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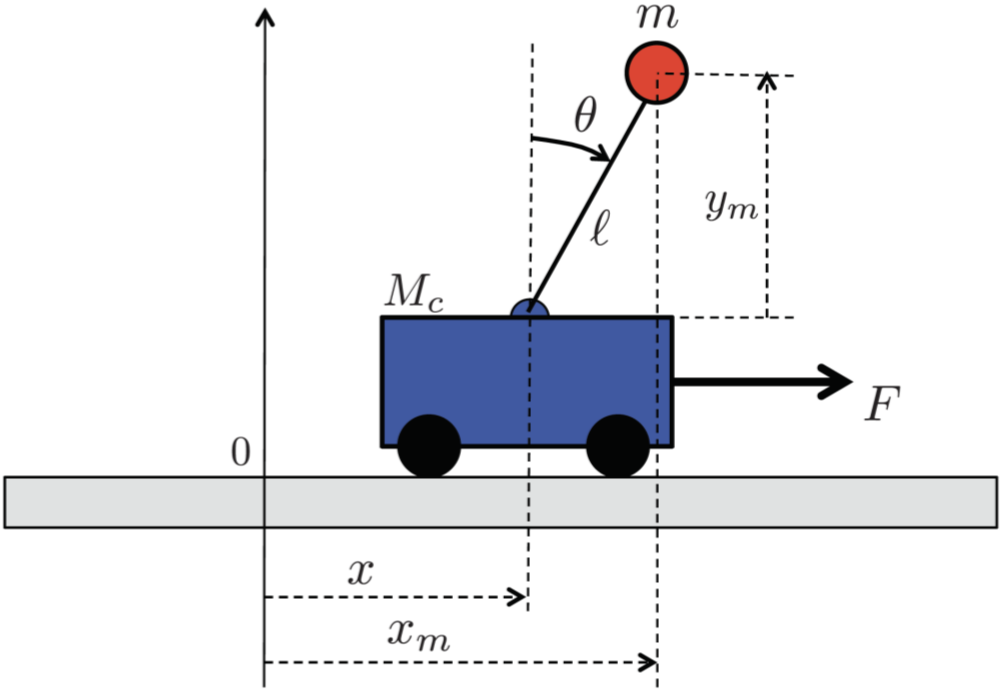
Modern Control EGH445

State-feedback control of the cart-pendulum system



Abstract

This report stages the process taken to design state-feedback controllers for the cart-pendulum system and simulate the closed loop system using Matlab-Simulink. Using the inverted pendulum's equations of motion, a nonlinear and a linearized state-space matrix equation will be computed. The matrices will then be used to check if the systems are controllable and then compute a controller gain for specified eigenvalues. A Simulink model will be built for each system using the computed matrices and gains. The simulation data, exported to Matlab, will then be used to animate and plot the states of both nonlinear and linearized systems. The plots are then compared to form an analysis.

Introduction

The cart-pendulum system is a benchmark that has been widely used to study control system designs. We will use it to design both linearized and nonlinear controllers based on state feedback. The controllers goal is to passage the pendulum from the initial position $x_0 = \begin{bmatrix} 0.2 & \frac{20\pi}{180} & 0 & 0 \end{bmatrix}^T$ to the upright equilibrium position located at $x = 0$.

The cart-pendulum system consists of a pendulum of mass m and length ℓ attached to a cart of mass M_c. The cart moves on the horizontal direction and is actuated by the input force F. The equations of motion of the system are as follows:

$$(Mc + m)\,\ddot{q}_1 + m\,\ell\,cos(q_2)\,\ddot{q}_2 - m\,\ell\,sin(q_2)\,\dot{q}_2^2 = F \tag{1}$$

$$\mathrm{m}\,\ell\,\cos(\mathrm{q}_2)\,\ddot{\mathrm{q}}_1 + \mathrm{m}\,\ell^2\,\ddot{\mathrm{q}}_2 - \mathrm{m}\,\mathrm{g}\,\ell\,\sin(\mathrm{q}_2) = 0 \tag{2}$$

where q_1 and $q_2$are the position of the cart and the angle of the pendulum respectively, and g is the gravitational constant. We consider the values of the model parameters given in Table 1.

Table 1 Model Parameters

Parameter	Value
m	$0.15\ Kg$
M_c	$0.4\ Kg$
ℓ	$0.2\ m$
g	$9.81\ m/s^2$

The mathematical model for the cart-pendulum

Non-Linear System

Using equation1 & 2, a non-linear state space module can be written

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \frac{x_4^2 \ell m \sin(x_2) - g m \sin(x_2)\cos(x_2) + F}{M_c + m - m\cos^2(x_2)} \\ \frac{-\ell m \sin(x_2)\cos(x_2) x_4^2 + g(M_c+m)\sin(x_2) - \cos(x_2) F}{\ell[M_c + m - m\cos^2(x_2)]} \end{bmatrix}, \tag{3}$$

where the states are:

- x1: the position of the cart,
- x2: the angle of the pendulum,
- x3: the velocity of the cart,
- x4: the angular velocity of the pendulum.

Linearized System

Although the system has 2 equilibrium points $\bar{x}_a$ & $\bar{x}_b$,

$$\bar{x}_a = \begin{bmatrix} \bar{x}_{1a} \\ \bar{x}_{2a} \\ \bar{x}_{3a} \\ \bar{x}_{4a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \qquad \bar{x}_b = \begin{bmatrix} \bar{x}_{1b} \\ \bar{x}_{2b} \\ \bar{x}_{3b} \\ \bar{x}_{4b} \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \\ 0 \\ 0 \end{bmatrix} . \tag{4}$$

stabilisation is only required at the upright position $\bar{x}_a$ only.

To linearized module of the cart-pendulum about $\bar{x}_a$, we define the deviation variable $\tilde{x}_a \triangleq x_a - \bar{x}_a$. Then $\dot{\tilde{x}}_a = A_a\tilde{x}_a + B_aF$ *and* $y = C_a\tilde{x}_a + D_aF$ where A_a and B_a are derived using the jacobians $\left(\frac{\partial f}{\partial x}\right)^T\Big|_{x=\bar{x},F=\bar{F}}$ and $\left(\frac{\partial f}{\partial F}\right)^T\Big|_{x=\bar{x},F=\bar{F}}$ respectively to get :

$$A_a = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m\,g}{M_c} & 0 & 0 \\ 0 & \frac{g(M_c+m)}{\ell M_c} & 0 & 0 \end{bmatrix} ; \quad B_a = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_c} \\ -\frac{1}{\ell M_c} \end{bmatrix} ; \quad C_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad D_a = 0_{4\times 1}. \tag{5}$$

where C_a and D_a are calculated to make $y = \tilde{x}_a$.

The design of the state-feedback controller

The state-feedback controller will be in the form $u = -Kx$, where the input u is the control force F, but before calculating K, we should determine if the linearized module is controllable by computing the controllability matrix C_{AB} and checking its rank. C_{AB} can be computed using the command **ctrb** on Matlab, and the rank can be computed using the command **rank**.

```
C_AaBa=ctrb(Aa,Ba);
Rank =rank(C_AaBa);
if Rank == min(size(C_AaBa))
    disp('the linearised model is controllable')
end
```

The system is controllable.

Next step is computing controller $u = -K_a\tilde{x}_a$ such that the eigenvalues of the (linearized) closed loop are $\lambda 1 = -3$, $\lambda 2 = -4$, $\lambda 3 = -5$ and $\lambda 4 = -6$. This is done on Matlab using the command **place**.

```
Pa=[-3 -4 -5 -6];
Ka=place(Aa,Ba,Pa);
```

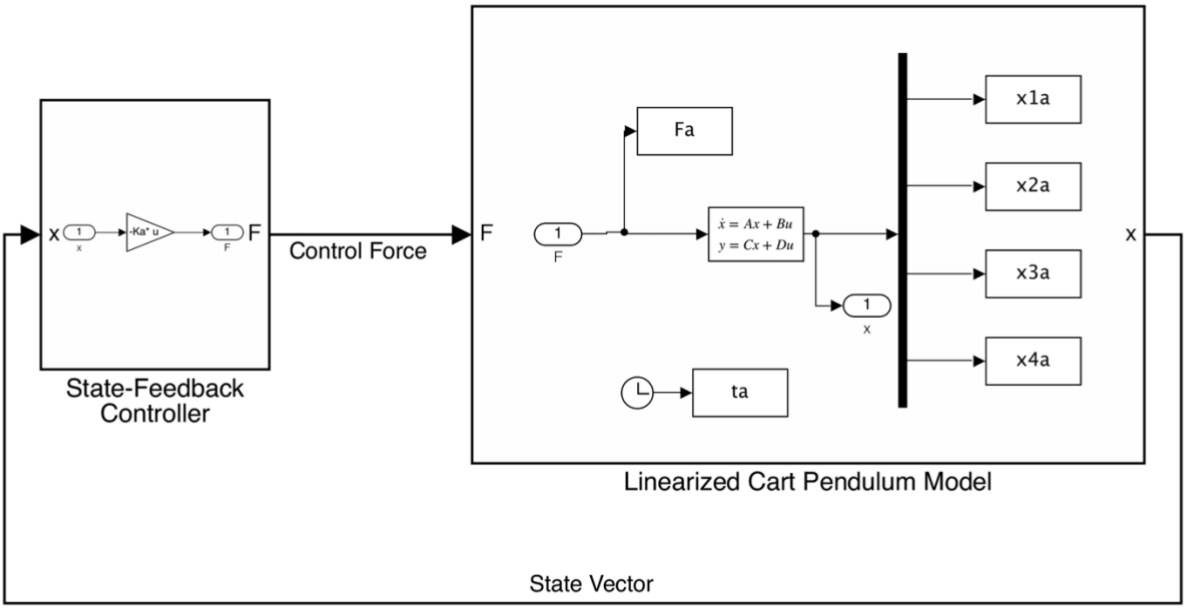
Where K_a is $[-2.9358 \quad -15.5027 \quad -2.7890 \quad -1.9978]$.

For the nonlinear system, the gain $K_{SF} = K_a$.

The Matlab-Simulink implementation of the control system

Linearized System Control Model

Using the matrices of the linearized model from equation 5 and the gain K_a, the following Simulink model was created:

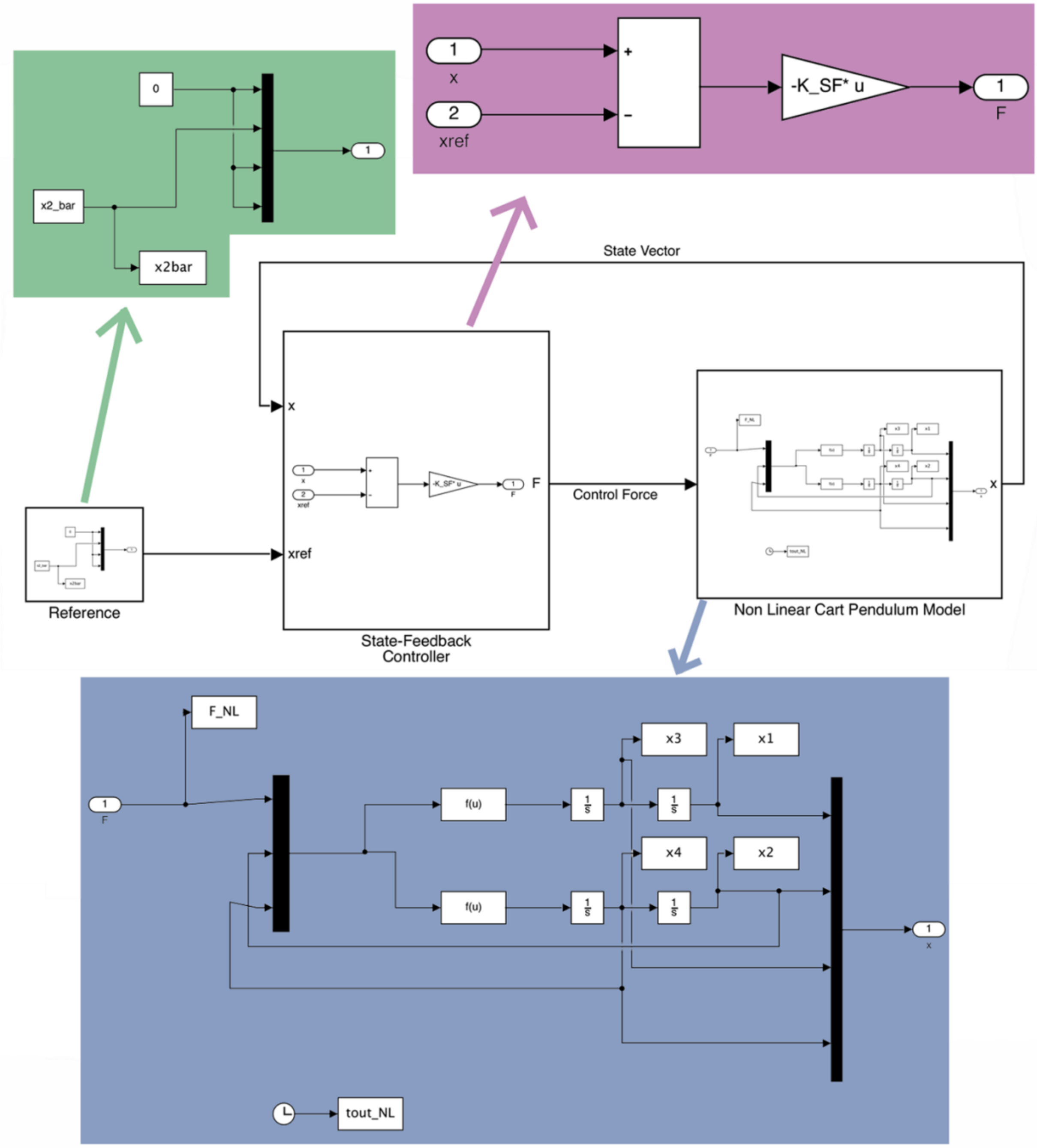


where the initial conditions in the state-space block are $x_0 = \begin{bmatrix} 0.2 & \frac{20\pi}{180} & 0 & 0 \end{bmatrix}^T$.

The simulation ran using a the fixed-step solver **ode4** with a $step\ time = 0.02$ for 5 seconds.

Non-Linear System Control Model

Using the matrix of the non-linear model from equation 3 and the gain K_{SF}, the following Simulink model was created:



where the initial conditions in the integrator blocks are $x_0 = \begin{bmatrix} 0.2 & \frac{20\pi}{180} & 0 & 0 \end{bmatrix}^T$.

The simulation ran using a the fixed-step solver **ode4** with a $step\ time = 0.02$ for 5 seconds.

The states, forces and time from both linearized and nonlinear models are then exported to Matlab using the block to Workspace as arrays. These arrays are then used to plot the following figure:

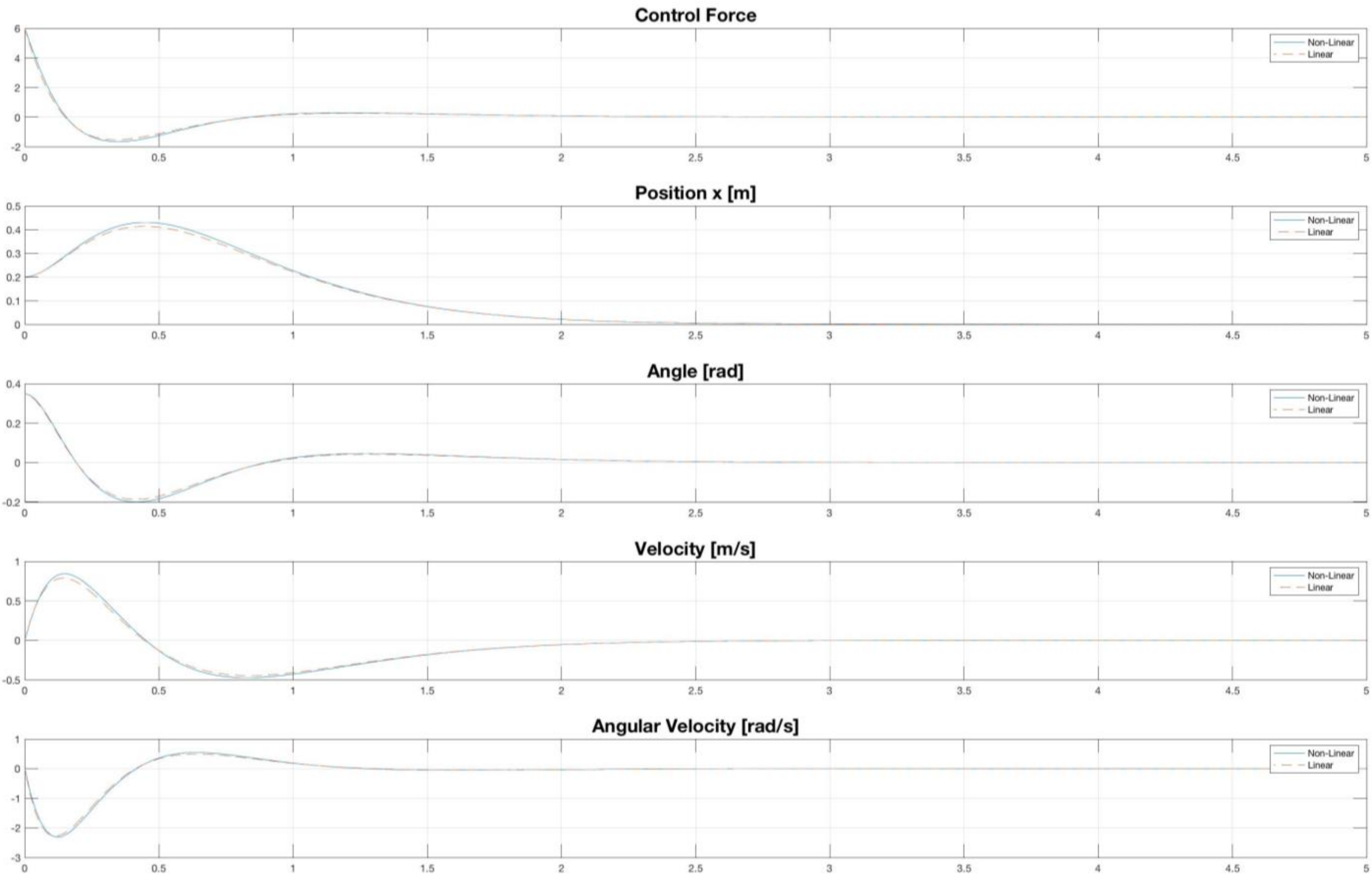


Figure 1 Time histories of the states and input.

Both models reached the equilibrium point at approximately similar times, yet the linearized model used slightly less Force to reach the desired position.

Conclusion

We used the inverted pendulum's equations of motion to compute a nonlinear and a linearized state-space matrix equation. The matrices were then used to check if the systems are controllable, which they were. A controller gain with specified eigenvalues was then computed. A Simulink model was built for each system using the computed matrices and gains. The simulation datum, exported to Matlab, were then used to animate and plot the states of both nonlinear and linearized systems. The plots showed approximately similar behavior where both models succeeded in reaching the goal point.