectangles that contain any points outside of R. This leaves only rectangles that are subsets of R (Figure 14.1.2). Assume that there are n such rectangles, and denote the area of the kth such rectangle by ΔA_k.
- Choose any arbitrary point in each subrectangle, and denote the point in the kth subrectangle by (x_k^*, y_k^*). As shown in Figure 14.1.3, the product $f(x_k^*, y_k^*)\Delta A_k$ is the volume of a rectangular parallelepiped with base area ΔA_k and height $f(x_k^*, y_k^*)$, so the sum

$$\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

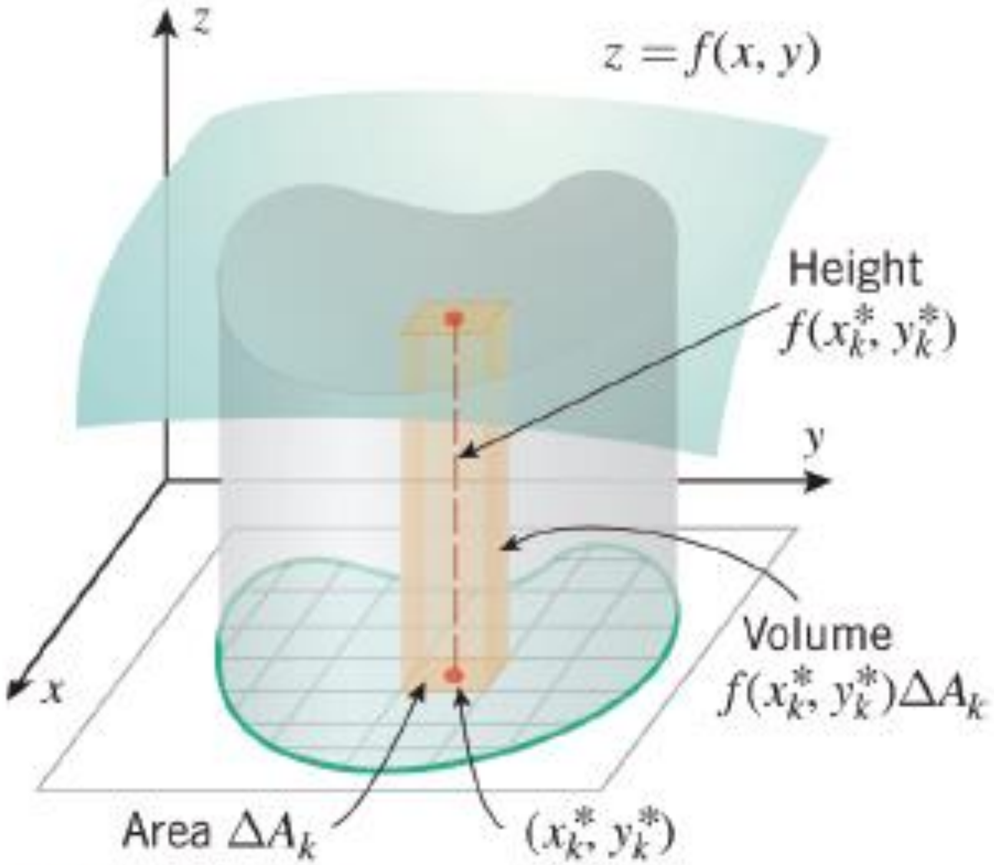
can be viewed as an approximation to the volume V of the entire solid.

- There are two sources of error in the approximation: first, the parallelepipeds have flat tops, whereas the surface $z = f(x, y)$ may be curved; second, the rectangles that form the bases of the parallelepipeds may not completely cover the region R. However, if we repeat the above process with more and more subdivisions in such a way that both the lengths and the widths of the subrectangles approach zero, then it is plausible that the errors of both types approach zero, and the exact volume of the solid will be

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$$



▲ **Figure 14.1.2**



▲ **Figure 14.1.3**

This suggests the following definition.

14.1.2 DEFINITION (*Volume Under a Surface*) If f is a function of two variables that is continuous and nonnegative on a region R in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R is defined by

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{2}$$

Here, $n \to +\infty$ indicates the process of increasing the number of subrectangles of the rectangle enclosing R in such a way that both the lengths and the widths of the subrectangles approach zero.

$$\boxed{\iint\limits_R f(x, y)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k} \tag{4}$$

which is called the ***double integral*** of $f(x, y)$ over R.

If f is continuous and nonnegative on the region R, then the volume formula in (2) can be expressed as

$$\boxed{V = \iint\limits_R f(x, y)\, dA} \tag{5}$$

$$\boxed{\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy} \tag{6}$$

$$\boxed{\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx} \tag{7}$$

Evaluate

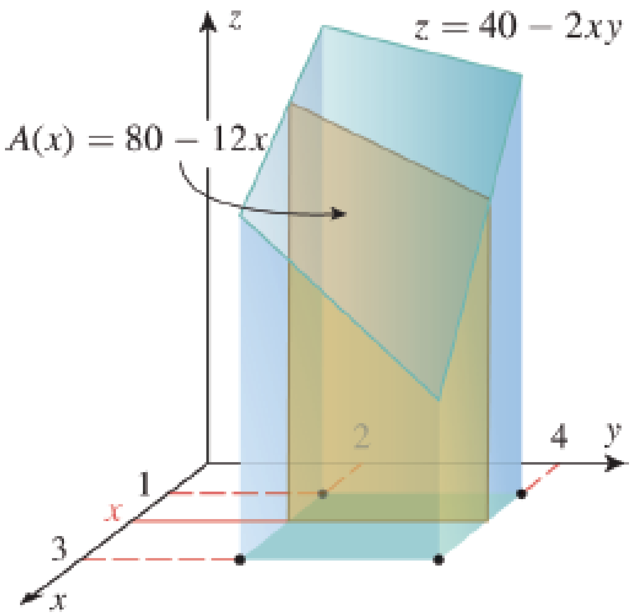
(a) $\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx$ (b) $\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$

Solution (*a*).

$$
\begin{aligned}
\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx &= \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy\right] dx \\
&= \int_1^3 (40y - xy^2)\Big]_{y=2}^4 \, dx \\
&= \int_1^3 [(160 - 16x) - (80 - 4x)]\, dx \\
&= \int_1^3 (80 - 12x)\, dx \\
&= (80x - 6x^2)\Big]_1^3 = 112
\end{aligned}
$$

Solution (b).

$$\begin{aligned}
\int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy &= \int_2^4 \left[\int_1^3 (40 - 2xy)\,dx\right] dy \\
&= \int_2^4 (40x - x^2y)\Big]_{x=1}^{3} dy \\
&= \int_2^4 [(120 - 9y) - (40 - y)]\,dy \\
&= \int_2^4 (80 - 8y)\,dy \\
&= (80y - 4y^2)\Big]_2^4 = 112 \blacktriangleleft
\end{aligned}$$



▲ **Figure 14.1.4**

It is no accident that both parts of Example 2 produced the same answer. Consider the solid S bounded above by the surface $z = 40 - 2xy$ and below by the rectangle R defined by $1 \le x \le 3$ and $2 \le y \le 4$. By the method of slicing discussed in Section 5.2, the volume of S is given by

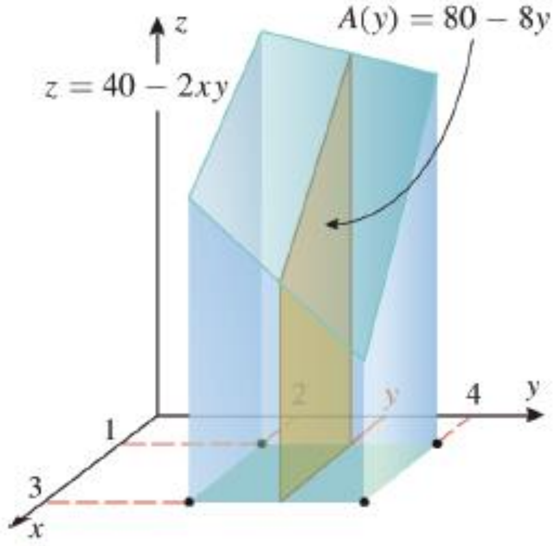
$$V = \int_1^3 A(x)\, dx$$

where $A(x)$ is the area of a vertical cross section of S taken perpendicular to the x-axis (Figure 14.1.4). For a fixed value of x, $1 \le x \le 3$, $z = 40 - 2xy$ is a function of y, so the integral

$$A(x) = \int_2^4 (40 - 2xy)\, dy$$

represents the area under the graph of this function of y. Thus,

$$V = \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy \right] dx = \int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx$$



▲ **Figure 14.1.5**

is the volume of S. Similarly, by the method of slicing with cross sections of S taken perpendicular to the y-axis, the volume of S is given by

$$V = \int_2^4 A(y)\,dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\,dx\right] dy = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

(Figure 14.1.5). Thus, the iterated integrals in parts (a) and (b) of Example 2 both measure the volume of S, which by Formula (5) is the double integral of $z = 40 - 2xy$ over R. That is,

$$\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx = \iint_R (40 - 2xy)\,dA = \int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy$$

The geometric argument above applies to any continuous function $f(x, y)$ that is nonnegative on a rectangle $R = [a, b] \times [c, d]$, as is the case for $f(x, y) = 40 - 2xy$ on $[1, 3] \times [2, 4]$. The conclusion that the double integral of $f(x, y)$ over R has the same value as either of the two possible iterated integrals is true even when f is negative at some points in R. We state this result in the following theorem and omit a formal proof.

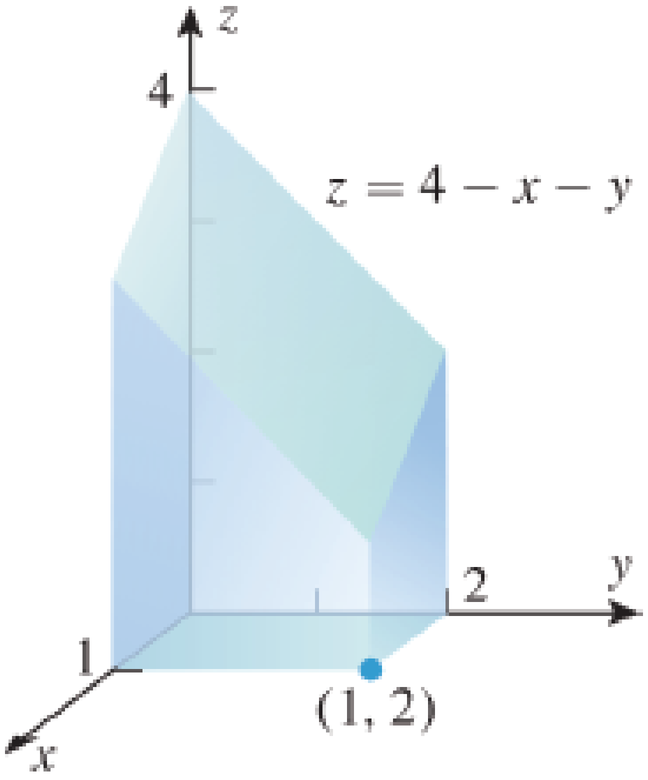
14.1.3 THEOREM (Fubini's Theorem) *Let R be the rectangle defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

► Example 3 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 14.1.6).



▲ **Figure 14.1.6**

Solution. The volume is the double integral of $z = 4 - x - y$ over R. Using Theorem 14.1.3, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\,dx\,dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\,dy\,dx \tag{8}$$

Using the first of these, we obtain

$$\begin{aligned} V &= \iint_R (4 - x - y)\,dA = \int_0^2 \int_0^1 (4 - x - y)\,dx\,dy \\ &= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y\right) dy \\ &= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5 \end{aligned}$$

You can check this result by evaluating the second integral in (8). ◀

▶ Example 4 Evaluate the double integral

$$\iint_R y^2 x \, dA$$

over the rectangle $R = \{(x, y) : -3 \le x \le 2, 0 \le y \le 1\}$.

Solution. In view of Theorem 14.1.3, the value of the double integral can be obtained by evaluating one of two possible iterated double integrals. We choose to integrate first with respect to x and then with respect to y.

$$\iint_R y^2x\,dA = \int_0^1 \int_{-3}^2 y^2x\,dx\,dy = \int_0^1 \left[\frac{1}{2}y^2x^2\right]_{x=-3}^2 dy$$
$$= \int_0^1 \left(-\frac{5}{2}y^2\right) dy = -\frac{5}{6}y^3\Big]_0^1 = -\frac{5}{6}$$

The integral in Example 4 can be interpreted as the net signed volume between the rectangle $[-3, 2] \times [0, 1]$ and the surface $z = y^2x$. That is, it is the volume below $z = y^2x$ and above $[0, 2] \times [0, 1]$ minus the volume above $z = y^2x$ and below $[-3, 0] \times [0, 1]$ (Figure 14.1.7).

PROPERTIES OF DOUBLE INTEGRALS

To distinguish between double integrals of functions of two variables and definite integrals of functions of one variable, we will refer to the latter as ***single integrals***. Because double integrals, like single integrals, are defined as limits, they inherit many of the properties of limits. The following results, which we state without proof, are analogs of those in Theorem 4.5.4.

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA \quad (c \text{ a constant}) \tag{9}$$

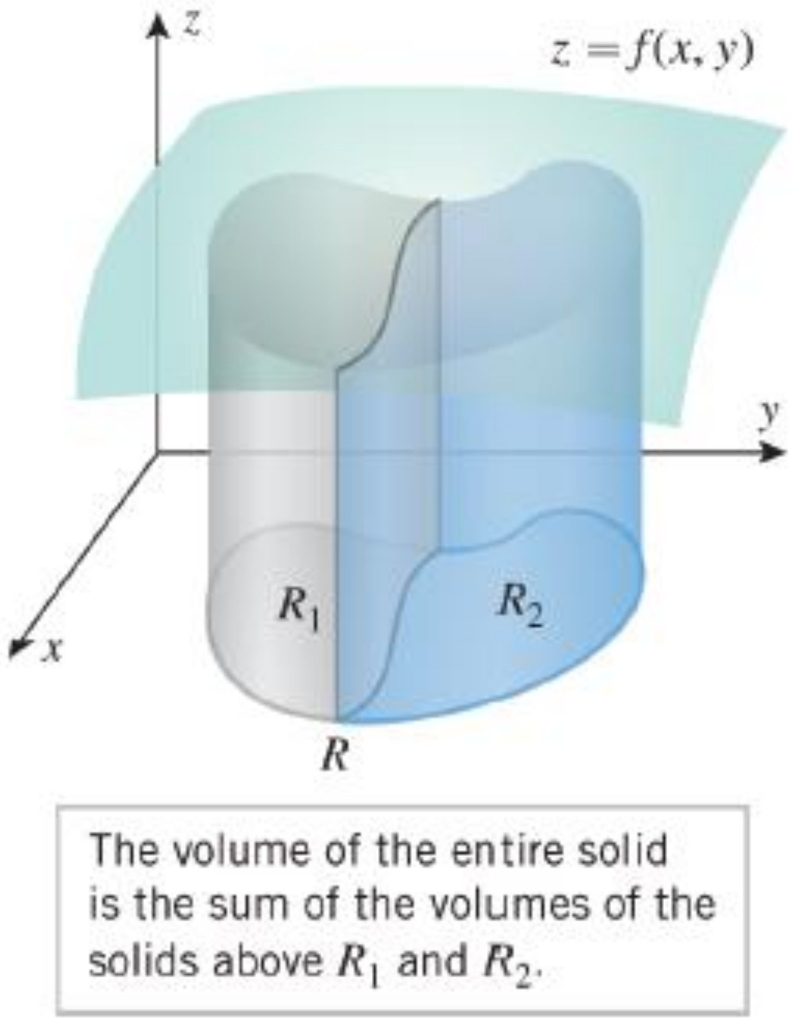
$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA \tag{10}$$

$$\iint_R [f(x,y)-g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA \tag{11}$$

Figure 14.1.8 illustrates the result that if $f(x, y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect of subdividing the solid between R and $z = f(x, y)$ into two solids, the sum of whose volumes is the volume of the entire solid. This suggests the following result, which holds even if f has negative values:

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA \tag{12}$$

The proof of this result will be omitted.


z
z = f(x, y)
y
x
R_1
R_2
R
The volume of the entire solid is the sum of the volumes of the solids above R_1 and R_2.

▲ **Figure 14.1.8**

EXERCISE SET 14.1 ☐C CAS

1–12 Evaluate the iterated integrals.

1. $\displaystyle\int_0^1 \int_0^2 (x+3)\,dy\,dx$

2. $\displaystyle\int_1^3 \int_{-1}^1 (2x-4y)\,dy\,dx$

3. $\displaystyle\int_2^4 \int_0^1 x^2 y\,dx\,dy$

4. $\displaystyle\int_{-2}^0 \int_{-1}^2 (x^2+y^2)\,dx\,dy$

5. $\displaystyle\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\,dy\,dx$

6. $\displaystyle\int_0^2 \int_0^1 y\sin x\,dy\,dx$

7. $\displaystyle\int_{-1}^0 \int_2^5 dx\,dy$

8. $\displaystyle\int_4^6 \int_{-3}^7 dy\,dx$

9. $\displaystyle\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx$

10. $\displaystyle\int_{\pi/2}^{\pi} \int_1^2 x\cos xy\,dy\,dx$

11. $\displaystyle\int_0^{\ln 2} \int_0^1 xye^{y^2x}\,dy\,dx$

12. $\displaystyle\int_3^4 \int_1^2 \frac{1}{(x+y)^2}\,dy\,dx$

13–16 Evaluate the double integral over the rectangular region R. ■

13. $\iint\limits_R 4xy^3\, dA;\ R = \{(x, y) : -1 \le x \le 1, -2 \le y \le 2\}$

14. $\iint\limits_R \dfrac{xy}{\sqrt{x^2 + y^2 + 1}}\, dA;$

$R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$

15. $\iint\limits_R x\sqrt{1 - x^2}\, dA;\ R = \{(x, y) : 0 \le x \le 1, 2 \le y \le 3\}$

16. $\iint\limits_R (x \sin y - y \sin x)\, dA;$

$R = \{(x, y) : 0 \le x \le \pi/2, 0 \le y \le \pi/3\}$

Solution:

Exercise Set 14.1

1. $\int_0^1 \int_0^2 (x+3)\,dy\,dx = \int_0^1 (2x+6)\,dx = 7.$

2. $\int_1^3 \int_{-1}^1 (2x-4y)\,dy\,dx = \int_1^3 4x\,dx = 16.$

3. $\int_2^4 \int_0^1 x^2 y\,dx\,dy = \int_2^4 \frac{1}{3} y\,dy = 2.$

4. $\int_{-2}^0 \int_{-1}^2 (x^2+y^2)\,dx\,dy = \int_{-2}^0 (3+3y^2)\,dy = 14.$

5. $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\,dy\,dx = \int_0^{\ln 3} e^x\,dx = 2.$

6. $\int_0^2 \int_0^1 y \sin x\,dy\,dx = \int_0^2 \frac{1}{2} \sin x\,dx = \frac{1-\cos 2}{2}.$

8. $\int_4^6 \int_{-3}^7 dy\,dx = \int_4^6 10\,dx = 20.$

9. $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\,dy\,dx = \int_0^1 \left(1 - \frac{1}{x+1}\right) dx = 1 - \ln 2.$

10. $\int_{\pi/2}^{\pi} \int_1^2 x \cos xy\,dy\,dx = \int_{\pi/2}^{\pi} (\sin 2x - \sin x)\,dx = -2.$

11. $\int_0^{\ln 2} \int_0^1 xy\,e^{y^2 x}\,dy\,dx = \int_0^{\ln 2} \frac{1}{2}(e^x - 1)\,dx = \frac{1-\ln 2}{2}.$

12. $\displaystyle\int_3^4\int_1^2 \frac{1}{(x+y)^2}\,dy\,dx = \int_3^4\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx = \ln(25/24).$

13. $\displaystyle\int_{-1}^1\int_{-2}^2 4xy^3\,dy\,dx = \int_{-1}^1 0\,dx = 0.$

14. $\displaystyle\int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx = \int_0^1 [x(x^2+2)^{1/2} - x(x^2+1)^{1/2}]\,dx = \frac{1}{3}(3\sqrt{3}-4\sqrt{2}+1).$

15. $\displaystyle\int_0^1\int_2^3 x\sqrt{1-x^2}\,dy\,dx = \int_0^1 x(1-x^2)^{1/2}\,dx = \frac{1}{3}.$

16. $\displaystyle\int_0^{\pi/2}\int_0^{\pi/3} (x\sin y - y\sin x)dy\,dx = \int_0^{\pi/2}\left(\frac{x}{2}-\frac{\pi^2}{18}\sin x\right)dx = \frac{\pi^2}{144}.$