15.2

Line integral

THE IDEA OF A LINE INTEGRAL

Imagine that you are rowing on a river with a noticeable current. At times you may be working against the current and at other times you may be moving with it. At the end you have a sense of whether, overall, you were helped or hindered by the current. The line integral, defined in this section, measures the extent to which a curve in a vector field is, overall, going with the vector field or against it.

Orientation of a Curve

A curve can be traced out in two directions, as shown in Figure 18.1. We need to choose one direction before we can define a line integral.

A curve is said to be **oriented** if we have chosen a direction of travel on it.

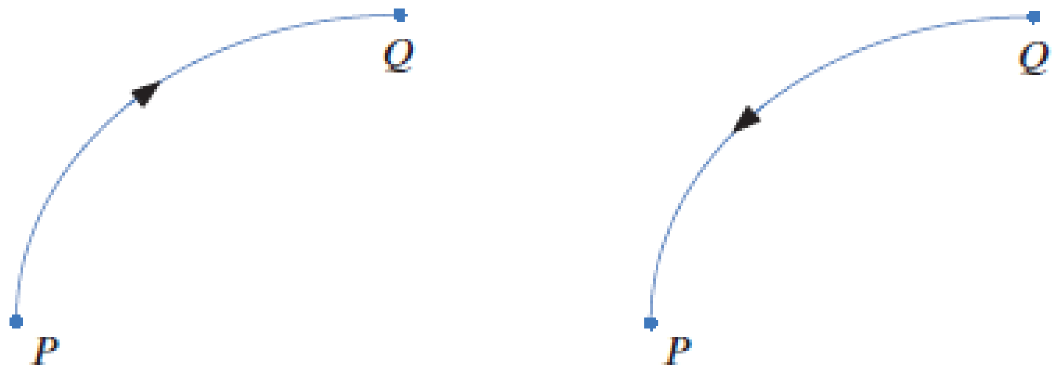


Figure 18.1: A curve with two different orientations represented by arrowheads

Definition of the Line Integral

Consider a vector field $\vec{F}$ and an oriented curve C. We begin by dividing C into n small, almost straight pieces along which $\vec{F}$ is approximately constant. Each piece can be represented by a displacement vector $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$ and the value of $\vec{F}$ at each point of this small piece of C is approximately $\vec{F}(\vec{r}_i)$. See Figures 18.2 and 18.3.

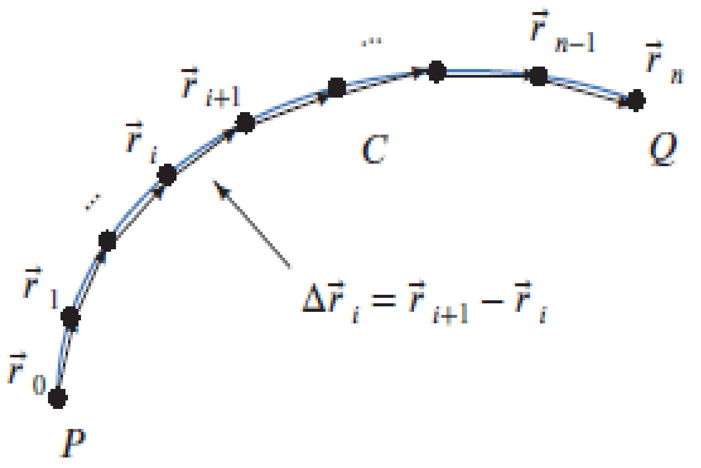


Figure 18.2: The curve C, oriented from P to Q, approximated by straight line segments represented by displacement vectors $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$

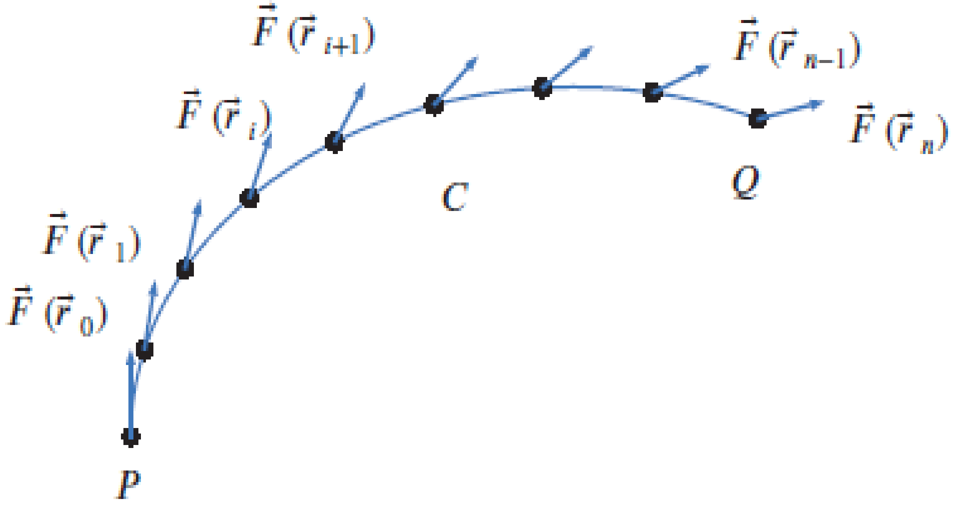


Figure 18.3: The vector field $\vec{F}$ evaluated at the points with position vector $\vec{r}_i$ on the curve C oriented from P to Q

Returning to our initial example, the vector field $\vec{F}$ represents the current and the oriented curve C is the path of the person rowing the boat. We wish to determine to what extent the vector field $\vec{F}$ helps or hinders motion along C. Since the dot product can be used to measure to what extent two vectors point in the same or opposing directions, we form the dot product $\vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$ for each point with position vector $\vec{r}_i$ on C. Summing over all such pieces, we get a Riemann sum:

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

We define the line integral, written $\int_C \vec{F} \cdot d\vec{r}$, by taking the limit as $\|\Delta\vec{r}_i\| \to 0$. Provided the limit exists, we make the following definition:

The **line integral** of a vector field $\vec{F}$ along an oriented curve C is

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\Delta\vec{r}_i\| \to 0} \sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i.$$

COMPUTING LINE INTEGRALS OVER PARAMETERIZED CURVES

The goal of this section is to show how to use a parameterization of a curve to convert a line integral
into an ordinary one-variable integral.

Using a Parameterization to Evaluate a Line Integral

Recall the definition of the line integral,

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i,$$

where the $\vec{r}_i$ are the position vectors of points subdividing the curve into short pieces. Now suppose we have a smooth parameterization, $\vec{r}(t)$, of C for $a \le t \le b$, so that $\vec{r}(a)$ is the position vector of the starting point of the curve and $\vec{r}(b)$ is the position vector of the end. Then we can divide C into n pieces by dividing the interval $a \le t \le b$ into n pieces, each of size $\Delta t = (b - a)/n$. See Figures 18.19 and 18.20.

At each point $\vec{r}_i = \vec{r}(t_i)$ we want to compute

$$\vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i.$$

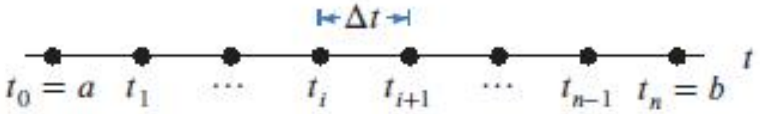


Figure 18.19: Subdivision of the interval $a \leq t \leq b$

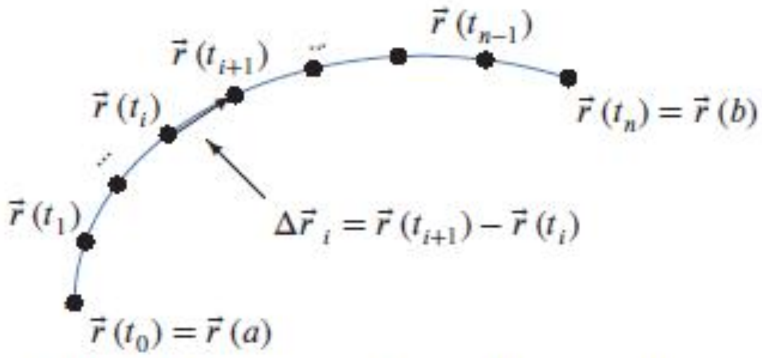


Figure 18.20: Corresponding subdivision of the parameterized path C

Since $t_{i+1} = t_i + \Delta t$, the displacement vectors $\Delta\vec{r}_i$ are given by

$$\begin{aligned}
\Delta\vec{r}_i &= \vec{r}\,(t_{i+1}) - \vec{r}\,(t_i) \\
&= \vec{r}\,(t_i + \Delta t) - \vec{r}\,(t_i) \\
&= \frac{\vec{r}\,(t_i + \Delta t) - \vec{r}\,(t_i)}{\Delta t} \cdot \Delta t \\
&\approx \vec{r}\,'(t_i)\Delta t,
\end{aligned}$$

where we use the facts that Δt is small and that $\vec{r}\,(t)$ is differentiable to obtain the last approximation.

Therefore,

$$\int_C \vec{F} \cdot d\vec{r} \approx \sum \vec{F}\,(\vec{r}_i) \cdot \Delta\vec{r}_i \approx \sum \vec{F}\,(\vec{r}\,(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t.$$

Notice that $\vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)$ is the value at t_i of a one-variable function of t, so this last sum is really a one-variable Riemann sum. In the limit as $\Delta t \to 0$, we get a definite integral:

$$\lim_{\Delta t \to 0} \sum \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt.$$

Thus, we have the following result:

If $\vec{r}(t)$, for $a \le t \le b$, is a smooth parameterization of an oriented curve C and $\vec{F}$ is a vector field which is continuous on C, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt.$$

In words: To compute the line integral of $\vec{F}$ over C, take the dot product of $\vec{F}$ evaluated on C with the velocity vector, $\vec{r}\,'(t)$, of the parameterization of C, then integrate along the curve.

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\|\mathbf{r}'(t)\|\, dt$$

$$IF\ \ P(x_0, y_0, , z_0)\ and\ Q(x_1, y_1, z_1)$$

Parametric : $x(t) = x_0 + (x_1 - x_0)t$

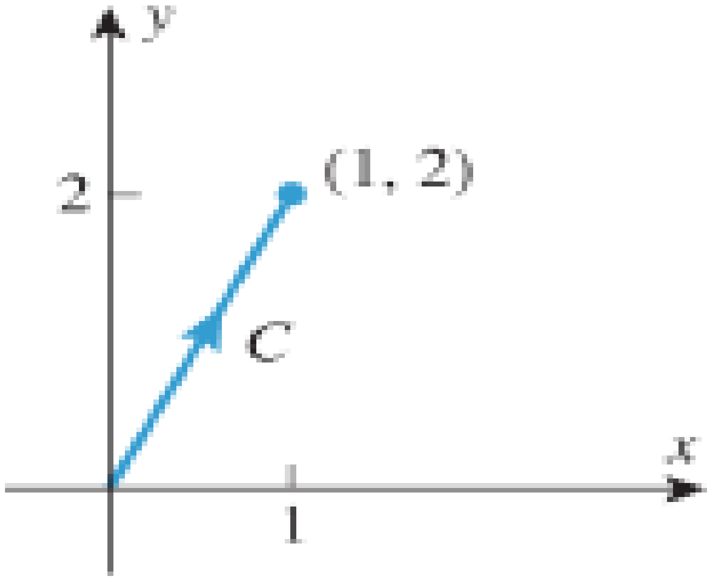
: y$(t) = y_0 + (y_1 - y_0)t$:

z$(t) = z_0 + (z_1 - z_0)t$

► **Example 1** Using the given parametrization, evaluate the line integral $\int_C (1 + xy^2)\, ds$.

(a) $C : \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} \quad (0 \le t \le 1)$ (see Figure 15.2.6a)

(b) $C : \mathbf{r}(t) = (1 - t)\mathbf{i} + (2 - 2t)\mathbf{j} \quad (0 \le t \le 1)$ (see Figure 15.2.6b)



(a)

$$r(t) = ti + 2tj \quad (0 \le t \le 1)$$

Put $t = 0$, $r(0) = 0 = (0,0)$

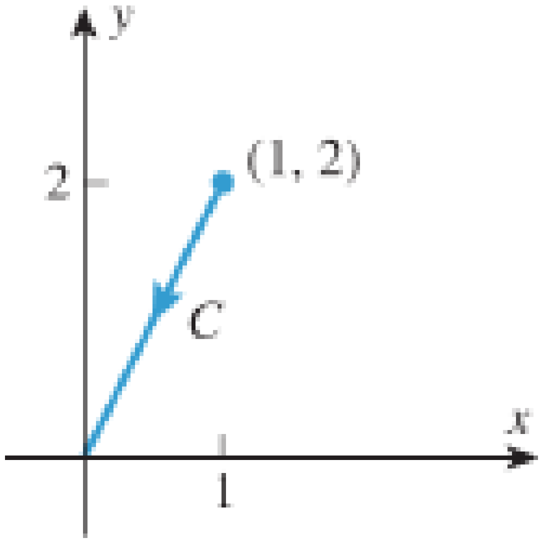
$put\ t = 1$, $r(1) = i + 2j = (1,2)$

Solution (a). Since $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j}$, we have $\|\mathbf{r}'(t)\| = \sqrt{5}$ and it follows from Formula (9) that

$$\begin{aligned}
\int_C (1 + xy^2)\, ds &= \int_0^1 [1 + t(2t)^2]\sqrt{5}\, dt \\
&= \int_0^1 (1 + 4t^3)\sqrt{5}\, dt \\
&= \sqrt{5}\left[t + t^4\right]_0^1 = 2\sqrt{5}
\end{aligned}$$

Solution (b). Since $\mathbf{r}'(t) = -\mathbf{i} - 2\mathbf{j}$, we have $\|\mathbf{r}'(t)\| = \sqrt{5}$ and it follows from Formula (9) that

$$\begin{aligned}\int_C (1 + xy^2)\, ds &= \int_0^1 [1 + (1 - t)(2 - 2t)^2]\sqrt{5}\, dt \\ &= \int_0^1 [1 + 4(1 - t)^3]\sqrt{5}\, dt \\ &= \sqrt{5}\left[t - (1 - t)^4\right]_0^1 = 2\sqrt{5}\end{aligned}$$



(b)

7–10 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the line segment C from P to Q.

7. $\mathbf{F}(x, y) = 8\mathbf{i} + 8\mathbf{j};\ P(-4, 4), Q(-4, 5)$

8. $\mathbf{F}(x, y) = 2\mathbf{i} + 5\mathbf{j};\ P(1, -3), Q(4, -3)$

9. $\mathbf{F}(x, y) = 2x\mathbf{j};\ P(-2, 4), Q(-2, 11)$

10. $\mathbf{F}(x, y) = -8x\mathbf{i} + 3y\mathbf{j};\ P(-1, 0), Q(6, 0)$

7–10 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the line segment C from P to Q.

7. $\mathbf{F}(x, y) = 8\mathbf{i} + 8\mathbf{j}$; $P(-4, 4)$, $Q(-4, 5)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt.$$

We observe that the **x-coordinate is constant** (−4), and **y changes from 4 to 5.** So, we can parameterize the line segment as:

$$\mathbf{r}(t) = \langle -4, t \rangle \quad \text{for} \quad t \in [4, 5]$$

Then, the derivative is:

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \langle 0, 1 \rangle$$

Since $\mathbf{F}(x, y) = \langle 8, 8 \rangle$, and it's constant everywhere:

$$\mathbf{F}(\mathbf{r}(t)) = \langle 8, 8 \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle 8, 8 \rangle \cdot \langle 0, 1 \rangle = 8$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=4}^{5} 8\, dt = 8(t)\Big|_4^5 = 8(5 - 4) = 8$$

Q#10 $\mathbf{F}(x, y) = -8x\mathbf{i} + 3y\mathbf{j};\ P(-1, 0), Q(6, 0)$

SOL:

Since $y = 0$, and x goes from −1 to 6:

$$\mathbf{r}(t) = \langle t, 0 \rangle, \quad t \in [-1, 6]$$

Then:

$$\mathbf{r}'(t) = \langle 1, 0 \rangle$$

Substitute $y = 0$:

$$\mathbf{F}(\mathbf{r}(t)) = \langle -8t, 3 \cdot 0 \rangle = \langle -8t, 0 \rangle$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle -8t, 0 \rangle \cdot \langle 1, 0 \rangle = -8t$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\left(\vec{r}\,(t)\right) \cdot \vec{r}\,'(t)\, dt.$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{6} -8t\, dt = -8 \int_{-1}^{6} t\, dt$$

$$= -8 \left[\frac{t^2}{2} \right]_{-1}^{6} = -8 \left(\frac{6^2}{2} - \frac{(-1)^2}{2} \right)$$

$$= -8 \left(\frac{36 - 1}{2} \right) = -8 \cdot \frac{35}{2} = -140$$

Answer : Q8 and 9

8. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^4 (2\cdot 1 + 5\cdot 0)\,dt = 6.$

9. $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_4^{11} (0\cdot 0 + 2(-2)\cdot 1)\,dt = -28.$

19–22 Evaluate the line integral with respect to s along the curve C.

19. $\displaystyle\int_C \frac{1}{1+x}\,ds$

$C: \mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{j} \quad (0 \le t \le 3)$

20. $\displaystyle\int_C \frac{x}{1+y^2}\,ds$

$C: x = 1 + 2t,\ y = t \quad (0 \le t \le 1)$

21. $\displaystyle\int_C 3x^2yz\,ds$

$C: x = t,\ y = t^2,\ z = \frac{2}{3}t^3 \quad (0 \le t \le 1)$

22. $\displaystyle\int_C \frac{e^{-z}}{x^2+y^2}\,ds$

$C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k} \quad (0 \le t \le 2\pi)$

Answe Q19 to 22

19. $\displaystyle\int_0^3 \frac{\sqrt{1+t}}{1+t}\,dt = \int_0^3 (1+t)^{-1/2}\,dt = 2.$

20. $\displaystyle\sqrt{5}\int_0^1 \frac{1+2t}{1+t^2}\,dt = \sqrt{5}(\pi/4+\ln 2).$

21. $\displaystyle\int_0^1 3(t^2)(t^2)(2t^3/3)(1+2t^2)\,dt = 2\int_0^1 t^7(1+2t^2)\,dt = 13/20.$

22. $\displaystyle\frac{\sqrt{5}}{4}\int_0^{2\pi} e^{-t}\,dt = \sqrt{5}(1-e^{-2\pi})/4.$

37–40 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C.

37. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$
$C: \mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} \quad (0 \le t \le \pi)$

38. $\mathbf{F}(x, y) = x^2y\mathbf{i} + 4\mathbf{j}$
$C: \mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} \quad (0 \le t \le 1)$

39. $\mathbf{F}(x, y) = (x^2 + y^2)^{-3/2}(x\mathbf{i} + y\mathbf{j})$
$C: \mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} \quad (0 \le t \le 1)$

40. $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
$C: \mathbf{r}(t) = \sin t\mathbf{i} + 3\sin t\mathbf{j} + \sin^2 t\mathbf{k} \quad (0 \le t \le \pi/2)$

Answer Q3 t0 40

37. $\int_0^{\pi} (0)dt = 0.$

38. $\int_0^{1} (e^{2t} - 4e^{-t})dt = e^2/2 + 4e^{-1} - 9/2.$

39. $\int_0^{1} e^{-t}dt = 1 - e^{-1}$

40. $\int_0^{\pi/2} (7\sin^2 t\cos t + 3\sin t\cos t)dt = 23/6.$

Formula (9) has an alternative expression for a curve C in the xy-plane that is given by parametric equations

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

In this case, we write (9) in the expanded form

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{11}$$

Similarly, if C is a curve in 3-space that is parametrized by

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b)$$

then we write (10) in the form

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{12}$$

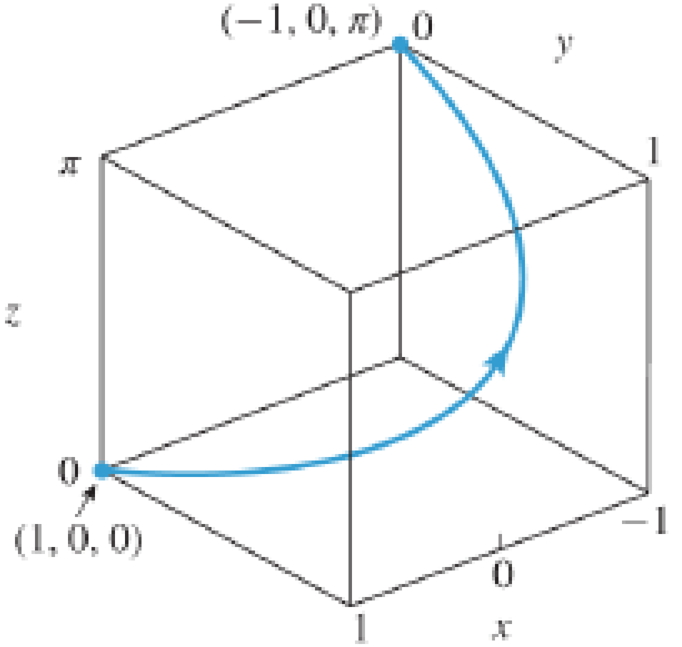
$$x = x(t), y = y(t)\ , z = z(t)\ \ where\, a \leq t \leq b$$
$$OR\quad r(t) = x(t)i + y(t)j + z(t)k$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

► Example 2 Evaluate the line integral $\int_C (xy + z^3)\,ds$ from $(1, 0, 0)$ to $(-1, 0, \pi)$ along the helix C that is represented by the parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = t \qquad (0 \le t \le \pi)$$

(Figure 15.2.7).



▲ **Figure 15.2.7**

Solution. From (12)

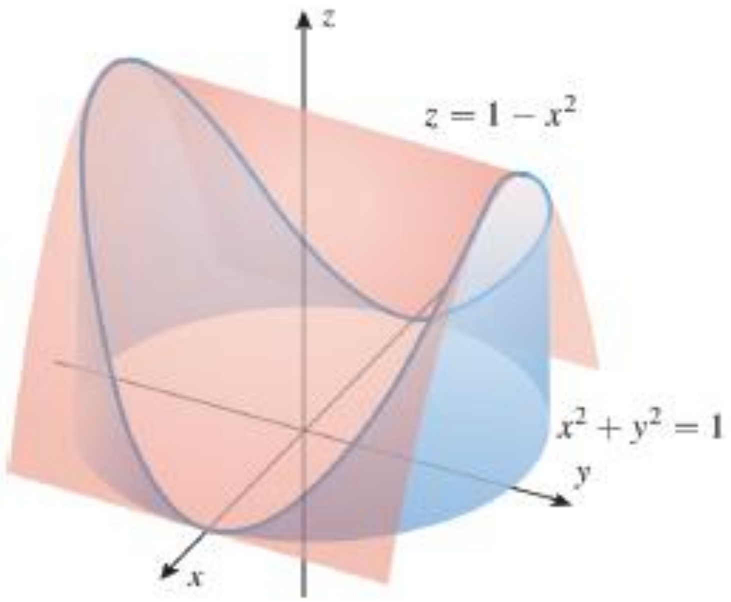
$$\int_C (xy + z^3)\,ds = \int_0^{\pi} (\cos t \sin t + t^3)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi} (\cos t \sin t + t^3)\sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\,dt$$

$$= \sqrt{2}\int_0^{\pi} (\cos t \sin t + t^3)\,dt$$

$$= \sqrt{2}\left[\frac{\sin^2 t}{2} + \frac{t^4}{4}\right]_0^{\pi} = \frac{\sqrt{2}\pi^4}{4} \blacktriangleleft$$

Example 4 Find the area of the surface extending upward from the circle $x^2+y^2=1$ in the xy-plane to the parabolic cylinder $z=1-x^2$ (Figure 15.2.9).



► Figure 15.2.9

Solution. It follows from (7) that the area A of the surface can be expressed as the line integral

$$A = \int_C (1 - x^2)\, ds \tag{15}$$

where C is the circle $x^2 + y^2 = 1$. This circle can be parametrized in terms of arc length as

$$x = \cos s, \quad y = \sin s \qquad (0 \le s \le 2\pi)$$

Thus, it follows from (13) and (15) that

$$\begin{aligned} A &= \int_C (1 - x^2)\, ds = \int_0^{2\pi} (1 - \cos^2 s)\, ds \\ &= \int_0^{2\pi} \sin^2 s\, ds = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2s)\, ds = \pi \end{aligned}$$ ◀

LINE INTEGRALS WITH RESPECT TO *x*, *y*, AND *z*

We now describe a second type of line integral in which we replace the "ds" in the integral by dx, dy, or dz. For example, suppose that f is a function defined on a smooth curve C in the xy-plane and that partition points of C are denoted by $P_k(x_k, y_k)$. Letting

$$IF\ \ P(x_0, y_0, , z_0)\ and\ Q(x_1, y_1, z_1)$$

Parametric : $x(t) = x_0 + (x_1 - x_0)t$

: $\mathrm{y}(t) = y_0 + (y_1 - y_0)t$:

$\mathrm{z}(t) = z_0 + (z_1 - z_0)t$

$x = x(t), y = y(t)\ , z = z(t)\ where\, a \le t \le b$

$OR \quad r(t) = x(t)i + y(t)j + z(t)k$

$$\int_C \ f(x,y,z)dx = \int_a^b f\big(x(t), y(t), z(t)\big)x'(t)dt$$

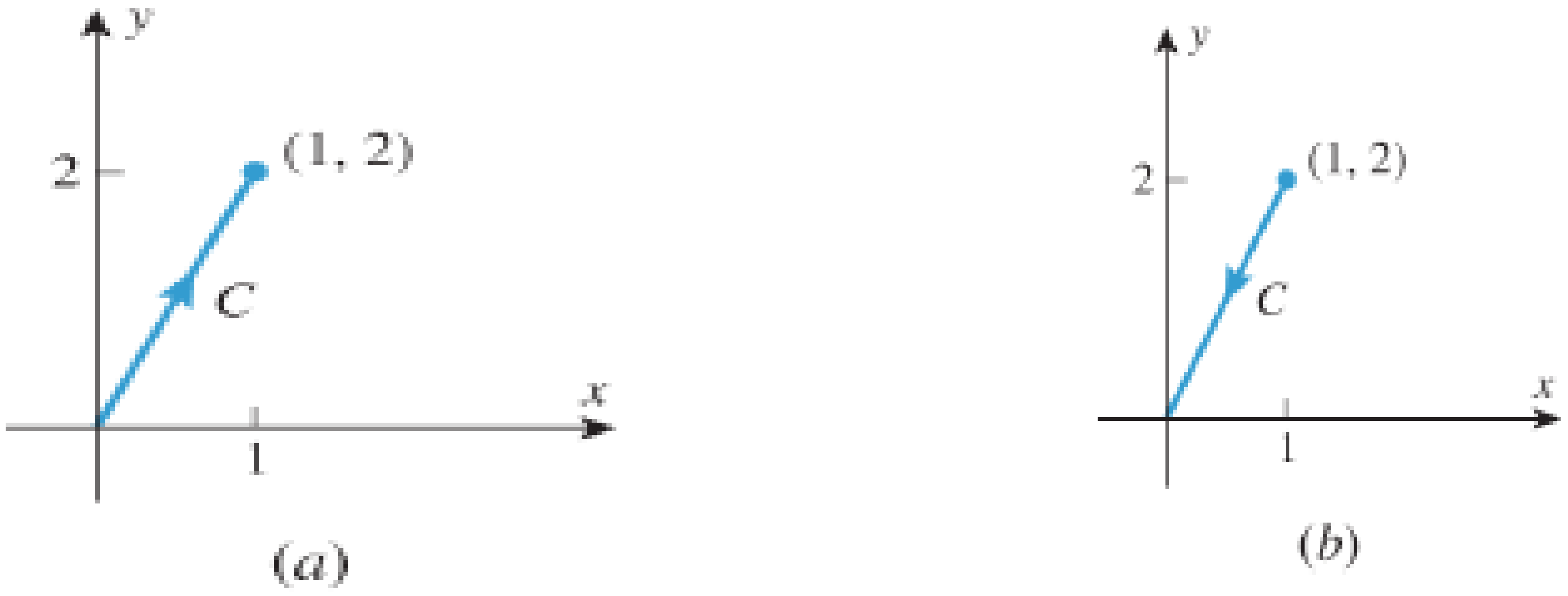
$$\int_C \ f(x,y,z)dy = \int_a^b f\big(x(t), y(t), z(t)\big)y'(t)dt$$

$$\int_C \ f(x,y,z)dz = \int_a^b f\big(x(t), y(t), z(t)\big)z'(t)dt$$

▶ **Example 5** Evaluate $\int_C 3xy\,dy$, where C is the line segment joining $(0,0)$ and $(1,2)$ with the given orientation.

(a) Oriented from $(0,0)$ to $(1,2)$ as in Figure 15.2.6*a*.

(b) Oriented from $(1,2)$ to $(0,0)$ as in Figure 15.2.6*b*.



$$IF \;\; P(x_0, y_0,, z_0) \; and \; Q(x_1, y_1, z_1)$$

Parametric : $x(t) = x_0 + (x_1 - x_0)t$

: $\mathrm{y}(t) = y_0 + (y_1 - y_0)t$:

$\mathrm{z}(t) = z_0 + (z_1 - z_0)t$

Solution ***(a).*** Using the parametrization

$$x = t, \quad y = 2t \qquad (0 \le t \le 1)$$

we have

$$\int_C 3xy\, dy = \int_0^1 3(t)(2t)(2)\, dt = \int_0^1 12t^2\, dt = 4t^3\Big]_0^1 = 4$$

$$\int_C \qquad \int_0 \qquad \int_0 \qquad \Big]_0$$

Solution (b). Using the parametrization

$$x = 1 - t, \quad y = 2 - 2t \qquad (0 \leq t \leq 1)$$

we have

$$\int_C 3xy\,dy = \int_0^1 3(1-t)(2-2t)(-2)\,dt = \int_0^1 -12(1-t)^2\,dt = 4(1-t)^3\Big]_0^1 = -4$$

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \quad \text{and} \quad \int_{-C} g(x,y)\,dy = -\int_C g(x,y)\,dy \tag{18–19}$$

while

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds \tag{20}$$

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_C f(x,y)\,dx + \int_C g(x,y)\,dy \tag{21}$$

Example 6 Evaluate

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy$$

along the circular arc C given by $x = \cos t, y = \sin t\,(0 \le t \le \pi/2)$ (Figure 15.2.11).

Solution. We have

$$\int_C 2xy\,dx = \int_0^{\pi/2} (2\cos t \sin t)\left[\frac{d}{dt}(\cos t)\right] dt$$

$$= -2\int_0^{\pi/2} \sin^2 t \cos t\,dt = -\frac{2}{3}\sin^3 t\,\Big]_0^{\pi/2} = -\frac{2}{3}$$

$$\int_C (x^2+y^2)\,dy = \int_0^{\pi/2} (\cos^2 t + \sin^2 t)\left[\frac{d}{dt}(\sin t)\right] dt$$

$$= \int_0^{\pi/2} \cos t\,dt = \sin t\,\Big]_0^{\pi/2} = 1$$

Thus, from (21)

$$\int_C 2xy\,dx + (x^2+y^2)\,dy = \int_C 2xy\,dx + \int_C (x^2+y^2)\,dy$$

$$= -\frac{2}{3} + 1 = \frac{1}{3} \blacktriangleleft$$

It can be shown that if f and g are continuous functions on C, then combinations of line integrals with respect to x and y can be expressed in terms of a limit and can be evaluated together in a single step. For example, we have

Example 7 Evaluate

$$\int_C (3x^2 + y^2)\,dx + 2xy\,dy$$

along the circular arc C given by $x = \cos t, y = \sin t\,(0 \le t \le \pi/2)$ (Figure 15.2.11).

Solution. From (23) we have

$$\begin{aligned}
\int_C (3x^2 + y^2)\,dx + 2xy\,dy &= \int_0^{\pi/2} [(3\cos^2 t + \sin^2 t)(-\sin t) + 2(\cos t)(\sin t)(\cos t)]\,dt \\
&= \int_0^{\pi/2} (-3\cos^2 t\sin t - \sin^3 t + 2\cos^2 t\sin t)\,dt \\
&= \int_0^{\pi/2} (-\cos^2 t - \sin^2 t)(\sin t)\,dt = \int_0^{\pi/2} -\sin t\,dt \\
&= \cos t\Big]_0^{\pi/2} = -1
\end{aligned}$$

11. Let C be the curve represented by the equations

$$x = 2t, \quad y = t^2 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\displaystyle\int_C (x - \sqrt{y})\, ds$

(b) $\displaystyle\int_C (x - \sqrt{y})\, dx$

(c) $\displaystyle\int_C (x - \sqrt{y})\, dy$

Answer Q11:

11. (a) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, so $\int_0^1 (2t - \sqrt{t^2})\sqrt{4+4t^2}\,dt = \int_0^1 2t\sqrt{1+t^2}\,dt = \frac{2}{3}(1+t^2)^{3/2}\Big]_0^1 = \frac{2}{3}(2\sqrt{2}-1)$.

(b) $\int_0^1 (2t - \sqrt{t^2})2\,dt = 1$.

(c) $\int_0^1 (2t - \sqrt{t^2})2t\,dt = \frac{2}{3}$.

12. Let C be the curve represented by the equations

$$x = t, \quad y = 3t^2, \quad z = 6t^3 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\int_C xyz^2\,ds$

(b) $\int_C xyz^2\,dx$

(c) $\int_C xyz^2\,dy$

(d) $\int_C xyz^2\,dz$

Answer 12

12. (a) $\displaystyle\int_0^1 t(3t^2)(6t^3)^2\sqrt{1+36t^2+324t^4}\,dt = \frac{864}{5}.$ (b) $\displaystyle\int_0^1 t(3t^2)(6t^3)^2\,dt = \frac{54}{5}.$

(c) $\displaystyle\int_0^1 t(3t^2)(6t^3)^2 6t\,dt = \frac{648}{11}.$ (d) $\displaystyle\int_0^1 t(3t^2)(6t^3)^2 18t^2\,dt = 162.$

13. In each part, evaluate the integral

$$\int_C (3x + 2y)\,dx + (2x - y)\,dy$$

along the stated curve.

(a) The line segment from $(0, 0)$ to $(1, 1)$.

(b) The parabolic arc $y = x^2$ from $(0, 0)$ to $(1, 1)$.

(c) The curve $y = \sin(\pi x/2)$ from $(0, 0)$ to $(1, 1)$.

(d) The curve $x = y^3$ from $(0, 0)$ to $(1, 1)$.

Answer Q13

13. **(a)** $C: x=t,\ y=t,\ 0\le t\le 1;\ \int_0^1 6t\,dt=3.$

(b) $C: x=t,\ y=t^2,\ 0\le t\le 1;\ \int_0^1 (3t+6t^2-2t^3)dt=3.$

(c) $C: x=t,\ y=\sin(\pi t/2),\ 0\le t\le 1;\ \int_0^1 [3t+2\sin(\pi t/2)+\pi t\cos(\pi t/2)-(\pi/2)\sin(\pi t/2)\cos(\pi t/2)]dt=3.$

(d) $C: x=t^3,\ y=t,\ 0\le t\le 1;\ \int_0^1 (9t^5+8t^3-t)dt=3.$

14. In each part, evaluate the integral

$$\int_C y\,dx + z\,dy - x\,dz$$

along the stated curve.

(a) The line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

(b) The twisted cubic $x = t$, $y = t^2$, $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$.

(c) The helix $x = \cos \pi t$, $y = \sin \pi t$, $z = t$ from $(1, 0, 0)$ to $(-1, 0, 1)$.

Answer Q14

14. **(a)** $C: x = t, y = t, z = t, 0 \le t \le 1; \int_0^1 (t + t - t)\,dt = \frac{1}{2}.$

(b) $C: x = t, y = t^2, z = t^3, 0 \le t \le 1; \int_0^1 (t^2 + t^3(2t) - t(3t^2))\,dt = -\frac{1}{60}.$

(c) $C: x = \cos \pi t, y = \sin \pi t, z = t, 0 \le t \le 1; \int_0^1 (-\pi \sin^2 \pi t + \pi t \cos \pi t - \cos \pi t)\,dt = -\frac{\pi}{2} - \frac{2}{\pi}.$

23–30 Evaluate the line integral along the curve C.

23. $\displaystyle\int_C (x + 2y)\,dx + (x - y)\,dy$

$C : x = 2\cos t,\ \ y = 4\sin t \quad (0 \le t \le \pi/4)$

24. $\displaystyle\int_C (x^2 - y^2)\,dx + x\,dy$

$C : x = t^{2/3},\ \ y = t \quad (-1 \le t \le 1)$

25. $\displaystyle\int_C -y\,dx + x\,dy$

$C : y^2 = 3x$ from $(3, 3)$ to $(0, 0)$

26. $\displaystyle\int_C (y - x)\,dx + x^2y\,dy$

$C : y^2 = x^3$ from $(1, -1)$ to $(1, 1)$

27. $\displaystyle\int_C (x^2 + y^2)\,dx - x\,dy$

$C: x^2 + y^2 = 1$, counterclockwise from $(1, 0)$ to $(0, 1)$

28. $\displaystyle\int_C (y - x)\,dx + xy\,dy$

C: the line segment from $(3, 4)$ to $(2, 1)$

29. $\displaystyle\int_C yz\,dx - xz\,dy + xy\,dz$

$C: x = e^t,\ y = e^{3t},\ z = e^{-t} \quad (0 \le t \le 1)$

30. $\displaystyle\int_C x^2\,dx + xy\,dy + z^2\,dz$

$C: x = \sin t,\ y = \cos t,\ z = t^2 \quad (0 \le t \le \pi/2)$

Answer Q23 to 30

23. $\displaystyle\int_0^{\pi/4} (8\cos^2 t - 16\sin^2 t - 20\sin t\cos t)dt = 1 - \pi.$

24. $\displaystyle\int_{-1}^{1} \left(\frac{2}{3}t - \frac{2}{3}t^{5/3} + t^{2/3}\right) dt = 6/5.$

25. $C : x = (3-t)^2/3,\ y = 3-t,\ 0 \le t \le 3;\ \displaystyle\int_0^3 \frac{1}{3}(3-t)^2 dt = 3.$

26. $C : x = t^{2/3},\ y = t,\ -1 \le t \le 1;\ \displaystyle\int_{-1}^{1} \left(\frac{2}{3}t^{2/3} - \frac{2}{3}t^{1/3} + t^{7/3}\right) dt = 4/5.$

27. $C : x = \cos t,\ y = \sin t,\ 0 \le t \le \pi/2;\ \displaystyle\int_0^{\pi/2} (-\sin t - \cos^2 t)dt = -1 - \pi/4.$

28. $C: x = 3 - t,\ y = 4 - 3t,\ 0 \leq t \leq 1;\ \int_0^1 (-37 + 41t - 9t^2)dt = -39/2.$

29. $\int_0^1 (-3)e^{3t}dt = 1 - e^3.$

30. $\int_0^{\pi/2} (\sin^2 t \cos t - \sin^2 t \cos t + t^4(2t))\, dt = \frac{\pi^6}{192}.$