Mathematical methods: Computational Tensor Algebra

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Calculus of variations and cosmological perturbation theory

1. Calculus of variations
   - Minimization of functionals and Euler-Lagrange equations
   - Noether Theorem and Conservation laws
   - Introduction to xAct tor action variation action)
2. Cosmological Perturbation theory
   - Introduction to xPand
   - Cosmological Perturbation on a homogeneous background

Relationship between Mathematics and Physics

- Physics problems are formulated in the language of calculus or functional or action
    1. Principle of least action
    2. Principle of relativity
    3. Principle of gauge invariance (coordinate invariance)

Definition of Functional derivative

- Consider a

$$J[y] = \int_{x_1}^{x_2} f(x, y, y', y'', \cdots, y^{(n)}) \mathrm{d}x \tag{1}$$

- where f depends on the value of $y(x)$ and finite number its derivatives.
- Locality: $J[y]$ is called local, if a small change in x leaves the $J[y]$ unchanged. (See also principle of locality in Physics).

Functional derivatives II

- Consider making an infinitesimal change

$$y(x) \to y(x) + \varepsilon\eta(x) \tag{2}$$

The functional changes as

$$\frac{\delta J}{\delta y(x)} = \int_{x_1}^{x_2} \left[f(x, y+\varepsilon\eta, y'+\varepsilon\eta') - f(x, yy') \right] \mathrm{d}x \tag{3}$$

$$= \left[(\varepsilon\eta) \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} (\varepsilon\eta(x) \left\{ \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right\} \mathrm{d}x \cdot \tag{4}$$

- Fixed Point: if $\eta(x_1) = \eta(x_2) = 0$, $[\cdots]_{x_2}^{x_1}$ vanishes
- where

$$\frac{\delta J}{\delta y(x)} \equiv \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \tag{5}$$

- $\delta J / \delta y(x)$ is called the functional derivative derivative of J with respective to $y(x)$.

Euler-Lagrange equation I

- By the principle of stationary action

$$\frac{\partial f}{\partial y(x)} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0\,, \qquad x_1 \leq x \leq x_2 \tag{6}$$

  This is called the Euler- Lagrange equations.

- If f depends on more than one function, y_i, the J is stationary under all possible variations

$$\frac{\delta J}{\delta y_i(x)} = \frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_i}\right) = 0 \tag{7}$$

  for each $y_i(x)$(multi-fields)

Euler-Lagrange equation II

- If f depends on higher derivatives y'', $y^{(3)}$, etc. then we have to integrate by parts more times

$$\frac{\delta J}{\delta y_i(x)} = \frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial f}{\partial y'''}\right) = 0 \quad (8)$$

- c.f higher derivative gravity theories.
- Physics notations

$$J[y] = \int_{x_1}^{x_2} f(x, y, y', y'')\mathrm{d}x \to S[q] = \int_{x_1}^{x_2} L(\tau, q, q', q'')\mathrm{d}x \quad (9)$$

where L is the Lagrangian.

Examples:Scalar field theory

Local field theory of a real scalar field $\Phi(x^i)$ defined on the spacetime

$$L = \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) \tag{10}$$

where $\partial_\mu = \nabla_\mu$ for scalars.

$$\frac{\partial}{\partial \Phi} = -\frac{\partial V}{\partial \Phi} \tag{11}$$

$$\frac{\partial L}{\partial (\nabla_\mu \Phi)} = \frac{\partial}{\partial (\nabla_\mu \Phi)} \left[\frac{1}{2} g^{\rho\sigma} \nabla_\rho \Phi \nabla_\sigma \Phi \right] = g^{\mu\nu} \nabla_{\nu\Phi} \tag{12}$$

The EOM becomes

$$\nabla_\mu \nabla^\mu \Phi + \frac{\mathrm{d}V}{\mathrm{d}\Phi} = 0 = \Box^2 + \frac{\mathrm{d}V}{\mathrm{d}\Phi} \tag{13}$$

where $\Box^2 = \nabla_\mu \nabla^\mu$. A simple choice for the potential is $V = m^2 \Phi^2 / 2$ where m is a constant parameter.

$$\nabla_\mu \nabla^\mu \Phi + m^2 \Phi = 0 \tag{14}$$

This is a well-known Klien-Gordon equation.

Vector field

- The Maxwell's action is given by

$$S = \int \mathcal{L} \mathrm{d}^4 x = \int_R \left[-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right] \sqrt{-g} \mathrm{d}^4 x \quad (15)$$

- Derive the equation of motion. The EL equation

$$\frac{\partial L}{\partial A_\nu} - \nabla_\mu \left[\frac{\partial L}{\partial (\nabla_\mu A_\nu)} \right] = 0 \quad (16)$$

$$\frac{\partial L}{\partial A_\nu} = -j^\mu \delta^\nu{}_\mu = -j^\nu \quad (17)$$

Vector field

$$
\begin{aligned}
\frac{\partial L}{\partial \nabla_\mu A_\nu} &= \frac{\partial}{\partial \nabla_\mu A_\nu}\left[-\frac{1}{4\mu_o} g^{\alpha\rho} g^{\beta\sigma} F_{\rho\sigma} F_{\alpha\beta}\right] \qquad (\\
&= -\frac{1}{4\mu_0}\left(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}\right) F_{\alpha\beta} - \frac{1}{4\mu_0}\left(g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma}\right) \\
&= -\frac{1}{4\mu_0}\left(F^{\mu\nu} - F^{\nu\mu}\right) - -\frac{1}{4\mu_0}\left(F^{\mu\nu} - F^{\mu\nu}\right) \qquad (
\end{aligned}
$$

The EL gives

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu \tag{20}$$

The anti-symmetry property leads to

$$\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0 \tag{21}$$

Gravity

The simplest non-trivial scalar that can be constructed from the metric and its derivatives is the Ricci R

$$S_{EH} = \int_{\mathcal{R}} R\sqrt{-g}\mathrm{d}^4 x \tag{22}$$

Where $g_{\mu\nu}$ is a dynamical field. EL

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \partial_\sigma \left[\frac{\partial \mathcal{L}}{\partial(\partial_\sigma g_{\mu\nu})}\right] + \partial_\rho \partial_\sigma \left[\frac{\mathcal{L}}{\partial(\partial_\rho \partial_\sigma g_{\mu\nu})}\right] = 0 \tag{23}$$

This is a non-trivial action to vary, we shall learn how to use the Mathematica to vary it.

Noether Theorem and Conservation laws

- The total energy of a a simple Harmonic osciallator is given by

$$E(x,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 \tag{24}$$

Where $\dot{x} = p$ and $\dot{p} = -\omega^2 x$

$$\begin{aligned} \frac{dE(x(t),p(t))}{dt} &= p\dot{p} + \omega^2 \dot{x} &(25)\\ &= p(-\omega^2 x) + \omega^2 xp = 0 &(26) \end{aligned}$$

- the energy of a simple harmonic oscillator is a first integral.

Noether theorem: First integral

- When f is of the form $f(y, y')$ i.e has no explicit dependence on x,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \cancelto{0}{\frac{\partial f}{\partial x}} \qquad (27)$$

We see that

$$\begin{aligned} \frac{d}{dx}\left(f - y'\frac{\mathrm{d}f}{dy'}\right) &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} - y''\frac{\partial f}{\partial y'} - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) \\ &= y'\left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] \end{aligned} \qquad (28)$$

This is zero if EL equation is satisfied.

$$I \equiv f - y'\frac{\partial f}{\partial y'} = f - \sum_i y_i'\frac{\partial f}{\partial y_i'} \qquad (29)$$

Noether theorem:Symmetry and Conservation law

- The time-independence of the first integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right] = 0 \tag{30}$$

- Noether showed that they are related to the underlying symmetry of of the system.

$$\text{Symmetry} \Leftrightarrow \text{Conservationlaw} \tag{31}$$

The energy-momentum tensor I

If we consider an action of the general form

$$S = \int \mathcal{L}(\Psi, \Phi_\mu) \mathrm{d}^{d+1} x \tag{32}$$

where $\mathcal{L}$ does not depend explcitly on any of the coordinates x^μ

$$\Psi(x^\mu) \to \Psi\left(x^\mu + \varepsilon^\mu(x)\right) = \Psi(x^\mu) + \varepsilon^\mu(x)\partial_\mu \Psi \tag{33}$$

The resulting variation leads to

$$\delta S \;\; = \;\; \int \varepsilon^\mu(x) \frac{\partial}{\partial x^\nu} \left[\mathcal{L} \delta^\nu{}_\mu - \frac{\partial \mathcal{L}}{\partial \Psi_\nu} \partial_\mu \Psi \right] \mathrm{d}^{d+1} x \tag{34}$$

where Ψ satisfies the equation of motion

$$\frac{\partial}{\partial x^\nu} \left[\mathcal{L} \delta^\nu{}_\mu - \frac{\partial \mathcal{L}}{\partial \Psi_\nu} \partial_\mu \Psi \right] = 0 \tag{35}$$

The energy-momentum tensor II

Thus we define the canonical energy-momentum tensor as

$$T^{\nu}{}_{\mu} \quad = \quad \mathcal{L}\delta^{\nu}{}_{\mu} - \frac{\partial \mathcal{L}}{\partial \Psi_{\nu}} \partial_{\mu} \Psi \tag{36}$$

And the conservation law

$$\partial_{\nu} T^{\nu}{}_{\mu} = 0 \tag{37}$$

Cosmological Perturbation theory

- The idea is to consider a background manifold $\overline{\mathcal{M}}$ along with its perturbed manifold $\mathcal{M}$ and then require that they are related by means of a diffeomorphism $\phi\colon \overline{\mathcal{M}} \to \mathcal{M}$.
- Perturbation of the metric

$$\phi^{\star}(\mathbf{g}) = \bar{\boldsymbol{g}} + \sum_{n=1}^{\infty} \frac{\Delta^n[\bar{\boldsymbol{g}}]}{n!} = \bar{\boldsymbol{g}} + \sum_{n=1}^{\infty} \frac{{}^{\{n\}}\boldsymbol{h}}{n!}$$

- Perturbation of the Christoffel symbols

$$\Delta^n \left[\Gamma^{\rho}_{\mu\nu}\right] = \sum_{(k_i)} (-1)^{m+1} \frac{n!}{k_1! \dots k_m!} \; {}^{\{k_m\}}h^{\rho\zeta_m} \; {}^{\{k_{m-1}\}}h_{\zeta_m}{}^{\zeta_{m-1}} \dots {}^{\{k_2\}}h_{\zeta_3}{}^{\zeta_2} \; {}^{\{k_1\}}h_{\zeta_2\mu\nu}$$

C. Pitrou, X. Roy and O. Umeh, Class. Quant. Grav. **30** *(2013), 165002 doi:10.1088/0264-9381/30/16/165002 [arXiv:1302.6174 [astro-ph.CO]].*