Homework 2

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Problem 1

Using the method of Lagrange multipliers, solve the following constrained optimization problem:

$$\max_{(x,y)} xy$$
$$\text{s.t. } 2x + y = 1$$

Indicate the optimal values of x and y as well as the optimal value of the objective.

Answer to problem 1

Lagrangian function is given by:

$$L(x, y, \lambda) = xy + \lambda(2x + y - 1)$$

Critical points of the Lagrangian function by taking the partial derivatives with respect to x, y, and λ and setting them equal to zero:

$$\frac{\partial L}{\partial x} = y + 2\lambda = 0$$
$$\frac{\partial L}{\partial y} = x + \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 2x + y - 1 = 0$$

Ssolving equations (1) and (2) for x and y:
From (1) $y = -2\lambda$ (Equation 4)
From (2)$x = -\lambda$ (Equation 5)
Substitute Equations (4) and (5) into Equation (3):

$$2x + y - 1 = 2(-\lambda) + (-2\lambda) - 1 = -3\lambda - 1 = 0$$

Solving for λ:

$$-3\lambda - 1 = 0$$
$$-3\lambda = 1$$
$$\lambda = -\frac{1}{3}$$

Thats the value of λ, now using Equations (4) and (5) to find x and y:
Equation (4): $y = -2\lambda = -2 \cdot \left(-\frac{1}{3}\right) = \frac{2}{3}$
Equation (5): $x = -\lambda = -\left(-\frac{1}{3}\right) = \frac{1}{3}$
So, the optimal values are: $x = \frac{1}{3}$ $y = \frac{2}{3}$
Substitute these values into the objective function:

$$xy = \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right) = \frac{2}{9}$$

The optimal value of the objective function is $\frac{2}{9}$, and the optimal values of x and y are $x = \frac{1}{3}$ and $y = \frac{2}{3}$.

Problem 2

Using the same method, solve the following constrained optimization problem:

$$\begin{aligned} \min_{(x,y)} \quad & x^2 + 3y^2 \\ \text{subject to} \quad & x - y \geq 1 \end{aligned}$$

Indicate the optimal values of x and y as well as the optimal value of the objective.

Answer to problem 2

The Lagrangian function is given by:

$$L(x, y, \lambda) = x^2 + 3y^2 + \lambda(x - y - 1)$$

Find the critical points of the Lagrangian function:

$$\frac{\partial L}{\partial x} = 2x + \lambda = 0$$
$$\frac{\partial L}{\partial y} = 6y - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x - y - 1 = 0$$

Solving equations (1) and (2) for x and y:
From (1): $2x + \lambda = 0$
From (2): $6y - \lambda = 0$
Finding expressions for x and y in terms of λ:

$$x = -\frac{\lambda}{2}$$
$$y = \frac{\lambda}{6}$$

Substitute these expressions into Equation (3):

$$-\frac{\lambda}{2} - \frac{\lambda}{6} - 1 = 0$$

Solve for λ:

$$-\frac{3\lambda}{6} - \frac{\lambda}{6} - 1 = 0$$
$$-\frac{4\lambda}{6} - 1 = 0$$
$$-\frac{2\lambda}{3} - 1 = 0$$
$$-\frac{2\lambda}{3} = 1$$
$$\lambda = -\frac{3}{2}$$

Find their values:
From $x = -\frac{\lambda}{2}$: $x = -\left(-\frac{3}{2}\right)/2 = \frac{3}{4}$
From $y = \frac{\lambda}{6}$: $y = \frac{-\frac{3}{2}}{6} = -\frac{1}{4}$
The optimal values are: $x = \frac{3}{4}$ $y = -\frac{1}{4}$
Find the optimal value of the objective $(x^2 + 3y^2)$

$$x^2 + 3y^2 = \left(\frac{3}{4}\right)^2 + 3\left(-\frac{1}{4}\right)^2 = \frac{9}{16} + \frac{3}{16} = \frac{12}{16} = \frac{3}{4}$$

The optimal value of the objective function is $\frac{3}{4}$, and the optimal values of x and y are $x = \frac{3}{4}$ and $y = -\frac{1}{4}$.