The Best Ever Solutions to Sets of Riccati Equations

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Abstract

This paper introduces for the first time in general forms, the ideal and exact solutions to equations of the form:

$$y'' + \frac{Ax}{(1-x^2)}y' + \frac{B}{(1-x^2)}y = 0$$

with the solutions containing pairs of fundamental solutions consisting of only terminating series. An example of this type of equation is the Legendre equation. The concept of having only terminating series brings about the exactitude of the solutions. These solutions become so much important when the value of x is far away from the center of convergence, then the tendency of error in approximation is larger. It is certain that these solutions would replace other contemporary solutions, ushering in advancement in the fields of electronics, fluid mechanics, heat conduction etc. An illustration is also given in the appendix section, on how to use an online calculator developed by the author in solving these equations.

Keywords: Variable coefficients differential equation, Riccati equation, Legendre equation.

1.0 Introduction

The Riccati differential equation can generally be written as:

$$y'' + P(x)y' + Q(x)y = 0 \tag{1.1}$$

Where $P(x)$ and $Q(x)$ are both functions of x.

This paper deals with cases where $P(x) = \frac{Ax}{(1-x^2)}$ and $Q(x) = \frac{B}{(1-x^2)}$; hence Eq. (1.1) becomes:

$$y'' + \frac{Ax}{(1-x^2)}y' + \frac{B}{(1-x^2)}y = 0 \tag{1.2}$$

Which can also be written as:

$$(1-x^2)y'' + Axy' + By = 0 \tag{1.3}$$

Where A must be an integer. The study is divided into four parts (4 sets of equations) and they are as follows:

- The first, with the value of **A** been a negative odd integer
- The second, with the value of **A** been a positive odd integer
- The third, with the value of **A** been a negative even integer, an example of which is the Legendre equation
- The fourth, with the value of **A** been a positive even integer.

2.0 The value of A been a negative integer, while B is independent of A

There are two solutions provided in this study for this set of equation: the first solution in complex number form, and then the second in trigonometric form. To ease simplicity in presentation of solutions, a constant k was introduced; note that the value of k is given as,

$$k = B + \left(\frac{A+1}{2}\right)^2 \tag{2.1}$$

For A = -1,

$$(1-x^2)y'' - xy' + By = 0 \tag{2.2}$$

In the equation above, $k = B + \left(\frac{-1+1}{2}\right)^2 = B$. The general solution can be given as:

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}} + C_2\left[x - i\sqrt{1-x^2}\right]^{-\sqrt{k}} \tag{2.3}$$

Note that,

$$\left[x - i\sqrt{1-x^2}\right]^{-\sqrt{k}} = \frac{1}{\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}} = \frac{\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}}{\left[(x - i\sqrt{1-x^2})(x + i\sqrt{1-x^2})\right]^{\sqrt{k}}} = \frac{\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}}{[1]^{\sqrt{k}}}$$
$$= \left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}$$

Then, Eq. (2.3) can also be re-written as:

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{B}} + C_2\left[x + i\sqrt{1-x^2}\right]^{\sqrt{B}} \tag{2.4}$$

Example: $(1-x^2)y'' - xy' - 36y = 0$

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{6i} + C_2\left[x + i\sqrt{1-x^2}\right]^{6i}$$

For A = -3,

$$(1-x^2)y'' - 3xy' + By = 0 \tag{2.5}$$

In the equation above, $k = B + \left(\frac{-3+1}{2}\right)^2 = B + 1$. The general solution can be given as:

$$y = \frac{C_1}{\sqrt{1-x^2}}\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}} + \frac{C_2}{\sqrt{1-x^2}}\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}} \tag{2.6}$$

Example: $(1-x^2)y'' - 3xy' + 8y = 0$

$$y = \frac{C_1}{\sqrt{1-x^2}}\left[x - i\sqrt{1-x^2}\right]^{3} + \frac{C_2}{\sqrt{1-x^2}}\left[x + i\sqrt{1-x^2}\right]^{3}$$

For A = -5,

$$(1-x^2)y'' - 5xy' + By = 0 \tag{2.7}$$

In the equation above, $k = B + \left(\frac{-5+1}{2}\right)^2 = B + 4$. The general solution can be given as:

$$y = \frac{C_1}{\left(\sqrt{1-x^2}\right)^3}\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[x + i\sqrt{k(1-x^2)}\right] + \frac{C_2}{\left(\sqrt{1-x^2}\right)^3}\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[x - i\sqrt{k(1-x^2)}\right] \tag{2.8}$$

Example: $(1-x^2)y'' - 5xy' + 7y = 0$

$$y = \frac{C_1}{(\sqrt{1-x^2})^3}\left[x - i\sqrt{1-x^2}\right]^{\sqrt{11}}\left[x + i\sqrt{11(1-x^2)}\right] + \frac{C_2}{\left(\sqrt{1-x^2}\right)^3}\left[x + i\sqrt{1-x^2}\right]^{\sqrt{11}}[x - i\sqrt{11(1-x^2)}]$$

For A = -7,

$$(1-x^2)y'' - 7xy' + By = 0 \tag{2.9}$$

In the equation above, $k = B + \left(\frac{-7+1}{2}\right)^2 = B + 9$. The general solution can be given as:

$$y = \frac{C_1}{\left(\sqrt{1-x^2}\right)^5}\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[3x^2 + 3ix\sqrt{k(1-x^2)} - (k-1)(1-x^2)\right] + \frac{C_2}{\left(\sqrt{1-x^2}\right)^5}\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[3x^2 - 3ix\sqrt{k(1-x^2)} - (k-1)(1-x^2)\right] \tag{2.10}$$

For A = -9,

$$(1-x^2)y'' - 9xy' + By = 0 \tag{2.11}$$

In the equation above, $k = B + \left(\frac{-9+1}{2}\right)^2 = B + 16$. The general solution can be given as:

$$y = \frac{C_1}{\left(\sqrt{1-x^2}\right)^7}\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[15x^3 + 15ix^2\sqrt{k(1-x^2)} - (6k-9)(1-x^2)x - (k-4)\sqrt{k}i\left(\sqrt{1-x^2}\right)^3\right] + \frac{C_2}{\left(\sqrt{1-x^2}\right)^7}\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[15x^3 - 15ix^2\sqrt{k(1-x^2)} - (6k-9)(1-x^2)x + (k-4)\sqrt{k}i\left(\sqrt{1-x^2}\right)^3\right] \tag{2.12}$$

For A = -11,

$$(1-x^2)y''-11xy'+By=0 \tag{2.13}$$

In the equation above, $k=B+\left(\frac{-11+1}{2}\right)^2=B+25$. The general solution can be given as:

$$\begin{aligned} y=&\frac{C_1}{\left(\sqrt{1-x^2}\right)^9}\left[x-i\sqrt{1-x^2}\right]^{\sqrt{k}}\Big[105x^4+105ix^3\sqrt{k(1-x^2)}-(45k-90)(1-x^2)x^2\\ &-(10k-55)\sqrt{k}ix\left(\sqrt{1-x^2}\right)^3+(k^2-10k+9)(1-x^2)^2\Big]\\ &+\frac{C_2}{\left(\sqrt{1-x^2}\right)^9}\left[x-i\sqrt{1-x^2}\right]^{\sqrt{k}}\Big[105x^4-105ix^3\sqrt{k(1-x^2)}-(45k-90)(1-x^2)x^2\\ &+(10k-55)\sqrt{k}ix\left(\sqrt{1-x^2}\right)^3+(k^2-10k+9)(1-x^2)^2\Big] \end{aligned} \tag{2.14}$$

2.1 Proof

Proving all solutions given above would make this study unnecessarily lengthy, however, to demonstrate that these solutions are true and authentic for the respective differential equations, the study will prove for one of the equations. Consider the equation below:

$$(1-x^2)y''-xy'+By=0 \tag{2.2}$$

Choosing a fundamental solution of the equation from its general solution in Eq. (2.4),

$$y=\left[x-i\sqrt{1-x^2}\right]^{\sqrt{B}} \tag{2.15 }$$

Differentiating,

$$y'=\left[x-i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{\sqrt{B}+\frac{\sqrt{B}ix}{\sqrt{1-x^2}}}{x-i\sqrt{1-x^2}}\right]=\left[x-i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{\left(\frac{\sqrt{B}i}{\sqrt{1-x^2}}\right)\left(-i\sqrt{1-x^2}+x\right)}{x-i\sqrt{1-x^2}}\right]$$

$$=\left[x-i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{\sqrt{B}i}{\sqrt{1-x^2}}\right]$$

$$\therefore\ -xy'=\left[x-i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{-\sqrt{B}ix}{\sqrt{1-x^2}}\right] \tag{2.16}$$

Differentiating y' in order to get y'',

$$y''=\left[x-i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{\sqrt{B}ix}{\left(\sqrt{1-x^2}\right)^3}-\frac{B}{1-x^2}\right]$$

$$\therefore\ (1-x^2)y''=\left[x-i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{\sqrt{B}ix}{\sqrt{1-x^2}}-B\right] \tag{2.17}$$

Adding Eq.s (2.16) and (2.17),

$$(1-x^2)y" - xy' = \left[x - i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{\sqrt{B}ix}{\sqrt{1-x^2}} - B\right] + \left[x - i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{-\sqrt{B}ix}{\sqrt{1-x^2}}\right]$$

$$\therefore (1-x^2)y" - xy' = \left[x - i\sqrt{1-x^2}\right]^{\sqrt{B}}\left[\frac{\sqrt{B}ix - \sqrt{B}ix}{\sqrt{1-x^2}} - B\right]$$

$$(1-x^2)y" - xy' = -B\left[x - i\sqrt{1-x^2}\right]^{\sqrt{B}}$$

$$\therefore (1-x^2)y" - xy' = -By$$

$$\therefore (1-x^2)y" - xy' + By = 0$$

Hence, it has been proven to be a solution to the differential equation.

2.2 Alternative Solutions

Recall that k has been introduced as;

$$k = B + \left(\frac{A+1}{2}\right)^2 \tag{2.1}$$

Also, introducing **C** and **S,** where;

$$\boldsymbol{C} = \cos\left(\sqrt{k}cos^{-1}x\right) \qquad and \qquad \boldsymbol{S} = \sin\left(\sqrt{k}cos^{-1}x\right)$$

The equations and their respective solutions are as follows;

$$(1-x^2)y'' - xy' + By = 0 \tag{2.2}$$

$$y = C_1\left[\cos\left(\sqrt{k}cos^{-1}x\right)\right] + C_2\left[\sin\left(\sqrt{k}cos^{-1}x\right)\right] \tag{2.18}$$

$$(1-x^2)y'' - 3xy' + By = 0 \tag{2.5}$$

$$y = \frac{C_1}{\sqrt{1-x^2}}\left[\cos\left(\sqrt{k}cos^{-1}x\right)\right] + \frac{C_2}{\sqrt{1-x^2}}\left[\sin\left(\sqrt{k}cos^{-1}x\right)\right] \tag{2.19}$$

$$(1-x^2)y'' - 5xy' + By = 0 \tag{2.7}$$

$$y = \frac{C_1}{\left(\sqrt{1-x^2}\right)^3}\left[x.\boldsymbol{S} - \sqrt{k(1-x^2)}.\boldsymbol{C}\right] + \frac{C_2}{\left(\sqrt{1-x^2}\right)^3}\left[-x.\boldsymbol{C} - \sqrt{k(1-x^2)}.\boldsymbol{S}\right] \tag{2.20}$$

$$(1-x^2)y'' - 7xy' + By = 0 \tag{2.9}$$

$$\begin{aligned} y = {} & \frac{C_1}{\left(\sqrt{1-x^2}\right)^5}\left[3x^2.\boldsymbol{S} - 3x\sqrt{k(1-x^2)}.\boldsymbol{C} - (k-1)(1-x^2)\boldsymbol{S}\right] \\ & + \frac{C_2}{\left(\sqrt{1-x^2}\right)^5}\left[-3x^2.\boldsymbol{C} - 3x\sqrt{k(1-x^2)}.\boldsymbol{S} + (k-1)(1-x^2)\boldsymbol{C}\right] \end{aligned} \tag{2.21}$$

$$(1-x^2)y'' - 9xy' + By = 0 \tag{2.11}$$

$$y = \frac{C_1}{\left(\sqrt{1-x^2}\right)^7}\left[15x^3.\boldsymbol{S} - 15x^2\sqrt{k(1-x^2)}.\boldsymbol{C} - (6k-9)(1-x^2)x.\boldsymbol{S} + (k-4)\sqrt{k(1-x^2)^3}.\boldsymbol{C}\right]$$
$$+ \frac{C_2}{\left(\sqrt{1-x^2}\right)^7}\left[-15x^3.\boldsymbol{C} - 15x^2\sqrt{k(1-x^2)}.\boldsymbol{S} + (6k-9)(1-x^2)x.\boldsymbol{C}\right.$$
$$\left. + (k-4)\sqrt{k(1-x^2)^3}.\boldsymbol{S}\right] \tag{2.21}$$

$$(1-x^2)y'' - 11xy' + By = 0 \tag{2.13}$$

$$y = \frac{C_1}{\left(\sqrt{1-x^2}\right)^9}\left[105x^3.\boldsymbol{S} - 105x^2\sqrt{k(1-x^2)}.\boldsymbol{C} - (45k-90)(1-x^2)x.\boldsymbol{S}\right.$$
$$\left. + (10k-55)\sqrt{k(1-x^2)^3}.\boldsymbol{C} + (k^2-10k+9)(1-x^2)^2.\boldsymbol{S}\right]$$
$$+ \frac{C_2}{\left(\sqrt{1-x^2}\right)^9}\left[-105x^3.\boldsymbol{C} - 105x^2\sqrt{k(1-x^2)}.\boldsymbol{S} + (45k-90)(1-x^2)x.\boldsymbol{C}\right.$$
$$\left. + (10k-55)\sqrt{k(1-x^2)^3}.\boldsymbol{S} - (k^2-10k+9)(1-x^2)^2.\boldsymbol{C}\right] \tag{2.22}$$

3.0 The value of A been a positive integer, while B is independent of the value of A

As in the previous section, there are two solutions provided in this study for this set of equation: the first solution in complex number form, and then the second in trigonometric form.

For A = 1,

$$(1-x^2)y'' + xy' + By = 0 \tag{3.1}$$

Let $k = B + \left(\frac{A+1}{2}\right)^2 = B + \left(\frac{1+1}{2}\right)^2 = B + 1$. The general solution can be given as:

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[x + i\sqrt{k(1-x^2)}\right] + C_2\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[x - i\sqrt{k(1-x^2)}\right] \tag{3.2}$$

Example: $(1-x^2)y'' + xy' + 27y = 0$

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{28}}\left[x + i\sqrt{28(1-x^2)}\right] + C_2\left[x + i\sqrt{1-x^2}\right]^{\sqrt{28}}\left[x - i\sqrt{28(1-x^2)}\right]$$

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{2\sqrt{7}}\left[x + 2i\sqrt{7(1-x^2)}\right] + C_2\left[x + i\sqrt{1-x^2}\right]^{2\sqrt{7}}\left[x - 2i\sqrt{7(1-x^2)}\right]$$

For A = 3,

$$(1-x^2)y'' + 3xy' + By = 0 \tag{3.3}$$

Let $k = B + \left(\frac{A+1}{2}\right)^2 = B + \left(\frac{3+1}{2}\right)^2 = B + 4$. The general solution can be given as:

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[3x^2 + 3\sqrt{k}ix\sqrt{1-x^2} - (k-1)(1-x^2)\right] + C_2\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[3x^2 - 3\sqrt{k}ix\sqrt{1-x^2} - (k-1)(1-x^2)\right] \tag{3.4}$$

Example: $(1-x^2)y'' + 3xy' - 8y = 0$

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{2i}\left[3x^2 - 6x\sqrt{1-x^2} - (2i-1)(1-x^2)\right] + C_2\left[x + i\sqrt{1-x^2}\right]^{2i}\left[3x^2 + 6x\sqrt{1-x^2} - (2i-1)(1-x^2)\right]$$

***For A* = 5,**

$$(1-x^2)y'' + 5xy' + By = 0 \tag{3.5}$$

Let $k = B + \left(\frac{A+1}{2}\right)^2 = B + \left(\frac{5+1}{2}\right)^2 = B + 9$. The general solution can be given as:

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[15x^3 + 15\sqrt{k}ix^2\sqrt{1-x^2} - (6k-9)(1-x^2)x - (k-4)\sqrt{k}i\left(\sqrt{1-x^2}\right)^3\right] + C_2\left[x + i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[15x^3 - 15\sqrt{k}ix^2\sqrt{1-x^2} - (6k-9)(1-x^2)x + (k-4)\sqrt{k}i\left(\sqrt{1-x^2}\right)^3\right] \tag{3.6}$$

***For A* = 7,**

$$(1-x^2)y'' + 7xy' + By = 0 \tag{3.7}$$

Let $k = B + \left(\frac{A+1}{2}\right)^2 = B + \left(\frac{7+1}{2}\right)^2 = B + 16$. The general solution can be given as:

$$y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[105x^4 + 105\sqrt{k}ix^3\sqrt{1-x^2} - (45k-90)(1-x^2)x^2 - (10k-55)\sqrt{k}ix\left(\sqrt{1-x^2}\right)^3 + (k^2-10k+9)(1-x^2)^2\right] + C_2\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}}\left[105x^4 - 105\sqrt{k}ix^3\sqrt{1-x^2} - (45k-90)(1-x^2)x^2 + (10k-55)\sqrt{k}ix\left(\sqrt{1-x^2}\right)^3 + (k^2-10k+9)(1-x^2)^2\right] \tag{3.8}$$

***For A* = 9,**

$$(1-x^2)y'' + 9xy' + By = 0 \tag{3.9}$$

Let $k = B + \left(\frac{A+1}{2}\right)^2 = B + \left(\frac{9+1}{2}\right)^2 = B + 25$. The general solution can be given as:

$$\begin{aligned}
y = C_1\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}} \Big[&945x^5 + 945\sqrt{k}ix^4\sqrt{1-x^2} - (420k-1050)(1-x^2)x^3 \\
&- (105k-735)\sqrt{k}ix^2\left(\sqrt{1-x^2}\right)^3 + (15k^2-195k+225)x(1-x^2)^2 \\
&+ (k^2-20k+64)\sqrt{k}i\left(\sqrt{1-x^2}\right)^5\Big] \\
+\ C_2\left[x - i\sqrt{1-x^2}\right]^{\sqrt{k}} \Big[&945x^5 + 945\sqrt{k}ix^4\sqrt{1-x^2} - (420k-1050)(1-x^2)x^3 \\
&- (105k-735)\sqrt{k}ix^2\left(\sqrt{1-x^2}\right)^3 + (15k^2-195k+225)x(1-x^2)^2 \\
&+ (k^2-20k+64)\sqrt{k}i\left(\sqrt{1-x^2}\right)^5\Big]
\end{aligned} \tag{3.10}$$

3.2 Alternative Solutions

Recall that k has been introduced as;

$$k = B + \left(\frac{A+1}{2}\right)^2 \tag{2.1}$$

Introducing **C** and **S,** where;

$$\boldsymbol{C} = \cos\left(\sqrt{k}cos^{-1}x\right) \qquad \textit{and} \qquad \boldsymbol{S} = \sin\left(\sqrt{k}cos^{-1}x\right)$$

The equations and their respective solutions are as follows;

$$(1-x^2)y'' + xy' + By = 0 \tag{3.2}$$

$$y = C_1\left[x.\boldsymbol{S} - \sqrt{k(1-x^2)}.\boldsymbol{C}\right] + \ C_2\left[-x.\boldsymbol{C} - \sqrt{k(1-x^2)}.\boldsymbol{S}\right] \tag{3.10}$$

$$(1-x^2)y'' + 3xy' + By = 0 \tag{3.4}$$

$$\begin{aligned}
y = C_1&\left[3x^2.\boldsymbol{S} - 3x\sqrt{k(1-x^2)}.\boldsymbol{C} - (k-1)(1-x^2)\boldsymbol{S}\right] \\
&+\ C_2\left[-3x^2.\boldsymbol{C} - 3x\sqrt{k(1-x^2)}.\boldsymbol{S} + (k-1)(1-x^2)\boldsymbol{C}\right]
\end{aligned} \tag{3.12}$$

$$(1-x^2)y'' + 5xy' + By = 0 \tag{3.6}$$

$$\begin{aligned}
y = C_1&\left[15x^3.\boldsymbol{S} - 15x^2\sqrt{k(1-x^2)}.\boldsymbol{C} - (6k-9)(1-x^2)x.\boldsymbol{S} + (k-4)\sqrt{k(1-x^2)^3}.\boldsymbol{C}\right] \\
&+\ C_2\Big[-15x^3.\boldsymbol{C} - 15x^2\sqrt{k(1-x^2)}.\boldsymbol{S} + (6k-9)(1-x^2)x.\boldsymbol{C} \\
&+ (k-4)\sqrt{k(1-x^2)^3}.\boldsymbol{S}\Big]
\end{aligned} \tag{3.13}$$

$$(1-x^2)y'' + 7xy' + By = 0 \tag{3.8}$$

$$y = C_1\left[105x^3.\boldsymbol{S} - 105x^2\sqrt{k(1-x^2)}.\boldsymbol{C} - (45k-90)(1-x^2)x.\boldsymbol{S} + (10k-55)\sqrt{k(1-x^2)^3}.\boldsymbol{C} + (k^2-10k+9)(1-x^2)^2.\boldsymbol{S}\right] + C_1\left[-105x^3.\boldsymbol{C} - 105x^2\sqrt{k(1-x^2)}.\boldsymbol{S} + (45k-90)(1-x^2)x.\boldsymbol{C} + (10k-55)\sqrt{k(1-x^2)^3}.\boldsymbol{S} - (k^2-10k+9)(1-x^2)^2.\boldsymbol{C}\right] \quad (3.14)$$

4.0 The value of A been a negative integer, while B depends on the value A

This section deals with those that fall under the equation below:

$$(1-x^2)y'' - 2(m+1)xy' + (n-m)(n+m+1)y = 0 \quad (4.1)$$

Where m and n are non-negative integers. Comparing Eq. (4.1) to Eq. (1.3), it is noted that $A = -2(m+1)$ and $B = (n-m)(n+m+1)$. Hence, it can be deduced that unlike other cases that has been treated before in this study, B depends on the value of A since they both depend on the value of the integer m.

A very important equation under this set is the Legendre equation, which sets m to be equal zero.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad (4.2)$$

4.1 Legendre Equation

Legendre equation occur in many areas of physics, applied mathematics and chemistry in physical situation with a spherical geometry such as flow of an ideal fluid past a sphere, the determination of the electric field due to a charged sphere and the determination of the temperature distribution in a sphere given its surface. Legendre differential equation takes the form of Eq. (4.2). The general solution of the above equation where n = 0,1,2,3, … (a positive integer) is given by:

$$y = C_1\, P_n(x) + C_2\, Q_n(x) \quad (4.3)$$

where are C_1 and C_2 are arbitrary constants,

$$P(x) = \frac{1}{2^n}\sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-2r)!}{(n-r)!\,(n-2r)!\,r!} x^{n-2r} \quad (4.4)$$

$$Q_n(x) = [P_n(x)]tanh^{-1}(x) + \frac{1}{2^n}\sum_{r=0}^{\frac{n-1}{2}}(-1)^{r+1}\left[\sum_{h=0}^{r}\frac{(2n-2h)!\,(2h)!}{((n-h)!\,h!)^2}\right]\frac{(n-1-r)!\,r!}{(2r+1)!\,(n-1-2r)!}x^{n-1-2r} \quad (4.5)$$

Note that $P_n(x)$is the Legendre polynomial (polynomial solution with even exponent) while $Q_n(x)$is the sum of an odd exponent polynomial series and the product between the Legendre polynomial and $tanh^{-1}(x)$.

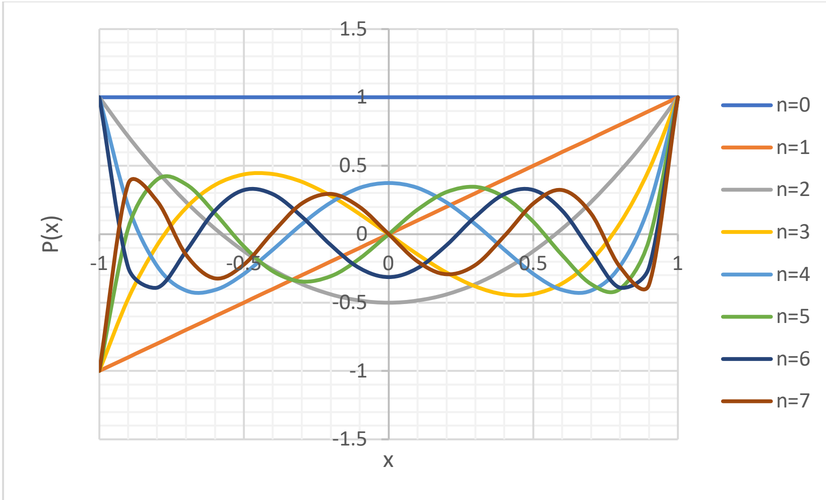


Fig 1: Graph of P(x) against x

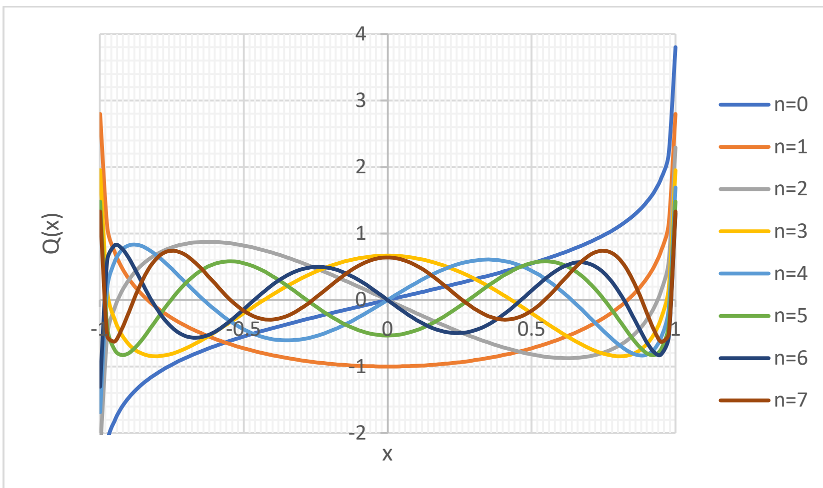


Fig 2: Graph of Q(x) against x

Example: $(1-x^2)y'' - 2xy' + 20y = 0$

$$\therefore\ (1-x^2)y'' - 2xy' + 4(4+1)y = 0$$

$$y = \frac{C_1}{8}[35x^4 - 30x^2 + 3] + \frac{C_2}{8}\left[(35x^4 - 30x^2 + 3)tanh^{-1}(x) - 35x^3 + \frac{55}{3}x\right]$$

4.2 General solutions to other forms of equation in this category

For the equation below:

$$(1-x^2)y'' - 2(m+1)xy' + (n-m)(n+m+1)y = 0 \qquad n \geq m \qquad (4.1)$$

The general solution can be given as:

$$y = C_1\, P_n(x,m) + C_2\, Q_n(x,m) \qquad (4.6)$$

where are C_1 and C_2 are arbitrary constants,

$$P_n(x,m) = \frac{1}{2^n}\sum_{r=0}^{\frac{n-m}{2}} \frac{(-1)^r.(2n-2r)!}{(n-r)!\,(n-m-2r)!\,r!}x^{n-m-2r} \tag{4.7}$$

$$Q_n(x,m) = [P_n(x,m)]tanh^{-1}(x) + \sum_{r=1}^{m}\frac{m!}{(m-r)!\,r}\left[\sum_{h=0}^{\frac{r-1}{2}}\frac{r!}{(r-2h-1)!\,(2h+1)!}x^{r-2h-1}\right]\frac{P_n(x,m-r)}{(1-x^2)^r}$$
$$+\frac{1}{2^n}\sum_{r=0}^{\frac{n-m-1}{2}}(-1)^{r+1}\left[\sum_{h=0}^{r}\frac{(2n-2h)!\,(2h)!}{((n-h)!\,h!)^2}\right]\frac{(n-1-r)!\,r!}{(2r+1)!\,(n-m-2r-1)!}x^{n-m-2r-1} \tag{4.8}$$

Example: $(1-x^2)y'' - 4xy' + 28y = 0$

$$\therefore\ (1-x^2)y'' - 2(1+1)xy' + (5-1)(5+1+1)y = 0$$

Hence, m=1 and n=5

$$y = C_1\,P_5(x,1) + C_2\,Q_5(x,1)$$

$$P_5(x,1) = \frac{1}{2^5}\sum_{r=0}^{2}\frac{(-1)^r\,(10-2r)!}{(5-r)!\,(4-2r)!\,r!}x^{4-2r} = \frac{1}{2^5}\left[\frac{10!}{5!\,4!\,0!}x^4 - \frac{8!}{4!\,2!\,1!}x^2 + \frac{6!}{3!\,0!\,2!}\right]$$
$$= \frac{1}{8}[315x^4 - 210x^2 + 15]$$

$$Q_5(x,1) = [P_5(x,1)]tanh^{-1}(x) + \sum_{r=1}^{1}\frac{1!}{(1-r)!\,r}\left[\sum_{h=0}^{\frac{r-1}{2}}\frac{r!}{(r-2h-1)!\,(2h+1)!}x^{r-2l-1}\right]\frac{P_5(x,1-r)}{(1-x^2)^r}$$
$$+\frac{1}{2^5}\sum_{r=0}^{\frac{3}{2}}(-1)^{r+1}\left[\sum_{h=0}^{r}\frac{(10-2h)!\,(2h)!}{((5-h)!\,h!)^2}\right]\frac{(4-r)!\,r!}{(2r+1)!\,(3-2r)!}x^{3-2r}$$

$$\therefore Q_5(x,1) = \frac{1}{8}[315x^4 - 210x^2 + 15]tanh^{-1}(x)$$
$$+\frac{1}{2^5}\frac{1!}{0!}\left[\frac{1!}{0!\,1!}x^0\right]\left[\left(\frac{10!}{5!\,5!\,0!}x^5 - \frac{8!}{3!\,4!\,1!}x^3 + \frac{6!}{1!\,3!\,2!}x\right)\frac{1}{(1-x^2)}\right]$$
$$+\frac{1}{2^5}\left[-\left[\frac{10!\,0!}{(5!\,0!)^2}\right]\frac{4!\,0!}{1!\,3!}x^3 + \left[\frac{10!\,0!}{(5!\,0!)^2} + \frac{8!\,2!}{(4!\,1!)^2}\right]\frac{3!\,1!}{3!\,1!}x\right]$$

$$\therefore Q_5(x,1) = \frac{1}{8}[315x^4 - 210x^2 + 15]tanh^{-1}(x) + \frac{1}{8}\left[\frac{63x^5 - 70x^3 + 15x}{(1-x^2)}\right] + \frac{1}{8}[-252x^3 + 98x]$$

5.0 The value of A been a negative integer, while B depends on the value A

This section deals with those that fall under the equation below:

$$(1-x^2)y'' + 2mxy' + (n-m)(n+m+1)y = 0 \tag{5.1}$$

Where both m and n are non-negative integers, and $n > m$,

The general solution can be given as:

$$y = (1-x^2)^{m+1}[C_1\, \overline{P_n}(x,m) + C_2\, \overline{Q_n}(x,m)] \tag{5.2}$$

where are C_1 and C_2 are arbitrary constants,

$$\overline{P_n}(x,m) = \frac{1}{2^n} \sum_{r=0}^{\frac{n-m-1}{2}} \frac{(-1)^r.(2n-2r)!}{(n-r)!\,(n-m-2r-1)!\,r!} x^{n-m-2r-1} \tag{5.3}$$

$$\begin{aligned}\overline{Q_n}(x,m) = {} & [\overline{P_n}(x,m)]tanh^{-1}(x) \\ & + \sum_{r=1}^{m} \frac{(m+1)!}{(m+1-r)!\,r} \left[\sum_{h=0}^{\frac{r-1}{2}} \frac{r!}{(r-2h-1)!\,(2h+1)!} x^{r-2h-1} \right] \frac{\overline{P_n}(x,m-r)}{(1-x^2)^r} \\ & + \frac{1}{2^n} \sum_{r=0}^{\frac{n-m-2}{2}} (-1)^{r+1} \left[\sum_{h=0}^{r} \frac{(2n-2h)!\,(2h)!}{((n-h)!\,h!)^2} \right] \frac{(n-1-r)!\,r!}{(2r+1)!\,(n-m-2r-2)!} x^{n-m-2r-2}\end{aligned} \tag{5.4}$$

A special case of Eq. (5.1) is Eq. (5.5) below, where m=0, n>0;

$$(1-x^2)y'' + n(n+1)y = 0 \tag{5.5}$$

The general solution becomes,

$$y = (1-x^2)[C_1\, \overline{P_n}(x) + C_2\, \overline{Q_n}(x)] \tag{5.6}$$

$$\overline{P_n}(x) = \frac{1}{2^n} \sum_{r=0}^{\frac{n-1}{2}} \frac{(-1)^r.(2n-2r)!}{(n-r)!\,(n-2r-1)!\,r!} x^{n-2r-1} \tag{5.7}$$

$$\overline{Q_n}(x) = [\overline{P_n}(x)]tanh^{-1}(x) + \frac{1}{2^n} \sum_{r=0}^{\frac{n-1}{2}} (-1)^{r+1} \left[\sum_{h=0}^{r} \frac{(2n-2h)!\,(2h)!}{((n-h)!\,h!)^2} \right] \frac{(n-1-r)!\,r!}{(2r+1)!\,(n-2r-2)!} x^{n-2r-2} \tag{5.8}$$

Example: $(1-x^2)y'' + 6xy' + 8y = 0$

$$\therefore\ (1-x^2)y'' + 2(3)xy' + (4-3)(4+3+1)y = 0$$

Hence, m=3 and n=4,

The general solution can be given as:

$$y = (1-x^2)^4[C_1\, \overline{P_4}(x,3) + C_2\, \overline{Q_4}(x,3)]$$

where are C_1 and C_2 are arbitrary constants,

$$\overline{P_4}(x,3) = \frac{1}{2^4}\sum_{r=0}^{0} \frac{(-1)^r.(0-2r)!}{(0-r)!(0-2r)!r!} x^{0-2r} = \frac{1}{16}$$

$$\overline{Q_4}(x,3) = [\overline{P_4}(x,3)]tanh^{-1}(x) + \sum_{r=1}^{3} \frac{4!}{(4-r)!r} \left[\sum_{h=0}^{\frac{r-1}{2}} \frac{r!}{(r-2h-1)!(2h+1)!} x^{r-2l-1}\right] \frac{\overline{P_4}(x,3-r)}{(1-x^2)^r}$$

$$+\frac{1}{2^4}\sum_{r=0}^{-\frac{1}{2}} (-1)^{r+1}\left[\sum_{h=0}^{r} \frac{(8-2h)!(2h)!}{((4-h)!h!)^2}\right] \frac{(3-r)!r!}{(2r+1)!(-2r-1)!} x^{-2r-1}$$

The series $\frac{1}{2^4}\sum_{r=0}^{-\frac{1}{2}}(-1)^{r+1}\left[\sum_{h=0}^{r}\frac{(8-2h)!(2h)!}{((4-h)!h!)^2}\right]\frac{(3-r)!r!}{(2r+1)!(-2r-1)!}x^{-2r-1}$ becomes zero, because the summation ends at $r = -\frac{1}{2}$, hence it does not at $r = 0$. Hence,

$$\overline{Q_4}(x,3) = \frac{1}{16}\ tanh^{-1}(x) + \sum_{r=1}^{3} \frac{4!}{(4-r)!r} \left[\sum_{h=0}^{\frac{r-1}{2}} \frac{r!}{(r-2h-1)!(2h+1)!} x^{r-2h-}\ \right] \frac{\overline{P_4}(x,3-r)}{(1-x^2)^r}$$

$$\overline{P_4}(x,3-r) = \frac{1}{2^4}\sum_{r=0}^{0} \frac{(-1)^r.(8-2r)!}{(4-r)!(0-r)!r!} x^{0-r} = \frac{1}{16}$$

$$\overline{Q_4}(x,3) = \frac{1}{16}\ tanh^{-1}(x) + \frac{1}{16}.\frac{4!}{3!}\left[\frac{1!}{0!\,1!}x^0\right]\frac{1}{(1-x^2)} + \frac{1}{16}.\frac{4!}{2!.2}\left[\frac{2!}{1!\,1!}x^1\right]\frac{1}{(1-x^2)^2}$$
$$+\frac{1}{16}.\frac{4!}{1!.3}\left[\frac{3!}{2!\,1!}x^2 + \frac{3!}{0!\,3!}\right]\frac{1}{(1-x^2)^3}$$

$$\overline{Q_4}(x,3) = \frac{1}{16}\ tanh^{-1}(x) + \frac{1}{4(1-x^2)} + \frac{3x}{4(1-x^2)^2} + \frac{3x^2+1}{2(1-x^2)^3}$$

6.0 Conclusion

Quoting from Dawkins [1], a math tutor, "*just because we know that a solution to a differential equation exists does not mean that we will be able to find it*." This saying has proven to be true as many have resorted to the use of non-analytic methods in the solution of variable coefficients differential equations, however, there are some equations which may not seem to have a simple solution, but they do.

This research work has introduced and tested new solutions to some differential equations which were previously solved only by the power series' method and some other numerical methods. Below are the summarized advantages of these new solutions over other contemporary methods:

- They are differentiable unlike the finite difference solution.
- They permit the study of the analytic properties
- They converge for all values of x.
- They reduce computational cost of the calculation.
- They are most accurate when comparing with the already known methods.

The solutions may seem ambiguous to produce for a given differential equation, however, a website (www.ultimate-calc.vercel.app) has been developed to erase every atom of difficulty. The equation is inputted and the solved solution is produced by the web calculator in split seconds. An example of this is shown in the appendix section.

The new solutions are flexible enough to allow a practically infinite list of examples, but at this point the construction of further examples might sound an exercise lacking interest. It is hoped however that the techniques that have been proposed will stimulate further work in this direction, and most importantly, the discovery of newer solutions to more equations of practical interests.

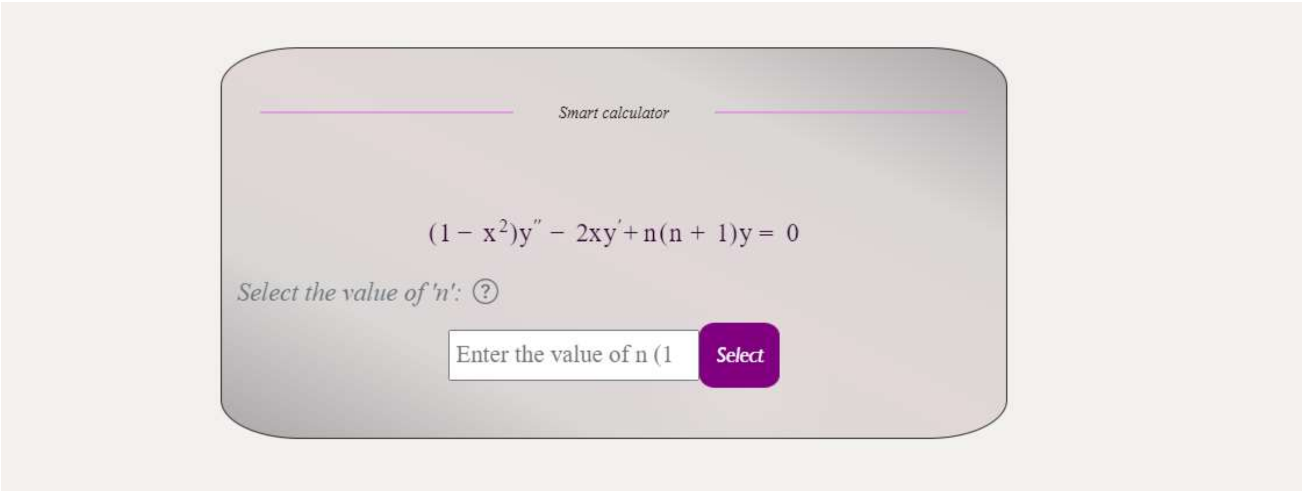
References

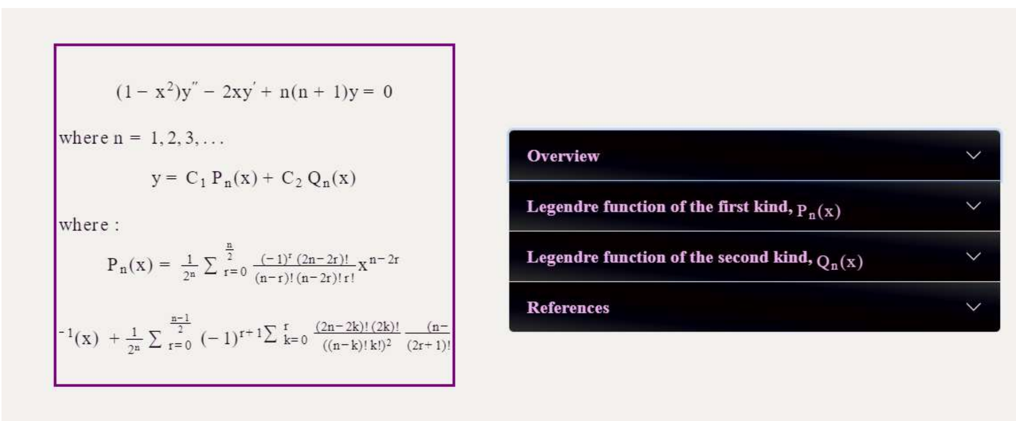
1. Dawkins, P. (2018, June 3). Differential Equations. *Tutorial.math.lamar.edu*. Retrieved October 16, 2021, from https://tutorial.math.lamar.edu/classes/de/finalthoughts.aspx

Appendix

Title: Obasi-Legendre.py (Legendre equation solver on python)

The following are images from the web calculator ((www.ultimate-calc.vercel.app)





Smart calculator

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

Is the above equation the intended equation?

Yes No

UltimateCalc Home Login/ SignUp Get in touch

Full Solution ***Simulation*** ***Solution*** ***Calculator***

Inputed equation:

$$(1 - x^2)y'' - 2xy' + 6y = 0$$

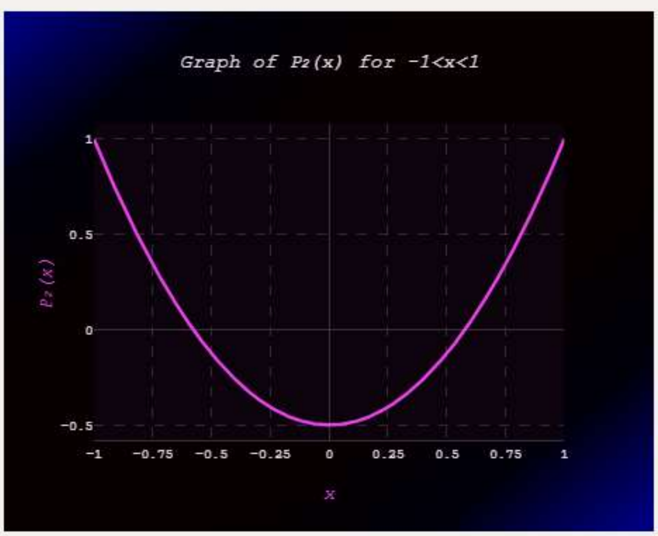
The General solution:

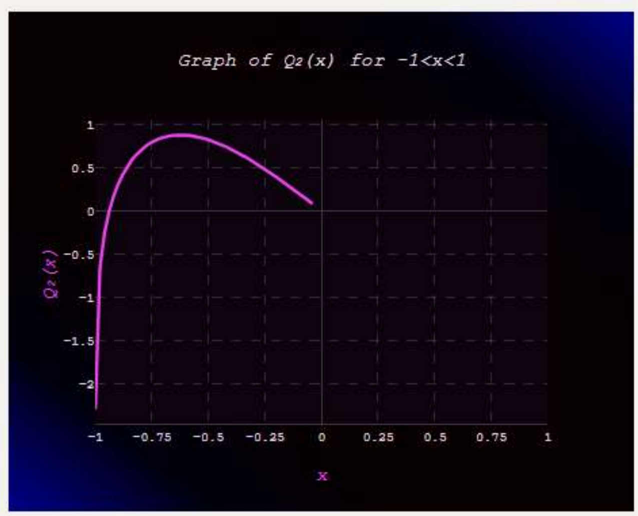
$$y = C_1 P_2(x) + C_2 Q_2(x)$$

Where:

C_1 and C_2 are constants

$$P_2(x) = \frac{1}{2}\left[3x^2 - 1\right]$$





$$Q_2(x) = \frac{1}{2}\left[3x^2 - 1\right]\tanh^{-1}(x) - \frac{3}{2}x^1$$