New Analytical Solutions to Variable Coefficients Differential Equations.

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Abstract

This paper introduces new solutions to two different types of variable coefficients differential equations. Each solution consists of two linearly independent solutions. This research work presents the newest and most accurate solutions to sets of differential equations which prior before now, are known to be solved by the power series' method and some other numerical methods.

Keywords: Variable coefficients differential equation, Analytical solutions.

1.0 Introduction

As Dawkins noted [1], just because we know that a solution to a differential equation exists does not mean that we will be able to find it. This saying has proven to be true as many have resorted to the use of non-analytic methods in the solution of variable coefficients differential equations, however, there are some equations which may not seem to have a simple solution, but they do. This research work presents two sets of variable coefficients differential equation which fall under this category.

2.0 The equation of the form $(x^2 + a)y'' - n(n+1)y = 0$

For the equation below:

$$(x^2 + a)y'' - n(n+1)y = 0 \tag{1}$$

Where n is an integer > 0, and a≠0

$$y = C_1\, \beta^{(1)}(x) + C_2\, \beta^{(2)}(x) \tag{2}$$

$$\beta^{(1)}(x) = (x^2 + a)\left[\sum_{r=0}^{\frac{n-1}{2}} \frac{a^r.(n-1)!\,(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!}\, x^{n-1-2r}\right] \tag{3}$$

$$\beta^{(2)}(x) = \beta^{(1)}(x) \int \frac{dx}{(\beta^{(1)}(x))^2} \tag{4}$$

Hence, applying the above formula;

For n=1,

$$(x^2 + a)y'' - 2y = 0$$

$$y = C_1\,(x^2 + a) + C_2\,(x^2 + a)\left[\frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2 + a}\right]$$

For n=2,

$$(x^2 + a)y'' - 6y = 0$$

$$y = C_1\,(3x^3 + 3ax) + C_2\,(3x^3 + 3ax)\left[\frac{3}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2 + a} + \frac{2}{x}\right]$$

For n=3,

$$(x^2 + a)y'' - 12y = 0$$

$$y = C_1\,(x^2 + a)(60x^2 + 12a) + C_2\,(x^2 + a)(60x^2 + 12a)\left[\frac{6}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2 + a} + \frac{25x}{5x^2 + a}\right]$$

For n=4,

$$(x^2 + a)y'' - 20y = 0$$

$$y = C_1\,(x^2 + a)(840x^3 + 360ax) + C_2\,(x^2 + a)(840x^3 + 360ax)\left[\frac{10}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2 + a} + \frac{686x}{9(7x^2 + 3a)} + \frac{32}{9x}\right]$$

For n=5,

$$(x^2 + a)y'' - 30y = 0$$

$$y = C_1\,6.5!\,(x^2 + a)(21x^4 + 14ax^2 + a^2) + C_2\,6.5!\,(x^2 + a)(21x^4 + 14ax^2 + a^2)\left[\frac{15}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2 + a} + \frac{14(21x^3 + 8ax)}{21x^4 + 14ax^2 + a^2}\right]$$

And so on…

2.1 Verification of solutions

For verification, a random equation is selected, this is because selecting more than one would make the publication too lengthy.

For the differential equation below,

$$(x^2 + a)y'' - 2y = 0$$

$$y = C_1\,(x^2+a) + C_2\,(x^2+a)\left[\frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2+a}\right]$$

$$\therefore y = C_1\,(x^2+a) + C_2\,\left[\frac{x^2+a}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + x\right]$$

Differentiating,

$$y' = C_1\,[2x] + C_2\,\left[\frac{2x}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x^2+a}{\sqrt{a}}.\frac{\sqrt{a}}{x^2+a} + 1\right]$$

$$y' = C_1\,[2x] + C_2\,\left[\frac{2x}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + 2\right]$$

Differentiating further,

$$y'' = C_1\,[2] + C_2\,\left[\frac{2}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{2x}{\sqrt{a}}.\frac{\sqrt{a}}{x^2+a}\right]$$

$$y'' = C_1\,[2] + C_2\,\left[\frac{2}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{2x}{x^2+a}\right]$$

$$\therefore y'' = 2\left[C_1 + C_2\,\left[\frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2+a}\right]\right]$$

$$\therefore (x^2+a)y'' = 2\left[C_1(x^2+a) + C_2\,(x^2+a)\left[\frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2+a}\right]\right]$$

Recall,

$$y = C_1\,(x^2+a) + C_2\,(x^2+a)\left[\frac{1}{\sqrt{a}}\tan^{-1}\left(\frac{x}{\sqrt{a}}\right) + \frac{x}{x^2+a}\right]$$

$$\therefore (x^2+a)y'' = 2y$$

$$\therefore (x^2+a)y'' - 2y = 0$$

Hence, it is proven for n=1.

Other positive integer values of n will produce the same outcome, however, the proving of these would make this paper ambiguous. The above was done to show that Eqn (2) is the general solution to Eqn (1) for all positive values of n.

2.2 Derivation of other solutions

For the equation below:

$$(x^2 + Ax + B)y'' - n(n+1)y = 0 \qquad (5)$$

Where n is an integer > 0, and a≠0

$$y = C_1\, \beta^{(1)}\left(x + {}^{A}/_{2}\right) + C_2\, \beta^{(2)}\left(x + {}^{A}/_{2}\right) \tag{6}$$

$$\beta^{(1)}\left(x + {}^{A}/_{2}\right) = (x^2 + Ax + B) \times \left[\sum_{r=0}^{\frac{n-1}{2}} \frac{(A^2 - 4B)^r . (n-1)!\,(2n-1-2r)!}{2^r (n-1-r)!\,(n-1-2r)!\,r!} \left(x + {}^{A}/_{2}\right)^{n-1-2r}\right] \tag{7}$$

$$\beta^{(2)}\left(x + {}^{A}/_{2}\right) = \beta^{(1)}\left(x + {}^{A}/_{2}\right) \times \int \frac{dx}{\left(\beta^{(1)}\left(x + {}^{A}/_{2}\right)\right)^2} \tag{8}$$

Hence, applying the above formula;

For n=1,

$$(x^2 + Ax + B)y'' - 2y = 0$$

$$y = C_1\, (x^2 + Ax + B) + C_2\, (x^2 + Ax + B)\left[\frac{2}{\sqrt{A^2 - 4B}} \tan^{-1}\left(\frac{2x + A}{\sqrt{A^2 - 4B}}\right) + \frac{x + {}^{A}/_{2}}{x^2 + Ax + B}\right]$$

For n=2,

$$(x^2 + a)y'' - 6y = 0$$

$$y = C_1\, (3x + {}^{3A}/_{2})(x^2 + Ax + B) + C_2\, (3x + {}^{3A}/_{2})(x^2 + Ax + B) \times \left[\frac{6}{\sqrt{A^2 - 4B}} \tan^{-1}\left(\frac{2x + A}{\sqrt{A^2 - 4B}}\right) + \frac{x + {}^{A}/_{2}}{x^2 + Ax + B} + \frac{2}{x + {}^{A}/_{2}}\right]$$

3.0 The equation of the form $(x^2 + a)y'' + xy' - (n+1)^2 y = 0$

For the equation below:

$$(x^2 + a)y'' + xy' - (n+1)^2 y = 0 \tag{9}$$

$$y = C_1\, \in^{(1)}(x) + C_2\, \in^{(2)}(x) \tag{10}$$

$$\in^{(1)}(x) = (x^2 + a)\left[\frac{d^n}{dx^n}\left(x(x^2 + a)^{n - {}^{1}/_{2}}\right)\right] \tag{11}$$

$$\in^{(2)}(x) = \in^{(1)}(x) \int \frac{dx}{\sqrt{x^2 + a}\,(\in^{(1)}(x))^2} \tag{12}$$

For n=0,

$$(x^2 + a)y'' + xy' - y = 0$$

$$y = C_1 x + C_2 \sqrt{x^2 + a}$$

For n=1,

$$(x^2+a)y''+xy'-4y=0$$

$$y=C_1[2x^2+a]+C_2x\sqrt{x^2+a}$$

For n=2,

$$(x^2+a)y''+xy'-9y=0$$

$$y=C_1[12x^3+9ax]+C_2[4x^2+a]\sqrt{x^2+a}$$

For n=3,

$$(x^2+a)y''+xy'-16y=0$$

$$y=C_1[120x^4+132ax^2+21a^2]+C_2[2x^3+ax]\sqrt{x^2+a}$$

For n=4,

$$(x^2+a)y''+xy'-25y=0$$

$$y=C_1[16x^5+20ax^3+5a^2x]+C_2[16x^4+12ax^2+a^2]\sqrt{x^2+a}$$

And so on…

3.1 Verification of solutions

For verification, a random equation is selected, this is because selecting more than one would make the publication too lengthy.

For the differential equation below,

$$(x^2+a)y''+xy'-9y=0$$

$$y=C_1[12x^3+9ax]+C_2[4x^2+a]\sqrt{x^2+a}$$

Differentiating,

$$y'=C_1[36x^2+9a]+C_2\left[8x\sqrt{x^2+a}+\frac{4x^3+ax}{\sqrt{x^2+a}}\right]$$

Differentiating further,

$$y''=C_1[72x]+C_2\left[8\sqrt{x^2+a}+\frac{8x^2+12x^2+a}{\sqrt{x^2+a}}+\frac{-4x^4-ax^2}{(x^2+a)^{3/2}}\right]$$

$$y''=C_1[72x]+C_2\left[8\sqrt{x^2+a}+\frac{20x^2+a}{\sqrt{x^2+a}}+\frac{-4x^4-ax^2}{(x^2+a)^{3/2}}\right]$$

$$y''=C_1[72x]+C_2\left[8\sqrt{x^2+a}+\frac{20x^4+20ax^2+ax^2+a^2-4x^4-ax^2}{(x^2+a)^{3/2}}\right]$$

$$y'' = C_1[72x] + C_2\left[8 + \frac{16x^4 + 20ax^2 + a^2}{(x^2+a)^2}\right]\sqrt{x^2+a}$$

$$y'' = C_1[72x] + C_2\left[\frac{8(x^4 + 2ax^2 + a^2) + 16x^4 + 20ax^2 + a^2}{(x^2+a)^2}\right]\sqrt{x^2+a}$$

$$y'' = C_1[72x] + C_2\left[\frac{8x^4 + 16ax^2 + 8a^2 + 16x^4 + 20ax^2 + a^2}{(x^2+a)^2}\right]\sqrt{x^2+a}$$

$$\therefore y'' = C_1[72x] + C_2\left[\frac{24x^4 + 36ax^2 + 9a^2}{(x^2+a)^2}\right]\sqrt{x^2+a}$$

$$\therefore (x^2+a)y'' = C_1[72x^3 + 72ax] + C_2\left[\frac{24x^4 + 36ax^2 + 9a^2}{x^2+a}\right]\sqrt{x^2+a}$$

$$xy' = C_1[36x^3 + 9ax] + C_2\left[8x^2 + \frac{4x^4 + ax^2}{x^2+a}\right]\sqrt{x^2+a}$$

$$\therefore xy' = C_1[36x^3 + 9ax] + C_2\left[\frac{12x^4 + 9ax^2}{x^2+a}\right]\sqrt{x^2+a}$$

Adding $(x^2+a)y''$ and xy',

$$\begin{aligned}(x^2+a)y'' + xy' \\ &= C_1[72x^3 + 72ax + 36x^3 + 9ax] \\ &\quad + C_2\left[\frac{24x^4 + 36ax^2 + 9a^2 + 12x^4 + 9ax^2}{x^2+a}\right]\sqrt{x^2+a}\end{aligned}$$

$$(x^2+a)y'' + xy' = C_1[108x^3 + 81ax] + C_2\left[\frac{36x^4 + 45ax^2 + 9a^2}{x^2+a}\right]\sqrt{x^2+a}$$

$$(x^2+a)y'' + xy' = 9\left(C_1[12x^3 + 9ax] + C_2\left[\frac{4x^4 + 5ax^2 + a^2}{x^2+a}\right]\sqrt{x^2+a}\right)$$

$$(x^2+a)y'' + xy' = 9\left(C_1[12x^3 + 9ax] + C_2\left[\frac{(4x^2+a)(x^2+a)}{x^2+a}\right]\sqrt{x^2+a}\right)$$

$$(x^2+a)y'' + xy' = 9\left(C_1[12x^3 + 9ax] + C_2[4x^2+a]\sqrt{x^2+a}\right)$$

Recall,

$$y = C_1[12x^3 + 9ax] + C_2[4x^2+a]\sqrt{x^2+a}$$

$$(x^2+a)y'' + xy' = 9y$$

$$(x^2+a)y'' + xy' - 9y = 0$$

Hence, it is proven for n=2.

Other positive integer values of n will produce the same outcome, however, the proving of these would make this paper ambiguous. The above was done to show that Eqn (10) is the general solution to Eqn (9) for all positive values of n.

4.0 Conclusion

This research work has introduced and tested new solutions to some differential equations which were previously solved only by the power series' method and some other numerical methods; showing the advantages of the new solutions over the power series. Below are the summarized advantages:

- They permit the study of the analytic properties
- They are differentiable unlike the finite difference solution.
- They are most accurate when comparing with the already known methods.
- They converge for all values of x.
- They reduce computational cost of the calculation.

The new solutions are flexible enough to allow a practically infinite list of examples, but at this point the construction of further examples might sound an exercise lacking interest. It is hoped however that the techniques that have been proposed will stimulate further work in this direction, and most importantly, the discovery of newer solutions to more equations of practical interests.

References

1. Dawkins, P. (2018, June 3). Differential Equations. *Tutorial.math.lamar.edu*. Retrieved October 16, 2021, from https://tutorial.math.lamar.edu/classes/de/finalthoughts.aspx