Mathematical methods: Computational Tensor Algebra

Obinna Umeh

[PhD Lectures 2023]

November 3, 2024

Who is this?

- “Research Software Engineer”
- Office address: DS 1.11
- Area of specialization:
  - Fish biology
  - Causal structure of spacetime
  - Signature of general relativistic in large scale structures
  - AdS/CFT in String theory (Gravity/condensed matter physics correspondence)
  - Pulsar Astrophysics(quantum gravity lab)
- Materials for this course will be available on GitHub:https://github.com/Obinna/

Structure of the lecture

- `2:00 --- 2:45`

$$\text{Mathematica} = \begin{cases} \text{Basic intro.} & 2:00 \leq t \leq 2:30 \\ \text{Exercise} & 2:30 \leq t \leq 2:45 \end{cases}$$

- `2:45 --- 3:00` Break, coffee and catch-up questions.
- `3:00 --- 3:50`

$$\text{xAct} = \begin{cases} \text{Basic intro.} & 3:00 \leq t \leq 3:30 \\ \text{Exercise} & 3:30 \leq t \leq 3:45 \end{cases}$$

What is Computational Tensor Algebra?

Living Reviews in Relativity (2018) 21:6
https://doi.org/10.1007/s41114-018-0015-6

REVIEW ARTICLE

Computer algebra in gravity research

Malcolm A. H. MacCallum[1]

Received: 9 April 2018 / Accepted: 13 July 2018 / Published online: 20 August 2018

Abstract
The complicated nature of calculations in general relativity was one of the driving forces in the early development of computer algebra (CA). CA has become widely used in gravity research (GR) and its use can be expected to grow further. Here the general nature of computer algebra is discussed, along with some aspects of CA system design; features particular to GR's requirements are considered; information on packages for CA in GR is provided, both for those packages currently available and for their predecessors; and applications of CA in GR are outlined.

Keywords Computer algebra · General relativity · Gravitation theory · Algorithms · Exact solutions · Programs

Why Computational Tensor Algebra?

- Advantages
  1. Accuracy
  2. Speed
  3. Repetative tasks
  4. Some calculations are infeasible by hand
- Disadvantages
  1. Computational cost
  2. Time and energy spent learning new things

Computational Tensor Algebra systems

1. Axiom: http://www.axiom-developer.org/
2. Maple: https://www.maplesoft.com/ns/maple/cas/...
3. Mathematica: https://www.wolfram.com/mathematica/
4. Sage: https://www.sagemath.org/
5. Reduce: https://reduce-algebra.sourceforge.io/
6. Maxima: https://maxima.sourceforge.io/
7. Symbolic Python: https://www.sympy.org/en/index.html
8. Mathlab: https://www.mathworks.com/
9. Cadabra: https://cadabra.science/

Why Mathematica? Many developers

- Abstract:
  1. xAct: http://www.xact.es/
  2. MathTensor: https://library.wolfram.com/infocenter/TechNotes/4697/
  3. Tensor: https://reference.wolfram.com/language/guide/Tensors.html
  4. TTC: https://library.wolfram.com/infocenter/Articles/2520/
  5. Ricci: https://sites.math.washington.edu// lee/Ricci/
  6. Atlas: https://mathworld.wolfram.com/Atlas.html
- Components:
  1. xAct(xCode)
  2. GRTensorM
  3. MathGR
  4. RGTC
  5. Tensorial
  6. GREAT
  7. Tensorialcalc

Structure of xAct

There are four packages acting as a kernel for the rest:

- xCore: generic programming tools
- xPerm: manipulation of large groups of permutations
- xTensor: abstract tensor computations, the flagship of the system
- xCoba: component tensor computations

Other packages include

- Invar: polynomial invariants of the Riemann tensor, by JMMG, David Yllanes, Renato Portugal and Leon Manssur.
- Spinors: spinor computations in GR, by Alfonso García-Parrado and JMMG.
- SymManipulator: Symmetrized expressions in xAct, by Thomas Bäckdahl.
- xTerior: Exterior calculus, by Alfonso García-Parrado and Leo C. Stein.
- SpinFrames: NP and GHP form of spinor equations, by Thomas Bäckdahl and Steffen Aksteiner.
- bimEX: $3+1$ Bimetric relativity, by Francesco Torsello.
- FieldsX: Fermions, gauge fields and BRST cohomology, by Markus B. Fröb.
- SymSpin: Symmetric spinors, by Steffen Aksteiner and Thomas Bäckdahl.
- HiGGS: Hamiltonian analysis of Poincaré gauge theory, by W. E. V. Barker.

Other packages include

- xPert: high-order perturbation theory in GR, by David Brizuela, JMMG and Guillermo Mena Marugán.
- xPand: Cosmological perturbation theory, by Cyril Pitrou, Xavier Roy and Obinna Umeh.
- xIST/COPPER: General scalar-tensor theories and perturbations, by Johannes Noller.
- xPPN: Parametrized post-Newtonian formalism, by Manuel Hohmann.
- EFTofPNG:: Effective field theory of post-Newtonian gravity, by Michele Levi and Jan Steinhoff.
- xTras: Additions to xAct, by Teake Nutma.

xAct basic design architecture

- Manifold: We consider a spacetime (M, g), where M is a real smooth manifold of dimension $d = D + 1$ and g is a metric on M, (M, g) is orientable (Geometry of hypersurfaces in two parts, past and future).
- Tangent space: At a given point $p \in M$, we denote by $T_p(M)$ the tangent space, i.e.the space of vectors at p.
- Choice of indices Greek letters, English alphabets, etc can use numbers if enough indices is not provided.
- Define a metric tensor $g_{\alpha\beta}$
- Christoffel symbols of the first kind

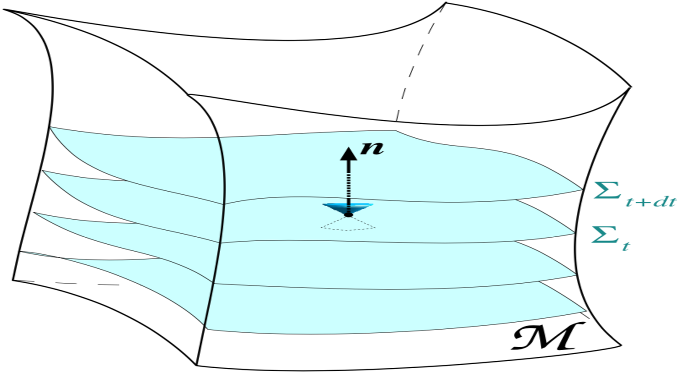
$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right)$$

- Curvature tensors:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$$

Threading of spacetime

- Threading of spacetime (1+D) decomposition: A geometric approach to studying the dynamics of spacetime by decomposing spacetime into a sequence of spatial hypersurfaces using a timelike vector field



Threading of spacetime (1+D) decomposition

- The four-velocity vector of the observer u^a

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau} \, .$$

- Introduce the metric on the hypersurface orthogonal to u^a: h_{ab}.

$$h_{ab} = g_{ab} + u_a u_b, \qquad \epsilon_{abc} = \eta_{abcd} u^d \, ,$$

- Irreducible decomposition of $\nabla_b u_a$ is given by

$$\nabla_b u_a = -u_b A_a + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} \, .$$

- A^a, Θ, σ_{ab} and ω_{ab} are acceleration, expansion, shear and vorticity respectively.

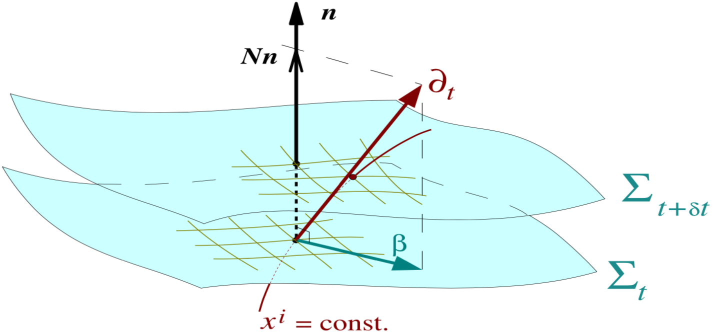
Resulting equations

Using the Ricci identity

$$
\begin{aligned}
\frac{\mathrm{D}\Theta}{\mathrm{D}\tau} &= -\frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} - \omega_{ab}\omega^{ab} \\
&\quad + \mathrm{D}_a A^a + A_a A^a - R_{ab} u^a u^b \,, \\
\frac{\mathrm{D}\sigma_{ab}}{\mathrm{D}\tau} &= -\frac{2}{3}\Theta\sigma_{ab} - \sigma^c{}_{\langle a}\sigma_{b\rangle c} - \omega^c{}_{\langle a}\omega_{b\rangle c} \\
&\quad + \mathrm{D}_{\langle a} A_{b\rangle} + A_{\langle a} A_{b\rangle} - C_{acbd} u^c u^d \,, \\
\frac{\mathrm{D}\omega_{ab}}{\mathrm{D}\tau} &= -\frac{2}{3}\Theta\omega_{ab} + \sigma^c{}_{[a}\omega_{b]c} - \mathrm{D}_{[a} A_{b]} \,.
\end{aligned}
$$

Slicing of spacetime

- Slicing of spacetime (D+1): A process of decomposing spacetime into a sequence of spatial hypersurfaces by choosing a timelike vector field and using it to define the normal vectors to the spatial hypersurfaces



Slicing of spacetime (D+1) decomposition

- Here the spacetime is foliated into a family of spacelike surfaces Σ_t, labeled by their time coordinate t, and with coordinates on each slice given by x^i

$$\mathrm{d}s^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j$$

- There are four Lagrange multipliers: the lapse function, N, and components of shift vector field, N_i h_{ij} is the dynamical field.
- n^a is defined with respect to scalar field ϕ, the gradient is orthogonal to the hypersurface h_{ab} and it is timelike $n^a n_a = -1$,

$$n^a = \frac{1}{N}(1, -N^i),\ n_a = N(-1, 0, 0, 0)$$

- Decomposition of the covariant derivative of $\nabla_a n_b$ is given by

$$\nabla_a n_b \quad = \quad -n_a \dot{n}_b - K_{ab} = -n_a \dot{n}_b + \frac{1}{3}\xi h_{ab} + \Sigma_{ab}$$

- The second fundamental form is given by $K_{ab} \equiv -h_a^c h_b^d n_{c;d}$

Key equations

Propagation equation

$$
\begin{aligned}
\frac{1}{N}\partial_t h_{ij} &= \frac{2}{N}\mathrm{D}_{(j}N_{i)} - 2NK_{ij}\ , \\
\frac{1}{N}\partial_t K^i_j &= \mathcal{R}^i_j + KK^i_j - \Lambda\delta^i_j - \frac{1}{N}\mathrm{D}_j\mathrm{D}^i N \\
&\quad + \frac{1}{N}\left(K^i_k\mathrm{D}_j N^k - K^k_j\mathrm{D}_k N^i + N^k\mathrm{D}_k K^i_j\right) \\
&\quad - 8\pi G\left(S^i_j + \frac{1}{2}(\epsilon - S^k_k)\delta^i_j\right)\ ,
\end{aligned}
$$

Constraint equations

$$
\begin{aligned}
\mathrm{D}_i K^i_j - \mathrm{D}_j K &= 8\pi G J_j\ , \\
R + K^2 - K_{ij}K^{ij} &= 16\pi\epsilon\ ,
\end{aligned}
$$

Cosmological Perturbation theory

- The idea is to consider a background manifold $\overline{\mathcal{M}}$ along with its perturbed manifold $\mathcal{M}$ and then require that they are related by means of a diffeomorphism $\phi \colon \overline{\mathcal{M}} \to \mathcal{M}$.
- Perturbation of the metric

$$\phi^\star(\mathbf{g}) = \bar{\boldsymbol{g}} + \sum_{n=1}^{\infty} \frac{\Delta^n[\bar{\boldsymbol{g}}]}{n!} = \bar{\boldsymbol{g}} + \sum_{n=1}^{\infty} \frac{{}^{\{n\}}\boldsymbol{h}}{n!}$$

- Perturbation of the Christoffel symbols

$$\Delta^n \left[\Gamma^\rho_{\mu\nu}\right] = \sum_{(k_i)} (-1)^{m+1} \frac{n!}{k_1! \dots k_m!} \; {}^{\{k_m\}}h^{\rho\zeta_m} \; {}^{\{k_{m-1}\}}h_{\zeta_m}{}^{\zeta_{m-1}} \dots {}^{\{k_2\}}h_{\zeta_3}{}^{\zeta_2} \; {}^{\{k_1\}}h_{\zeta_2\mu\nu}$$

C. Pitrou, X. Roy and O. Umeh, Class. Quant. Grav. **30** *(2013), 165002 doi:10.1088/0264-9381/30/16/165002 [arXiv:1302.6174 [astro-ph.CO]].*

Example

- https://github.com/xAct-contrib/examples
- Description: https://github.com/xAct-contrib/examples/blob/master/README.md
- Black hole ring down calculations: https://github.com/sergisl/ringdown-calculations
- Consult xAct documentation in doubt