Ex#13.8

Extreme value of the function of two variables. Absolute & Relative Extrema, Extreme Value theorem, The second order Partials test

Q#1,2 and 9 to 18

Functions of several variables, like functions of one variable, can have local and global extrema. (That is, local and global maxima and minima.) A function has a local extremum at a point where it takes on the largest or smallest value in a small region around the point. Global extrema are the largest or smallest values anywhere on the domain under consideration. (See Figures 15.1 and 15.2.

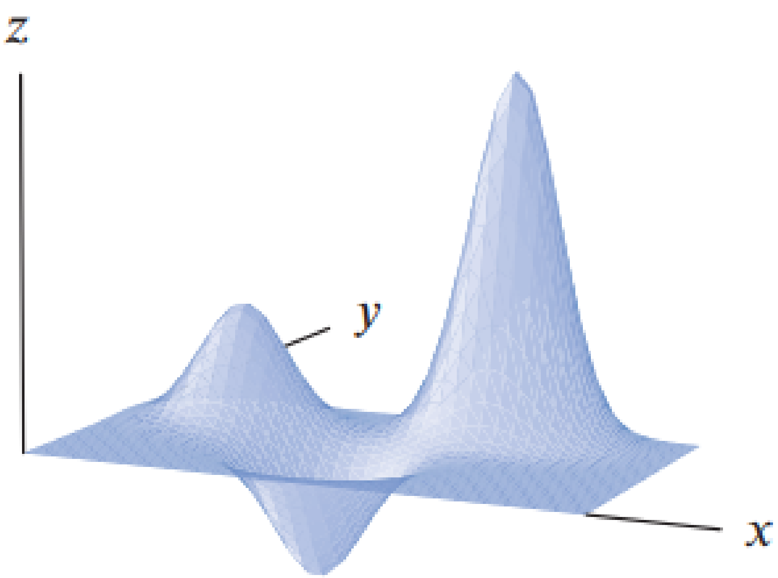


Figure 15.1: Local and global extrema for a function of two variables on $0 \le x \le a$, $0 \le y \le b$

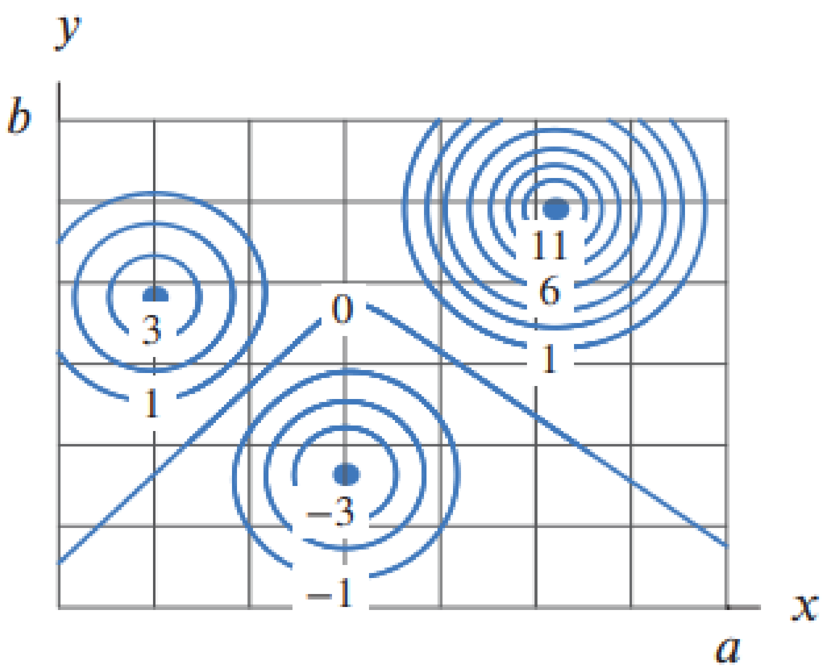


Figure 15.2: Contour map of the function in Figure 15.1

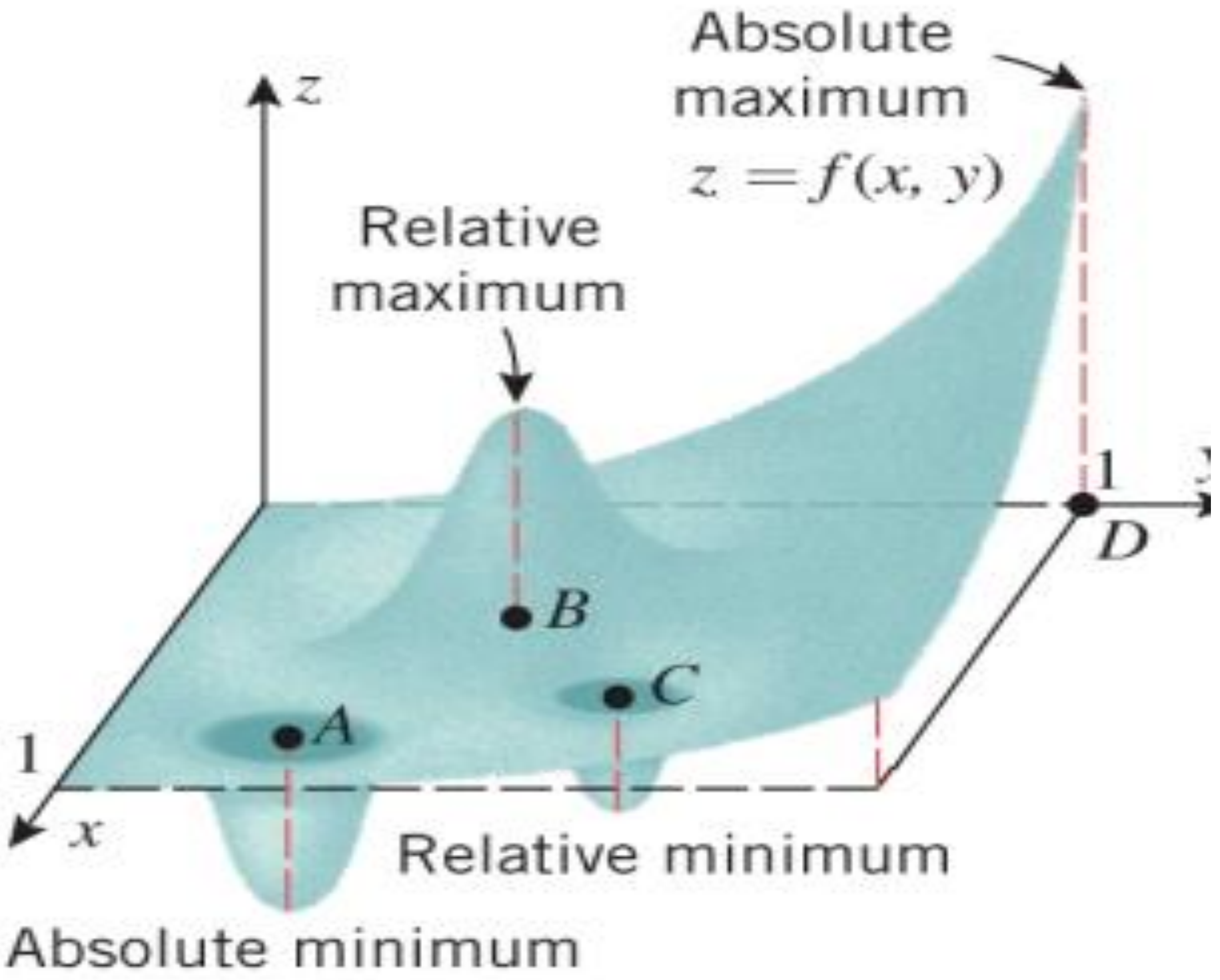
More precisely, considering only points at which f is defined, we say:

- f has a **local maximum** at the point P_0 if $f(P_0) \geq f(P)$ for all points P near P_0.
- f has a **local minimum** at the point P_0 if $f(P_0) \leq f(P)$ for all points P near P_0.

For example, the function whose contour map is shown in Figure 15.2 has a local minimum value of -3 and local maximum values of 3 and 11 in the rectangle shown.

13.8.1 DEFINITION A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

13.8.2 DEFINITION A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.


Absolute maximum
$z = f(x, y)$
z
Relative maximum
1
D
B
C
A
1
x
Relative minimum
Absolute minimum

FINDING RELATIVE EXTREMA

Recall that if a function g of one variable has a relative extremum at a point x_0 where g is differentiable, then $g'(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 13.8.4), so

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

13.8.4 THEOREM *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

13.8.5 DEFINITION A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

Example 1 Find and analyze the critical points of $f(x, y) = x^2 - 2x + y^2 - 4y + 5$.

Solution To find the critical points, we set both partial derivatives equal to zero:

$$f_x(x, y) = 2x - 2 = 0$$
$$f_y(x, y) = 2y - 4 = 0.$$

Solving these equations gives $x = 1$, $y = 2$. Hence, f has only one critical point, namely $(1, 2)$. To see the behavior of f near $(1, 2)$, look at the values of the function in Table 15.1.

Table 15.1 *Values of $f(x, y)$ near the point $(1, 2)$*

		x				
		0.8	0.9	1.0	1.1	1.2
	1.8	0.08	0.05	0.04	0.05	0.08
	1.9	0.05	0.02	0.01	0.02	0.05
y	2.0	0.04	0.01	0.00	0.01	0.04
	2.1	0.05	0.02	0.01	0.02	0.05
	2.2	0.08	0.05	0.04	0.05	0.08

The table suggests that the function has a local minimum value of 0 at (1, 2). We can confirm

13.8.2 DEFINITION A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

this by completing the square:

$$f(x, y) = x^2 - 2x + y^2 - 4y + 5 = (x-1)^2 + (y-2)^2.$$

Figure 15.5 shows that the graph of f is a paraboloid with vertex at the point $(1, 2, 0)$. It is the same shape as the graph of $z = x^2 + y^2$ (see Figure 12.12 on page 661), except that the vertex has been shifted to $(1, 2)$. So the point $(1, 2)$ is a local minimum of f (as well as a global minimum).

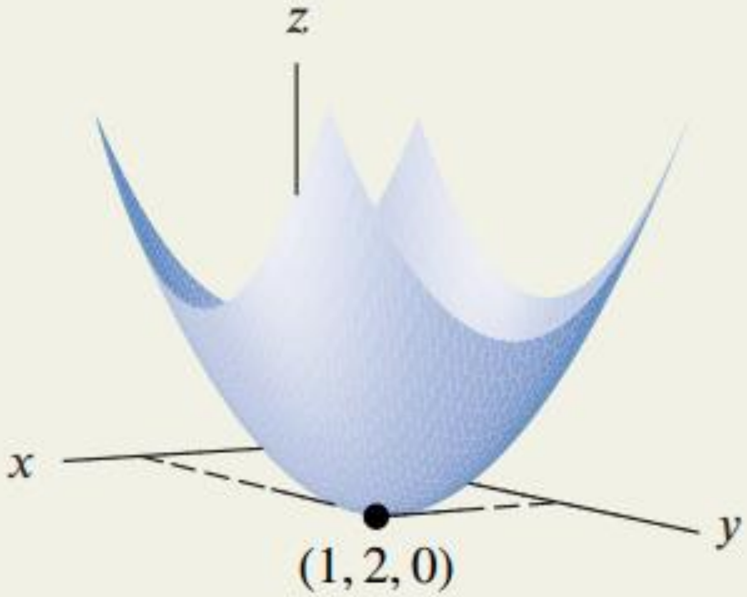


Figure 15.5: The graph of $f(x, y) = x^2 - 2x + y^2 - 4y + 5$ with a local minimum at the point $(1, 2)$

Example 2 Find and analyze any critical points of $f(x, y) = -\sqrt{x^2 + y^2}$.

Solution We look for points where grad $f = \vec{0}$ or is undefined. The partial derivatives are given by

$$f_x(x, y) = -\frac{x}{\sqrt{x^2 + y^2}},$$

$$f_y(x, y) = -\frac{y}{\sqrt{x^2 + y^2}}.$$

These partial derivatives are never simultaneously zero, but they are undefined at $x = 0, y = 0$. Thus, $(0, 0)$ is a critical point and a possible extreme point. The graph of f (see Figure 15.6) is a cone, with vertex at $(0, 0)$. So f has a local and global maximum at $(0, 0)$.

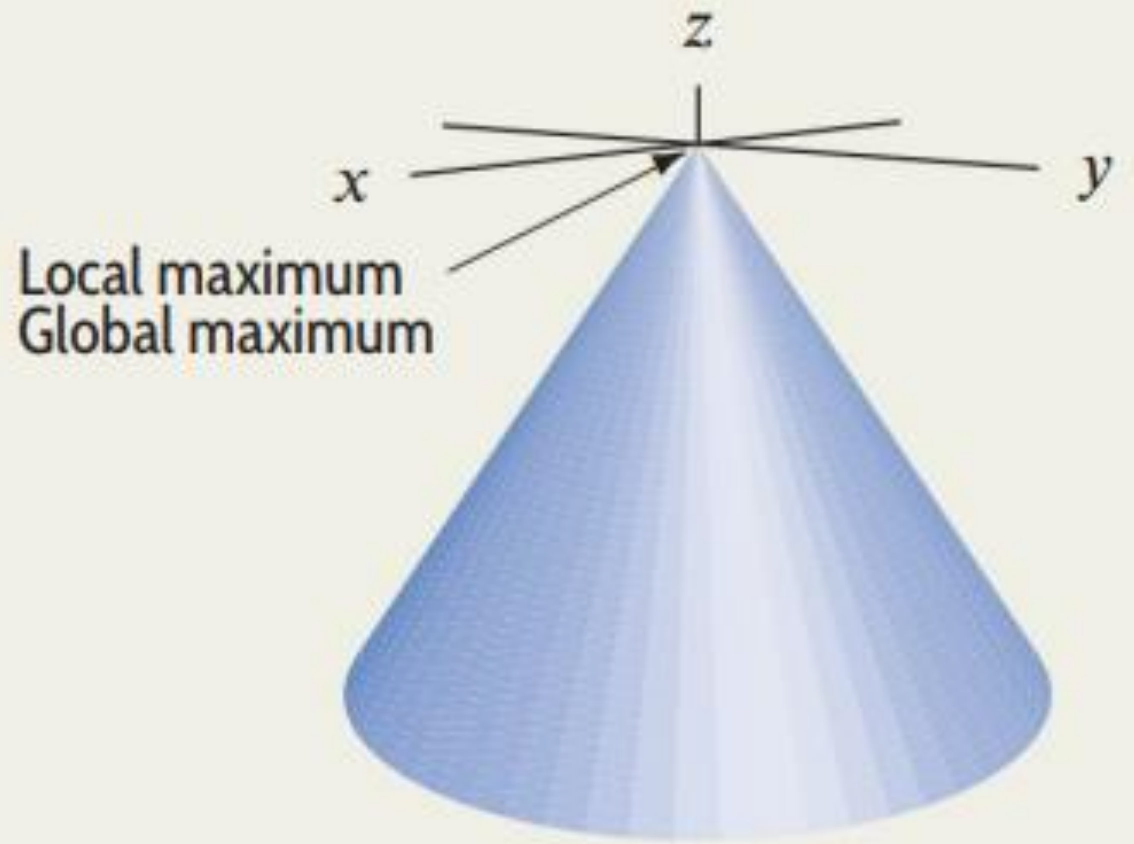


Figure 15.6: Graph of $f(x, y) = -\sqrt{x^2 + y^2}$

Example 3 Find and analyze any critical points of $g(x, y) = x^2 - y^2$.

Solution To find the critical points, we look for points where both partial derivatives are zero:

$$g_x(x, y) = 2x = 0$$
$$g_y(x, y) = -2y = 0.$$

Solving gives $x = 0$, $y = 0$, so the origin is the only critical point.

Figure 15.7 shows that near the origin g takes on both positive and negative values. Since $g(0, 0) = 0$, the origin is a critical point which is neither a local maximum nor a local minimum. The graph of g looks like a saddle.

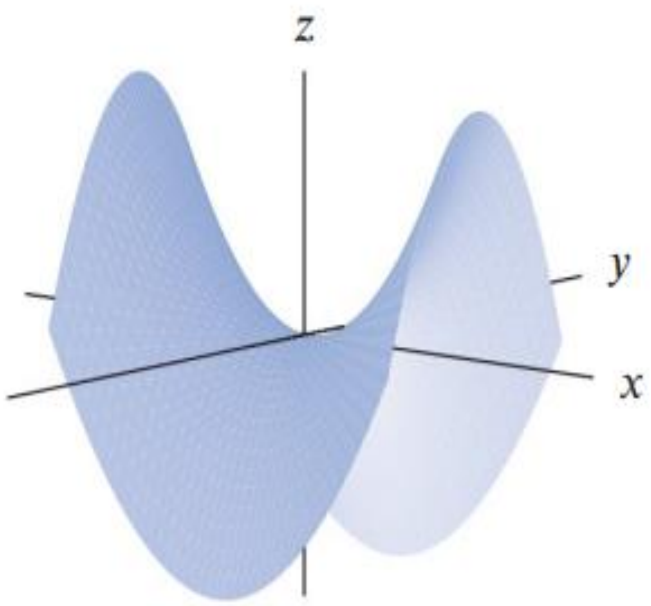


Figure 15.7: Graph of $g(x, y) = x^2 - y^2$, showing saddle shape at the origin

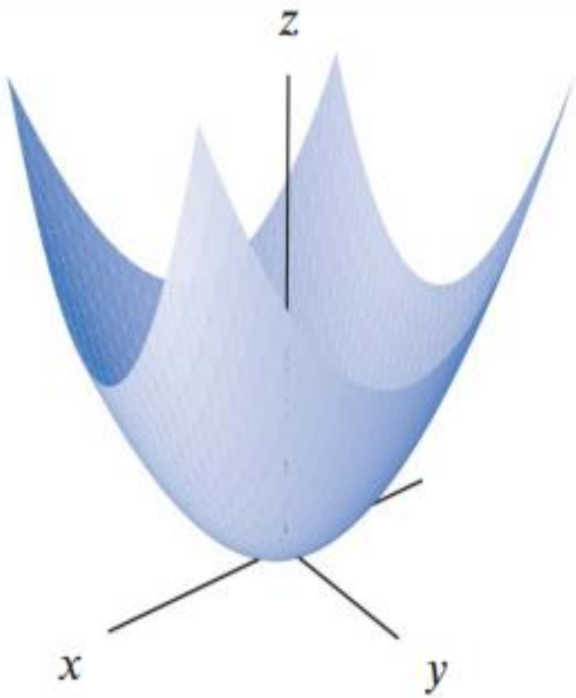


Figure 15.8: Graph of $h(x, y) = x^2 + y^2$, showing minimum at the origin

The previous examples show that critical points can occur at local maxima or minima, or at points which are neither: The functions g and h in Figures 15.7 and 15.8 both have critical points at the origin. Figure 15.9 shows level curves of g. They are hyperbolas showing both positive and negative values of g near $(0, 0)$. Contrast this with the level curves of h near the local minimum in Figure 15.10.

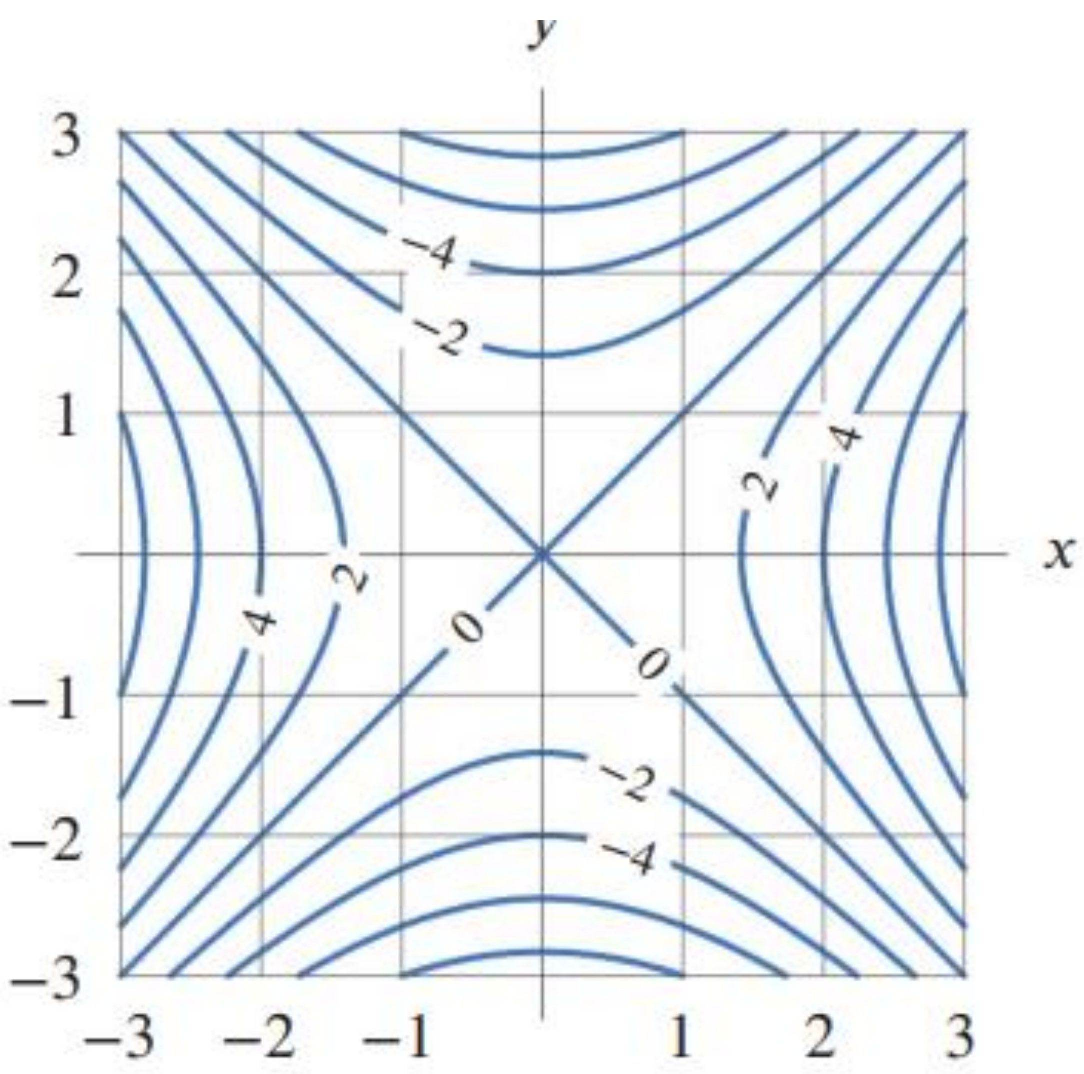
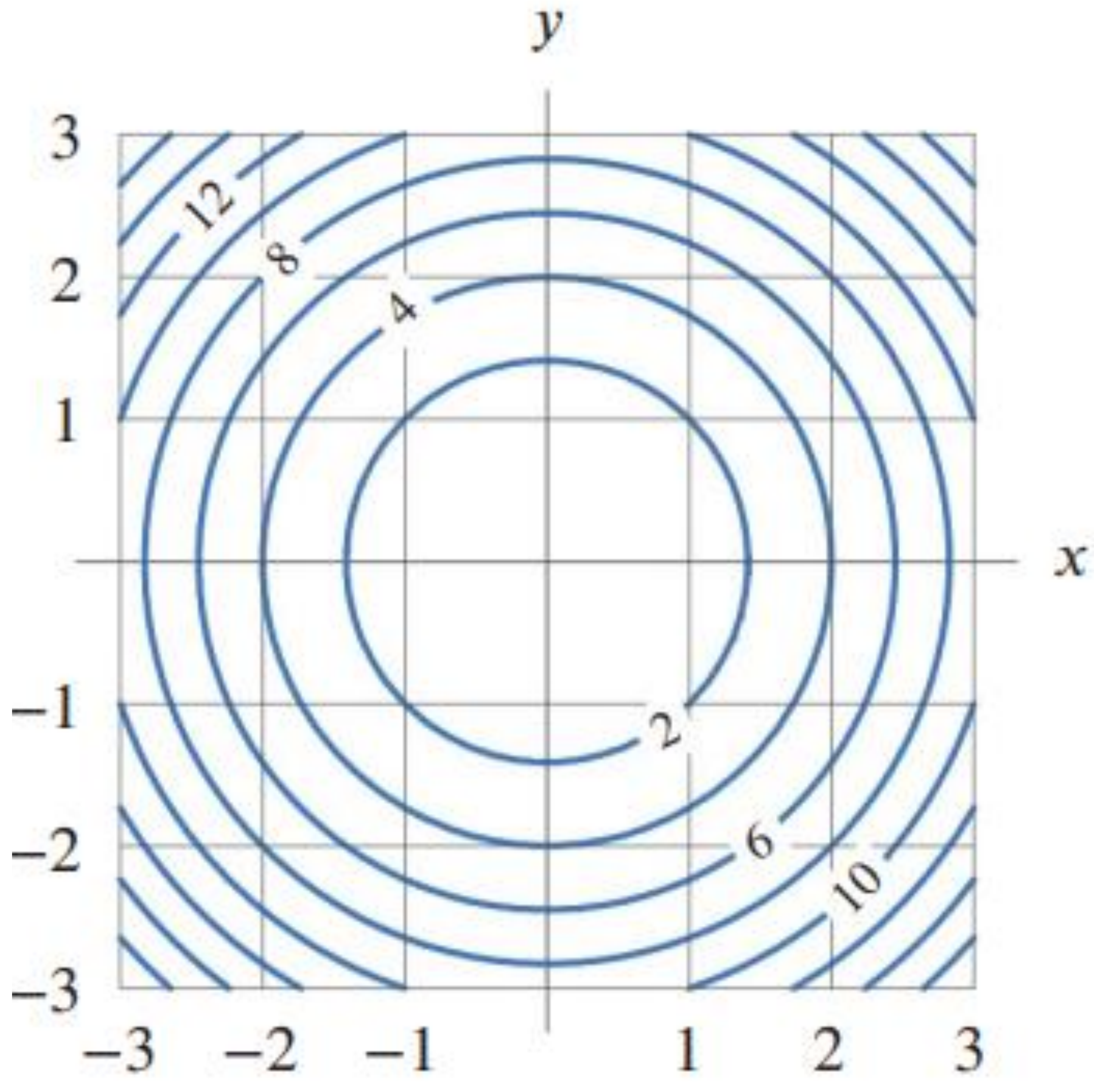


Figure 15.9: Contours of $g(x, y) = x^2 - y^2$, showing a saddle shape at the origin



igure 15.10: Contours of $h(x, y) = x^2 + y^2$, showing a local minimum at the origin

Example 4 Find the local extrema of the function $f(x, y) = 8y^3 + 12x^2 - 24xy$.

We begin by looking for critical points:

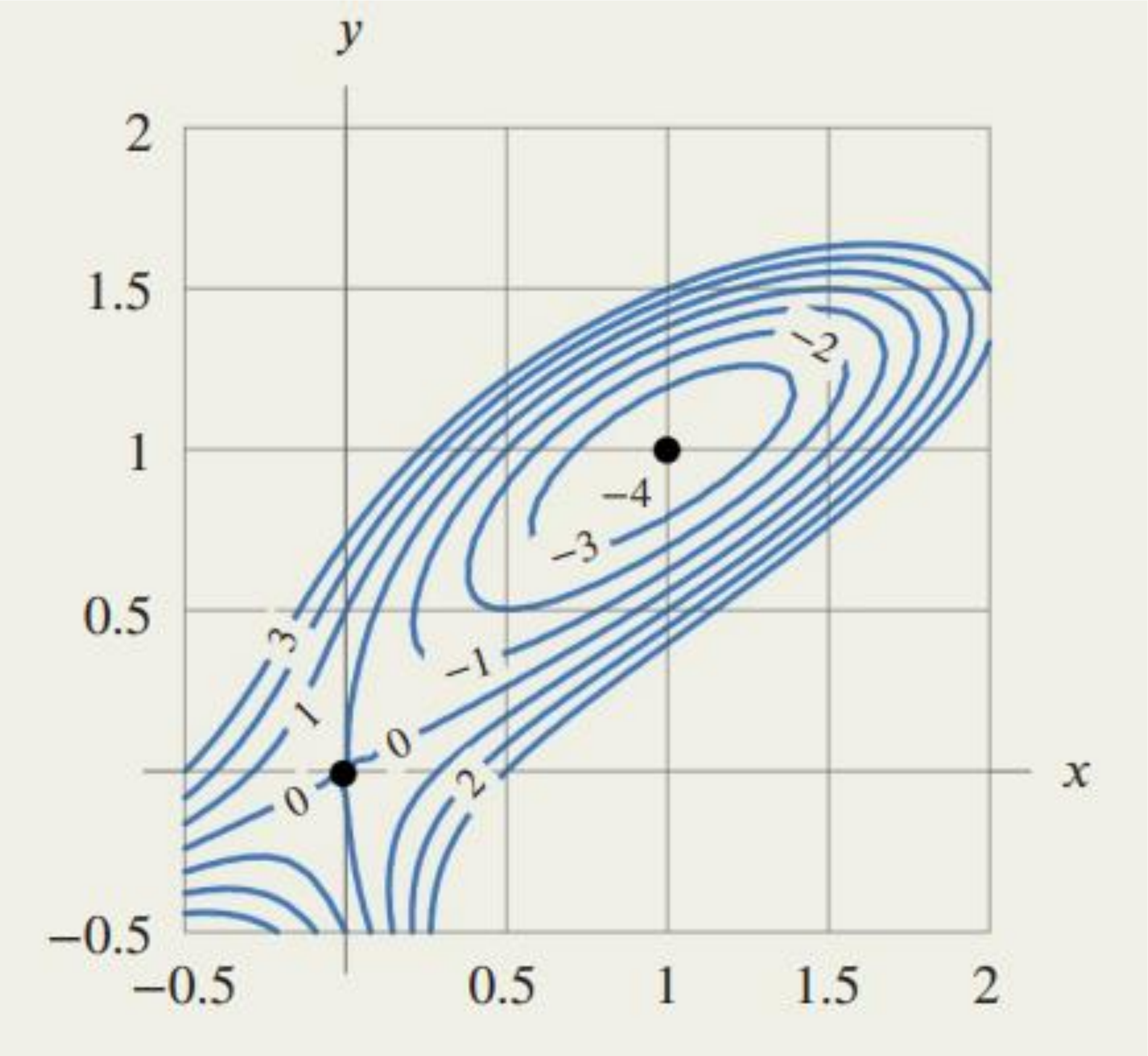
$$f_x(x, y) = 24x - 24y,$$
$$f_y(x, y) = 24y^2 - 24x.$$

Setting these expressions equal to zero gives the system of equations

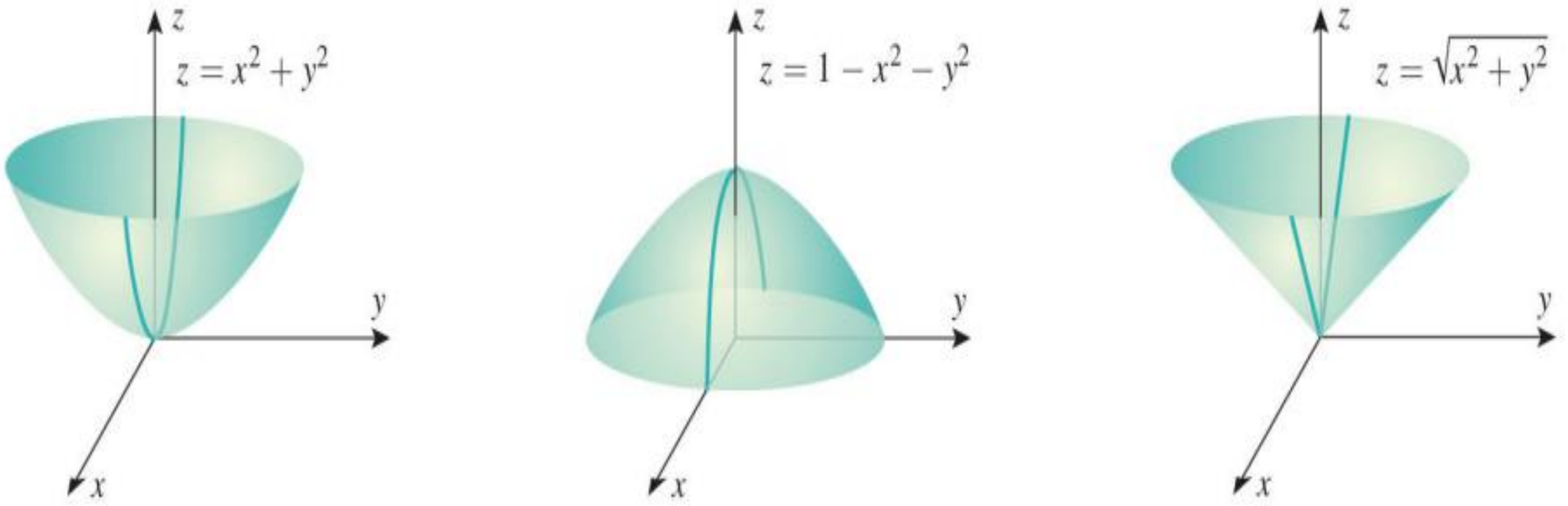
$$x = y, \qquad x = y^2,$$

which has two solutions, $(0, 0)$ and $(1, 1)$. Are these local maxima, local minima or neither? Figure 15.11 shows contours of f near the points. Notice that $f(1, 1) = -4$ and the contours at nearby points have larger function values. This suggests f has a local minimum at $(1, 1)$.

We have $f(0, 0) = 0$ and the contours near $(0, 0)$ show that f takes both positive and negative values nearby. This suggests that $(0, 0)$ is a critical point which is neither a local maximum nor a local minimum.



y
x
2
1.5
1
0.5
−0.5
−0.5
0.5
1
1.5
2
−2
−4
−3
3
−1
1
0
0
2



$f_x(0, 0) = f_y(0, 0) = 0$
relative and absolute min at (0, 0)

(*a*)

$f_x(0, 0) = f_y(0, 0) = 0$
relative and absolute max at (0, 0)

(*b*)

$f_x(0, 0)$ and $f_y(0, 0)$ do not exist
relative and absolute min at (0, 0)

(*c*)

1–2 Locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

1. (a) $f(x, y) = (x - 2)^2 + (y + 1)^2$
(b) $f(x, y) = 1 - x^2 - y^2$
(c) $f(x, y) = x + 2y - 5$

2. (a) $f(x, y) = 1 - (x + 1)^2 - (y - 5)^2$
(b) $f(x, y) = e^{xy}$
(c) $f(x, y) = x^2 - y^2$

$Q1a$: This function is a sum of squares, which means it represents a paraboloid. Since squared terms are always
− negative, the function reaches its **minimum** where both squared terms are zero:

- $(x-2)^2 = 0$ when $x = 2$ and $(y+1)^2 = 0$ when $y = -1$

Thus, the minimum value occurs at $(2, -1)$, where:

$$f(2,-1) = (2-2)^2 + (-1+1)^2 = 0$$

Since squares are always non − negative, $f(x, y) \geq 0$

0 is the absolute minimum.

There is **no absolute maximum** since the function increases indefinitely.

Q2b: Analysis of the Function $f(x, y) = 1 - x^2 - y^2$

This function represents a downward-opening paraboloid, as both x^2 and y^2 are squared terms being subtracted from 1. The highest value occurs where $x^2 + y^2$ is minimized, which happens at $(x, y) = (0,0)$.

Finding the Maximum

At the point (0,0), we calculate the function's value: $f(0,0) = 1 - 0^2 - 0^2 = 1$.

Since squared terms are always non-negative ($x^2, y^2 \geq 0$), it follows that $f(x, y) \leq 1$ for all (x, y). This confirms that $f(0,0) = 1$ is the absolute maximum of the function.

Behavior at Infinity

As $x^2 + y^2 \to \infty$, the function decreases indefinitely:

$f(x, y) \to -\infty$.

This indicates that the function has no absolute minimum, as it keeps decreasing without bound.

Conclusion

- Absolute Maximum: $f(0,0) = 1$ at (0,0).

- No Absolute Minimum (the function decreases indefinitely as $|x|, |y| \to \infty$).

Q1c) Analysis of the Function $f(x, y) = x + 2y - 5$

Step 1: Inspection

This is a linear function, meaning it represents a plane in 3D space. The *equation*

$$f(x, y) = x + 2y - 5$$

has no squared or higher-order terms, so the function does not curve or bend. It is a straight plane, and thus:

- The function either increases or decreases infinitely depending on the direction.

- Since the function is unbounded in all directions, it does not have an absolute maximum or minimum.

Conclusion

In conclusion, there is no maximum or minimum for this function.

THE SECOND PARTIALS TEST

For functions of one variable the second derivative test (Theorem 3.2.4) was used to determine the behavior of a function at a critical point. The following theorem, which is usually proved in advanced calculus, is the analog of that theorem for functions of two variables.

13.8.6 THEOREM (*The Second Partials Test*) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(a) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(b) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(c) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(d) *If $D = 0$, then no conclusion can be drawn.*

▶ Example 3 Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$\begin{aligned} 6x - 2y &= 0 \\ -2x + 2y - 8 &= 0 \end{aligned}$$

Solving these for x and y yields $x = 2, y = 6$ (verify), so $(2, 6)$ is the only critical point. To apply Theorem 13.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

At the point $(2, 6)$ we have

$$D = f_{xx}(2, 6)f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at $(2, 6)$ by part (a) of the second partials test. Figure 13.8.7 shows a graph of f in the vicinity of the relative minimum. ◀

Example 4 Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution. Since

$$f_x(x,y) = 4y - 4x^3$$
$$f_y(x,y) = 4x - 4y^3 \tag{1}$$

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or, equivalently, $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0, x = 1, x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0, y = 1, y = -1$. Thus, the critical points of f are $(0,0), (1,1)$, and $(-1,-1)$.

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
$(0, 0)$	0	0	4	-16
$(1, 1)$	-12	-12	4	128
$(-1, -1)$	-12	-12	4	128

At the points $(1, 1)$ and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At $(0, 0)$ there is a saddle point since $D < 0$. The surface and a contour plot are shown in Figure 13.8.8. ◀

Example 6 Find the local maxima, minima, and saddle points of $f(x, y) = \frac{1}{2}x^2 + 3y^3 + 9y^2 - 3xy + 9y - 9x$.

Solution Setting the partial derivatives of f to zero gives

$$f_x(x, y) = x - 3y - 9 = 0,$$
$$f_y(x, y) = 9y^2 + 18y - 3x + 9 = 0.$$

Eliminating x gives $9y^2 + 9y - 18 = 0$, with solutions $y = -2$ and $y = 1$. The corresponding values of x are $x = 3$ and $x = 12$, so the critical points of f are $(3, -2)$ and $(12, 1)$. The discriminant is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (1)(18y + 18) - (-3)^2 = 18y + 9.$$

Since $D(3, -2) = -36 + 9 < 0$, we know that $(3, -2)$ is a saddle point of f. Since $D(12, 1) = 18 + 9 > 0$ and $f_{xx}(12, 1) = 1 > 0$, we know that $(12, 1)$ is a local minimum of f.

9–20 Locate all relative maxima, relative minima, and saddle points, if any.

9. $f(x, y) = y^2 + xy + 3y + 2x + 3$

10. $f(x, y) = x^2 + xy - 2y - 2x + 1$

11. $f(x, y) = x^2 + xy + y^2 - 3x$

12. $f(x, y) = xy - x^3 - y^2$

13. $f(x, y) = x^2 + y^2 + \dfrac{2}{xy}$

14. $f(x, y) = xe^y$

15. $f(x, y) = x^2 + y - e^y$

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

17. $f(x, y) = e^x \sin y$

18. $f(x, y) = y \sin x$

19. $f(x, y) = e^{-(x^2+y^2+2x)}$

Solution:

Exercise Set 13.8

1. **(a)** Minimum at $(2, -1)$, no maxima. **(b)** Maximum at $(0, 0)$, no minima. **(c)** No maxima or minima.

2. **(a)** Maximum at $(-1, 5)$, no minima. **(b)** No maxima or minima. **(c)** No maxima or minima.

3. $f(x, y) = (x - 3)^2 + (y + 2)^2$, minimum at $(3, -2)$, no maxima.

4. $f(x, y) = -(x + 1)^2 - 2(y - 1)^2 + 4$, maximum at $(-1, 1)$, no minima.

5. $f_x = 6x + 2y = 0$, $f_y = 2x + 2y = 0$; critical point (0,0); $D = 8 > 0$ and $f_{xx} = 6 > 0$ at (0,0), relative minimum.

6. $f_x = 3x^2 - 3y = 0$, $f_y = -3x - 3y^2 = 0$; critical points (0,0) and $(-1, 1)$; $D = -9 < 0$ at (0,0), saddle point; $D = 27 > 0$ and $f_{xx} = -6 < 0$ at $(-1, 1)$, relative maximum.

7. $f_x = 2x - 2xy = 0$, $f_y = 4y - x^2 = 0$; critical points (0,0) and $(\pm 2, 1)$; $D = 8 > 0$ and $f_{xx} = 2 > 0$ at (0,0), relative minimum; $D = -16 < 0$ at $(\pm 2, 1)$, saddle points.

8. $f_x = 3x^2 - 3 = 0$, $f_y = 3y^2 - 3 = 0$; critical points $(-1, \pm 1)$ and $(1, \pm 1)$; $D = -36 < 0$ at $(-1, 1)$ and $(1, -1)$, saddle points; $D = 36 > 0$ and $f_{xx} = 6 > 0$ at (1,1), relative minimum; $D = 36 > 0$ and $f_{xx} = -36 < 0$ at $(-1, -1)$, relative maximum.

9. $f_x = y + 2 = 0$, $f_y = 2y + x + 3 = 0$; critical point $(1, -2)$; $D = -1 < 0$ at $(1, -2)$, saddle point.

10. $f_x = 2x + y - 2 = 0$, $f_y = x - 2 = 0$; critical point $(2, -2)$; $D = -1 < 0$ at $(2, -2)$, saddle point.

11. $f_x = 2x + y - 3 = 0$, $f_y = x + 2y = 0$; critical point $(2, -1)$; $D = 3 > 0$ and $f_{xx} = 2 > 0$ at $(2, -1)$, relative minimum.

12. $f_x = y - 3x^2 = 0$, $f_y = x - 2y = 0$; critical points (0,0) and $(1/6, 1/12)$; $D = -1 < 0$ at (0,0), saddle point; $D = 1 > 0$ and $f_{xx} = -1 < 0$ at $(1/6, 1/12)$, relative maximum.

13. $f_x = 2x - 2/\left(x^2 y\right) = 0$, $f_y = 2y - 2/\left(xy^2\right) = 0$; critical points $(-1, -1)$ and $(1, 1)$; $D = 32 > 0$ and $f_{xx} = 6 > 0$ at $(-1, -1)$ and $(1, 1)$, relative minima.

14. $f_x = e^y = 0$ is impossible, no critical points.

15. $f_x = 2x = 0$, $f_y = 1 - e^y = 0$; critical point $(0, 0)$; $D = -2 < 0$ at $(0, 0)$, saddle point.

16. $f_x = y - 2/x^2 = 0$, $f_y = x - 4/y^2 = 0$; critical point (1,2); $D = 3 > 0$ and $f_{xx} = 4 > 0$ at $(1, 2)$, relative minimum.

17. $f_x = e^x \sin y = 0$, $f_y = e^x \cos y = 0$, $\sin y = \cos y = 0$ is impossible, no critical points.

18. $f_x = y \cos x = 0$, $f_y = \sin x = 0$; $\sin x = 0$ if $x = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$ and $\cos x \neq 0$ for these values of x so $y = 0$; critical points $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$; $D = -1 < 0$ at $(n\pi, 0)$, saddle points.

19. $f_x = -2(x+1)e^{-(x^2+y^2+2x)} = 0$, $f_y = -2ye^{-(x^2+y^2+2x)} = 0$; critical point $(-1, 0)$; $D = 4e^2 > 0$ and $f_{xx} = -2e < 0$ at $(-1, 0)$, relative maximum.

20. $f_x = y - a^3/x^2 = 0$, $f_y = x - b^3/y^2 = 0$; critical point $\left(a^2/b, b^2/a\right)$; if $ab > 0$ then $D = 3 > 0$ and $f_{xx} = 2b^3/a^3 > 0$ at $\left(a^2/b, b^2/a\right)$, relative minimum; if $ab < 0$ then $D = 3 > 0$ and $f_{xx} = 2b^3/a^3 < 0$ at $\left(a^2/b, b^2/a\right)$, relative maximum.

A model for the yield Y of an agricultural crop as a function of the nitrogen level N and phosphorus level P in the soil (measured in appropriate units) is

$$\boldsymbol{Y(N,P) = kNPe^{(-N-P)}}$$

where k is a positive constant. What levels of nitrogen and phosphorus result in the best yield?

A missile has a guidance device which is sensitive to both temperature, t ◦C, and humidity, h. The range in km over which the missile can be controlled is given by

$$\boldsymbol{Range = 27,800 - 5t^2 - 6ht - 3h^2 + 400t + 300h}.$$

What are the optimal atmospheric conditions for controlling the missile?

Suppose the cost function of manufacturing cost for a certain product be approximated by

$$\boldsymbol{C(x, y) = 3x^2 + y^2 - x - y - 3xy + 100},$$

where *x* is the cost of labor per hour and *y* is the cost of materials per unit. Find values of *x* and *y* that minimize the cost function. Find the minimum cost

A flat metal plate is located on a coordinate plane. The temperature of the plate, in degrees Fahrenheit, at point is given by

$$\boldsymbol{T(x, y) = x^2 + 2y^2 - 8x + 4y}.$$

Find the minimum temperature and where it occurs. Is there a maximum temperature?