The Ideal and Exact Analytic Solution to the Legendre Equation.

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Abstract

This research work introduces for the first time since the invention of the Legendre equation in the 18th century, the ideal and exact solution to the Legendre equation. The solution comprises of two linearly independent fundamental solutions, ${L_n}^{(1)}(x)$ and ${L_n}^{(2)}(x)$. ${L_n}^{(1)}(x)$ is a terminating series analogous to the afore-known Legendre polynomials while ${L_n}^{(2)}(x)$ is the sum of a terminating series and the product of ${L_n}^{(1)}(x)$ and the function $tanh^{-1}(x)$. The distinction between this new solution and the previous solution is that this solution is a terminating series while the other is an infinite series. This implies that the new solution gives the exact value while the other gives an approximate value; in fact, the contemporary solution is a constant multiple of the Taylor series' expansion of this new solution. This solution becomes indispensable when the value of x is far away from the center of convergence, then the tendency of error in approximation with other solutions is larger. The proof of this solution is given in details as well as its utilization in the proposed ideal solution to the Laplace equation for a sphere with a potential independent of $\emptyset$. A programming code in Python was also developed to show that the new solution is easily programmable and convenient to use. It is certain that this solution would replace other contemporary solutions, ushering in advancement in the fields of electronics, fluid mechanics, heat and mass transfer, electromagnetism etc.

Keywords: Legendre equation, Laplace equation

1.0 Introduction

The Legendre equation arises in mathematical models of heat conduction in spherical geometries and expansion of electromagnetic potential. It is encountered in situations where one has to solve partial differential equations containing the Laplacian polar coordinates. Legendre differential equation was introduced by Legendre in the 18th century and takes the form of

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (1)$$

The general solution of the above equation in case where n = 0,1,2, 3… (a positive integer) is given by,

$$y = C_1\, P_n(x) + C_2\, Q_n(x) \qquad (2)$$

where C_1 and C_2 are constants, $P_n{}^{(1)}(x)$ is Legendre polynomial (polynomial solution with even exponent) and $Q_n{}^{(2)}(x)$ is the Legendre polynomial (polynomial solution with odd exponent) [1], and both of which converge for $-1 < x < 1$.

$$P_n(x) = 1 + \sum_{r=1}^{\infty} (-1)^r \frac{n(n-2) \dots (n-2r+2)(n+1) \dots (n+2r-1)}{(2r)!} x^{2r} \tag{3}$$

$$Q_n(x) = x + \sum_{r=1}^{\infty} (-1)^r \frac{(n-1)(n-3) \dots (n-2r+1)(n+2) \dots (n+2r)}{(2r)!} x^{2r+1} \tag{4}$$

In addition, if the value of n is even then the even series will contain a finite number of terms, as will the odd series when n is odd. These truncated series are known as the Legendre polynomials. Also, we can obtain the Legendre polynomials by the use of the Rodrigues' formula. However, the series from the other linearly independent solution is an infinite series, thus making the general solution (with the use of the power series' method) to be an infinite series needing approximation.

The Laplace equation is arguably the most important differential equation in all of applied mathematics. Laplace equation is the simplest elliptic partial differential equation modelling a plethora of steady state phenomena [2]. Despite strong interests, very few analytical solutions of the Laplace equation for a sphere are known [3][4]. The Laplace equation is solved by complicated and time-consuming methods [5]. The solution to the Laplace equation comprises of solutions to the Cauchy-Euler equation and the Legendre equation. Hence, the exact solution to the Legendre equation will aid in getting the best solution to the Laplace equation.

The objective of this research work is to discourage the use of an infinite series as the solution to the Legendre equation by introducing the ideal and exact solution to the Legendre equation. This solution will in-turn help in getting the best approximate solution to the Laplace equation.

The new solution is introduced in Section 2, while the two linearly independent fundamental solutions that make up this new solution is verified and proven to be fundamental solutions in Section 3. Also, a derivation of another is done in the fourth section to show other solutions can also be derived from it. Section 5 shows a proposed ideal solution developed in this research work for the Laplace equation. The research work is concluded in the last section by stating the advantages of this new solution to other contemporary solutions.

2.0 New Solution to the Legendre Differential Equation

For the equation below:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

Where n is non-negative integer (i.e. 0,1,2,3…)

$$y = C_1\, {L_n}^{(1)}(x) + C_2\, {L_n}^{(2)}(x) \tag{5}$$

Where ${L_n}^{(1)}(x)$ and ${L_n}^{(2)}(x)$ are linearly independent solutions. The first of the two, ${L_n}^{(1)}(x)$ is defined as:

$${L_n}^{(1)}(x) = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.\,(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,r!} x^{n-2r} \tag{6}$$

$${L_0}^{(1)}(x) = \frac{(-1)^0.\,(-1)!}{(-1)!\,(0)!\,0!} x^0 = 1$$

$${L_1}^{(1)}(x) = x$$

$${L_2}^{(1)}(x) = 3x^2 - 1$$

$${L_3}^{(1)}(x) = 10x^3 - 6x$$

$${L_4}^{(1)}(x) = 35x^4 - 30x^2 + 3$$

$${L_5}^{(1)}(x) = 126x^5 - 140x^3 + 30x$$

$${L_6}^{(1)}(x) = 462x^6 - 630x^4 + 210x^2 - 10$$

$${L_7}^{(1)}(x) = 1716x^7 - 2772x^5 + 1260x^3 - 140x$$

And so on…

Comparing ${L_n}^{(1)}(x)$ with the afore-known Legendre polynomials $P_n(x)$.

$$P_n(x) = \begin{cases} 1, & n = 0 \\ 2^{1-n} \times {L_n}^{(1)}(x), & n > 0 \end{cases} \tag{7}$$

The figure below shows the graph of ${L_n}^{(1)}(x)$ against x as x ranges from -1 to 1.

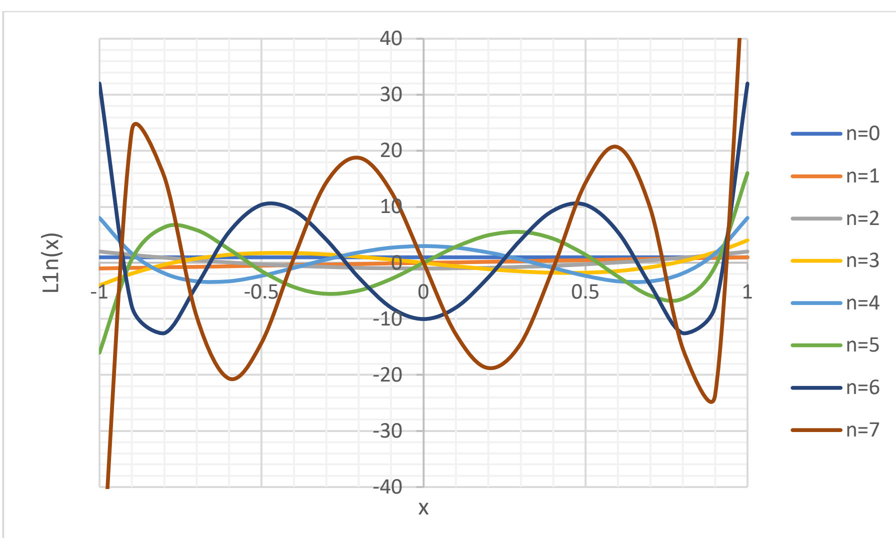

Fig. 1: Graph of ${L_n}^{(1)}(x)$ against x

The second fundamental solution, ${L_n}^{(2)}(x)$ is defined as:

$$ {L_n}^{(2)}(x) = \left[{L_n}^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r} \tag{8} $$

Where,

$$ a_0 = -\frac{(2n-1)!}{n!\,(n-1)!} \tag{9} $$

And the recurrence relation,

$$ a_{r+1} = -\frac{1}{2r+3}\left[\frac{(n-1-2r)(n-2-2r)}{2(n-1-r)}a_r - \frac{(-1)^r.\,(2n-3-2r)!}{(n-1-r)!\,(n-3-2r)!\,(r+1)!}\right] \tag{10} $$

Note that the series $\sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r}$ becomes zero for n=0.

Defining a_r from Eq.s (9) and (10) for ease of calculation,

$$ a_r = \sum_{r=0}^{\frac{n-1}{2}} \frac{(-1)^{r+1}}{2}\left[\sum_{k=0}^{r}\frac{(2n-2k)!\,(2k)!}{((n-k)!\,k!)^2}\right]\frac{(n-1-r)!\,r!}{(2r+1)!\,(n-1-2r)!} \tag{11} $$

Therefore,

$$ \begin{aligned} {L_n}^{(2)}(x) &= \left[{L_n}^{(1)}(x)\right]tanh^{-1}(x) \\ &\quad + \sum_{r=0}^{\frac{n-1}{2}} \frac{(-1)^{r+1}}{2}\left[\sum_{k=0}^{r}\frac{(2n-2k)!\,(2k)!}{((n-k)!\,k!)^2}\right]\frac{(n-1-r)!\,r!}{(2r+1)!\,(n-1-2r)!}x^{n-1-2r} \end{aligned} \tag{12} $$

The above equation in Eq. (12) is also a terminating series. ${L_n}^{(2)}(x)$ is the sum of finite series and the product of ${L_n}^{(1)}(x)$ and the function $tanh^{-1}(x)$. The function $tanh^{-1}(x)$ is analytic in the region of $|x| < 1$ with -1 and +1 been singular points, hence, ${L_n}^{(2)}(x)$ is only analytic at this same region.

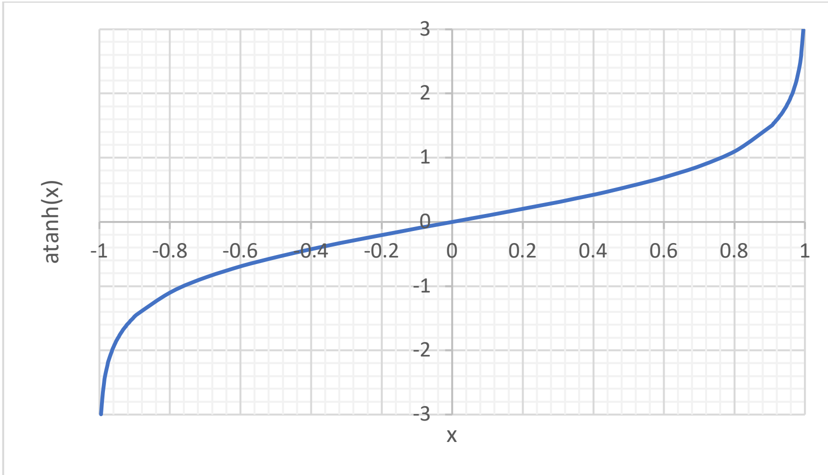

Fig. 2: Graph of $tanh^{-1}(x)$ against x

Utilizing the above formula,

$$L_0^{(2)}(x) = tanh^{-1}(x)$$

$$L_1^{(2)}(x) = xtanh^{-1}(x) - \frac{1!}{1!\,0!} = xtanh^{-1}(x) - 1$$

$$L_2^{(2)}(x) = [3x^2 - 1]tanh^{-1}(x) - \frac{3!}{2!\,1!} = [3x^2 - 1]tanh^{-1}(x) - 3x$$

$$L_3^{(2)}(x) = [10x^3 - 6x]tanh^{-1}(x) - \frac{5!}{3!\,2!}x^2 + \frac{1}{2}\left(\frac{6!}{(3!)^2} + \frac{4!\,2!}{(2!\,1!)^2}\right)\frac{1!\,1!}{3!\,0!}$$

$$= [10x^3 - 6x]tanh^{-1}(x) - \frac{5!}{3!\,2!}x^2 + \frac{1}{2}(20 + 12)\frac{1}{6}$$

$$= [10x^3 - 6x]tanh^{-1}(x) - 10x^2 + \frac{8}{3}$$

$$L_4^{(2)}(x) = [35x^4 - 30x^2 + 3]tanh^{-1}(x) - 35x^3 + \frac{55}{3}x$$

$$L_5^{(2)}(x) = [126x^5 - 140x^3 + 30x]tanh^{-1}(x) - 126x^4 + 98x^2 - \frac{128}{15}$$

$$L_6^{(2)}(x) = [462x^6 - 630x^4 + 210x^2 - 10]tanh^{-1}(x) - 462x^5 + 476x^3 - \frac{462}{5}x$$

$$L_7^{(2)}(x) = [1716x^7 - 2772x^5 + 1260x^3 - 140x]tanh^{-1}(x) - 1716x^6 + 2200x^4 - \frac{3396}{5}x^2 + \frac{1024}{35}$$

And so on…

The figure below shows the graph of $L_n^{(2)}(x)$ against x is bounded within the region of

$-1 < x < 1.$

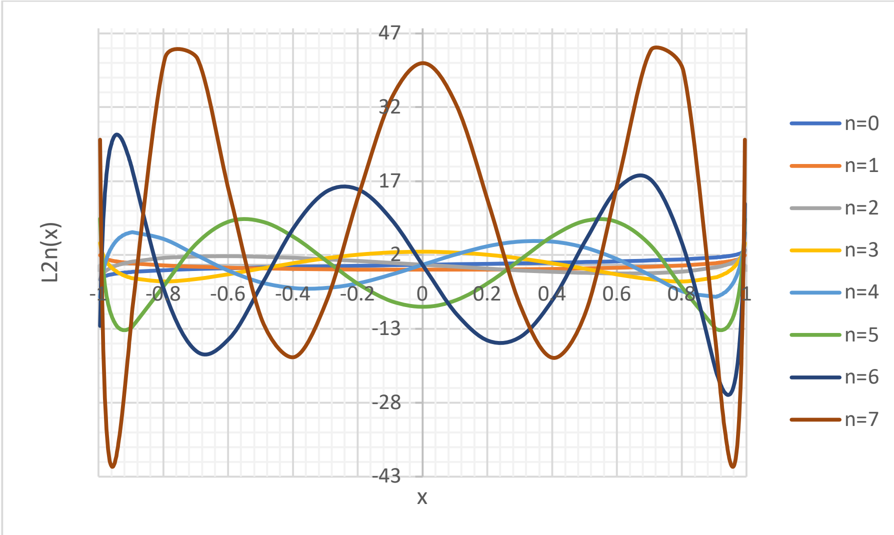

Fig. 3: Graph of $L_n{}^{(2)}(x)$ against x

3.0 Verification of New solution

Objective I: To proof that $\boldsymbol{L_n{}^{(1)}(x)}$ is a solution to the Legendre equation.

Recall the Legendre differential equation,

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

Putting $y = L_n{}^{(1)}(x)$,

$$\therefore y = L_n{}^{(1)}(x) = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,r!} x^{n-2r} \tag{13}$$

Differentiating y,

$$y' = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!} x^{n-1-2r}$$

Differentiating further,

$$y'' = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r}$$

Therefore,

$$(1-x^2)y'' = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r} + \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^{r+1}.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2r}$$

Note that the series $\sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!(n-2-2r)!r!} x^{n-2-2r} \equiv \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!(n-2-2r)!r!} x^{n-2-2r}$, this is because the last term of the series is zero.

$$\therefore (1-x^2)y'' = \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r} + \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^{r+1}.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2r}$$

$$-2xy' = \sum_{r=0}^{\frac{n}{2}} \frac{2(-1)^{r+1}.(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!} x^{n-2r}$$

Adding $(1-x^2)y''$ and $-2xy'$,

$$(1-x^2)y'' - 2xy' = \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r} + \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^{r+1}.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} \left[\frac{2}{n-1-r} + 1\right] x^{n-2r}$$

$$= \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r} + \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^{r+1}.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} \left[\frac{2+n-1-2r}{n-1-2r}\right] x^{n-2r}$$

$$= \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r} + \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^{r+1}(n+1-2r).(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!} x^{n-2r}$$

$$= \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r} + \frac{(2n-1)!\,(n+1)}{(n-1)!\,(n-1)!} x^n + \sum_{r=1}^{\frac{n}{2}} \frac{(-1)^{r+1}(n+1-r).(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!} x^{n-2r}$$

$$= \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2-2r)!\,r!} x^{n-2-2r} + \frac{-(2n-1)!\,(n+1)}{(n-1)!\,(n-1)!} x^n + \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r(n-1-r).(2n-3-2r)!}{(n-2-r)!\,(n-3-2r)!\,(r+1)!} x^{n-2-2r}$$

$$= \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-3-2r)!}{(n-2-r)!\,(n-3-2r)!\,r!} \left[\frac{(2n-1-2r)(2n-2-2r)}{(n-1-r)(n-2-2r)} + \frac{n-1-2r}{r+1}\right] x^{n-2-2r} - \frac{(2n-1)!\,(n+1)}{(n-1)!\,(n-1)!} x^n$$

$$= \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-3-2r)! \begin{bmatrix} 4rn+4n-2r-2-4r^2-4r+n^2 \\ -2n-2rn-n+2+2r-2rn+4r+4r^2 \end{bmatrix}}{(n-2-r)!\,(n-2-2r)!\,(r+1)!} x^{n-2-2r} - \frac{(2n-1)!\,(n+1)}{(n-1)!\,(n-1)!} x^n$$

$$= \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-3-2r)!\,[n^2+n]}{(n-2-r)!\,(n-2-2r)!\,(r+1)!} x^{n-2-2r} - \frac{(2n-1)!\,(n+1)}{(n-1)!\,(n-1)!} x^n$$

$$= -n(n+1)\left[\frac{(2n-1)!}{n!\,(n-1)!} x^n - \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r.(2n-3-2r)!}{(n-2-r)!\,(n-2-2r)!\,(r+1)!} x^{n-2-2r}\right]$$

$$= -n(n+1)\left[\frac{(2n-1-0)!}{(n-0)!\,(n-1-0)!\,0!} x^{n-0} + \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^{r+1}.(2n-3-2r)!}{(n-2-r)!\,(n-2-2r)!\,(r+1)!} x^{n-2-2r}\right]$$

$$= -n(n+1)\left[\frac{(2n-1-0)!}{(n-0)!\,(n-1-0)!\,0!} x^n + \sum_{r=1}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,(r)!} x^{n-2r}\right]$$

$$= -n(n+1)\left[\sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,(r)!} x^{n-2r}\right]$$

Recall from Eq. (13)

$$y = L_n{}^{(1)}(x) = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,r!} x^{n-2r} \tag{13}$$

$$(1-x^2)y'' - 2xy' = -n(n+1)y$$

$$\therefore (1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Hence it is proven for all non-negative integer values of n.

Objective II: To proof that $\boldsymbol{L_n{}^{(2)}(x)}$ is a solution to the Legendre equation.

Recall the Legendre differential equation,

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

Putting $y = L_n{}^{(2)}(x)$

$$y = L_n{}^{(2)}(x) = \left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r} \tag{14}$$

Let $\Sigma = \sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r}$

$$\therefore y = \left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) + \Sigma$$

Differentiating y,

$$y' = \left[\frac{d}{dx}(L_n{}^{(1)}(x))\right]tanh^{-1}(x) + \frac{L_n{}^{(1)}(x)}{1-x^2} + \Sigma'$$

Differentiating further,

$$y'' = \left[\frac{d^2}{dx^2}(L_n{}^{(1)}(x))\right]tanh^{-1}(x) + \frac{2\frac{d}{dx}(L_n{}^{(1)}(x))}{1-x^2} + \frac{2x.L_n{}^{(1)}(x)}{(1-x^2)^2} + \Sigma''$$

$$(1-x^2)y'' = \left[(1-x^2)\frac{d^2}{dx^2}(L_n{}^{(1)}(x))\right]tanh^{-1}(x) + 2\frac{d}{dx}(L_n{}^{(1)}(x)) + \frac{2x.L_n{}^{(1)}(x)}{1-x^2} + (1-x^2)\Sigma''$$

Also,

$$-2xy' = \left[-2x\frac{d}{dx}\left(L_n{}^{(1)}(x)\right)\right]tanh^{-1}(x) - \frac{2x.L_n{}^{(1)}(x)}{1-x^2} - 2x.\Sigma'$$

Therefore,

$$\begin{aligned}(1-x^2)y'' - 2xy' \\ &= \left[(1-x^2)\frac{d^2}{dx^2}\left(L_n{}^{(1)}(x)\right) - 2x\frac{d}{dx}\left(L_n{}^{(1)}(x)\right)\right]tanh^{-1}(x) \\ &+ 2\frac{d}{dx}\left(L_n{}^{(1)}(x)\right) + \ (1-x^2)\Sigma'' - 2x.\Sigma'\end{aligned}$$

Recall that $L_n{}^{(1)}(x)$ has been proven to be a solution to the Legendre equation, this implies that;

$$(1-x^2)\frac{d^2}{dx^2}\left(L_n{}^{(1)}(x)\right) - 2x\frac{d}{dx}\left(L_n{}^{(1)}(x)\right) + n(n+1)L_n{}^{(1)}(x) = 0$$

$$\therefore\ (1-x^2)\frac{d^2}{dx^2}\left(L_n{}^{(1)}(x)\right) - 2x\frac{d}{dx}\left(L_n{}^{(1)}(x)\right) = -n(n+1)L_n{}^{(1)}(x)$$

Therefore,

$$\begin{aligned}(1-x^2)y'' - 2xy' \\ &= \left[-n(n+1)L_n{}^{(1)}(x)\right]tanh^{-1}(x) + 2\frac{d}{dx}\left(L_n{}^{(1)}(x)\right) + \ (1-x^2)\Sigma'' - 2x.\Sigma'\end{aligned}$$

Recall,

$$\Sigma = \sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r}$$

Differentiating,

$$\Sigma' = \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)a_r x^{n-2-2r}$$

$$\Sigma'' = \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r}$$

$$\begin{aligned}(1-x^2)\Sigma'' &= \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r} \\ &- \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-1-2r}\end{aligned}$$

$$-2x.\Sigma' = -\sum_{r=0}^{\frac{n-1}{2}} 2(n-1-2r)a_r x^{n-1-2r}$$

Case I: For n=0, ${L_n}^{(1)}(x)$ =1, $\frac{d}{dx}({L_n}^{(1)}(x)) = 0$

the series $\sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r} = 0$

$$\therefore (1-x^2)y'' - 2xy' = [-n(n+1)]tanh^{-1}(x) + 2[0] + (1-x^2)[0] - 2x[0]$$

$$(1-x^2)y'' - 2xy' = -n(n+1).tanh^{-1}(x)$$

Recall that,

$$y = \left[{L_n}^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r} \qquad (14)$$

$$y = tanh^{-1}(x)$$

$$(1-x^2)y'' - 2xy' = -n(n+1)y$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Hence it is proven for n=0

Case II: For $n \neq 0$

$${L_n}^{(1)}(x) = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,r!}x^{n-2r} \equiv \sum_{r=0}^{\frac{n-1}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,r!}x^{n-2r}$$

$$2\frac{d}{dx}({L_n}^{(1)}(x)) = \sum_{r=0}^{\frac{n-1}{2}} \frac{2(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!}x^{n-1-2r}$$

Therefore,

$$\begin{aligned}(1-x^2)y'' - 2xy' &\\ &= -n(n+1)\left[{L_n}^{(1)}(x)\right]tanh^{-1}(x)\\ &+ \sum_{r=0}^{\frac{n-1}{2}} \frac{2(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!}x^{n-1-2r}\\ &+ \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r}\\ &- \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-1-2r} - \sum_{r=0}^{\frac{n-1}{2}} 2(n-1-2r)a_r x^{n-1-2r}\end{aligned}$$

$$= -n(n+1)\left[L_n^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} \frac{2(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!} x^{n-1-2r}$$
$$+ \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r}$$
$$+ \sum_{r=0}^{\frac{n-1}{2}} [-(n-1-2r)(n-2-2r) - 2(n-1-2r)]a_r x^{n-1-2r}$$

$$= -n(n+1)\left[L_n^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r}$$
$$+ \sum_{r=0}^{\frac{n-1}{2}} \left[\frac{2(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!} - (n-2r)(n-1-2r)a_r\right] x^{n-1-2r}$$

$$= -n(n+1)\left[L_n^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r}$$
$$+ \left[\frac{2(2n-1)!}{(n-1)!\,(n-1)!} - n(n-1)a_0\right] x^{n-1}$$
$$+ \sum_{r=1}^{\frac{n-1}{2}} \left[\frac{2(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-1-2r)!\,r!} - (n-2r)(n-1-2r)a_r\right] x^{n-1-2r}$$

Note that,

$$\sum_{r=0}^{\frac{n-1}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r} \equiv \sum_{r=0}^{\frac{n-3}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r}$$

This is because the last term of the series is zero.

$$\therefore (1-x^2)y'' - 2xy'$$
$$= -n(n+1)\left[L_n^{(1)}(x)\right]tanh^{-1}(x)$$
$$+ \sum_{r=0}^{\frac{n-3}{2}} (n-1-2r)(n-2-2r)a_r x^{n-3-2r}$$
$$+ \left[\frac{2(2n-1)!}{(n-1)!\,(n-1)!} - n(n-1)a_0\right] x^{n-1}$$
$$+ \sum_{r=0}^{\frac{n-3}{2}} \left[\begin{array}{c} \dfrac{2(-1)^{r+1}.(2n-3-2r)!}{(n-2-r)!\,(n-3-2r)!\,r!} \\ -(n-2-2r)(n-3-2r)a_{r+1} \end{array}\right] x^{n-3-2r}$$

$$= -n(n+1)\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) + \left[\frac{2(2n-1)!}{(n-1)!\,(n-1)!} - n(n-1)a_0\right]x^{n-1}$$

$$+\sum_{r=0}^{\frac{n-3}{2}}\left[\begin{array}{c}\frac{2(-1)^{r+1}.\,(2n-3-2r)!}{(n-2-r)!\,(n-3-2r)!\,r!} - (n-2-2r). \\ (n-3-2r)a_{r+1} + (n-1-2r)(n-2-2r)a_r\end{array}\right]x^{n-3-2r}$$

Recall from Eq,s (9) and (10),

$$a_0 = -\frac{(2n-1)!}{n!\,(n-1)!} \tag{9}$$

$$a_{r+1} = -\frac{1}{2r+3}\left[\frac{(n-1-2r)(n-2-2r)}{2(n-1-r)}a_r - \frac{(-1)^r.\,(2n-3-2r)!}{(n-1-r)!\,(n-3-2r)!\,(r+1)!}\right] \tag{10}$$

Therefore,

$$\frac{(2n-1)!}{n!\,(n-1)!} = -a_0$$

$$(n-1-2r)(n-2-2r)a_r$$
$$= \frac{-2(-1)^{r+1}.\,(2n-3-2r)!}{(n-2-r)!\,(n-3-2r)!\,(r+1)!} - 2(n-1-r)(2r+3)a_{r+1}$$

Substituting these into the equation,

$$(1-x^2)y'' - 2xy'$$
$$= -n(n+1)\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) + [-n(n+1)a_0]x^{n-1}$$
$$+\sum_{r=0}^{\frac{n-3}{2}}\left[\begin{array}{c}\frac{2(-1)^{r+1}.\,(2n-3-2r)!}{(n-2-r)!\,(n-3-2r)!\,r!} - (n-2-2r). \\ (n-3-2r)a_{r+1} + \frac{-2(-1)^{r+1}.\,(2n-3-2r)!}{(n-2-r)!\,(n-3-2r)!\,(r+1)!} \\ -2(n-1-r)(2r+3)a_{r+1}\end{array}\right]x^{n-3-2r}$$

$$= -n(n+1)\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) - [n(n+1)a_0]x^{n-1}$$
$$-\sum_{r=0}^{\frac{n-3}{2}}\left[\begin{array}{c}(n^2-3n-2rn-2rn+6r+4r^2-2n+6+4r) + \\ (6n+4rn-6-4r-6r-4r^2)\end{array}\right]a_{r+1}x^{n-3-2r}$$

$$= -n(n+1)\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) - [n(n+1)a_0]x^{n-1} - \sum_{r=0}^{\frac{n-3}{2}}[n(n+1)]a_{r+1}x^{n-3-2r}$$

$$= -n(n+1)\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) - n(n+1)\left[a_0x^{n-1} + \sum_{r=0}^{\frac{n-3}{2}}a_{r+1}x^{n-3-2r}\right]$$

$$= -n(n+1)\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) - n(n+1)\left[a_0x^{n-1} + \sum_{r=1}^{\frac{n-1}{2}} a_r x^{n-1-2r}\right]$$

$$= -n(n+1)\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) - n(n+1)\left[\sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r}\right]$$

$$\therefore (1-x^2)y'' - 2xy' = -n(n+1)\left[\left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r}\right]$$

Recall that,

$$y = L_n{}^{(2)}(x) = \left[L_n{}^{(1)}(x)\right]tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} a_r x^{n-1-2r} \qquad (14)$$

$$\therefore (1-x^2)y'' - 2xy' = -n(n+1)y$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Hence, it is proven for all positive integer values of n.

Therefore, it has been proven that $L_n{}^{(1)}(x)$ and $L_n{}^{(2)}(x)$ are solutions to the Legendre equation for all non-negative values of n.

So for example, the Legendre equation below:

$$(1-x^2)y'' - 2xy' + 20y = 0 \qquad where\; n = 4$$

Has a general solution,

$$y = C_1\, L_4{}^{(1)}(x) + C_2\, L_4{}^{(2)}(x)$$

$$y = C_1\,[35x^4 - 30x^2 + 3] + C_2\left[(35x^4 - 30x^2 + 3)tanh^{-1}(x) - 35x^3 + \frac{55}{3}x\right]$$

4.0 Derivation of another solution

Some other solutions can also be gotten from the solution to Legendre equation, for example;

For the equation below:

$$(A - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (15)$$

Where n is a non-negative integer.

$$y = C_1\, L_n{}^{(1)}\left(\frac{x}{\sqrt{A}}\right) + C_2\, L_n{}^{(2)}\left(\frac{x}{\sqrt{A}}\right) \qquad (16)$$

5.0 Proposed Ideal Solution to the Laplace Equation for a Sphere with a Potential Independent of $\emptyset$ (Azimuthally Invariant)

The Laplace equation can be expressed as,

$$\nabla^2 u = 0 \qquad (17)$$

Using spherical coordinates in Eq. (17), we have

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2 u}{\partial \emptyset^2}\right] = 0 \qquad (18)$$

One of the methods to solve Eq. (18) is the separation of the variables. We suppose that the potential function is followed by [5],

$$u = R(r).P(\theta).Q(\emptyset) \qquad (19)$$

where r, θ and ∅ are radius, the angle between a vector and the z-axis and the angle of vector projection onto xy plane with the positive x-axis, respectively. Substituting Eq. (19) into Eq. (18) and by using the direction symmetry condition as a boundary condition, Eq. (18) is transformed into three ordinary differential equations in which direction the solution is symmetry.

$$\frac{d^2Q}{d\emptyset^2} = -m^2Q \qquad (20)$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \qquad (21)$$

$$\frac{1}{sin\theta}.\frac{d}{d\theta}\left(sin\theta\frac{dP}{d\theta}\right) + n(n+1)P = 0 \qquad (22)$$

where Q(∅) is constant and m^2 = 0. Also, m and n are parameters for solving differential equations in spherical coordinates. Using a new variable x = cos θ, Eq. (22) is written as follows [4],

$$sin^2\theta\frac{d^2P}{dx^2} - 2\cos\theta\frac{dP}{dx} + n(n+1)P = 0 \qquad (23)$$

Therefore,

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + n(n+1)P = 0 \qquad (24)$$

The general solutions for Eq. (23) and (24) are as follows

$$R = a_n\, r^n + b_n\, r^{-(n+1)} \qquad (25)$$

$$P(x) = A_n\, {L_n}^{(1)}(x) + B_n\, {L_n}^{(2)}(x) \qquad (26)$$

Where,

a_n, b_n, A_n and B_n are constants which will be determined once we apply specific boundary equations.

$$L_n^{(1)}(x) = \sum_{r=0}^{\frac{n}{2}} \frac{(-1)^r.(2n-1-2r)!}{(n-1-r)!\,(n-2r)!\,r!} x^{n-2r} \qquad (6)$$

$$L_n^{(2)}(x) = \left[L_n^{(1)}(x)\right] tanh^{-1}(x) + \sum_{r=0}^{\frac{n-1}{2}} \frac{(-1)^{r+1}}{2} \left[\sum_{k=0}^{r} \frac{(2n-2k)!\,(2k)!}{((n-k)!\,k!)^2}\right] \frac{(n-1-r)!\,r!}{(2r+1)!\,(n-1-2r)!} x^{n-1-2r} \qquad (12)$$

Note that $x = \cos\theta$, $Q(\emptyset)$ is constant and $u = Q(\emptyset).R(r).P(\theta)$

Putting $Q(\emptyset) = 1$, therefore the general solution can be taken to be the summation of successive solutions as described below,

$$u = \sum_{n=0}^{\infty} \left[a_n r^n + b_n r^{-(n+1)}\right]\left[A_n\, L_n^{(1)}(\cos\theta) + B_n\, L_n^{(2)}(\cos\theta)\right] \qquad (27)$$

$$u \cong \sum_{n=0}^{t} \left[a_n r^n + b_n r^{-(n+1)}\right]\left[A_n\, L_n^{(1)}(\cos\theta) + B_n\, L_n^{(2)}(\cos\theta)\right] \qquad (28)$$

Where t is sufficiently large, and for every value of n, there are corresponding values of a_n, b_n, A_n, B_n (constants which will be determined once we apply specific boundary equations) and also $L_n^{(1)}(\cos\theta)$, $L_n^{(2)}(\cos\theta)$ (fundamental solutions to the Legendre equation). The above solution unlike other contemporary solutions regards the second fundamental solution of the Legendre equation.

Solving the ordinary and partial differential equations is one of the advantages of this method. It helps to obtain the ideal solutions of ordinary and partial differential equations related to the Legendre equation.

6.0 Conclusion

As Dawkins noted [6], just because we know that a solution to a differential equation exists does not mean that we will be able to find it. This saying has proven to be true as many have resorted to the use of non-analytic methods in the solution of variable coefficients differential equations, however, there are some equations which may not seem to have a simple solution, but they do.

This research work has introduced and proven the exact solution to the Legendre equation which was previously solved by methods yielding approximate solutions; showing the advantages of the exact solution over all other contemporary solutions. Below are the summarized advantages:

- They are most accurate when comparing with the already known methods
- They permit the study of the analytic properties
- They are differentiable unlike the finite difference solution
- They are 100% accurate for the region $-1 < x < 1$
- They reduce computational cost of the calculation.

The approximate solution of the Laplace equation is ideal and further work on it will help to bring about an exact solution. The new solutions are flexible enough to allow a practically infinite list of examples, but at this point the construction of further examples might sound an exercise lacking interest. It is hoped however that new solutions that have been proposed will stimulate further work in this direction, and most importantly, the advancement in the fields of electronics, fluid mechanics, heat and mass transfer, electromagnetism etc.

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Appendix

Title: Obasi-Legendre.py (Legendre equation solver on python)

```
import math
from fractions import Fraction
print("""\nTo solve the differential equation:
    \n\t\t\t\t\t\t(1-x^2)y" - 2xy\' + n(n+1)y = 0 \nwhere,
    n must be a non-negative integer""")
"""
Input the values of n for the solution of the equation. Verify the inputs if it forms the
intended equation.
"""
intended='no'
while intended=='no':
  n1=''
  while n1.isnumeric()==False:
    try:
      nn=int(input('Enter the value of n : '))
      if nn<0:
        print('\t\t\t\t\tError! input must be a non-negative integer (e.g. 0,1,2,3...)')
        n1=''
      else:
        n=nn
        break
    except:
      print('\t\t\t\t\tError! input must be a non-negative integer (e.g. 0,1,2,3...)')
      n1=''
# to display the inputed equation for verification
  n2=str(n*(n+1))
  print('\nInputed equation:\n\t\t\t\t(1-x^2)y" - 2xy\' + '+n2+'y = 0')
  if n%2==1:
    N=(n-1)/2
  else:
    N=n/2
  N=int(N)
  if n%2==1:
```

```
    N1=(n-1)/2
  else:
    N1=(n-2)/2
  N1=int(N1)
  st=''
  while st=='':
    Q1=input('Is the above equation the intended equation you had in mind to be solved? (yes/no): ')
    if Q1.lower()=='yes':
      intended='yes'
      st=0
      def L1(n):
        s=''
        if n==0:
          s='1'
        elif n==1:
          s='x'
        else:
          for r in range(N+1):
            s1=((-1)**r*math.factorial(2*n-1-2*r))/(math.factorial(n-1-r)*math.factorial(n-
2*r)*math.factorial(r))
            tt=n-2*r
            if tt==0:
              s1='('+str(Fraction(s1))+')'
            elif tt==1:
              s1='('+str(Fraction(s1))+')x'
            else:
              s1='('+str(Fraction(s1))+')x^'+str(tt)
            s+=' + '+s1
          s=s[3:]
        return s
      def L2(n):
        s='('+L1(n)+').atanh(x)'
        if n==0:
          s=s
        else:
          for r in range(N1+1):
```

```
                s00=0
                for k in range(r+1):
                    s00+=Fraction(math.factorial(2*n-2*k)*math.factorial(2*k))/Fraction((math.factorial(n-k)*math.factorial(k))**2)
                A=Fraction((-1)**(r+1)*math.factorial(n-1-r)*math.factorial(r))/Fraction(2*math.factorial(2*r+1)*math.factorial(n-1-2*r))
                s0=s00*A
                tt=n-1-2*r
                if tt==0:
                    s1='('+str(s0)+')'
                elif tt==1:
                    s1='('+str(s0)+')x'
                else:
                    s1='('+str(s0)+')x^'+str(tt)
                s+=' + '+s1
            return s
        GS='\nThe general solution:\ny = C1['+L1(n)+'] + C2['+L2(n)+']'
        print('L1_'+str(n)+'(x) = '+L1(n))
        print('\nL2_'+str(n)+'(x) = '+L2(n))
        print(GS)
    elif Q1.lower()=='no':
        st=0
        print('\nThen you will have to input again.')
        intended='no'
    else:
        print('\t\t\t\t\tError! you are only allowed to select either yes or no.')
        st=''
```