Mathematical methods for graduate students in Cosmology and Gravitation.

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A gentle introduction to basic mathematical methods for graduate students in Cosmology and gravitation.

1. SUMMARY OF LECTURE ONE

- Euler-Lagrange equation

$$\frac{\partial f}{\partial y(x)} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{1}$$

- First integral

$$\frac{\partial}{\partial x^{\nu}}\left[\mathcal{L}\delta^{\nu}{}_{\mu} - \frac{\partial \mathcal{L}}{\partial \Psi_{\nu}}\partial_{\mu}\Psi\right] = 0 \tag{2}$$

- Energy-Momentum tensor

$$T^{\nu}{}_{\mu} \;=\; \mathcal{L}\delta^{\nu}{}_{\mu} - \frac{\partial \mathcal{L}}{\partial \Psi_{\nu}}\partial_{\mu}\Psi \tag{3}$$

- The conservation law

$$\partial_{\nu}T^{\nu}{}_{\mu} = 0 \tag{4}$$

- Noether current: If there exist a transformation $\Psi(x) \to \Psi(x) + \varepsilon\delta\Psi(x)$, such that

$$j^{\alpha} \equiv \frac{\partial \mathcal{L}}{\partial \nabla_{\alpha}\Psi}\delta\Psi \tag{5}$$

which satisfies the EOM then

$$\nabla_{\alpha} j^{\alpha} = \delta\mathcal{L} \tag{6}$$

If a set of transformation leaves the Lagrangian unchanged i.e $\delta\mathcal{L} = 0$, then the Noether current is conserved

$$\nabla_{\alpha} j^{\alpha} = 0 \tag{7}$$

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2. EXERCISE

1. The Lagrangian density for a real scalar with a vanishing potential is given by

$$L = \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi \tag{8}$$

where the associated energy-Momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \Phi \partial^\rho \Phi \tag{9}$$

Show that the divergence vanishes, i.e $\partial^\mu T_{\mu\nu} = 0$ (Hint: using the EoM. on-shell)

2. Given the Lagrangian density of a real scalar field with a potential $V(\Phi)$

$$L = \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) \tag{10}$$

Based on Noether's theorem, derive the energy-momentum tensor. *Hint* 1: The first-integral is given by

$$T^\nu{}_\mu \; = \; \frac{\partial \mathcal{L}}{\partial \partial_\nu \Psi} \partial_\mu \Psi - \mathcal{L} \delta^\nu{}_\mu \tag{11}$$

Compare your result with equation 5.3 of Malik and Wands 2008.

3. Given the action of a free Maxwell theory

$$S = \int \mathcal{L} \mathrm{d}^4 x = -\frac{1}{4} \int \; F_{\mu\nu} F^{\mu\nu} \sqrt{-g} \mathrm{d}^4 x \tag{12}$$

where electromagnetic field strength tensor is given by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{13}$$

Let's define the electric and the magnetic field as

$$E^i = -F^{oi} \, , \qquad \varepsilon^{ijk} B_k = -F^{ij} \tag{14}$$

Using Noether theorem, derive the energy-momentum tensor and show that the energy and momentum density is given by

$$\varepsilon \; = \; \frac{1}{2} \left(E^2 + B^2 \right) \tag{15}$$

$$s \; = \; \mathbf{E} \times \mathbf{B} \tag{16}$$

4. Dimastrogiovannia-Matteo-Fasiellob-Fujita model of inflation(see 1608.04216)

$$S = \int d^4 x \sqrt{-g} \left[\frac{M_{\rm pl}^2}{2} R - \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi \right) - V(\phi) - \frac{1}{2} \left(\partial_\mu \chi \partial^\mu \chi \right) - U(\chi) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{\lambda \chi}{4 f} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \right] \tag{17}$$

Where

$$\tilde{F}^a_{\mu\nu} \; = \; \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma} \tag{18}$$

$$F^a_{\mu\nu} \; = \; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g \varepsilon_{abc} A^b_\mu A^c_\nu \tag{19}$$

and

- χ is a pseudo-scalar field(Axion),
- ϕ inflation,
- $U(\chi)$ axion potential,
- $F^a_{\mu\nu}$ is the field strength of an $SU(2)$ gauge field A^a_μ,
- f is sub-Planckian decay constant.

Derive the the EoM for all the fields and the total energy-momentum tensor. If you get confused ask Matteo(He proposed it). On the EMT see also 1605.01121.

3. SOLUTION TO 3

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{20}$$

From the first integral(Noether theorem)

$$T^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial \partial_{\mu} A_{\lambda}} \partial_{\mu} A_{\lambda} - \delta^{\mu}{}_{\nu} \mathcal{L} \tag{21}$$

Expand the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \left(\partial^{\mu} A^{\nu} \partial_{\mu} A_{\nu} - \partial^{\mu} A^{\nu} \partial_{\nu} A_{\mu} \right) \tag{22}$$

Calculate

$$\frac{\partial \mathcal{L}}{\partial \partial_{\mu} A_{\lambda}} = -F_{\mu\nu} \tag{23}$$

Giving

$$T^{\mu\nu} = -F^{\mu\lambda} \partial_{\nu} A_{\lambda} + \frac{1}{4} \eta_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \tag{24}$$

At this point $T^{\mu\nu}$ is not symmetric in $\mu\nu$, to make it symmetric, we perform a trick. The trick involves adding a total derivative term $\partial_{\lambda} K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in the first two indices. Take

$$\partial_{\lambda} K^{\lambda\mu\nu} = \partial_{\lambda} \left(F^{\mu\lambda} A^{\nu} \right) = \left(\partial_{\lambda} F^{\mu\lambda} \right) A^{\nu} + F^{\mu\lambda} \partial_{\lambda} A^{\nu} \tag{25}$$

where $\partial_{\lambda} F^{\mu\lambda} = 0$ via eom. Adding this to equation (24) leads to

$$T^{\mu\nu} = F^{\mu\lambda} F^{\nu}{}_{\lambda} + \frac{1}{4} \eta_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \tag{26}$$

$T^{\mu\nu}$ is now symmetric in $\mu\nu$.

Compute the energy density

$$\begin{aligned} \varepsilon = T^{00} &= F^{0i} F^{0}_{i} + \frac{1}{4} \left(2F^{0i} F_{0i} + F_{ij} F^{ij} \right) && (27) \\ &= \frac{1}{2} F^{0i} F^{0i} + \frac{1}{4} F^{ij} F_{ij} && (28) \\ &= \frac{1}{2} \left(E^2 + B^2 \right) && (29) \end{aligned}$$

The momentum density

$$s^i = T^{01} = F^{0k} F_{k}{}^{i} = -E^k \varepsilon^{ikl} B_l = (E \times B)^i \tag{30}$$