14

PARTIAL DERIVATIVES

PARTIAL DERIVATIVES

So far, we have dealt with the calculus of functions of a single variable.

However, in the real world, physical quantities often depend on two or more variables.

So, in this chapter, we:

- Turn our attention to functions of several variables.
- Extend the basic ideas of differential calculus to such functions.

13.1
Functions of Several Variables

In this section, we will learn about:

Functions of two or more variables and how to produce their graphs.

FUNCTIONS OF SEVERAL VARIABLES

In this section, we study functions of two or more variables from four points of view:

- Verbally (a description in words)
- Numerically (a table of values)
- Algebraically (an explicit formula)
- Visually (a graph or level curves)

FUNCTIONS OF TWO VARIABLES

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.

- We can think of T as being a function of the two variables x and y, or as a function of the pair (x, y).
- We indicate this functional dependence by writing:

$$T = f(x, y)$$

The volume V of a circular cylinder depends on its radius r and its height h.

- In fact, we know that $V = \pi r^2 h$.
- We say that V is a function of r and h.
- We write $V(r, h) = \pi r^2 h$.

FUNCTION OF TWO VARIABLES

A function *f* of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by (x, y).

- The set D is the domain of *f.*
- Its range is the set of values that *f* takes on, that is,

$$\{f(x, y) \mid (x, y) \in D\}$$

FUNCTIONS OF TWO VARIABLES

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y).

- The variables x and y are independent variables.
- z is the dependent variable.
- Compare this with the notation $y = f(x)$ for functions of a single variable.

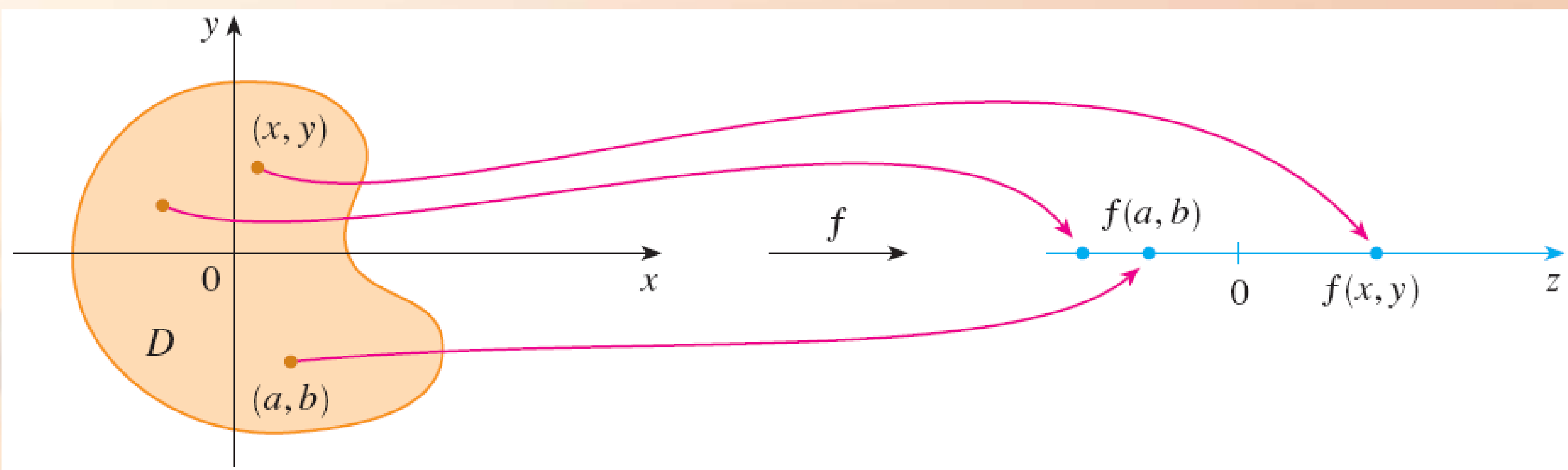
FUNCTIONS OF TWO VARIABLES

A function of two variables is just a function whose:

- Domain is a subset of R^2
- Range is a subset of R

FUNCTIONS OF TWO VARIABLES

One way of visualizing such a function is by means of an arrow diagram, where the domain *D* is represented as a subset of the *xy*-plane.



FUNCTIONS OF TWO VARIABLES

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be:

- The set of all pairs (x, y) for which the given expression is a well-defined real number.

 Example 1

For each of the following functions, evaluate $f(3, 2)$ and find the domain.

a. $$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$$

b. $$f(x, y) = x \ln(y^2 - x)$$

FUNCTIONS OF TWO VARIABLES **Example 1 a**

$$f(3,2)=\frac{\sqrt{3+2+1}}{3-1}=\frac{\sqrt{6}}{2}$$

- The expression for *f* makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative.

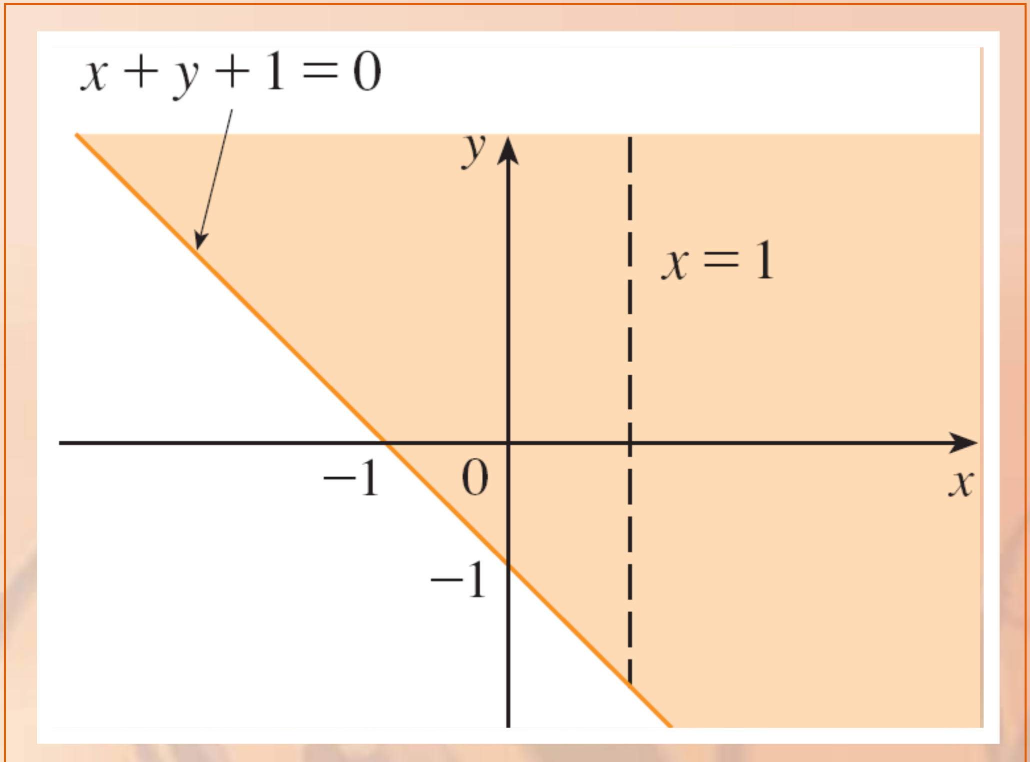
- So, the domain of *f* is:

$$D=\{(x,y)\mid x+y+1\geq 0,\ x\neq 1\}$$

 Example 1 a

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$

- $x \neq 1$ means that the points on the line $x = 1$ must be excluded from the domain.



 Example 1 b

$$f(3, 2) = 3 \ln(2^2 - 3)$$

$$= 3 \ln 1$$

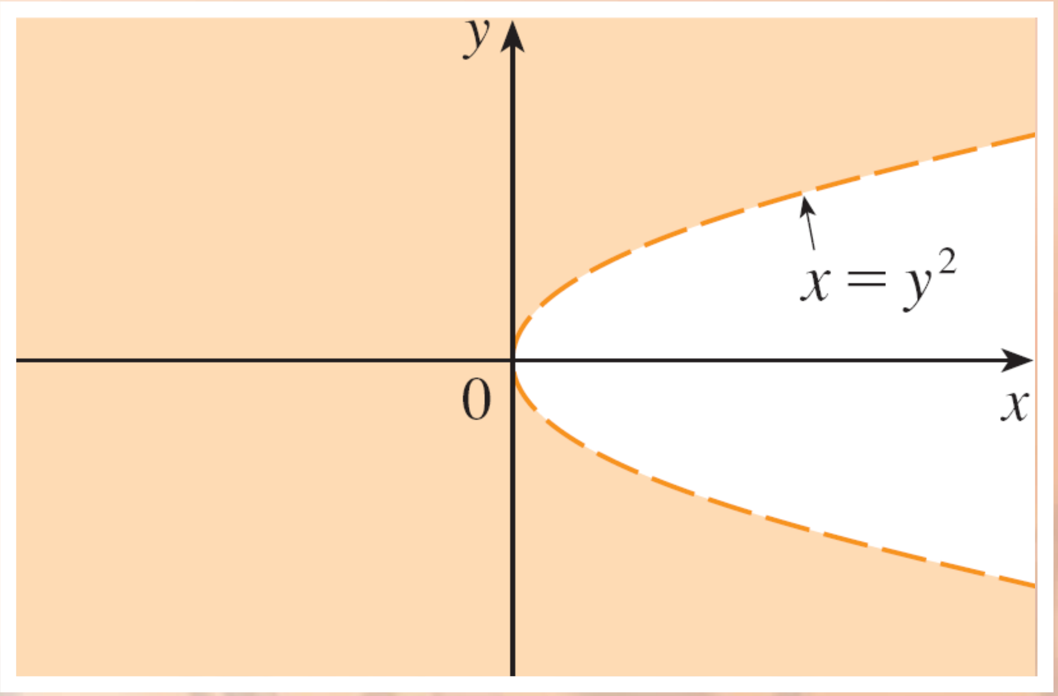
$$= 0$$

- Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is:

$$D = \{(x, y) | \, x < y^2\}$$

 Example 1 b

This is the set of points to the left of the parabola $x = y^2$.



FUNCTIONS OF TWO VARIABLES **Example 2**

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

For instance, the table shows that, if the temperature is –5°C and the wind speed is 50 km/h, then subjectively it would feel as cold as a temperature of about –15°C with no wind.

- Therefore,

$$f(-5, 50) = -15$$

Find the domain and range of:

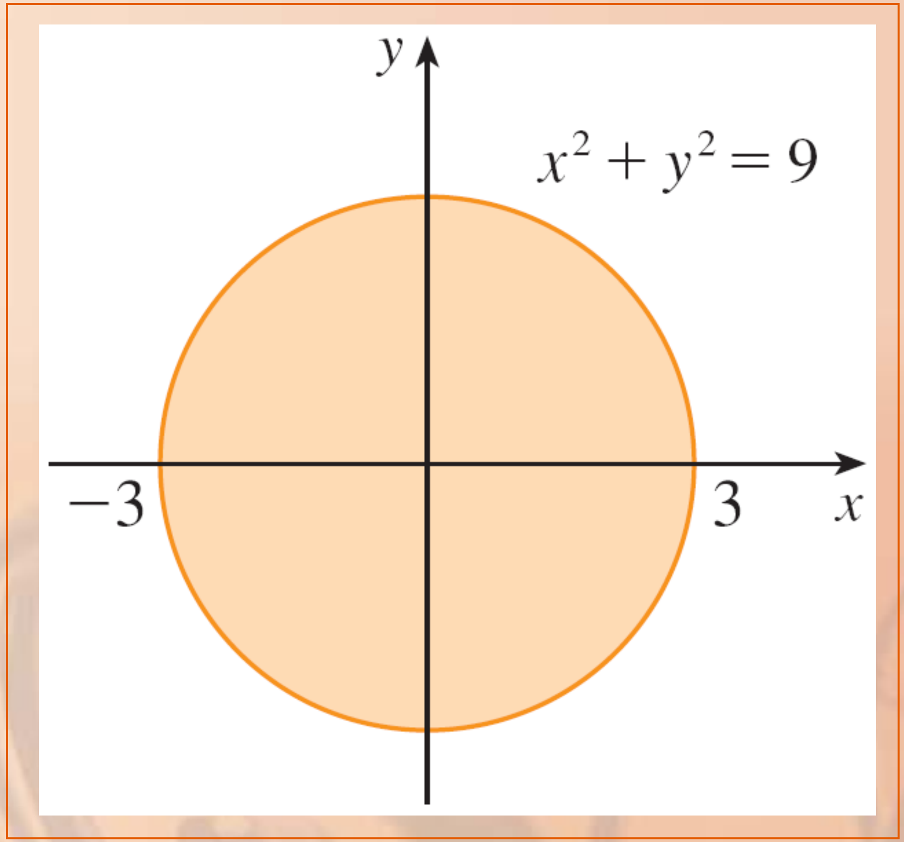
$$g(x,y)=\sqrt{9-x^2-y^2}$$

- The domain of *g* is:

$$D = \{(x, y) | \ 9 - x^2 - y^2 \geq 0\}$$
$$= \{(x, y) | \ x^2 + y^2 \leq 9\}$$

 Example 4

This is the disk with center (0, 0) and radius 3.



The range of *g* is:

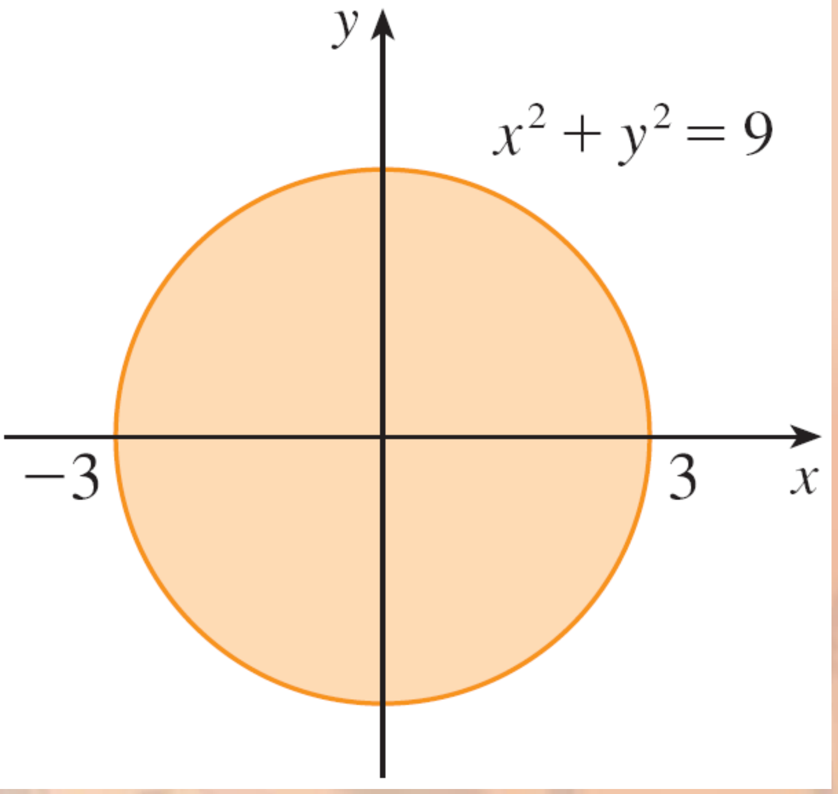
$$\{z \mid z = \sqrt{9 - x^2 - y^2}, (x, y) \in D\}$$

- Since *z* is a positive square root, $z \geq 0$.
- Also,

$$9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$$

So, the range is:

$$\{z \mid 0 \le z \le 3\} = [0, 3]$$



GRAPHS

Another way of visualizing the behavior of a function of two variables is to consider its graph.

GRAPH

If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that

$z = f(x, y)$ and (x,y) is in D.

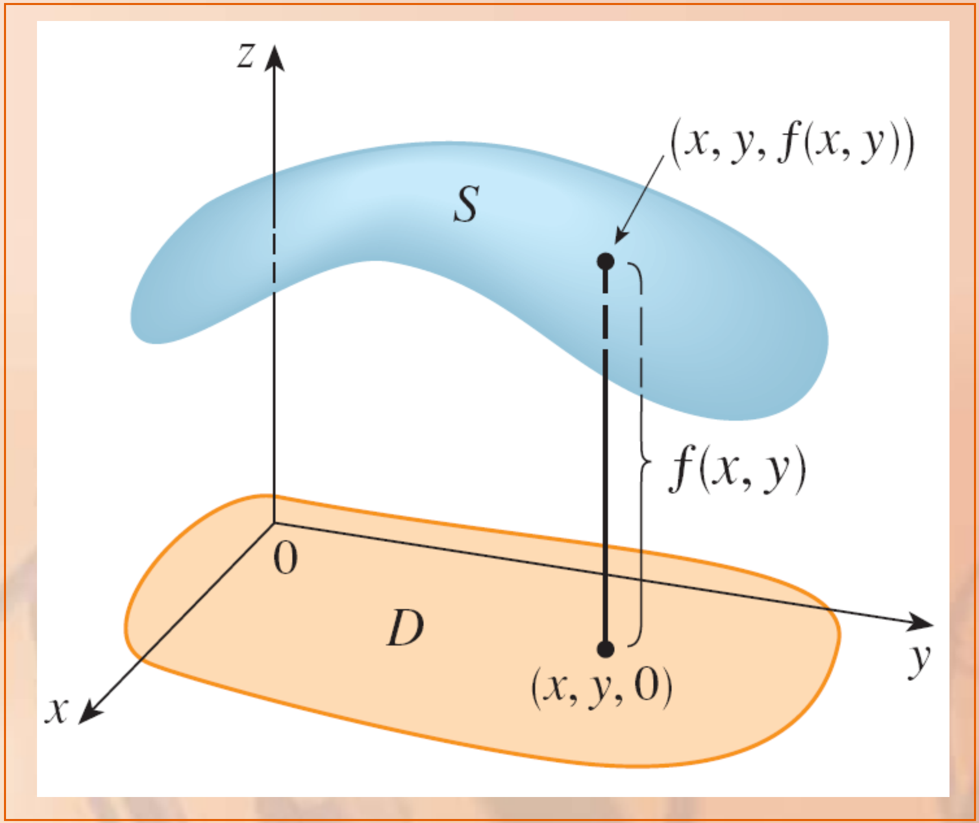
GRAPHS

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is:

- A surface S with equation $z = f(x, y)$

GRAPHS

We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane.



Sketch the graph of the function

$$f(x, y) = 6 - 3x - 2y$$

- The graph of f has the equation

$$z = 6 - 3x - 2y$$

or

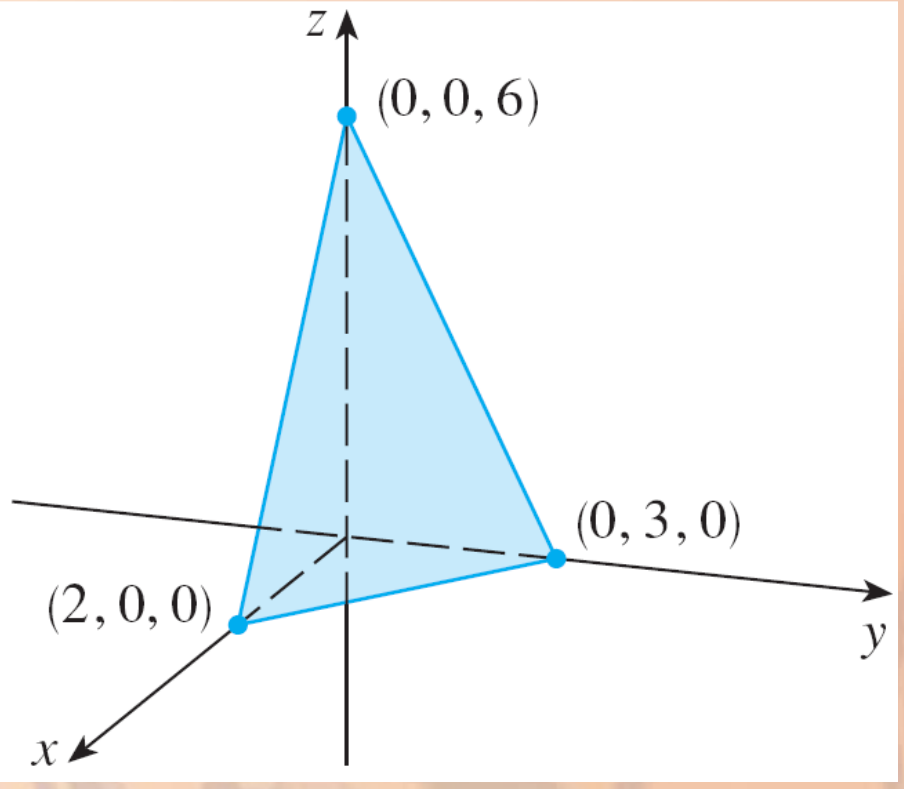
$$3x + 2y + z = 6$$

- This represents a plane.

To graph the plane, we first find the intercepts.

- Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept.
- Similarly, the y-intercept is 3 and the z-intercept is 6.

This helps us sketch the portion of the graph that lies in the first octant.



LINEAR FUNCTION

The function in Example 5 is a special case of the function

$$f(x, y) = ax + by + c$$

- It is called a linear function.

The graph of such a function has the equation

$$z = ax + by + c$$

or

$$ax + by - z + c = 0$$

- Thus, it is a plane.

In much the same way that linear functions of one variable are important in single-variable calculus, we will see that:

- Linear functions of two variables play a central role in multivariable calculus.

Sketch the graph of

$$g(x, y) = \sqrt{9 - x^2 - y^2}$$

- The graph has equation

$$z = \sqrt{9 - x^2 - y^2}$$

We square both sides of the equation to obtain:

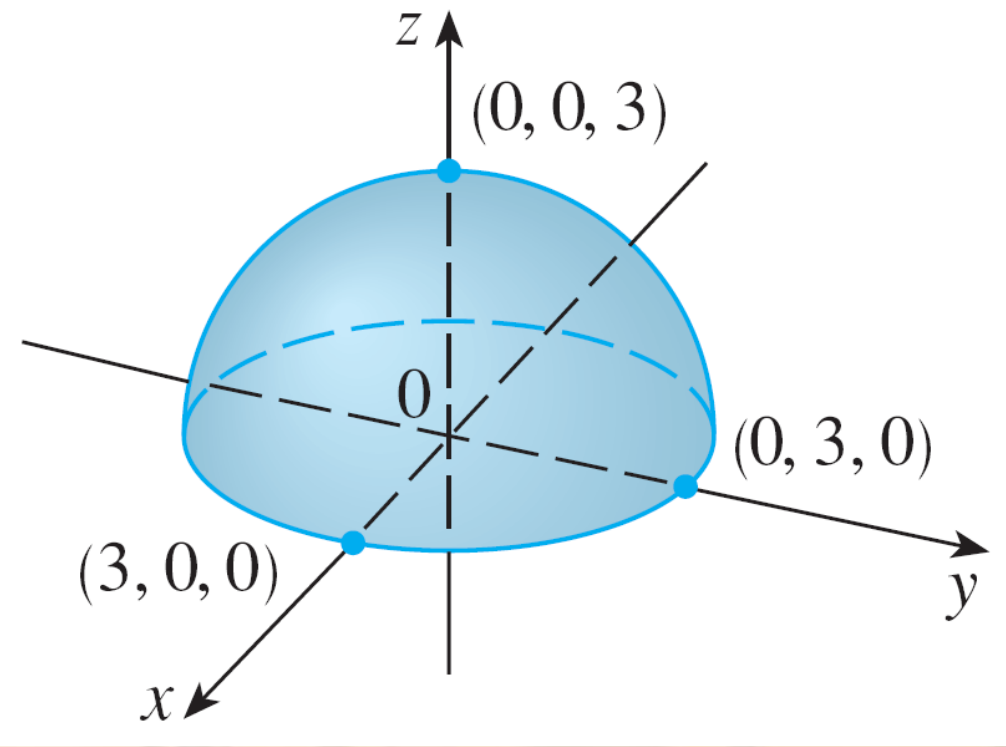
$$z^2 = 9 - x^2 - y^2$$

or

$$x^2 + y^2 + z^2 = 9$$

- We recognize this as an equation of the sphere with center the origin and radius 3.

However, since $z \geq 0$, the graph of g is just the top half of this sphere.



Note

An entire sphere can't be represented by a single function of *x* and *y*.

- As we saw in Example 6, the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 9$ is represented by the function
$$g(x, y) = \sqrt{9 - x^2 - y^2}$$
- The lower hemisphere is represented by the function
$$h(x, y) = -\sqrt{9 - x^2 - y^2}$$

Notice that $h(x, y)$ is defined for all possible ordered pairs of real numbers (x,y).

- So, the domain is $\mathbb{R}^2$, the entire xy-plane.

The range of h is the set $[0, \infty)$ of all nonnegative real numbers.

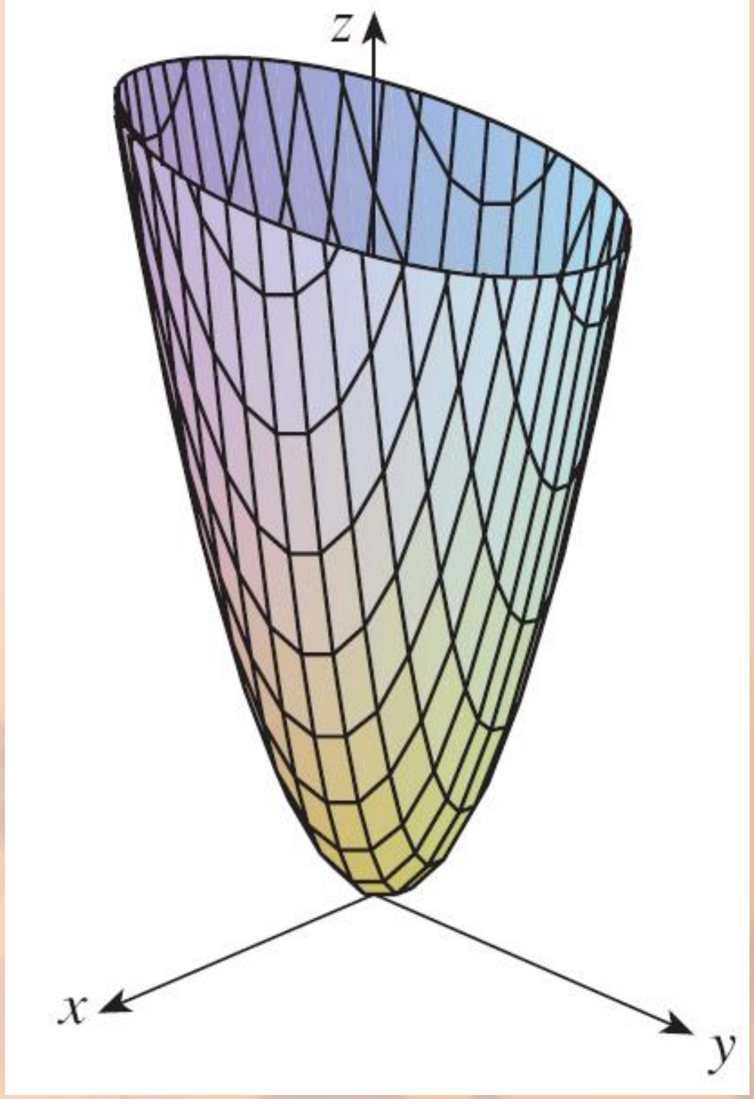
- Notice that $x^2 \geq 0$ and $y^2 \geq 0$.
- So, $h(x, y) \geq 0$ for all x and y.

The graph of *h* has the equation

$$z = 4x^2 + y^2$$

- This is the elliptic paraboloid that we sketched in Example 4 in Section 12.6

Horizontal traces are ellipses and vertical traces are parabolas.



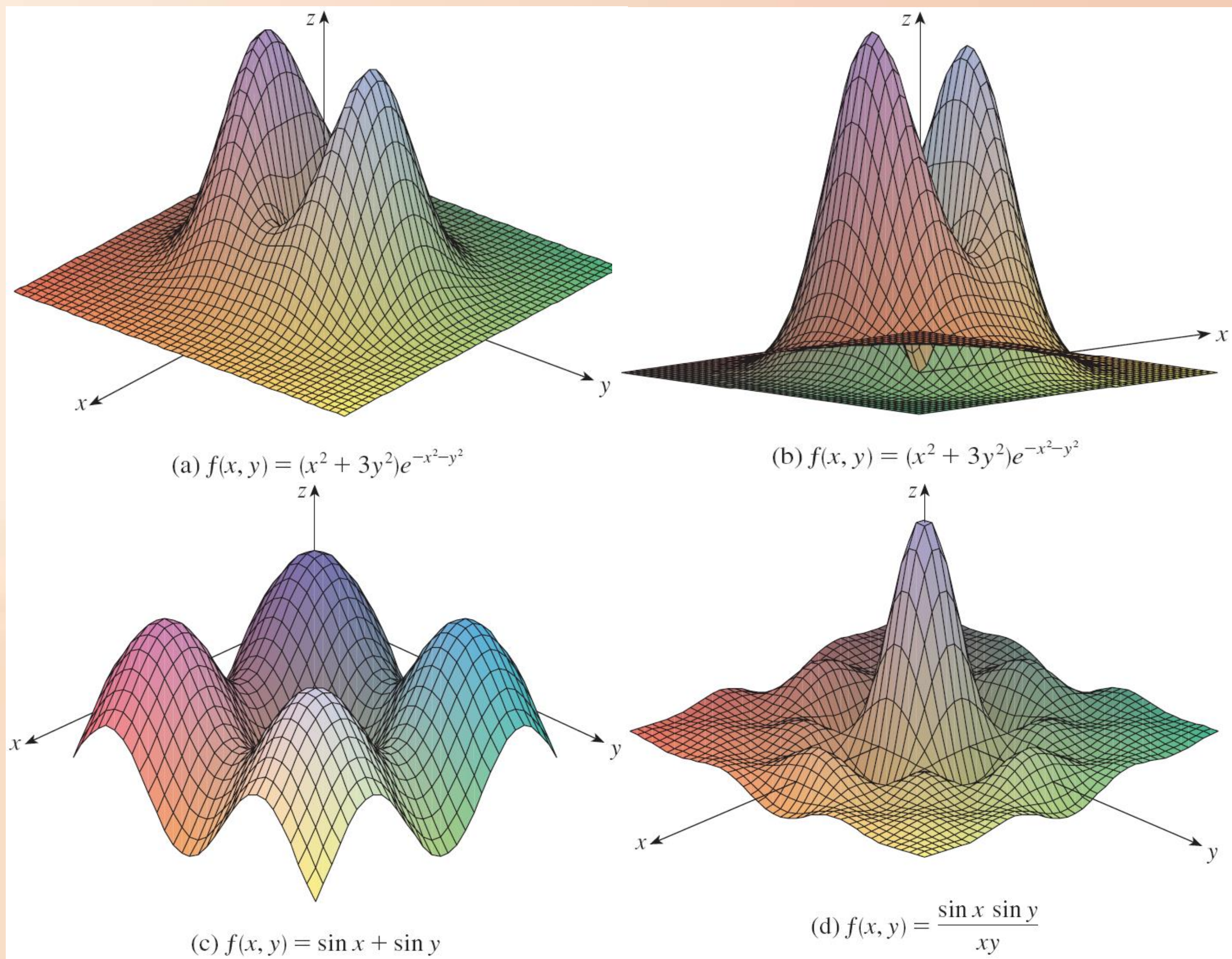
Computer programs are readily available for graphing functions of two variables.

In most such programs,

- Traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k.
- Parts of the graph are eliminated using hidden line removal.

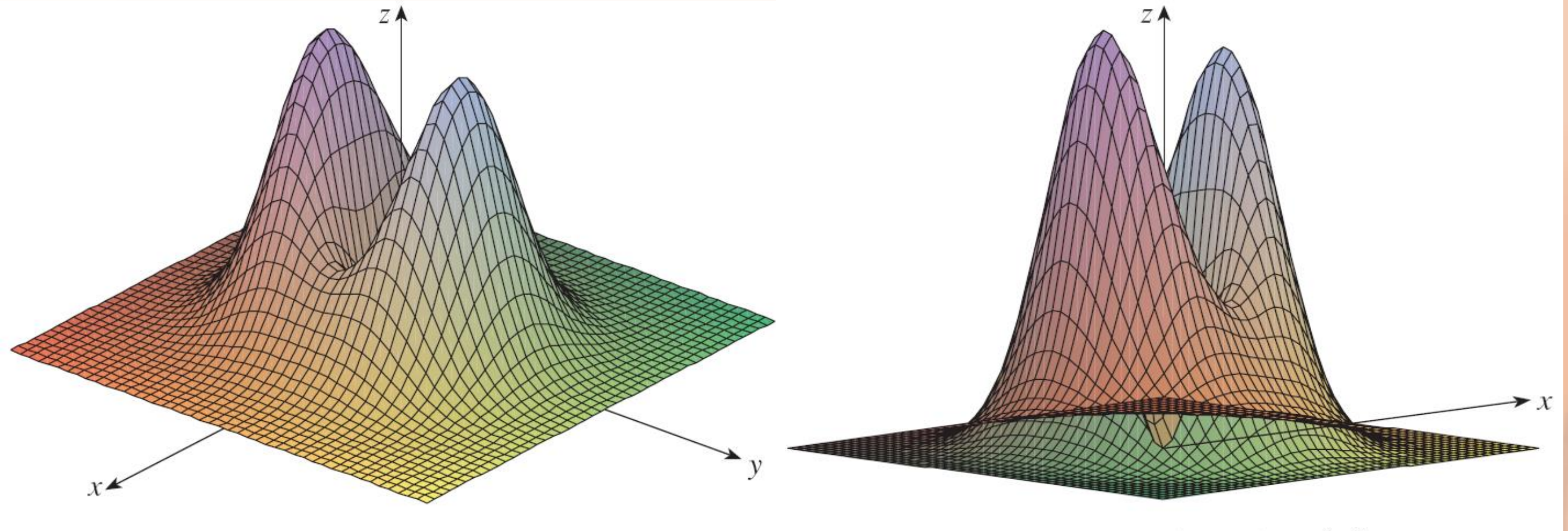
GRAPHS BY COMPUTERS

The figure shows computer-generated graphs of several functions.



GRAPHS BY COMPUTERS

Notice that we get an especially good picture of a function when rotation is used to give views from different vantage points.



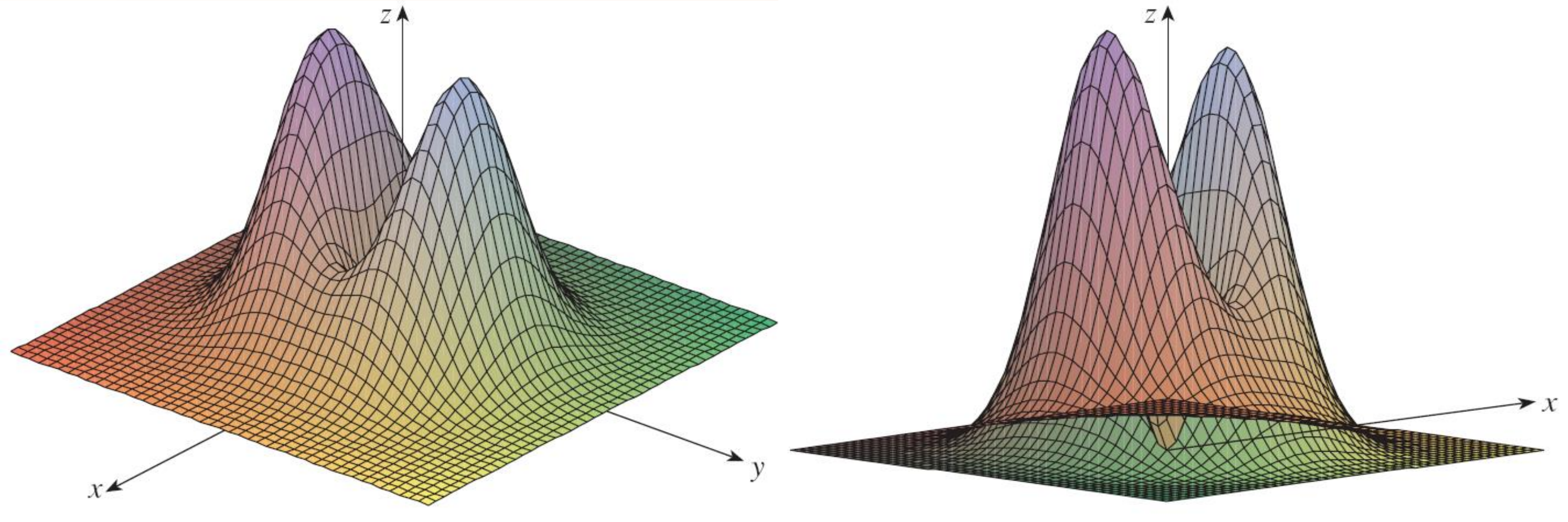
(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

GRAPHS BY COMPUTERS

In (a) and (b), the graph of *f* is very flat and close to the *xy*-plane except near the origin.

- This is because $e^{-x^2-y^2}$ is small when *x* or *y* is large.



(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

LEVEL CURVES

So far, we have two methods for visualizing functions, arrow diagrams and graphs.

- A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form contour curves, or level curves.

Definition

The level curves of a function f of two variables are the curves with equations

$$f(x, y) = k$$

where k is a constant
(in the range of f).

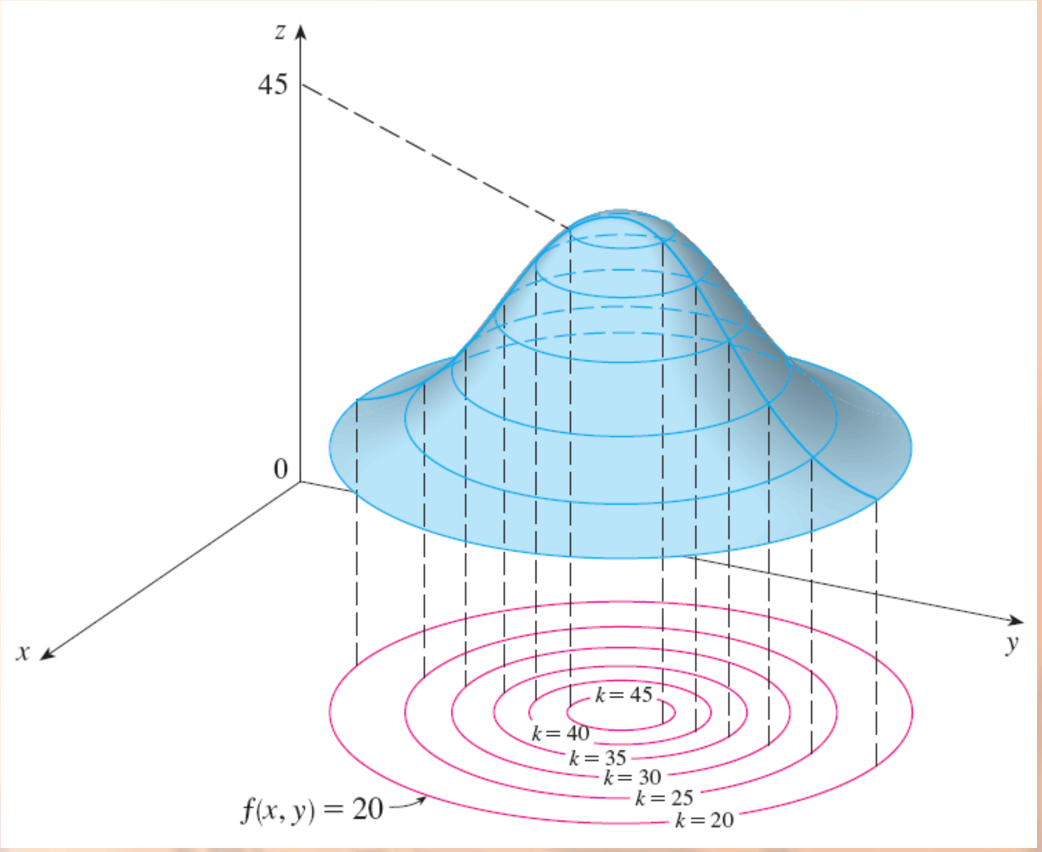
LEVEL CURVE

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k.

- That is, it shows where the graph of f has height k.

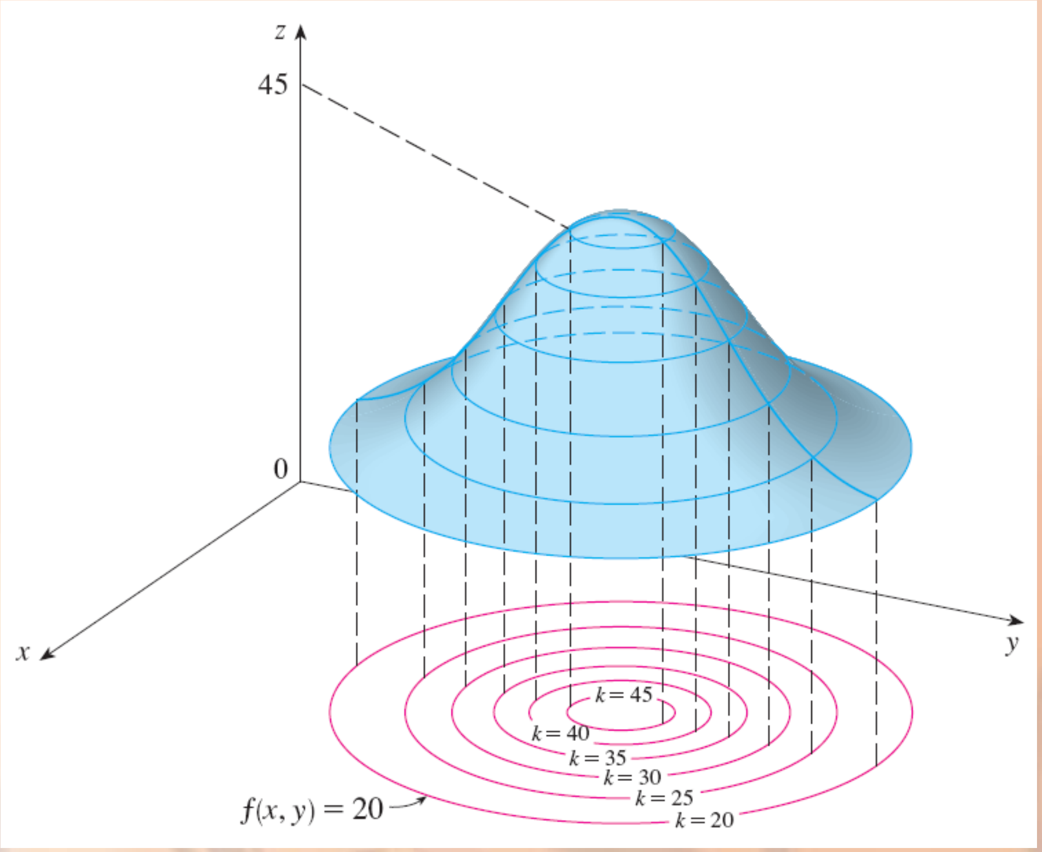
LEVEL CURVES

You can see from the figure the relation between level curves and horizontal traces.



LEVEL CURVES

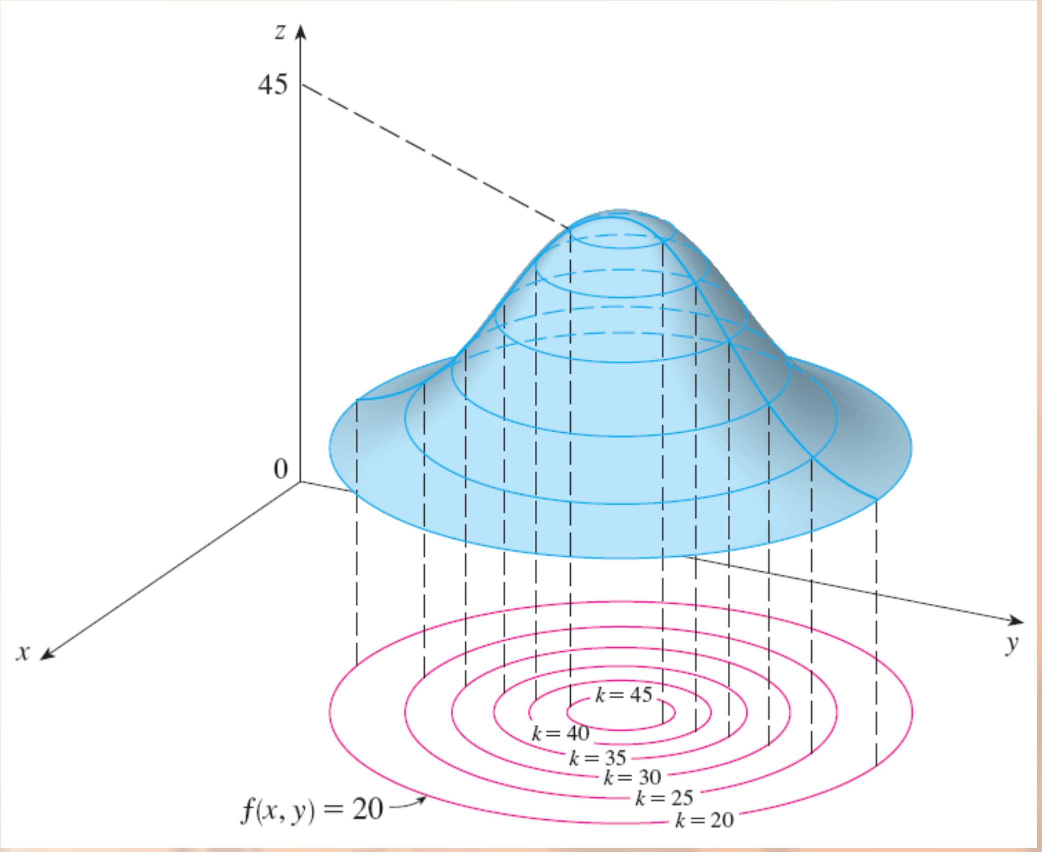
The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.



LEVEL CURVES

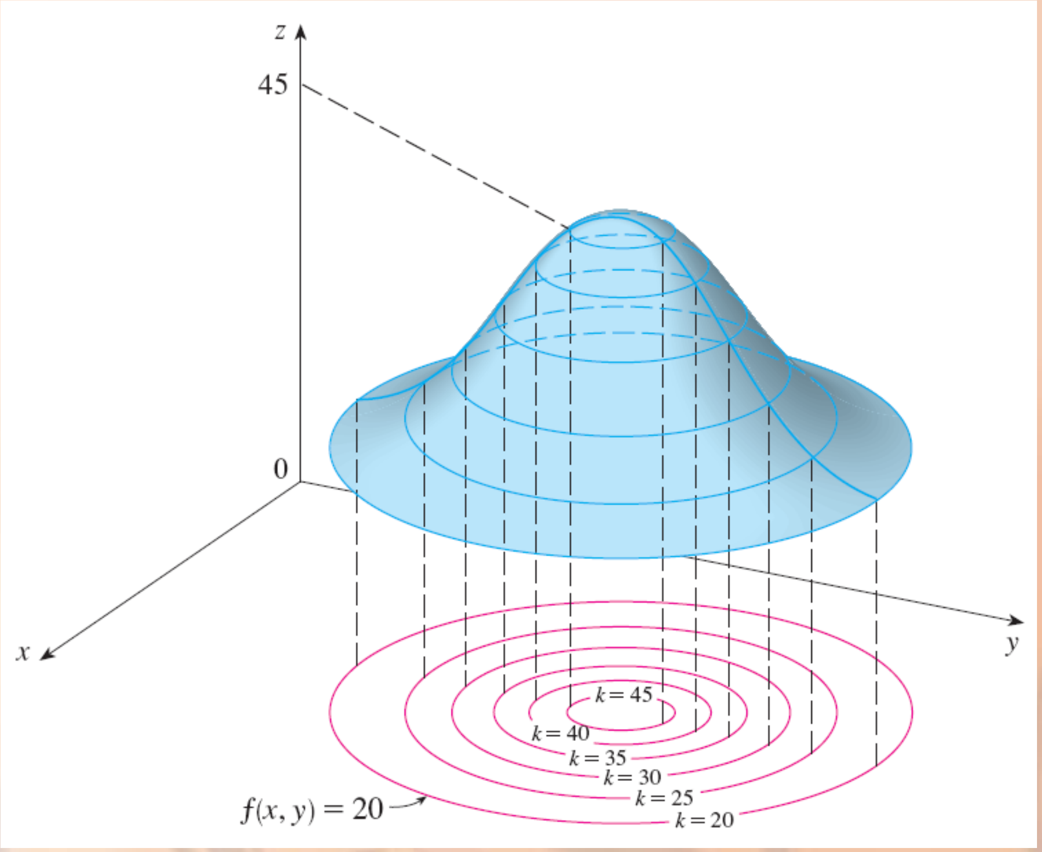
So, suppose you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height.

- Then, you can mentally piece together a picture of the graph.



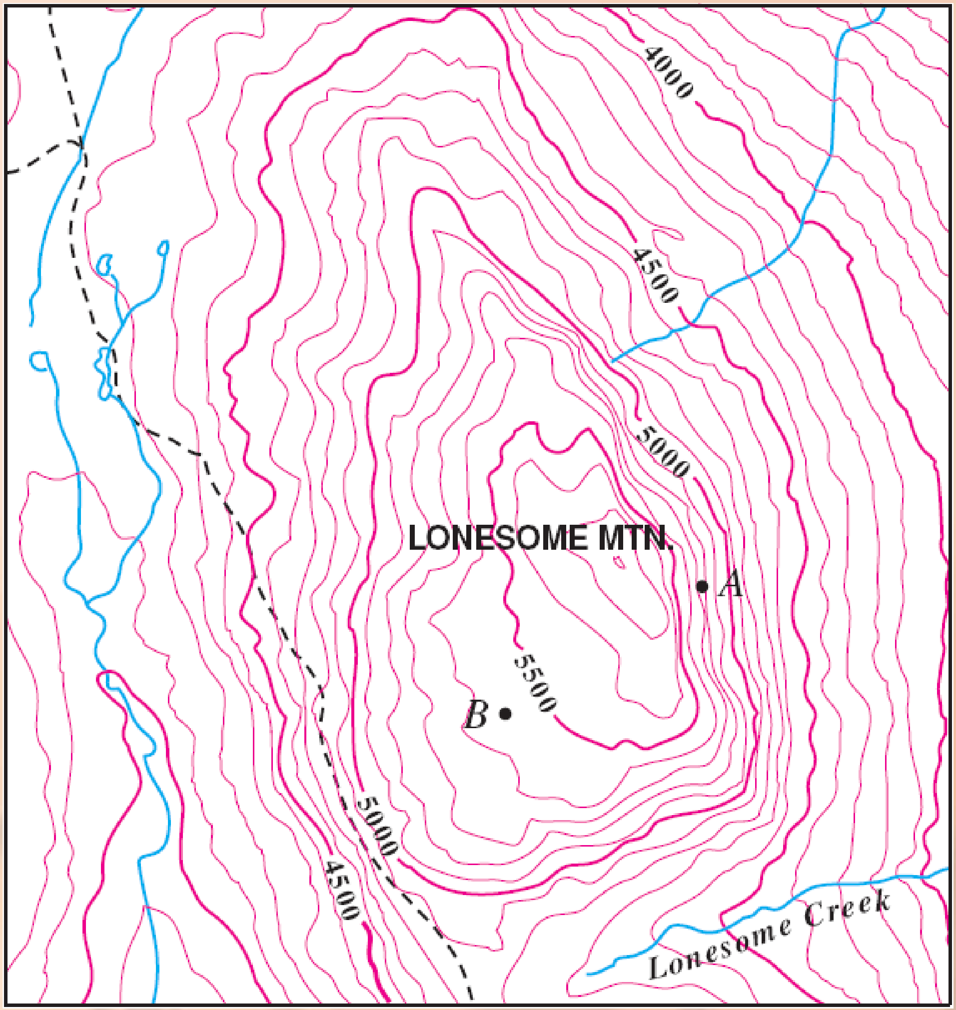
The surface is:

- Steep where the level curves are close together.
- Somewhat flatter where the level curves are farther apart.



LEVEL CURVES

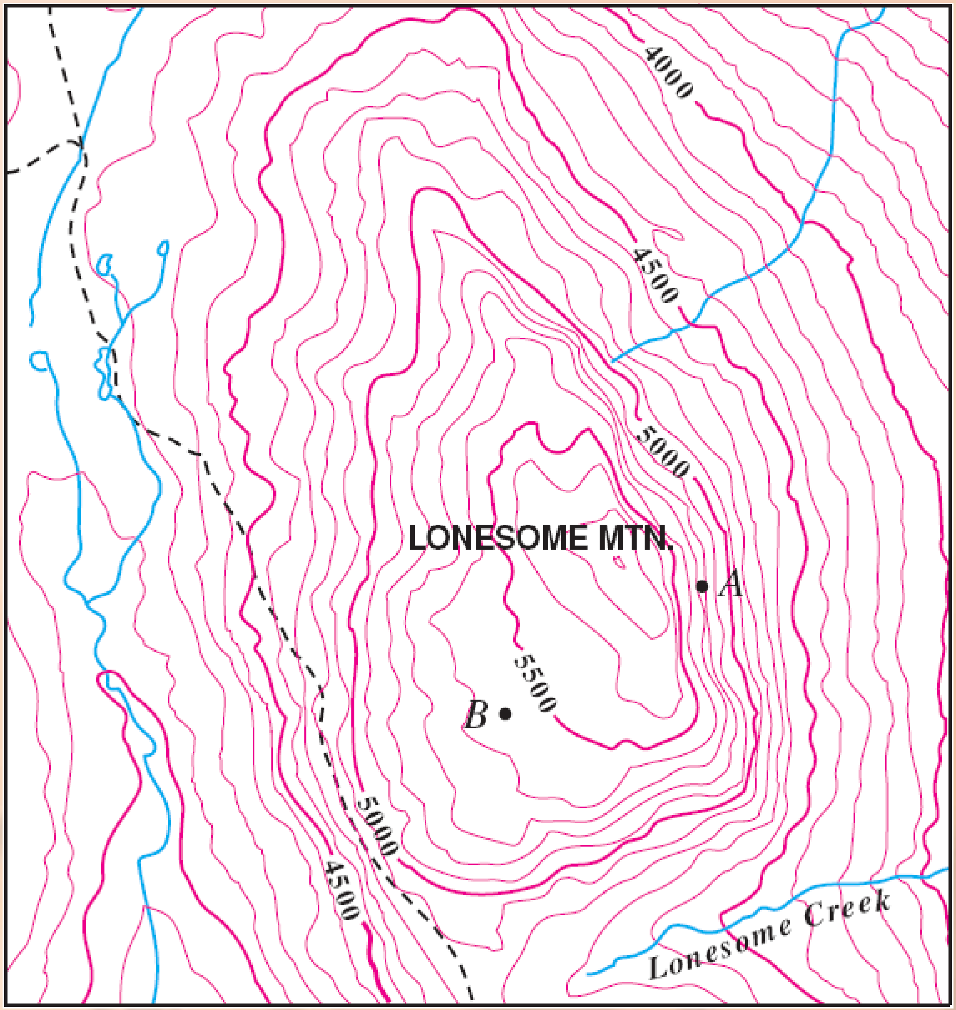
One common example of level curves occurs in topographic maps of mountainous regions, such as shown.



LEVEL CURVES

The level curves are curves of constant elevation above sea level.

- If you walk along one of these contour lines, you neither ascend nor descend.

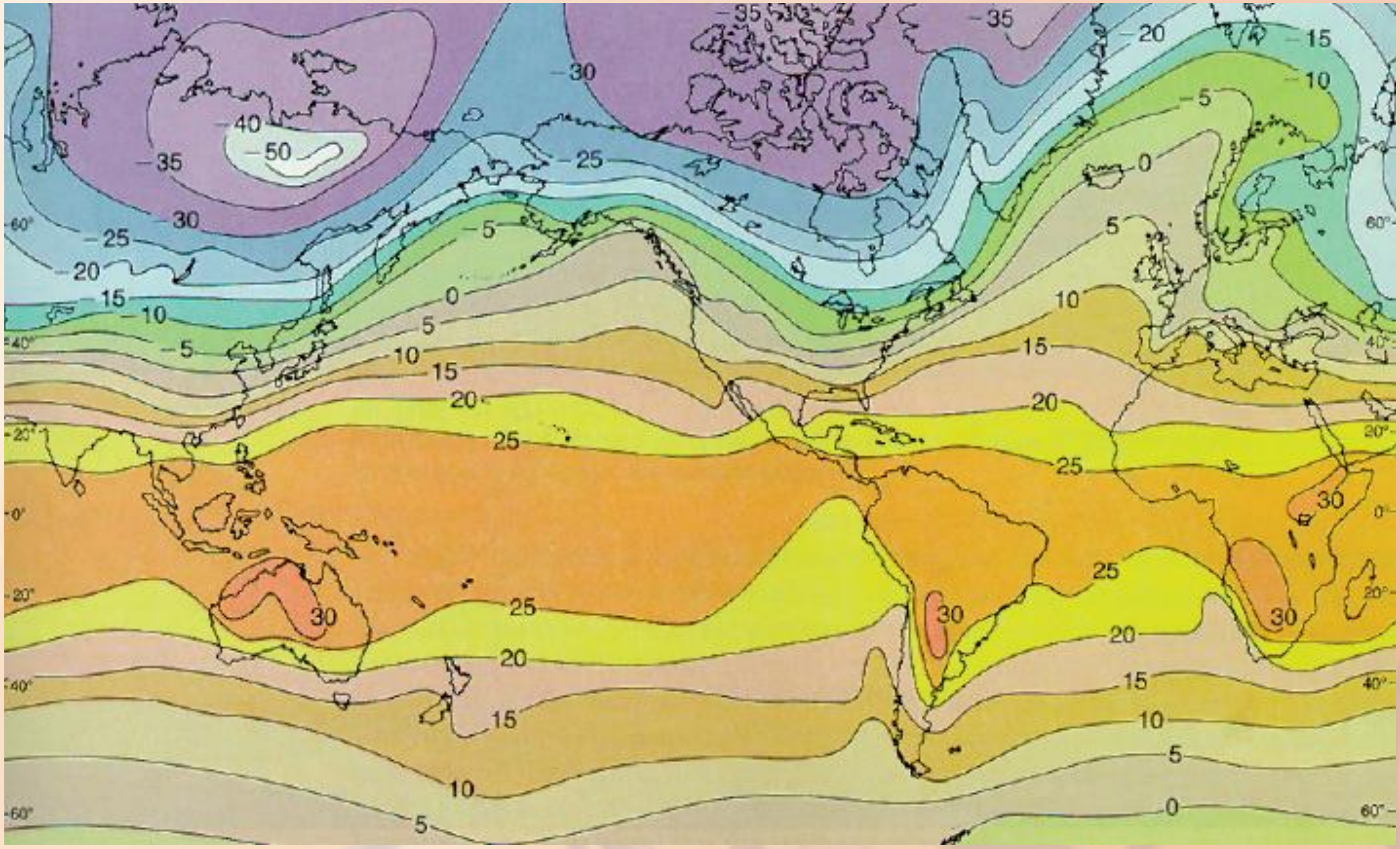


Another common example is the temperature function introduced in the opening paragraph of the section.

- Here, the level curves are called isothermals.
- They join locations with the same temperature.

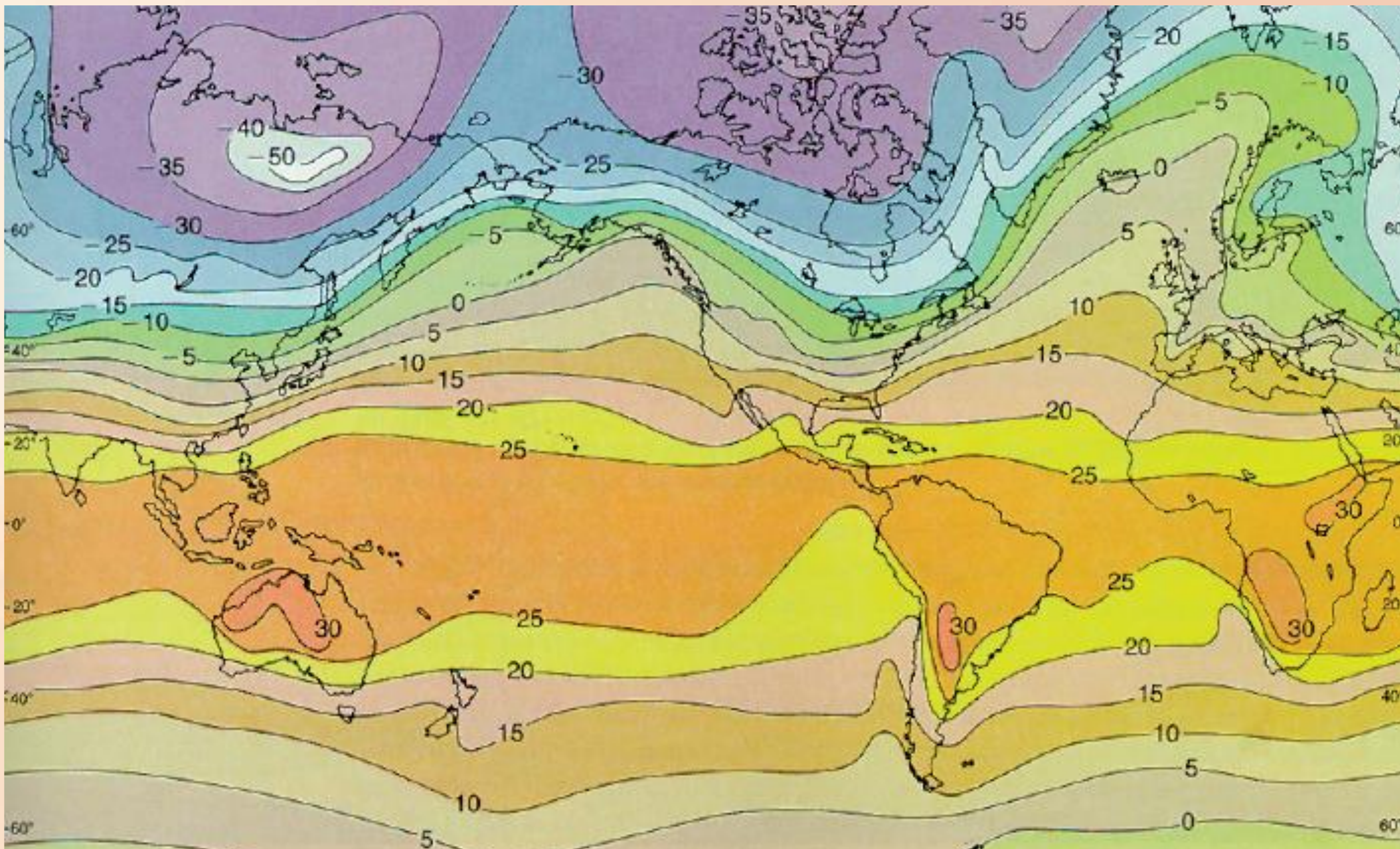
LEVEL CURVES

The figure shows a weather map of the world indicating the average January temperatures.



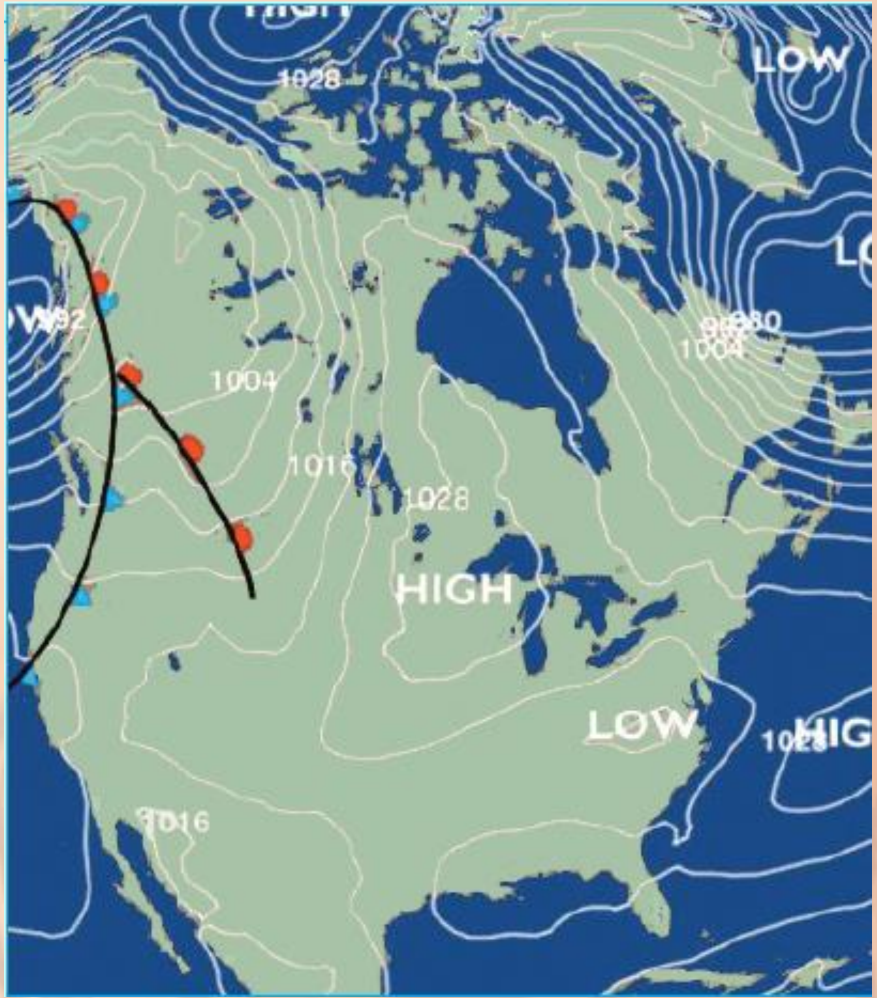
LEVEL CURVES

The isothermals are the curves that separate the colored bands.



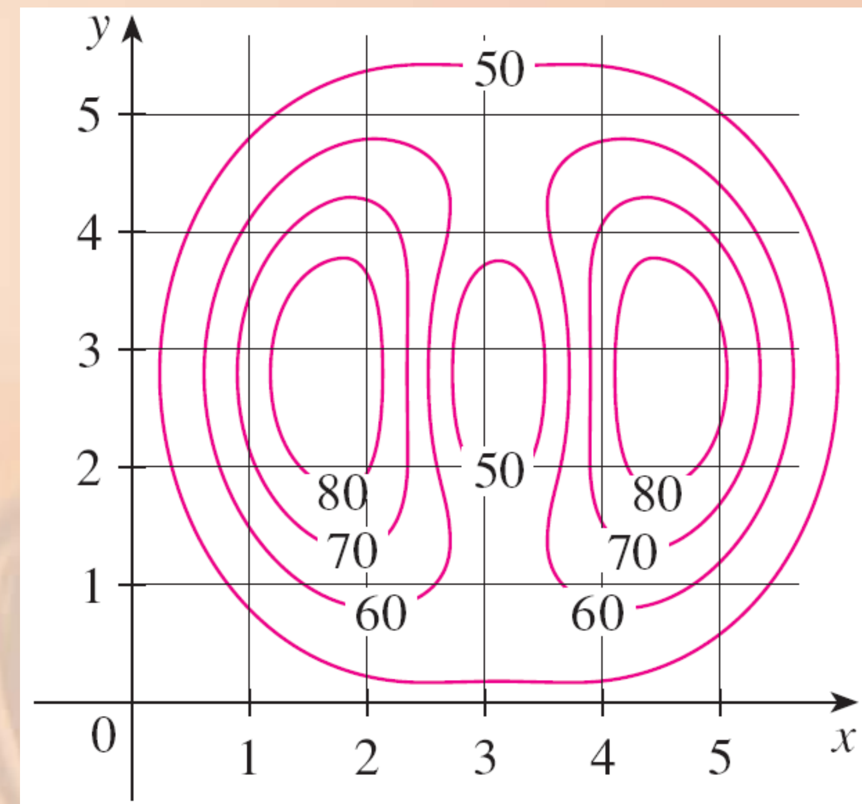
LEVEL CURVES

The isobars in this atmospheric pressure map provide another example of level curves.



A contour map for a function *f* is shown.

- Use it to estimate the values of *f*(1, 3) and *f*(4, 5).



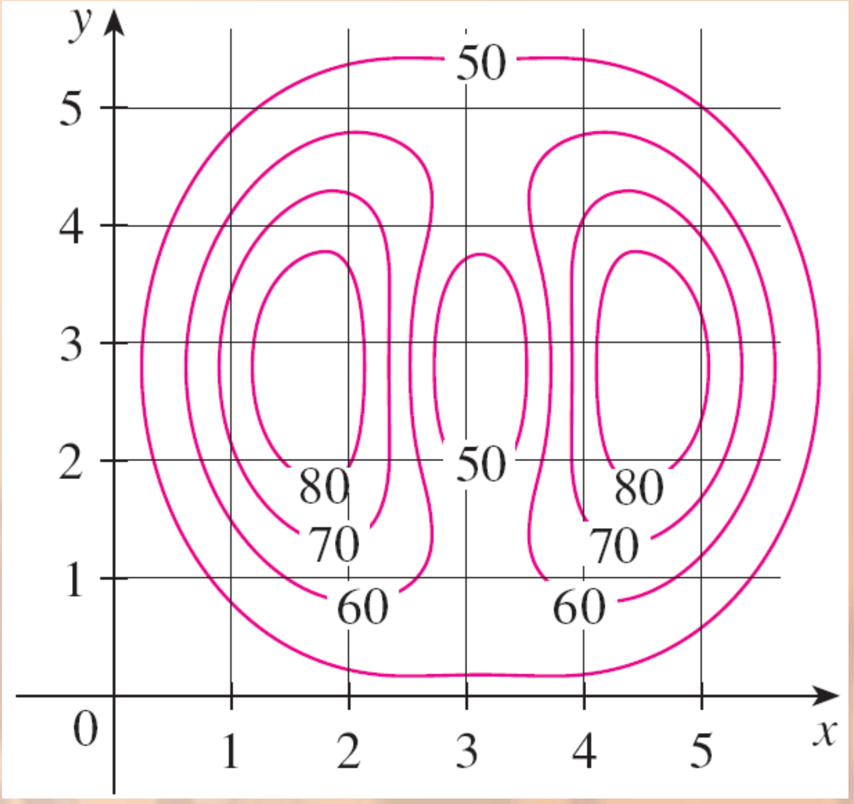
The point (1, 3) lies partway between the level curves with *z*-values 70 and 80.

- We estimate that:

  $f(1, 3) \approx 73$

- Similarly, we estimate that:

  $f(4, 5) \approx 56$



Sketch the level curves of the function

$$f(x, y) = 6 - 3x - 2y$$

for the values

$$k = -6, 0, 6, 12$$

The level curves are:

$$6 - 3x - 2y = k$$

or

$$3x + 2y + (k - 6) = 0$$

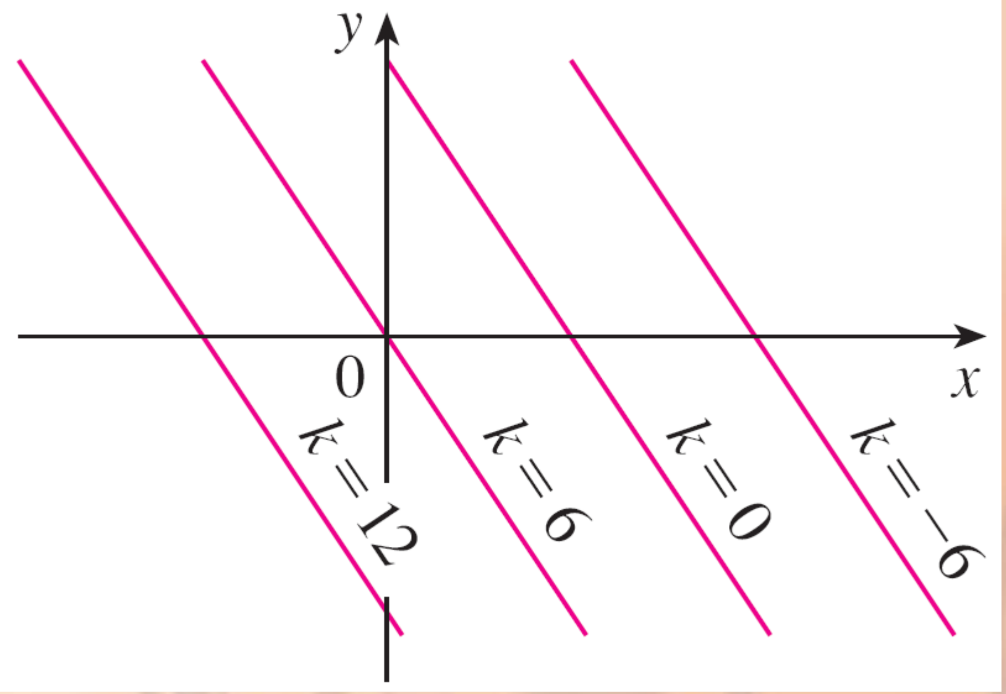
This is a family of lines with slope –3/2.

The four particular level curves with

k = –6, 0, 6, 12

are:

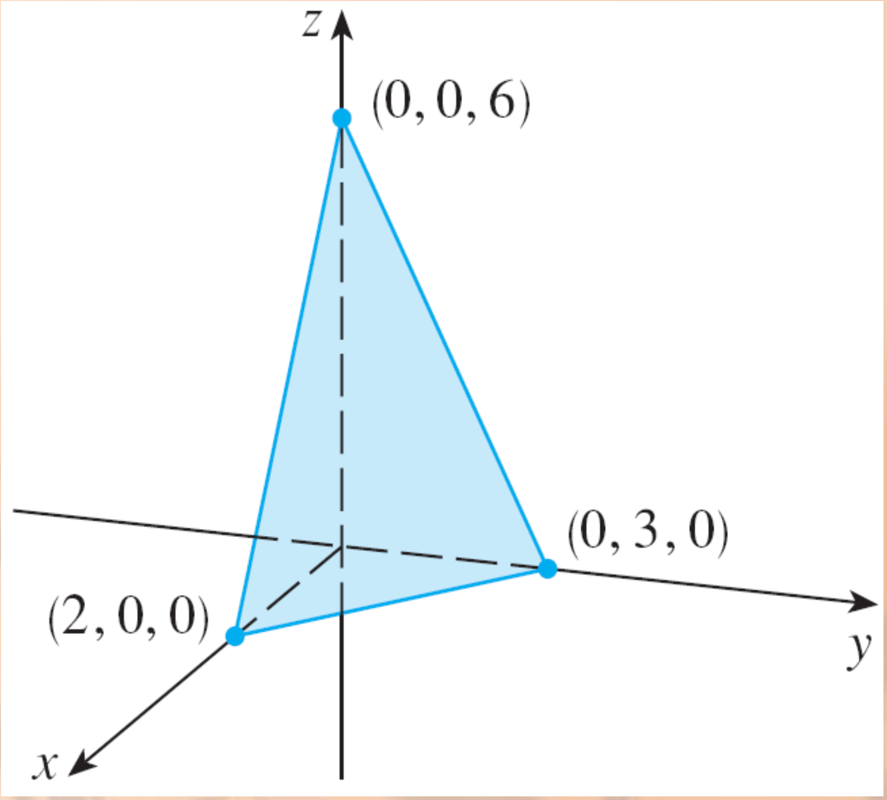
- $3x + 2y - 12 = 0$
- $3x + 2y - 6 = 0$
- $3x + 2y = 0$
- $3x + 2y + 6 = 0$



LEVEL CURVES

Example 10

The level curves are equally spaced parallel lines because the graph of *f* is a plane.



Sketch the level curves of the function

$$g(x,y)=\sqrt{9-x^2-y^2}$$

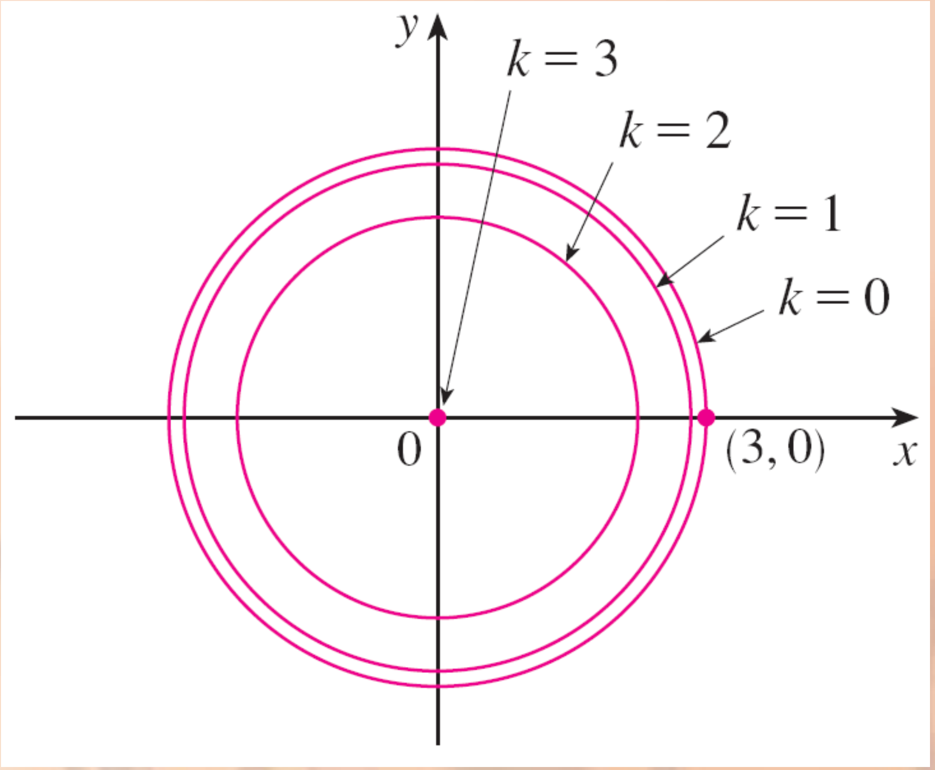
for k = 0, 1, 2, 3

- The level curves are:

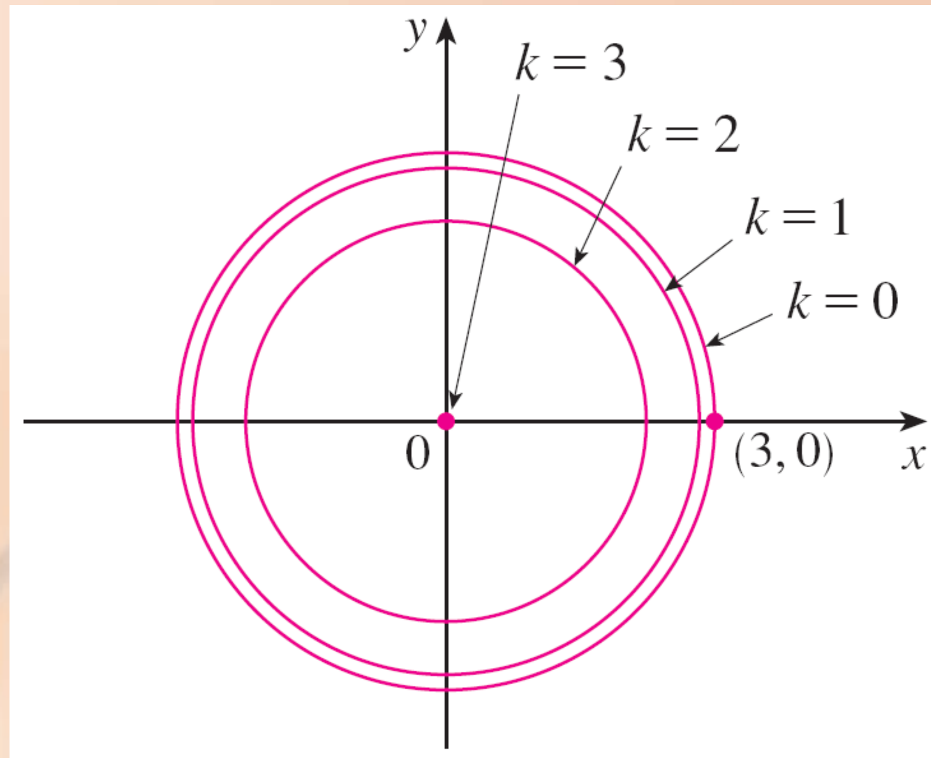
$$\sqrt{9-x^2-y^2}=k \quad \text{or} \quad x^2+y^2=9-k^2$$

This is a family of concentric circles with center (0, 0) and radius $\sqrt{9-k^2}$

- The cases k = 0, 1, 2, 3 are shown.

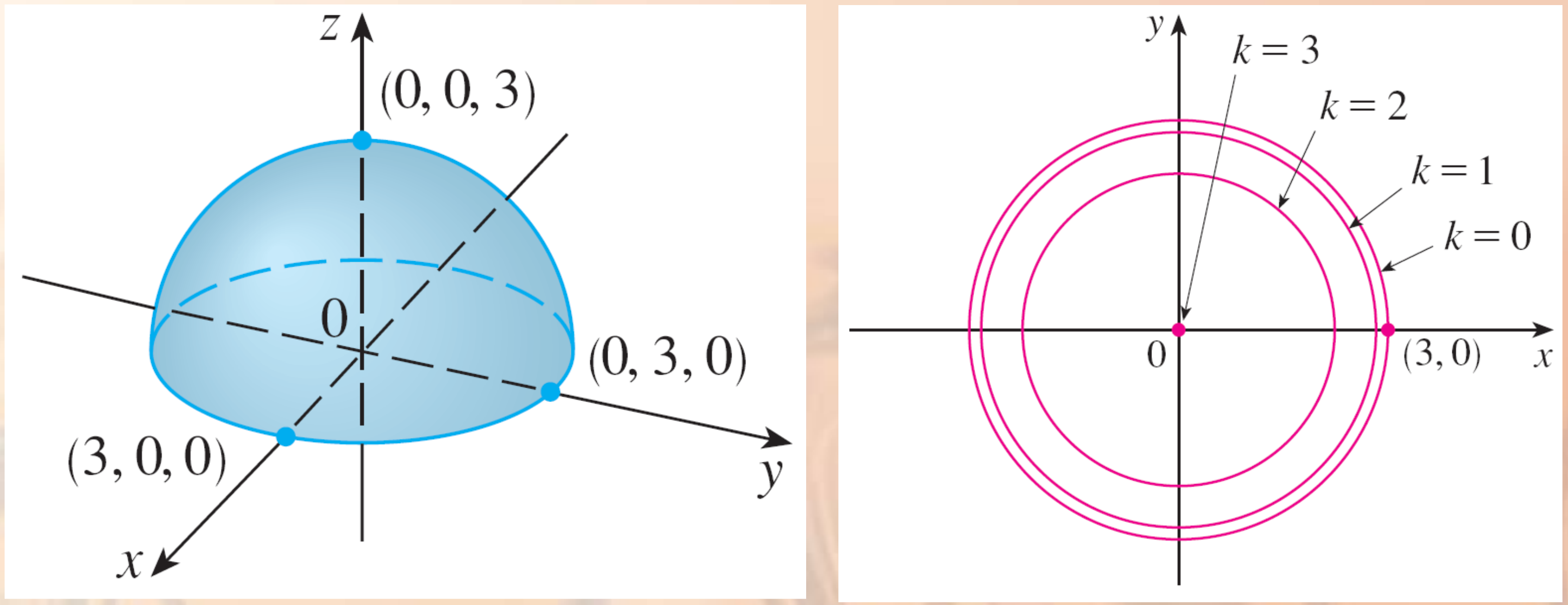


Try to visualize these level curves lifted up to form a surface.



Example 11

Then, compare the formed surface with the graph of *g* (a hemisphere), as in the other figure.



Sketch some level curves of the function

$$h(x, y) = 4x^2 + y^2$$

- The level curves are:

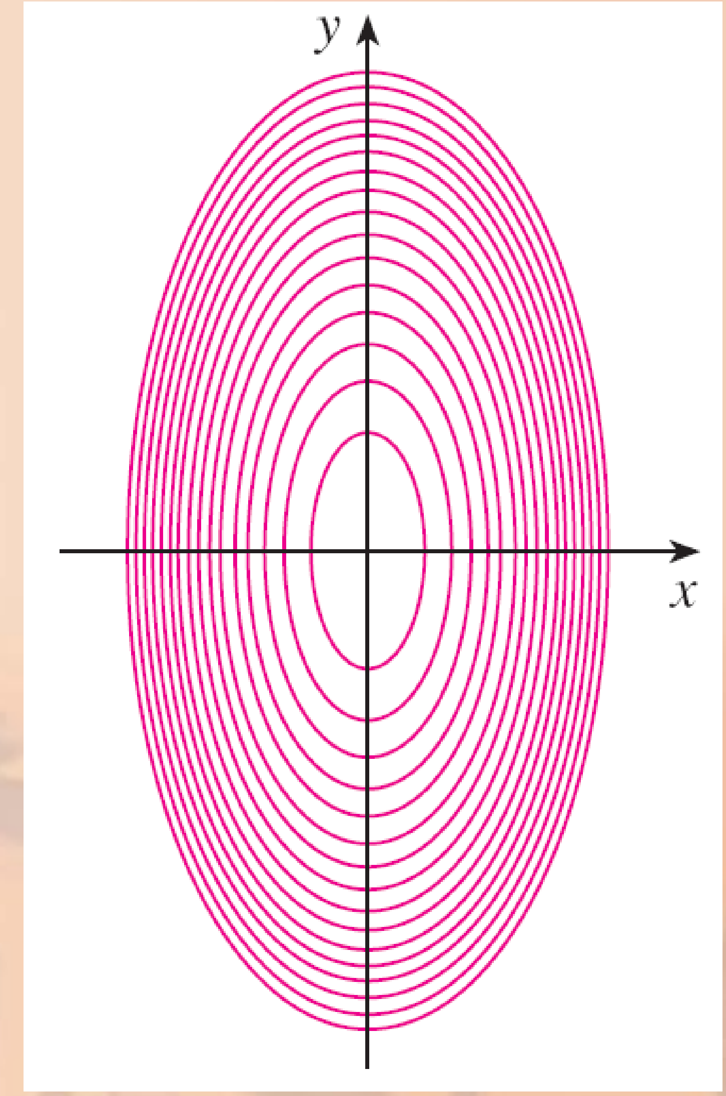
$$4x^2 + y^2 = k \quad \text{or} \quad \frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

- For $k > 0$, this describes a family of ellipses with semiaxes $\sqrt{k}/2$ and $\sqrt{k}$.

Example 12

The figure shows a contour map of *h* drawn by a computer with level curves corresponding to:

$k = 0.25, 0.5, 0.75, \dots, 4$



Example 12

This figure shows those level curves lifted up to the graph of *h* (an elliptic paraboloid) where they become horizontal traces.