PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

If $z = f(x, y)$, then one can inquire how the value of z changes if y is held fixed and x is allowed to vary, or if x is held fixed and y is allowed to vary. For example, the ideal gas law in physics states that under appropriate conditions the pressure exerted by a gas is a function of the volume of the gas and its temperature. Thus, a physicist studying gases might be interested in the rate of change of the pressure if the volume is held fixed and the temperature is allowed to vary, or if the temperature is held fixed and the volume is allowed to vary. We now define a derivative that describes such rates of change.

Suppose that (x_0, y_0) is a point in the domain of a function $f(x, y)$. If we fix $y = y_0$, then $f(x, y_0)$ is a function of the variable x alone. The value of the derivative

$$\frac{d}{dx}[f(x, y_0)]$$

at x_0 then gives us a measure of the instantaneous rate of change of f with respect to x at the point (x_0, y_0). Similarly, the value of the derivative

$$\frac{d}{dy}[f(x_0, y)]$$

at y_0 gives us a measure of the instantaneous rate of change of f with respect to y at the point (x_0, y_0). These derivatives are so basic to the study of differential calculus of multivariable functions that they have their own name and notation.

13.3.1 DEFINITION If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f, then the ***partial derivative of f with respect to x*** at (x_0, y_0) [also called the ***partial derivative of z with respect to x*** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \tag{1}$$

Similarly, the ***partial derivative of f with respect to y*** at (x_0, y_0) [also called the ***partial derivative of z with respect to y*** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \tag{2}$$

Example 1 Find $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

Solution. Since

$$f_x(x, 3) = \frac{d}{dx}[f(x, 3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

we have $f_x(1, 3) = 54 + 4 = 58$. Also, since

$$f_y(1, y) = \frac{d}{dy}[f(1, y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

we have $f_y(1, 3) = 4(3) + 2 = 14$. ◀

THE PARTIAL DERIVATIVE FUNCTIONS

Formulas (1) and (2) define the partial derivatives of a function at a specific point (x_0, y_0). However, often it will be desirable to omit the subscripts and think of the partial derivatives as functions of the variables x and y. These functions are

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \qquad f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

The following example gives an alternative way of performing the computations in Example 1.

▶ **Example 2** Find $f_x(x, y)$ and $f_y(x, y)$ for $f(x, y) = 2x^3y^2 + 2y + 4x$, and use those partial derivatives to compute $f_x(1, 3)$ and $f_y(1, 3)$.

Solution. Keeping y fixed and differentiating with respect to x yields

$$f_x(x, y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

and keeping x fixed and differentiating with respect to y yields

$$f_y(x, y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

Thus,

$$f_x(1, 3) = 6(1^2)(3^2) + 4 = 58 \quad \text{and} \quad f_y(1, 3) = 4(1^3)3 + 2 = 14$$

which agree with the results in Example 1. ◀

PARTIAL DERIVATIVE NOTATION

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

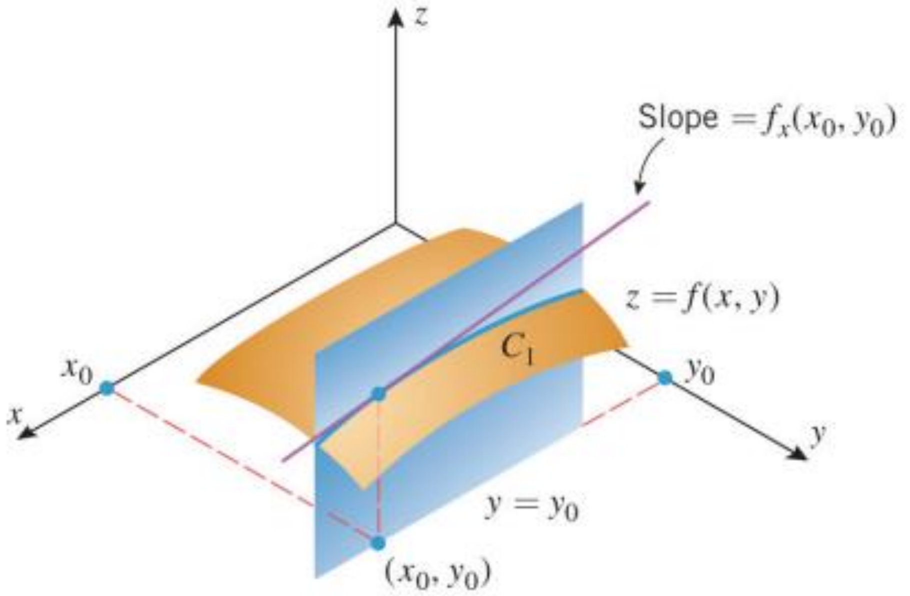
Example 3 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

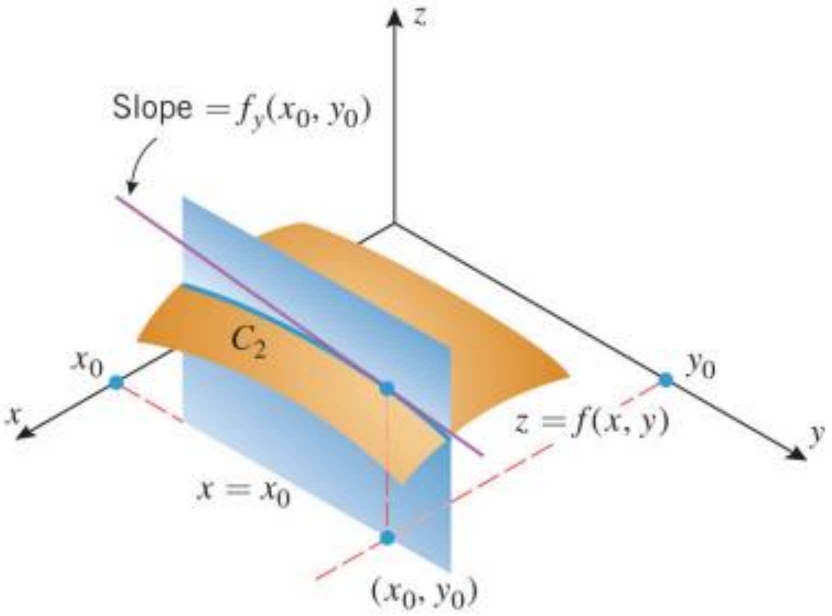
Solution.

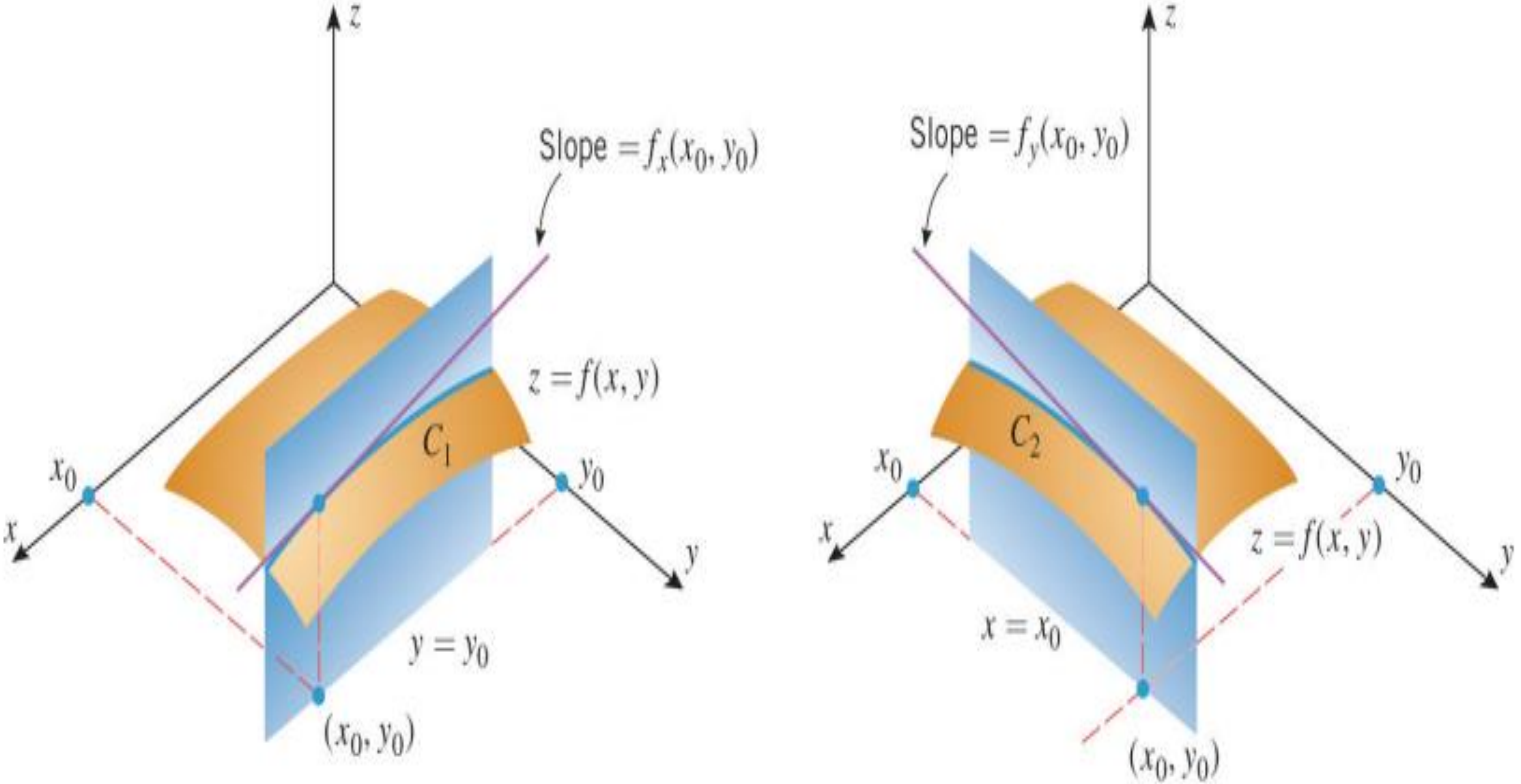
$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4) \\
&= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3 = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3) \\
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4) \\
&= x^4 \cos(xy^3) \cdot 3xy^2 + \sin(xy^3) \cdot 0 = 3x^5 y^2 \cos(xy^3)
\end{aligned}$$

PARTIAL DERIVATIVES VIEWED AS RATES OF CHANGE AND SLOPES

Recall that if $y = f(x)$, then the value of $f'(x_0)$ can be interpreted either as the rate of change of y with respect to x at x_0 or as the slope of the tangent line to the graph of f at x_0. Partial derivatives have analogous interpretations. To see that this is so, suppose that C_1 is the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$ and that C_2 is its intersection with the plane $x = x_0$ (Figure 13.3.1). Thus, $f_x(x, y_0)$ can be interpreted as the rate of change of z with respect to x along the curve C_1, and $f_y(x_0, y)$ can be interpreted as the rate of change of z with respect to y along the curve C_2. In particular, $f_x(x_0, y_0)$ is the rate of change of z with respect to x along the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ is the rate of change of z with respect to y along the curve C_2 at the point (x_0, y_0).






z
Slope $= f_x(x_0, y_0)$
$z = f(x, y)$
C_1
x_0
y_0
x
y
$y = y_0$
(x_0, y_0)
z
Slope $= f_y(x_0, y_0)$
C_2
x_0
y_0
x
y
$z = f(x, y)$
$x = x_0$
(x_0, y_0)

▶ **Example 4** Recall that the wind chill temperature index is given by the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16}$$

Compute the partial derivative of W with respect to v at the point $(T, v) = (25, 10)$ and interpret this partial derivative as a rate of change.

Solution. Holding T fixed and differentiating with respect to v yields

$$\frac{\partial W}{\partial v}(T, v) = 0 + 0 + (0.4275T - 35.75)(0.16)v^{0.16-1} = (0.4275T - 35.75)(0.16)v^{-0.84}$$

Since W is in degrees Fahrenheit and v is in miles per hour, a rate of change of W with respect to v will have units °F/(mi/h) (which may also be written as °F·h/mi). Substituting $T = 25$ and $v = 10$ gives

$$\frac{\partial W}{\partial v}(25, 10) = (-4.01)10^{-0.84} \approx -0.58\ \frac{^\circ\mathrm{F}}{\mathrm{mi/h}}$$

as the instantaneous rate of change of W with respect to v at $(T, v) = (25, 10)$. We conclude that if the air temperature is a constant 25°F and the wind speed changes by a small amount from an initial speed of 10 mi/h, then the ratio of the change in the wind chill index to the change in wind speed should be about −0.58°F/(mi/h). ◀

Example 5 Let $f(x, y) = x^2y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(1, -2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(1, -2)$.

Solution (a). Differentiating f with respect to x with y held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1, -2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

Solution (b). Differentiating f with respect to y with x held fixed yields

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y-direction is $f_y(1, -2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y. ◀

EXERCISE SET 13.3 Graphing Utility

1. Let $f(x, y) = 3x^3y^2$. Find

(a) $f_x(x, y)$ (b) $f_y(x, y)$ (c) $f_x(1, y)$

(d) $f_x(x, 1)$ (e) $f_y(1, y)$ (f) $f_y(x, 1)$

(g) $f_x(1, 2)$ (h) $f_y(1, 2)$.

2. Let $z = e^{2x} \sin y$. Find

(a) $\partial z/\partial x$ (b) $\partial z/\partial y$ (c) $\partial z/\partial x|_{(0,y)}$

(d) $\partial z/\partial x|_{(x,0)}$ (e) $\partial z/\partial y|_{(0,y)}$ (f) $\partial z/\partial y|_{(x,0)}$

(g) $\partial z/\partial x|_{(\ln 2,0)}$ (h) $\partial z/\partial y|_{(\ln 2,0)}$.

3–10 Evaluate the indicated partial derivatives.

3. $z = 9x^2y - 3x^5y; \ \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

4. $f(x, y) = 10x^2y^4 - 6xy^2 + 10x^2; \ f_x(x, y), \ f_y(x, y)$

5. $z = (x^2 + 5x - 2y)^8; \ \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

6. $f(x, y) = \dfrac{1}{xy^2 - x^2y}; \ f_x(x, y), \ f_y(x, y)$

7. $\dfrac{\partial}{\partial p}(e^{-7p/q}), \ \dfrac{\partial}{\partial q}(e^{-7p/q})$

8. $\dfrac{\partial}{\partial x}(xe^{\sqrt{15xy}}), \ \dfrac{\partial}{\partial y}(xe^{\sqrt{15xy}})$

9. $z = \sin(5x^3y + 7xy^2); \ \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

10. $f(x, y) = \cos(2xy^2 - 3x^2y^2); \ f_x(x, y), \ f_y(x, y)$

11. Let $f(x, y) = \sqrt{3x + 2y}$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(4, 2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(4, 2)$.

12. Let $f(x, y) = xe^{-y} + 5y$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(3, 0)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(3, 0)$.

13. Let $z = \sin(y^2 - 4x)$.

(a) Find the rate of change of z with respect to x at the point $(2, 1)$ with y held fixed.

(b) Find the rate of change of z with respect to y at the point $(2, 1)$ with x held fixed.

14. Let $z = (x + y)^{-1}$.

(a) Find the rate of change of z with respect to x at the point $(-2, 4)$ with y held fixed.

(b) Find the rate of change of z with respect to y at the point $(-2, 4)$ with x held fixed.

ESTIMATING PARTIAL DERIVATIVES FROM TABULAR DATA

For functions that are presented in tabular form, we can estimate partial derivatives by using adjacent entries within the table.

Example 6 Use the values of the wind chill index function $W(T, v)$ displayed in Table 13.3.1 to estimate the partial derivative of W with respect to v at $(T, v) = (25, 10)$. Compare this estimate with the value of the partial derivative obtained in Example 4.

Table 13.3.1

WIND SPEED v (mi/h) \ TEMPERATURE T (°F)	20	25	30	35
5	13	19	25	31
10	9	15	21	27
15	6	13	19	25
20	4	11	17	24

Solution. Since

$$\frac{\partial W}{\partial v}(25, 10) = \lim_{\Delta v \to 0} \frac{W(25, 10 + \Delta v) - W(25, 10)}{\Delta v} = \lim_{\Delta v \to 0} \frac{W(25, 10 + \Delta v) - 15}{\Delta v}$$

we can approximate the partial derivative by

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 + \Delta v) - 15}{\Delta v}$$

With $\Delta v = 5$ this approximation is

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 + 5) - 15}{5} = \frac{W(25, 15) - 15}{5} = \frac{13 - 15}{5} = -\frac{2}{5} \, \frac{{}^\circ\mathrm{F}}{\mathrm{mi/h}}$$

and with $\Delta v = -5$ this approximation is

$$\frac{\partial W}{\partial v}(25, 10) \approx \frac{W(25, 10 - 5) - 15}{-5} = \frac{W(25, 5) - 15}{-5} = \frac{19 - 15}{-5} = -\frac{4}{5} \, \frac{{}^\circ\mathrm{F}}{\mathrm{mi/h}}$$

We will take the average, $-\frac{3}{5} = -0.6^\circ\mathrm{F/(mi/h)}$, of these two approximations as our estimate of $(\partial W/\partial v)(25, 10)$. This is close to the value

$$\frac{\partial W}{\partial v}(25, 10) = (-4.01)10^{-0.84} \approx -0.58 \, \frac{{}^\circ\mathrm{F}}{\mathrm{mi/h}}$$

found in Example 4. ◀

17. Suppose that Nolan throws a baseball to Ryan and that the baseball leaves Nolan's hand at the same height at which it is caught by Ryan. It we ignore air resistance, the horizontal range r of the baseball is a function of the initial speed v of the ball when it leaves Nolan's hand and the angle θ above the horizontal at which it is thrown. Use the accompanying table and the method of Example 6 to estimate

(a) the partial derivative of r with respect to v whe $v = 80$ ft/s and $\theta = 40°$

(b) the partial derivative of r with respect to θ when $v = 80$ ft/s and $\theta = 40°$.

	SPEED v (ft/s)			
ANGLE θ (degrees)	75	80	85	90
35	165	188	212	238
40	173	197	222	249
45	176	200	226	253
50	173	197	222	249

Table Ex

18. Use the table in Exercise 17 and the method of Example 6 to estimate

(a) the partial derivative of r with respect to v when $v = 85$ ft/s and $\theta = 45°$

(b) the partial derivative of r with respect to θ when $v = 85$ ft/s and $\theta = 45°$.

ANGLE θ (degrees) \ SPEED v (ft/s)	75	80	85	90
35	165	188	212	238
40	173	197	222	249
45	176	200	226	253
50	173	197	222	249

Table Ex-17

25–30 Find $\partial z/\partial x$ and $\partial z/\partial y$.

25. $z = 4e^{x^2y^3}$

26. $z = \cos(x^5y^4)$

27. $z = x^3 \ln(1 + xy^{-3/5})$

28. $z = e^{xy} \sin 4y^2$

29. $z = \dfrac{xy}{x^2 + y^2}$

30. $z = \dfrac{x^2y^3}{\sqrt{x + y}}$

31–36 Find $f_x(x, y)$ and $f_y(x, y)$.

31. $f(x, y) = \sqrt{3x^5y - 7x^3y}$

32. $f(x, y) = \dfrac{x + y}{x - y}$

33. $f(x, y) = y^{-3/2} \tan^{-1}(x/y)$

34. $f(x, y) = x^3e^{-y} + y^3 \sec \sqrt{x}$

35. $f(x, y) = (y^2 \tan x)^{-4/3}$

36. $f(x, y) = \cosh(\sqrt{x}) \sinh^2(xy^2)$

37–40 Evaluate the indicated partial derivatives.

37. $f(x, y) = 9 - x^2 - 7y^3$; $f_x(3, 1)$, $f_y(3, 1)$

38. $f(x, y) = x^2ye^{xy}$; $\partial f/\partial x(1, 1)$, $\partial f/\partial y(1, 1)$

39. $z = \sqrt{x^2 + 4y^2}$; $\partial z/\partial x(1, 2)$, $\partial z/\partial y(1, 2)$

40. $w = x^2 \cos xy$; $\partial w/\partial x\left(\frac{1}{2}, \pi\right)$, $\partial w/\partial y\left(\frac{1}{2}, \pi\right)$

41. Let $f(x, y, z) = x^2y^4z^3 + xy + z^2 + 1$. Find

(a) $f_x(x, y, z)$ (b) $f_y(x, y, z)$ (c) $f_z(x, y, z)$
(d) $f_x(1, y, z)$ (e) $f_y(1, 2, z)$ (f) $f_z(1, 2, 3)$.

42. Let $w = x^2y \cos z$. Find

(a) $\partial w/\partial x(x, y, z)$ (b) $\partial w/\partial y(x, y, z)$
(c) $\partial w/\partial z(x, y, z)$ (d) $\partial w/\partial x(2, y, z)$
(e) $\partial w/\partial y(2, 1, z)$ (f) $\partial w/\partial z(2, 1, 0)$.

43–46 Find f_x, f_y, and f_z.

43. $f(x, y, z) = z \ln(x^2 y \cos z)$

44. $f(x, y, z) = y^{-3/2} \sec\left(\dfrac{xz}{y}\right)$

45. $f(x, y, z) = \tan^{-1}\left(\dfrac{1}{xy^2z^3}\right)$

46. $f(x, y, z) = \cosh(\sqrt{z}) \sinh^2(x^2yz)$

47–50 Find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

47. $w = ye^z \sin xz$

48. $w = \dfrac{x^2 - y^2}{y^2 + z^2}$

49. $w = \sqrt{x^2 + y^2 + z^2}$

50. $w = y^3 e^{2x+3z}$

51. Let $f(x, y, z) = y^2 e^{xz}$. Find
(a) $\partial f/\partial x|_{(1,1,1)}$ (b) $\partial f/\partial y|_{(1,1,1)}$ (c) $\partial f/\partial z|_{(1,1,1)}$.

52. Let $w = \sqrt{x^2 + 4y^2 - z^2}$. Find
(a) $\partial w/\partial x|_{(2,1,-1)}$ (b) $\partial w/\partial y|_{(2,1,-1)}$
(c) $\partial w/\partial z|_{(2,1,-1)}$.

57. A point moves along the intersection of the plane $y = 3$ and the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is z changing with respect to x when the point is at $(4, 3, 2)$?

58. Find the slope of the tangent line at $(-1, 1, 5)$ to the curve of intersection of the surface $z = x^2 + 4y^2$ and
(a) the plane $x = -1$ (b) the plane $y = 1$.

59. The volume V of a right circular cylinder is given by the formula $V = \pi r^2 h$, where r is the radius and h is the height.

(a) Find a formula for the instantaneous rate of change of V with respect to r if r changes and h remains constant.

(b) Find a formula for the instantaneous rate of change of V with respect to h if h changes and r remains constant.

(c) Suppose that h has a constant value of 4 in, but r varies. Find the rate of change of V with respect to r at the point where $r = 6$ in.

(d) Suppose that r has a constant value of 8 in, but h varies. Find the instantaneous rate of change of V with respect to h at the point where $h = 10$ in.

with respect to h at the point where $h = 10$ in.

60. The volume V of a right circular cone is given by

$$V = \frac{\pi}{24} d^2 \sqrt{4s^2 - d^2}$$

where s is the slant height and d is the diameter of the base.

(a) Find a formula for the instantaneous rate of change of V with respect to s if d remains constant.

(b) Find a formula for the instantaneous rate of change of V with respect to d if s remains constant.

(c) Suppose that d has a constant value of 16 cm, but s varies. Find the rate of change of V with respect to s when $s = 10$ cm.

(d) Suppose that s has a constant value of 10 cm, but d varies. Find the rate of change of V with respect to d when $d = 16$ cm.

61. According to the ideal gas law, the pressure, temperature, and volume of a gas are related by $P = kT/V$, where k is a constant of proportionality. Suppose that V is measured in cubic inches (in^3), T is measured in kelvins (K), and that for a certain gas the constant of proportionality is $k = 10\ \text{in}\cdot\text{lb/K}$.

(a) Find the instantaneous rate of change of pressure with respect to temperature if the temperature is 80 K and the volume remains fixed at 50 in^3.

(b) Find the instantaneous rate of change of volume with respect to pressure if the volume is 50 in^3 and the temperature remains fixed at 80 K.

62. The temperature at a point (x, y) on a metal plate in the xy-plane is $T(x, y) = x^3 + 2y^2 + x$ degrees Celsius. Assume that distance is measured in centimeters and find the rate at which temperature changes with respect to distance if we start at the point $(1, 2)$ and move

(a) to the right and parallel to the x-axis

(b) upward and parallel to the y-axis.

63. The length, width, and height of a rectangular box are $l = 5$, $w = 2$, and $h = 3$, respectively.

(a) Find the instantaneous rate of change of the volume of the box with respect to the length if w and h are held constant.

(b) Find the instantaneous rate of change of the volume of the box with respect to the width if l and h are held constant.

(c) Find the instantaneous rate of change of the volume of the box with respect to the height if l and w are held constant.

held constant.

64. The area A of a triangle is given by $A = \frac{1}{2}ab \sin\theta$, where a and b are the lengths of two sides and θ is the angle between these sides. Suppose that $a = 5$, $b = 10$, and $\theta = \pi/3$.

(a) Find the rate at which A changes with respect to a if b and θ are held constant.

(b) Find the rate at which A changes with respect to θ if a and b are held constant.

(c) Find the rate at which b changes with respect to a if A and θ are held constant.

65. The volume of a right circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Show that if the height remains constant while the radius changes, then the volume satisfies

$$\frac{\partial V}{\partial r} = \frac{2V}{r}$$

IMPLICIT PARTIAL DIFFERENTIATION

Example 7 Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ (Figure 13.3.2).

IMPLICIT PARTIAL DIFFERENTIATION

Example 7 Find the slope of the sphere $x^2+y^2+z^2=1$ in the y-direction at the points $\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$ and $\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)$ (Figure 13.3.2).

Solution. The point $\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$ lies on the upper hemisphere $z=\sqrt{1-x^2-y^2}$, and the point $\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)$ lies on the lower hemisphere $z=-\sqrt{1-x^2-y^2}$. We could find the slopes by differentiating each expression for z separately with respect to y and then evaluating the derivatives at $x=\frac{2}{3}$ and $y=\frac{1}{3}$. However, it is more efficient to differentiate the given equation

$$x^2+y^2+z^2=1$$

implicitly with respect to y, since this will give us both slopes with one differentiation. To perform the implicit differentiation, we view z as a function of x and y and differentiate both sides with respect to y, taking x to be fixed. The computations are as follows:

$$\frac{\partial}{\partial y}[x^2+y^2+z^2]=\frac{\partial}{\partial y}[1]$$

$$0+2y+2z\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial y}=-\frac{y}{z}$$

Substituting the y- and z-coordinates of the points $\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$ and $\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)$ in this expression, we find that the slope at the point $\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$ is $-\frac{1}{2}$ and the slope at $\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)$ is $\frac{1}{2}$.

► Example 8 Suppose that $D = \sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths x and y that are allowed to vary. Find a formula for the rate of change of D with respect to x if x varies with y held constant, and use this formula to find the rate of change of D with respect to x at the point where $x = 3$ and $y = 4$.

Solution. Differentiating both sides of the equation $D^2 = x^2 + y^2$ with respect to x yields

$$2D\frac{\partial D}{\partial x} = 2x \quad \text{and thus} \quad D\frac{\partial D}{\partial x} = x$$

Since $D = 5$ when $x = 3$ and $y = 4$, it follows that

$$5\left.\frac{\partial D}{\partial x}\right|_{x=3,y=4} = 3 \quad \text{or} \quad \left.\frac{\partial D}{\partial x}\right|_{x=3,y=4} = \frac{3}{5}$$

Thus, D is increasing at a rate of $\frac{3}{5}$ unit per unit increase in x at the point $(3, 4)$. ◀

PARTIAL DERIVATIVES AND CONTINUITY

In contrast to the case of functions of a single variable, the existence of partial derivatives for a multivariable function does not guarantee the continuity of the function. This fact is shown in the following example.

▶ Example 9 Let

$$f(x, y) = \begin{cases} -\dfrac{xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \tag{3}$$

(a) Show that $f_x(x, y)$ and $f_y(x, y)$ exist at all points (x, y).

(b) Explain why f is not continuous at $(0, 0)$.

Solution (a). Figure 13.3.3 shows the graph of f. Note that f is similar to the function considered in Example 1 of Section 13.2, except that here we have assigned f a value of 0 at $(0, 0)$. Except at this point, the partial derivatives of f are

$$f_x(x, y) = -\frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{x^2y - y^3}{(x^2 + y^2)^2} \tag{4}$$

$$f_y(x, y) = -\frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{xy^2 - x^3}{(x^2 + y^2)^2} \tag{5}$$

It is not evident from Formula (3) whether f has partial derivatives at $(0, 0)$, and if so, what the values of those derivatives are. To answer that question we will have to use the definitions of the partial derivatives (Definition 13.3.1). Applying Formulas (1) and (2) to (3) we obtain

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

This shows that f has partial derivatives at $(0, 0)$ and the values of both partial derivatives are 0 at that point.

Solution (b). We saw in Example 3 of Section 13.2 that

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist. Thus, f is not continuous at $(0,0)$. ◀

We will study the relationship between the continuity of a function and the properties of its partial derivatives in the next section.

PARTIAL DERIVATIVES OF FUNCTIONS WITH MORE THAN TWO VARIABLES

For a function $f(x, y, z)$ of three variables, there are three ***partial derivatives***:

$$f_x(x, y, z), \quad f_y(x, y, z), \quad f_z(x, y, z)$$

The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. For f_y the variables x and z are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w = f(x, y, z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \text{and} \quad \frac{\partial w}{\partial z}$$

Example 10 If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, then

$$f_x(x, y, z) = 3x^2y^2z^4 + 2y$$
$$f_y(x, y, z) = 2x^3yz^4 + 2x$$
$$f_z(x, y, z) = 4x^3y^2z^3 + 1$$
$$f_z(-1, 1, 2) = 4(-1)^3(1)^2(2)^3 + 1 = -31$$

Example 11 If $f(\rho, \theta, \phi) = \rho^2 \cos\phi \sin\theta$, then

$$f_\rho(\rho, \theta, \phi) = 2\rho \cos\phi \sin\theta$$

$$f_\theta(\rho, \theta, \phi) = \rho^2 \cos\phi \cos\theta$$

$$f_\phi(\rho, \theta, \phi) = -\rho^2 \sin\phi \sin\theta$$

HIGHER-ORDER PARTIAL DERIVATIVES

Suppose that f is a function of two variables x and y. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of f, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to x.

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to x and then with respect to y.

Differentiate first with respect to y and then with respect to x.

The last two cases are called the ***mixed second-order partial derivatives*** or the ***mixed second partials***. Also, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are often called the ***first-order partial derivatives*** when it is necessary to distinguish them from higher-order partial derivatives. Similar conventions apply to the second-order partial derivatives of a function of three variables.

WARNING Observe that the two notations for the mixed second partials have opposite conventions for the order of differentiation. In the "∂" notation the derivatives are taken right to left, and in the "subscript" notation they are taken left to right. The conventions are logical if you insert parentheses:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Right to left. Differentiate inside the parentheses first.

$$f_{xy} = (f_x)_y$$

Left to right. Differentiate inside the parentheses first.

▶ Example 12 Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution. We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

so that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3 \quad ◀$$

▶ Example 13 Let $f(x, y) = y^2e^x + y$. Find f_{xyy}.

Solution.

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y^2}(y^2e^x) = \frac{\partial}{\partial y}(2ye^x) = 2e^x \blacktriangleleft$$

EQUALITY OF MIXED PARTIALS

For a function $f(x, y)$ it might be expected that there would be four distinct second-order partial derivatives: f_{xx}, f_{xy}, f_{yx}, and f_{yy}. However, observe that the mixed second-order partial derivatives in Example 12 are equal. The following theorem (proved in Web Appendix L) explains why this is so.

13.3.2 THEOREM *Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.*

It follows from this theorem that if $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous everywhere, then $f_{xy}(x, y) = f_{yx}(x, y)$ for all values of x and y. Since polynomials are continuous everywhere, this explains why the mixed second-order partials in Example 12 are equal.

69–72 Calculate $\partial z/\partial x$ and $\partial z/\partial y$ using implicit differentiation. Leave your answers in terms of x, y, and z.

69. $(x^2 + y^2 + z^2)^{3/2} = 1$

70. $\ln(2x^2 + y - z^3) = x$

71. $x^2 + z \sin xyz = 0$

72. $e^{xy} \sinh z - z^2 x + 1 = 0$

73–76 Find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$ using implicit differentiation. Leave your answers in terms of x, y, z, and w.

73. $(x^2 + y^2 + z^2 + w^2)^{3/2} = 4$

74. $\ln(2x^2 + y - z^3 + 3w) = z$

75. $w^2 + w \sin xyz = 1$

76. $e^{xy} \sinh w - z^2 w + 1 = 0$

$\int_{0}$ $\int_{x+y}$

81. Let $z = \sqrt{x}\cos y$. Find
(a) $\partial^2 z/\partial x^2$ (b) $\partial^2 z/\partial y^2$
(c) $\partial^2 z/\partial x \partial y$ (d) $\partial^2 z/\partial y \partial x$.

82. Let $f(x, y) = 4x^2 - 2y + 7x^4y^5$. Find
(a) f_{xx} (b) f_{yy} (c) f_{xy} (d) f_{yx}.

83. Let $f(x, y) = \sin(3x^2 + 6y^2)$. Find
(a) f_{xx} (b) f_{yy} (c) f_{xy} (d) f_{yx}.

84. Let $f(x, y) = xe^{2y}$. Find
(a) f_{xx} (b) f_{yy} (c) f_{xy} (d) f_{yx}.

85–92 Confirm that the mixed second-order partial derivatives of f are the same.

85. $f(x, y) = 4x^2 - 8xy^4 + 7y^5 - 3$

86. $f(x, y) = \sqrt{x^2 + y^2}$

87. $f(x, y) = e^x \cos y$

88. $f(x, y) = e^{x-y^2}$

89. $f(x, y) = \ln(4x - 5y)$

90. $f(x, y) = \ln(x^2 + y^2)$

91. $f(x, y) = (x - y)/(x + y)$

92. $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$

93. Express the following derivatives in "∂" notation.

(a) f_{xxx} (b) f_{xyy} (c) f_{yyxx} (d) f_{xyyy}

94. Express the derivatives in "subscript" notation.

(a) $\dfrac{\partial^3 f}{\partial y^2 \partial x}$ (b) $\dfrac{\partial^4 f}{\partial x^4}$ (c) $\dfrac{\partial^4 f}{\partial y^2 \partial x^2}$ (d) $\dfrac{\partial^5 f}{\partial x^2 \partial y^3}$

95. Given $f(x, y) = x^3 y^5 - 2x^2 y + x$, find

(a) f_{xxy} (b) f_{yxy} (c) f_{yyy}.

96. Given $z = (2x - y)^5$, find

(a) $\dfrac{\partial^3 z}{\partial y \partial x \partial y}$ (b) $\dfrac{\partial^3 z}{\partial x^2 \partial y}$ (c) $\dfrac{\partial^4 z}{\partial x^2 \partial y^2}$.

97. Given $f(x, y) = y^3 e^{-5x}$, find

(a) $f_{xyy}(0, 1)$ (b) $f_{xxx}(0, 1)$ (c) $f_{yyxx}(0, 1)$.

98. Given $w = e^y \cos x$, find

(a) $\left.\dfrac{\partial^3 w}{\partial y^2 \partial x}\right|_{(\pi/4,0)}$ (b) $\left.\dfrac{\partial^3 w}{\partial x^2 \partial y}\right|_{(\pi/4,0)}$

99. Let $f(x, y, z) = x^3 y^5 z^7 + xy^2 + y^3 z$. Find

(a) f_{xy} (b) f_{yz} (c) f_{xz} (d) f_{zz}

(e) f_{zyy} (f) f_{xxy} (g) f_{zyx} (h) f_{xxyz}.

100. Let $w = (4x - 3y + 2z)^5$. Find

(a) $\dfrac{\partial^2 w}{\partial x \partial z}$ (b) $\dfrac{\partial^3 w}{\partial x \partial y \partial z}$ (c) $\dfrac{\partial^4 w}{\partial z^2 \partial y \partial x}$.