HW1-Introduction to Machine Learning

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Question 4.1:

According to base rule, sample x belongs to class w_i when

$$\forall j \neq i : P(w_i|x) \geq P(w_j|x)$$

According to Bayes' theorem:

$$P(w_i|x) = \frac{P(x|w_i) * P(w_i)}{P(x)}$$

Then, x belongs to class w_i when:

$$\forall j \neq i : \frac{P(x|w_i) * P(w_i)}{P(x)} \geq \frac{P(x|w_j) * P(w_j)}{P(x)}$$

Since all the classes share the same $P(x)$, the rule can be reduced to be:

$$\forall j \neq i : P(x|w_i) * P(w_i) \geq P(x|w_j) * P(w_j)$$

According to Parzen windows estimation, the density estimation formula is:

$$p_\varphi(x) = \frac{1}{n_i * h^d} * \sum_{i=1}^{n_i} \varphi\left(\frac{x_i^k - x}{h}\right)$$

Placing $p_\varphi(x)$ in the decision rule for each class will lead us to the following:

$$\forall j \neq i : \frac{1}{n_i * h^d} * \sum_{i=1}^{n_i} \varphi\left(\frac{x_i^k - x}{h}\right) * P(w_i)$$
$$\geq \frac{1}{n_j * h^d} * \sum_{i=1}^{n_j} \varphi\left(\frac{x_j^k - x}{h}\right) * P(w_j)$$

Notice that $P(w_i)$ is the percentage of the samples belonging to w_i among all samples, which is formally:

$$P(w_i) = \frac{n_i}{n}$$

Placing $\frac{n_i}{n}$ in the decision rule will get us the following:

$$\forall j \neq i : \frac{1}{n_i * h^d} * \sum_{i=1}^{n_i} \varphi\left(\frac{x_i^k - x}{h}\right) * \frac{n_i}{n} \geq \frac{1}{n_j * h^d} * \sum_{i=1}^{n_j} \varphi\left(\frac{x_j^k - x}{h}\right) * \frac{n_j}{n}$$

Both $\frac{1}{h^d}$ and $\frac{1}{n}$ are shared by the two sides, then we can reduce the rule to be:

$$\forall j \neq i : \frac{n_i}{n_i} * \sum_{i=1}^{n_i} \varphi\left(\frac{x_i^k - x}{h}\right) \geq \frac{n_j}{n_j} * \sum_{i=1}^{n_j} \varphi\left(\frac{x_j^k - x}{h}\right)$$

$$\rightarrow \ \forall j \neq i : \sum_{i=1}^{n_i} \varphi\left(\frac{x_i^k - x}{h}\right) \geq \sum_{i=1}^{n_j} \varphi\left(\frac{x_j^k - x}{h}\right)$$

Question 4.2:

In order to build the distribution we'll find for each point x (total of 10 points) the number of samples which are inside the hypercube with side h=4 and centered at x, using the following formula:

$$p_\varphi(x) = \frac{1}{n * h^d}\sum_{i=1}^{n}\varphi\left(\frac{x - x_i}{h}\right) = \frac{1}{10 * 4}\sum_{i=1}^{n}\varphi\left(\frac{x - x_i}{4}\right)$$

Where:

$$\varphi(x) = \begin{cases} 1, & |x| \le \frac{1}{2} \\ 0, & otherwise \end{cases}$$

- $For\ x \le -10: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \varphi\left(\frac{x-(-1)}{4}\right) + \varphi\left(\frac{x-0}{4}\right) + \varphi\left(\frac{x-1}{4}\right) + \varphi\left(\frac{x-2}{4}\right) + \varphi\left(\frac{x-4}{4}\right) + \varphi\left(\frac{x-5}{4}\right)\right) = \frac{1}{40}(0 + 0 + \cdots + 0) = 0$
- $For - 10 \le x \le -5: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \varphi\left(\frac{x-(-1)}{4}\right) + \varphi\left(\frac{x-0}{4}\right) + \varphi\left(\frac{x-1}{4}\right) + \varphi\left(\frac{x-2}{4}\right) + \varphi\left(\frac{x-4}{4}\right) + \varphi\left(\frac{x-5}{4}\right)\right) = \frac{1}{40}(1 + 0 + \cdots + 0) = \frac{1}{40}$
- $For - 5 \le x \le -4: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right) =$
  $\frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \right.$

$\varphi\left(\frac{x-(-1)}{4}\right)+\varphi\left(\frac{x-0}{4}\right)+\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+$

$\left.\varphi\left(\frac{x-5}{4}\right)\right)= \frac{1}{40}(0+1+1+0+\cdots+0)= \frac{2}{40}$

- $For -4 \le x \le -3: p_{\varphi}(x) = \frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right) =$

  $\frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right)+\varphi\left(\frac{x-(-4)}{4}\right)+\varphi\left(\frac{x-(-3)}{4}\right)+\varphi\left(\frac{x-(-2)}{4}\right)+\right.$

  $\varphi\left(\frac{x-(-1)}{4}\right)+\varphi\left(\frac{x-0}{4}\right)+\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+$

  $\left.\varphi\left(\frac{x-5}{4}\right)\right)= \frac{1}{40}(0+1+1+1+0+\cdots+0)= \frac{3}{40}$

- $For -3 \le x \le -2: p_{\varphi}(x) = \frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right) =$

  $\frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right)+\varphi\left(\frac{x-(-4)}{4}\right)+\varphi\left(\frac{x-(-3)}{4}\right)+\varphi\left(\frac{x-(-2)}{4}\right)+\right.$

  $\varphi\left(\frac{x-(-1)}{4}\right)+\varphi\left(\frac{x-0}{4}\right)+\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+$

  $\left.\varphi\left(\frac{x-5}{4}\right)\right)= \frac{1}{40}(0+1+1+1+1+0+\cdots+0)= \frac{4}{40}$

- $For -2 \le x \le -1: p_{\varphi}(x) = \frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right) =$

  $\frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right)+\varphi\left(\frac{x-(-4)}{4}\right)+\varphi\left(\frac{x-(-3)}{4}\right)+\varphi\left(\frac{x-(-2)}{4}\right)+\right.$

  $\varphi\left(\frac{x-(-1)}{4}\right)+\varphi\left(\frac{x-0}{4}\right)+\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+$

  $\left.\varphi\left(\frac{x-5}{4}\right)\right)= \frac{1}{40}(0+0+1+1+1+1+0+\cdots+0)= \frac{4}{40}$

- $For -1 \le x \le 0: p_{\varphi}(x) = \frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right)+\right.$

  $\varphi\left(\frac{x-(-4)}{4}\right)+\varphi\left(\frac{x-(-3)}{4}\right)+\varphi\left(\frac{x-(-2)}{4}\right)+\varphi\left(\frac{x-(-1)}{4}\right)+\varphi\left(\frac{x-0}{4}\right)+$

  $\left.\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+\varphi\left(\frac{x-5}{4}\right)\right)= \frac{1}{40}(0+0+0+1+$

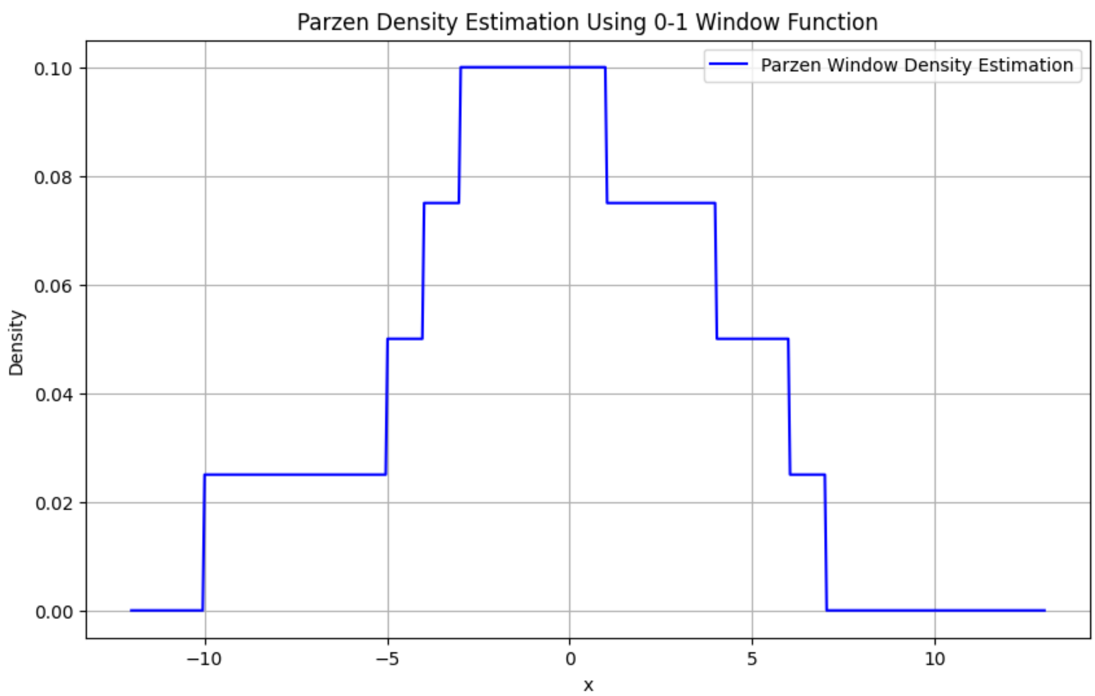
  $1+1+1+0+\cdots+0)= \frac{4}{40}$

- $For\ 0 \le x \le 1: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n} \varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \varphi\left(\frac{x-(-1)}{4}\right) + \varphi\left(\frac{x-0}{4}\right) + \varphi\left(\frac{x-1}{4}\right) + \varphi\left(\frac{x-2}{4}\right) + \varphi\left(\frac{x-4}{4}\right) + \varphi\left(\frac{x-5}{4}\right)\right) = \frac{1}{40}(0 + 0 + 0 + 0 + 1 + 1 + 1 + 1 + 0 + 0) = \frac{4}{40}$
- $For\ 1 \le x \le 2: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n} \varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \varphi\left(\frac{x-(-1)}{4}\right) + \varphi\left(\frac{x-0}{4}\right) + \varphi\left(\frac{x-1}{4}\right) + \varphi\left(\frac{x-2}{4}\right) + \varphi\left(\frac{x-4}{4}\right) + \varphi\left(\frac{x-5}{4}\right)\right) = \frac{1}{40}(0 + \cdots + 0 + 1 + 1 + 1 + 0 + 0) = \frac{3}{40}$
- $For\ 2 \le x \le 3: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n} \varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \varphi\left(\frac{x-(-1)}{4}\right) + \varphi\left(\frac{x-0}{4}\right) + \varphi\left(\frac{x-1}{4}\right) + \varphi\left(\frac{x-2}{4}\right) + \varphi\left(\frac{x-4}{4}\right) + \varphi\left(\frac{x-5}{4}\right)\right) = \frac{1}{40}(0 + \cdots + 0 + 1 + 1 + 1 + 0) = \frac{3}{40}$
- $For\ 3 \le x \le 4: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n} \varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \varphi\left(\frac{x-(-1)}{4}\right) + \varphi\left(\frac{x-0}{4}\right) + \varphi\left(\frac{x-1}{4}\right) + \varphi\left(\frac{x-2}{4}\right) + \varphi\left(\frac{x-4}{4}\right) + \varphi\left(\frac{x-5}{4}\right)\right) = \frac{1}{40}(0 + \cdots + 0 + 1 + 1 + 1) = \frac{3}{40}$
- $For\ 4 \le x \le 6: p_\varphi(x) = \frac{1}{40}\sum_{i=1}^{n} \varphi\left(\frac{x-x_i}{4}\right) = \frac{1}{40}\left(\varphi\left(\frac{x-(-8)}{4}\right) + \varphi\left(\frac{x-(-4)}{4}\right) + \varphi\left(\frac{x-(-3)}{4}\right) + \varphi\left(\frac{x-(-2)}{4}\right) + \varphi\left(\frac{x-(-1)}{4}\right) + \varphi\left(\frac{x-0}{4}\right) +\right.$

$\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+\varphi\left(\frac{x-5}{4}\right)\Big)=\frac{1}{40}(0+\cdots+0+1+1)=\frac{2}{40}$

- $For\ 6\leq x\leq 7: p_\varphi(x)=\frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right)=\frac{1}{40}\Big(\varphi\left(\frac{x-(-8)}{4}\right)+\varphi\left(\frac{x-(-4)}{4}\right)+\varphi\left(\frac{x-(-3)}{4}\right)+\varphi\left(\frac{x-(-2)}{4}\right)+\varphi\left(\frac{x-(-1)}{4}\right)+\varphi\left(\frac{x-0}{4}\right)+\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+\varphi\left(\frac{x-5}{4}\right)\Big)=\frac{1}{40}(0+\cdots+0+1)=\frac{1}{40}$
- $For\ x\geq 7: p_\varphi(x)=\frac{1}{40}\sum_{i=1}^{n}\varphi\left(\frac{x-x_i}{4}\right)=\frac{1}{40}\Big(\varphi\left(\frac{x-(-8)}{4}\right)+\varphi\left(\frac{x-(-4)}{4}\right)+\varphi\left(\frac{x-(-3)}{4}\right)+\varphi\left(\frac{x-(-2)}{4}\right)+\varphi\left(\frac{x-(-1)}{4}\right)+\varphi\left(\frac{x-0}{4}\right)+\varphi\left(\frac{x-1}{4}\right)+\varphi\left(\frac{x-2}{4}\right)+\varphi\left(\frac{x-4}{4}\right)+\varphi\left(\frac{x-5}{4}\right)\Big)=\frac{1}{40}(0+0+\cdots+0)=0$

The graph which formed by these $p_\varphi(x)$'s is the following:



(a) According to the graph, the data appears to follow a normal distribution, as evidenced by the bell-shaped curve with a prominent peak. The data seems to be symmetrically distributed around this peak. However, it is important to note that the small number of data samples limits our ability to generalize these observations.

(b) Since we received a data that seems to be distributed normally, we'll use the following univariate normal distribution PDF in order to estimate the distribution using MLE:

$$p(x) = \frac{1}{\sqrt{2 * \pi} * \sigma} * e^{-\frac{1}{2}*\left(\frac{x-\mu}{\sigma}\right)^2}$$

We'll find μ and σ:

$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

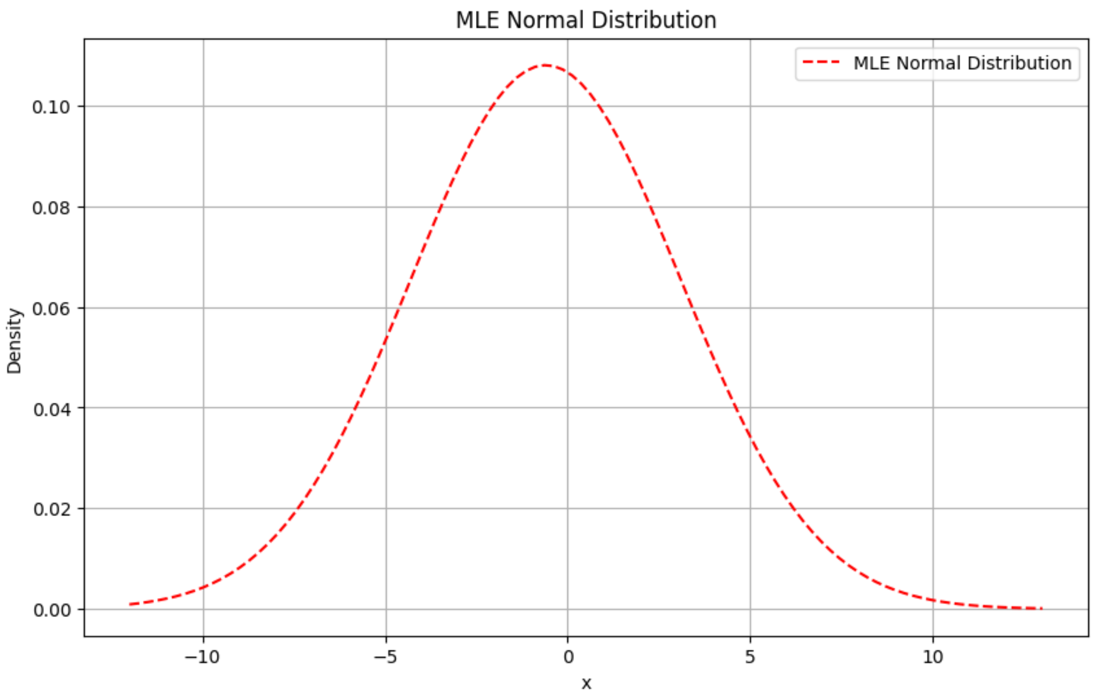
$$= \frac{1}{10}\big((-8) + (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 4 + 5\big) = -\mathbf{0.6}$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 = \frac{1}{10}\sum_{i=1}^{10}(x_i + 0.6)^2$$

$$= \frac{1}{10}\big((-8 + 0.6)^2 + (-4 + 0.6)^2 + (-3 + 0.6)^2 + (-2 + 0.6)^2 + (-1 + 0.6)^2 + (0 + 0.6)^2 + (1 + 0.6)^2 + (2 + 0.6)^2 + (4 + 0.6)^2 + (5 + 0.6)^2\big) = 13.64$$

$$\rightarrow \sigma = \sqrt{13.64} \approx \mathbf{3.69}$$

Substituting into the formula gives us the following graph:



The similarities between the two graphs lie primarily in their overall structure and visual presentation. Both exhibit a bell-shaped curve, though it is more pronounced and smoother in the second graph. Additionally, the range of the data and their distribution appear similar in both graphs.