# 13.4 DIFFERENTIABILITY, DIFFERENTIALS, AND LOCAL LINEARITY

*In this section we will extend the notion of differentiability to functions of two or three variables. Our definition of differentiability will be based on the idea that a function is differentiable at a point provided it can be very closely approximated by a linear function near that point. In the process, we will expand the concept of a "differential" to functions of more than one variable and define the "local linear approximation" of a function.*

## DIFFERENTIABILITY

Recall that a function $f$ of one variable is called differentiable at $x_0$ if it has a derivative at $x_0$, that is, if the limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{1}$$

exists. As a consequence of (1) a differentiable function enjoys a number of other important properties:

- The graph of $y = f(x)$ has a nonvertical tangent line at the point $(x_0, f(x_0))$;
- $f$ may be closely approximated by a linear function near $x_0$ (Section 2.9);
- $f$ is continuous at $x_0$.

**13.4.1 DEFINITION** A function $f$ of two variables is said to be ***differentiable*** at $(x_0, y_0)$ provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \tag{4}$$

**13.4.2 DEFINITION** A function $f$ of three variables is said to be ***differentiable*** at $(x_0, y_0, z_0)$ provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z) \to (0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0 \quad (6)$$

## DIFFERENTIABILITY AND CONTINUITY

Recall that we want a function to be continuous at every point at which it is differentiable. The next result shows this to be the case.

**13.4.3 THEOREM** *If a function is differentiable at a point, then it is continuous at that point.*

**13.4.4 THEOREM** *If all first-order partial derivatives of $f$ exist and are continuous at a point, then $f$ is differentiable at that point.*

For example, consider the function

$$f(x, y, z) = x + yz$$

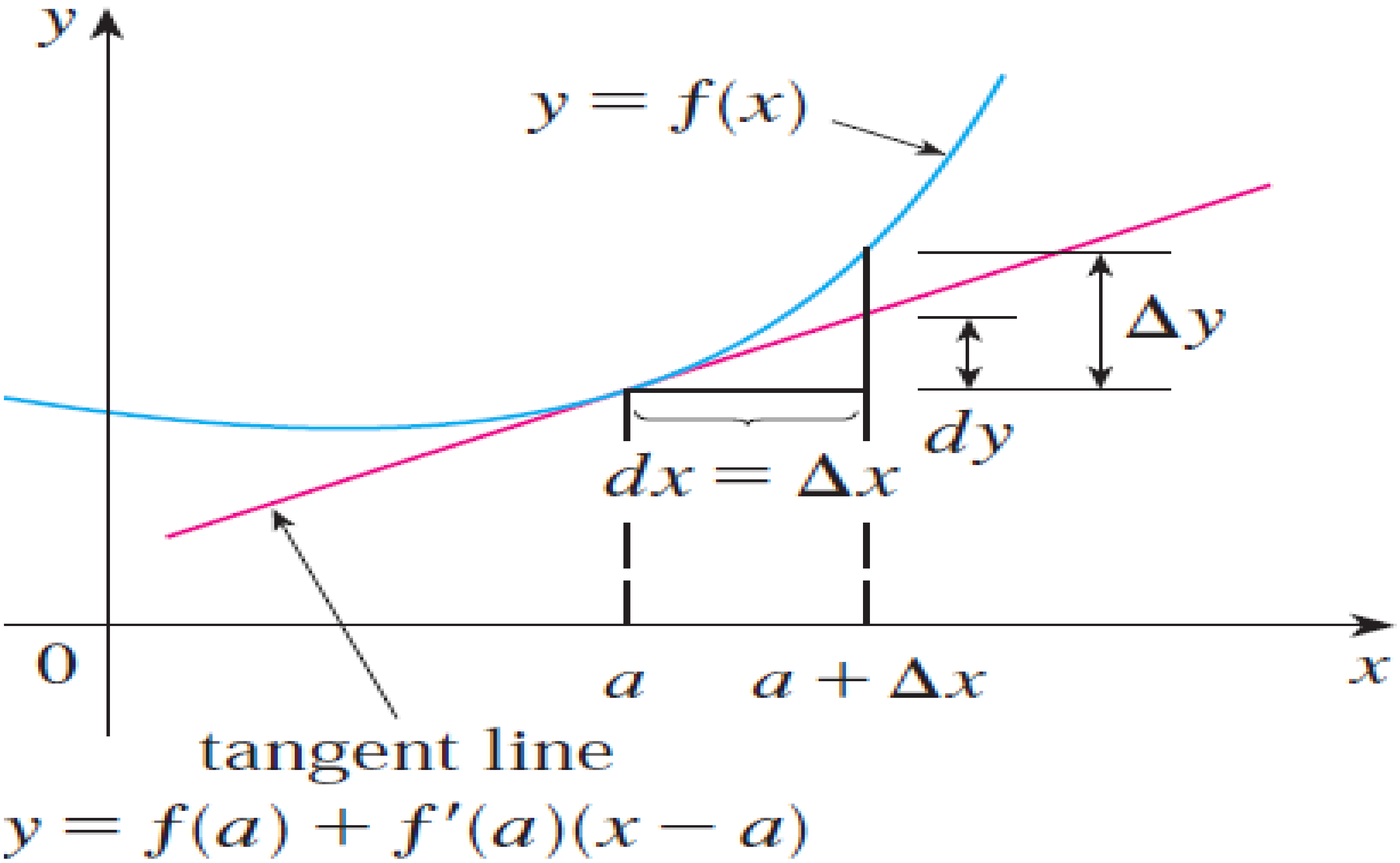
Since $f_x(x, y, z) = 1$, $f_y(x, y, z) = z$, and $f_z(x, y, z) = y$ are defined and continuous everywhere, we conclude from Theorem 13.4.4 that $f$ is differentiable everywhere.

# TOTAL DIFFERENTIAL

## Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential $dx$ to be an independent variable; that is, $dx$ can be given the value of any real number. The differential of $y$ is then defined as

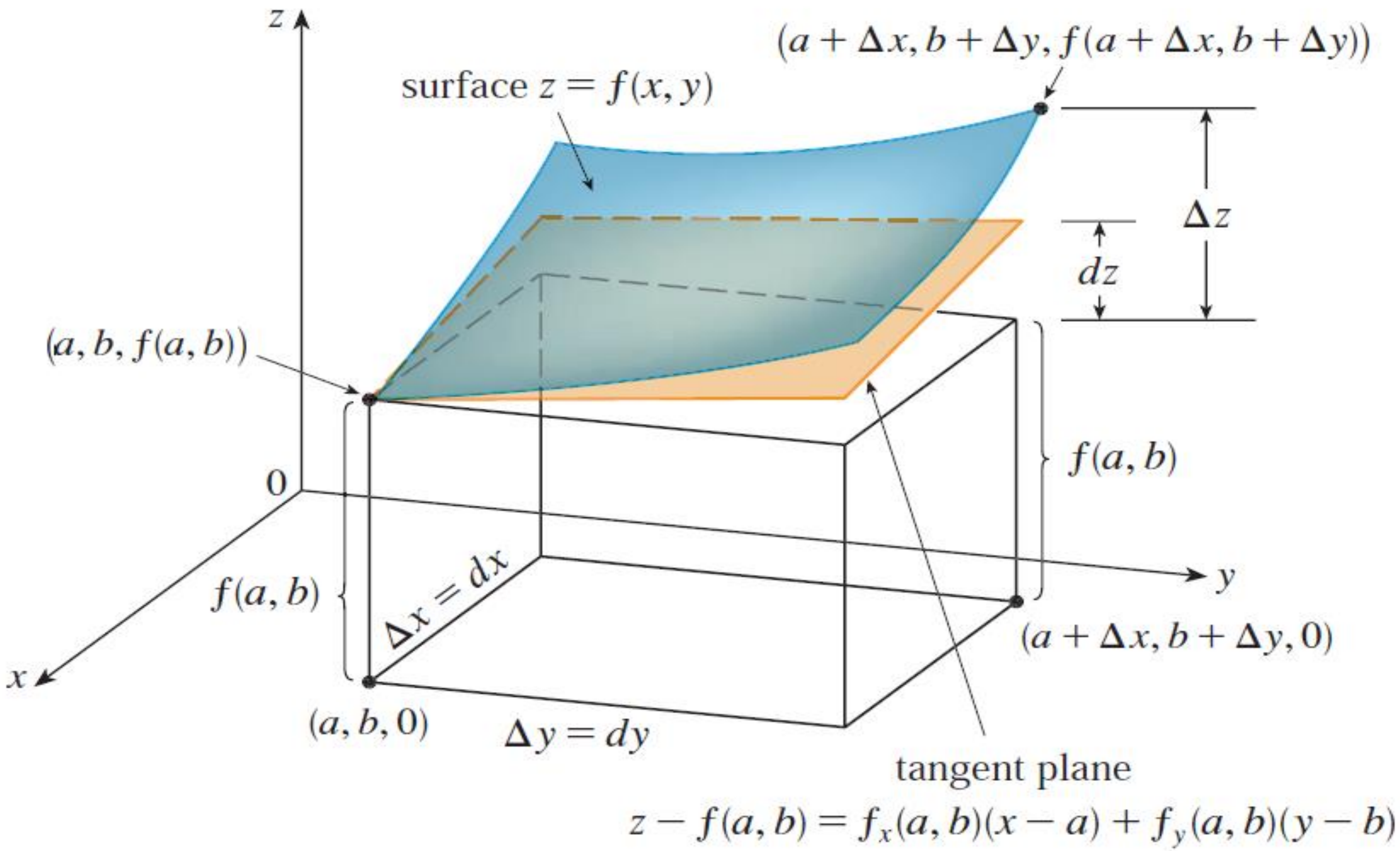
$$dy = f'(x)\,dx \tag{9}$$



y
y = f(x)
Δy
dy
dx = Δx
0
a
a + Δx
x
tangent line
y = f(a) + f′(a)(x − a)

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** $dx$ and $dy$ to be independent variables; that is, they can be given any values. Then the **differential** $dz$, also called the **total differential**, is defined by

$$\boxed{10} \qquad \boxed{dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy}$$



$z$
$(a + \Delta x, b + \Delta y, f(a + \Delta x, b + \Delta y))$
surface $z = f(x, y)$
$\Delta z$
$dz$
$(a, b, f(a, b))$
$f(a, b)$
0
$y$
$f(a, b)$
$\Delta x = dx$
$(a + \Delta x, b + \Delta y, 0)$
$x$
$(a, b, 0)$
$\Delta y = dy$
tangent plane
$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential $dz$.
(b) If $x$ changes from 2 to 2.05 and $y$ changes from 3 to 2.96, compare the values of $\Delta z$ and $dz$.

SOLUTION

(a) Definition 10 gives

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of $z$ is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

Notice that $\Delta z \approx dz$ but $dz$ is easier to compute.

**Q1** The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

SOLUTION The volume $V$ of a cone with base radius $r$ and height $h$ is $V = \pi r^2 h/3$. So the differential of $V$ is

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh = \frac{2\pi rh}{3}\,dr + \frac{\pi r^2}{3}\,dh$$

Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about $20\pi \text{ cm}^3 \approx 63 \text{ cm}^3$.

**Q2** The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

**SOLUTION** If the dimensions of the box are $x$, $y$, and $z$, its volume is $V = xyz$ and so

$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz = yz\,dx + xz\,dy + xy\,dz$$

in the volume, we therefore use $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:

$$\Delta V \approx dV = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of approximately 1980 $cm^3$ in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.

## DIFFERENTIALS

As with the one-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

for a function of two variables and the approximation

$$\Delta f \approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z \tag{7}$$

for a function of three variables have a convenient formulation in the language of differentials. If $z = f(x, y)$ is differentiable at a point $(x_0, y_0)$, we let

$$dz = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy \tag{8}$$

denote a new function with dependent variable $dz$ and independent variables $dx$ and $dy$. We refer to this function (also denoted $df$) as the ***total differential of z*** at $(x_0, y_0)$ or as the ***total differential of f*** at $(x_0, y_0)$. Similarly, for a function $w = f(x, y, z)$ of three variables we have the ***total differential of w*** at $(x_0, y_0, z_0)$,

$$dw = f_x(x_0, y_0, z_0)\, dx + f_y(x_0, y_0, z_0)\, dy + f_z(x_0, y_0, z_0)\, dz \tag{9}$$

which is also referred to as the ***total differential of f*** at $(x_0, y_0, z_0)$. It is common practice to omit the subscripts and write Equations (8) and (9) as

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy \tag{10}$$

and

$$dw = f_x(x, y, z)\, dx + f_y(x, y, z)\, dy + f_z(x, y, z)\, dz \tag{11}$$

In the two-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

can be written in the form

$$\Delta f \approx df \tag{12}$$

for $dx = \Delta x$ and $dy = \Delta y$. Equivalently, we can write approximation (12) as

$$\Delta z \approx dz \tag{13}$$

In other words, we can estimate the change $\Delta z$ in $z$ by the value of the differential $dz$ where $dx$ is the change in $x$ and $dy$ is the change in $y$. Furthermore, it follows from (4) that if $\Delta x$ and $\Delta y$ are close to 0, then the magnitude of the error in approximation (13) will be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between $(x_0, y_0)$ and $(x_0 + \Delta x, y_0 + \Delta y)$.

▶ **Example 2** Use (13) to approximate the change in $z = xy^2$ from its value at $(0.5, 1.0)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in this approximation with the distance between the points $(0.5, 1.0)$ and $(0.503, 1.004)$.

***Solution.*** For $z = xy^2$ we have $dz = y^2\,dx + 2xy\,dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$, $dx = \Delta x = 0.503 - 0.5 = 0.003$, and $dy = \Delta y = 1.004 - 1.0 = 0.004$ yields

$$dz = 1.0^2(0.003) + 2(0.5)(1.0)(0.004) = 0.007$$

Since $z = 0.5$ at $(x, y) = (0.5, 1.0)$ and $z = 0.507032048$ at $(x, y) = (0.503, 1.004)$, we have

$$\Delta z = 0.507032048 - 0.5 = 0.007032048$$

and the error in approximating $\Delta z$ by $dz$ has magnitude

$$|dz - \Delta z| = |0.007 - 0.007032048| = 0.000032048$$

Since the distance between $(0.5, 1.0)$ and $(0.503, 1.004) = (0.5 + \Delta x, 1.0 + \Delta y)$ is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = \sqrt{0.000025} = 0.005$$

we have

$$\frac{|dz - \Delta z|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{0.000032048}{0.005} = 0.0064096 < \frac{1}{150}$$

Thus, the magnitude of the error in our approximation is less than $\frac{1}{150}$ of the distance between the two points. ◀

▶ **Example 3** The length, width, and height of a rectangular box are measured with an error of at most 5%. Use a total differential to estimate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

***Solution.*** The diagonal $D$ of a box with length $x$, width $y$, and height $z$ is given by

$$D = \sqrt{x^2 + y^2 + z^2}$$

Let $x_0$, $y_0$, $z_0$, and $D_0 = \sqrt{x_0^2 + y_0^2 + z_0^2}$ denote the actual values of the length, width, height, and diagonal of the box. The total differential $dD$ of $D$ at $(x_0, y_0, z_0)$ is given by

$$dD = \frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dx + \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dy + \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}\,dz$$

If $x$, $y$, $z$, and $D = \sqrt{x^2 + y^2 + z^2}$ are the measured and computed values of the length, width, height, and diagonal, respectively, then

$$\Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta z = z - z_0$$

and

$$\left|\frac{\Delta x}{x_0}\right| \le 0.05, \quad \left|\frac{\Delta y}{y_0}\right| \le 0.05, \quad \left|\frac{\Delta z}{z_0}\right| \le 0.05$$

We are seeking an estimate for the maximum size of $\Delta D / D_0$. With the aid of Equation (11) we have

$$\begin{aligned}\frac{\Delta D}{D_0} \approx \frac{dD}{D_0} &= \frac{1}{x_0^2 + y_0^2 + z_0^2}[x_0\Delta x + y_0\Delta y + z_0\Delta z] \\ &= \frac{1}{x_0^2 + y_0^2 + z_0^2}\left[x_0^2\frac{\Delta x}{x_0} + y_0^2\frac{\Delta y}{y_0} + z_0^2\frac{\Delta z}{z_0}\right]\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{dD}{D_0}\right| &= \frac{1}{x_0^2 + y_0^2 + z_0^2}\left|x_0^2\frac{\Delta x}{x_0} + y_0^2\frac{\Delta y}{y_0} + z_0^2\frac{\Delta z}{z_0}\right| \\ &\le \frac{1}{x_0^2 + y_0^2 + z_0^2}\left(x_0^2\left|\frac{\Delta x}{x_0}\right| + y_0^2\left|\frac{\Delta y}{y_0}\right| + z_0^2\left|\frac{\Delta z}{z_0}\right|\right) \\ &\le \frac{1}{x_0^2 + y_0^2 + z_0^2}\left(x_0^2(0.05) + y_0^2(0.05) + z_0^2(0.05)\right) = 0.05\end{aligned}$$

we estimate the maximum percentage error in $D$ to be 5%. ◀

## LOCAL LINEAR APPROXIMATIONS

We now show that if a function $f$ is differentiable at a point, then it can be very closely approximated by a linear function near that point. For example, suppose that $f(x, y)$ is differentiable at the point $(x_0, y_0)$. Then approximation (3) can be written in the form

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

If we let $x = x_0 + \Delta x$ and $y = x_0 + \Delta y$, this approximation becomes

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{14}$$

which yields a linear approximation of $f(x, y)$. Since the error in this approximation is equal to the error in approximation (3), we conclude that for $(x, y)$ close to $(x_0, y_0)$, the error in (14) will be much smaller than the distance between these two points. When $f(x, y)$ is differentiable at $(x_0, y_0)$ we let

$$\boxed{L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)} \tag{15}$$

# TANGENT PLANE AND LINEAR APROXIMATION

## Tangent Planes

$\boxed{2}$ Suppose $f$ has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

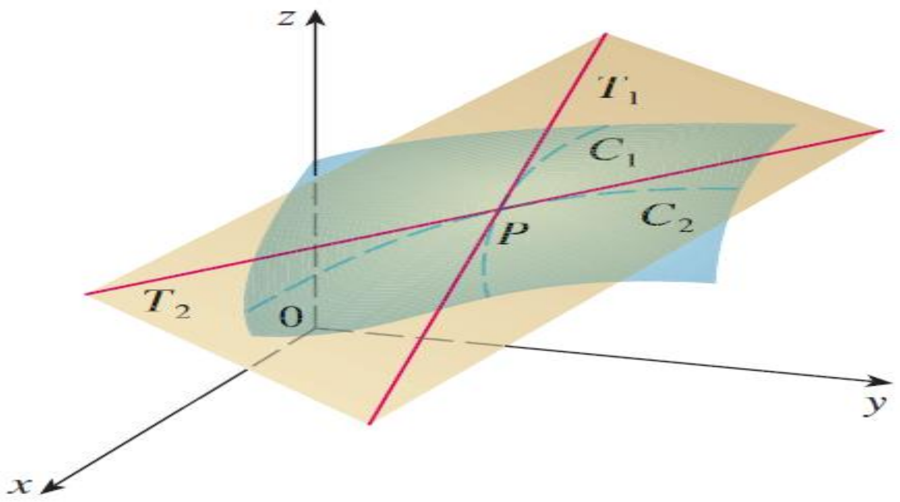



**FIGURE 1**
The tangent plane contains the tangent lines $T_1$ and $T_2$.

**1–6** Find an equation of the tangent plane to the given surface at the specified point.

**1.** $z = 3y^2 - 2x^2 + x, \quad (2, -1, -3)$

**2.** $z = 3(x - 1)^2 + 2(y + 3)^2 + 7, \quad (2, -2, 12)$

**3.** $z = \sqrt{xy}, \quad (1, 1, 1)$

**4.** $z = xe^{xy}, \quad (2, 0, 2)$

**5.** $z = x\sin(x + y), \quad (-1, 1, 0)$

**6.** $z = \ln(x - 2y), \quad (3, 1, 0)$

**linearization** of $f$ at $(a, b)$ and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation**

▶ **Example 4** Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ at the point $(3, 4)$. Compare the error in approximating

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$$

by $L(3.04, 3.98)$ with the distance between the points $(3, 4)$ and $(3.04, 3.98)$.

***Solution.*** We have

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

with $f_x(3,4) = \frac{3}{5}$ and $f_y(3,4) = \frac{4}{5}$. Therefore, the local linear approximation to $f$ at $(3,4)$ is given by

$$L(x,y) = 5 + \tfrac{3}{5}(x-3) + \tfrac{4}{5}(y-4)$$

Consequently,

$$f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \tfrac{3}{5}(0.04) + \tfrac{4}{5}(-0.02) = 5.008$$

Since

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$$

the error in the approximation is about $5.00819 - 5.008 = 0.00019$. This is less than $\frac{1}{200}$ of the distance

$$\sqrt{(3.04-3)^2 + (3.98-4)^2} \approx 0.045$$

between the points $(3,4)$ and $(3.04, 3.98)$. ◀

For a function $f(x, y, z)$ that is differentiable at $(x_0, y_0, z_0)$, the local linear approximation is

$$\begin{aligned} L(x, y, z) = {} & f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) \\ & + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \end{aligned} \tag{16}$$

**9–20** Compute the differential $dz$ or $dw$ of the function.

**9.** $z = 7x - 2y$ **10.** $z = e^{xy}$ **11.** $z = x^3y^2$

**12.** $z = 5x^2y^5 - 2x + 4y + 7$

**13.** $z = \tan^{-1} xy$ **14.** $z = e^{-3x} \cos 6y$

**15.** $w = 8x - 3y + 4z$ **16.** $w = e^{xyz}$

**17.** $w = x^3y^2z$

**18.** $w = 4x^2y^3z^7 - 3xy + z + 5$

**19.** $w = \tan^{-1}(xyz)$ **20.** $w = \sqrt{x} + \sqrt{y} + \sqrt{z}$

**21–26** Use a total differential to approximate the change in the values of $f$ from $P$ to $Q$. Compare your estimate with the actual change in $f$.

**21.** $f(x, y) = x^2 + 2xy - 4x$; $P(1, 2)$, $Q(1.01, 2.04)$

**22.** $f(x, y) = x^{1/3}y^{1/2}$; $P(8, 9)$, $Q(7.78, 9.03)$

**23.** $f(x, y) = \dfrac{x + y}{xy}$; $P(-1, -2)$, $Q(-1.02, -2.04)$

**24.** $f(x, y) = \ln\sqrt{1 + xy}$; $P(0, 2)$, $Q(-0.09, 1.98)$

**25.** $f(x, y, z) = 2xy^2z^3$; $P(1, -1, 2)$, $Q(0.99, -1.02, 2.02)$

**26.** $f(x, y, z) = \dfrac{xyz}{x + y + z}$; $P(-1, -2, 4)$, $Q(-1.04, -1.98, 3.97)$

differential.

**33–40** (a) Find the local linear approximation $L$ to the specified function $f$ at the designated point $P$. (b) Compare the error in approximating $f$ by $L$ at the specified point $Q$ with the distance between $P$ and $Q$.

**33.** $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$; $P(4, 3)$, $Q(3.92, 3.01)$

**34.** $f(x, y) = x^{0.5}y^{0.3}$; $P(1, 1)$, $Q(1.05, 0.97)$

**35.** $f(x, y) = x \sin y$; $P(0, 0)$, $Q(0.003, 0.004)$

**36.** $f(x, y) = \ln xy$; $P(1, 2)$, $Q(1.01, 2.02)$

**37.** $f(x, y, z) = xyz$; $P(1, 2, 3)$, $Q(1.001, 2.002, 3.003)$

**38.** $f(x, y, z) = \dfrac{x + y}{y + z}$; $P(-1, 1, 1), Q(-0.99, 0.99, 1.01)$

**39.** $f(x, y, z) = xe^{yz}$; $P(1, -1, -1), Q(0.99, -1.01, -0.99)$

**40.** $f(x, y, z) = \ln(x + yz)$; $P(2, 1, -1)$, $Q(2.02, 0.97, -1.01)$

**Problem 4.** If $z = f(x, y)$ and $z = 2x^3 \sin 2y$ find the rate of change of $z$, correct to 4 significant figures, when $x$ is 2 units and $y$ is $\pi/6$ radians and when $x$ is increasing at 4 units/s and $y$ is decreasing at 0.5 units/s.

Using equation (2), the rate of change of $z$,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Since $z = 2x^3 \sin 2y$, then

$$\frac{\partial z}{\partial x} = 6x^2 \sin 2y \text{ and } \frac{\partial z}{\partial y} = 4x^3 \cos 2y$$

Since $x$ is increasing at 4 units/s, $\frac{dx}{dt} = +4$

and since $y$ is decreasing at 0.5 units/s, $\frac{dy}{dt} = -0.5$

Hence $\frac{dz}{dt} = (6x^2 \sin 2y)(+4) + (4x^3 \cos 2y)(-0.5)$

$= 24x^2 \sin 2y - 2x^3 \cos 2y$

When $x = 2$ units and $y = \frac{\pi}{6}$ radians, then

$$\frac{dz}{dt} = 24(2)^2 \sin[2(\pi/6)] - 2(2)^3 \cos[2(\pi/6)]$$

$$= 83.138 - 8.0$$

Hence the rate of change of $z$, $\mathbf{\frac{dz}{dt} = 75.14}$ **units/s**, correct to 4 significant figures.

3. Find the rate of change of $k$, correct to 4 significant figures, given the following data: $k = f(a, b, c)$; $k = 2b \ln a + c^2 e^a$; $a$ is increasing at 2 cm/s; $b$ is decreasing at 3 cm/s; $c$ is decreasing at 1 cm/s; $a = 1.5$ cm, $b = 6$ cm and $c = 8$ cm. [515.5 cm/s]

**Problem 9.** Modulus of rigidity $G=(R^4\theta)/L$, where $R$ is the radius, $\theta$ the angle of twist and $L$ the length. Determine the approximate percentage error in $G$ when $R$ is increased by 2%, $\theta$ is reduced by 5% and $L$ is increased by 4%.

Using $\delta G \approx \dfrac{\partial G}{\partial R}\delta R + \dfrac{\partial G}{\partial \theta}\delta\theta + \dfrac{\partial G}{\partial L}\delta L$

Since $G = \dfrac{R^4\theta}{L}, \dfrac{\partial G}{\partial R} = \dfrac{4R^3\theta}{L}, \dfrac{\partial G}{\partial \theta} = \dfrac{R^4}{L}$

and $\dfrac{\partial G}{\partial L} = \dfrac{-R^4\theta}{L^2}$

Since $R$ is increased by 2%, $\delta R = \dfrac{2}{100}R = 0.02R$

Similarly, $\delta\theta = -0.05\theta$ and $\delta L = 0.04L$

$$\text{Hence } \delta G \approx \left(\frac{4R^3\theta}{L}\right)(0.02R) + \left(\frac{R^4}{L}\right)(-0.05\theta) + \left(-\frac{R^4\theta}{L^2}\right)(0.04L)$$

$$\approx \frac{R^4\theta}{L}[0.08 - 0.05 - 0.04] \approx -0.01\frac{R^4\theta}{L},$$

i.e. $\delta G \approx -\dfrac{1}{100}G$

**Hence the approximate percentage error in $G$ is a 1% decrease.**

**Problem 11.** The time of oscillation $t$ of a pendulum is given by $t = 2\pi\sqrt{\frac{l}{g}}$. Determine the approximate percentage error in $t$ when $l$ has an error of 0.2% too large and $g$ 0.1% too small.

Using equation (3), the approximate change in $t$,

$$\delta t \approx \frac{\partial t}{\partial l}\delta l + \frac{\partial t}{\partial g}\delta g$$

Since $t = 2\pi\sqrt{\frac{l}{g}}$, $\frac{\partial t}{\partial l} = \frac{\pi}{\sqrt{lg}}$

and $\frac{\partial t}{\partial g} = -\pi\sqrt{\frac{l}{g^3}}$ (from Problem 6, Chapter 34)

$$\delta l = \frac{0.2}{100}l = 0.002l \text{ and } \delta g = -0.001g$$

$$\text{hence } \delta t \approx \frac{\pi}{\sqrt{lg}}(0.002l) + -\pi\sqrt{\frac{l}{g^3}}(-0.001\,g)$$

$$\approx 0.002\pi\sqrt{\frac{l}{g}} + 0.001\pi\sqrt{\frac{l}{g}}$$

$$\approx (0.001)\left[2\pi\sqrt{\frac{l}{g}}\right] + 0.0005\left[2\pi\sqrt{\frac{l}{g}}\right]$$

$$\approx 0.0015t \approx \frac{0.15}{100}t$$

**Hence the approximate error in $t$ is a 0.15% increase.**

3. $f_r = \dfrac{1}{2\pi\sqrt{LC}}$ represents the resonant frequency of a series connected circuit containing inductance $L$ and capacitance $C$. Determine the approximate percentage change in $f_r$ when $L$ is decreased by 3% and $C$ is increased by 5%. [−1%]

4. The second moment of area of a rectangle about its centroid parallel to side $b$ is given by $I = bd^3/12$. If $b$ and $d$ are measured as 15 cm and 6 cm respectively and the measurement errors are +12 mm in $b$ and −1.5 mm in $d$, find the error in the calculated value of $I$. [$+1.35\,\mathrm{cm}^4$]