The Ideal Solution to the Legendre Equation and the Laplace Equation.

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Abstract

This research work presents the simplest form of the Legendre function of the second kind, $Q_n(x)$. Before now, it had been presented in form of a recursion formula. This research work also suggests that a more accurate solution for a sphere with a Potential independent of $\emptyset$ must include the Legendre function of the second kind, $Q_n(x)$, as the Legendre polynomials solely does not form the general solution to the Legendre equation.

Keywords: Legendre equation, Laplace equation

1.0 Introduction

The Legendre equation arises in mathematical models of heat conduction in spherical geometries and expansion of electromagnetic potential. It is encountered in situations where one has to solve partial differential equations containing the Laplacian polar coordinates. Legendre differential equation was introduced by Legendre in the 18th century [1] and takes the form of

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

The general solution of the above equation in case where n = 0,1,2, 3… (a positive integer) is given by,

$$y = C_1\, P_n(x) + C_2\, Q_n(x) \tag{2}$$

Where C_1 and C_2 are constants. $P_n(x)$ and $Q_n(x)$ are the two fundamental solutions, and both of which converge for $-1 < x < 1$. They are defined as follows;

$$P_n(x) = \frac{1}{2^n}\left[\sum_{r=0}^{n/2}(-1)^r \frac{(2n-2r)!}{(n-2r)!\,(n-r)!\,r!}x^{n-2r}\right] \tag{3}$$

Note: $P_n(x)$ is also referred to as Legendre polynomials.

$$Q_n(x) = [P_n(x)]tanh^{-1}(x) + \frac{1}{2^n}\left[\sum_{r=0}^{\frac{n-1}{2}}(-1)^{r+1}\left[\sum_{k=0}^{r}\frac{(2n-2k)!\,(2k)!}{((n-k)!\,k!)^2}\right]\frac{(n-1-r)!\,r!}{(n-1-2r)!\,(2r+1)!}x^{n-2r-1}\right] \tag{4}$$

Equation (4) is analogous to the recursion formula given by the French mathematician Pierre Ossian Bonnet [2], however, Equation (4) which is introduced in this research work, will undoubtedly prove to be more direct; especially when trying to determine the solution to a Legendre equation with a high value of n.

Utilizing Equation (3) to obtain the values of $P_n(x)$ for n = 0,1,2, 3…

$$P_0(x) = \frac{1}{2^0}\left[\frac{0!}{0!\,0!\,0!}x^0\right] = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}[3x^2 - 1]$$

$$P_3(x) = \frac{1}{2}[5x^3 - 3x]$$

$$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$$

$$P_5(x) = \frac{1}{8}[63x^5 - 70x^3 + 15]$$

$$P_6(x) = \frac{1}{16}[231x^6 - 315x^4 + 105x^2 - 5]$$

$$P_7(x) = \frac{1}{16}[429x^7 - 693x^5 + 315x^3 - 35x]$$

and so on …

Utilizing Equation (4) to obtain the values of $Q_n(x)$ for n = 0,1,2, 3…

$$Q_0(x) = tanh^{-1}(x)$$

Note: for n=0, the series $\left[\sum_{r=0}^{\frac{n-1}{2}}(-1)^r\left[\sum_{k=0}^{r}\frac{(2n-2k)!(2k)!}{((n-k)!k!)^2}\right]\frac{(n-1-r)!r!}{(n-1-2r)!(2r+1)!}x^{n-2r-1}\right]$ becomes zero, as n must be greater or equal to 1 for the series to be applicable.

$$Q_1(x) = xtanh^{-1}(x) - \frac{2!\,0!}{2.\,(1!\,0!)^2} = xtanh^{-1}(x) - \frac{2}{2} = xtanh^{-1}(x) - 1$$

$$Q_2(x) = \frac{1}{2}[3x^2 - 1]tanh^{-1}(x) - \frac{4!\,0!}{2^2.\,(2!\,0!)^2}x = \frac{1}{2}[3x^2 - 1]tanh^{-1}(x) - \frac{24}{16}x$$
$$= \frac{1}{2}[3x^2 - 1]tanh^{-1}(x) - \frac{3}{2}x$$

$$Q_3(x) = \frac{1}{2}[5x^3 - 3x]tanh^{-1}(x) - \frac{6!\,0!}{2^3.(3!\,0!)^2}x^2 + \frac{1}{2^3}\left[\frac{6!\,0!}{(3!\,0!)^2} + \frac{4!\,2!}{(2!\,1!)^2}\right]\frac{1!\,1!}{0!\,3!}$$
$$= \frac{1}{2}[5x^3 - 3x]tanh^{-1}(x) - \frac{5}{2}x^2 + \frac{2}{3}$$

$$Q_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]tanh^{-1}(x) - \frac{35}{8}x^3 + \frac{55}{24}x$$

$$Q_5(x) = \frac{1}{8}[63x^5 - 70x^3 + 15]tanh^{-1}(x) - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15}$$

$$Q_6(x) = \frac{1}{16}[231x^6 - 315x^4 + 105x^2 - 5]tanh^{-1}(x) - \frac{231}{16}x^5 + \frac{119}{8}x^3 - \frac{231}{80}x$$

$$Q_7(x) = \frac{1}{16}[429x^7 - 693x^5 + 315x^3 - 35x]tanh^{-1}(x) - \frac{429}{16}x^6 + \frac{275}{8}x^4 - \frac{849}{80}x^2 + \frac{16}{35}$$

and so on…

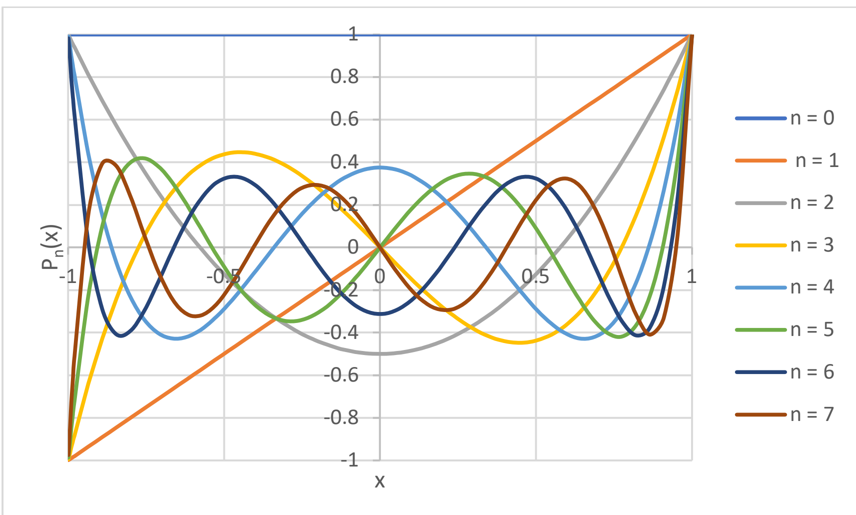

Figure 1: Graph of $P_n(x)$ against x for values of -1< x <1

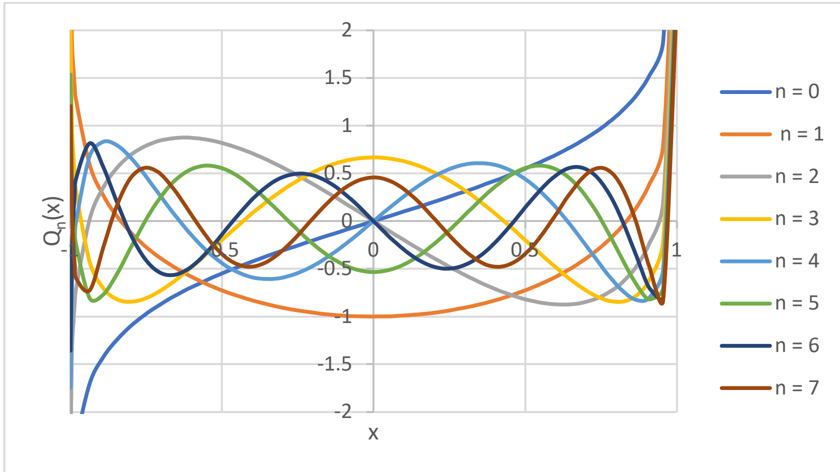

Figure 2: Graph of $Q_n(x)$ against x for values of -1< x <1

The Laplace equation is arguably the most important differential equation in all of applied mathematics. Laplace equation is the simplest elliptic partial differential equation modelling a plethora of steady state phenomena [3]. Despite strong interests, very few analytical solutions of the Laplace equation for a sphere are known [4][5]. The Laplace equation is solved by complicated and time-consuming methods [6]. The solution to the Laplace equation comprises of solutions to the Cauchy-Euler equation and the Legendre equation. Hence, the exact solution to the Legendre equation will aid in getting the best solution to the Laplace equation.

The objective of this research work is to encourage the use of $Q_n(x)$ in the applications of Legendre equation. This will also help in getting the best approximate solution to the Laplace equation.

2.0 Proposed Ideal Solution to the Laplace Equation for a Sphere with a Potential Independent of $\emptyset$ (Azimuthally Invariant)

The Laplace equation can be expressed as,

$$\nabla^2 u = 0 \tag{5}$$

Using spherical coordinates in Eq. (5), we have

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2 u}{\partial \emptyset^2}\right] = 0 \tag{6}$$

One of the methods to solve Eq. (6) is the separation of the variables. We suppose that the potential function is followed by [6],

$$u = R(r)\,\Theta(\theta)\,\Phi(\emptyset) \tag{7}$$

where r, θ and ∅ are radius, the angle between a vector and the z-axis and the angle of vector projection onto xy plane with the positive x-axis, respectively. Substituting Eq. (7) into Eq. (6) and by using the direction symmetry condition as a boundary condition, Eq. (6) is transformed into three ordinary differential equations in which direction the solution is symmetry.

$$\frac{d^2\Phi}{d\emptyset^2} = -m^2\Phi \qquad (8)$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \qquad (9)$$

$$\frac{1}{sin\theta}.\frac{d}{d\theta}\left(sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\Theta = 0 \qquad (10)$$

where $\Phi(\emptyset)$ is constant and $m^2 = 0$. Also, m and n are parameters for solving differential equations in spherical coordinates. Using a new variable x = cos θ, Eq. (10) is written as follows [4],

$$sin^2\theta\frac{d^2\Theta}{dx^2} - 2\cos\theta\frac{d\Theta}{dx} + n(n+1)\Theta = 0 \qquad (11)$$

Therefore,

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + n(n+1)\Theta = 0 \qquad (12)$$

The general solutions for Eq. (9) and (10) are as follows

$$R = a_n\, r^n + b_n\, r^{-(n+1)} \qquad (13)$$

$$\Theta(x) = A_n\, P_n(x) + B_n\, Q_n(x) \qquad (14)$$

Where a_n, b_n, A_n and B_n are constants which will be determined once we apply specific boundary equations.

$$P_n(x) = \frac{1}{2^n}\left[\sum_{r=0}^{n/2}(-1)^r\frac{(2n-2r)!}{(n-2r)!\,(n-r)!\,r!}x^{n-2r}\right] \qquad (3)$$

$$Q_n(x) = [P_n(x)]tanh^{-1}(x) + \frac{1}{2^n}\left[\sum_{r=0}^{\frac{n-1}{2}}(-1)^{r+1}\left[\sum_{k=0}^{r}\frac{(2n-2k)!\,(2k)!}{((n-k)!\,k!)^2}\right]\frac{(n-1-r)!\,r!}{(n-1-2r)!\,(2r+1)!}x^{n-2r-1}\right] \qquad (4)$$

Note that $x = \cos\theta$, $\Phi(\emptyset)$is constant and $u = R(r)\,\Theta(\theta)\,\Phi(\emptyset)$

Putting $\Phi(\emptyset) = 1$, therefore the general solution can be taken to be the summation of successive solutions as described below,

$$u(r,\theta) = \sum_{n=0}^{\infty}\left[a_nr^n + b_nr^{-(n+1)}\right]\left[A_n\, P_n(cos\theta) + B_n\, Q_n(cos\theta)\right] \qquad (15)$$

$$u(r,\theta) \cong \sum_{n=0}^{t} \left[a_n r^n + b_n r^{-(n+1)}\right]\left[A_n\, P_n(cos\theta) + B_n\, Q_n(cos\theta)\right] \tag{16}$$

Where t is sufficiently large, and for every value of n, there are corresponding values of $a_n, b_n, A_n,\ B_n$ (constants which will be determined once we apply specific boundary equations) and also $P_n(cos\theta)$, $Q_n(cos\theta)$ (fundamental solutions to the Legendre equation).

This research work suggests that a more accurate solution for a sphere with a Potential independent of $\emptyset$ must include the Legendre function of the second kind, $Q_n(x)$, as the Legendre polynomials solely does not form the general solution to the Legendre equation. Including $Q_n(x)$ in the solution might seem ambiguous, however, there are many computing tools capable of minimizing this kind of ambiguity, producing more accurate solutions.

3.0 Conclusion

This research work has introduced the simplest form of the Legendre function of the second kind, $Q_n(x)$; also suggesting that more accuracy will be achieved when $Q_n(x)$ is included in the solution for a sphere with a Potential independent of $\boldsymbol{\emptyset}$. It is hoped however that the suggestion proposed will stimulate further work in this direction, and most importantly, the advancement in the fields of electronics, fluid mechanics, heat and mass transfer, electromagnetism etc.

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