Convergence Study for Tetrahedrons with Composite, Quadratic, and Linear Shape Functions in Bending

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January 29, 2019

Abstract

The convergence of tetrahedral elements with 2^{nd} order shape functions are compared to those with composite and linear shape functions in a beam bending problem. A linear elastic material model was used and no time components were considered. The geometry was a beam with a square cross-section and aspect ratio of 100. The beam was cantilevered on one end and received a shearing traction at the other. The tip deflection of the finite element solutions were compared to that from Euler-Bernoulli beam theory. Tetrahedrons with 2^{nd} order and composite shape functions under-predicted the theoretical tip displacement by less than 1% with 2 elements through the thickness of the beam; elements with linear shape functions under-predicted the theoretical tip displacement by 84% with 8 elements through the thickness. Tetrahedrons with 2^{nd} order shape functions under-predicted the peak bending stress by 4% in tension and 7% in compression with 8 elements through the thickness. Tetrahedrons with composite shape functions under-predicted the peak bending stress by 10% in tension and 0.4% in compression with 8 elements through the thickness. Performance of iterative solvers for both the quadratic and composite element types are discussed.

1 Introduction

The performance of tetrahedron elements with 2^{nd} order shape functions was compared to that of elements with composite (discussed in [1]) and linear shape functions in a beam-bending problem. The mesh was refined such that the aspect ratio of the elements stayed at 10 as the number of elements through the thickness was increased. The theoretical deflection from Euler-Bernoulli beam theory was used to compared the finite element results.

2 Methods

The problem geometry was a beam with a square cross-section. The cross-section was 0.01 m $\times$ 0.01 m; the beam was 1 m long in the Y direction. The other edges of the beam were aligned with the X and Z axes.

A linear elastic material model was used. Young's Modulus was 80,000 Pa; Poisson's ratio was 0.25. No time components were considered and only the static solution was calculated.

The square face with normal in the negative Y direction had all degrees of freedom fixed to act as a cantilever support. The face on the other end of the beam received a traction of 0.01 $\frac{N}{m^2}$ in the Z direction. From Euler-Bernoulli beam theory, the tip deflection is modeled as:

$$\delta = \frac{FL^3}{3EI} \tag{1}$$

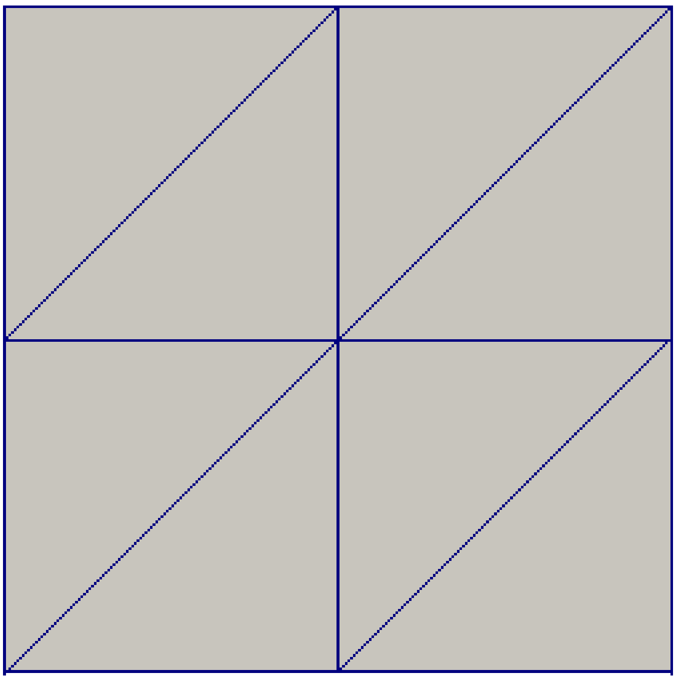
where F is the total force acting at the tip of the beam, L is the length of the beam, E is Young's Modulus for the beam's material, and I is the second moment of the cross-sectional area. Evaluating Equation (1) for the given parameters of the problem, the theoretical tip deflection is 5.0 mm. The peak bending stress, σ_b, was modeled as:

$$\sigma_b = \frac{Mc}{I} \tag{2}$$

where M is the bending moment, c is the maximum distance from the bending axis to a point on the beam. With the parameters of the problem, $\sigma_b = \pm 6$ with "+" taken as tension.

A mesh was constructed to approximate the beam geometry and used only tetrahedral elements. The convergence of tetrahedrons with 2$^{\text{nd}}$ order (quadratic tet10 elements), composite, and linear shape functions were separately measured. An element aspect ratio of 10 was used for all meshes and the number of elements through the thickness of the beam was increased. Figure 1 shows an example mesh with 2 elements through the thickness. Additional images of the other meshes are shown in Appendix B All systems were solved with iterative solvers.

(a) Side View.



(b) End View.

Figure 1: Mesh for beam of size $0.01 \times 0.01 \times 1$ with $2 \times 2 \times 20$ elements.

3 Results

The tip deflections from the finite element solutions are shown with respect to the number of degrees of freedom in Figure 2; a focused view of just the composite and quadratic elements is shown in Figure 3. Four meshes were constructed with 1, 2, 4, and 8 elements through the thickness.

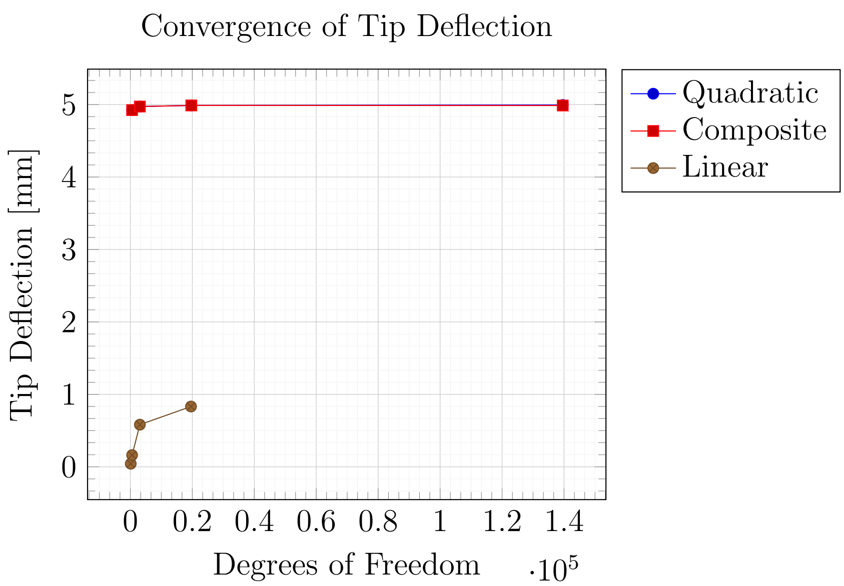


Figure 2: Tip deflection for beam geometry with composite, quadratic, and linear elements. The number of elements through the thickness was 1, 2, 4, and 8. Analytical solution is 5.0 mm.

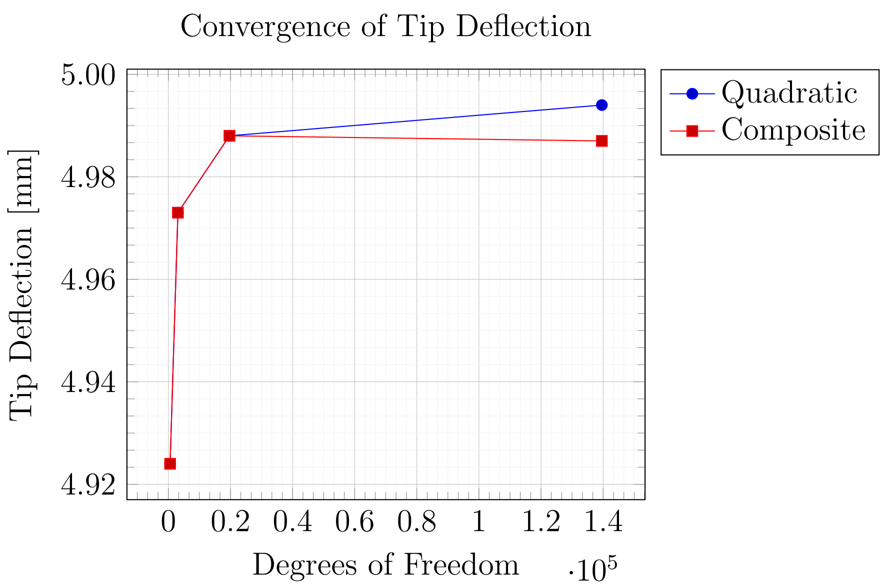


Figure 3: Tip deflection for beam geometry with composite and quadratic elements. The number of elements through the thickness was 1, 2, 4, and 8. Analytical solution is 0.1 m.

The quadratic and composite tet10 elements underestimated the tip deflection by

0.1% of the analytical solution with 4 elements through the thickness. The linear elements underestimated the analytical solution by more than 83.5% in all cases. The peak compression and bending stresses for the composite and quadratic elements are shown in Figure 4.

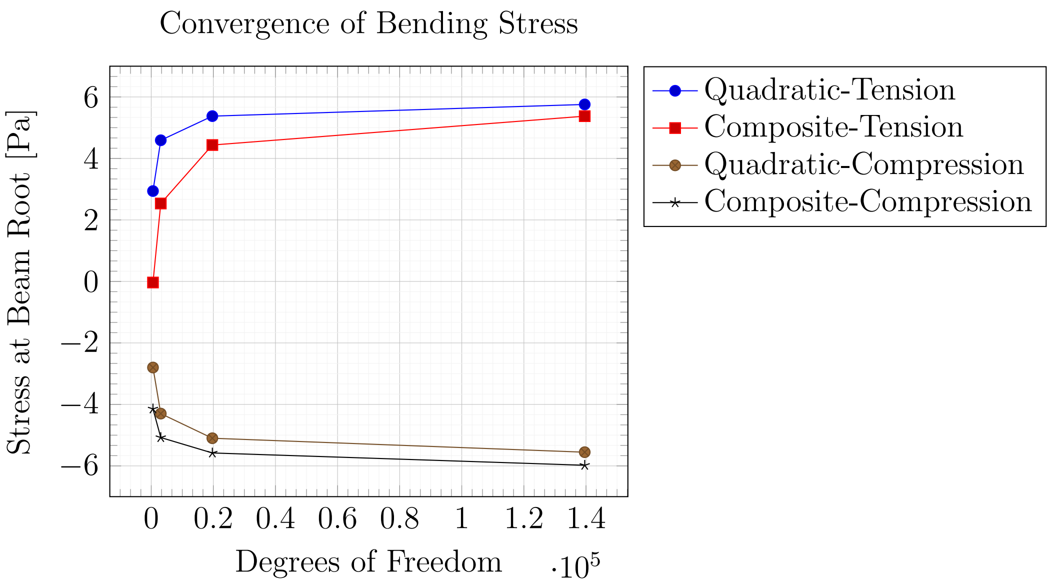


Figure 4: Bending stress for beam with composite and quadratic elements. The number of elements through the thickness was 1, 2, 4, and 8. Analytical solution is ± 6 Pa.

The quadratic elements underestimated the peak bending stress by 4% in tension and 7% in compression with 8 elements through the thickness. The composite elements underestimated the peak bending stress by 10% in tension and 0.4% in compression with 8 elements through the thickness. A `Block GMRES` method was used to solve the linear systems; a relative tolerance of 10^{-6} was used for all solutions. More details of the solver and preconditioner are shown in the sample input file in Appendix A. The number of steps to solve each system is shown in Figure 5.

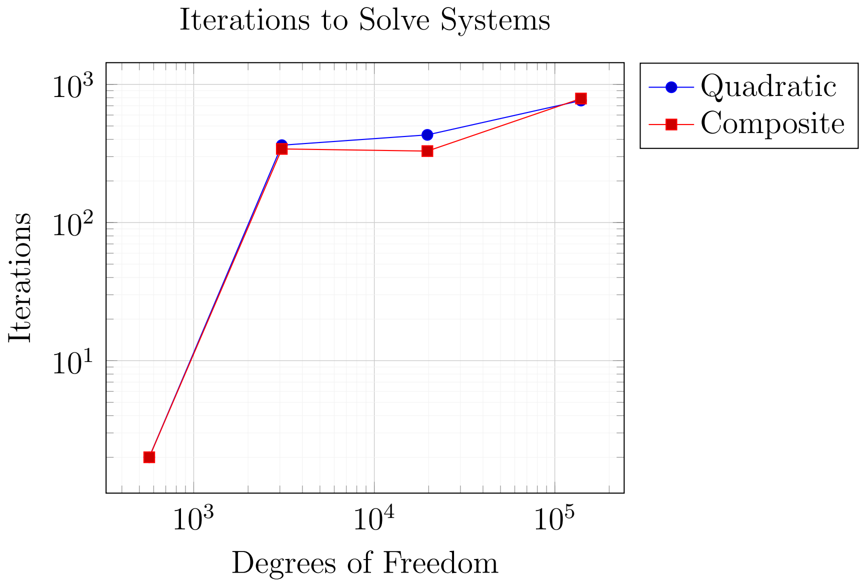


Figure 5: Number of linear solver iterations to achieve the final solution. The number of elements through the thickness was 1, 2, 4, and 8.

The number of linear solver iterations required to solve the system with elements of either 2^{nd} order or composite shape functions were approximately equal.

References

[1] J. T. Ostien, J. W. Foulk, A. Mota, M. G. Veilleux. A 10-node Composite tetrahedral finite element for solid mechanics. International Journal For Numerical Methods in Engineering. 2016:107:1145-1170.

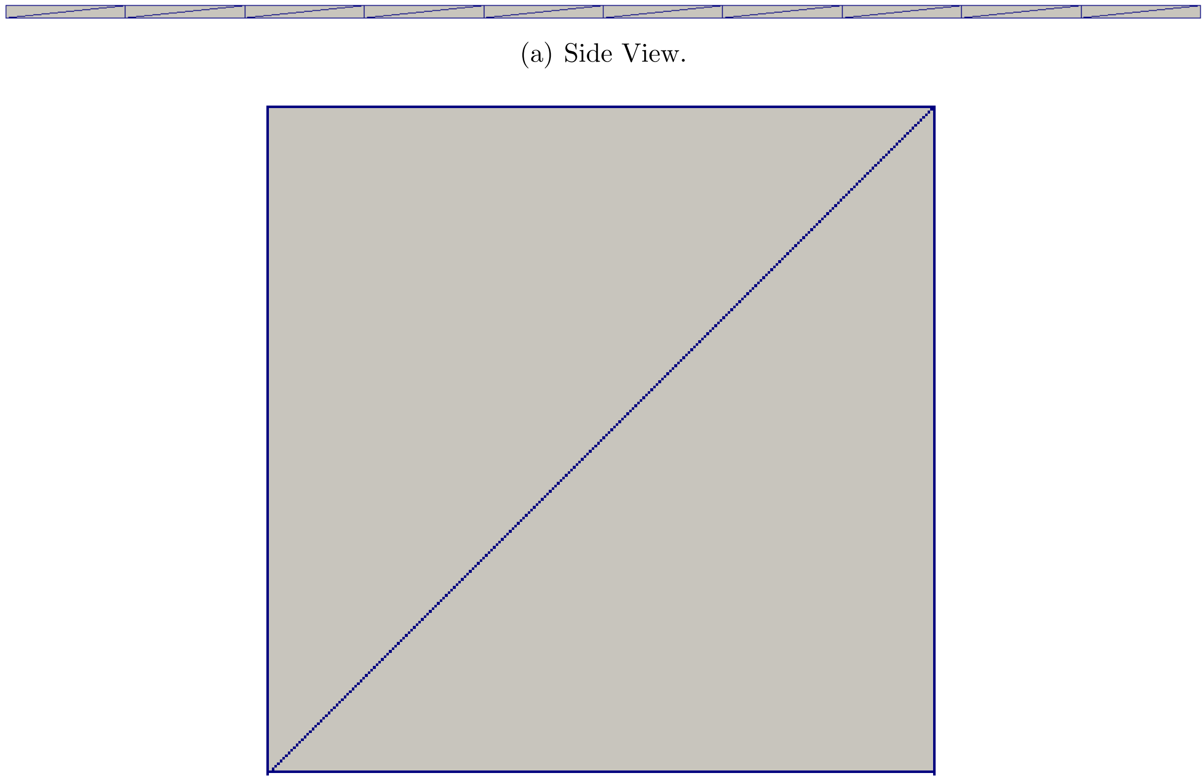
A Sample Input File

```yaml
%YAML 1.1
---
LCM:
  Debug Output:
    Write Jacobian to MatrixMarket: 0
    Compute Jacobian Condition Number: 0
    Write Residual to MatrixMarket: 0
    Write Solution to MatrixMarket: false
  Problem:
    Name: Mechanics 3D
    Solution Method: Steady
    Dirichlet BCs:
      SDBC on NS nsYmin for DOF X: 0.00000000e+00
      SDBC on NS nsYmin for DOF Y: 0.00000000e+00
      SDBC on NS nsYmin for DOF Z: 0.00000000e+00
    Neumann BCs:
      'NBC on SS ymax for DOF all set (t_x, t_y, t_z)': [0.0, 0.0, -0.01]
    MaterialDB Filename: 'material.yaml'
    Parameters:
      Number: 4
      Parameter 0: SDBC on NS nsYmin for DOF X
      Parameter 1: SDBC on NS nsYmin for DOF Y
      Parameter 2: SDBC on NS nsYmin for DOF Z
      Parameter 3: 'NBC on SS ymax for DOF all set (t_x, t_y, t_z)[0]'
    Response Functions:
      Number: 1
      Response 0: Solution Average
  Discretization:
    Method: PUMI
    PUMI Input File Name: test.smb
    Mesh Model Input File Name: test.dmg
    PUMI Output File Name: results.vtk
    Model Associations File Name: base_assoc.txt
  Piro:
    LOCA:
      Bifurcation: { }
      Constraints: { }
      Predictor:
        First Step Predictor: { }
        Last Step Predictor: { }
      Step Size: { }
      Stepper:
        Eigensolver: { }
    NOX:
```

```
    Direction:
      Method: Newton
      Newton:
        Forcing Term Method: Constant
        Rescue Bad Newton Solve: true
        Stratimikos Linear Solver:
          NOX Stratimikos Options: { }
          Stratimikos:
            Linear Solver Type: Belos
            Linear Solver Types:
              Belos:
                VerboseObject:
                  Verbosity Level: high
                Solver Type: Block GMRES
                Solver Types:
                  Block GMRES:
                    Convergence Tolerance: 1.00000000e-6
                    Output Frequency: 10
                    Output Style: 1
                    Verbosity: 33
                    Maximum Iterations: 500
                    Block Size: 1
                    Num Blocks: 500
                    Flexible Gmres: false
            Preconditioner Type: MueLu
            Preconditioner Types:
              MueLu:
                multigrid algorithm: sa
                cycle type: V
                max levels: 4
                'repartition:␣enable': true
                'repartition:␣min␣rows␣per␣proc': 1000
                'smoother:␣type': CHEBYSHEV
                'smoother:␣params':
                  'chebyshev:␣degree': 3
                  'chebyshev:␣ratio␣eigenvalue': 30.0
                'smoother:␣pre␣or␣post': both
                'coarse:␣max␣size': 1500
                number of equations: 3
                'coarse:␣type': Superlu
              Ifpack2:
                Overlap: 2
                Prec Type: ILUT
                Ifpack2 Settings:
                  'fact:␣drop␣tolerance': 0.00000000e+00
                  'fact:␣ilut␣level-of-fill': 1.00000000
                  'fact:␣level-of-fill': 1
    Line Search:
```

```
        Full Step:
          Full Step: 1.00000000
        Method: Backtrack
      Status Tests:
        Test Type: Combo
        Combo Type: OR
        Number of Tests: 2
        Test 0:
          Test Type: NormF
          Norm Type: Two Norm
          Tolerance: 1.0e-08
        Test 1:
          Test Type: MaxIters
          Maximum Iterations: 20
      Nonlinear Solver: Line Search Based
      Printing:
        Output Information:
          Error: true
          Warning: true
          Outer Iteration: true
          Parameters: true
          Details: true
          Linear Solver Details: true
          Stepper Iteration: true
          Stepper Details: true
          Stepper Parameters: true
        Output Precision: 3
        Output Processor: 0
      Solver Options:
        Status Test Check Type: Minimal
...
```

B Mesh Images

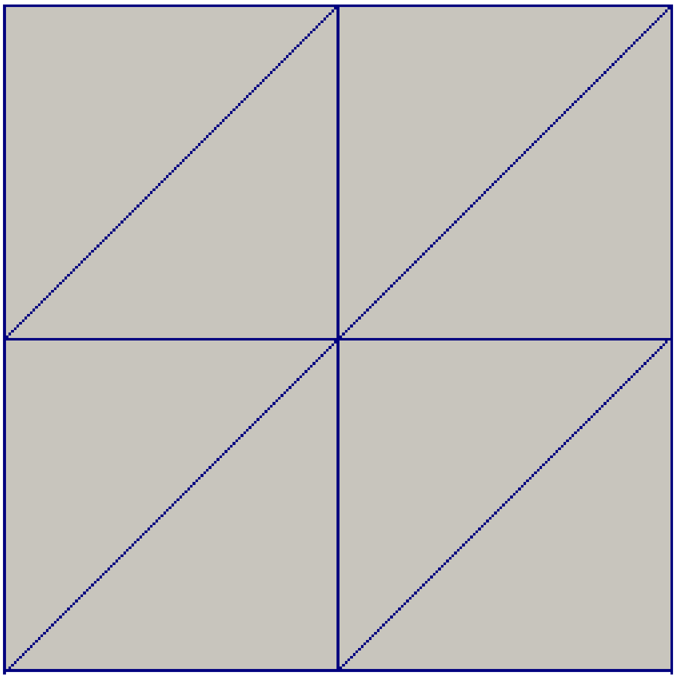


(a) Side View.

(b) End View.

Figure 6: Mesh for beam of size $0.01 \times 0.01 \times 1$ with $1 \times 1 \times 10$ elements.

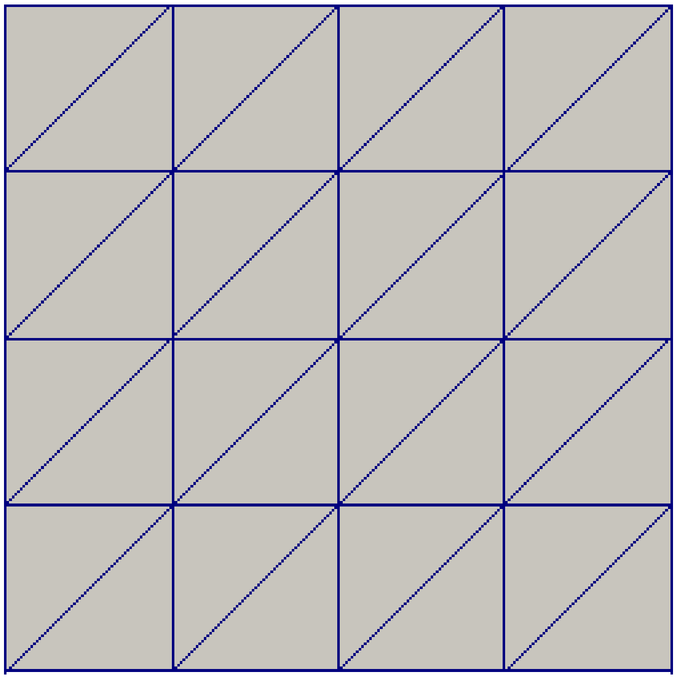
(a) Side View.



(b) End View.

Figure 7: Mesh for beam of size $0.01 \times 0.01 \times 1$ with $2 \times 2 \times 20$ elements. Duplicated from Figure 1.

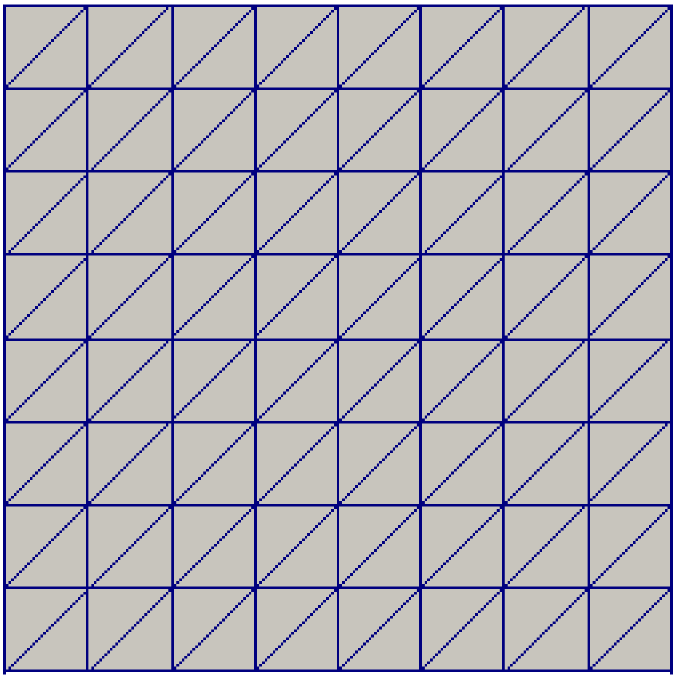
(a) Side View.



(b) End View.

Figure 8: Mesh for beam of size $0.01 \times 0.01 \times 1$ with $4 \times 4 \times 40$ elements.

(a) Side View.



(b) End View.

Figure 9: Mesh for beam of size $0.01 \times 0.01 \times 1$ with $8 \times 8 \times 80$ elements.