Mathematical methods for graduate students in Cosmology and Gravitation.

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A gentle introduction to basic mathematical methods for graduate students in Cosmology and gravitation.

1. TODAY: STRUCTURE OF THE LECTURE

- ***2:00 —2:45***

$$\text{Solution to ODE} = \begin{cases} \text{Variationof parameters} & 2:00 \leq t \leq 2:15 \\ \text{Exercises} & 2:15 \leq t \leq 2:30 \\ \text{Intro to Green's Functions} & 2:30 \leq t \leq 2:45 \end{cases}$$

- ***2:45 —3:00*** Break, coffee and catch-up questions.

- ***3:00 —3:50***

$$\text{Green's function contd} = \begin{cases} \text{Exercises.} & 3:00 \leq t \leq 3:15 \\ \text{Green's function in Sturm-Liouville basis} & 3:15 \leq t \leq 3:25 \\ \text{Exact ODE with variable coeff.} & 3:25 \leq t \leq 3:35 \end{cases}$$

- ***3:35 —3:50***

$$\text{Complex analysis} = \begin{cases} \text{Basic Intro.} & 3:35 \leq t \leq 3:45 \\ \text{Exercises.} & 3:45 \leq t \leq 3:50 \end{cases}$$

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2. INHOMOGENEOUS ODE

2.1. homogeneous ODE with constant coefficients: Complementary solution

Starting with a linear homogeneous differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0 \tag{1}$$

where $p_n, p_{n-1}, \ldots, p_1, p_0$ are constant coefficients it can be seen that if $y(x) = e^{rx}$, each term would be a constant multiple of e^{rx}

1. ***Real and distinct Roots***: If the characteristic equation has distinct real number roots $r_1, \ldots, r_n$, then the complementary solution will be of the form

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x} \tag{2}$$

2. ***Real and repeated Roots***:

$$y_c(x) = e^{r_1 x}(c_1 + c_2 x + \cdots + c_k x^{k-1}) \tag{3}$$

   where k is the degree of the polynomial.

3. ***Complex roots*** If the roots are of the form $r_1 = a + bi$ and $r_2 = a - bi$, then the general solution is accordingly

$$y(x) = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x} \tag{4}$$

   may be simplified further using

$$e^{i\theta} = \cos\theta + i \sin\theta \tag{5}$$

   this solution can be rewritten as

$$y(x) \;=\; (c_1 + c_2) e^{ax} \cos bx + i(c_1 - c_2) e^{ax} \sin bx \tag{6}$$

Example

1. Find the general solution to

$$y''' + 4y'' - 7y' - 10y = 0 \tag{7}$$

   solution Solve the characteristic equation

$$\lambda^3 + 4\lambda^2 - 7\lambda - 10 = 0 \tag{8}$$

   with the solution $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = -5$, thus the general solution becomes

$$y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-5t} \tag{9}$$

   Ex Find the general solution to

$$y^{(4)} + 8y'' + 16y = 0 \tag{10}$$

2.2. Particular solution

We will consider few methods of finding a full(complementary + particular) solution to an inhomogeneous ODE

2.2.1. *Method of undetermined coefficients*

It is involves making a guess of the form of the particular solution by looking at the form of the forcing function. For example

$$y'' - 4y' - 12y = 3\exp^{5t} \tag{11}$$

Find the full solution. Solution:(show on the board). Key bits are

- $y_c = c_1 \exp^{-2t} + c_2 \exp^{6t}$
- For the particular solution, try $y_p = A\exp^{5t}$
- sub this in the original ODE and solve for A(ans $A = -3/7$)
- particular solution $y_p = -3\exp^{5t}/7$.

Example Find the full solution to the following ODE

$$y'' - 2y' - 3y = 3t^2 + 4t - 5 \tag{12}$$

Here the forcing function is a Quadratic polynomial and it is smooth. Hence for the y_p should have the following form

$$y_p = At^2 + Bt + C \tag{13}$$

Students to do this.

Use this solution to show a possible break down of the rules and how to fix it

$$y'' - 2y' - 3y = 5\exp^{3t} \tag{14}$$

If you try $y_p = A\exp^{3t}$, the LHS leads to 0, remedy multiply by t: $y_p = At\exp^{3t}$. See table below for further guidelines

$g_i(t)$	$Y_i(t)$
$P_n(t)$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)$
$P_n(t)e^{at}$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)e^{at}$
$P_n(t)e^{at}\cos\mu t$ and/or $P_n(t)e^{at}\sin\mu t$	$t^s(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0)e^{at}\cos\mu t$ + $t^s(B_n t^n + B_{n-1} t^{n-1} + \ldots + B_0)e^{at}\sin\mu t$

Figure 1: *Left panel*:Guideline for the method of undetermined coefficients[cite:online]

2.2.2. Lagrange method or variation of parameters

Method of undetermined coefficients works main for simple forcing functions, for a complicated one, it could become messy. Method of variation of parameters or Lagrange method works for arbitray forcing function. Given a second order ODE of the form

$$y'' + a(x)y' + b(x)y = f(x) \tag{15}$$

The general solution is a sum of the complementary solution and the particular solution

$$y = c_1y_1 + c_2y_2 + \overbrace{v_1(x)y_1(x) + v_2(x)y_2(x)}^{\text{Particular solution}} \tag{16}$$

i.e we try to find a pair of functions v_1 and v_2 such that the particular solution is given by

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{17}$$

We solve for v_1 and v_2 since y_1 and y_2 are solutions of the homogeneous equation; i.e take first derivative of the particular solution

$$y' \;=\; v_1y_1' + v_2y_2' + [v_1'y_1 + v_2'y_2] \tag{18}$$

Set the square bracket to zero for convenient

$$[v_1'y_1 + v_2'y_2] = 0 \tag{19}$$

Take another derivative of it and plug the result in equation (15). Since y_1 and y_2 satisfies equation equation (15), we will be left with

$$v_1'y_1' + v_2'y_2' = f(x) \tag{20}$$

Solve equation (19) and equation (20) simultaneously leads to

$$v_1' = -\frac{y_2(x)f(x)}{W(x)} \qquad v_2' = \frac{y_1(x)f(x)}{W(x)} \tag{21}$$

Finally, you the general solution becomes

$$y = c_1y_1(x) + c_2y_2(x) - y_1\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x \tag{22}$$

The same procedure is easily extended to the n-th order linear differential equation .

1. Show that the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}x}{\mathrm{d}x} + y = \frac{e^x}{1+x^2} \tag{23}$$

   is

$$y = Ae^x + Vxe^x - \frac{1}{2}e^x\ln\left(1+x^2\right) + xe^x\tan^{-1}x \tag{24}$$

3. GREENS FUNCTIONS

Green's function is a solution to inhomogeneous differential equations

$$\mathcal{L}u(x) = f(x) \tag{25}$$

where $\mathcal{L}$ is a differential operator

$$\mathcal{L} = P_0(x)\frac{d^n}{dx^n} + P_1(x)\frac{\mathrm{d}^{n-1}}{dx^{n-1}} + \cdots P_{n-1}(x)\frac{d}{\mathrm{d}x} + P_n(x) \tag{26}$$

We focus on $n = 2$ case. Green's function method is one of the most effective ways of solving the ODE given some boundary conditions. The approach involves first understanding how equation (25) responds to a **unit impulse**

$$\mathcal{L}g = \delta(x - \xi) \tag{27}$$

where ξ is an arbitrary point of excitation of the unit impulse. The solution normally appears as an integral involving the Green's function $g(x, \xi)$ (or $g(x|\xi)$) and $f(x)$ that satisfies a homogeneous boundary conditions. $\delta(x - \xi)$ is a shifted Dirac delta function.

The Green's function is so powerful because given the solution for $g(x|\xi)$, we can immediately solve the general problem (equation (25)) for an arbitrary $f(u)$ by writing

$$u(x) = \int_a^b g(x|\xi)f(\xi)\mathrm{d}\xi \tag{28}$$

You can see that it solves equation (25) by plugging in equation (28) in equation (25)

$$\mathcal{L}u = \mathcal{L}\left[\int_a^b g(x|\xi)f(\xi)\mathrm{d}\xi\right] = \int_a^b \left[\mathcal{L}g(x|\xi)\right] f(\xi)\mathrm{d}\xi = \int_a^b \delta(x-\xi)f(\xi)\mathrm{d}\xi = f(x) \tag{29}$$

Basic properties of the delta function

$\delta(x - \xi)$ is 0 everywhere but at $x = \xi$. Its total area is 1

$$\int_c^d \delta(x)\mathrm{d}x = \begin{cases} 1 & \text{if} \quad c \le 0 \le d \\ 0 & 0 \quad \text{otherwise} \end{cases} \tag{30}$$

when shifted $\delta(x - \xi)$ its spike shifts to $x = \xi$

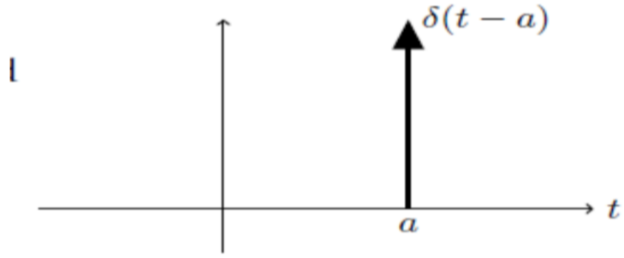

Figure 2: *Left panel*:Shifted delta function

$$\int_c^d f(x)\delta(x-a)\mathrm{d}x = \begin{cases} \text{f(a)} & \text{if} \quad c \le 0 \le d \\ 0 & 0 \quad \text{otherwise} \end{cases} \tag{31}$$

$\delta(x) = u'(x)$ where $u(x)$ is a unit step function.

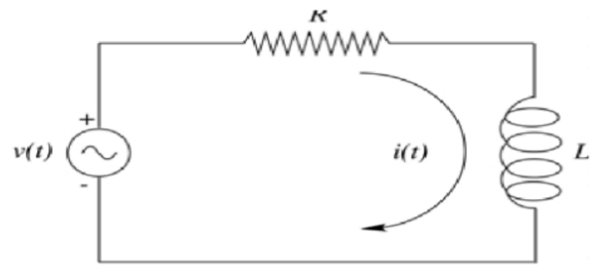

Figure 3: Rl electrical circuit with applied voltage

1. ***Electric Circuit problem***

   In electrical engineering, the equation describing current flow within a circuit is given by

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = v(t) \tag{32}$$

where $v(t)$ is the voltage source, R is the resistance, L is the inductance, i is the current. Now consider this: with the circuit initially dead, allow the voltage to become $V_0/\Delta\tau$ during a very short duration Δ starting at $t = \tau$.

At $t > \tau + \Delta\tau$, we have a homogeneous equation

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = 0\,, \qquad t > \tau + \Delta\tau \tag{33}$$

with a solution

$$i(t) = I_0 e^{-Rt/L}\,, t > \tau + \Delta\tau \tag{34}$$

where I_0 - constant and L/R is the time constant of the circuit. The voltage during $\tau < t < \tau + \Delta\tau$ is $V_0/\Delta\tau$, then

$$\int_{\tau}^{\tau+\Delta\tau} v(t)\mathrm{d}t = V_0 \tag{35}$$

Now we have to find the solution to equation (32) at the interval of the voltage surge $\tau < t < \tau + \Delta\tau$

$$L\int_{\tau}^{\tau+\Delta\tau} \mathrm{d}i + R\int_{\tau}^{\tau+\Delta\tau} i(t)\mathrm{d}t = \int_{\tau}^{\tau+\Delta\tau} \mathrm{d}v(t)\mathrm{d}t \tag{36}$$

which gives

$$L\left[i(\tau+\Delta\tau) - i(\tau)\right] + R\int_{\tau}^{\tau+\Delta\tau} i(t)\mathrm{d}t = V_0 \tag{37}$$

Condition 1: If $i(t)$ remains continous as $\Delta\tau$ becomes small, then

$$R\int_{\tau}^{\tau+\Delta\tau} i(t)\mathrm{d}t \approx 0 \tag{38}$$

then

$$i(\tau+\Delta\tau) = I_0 e^{-R(\tau+\Delta\tau)/L} = I_0 e^{-R\tau/L} \tag{39}$$

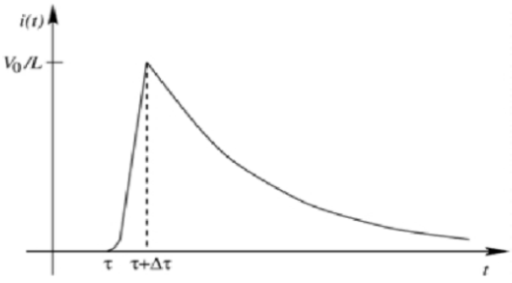

Figure 4: Current after a voltage surge

for $\Delta\tau$ small $LI_0 e^{-R\tau/L} = V_0$, which implies $I_0 = (V_0/L)e^{R\tau/L}$, we can finally write

$$i(t) = \begin{cases} 0 & t < \tau \\ \frac{V_0}{L} e^{-R(t-\tau)/L} & \tau \leq t \end{cases} \tag{40}$$

Now consider a situation whereby we have an N number of voltage surges

$$i(t) = \begin{cases} 0 & t < \tau_0 \\ \frac{V_0}{L} e^{-R(t-\tau)/L} & \tau_0 \leq t \leq \tau_1 \\ \frac{V_0}{L} e^{-R(t-\tau)/L} + \frac{V_1}{L} e^{-R(t-\tau_1)/L} & \tau_1 \leq t \leq \tau_2 \\ \vdots & \vdots \\ \sum_{i=0}^{N} \frac{V_i}{L} e^{R(t-\tau_i)/L} & \tau_N \leq t \leq \tau_{N+1} \end{cases} \tag{41}$$

For a continuous voltage surges within a very short duration

$$i(t) = \int_{\tau}^{t} \frac{v(t)}{L} e^{-R(t-\tau)/L} \mathrm{d}\tau = \int_{\tau}^{t} v(t) g(t|\tau) \mathrm{d}\tau \tag{42}$$

where

$$g(t|\tau) = \frac{1}{L} e^{R(t-\tau)/L}\,, \qquad \tau < t \tag{43}$$

Here $g(t|\tau)$ is called the Green's(Green) functions.

Lessons

- Because, the problem is a first order ODE, we had only one arbitrary constant to fix using the continuity of $i(t)$ at the $t = \tau$.

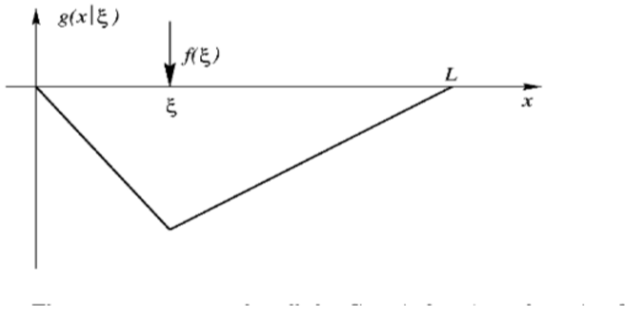

Figure 5: Downward acting external load

2. ***Load on a String problem***

Consider a string of length L connected at both ends of the support and subjected to an external force acting downwards:

$$T\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f(x) \tag{44}$$

where T denotes the uniform tensile force of the string. The string is stationary at both ends.

$$u(0) = u(L) = 0 \tag{45}$$

Instead of solving for $f(x)$ directly, we solve for its response to a unit load at a point $x = \xi$.

$$T\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \delta(x-\xi) \tag{46}$$

subject to the boundary conditions

$$g(0|\xi) = g(L|\xi) = 0 \tag{47}$$

$g(x|\xi)$ denotes the displacement of the string when it is subjected to a unit load at $x = \xi$.

As soon as $g(x|\xi)$ is known, the displacement at any other location is found by convolving $f(x)$ with $g(x|\xi)$.

At $x \neq \xi$, we have the homogeneous equation

$$T\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0 \tag{48}$$

which has the solution

$$g(x|\xi) = \begin{cases} ax+b & 0 \le x < \xi \\ cx+d & \xi < x \le \tau \end{cases} \tag{49}$$

There four arbitrary constants to fix.

- **Apply the stationary boundary conditions to** $g(x|\xi)$

$$g(0|\xi) = a.0 + b = b = 0\,, \qquad \text{and} \qquad g(L|\xi) = c.L + d = 0\,, \quad \text{or} \quad d = -cL \tag{50}$$

Applying these to equation (49) gives

$$g(x|\xi) = \begin{cases} ax & 0 \le x < \xi \\ c(x-L) & \xi < x \le \tau \end{cases} \tag{51}$$

where a and c are yet to be determined.

3. **Impose continuity at $x = \xi$ for $u(x)$ since the string is not broken**
   Continuity of $u(x)$ implies continuity of both solutions of $g(x|\xi)$
$$a\xi = c(\xi - L) \longrightarrow c = \frac{a\xi}{\xi - L} \tag{52}$$
   We still have to determine a.
4. **There is a discontinuity for first derivatives at $x = \xi$**
   Integrating equation (46) once gives
$$T\int_{\xi-\epsilon}^{\xi+\epsilon} \frac{\mathrm{d}^2 g(x|\xi)}{\mathrm{d}x^2}\mathrm{d}x = \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\mathrm{d}x \tag{53}$$
   As we approach $x = \xi$ from left and right, equation (53) becomes
$$\lim_{\epsilon\to 0}\left[\frac{\mathrm{d}g(\xi+\epsilon|\xi)}{\mathrm{d}x} - \frac{g(\xi-\epsilon|\xi)}{\mathrm{d}x}\right] = \frac{1}{T} = \frac{\mathrm{d}g(\xi_+|\xi)}{\mathrm{d}x} - \frac{g(\xi_-|\xi)}{\mathrm{d}x} = \frac{1}{T} \tag{54}$$
   where ξ_+ and ξ_- are nearest point to right and left of ξ. Using equation (51), we find
$$\frac{g(\xi_-|\xi)}{\mathrm{d}x} = a\,, \quad \text{and} \quad \frac{\mathrm{d}g(\xi_+|\xi)}{\mathrm{d}x} = c = \frac{a\xi}{\xi - L} \tag{55}$$
   Plugging this in the jump condition (equation (54)) gives
$$\frac{a\xi}{\xi - L} - a = \frac{1}{T} \Rightarrow \frac{aL}{\xi - T} = \frac{1}{T} \tag{56}$$
   Hence
$$g(x|\xi) = \begin{cases} \frac{x(\xi-L)}{LT} & 0 \le x < \xi \\ \frac{x(x-L)}{LT} & \xi < x \le \tau \end{cases} \tag{57}$$
   Finally
$$u(x) = \frac{(x-L)}{LT}\int_0^x f(\xi)\xi\mathrm{d}\xi + \frac{x}{LT}\int_x^L f(\xi)(\xi - L)\mathrm{d}\xi \tag{58}$$

Steps

1. Ensure $g(x|\xi)$ satisfies the homogeneous equation $f(x) = 0$ except at $x = \xi$.
2. $g(x|\xi)$ satisfies certain homogeneous boundary conditions
3. Ensure $g(x|\xi)$ is continuous at $x = \xi$.
4. Ensure derivative of $g(x|\xi)$ satisfies the jump condition for the first derivatives.

3.0.1. Exercise

1. Find the Green's function for the system
$$y'' - 3y' + 2y = f(t) \qquad \text{with} \qquad y(0) = y'(0) = 0 \tag{59}$$
   Ans:
$$g(t|\tau) = \left[e^{2(t-\tau)} - e^{(t-\tau)}\right] H(t-\tau) \tag{60}$$
   where $H(t-\tau)$ is Heavside function.

2. Damped harmonic oscillator

$$my'' + cy' = ky = f(t) \tag{61}$$

where k is the spring constant, m is the mass attached to the string and c is the damping coefficient. Find the Green's function for $g(0|\tau) = g'(0|\tau) = 0$ Ans

$$mg(t|\tau) = \frac{e^{-\gamma(t-\tau)}}{\sqrt{\omega_0^2 - \gamma^2}} \sin\left[(t-\tau)\sqrt{\omega_0^2 - \gamma^2}\right] \tag{62}$$

$\omega_0 = k/m$ and $\gamma = c/(2m)$

3.1. Green's functions with homogeneous solution in any basis

Consider a general linear second order differential equation

$$\mathcal{L}y(x) = \alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x}y + \gamma(x)y = f(x) \tag{63}$$

where α, β, γ are continuous functions on [a,b] and α is non zero. The forcing terms is bounded within the same range. Using Green's function technique to solve equation (63) compares exactly to using Eigenfunction expansion technique to invert the Strum-Liouville operators.

Firstly for $x \neq \xi$, we can solve $\mathcal{L}g(x|\xi) = 0$ for $x < \xi$ and $x > \xi$. We suppose that $\{y_1, y_2\}$ constitute the basis for the linearly independent solutions to the homogeneous equation. Thus, we can write

$$g(x|\xi) = \begin{cases} A(\xi)y_1(x) + B(\xi)y_2(x) & a \leq x < \xi \\ C(\xi)y_1(x) + D(\xi)y_2(x) & \xi < x \leq b \end{cases} \tag{64}$$

Apply stationary boundary conditions $y_1(a) = y_2(b) = 0$

$$g(a|\xi) = A(\xi)y_1(a) + B(\xi)y_2(x) = A(\xi).0 + B(\xi)y_2(x) = B(\xi)y_2(x) \tag{65}$$
$$g(b|\xi) = C(\xi)y_1(x) + D(\xi)y_2(b) = C(\xi)y_1(x) + D(\xi).0 = C(\xi)y_1(x) \tag{66}$$

Our solution now reduce to

$$g(x|\xi) = \begin{cases} B(\xi)y_2(x) & a \leq x < \xi \\ C(\xi)y_1(x) & \xi < x \leq b \end{cases} \tag{67}$$

Now we need to impose continuity condition at $x = \xi$ and and jump in derivative conditions as we approach to ξ from lefft and right

$$g(\xi_+|\xi) = g(\xi_-|\xi) \tag{68}$$
$$\frac{\partial g(\xi_+|\xi)}{\partial x} - \frac{\partial g(\xi_-|\xi)}{\partial x} = \frac{1}{\alpha(\xi)} \tag{69}$$

Applying these will lead to

$$B(\xi)y_2(\xi) = C(\xi)y_1(\xi) \tag{70}$$
$$B(\xi)y_2'(\xi) - C(\xi)y_1'(\xi) = \frac{1}{\alpha(\xi)} \tag{71}$$

These are two linear system of equations for B and C

$$C(\xi) = \frac{y_2(\xi)}{\alpha(\xi)W(\xi)} \quad \text{and} \quad B(\xi) = \frac{y_1(\xi)}{\alpha(\xi)W(\xi)} \tag{72}$$

where $W(x) = y_1y_2' - y_2y_1'$ is the Wronskian. It would be evaluate at $x = \xi$ in order to determine constants in equation (72).

$$g(x|\xi) = \begin{cases} \frac{y_1(\xi)y_2(x)}{\alpha(\xi)W(\xi)} & a \leq x < \xi \\ \frac{y_1(x)y_2(\xi)}{\alpha(\xi)W(\xi)} & \xi < x \leq b \end{cases} \tag{73}$$

$$= \frac{1}{\alpha(\xi)W(\xi)} \left[\Theta(\xi - x)y_1(\xi)y_2(x) + \Theta(x - \xi)y_1(x)y_2(\xi)\right] \tag{74}$$

where Θ is a step function The final solution is given by

$$y(x) = \int_a^b g(x|\xi)f(\xi)\mathrm{d}\xi \tag{75}$$

$$= y_2(x)\int_a^x \frac{y_1(\xi)}{\alpha(\xi)W(\xi)}f(\xi)\mathrm{d}\xi + y_1(x)\int_x^b \frac{y_2(\xi)}{\alpha(\xi)W(\xi)}f(\xi)\mathrm{d}\xi \tag{76}$$

The Green's function given in equation (74) is related to the eigenfunctions $\{Y_n(x)\}$ and eigenvalues $\{\lambda\}$of the Sturm-Liouville operators

$$g(x|\xi) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} Y_n(x)Y_n^*(\xi) \tag{77}$$

we won't go into further details.

3.1.1. More Exercises

1. Use the Green's function technique to solve the forced problem

$$\mathcal{L}y = -y'' - y = f(x) \tag{78}$$

on the interval $[0, 1]$ subject tot he boundary conditions $y(0) = y(1) = 0$.

Ans:

$$y(x) = \frac{\sin(1-x)}{\sin 1}\int_0^x f(\xi)\sin\xi\mathrm{d}\xi + \frac{\sin x}{\sin 1}\int_x^1 f(\xi)\sin(1-\xi)\mathrm{d}\xi \tag{79}$$

4. NON-CONSTANT COEFFICENT ODE

4.1. Missing a dependent function in the ODE

If an ODE is missing a dependent function, then it can be reduced to an ODE of one-order less, for example

$$ty'' + 4y' = t^2 \tag{80}$$

Set $y' = z$ which implies that $y'' = z'$, leads to a linear first order ODE.

4.2. Exact Equations

If the ODE can be written as a derivative of another ODE, we say it is exact

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x) \tag{81}$$

is exact if the LHS can be written as

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_0(x)y = \frac{\mathrm{d}}{dx}\left[b_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{dx^{n-1}} + \cdots + b_0(x)y\right] \tag{82}$$

This holds if

$$a_0(x) - a_1'(x) + a_2''(x) - \cdots + (-1)^2 a_n^{(n)}(x) = 0 \tag{83}$$

Example
Find the general solution to the ODE

$$(1-x^2)\frac{\mathrm{d}^2 y}{dx^2} - 3x\frac{\mathrm{d}x}{\mathrm{d}x} - y = 1 \tag{84}$$

solution
Comparing to equation (80), we find $a_2 = 1 - x^2$, $a_1 = -3x$, $a_0 = 1$.
Check that $a_0 - a_1' + a_2'' = 0$, hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[b_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b_0(x)y\right] = 0 \tag{85}$$

Matching

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[b_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b_0(x)y\right] = b_1\frac{\mathrm{d}^2 y}{dx^2} + (b_1' + b_0)\frac{\mathrm{d}x}{\mathrm{d}x} + b_1' y \tag{86}$$

We can then compute b_1 and b_0 by comparing with the original equation

$$b_1 = 1 - x^2\,, \quad b_1' + b_0 = -3x\,, \qquad b_0' = -1 \tag{87}$$

This implies that $b_1 = 1 - x^2$ ad $b_0 = -x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy\right] = 1 \tag{88}$$

Integrating give

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{x}{1-x^2}\right)y = \frac{x + c_1}{(1-x^2)} \tag{89}$$

Solving further gives

$$y = \frac{c_1 \sin^{-1} x + c_2}{\sqrt{1 - x^2}} - 1 \tag{90}$$

Ex
Is this ODE exact?

$$x(1 - x^2)\frac{\mathrm{d}^2 y}{dx^2} - 3x^2 \frac{\mathrm{d}x}{\mathrm{d}x} - xy = x \tag{91}$$