13.6
DIRECTIONAL DERIVATIVES AND GRADIENTS

13.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ represent the rates of change of $f(x, y)$ in directions parallel to the x- and y-axes. In this section we will investigate rates of change of $f(x, y)$ in other directions.

Example 1 Figure 14.29 shows the temperature, in °C, at the point (x, y). Estimate the average rate of change of temperature as we walk from point A to point B.

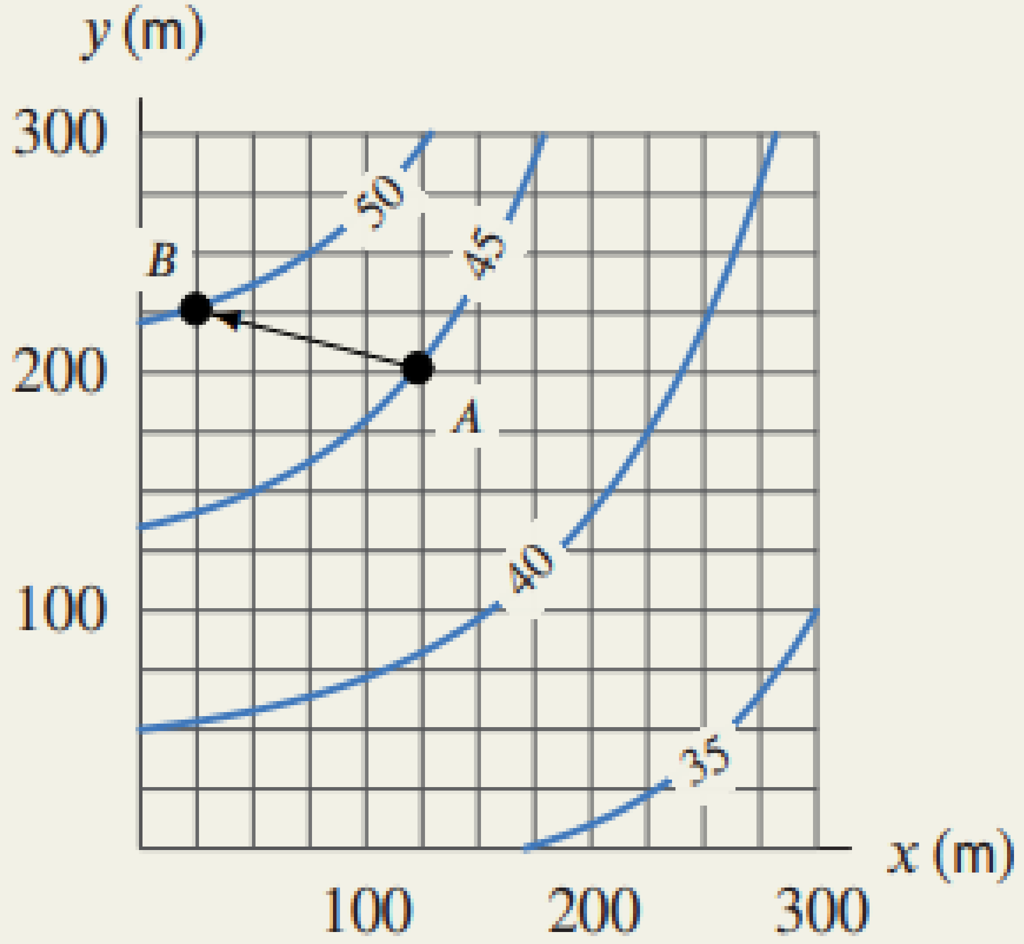


Figure 14.29: Estimating rate of change on a temperature map

Solution At the point A we are on the $H = 45°C$ contour. At B we are on the $H = 50°C$ contour. The displacement vector from A to B has x component approximately $-100\vec{i}$ and y component approximately $25\vec{j}$, so its length is $\sqrt{(-100)^2 + 25^2} \approx 103$. Thus, the temperature rises by 5°C as we move 103 meters, so the average rate of change of the temperature in that direction is about $5/103 \approx 0.05°C/m$.

Suppose we want to compute the rate of change of a function $f(x, y)$ at the point $P = (a, b)$ in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$. For $h > 0$, consider the point $Q = (a + hu_1, b + hu_2)$ whose displacement from P is $h\vec{u}$. (See Figure 14.30.) Since $\|\vec{u}\| = 1$, the distance from P to Q is h. Thus,

$$\begin{array}{l}\text{Average rate of change}\\ \text{in } f \text{ from } P \text{ to } Q\end{array} = \frac{\text{Change in } f}{\text{Distance from } P \text{ to } Q} = \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.$$

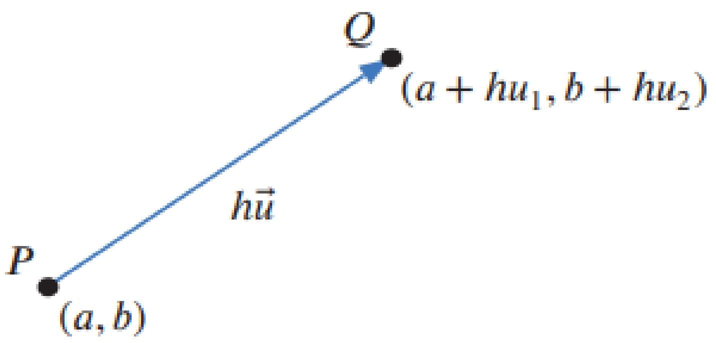
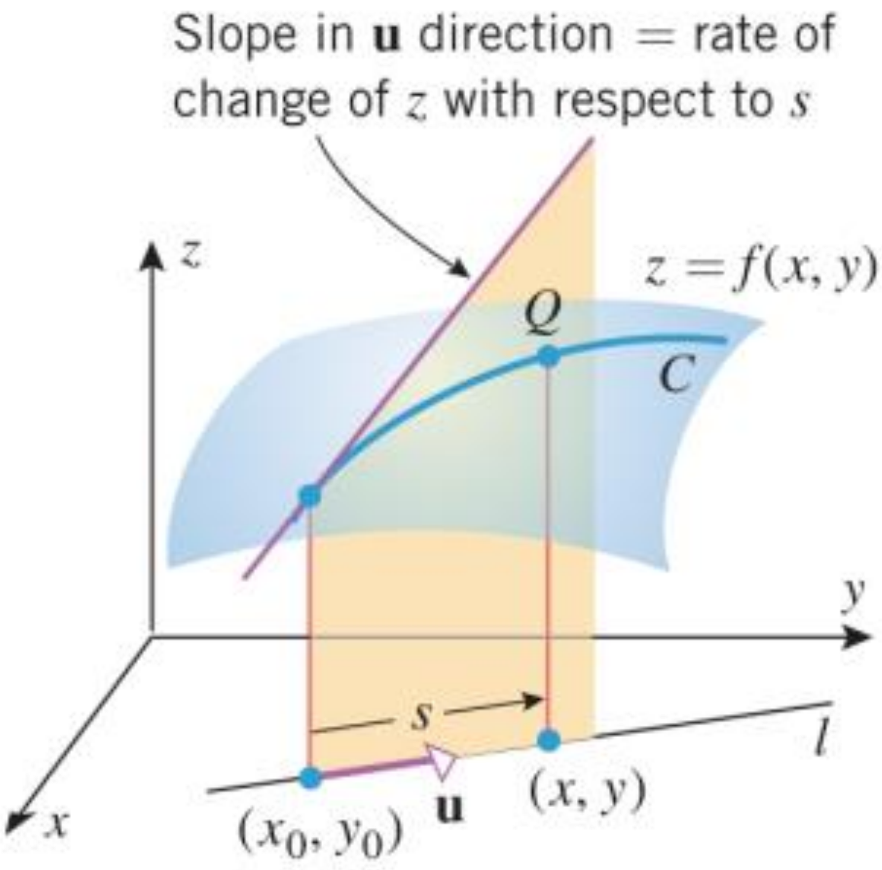
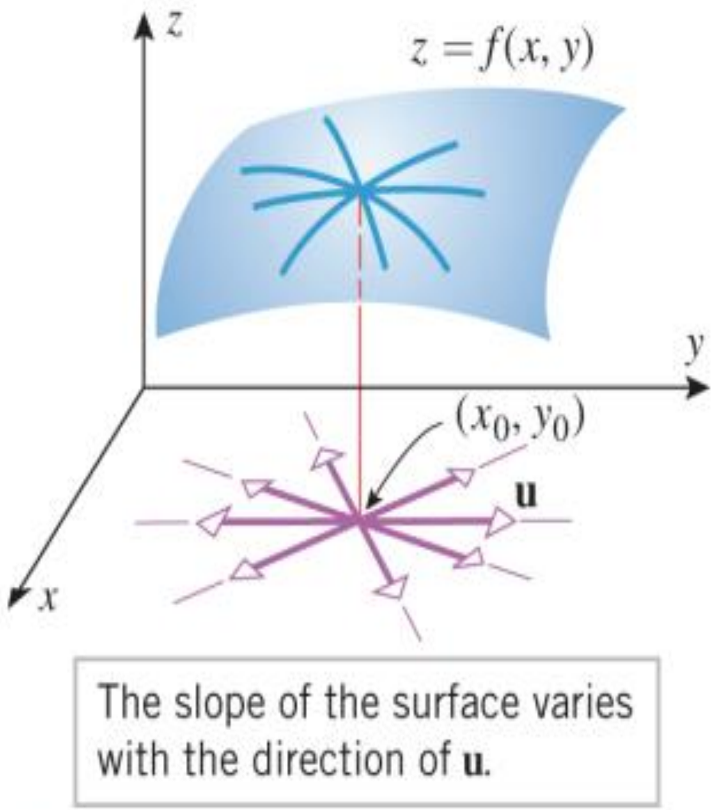


Figure 14.30: Displacement of $h\vec{u}$ from the point (a, b)

Taking the limit as $h \to 0$ gives the instantaneous rate of change and the following definition:



▲ **Figure 13.6.2**



▲ **Figure 13.6.3**

Directional Derivative of f at (a, b) in the Direction of a Unit Vector $\vec{u}$

If $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, we define the directional derivative, $f_{\vec{u}}$, by

$$f_{\vec{u}}(a, b) = \begin{array}{c}\text{Rate of change} \\ \text{of } f \text{ in direction} \\ \text{of } \vec{u} \text{ at } (a, b)\end{array} = \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists. Note that the directional derivative is a scalar.

Notice that if $\vec{u} = \vec{i}$, so $u_1 = 1, u_2 = 0$, then the directional derivative is f_x, since

$$f_{\vec{i}}(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h} = f_x(a, b).$$

Similarly, if $\vec{u} = \vec{j}$ then the directional derivative $f_{\vec{j}} = f_y$.

Example 2 For each of the functions f, g, and h in Figure 14.31, decide whether the directional derivative at the indicated point is positive, negative, or zero, in the direction of the vector $\vec{v} = \vec{i} + 2\vec{j}$, and in the direction of the vector $\vec{w} = 2\vec{i} + \vec{j}$.

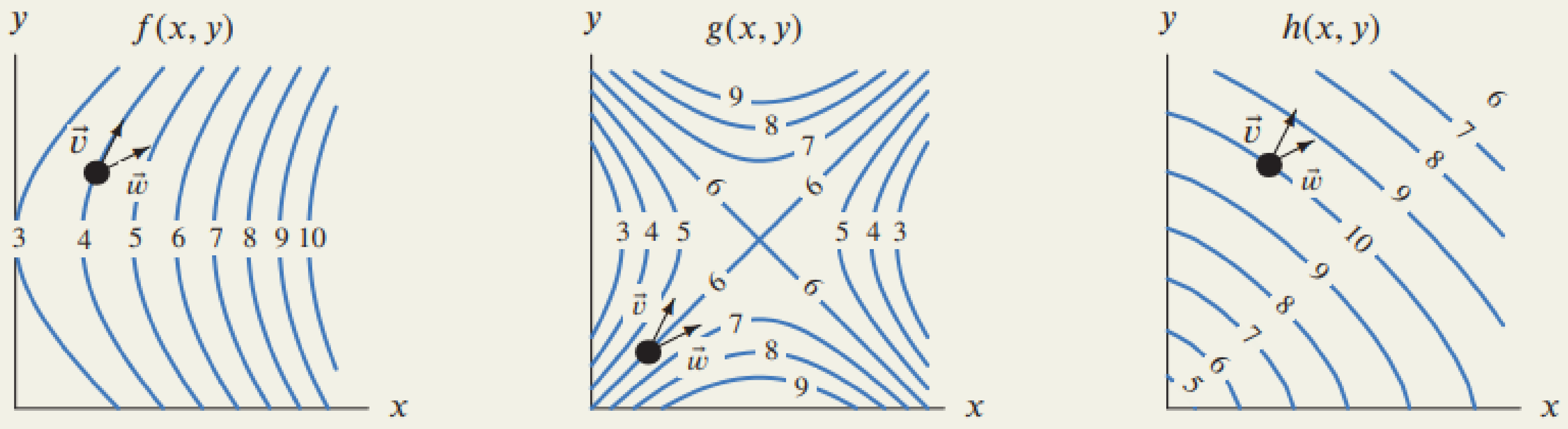
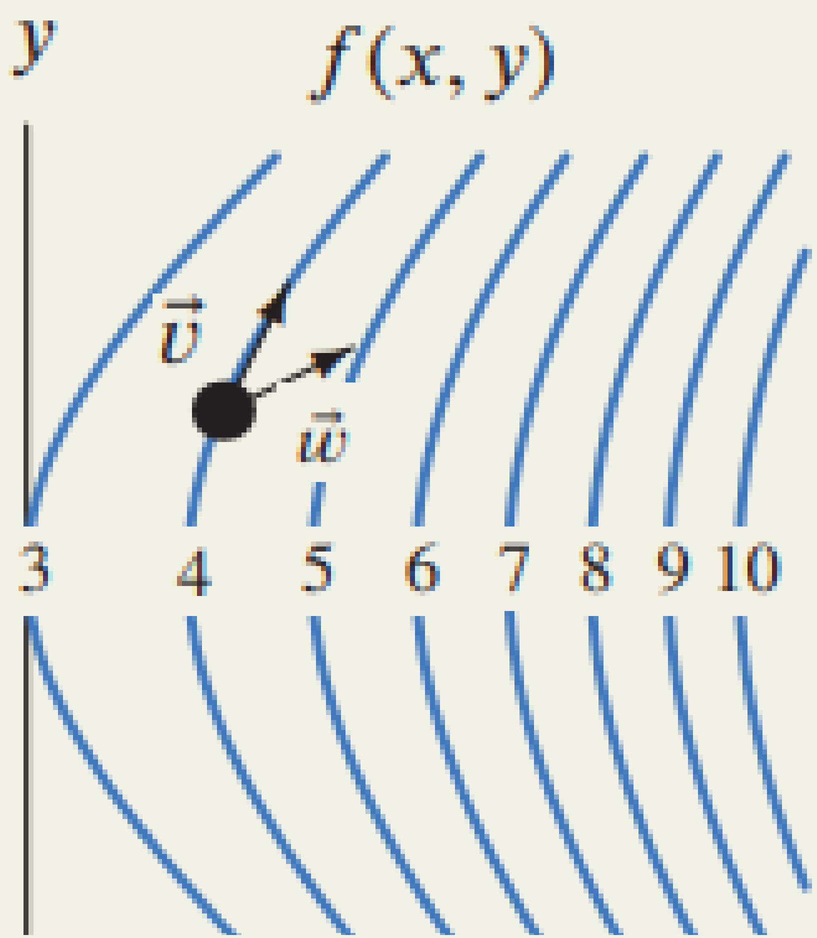
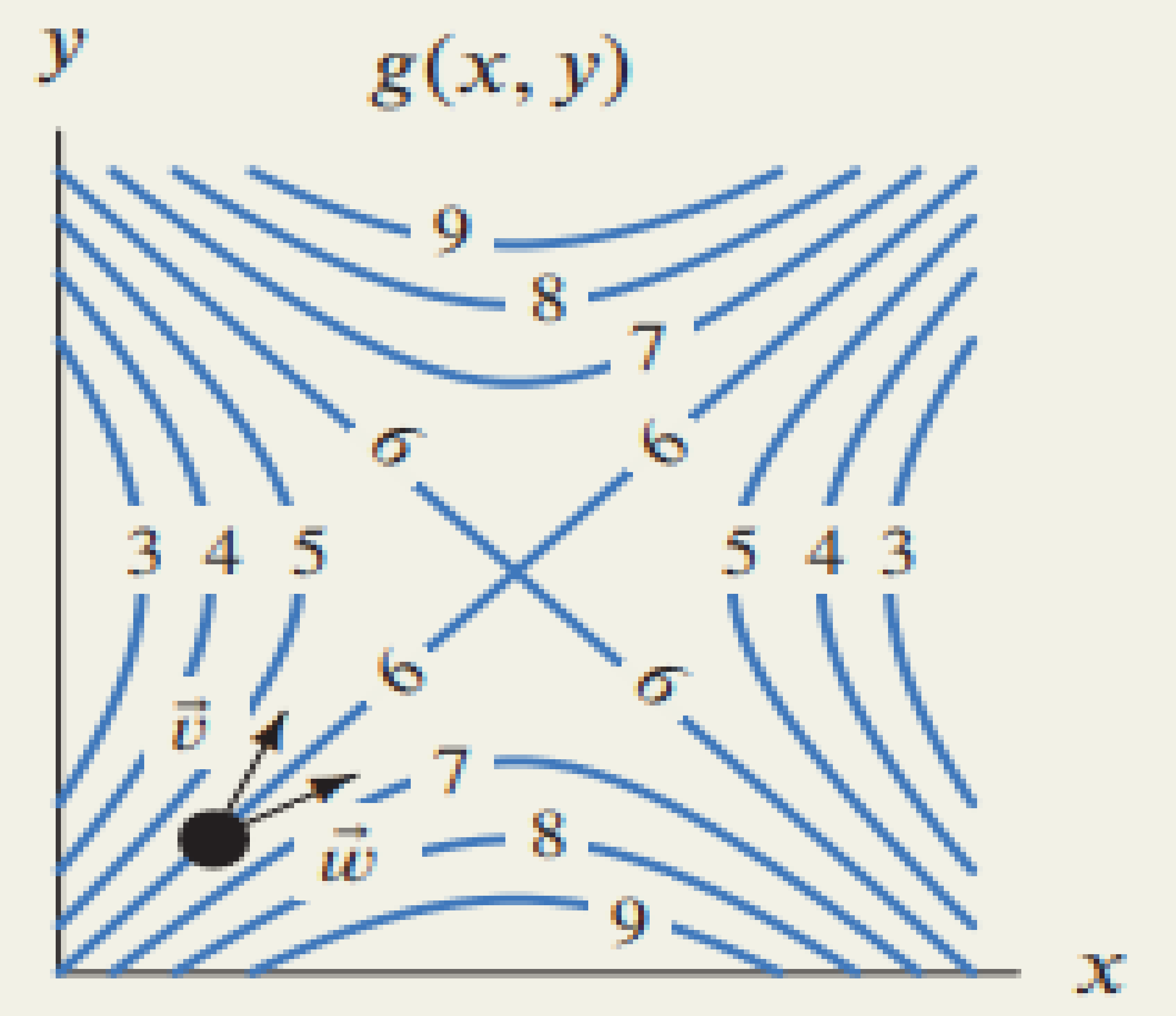


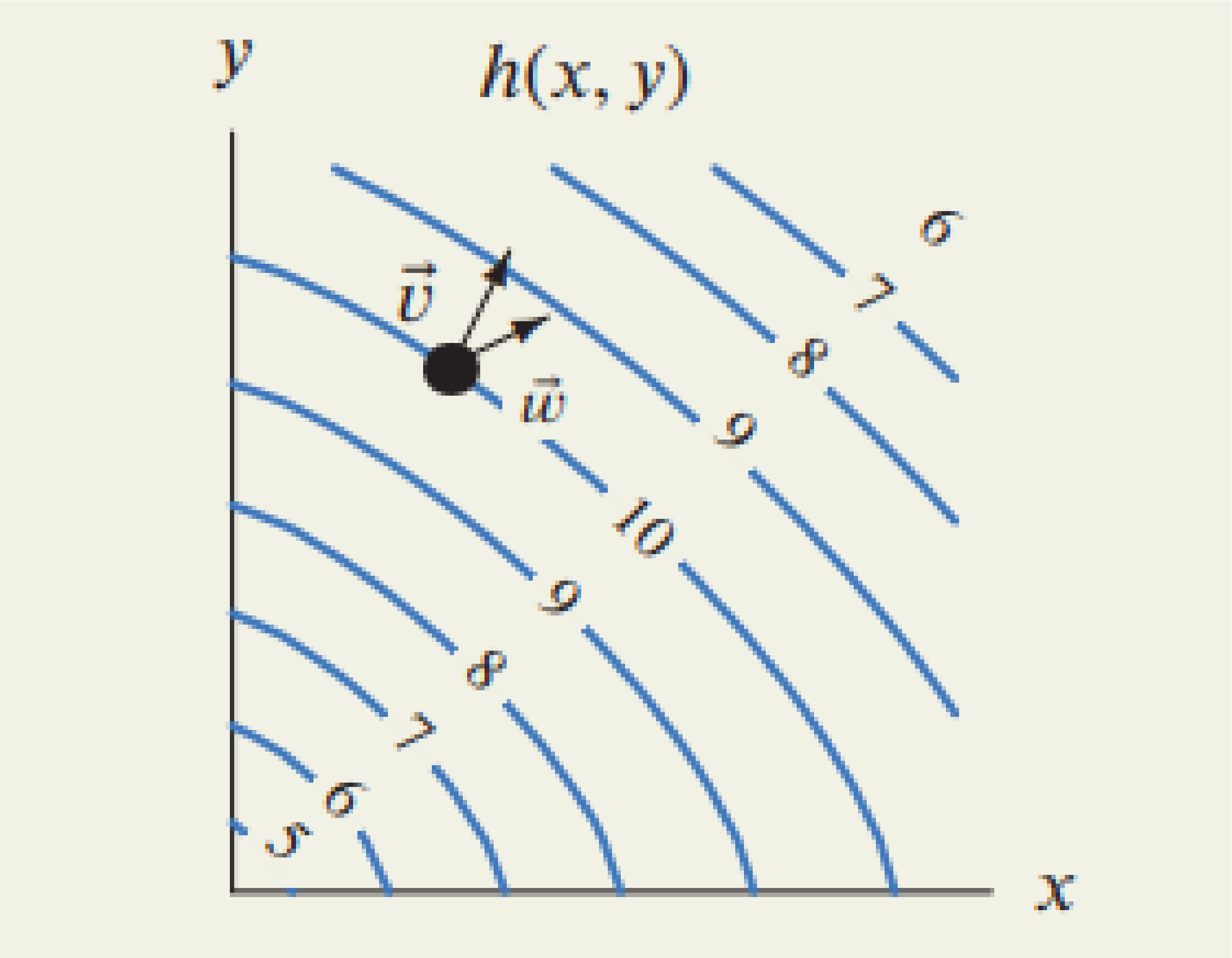
Figure 14.31: Contour diagrams of three functions with direction vectors $\vec{v} = \vec{i} + 2\vec{j}$ and $\vec{w} = 2\vec{i} + \vec{j}$ marked on each



On the contour diagram for f, the vector $\vec{v} = \vec{i} + 2\vec{j}$ appears to be tangent to the contour. Thus, in this direction, the value of the function is not changing, so the directional derivative in the direction of $\vec{v}$ is zero. The vector $\vec{w} = 2\vec{i} + \vec{j}$ points from the contour marked 4 toward the contour marked 5. Thus, the values of the function are increasing and the directional derivative in the direction of $\vec{w}$ is positive.



On the contour diagram for g, the vector $\vec{v} = \vec{i} + 2\vec{j}$ points from the contour marked 6 toward the contour marked 5, so the function is decreasing in that direction. Thus, the rate of change is negative. On the other hand, the vector $\vec{w} = 2\vec{i} + \vec{j}$ points from the contour marked 6 toward the contour marked 7, and hence the directional derivative in the direction of $\vec{w}$ is positive.



Finally, on the contour diagram for h, both vectors point from the $h = 10$ contour to the $h = 9$ contour, so both directional derivatives are negative.

13.6.1 DEFINITION If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the ***directional derivative of f in the direction of* u** at (x_0, y_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0)$ and is defined by

$$D_{\mathbf{u}}f(x_0, y_0) = \frac{d}{ds}\left[f(x_0 + su_1, y_0 + su_2)\right]_{s=0} \tag{2}$$

provided this derivative exists.

Geometrically, $D_{\mathbf{u}}f(x_0, y_0)$ can be interpreted as the ***slope of the surface z = f(x, y) in the direction of* u** at the point $(x_0, y_0, f(x_0, y_0))$ (Figure 13.6.2). Usually the value of $D_{\mathbf{u}}f(x_0, y_0)$ will depend on both the point (x_0, y_0) and the direction $\mathbf{u}$. Thus, at a fixed point the slope of the surface may vary with the direction (Figure 13.6.3). Analytically, the directional derivative represents the ***instantaneous rate of change of f(x, y) with respect to distance in the direction of* u** at the point (x_0, y_0).

13.6.3 THEOREM

(*a*) *If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \tag{4}$$

(*b*) *If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0, z_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3 \tag{5}$$

▶ Example 1 Let $f(x, y) = xy$. Find and interpret $D_{\mathbf{u}}f(1, 2)$ for the unit vector

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

We can use Theorem 13.6.3 to confirm the result of Example 1. For $f(x, y) = xy$ we have $f_x(1, 2) = 2$ and $f_y(1, 2) = 1$ (verify). With

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Equation (4) becomes

$$D_{\mathbf{u}}f(1, 2) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2} = \sqrt{3} + \frac{1}{2}$$

1–8 Find $D_{\mathbf{u}}f$ at P.

1. $f(x, y) = (1 + xy)^{3/2}$; $P(3, 1)$; $\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$

2. $f(x, y) = \sin(5x - 3y)$; $P(3, 5)$; $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

3. $f(x, y) = \ln(1 + x^2 + y)$; $P(0, 0)$;
$\mathbf{u} = -\dfrac{1}{\sqrt{10}}\mathbf{i} - \dfrac{3}{\sqrt{10}}\mathbf{j}$

4. $f(x, y) = \dfrac{cx + dy}{x - y}$; $P(3, 4)$; $\mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$

5. $f(x, y, z) = 4x^5y^2z^3$; $P(2, -1, 1)$; $\mathbf{u} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

6. $f(x, y, z) = ye^{xz} + z^2$; $P(0, 2, 3)$; $\mathbf{u} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$

7. $f(x, y, z) = \ln(x^2 + 2y^2 + 3z^2)$; $P(-1, 2, 4)$;
$\mathbf{u} = -\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$

8. $f(x, y, z) = \sin xyz$; $P\left(\frac{1}{2}, \frac{1}{3}, \pi\right)$;
$\mathbf{u} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$

9–18 Find the directional derivative of f at P in the direction of $\mathbf{a}$.

9. $f(x, y) = 4x^3y^2$; $P(2, 1)$; $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$
10. $f(x, y) = 9x^3 - 2y^3$; $P(1, 0)$; $\mathbf{a} = \mathbf{i} - \mathbf{j}$
11. $f(x, y) = y^2 \ln x$; $P(1, 4)$; $\mathbf{a} = -3\mathbf{i} + 3\mathbf{j}$
12. $f(x, y) = e^x \cos y$; $P(0, \pi/4)$; $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$
13. $f(x, y) = \tan^{-1}(y/x)$; $P(-2, 2)$; $\mathbf{a} = -\mathbf{i} - \mathbf{j}$
14. $f(x, y) = xe^y - ye^x$; $P(0, 0)$; $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$
15. $f(x, y, z) = xy + z^2$; $P(-3, 0, 4)$; $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
16. $f(x, y, z) = y - \sqrt{x^2 + z^2}$; $P(-3, 1, 4)$; $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
17. $f(x, y, z) = \dfrac{z - x}{z + y}$; $P(1, 0, -3)$; $\mathbf{a} = -6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
18. $f(x, y, z) = e^{x+y+3z}$; $P(-2, 2, -1)$; $\mathbf{a} = 20\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

Recall from Formula (13) of Section 11.2 that a unit vector $\mathbf{u}$ in the xy-plane can be expressed as

$$\mathbf{u} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \tag{6}$$

where ϕ is the angle from the positive x-axis to $\mathbf{u}$. Thus, Formula (4) can also be expressed as

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)\cos\phi + f_y(x_0, y_0)\sin\phi \tag{7}$$

Example 2 Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.

Solution. The partial derivatives of f are

$$f_x(x, y) = ye^{xy}, \quad f_y(x, y) = xe^{xy}$$
$$f_x(-2, 0) = 0, \quad f_y(-2, 0) = -2$$

The unit vector $\mathbf{u}$ that makes an angle of $\pi/3$ with the positive x-axis is

$$\mathbf{u} = \cos(\pi/3)\mathbf{i} + \sin(\pi/3)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Thus, from (7)

$$\begin{aligned} D_{\mathbf{u}}f(-2, 0) &= f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3) \\ &= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3} \end{aligned}$$

19–22 Find the directional derivative of f at P in the direction of a vector making the counterclockwise angle θ with the positive x-axis.

19. $f(x, y) = \sqrt{xy}$; $P(1, 4)$; $\theta = \pi/3$

20. $f(x, y) = \dfrac{x - y}{x + y}$; $P(-1, -2)$; $\theta = \pi/2$

21. $f(x, y) = \tan(2x + y)$; $P(\pi/6, \pi/3)$; $\theta = 7\pi/4$

22. $f(x, y) = \sinh x \cosh y$; $P(0, 0)$; $\theta = \pi$

THE GRADIENT

Formula (4) can be expressed in the form of a dot product as

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\ &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u} \end{aligned}$$

Similarly, Formula (5) can be expressed as

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}) \cdot \mathbf{u}$$

In both cases the directional derivative is obtained by dotting the direction vector **u** with a new vector constructed from the first-order partial derivatives of f.

13.6.4 DEFINITION

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \tag{8}$$

(b) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \tag{9}$$

The formula for the directional derivative can be written in terms of the gradient as follows:

The Directional Derivative and the Gradient

If f is differentiable at (a, b) and $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, then

$$f_{\vec{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2 = \operatorname{grad} f(a, b) \cdot \vec{u}\,.$$

The change in f corresponding to a small change $\Delta\vec{r} = \Delta x\vec{i} + \Delta y\vec{j}$ can be estimated using the gradient:

$$\boxed{\Delta f \approx \operatorname{grad} f \cdot \Delta\vec{r}\,.}$$

PROPERTIES OF THE GRADIENT

The gradient is not merely a notational device to simplify the formula for the directional derivative; we will see that the length and direction of the gradient ∇f provide important information about the function f and the surface $z = f(x, y)$. For example, suppose that

13.6.5 THEOREM *Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P.*

(*a*) *If $\nabla f = \mathbf{0}$ at P, then all directional derivatives of f at P are zero.*

(*b*) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.*

(*c*) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P.*

Find the gradient vector of $f(x, y) = x + e^y$ at the point $(1, 1)$.

Using the definition, we have

$$\operatorname{grad} f = f_x\vec{i} + f_y\vec{j} = \vec{i} + e^y\vec{j},$$

so at the point $(1, 1)$

$$\operatorname{grad} f(1, 1) = \vec{i} + e\vec{j}.$$

33–40 Find ∇z or ∇w.

33. $z = \sin(7y^2 - 7xy)$

34. $z = 7\sin(6x/y)$

35. $z = \dfrac{6x + 7y}{6x - 7y}$

36. $z = \dfrac{6xe^{3y}}{x + 8y}$

37. $w = -x^9 - y^3 + z^{12}$

38. $w = xe^{8y}\sin 6z$

39. $w = \ln\sqrt{x^2 + y^2 + z^2}$

40. $w = e^{-5x}\sec x^2yz$

41–46 Find the gradient of f at the indicated point.

41. $f(x, y) = 5x^2 + y^4;\ (4, 2)$

42. $f(x, y) = 5 \sin x^2 + \cos 3y;\ (\sqrt{\pi}/2, 0)$

43. $f(x, y) = (x^2 + xy)^3;\ (-1, -1)$

44. $f(x, y) = (x^2 + y^2)^{-1/2};\ (3, 4)$

45. $f(x, y, z) = y \ln (x + y + z);\ (-3, 4, 0)$

46. $f(x, y, z) = y^2 z \tan^3 x;\ (\pi/4, -3, 1)$

23. Find the directional derivative of

$$f(x, y) = \frac{x}{x + y}$$

at $P(1, 0)$ in the direction of $Q(-1, -1)$.

24. Find the directional derivative of $f(x, y) = e^{-x} \sec y$ at $P(0, \pi/4)$ in the direction of the origin.

25. Find the directional derivative of $f(x, y) = \sqrt{xy}e^y$ at $P(1, 1)$ in the direction of the negative y-axis.

26. Let

$$f(x, y) = \frac{y}{x + y}$$

Find a unit vector $\mathbf{u}$ for which $D_{\mathbf{u}}f(2, 3) = 0$.

27. Find the directional derivative of

$$f(x, y, z) = \frac{y}{x + z}$$

at $P(2, 1, -1)$ in the direction from P to $Q(-1, 2, 0)$.

28. Find the directional derivative of the function

$$f(x, y, z) = x^3y^2z^5 - 2xz + yz + 3x$$

at $P(-1, -2, 1)$ in the direction of the negative z-axis.

29. Suppose that $D_{\mathbf{u}} f(1, 2) = -5$ and $D_{\mathbf{v}} f(1, 2) = 10$, where $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$ and $\mathbf{v} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$. Find

(a) $f_x(1, 2)$ (b) $f_y(1, 2)$

(c) the directional derivative of f at $(1, 2)$ in the direction of the origin.

30. Given that $f_x(-5, 1) = -3$ and $f_y(-5, 1) = 2$, find the directional derivative of f at $P(-5, 1)$ in the direction of the vector from P to $Q(-4, 3)$.

▶ Example 4 Let $f(x, y) = x^2e^y$. Find the maximum value of a directional derivative at $(-2, 0)$, and find the unit vector in the direction in which the maximum value occurs.

Solution. Since

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2xe^y\mathbf{i} + x^2e^y\mathbf{j}$$

the gradient of f at $(-2, 0)$ is

$$\nabla f(-2, 0) = -4\mathbf{i} + 4\mathbf{j}$$

By Theorem 13.6.5, the maximum value of the directional derivative is

$$\|\nabla f(-2, 0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{\|\nabla f(-2, 0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad ◀$$

53–60 Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f at P in that direction.

53. $f(x, y) = 4x^3y^2$; $P(-1, 1)$

54. $f(x, y) = 3x - \ln y$; $P(2, 4)$

55. $f(x, y) = \sqrt{x^2 + y^2}$; $P(4, -3)$

56. $f(x, y) = \dfrac{x}{x + y}$; $P(0, 2)$

57. $f(x, y, z) = x^3z^2 + y^3z + z - 1$; $P(1, 1, -1)$

58. $f(x, y, z) = \sqrt{x - 3y + 4z}$; $P(0, -3, 0)$

59. $f(x, y, z) = \dfrac{x}{z} + \dfrac{z}{y^2}$; $P(1, 2, -2)$

60. $f(x, y, z) = \tan^{-1}\left(\dfrac{x}{y + z}\right)$; $P(4, 2, 2)$

61–66 Find a unit vector in the direction in which f decreases most rapidly at P, and find the rate of change of f at P in that direction.

61. $f(x, y) = 20 - x^2 - y^2;\ P(-1, -3)$

62. $f(x, y) = e^{xy};\ P(2, 3)$

63. $f(x, y) = \cos(3x - y);\ P(\pi/6, \pi/4)$

64. $f(x, y) = \sqrt{\dfrac{x - y}{x + y}};\ P(3, 1)$

65. $f(x, y, z) = \dfrac{x + z}{z - y};\ P(5, 7, 6)$

66. $f(x, y, z) = 4e^{xy} \cos z;\ P(0, 1, \pi/4)$

Q1: Suppose the temperature at (x, y, z) is given by
$T = xy + \sin(yz)$.
In what direction should you go from the point (1, 1, 0) to decrease the temperature as quickly as possible? What is the rate of change of temperature in this direction?

Solution:

The temperature function is given by:

$$T(x, y, z) = xy + \sin(yz)$$

The gradient of T is:

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right).$$

Computing the partial derivatives:

$$.\frac{\partial T}{\partial x} = y, \quad \frac{\partial T}{\partial y} = x + z\cos(yz), \quad \frac{\partial T}{\partial z} = y\cos(yz).$$

$$Substituting\,(x, y, z) = (1,1,0)\,\frac{\partial T}{\partial x} = 1, \quad \frac{\partial T}{\partial y} = 1 + 0 \cdot \cos(0) = 1,$$

$$Thus, the gradient at\ (1,1,0)\,is: \nabla T(1,1,0) = (1,1,1)$$

$$The\ temperature\ decreases\ most\ rapidly\ in\ the\ direction\ opposite\ to\ the\ gradient:$$
$$-\nabla T(1,1,0) = (-1,-1,-1).$$

$$The\ maximum\ rate\ of\ decrease\ is\ given\ by:$$

$$Maximum\ rate\ of\ decrease = -\parallel \nabla T(1,1,0) \parallel$$

$$Computing\ the\ magnitude:$$

$$|\nabla T(1,1,0)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

$$Thus, the\ rate\ of\ decrease\ is: -\sqrt{3}.$$

$\boldsymbol{Direction}: (-1,-1,-1)$

$\boldsymbol{RateofChange}: -\sqrt{3}$

GRADIENTS ARE NORMAL TO LEVEL CURVES

13.6.6 THEOREM *Assume that $f(x, y)$ has continuous first-order partial derivatives in an open disk centered at (x_0, y_0) and that $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then $\nabla f(x_0, y_0)$ is normal to the level curve of f through (x_0, y_0).*

When we examine a contour map, we instinctively regard the distance between adjacent contours to be measured in a normal direction. If the contours correspond to equally spaced values of f, then the closer together the contours appear to be, the more rapidly the values of f will be changing in that normal direction. It follows from Theorems 13.6.5 and 13.6.6 that this rate of change of f is given by $\|\nabla f(x, y)\|$. Thus, the closer together the contours appear to be, the greater the length of the gradient of f.