Mathematical Methods for Graduate Students in Gravitational Wave Astronomy, Astrophysics and Cosmology

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Who is this?

- I am a Research Felllow at the ICG
- Office address: Room 2.04
- Area of specilization:
  - General relativistic signature in large scale structures.
  - Foundational problems in Cosmology, such as the averaging problem, back reaction problem.
  - AdS/CFT in String theory (Played crucial role in resolving the Black hole information paradox) (https://www.quantamagazine.org/the-black-hole-information-paradox-comes-to-an-end-20201029/)
  - Pulsar Astrophysics as well.
- Some of the materials for this course will be available on GitHub:https://github.com/Obinna/Documentation and on the ICG website:http://www.icg.port.ac.uk/author/umeho/

Aim of the lecture

- A level playing field.
- Expose to new skills.
- Might find some useful tips for your research.

Course Outline: Monday

1. Calculus of variations
   - Minimization of functionals
   - Euler-Lagrange equations
   - Hamiltonian dynamices
   - Noether Theorem and Conservation laws
2. Computer algebra software in gravity research
   - Introduction to Mathematica
   - Introduction to xAct
   - Introduction to xPand

Course Outline: Wednesday

1. Differential equations
   - First-order differential equations
   - Second-order differential equations
   - Homogeneous second-order differential equations
2. Inhomogeneous differential equations
   - Inhomogeneous second-order differential equations
   - Applications of the Wronskian to ordinary differential equations
   - Use of the method of Green functions
   - Boundary value problems

Course Outline: Friday

1. Special Functions
   - Transforms and series solutions to differential equations
     - Spherical harmonics and convolution
     - Spherical Bessel functions
     - Fourier transforms and convolution
     - Numerical Integration using the FFTLog formalism.
   - Partial differential equations
     - Heat equation
     - Black–Scholes equation

Unable to cover

- Groups and representation theory
- Lie Groups
- Functions spaces (Hilbert space)
- Complex analysis
- Matrix operations

Structure of the lecture

- `2:00 ---2:45`

$$\text{Calculus of varations} = \begin{cases} \text{Basic intro.} & 2:00 \leq t \leq 2:30 \\ \text{Exercise} & 2:30 \leq t \leq 2:45 \end{cases}$$

- `2:45 ---3:00` Break, coffee and catch-up questions.

- `3:00 ---3:50`

$$\text{Mathematica} = \begin{cases} \text{Basic intro.} & 3:00 \leq t \leq 3:30 \\ \text{Exercise} & 3:30 \leq t \leq 3:45 \end{cases}$$

Books

1. Mathematical Methods for physicist by George Arfken Hans Weber Frank E. Harris, Academic Press (2012).
2. Introduction to Mathematical Physics by *M. T. Vaughn*, Wiley, (2007).
3. Mathematical Tools for Physicists. edited by *G. L. Trigg*,Wiley, (2005)
4. Mathematical Methods for Physics and Engineering by *K. F Riley*, *M. P Hobson* and *S. J. Bence*, Cambridge University Press(2006)

Relationship between Mathematics and Physics

Relationship between Mathematics and Physics

- Pythagoras: Mathematics is the language of nature
- Galileo Galilei: All is number
- Isaac Newton: Regarded geometry as a branch of mechanics.

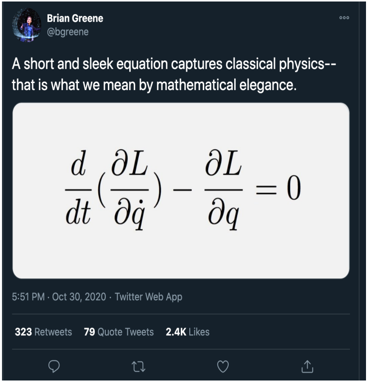
The Unreasonable Effectiveness of Mathematics in the Natural Sciences

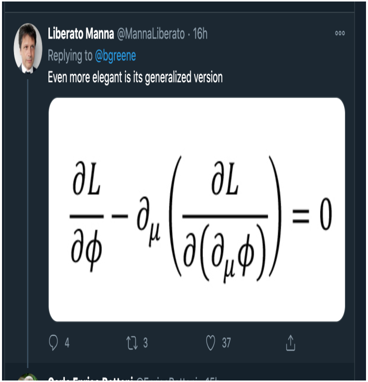
by

Eugene Wigner.

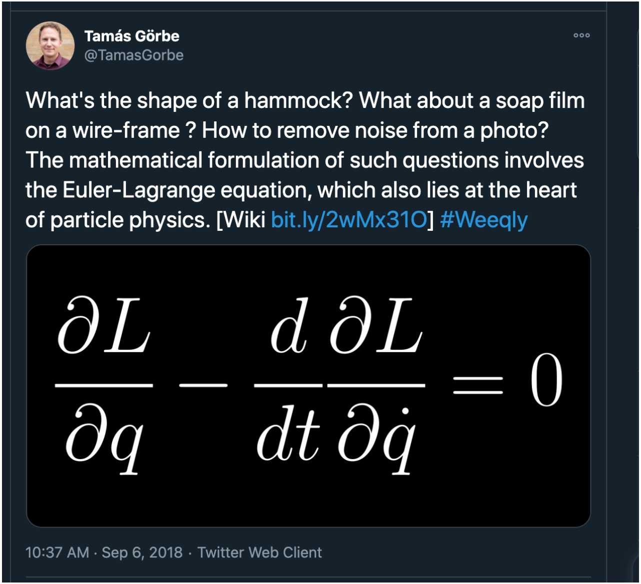
- The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it
- The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.
- We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.

How important is a short and sleek equation today?





Application of the Euler Lagrangian equations



Tamás Görbe
@TamasGorbe

What's the shape of a hammock? What about a soap film on a wire-frame ? How to remove noise from a photo? The mathematical formulation of such questions involves the Euler-Lagrange equation, which also lies at the heart of particle physics. [Wiki bit.ly/2wMx31O] #Weeqly

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

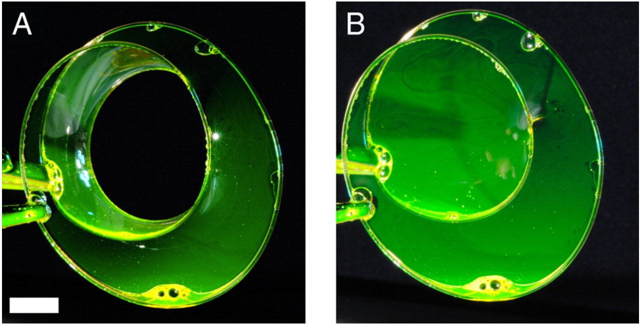
10:37 AM · Sep 6, 2018 · Twitter Web Client

Early application of the Euler Lagrangian equations

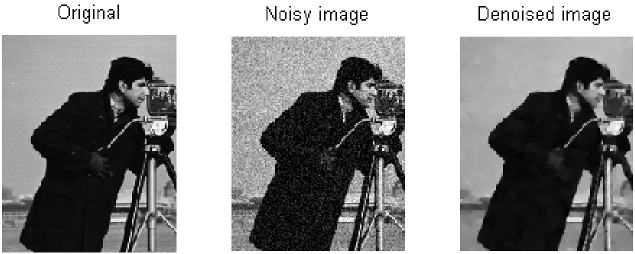
- What's the shape of a hammock?
- What about a soap film on a wire-frame ?
- How to remove noise from a photo?

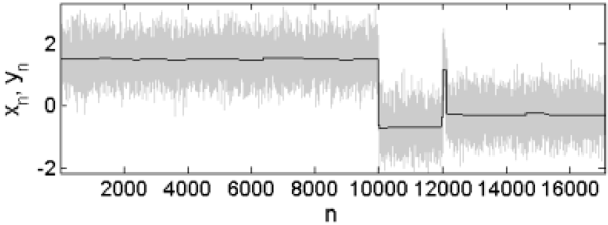
Swinging and balancing problem (Hammock)

Shapes and properties of soap films and bubbles



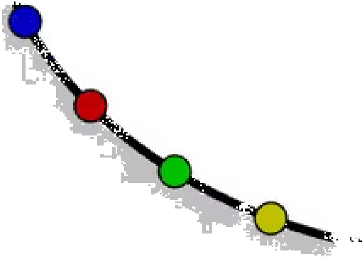
Denoising: Total variation denoising





Mathematical formulation

- The mathematical formulation of such questions involves the Euler-Lagrange equation, which lies at the heart of modern Physics.
- This mathematical formulation was provided by the mathematicians Joseph Louis Lagrange and Leonhard Euler, who provided an analytical solution to tautochrone problem.
- The tautochrone problem(from Greek prefixes tauto- meaning same or iso- equal, and chrono time):



How do you find it?

Find the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point

The tautochrone problem: Path independent of its starting point

- Assuming the particle's position is parametrized by the arclength from the from the lowest point $s(t)$, then the K.E is proportional to $\dot{s}^2$ and the P.E is proportional to the height $y(s)$.
- Ischrone requires that

$$y(s) = s^2 \tag{1}$$

- Line element in 2D: $dy = 2s\,ds$ and $dy^2 = 4s^2\,ds^2 = 4y\,(dx^2 + dy^2)$,
- Derive EoM

$$\frac{dx}{dy} = \frac{\sqrt{1-4y}}{2\sqrt{y}}, \tag{2}$$

- Solution using $\theta = \arcsin 2\sqrt{y}$ (this is a cycloid)

$$\begin{aligned} 8x &= 2\sin\theta\cos\theta + 2\theta = \sin 2\theta + 2\theta, && (3)\\ 8y &= 2\sin^2\theta = 1 - \cos 2\theta && (4) \end{aligned}$$

The tautochrone problem: Time under uniform gravity

- Since gravity acts vertically downwards, $F \sim g \sin\theta$
- Assume that the position of the particle along $s(t)$ is described by the force

$$\frac{d^2 s}{dt^2} = -\omega^2 s \tag{5}$$

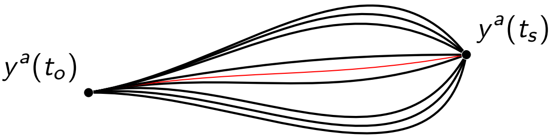
- Newtonia's second law of motion detects that $-g\sin\theta = -\omega^2 s$.
- This can be written as $ds = \frac{g}{\omega^2}\cos\theta \, d\theta$
- Perform change of coordinates $ds = dx/\cos\theta$ and $ds = dy/\sin\theta$
- The path and time under this condition becomes

$$x = r(2\theta + \sin 2\theta) \tag{6}$$

$$y = r(1 - \cos 2\theta) \tag{7}$$

$$T = \pi\sqrt{\frac{r}{g}} \tag{8}$$

Generalisation of the idea: Concept of action



Physical meaning

- Set of principles
  1. Principle of least action (states that the configurations $y(t)$ that are actually realised are those that extremise the action)
  2. Principle of gauge invariance (coordinate invariance)
  3. Principle of relativity (requirement that the equations describing the laws of physics have the same form in all admissible frames of reference)

Concept of Functional Derivative

- A function takes an input(a number) and pops out an output(a number) (e.g $y[x] = mx + c$) .
- A functional takes takes as an input a function(and its smooth derivatives) and gives out a number.
- Consider a definite integral

$$J[y] = \int_{x_1}^{x_2} L(x, y, y', y'', \cdots, y^{(n)}) \mathrm{d}^n x \tag{9}$$

Definition of Functional derivative and generalized coordinates

- where L depends on the value of $y(x)$ and finite number its derivatives $y(n) = d^{(n-1}y/d^n x$.
- Locality: $J[y]$ is called local, if a small change in x leaves the $J[y]$ unchanged.
- To find a a function that maximizes or minimizes a given functional $J(y)$, we need to define and evaluate its functional derivative.

Functional derivatives: Maximum two derivatives

- Consider making an infinitesimal change

$$y(x) \to y(x) + \varepsilon\eta(x) \tag{10}$$

The functional changes as

$$\begin{aligned}
\frac{\delta J}{\delta y(x)} &= \int_{x_1}^{x_2} \left[L(x, y+\varepsilon\eta, y'+\varepsilon\eta') - L(x, yy') \right] \mathrm{d}x \qquad (11) \\
&= \left[(\varepsilon\eta) \frac{\partial L}{\partial y'} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} (\varepsilon\eta(x) \left\{ \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) \right\} \mathrm{d}x \cdots (12)
\end{aligned}$$

- Fixed Point: if $\eta(x_1) = \eta(x_2) = 0$, $[\cdots]_{x_2}^{x_1}$ vanishes
- where

$$\frac{\delta J}{\delta y(x)} \equiv \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) \tag{13}$$

- $\delta J/\delta y(x)$ is called the functional derivative derivative of J with respective to $y(x)$.

Euler-Lagrange equation: Principle of least action

- Principle of stationary action: The path of a particle is the one that yields a stationary value of the action
- By the principle of stationary action(least action), $\delta J/\delta y(x)$ must vanish

$$\frac{\partial L}{\partial y(x)} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = 0\,, \qquad x_1 \leq x \leq x_2 \tag{14}$$

This is called the Euler- Lagrange equations.

- If L depends on more than one function, y_i, the J is stationary under all possible variations

$$\frac{\delta J}{\delta y_i(x)} = \frac{\partial L}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y_i}\right) = 0 \tag{15}$$

for each $y_i(x)$(like in the case of multi-fields inflation)

Euler-Lagrange equation: Greater than two derivatives

- If f depends on higher derivatives y'', $y^{(3)}$, etc. then we have to integrate by parts more times

$$\frac{\delta J}{\delta y_i(x)} = \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right) - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial L}{\partial y'''}\right) = 0 \quad (16)$$

- c.f higher derivative gravity theories like the f(R) theories or Beyond Horndeski theory.
- In classical Physics notations

$$J[y] = \int_{x_1}^{x_2} L(x, y, y', y'')\mathrm{d}x \rightarrow S[q] = \int_{\tau_1}^{\tau_2} L(\tau, q, q', q'')\mathrm{d}\tau \quad (17)$$

where L is the Lagrangian.

Few examples

- Shortest path between two points in 2D
- Dynamics of a real scalar field on a curved background
- Dynamics of a vector field on a curved background
- Dynamics of spacetime

Shortest path between two points:

- Consider a rela value function $y(x)$ on the interval $[a, b]$ such that $y(a) = c$ and $y(b) = d$ for which the path length along the curve traced by y is as short as possible

$$S = \int_a^b \sqrt{\mathrm{d}^2 x + \mathrm{d}^2 y} = \int_a^b \sqrt{1 + (y')^2}\mathrm{d}x \tag{18}$$

- The integrand

$$L(z, y, y') = \sqrt{1 + (y')^2} \tag{19}$$

- The partial derivative of L over

$$\frac{\partial L(z, y, y')}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}} \qquad \frac{\partial L(z, y, y')}{\partial y} = 0 \tag{20}$$

- HEnce

$$\frac{\partial L}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y_i}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{y'}{\sqrt{1 + (y')^2}} = 0 \tag{21}$$

- Performing the integration leads to a straight line $y(x) = Ax + B$

Examples:Scalar field theory

Local field theory of a real scalar field $\Phi(x^i)$ defined on the spacetime

$$L = \frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - V(\Phi) \tag{22}$$

where $\partial_\mu = \nabla_\mu$ for scalars.

$$\frac{\partial L}{\partial\Phi} = -\frac{\partial V}{\partial\Phi} \tag{23}$$

$$\frac{\partial L}{\partial(\nabla_\mu\Phi)} = \frac{\partial}{\partial(\nabla_\mu\Phi)}\left[\frac{1}{2}g^{\rho\sigma}\nabla_\rho\Phi\nabla_\sigma\Phi\right] = g^{\mu\nu}\nabla_\nu\Phi \tag{24}$$

The EOM becomes

$$\nabla_\mu\nabla^\mu\Phi + \frac{\mathrm{d}V}{\mathrm{d}\Phi} = 0 = \Box^2 + \frac{\mathrm{d}V}{\mathrm{d}\Phi} \tag{25}$$

where $\Box^2 = \nabla_\mu\nabla^\mu$. A simple choice for the potential is $V = m^2\Phi^2/2$ where m is a constant parameter.

$$\nabla_\mu\nabla^\mu\Phi + m^2\Phi = 0 \tag{26}$$

This is a well-known Klien-Gordon equation.

Vector field

- The Maxwell's action is given by

$$S = \int \mathcal{L} \mathrm{d}^4 x = \int_R \left[-\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right] \sqrt{-g} \mathrm{d}^4 x \tag{27}$$

- where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, j^μ is the four-current and A is the four-potential.
- Derive the equation of motion. The EL equation

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \nabla_\mu \left[\frac{\partial \mathcal{L}}{\partial (\nabla_\mu A_\nu)} \right] = 0 \tag{28}$$

- Take derivative of $\mathcal{L}$ wrt A_μ

$$\frac{\partial \mathcal{L}}{\partial A_\nu} \quad = \quad -j^\mu \delta^\nu{}_\mu = -j^\nu \tag{29}$$

Take derivative of $\mathcal{L}$ wrt $\nabla_\mu A_\nu$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \nabla_\mu A_\nu} &= \frac{\partial}{\partial \nabla_\mu A_\nu}\left[-\frac{1}{4\mu_o} g^{\alpha\rho} g^{\beta\sigma} F_{\rho\sigma} F_{\alpha\beta}\right] \qquad (30) \\
&= -\frac{1}{4\mu_0}\left(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}\right) F_{\alpha\beta} - \frac{1}{4\mu_0}\left(g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma}\right) F_{\rho\sigma} \\
&= -\frac{1}{4\mu_0}\left(F^{\mu\nu} - F^{\nu\mu}\right) - -\frac{1}{4\mu_0}\left(F^{\mu\nu} - F^{\mu\nu}\right) \qquad (31)
\end{aligned}
$$

The EL gives

$$
\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu \qquad (32)
$$

The anti-symmetry property (Bianchi identity)

$$
\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0 \qquad (33)
$$

Gravity

The simplest non-trivial scalar that can be constructed from the metric and its derivatives is the Ricci R

$$S_{EH} = \int_{\mathcal{R}} R\sqrt{-g}\mathrm{d}^4x \tag{34}$$

Where $g_{\mu\nu}$ is a dynamical field. EL

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \partial_\sigma \left[\frac{\partial \mathcal{L}}{\partial(\partial_\sigma g_{\mu\nu})}\right] + \partial_\rho\partial_\sigma \left[\frac{\mathcal{L}}{\partial(\partial_\rho\partial_\sigma g_{\mu\nu})}\right] = 0 \tag{35}$$

This is a non-trivial action to vary, we shall learn how to use the Mathematica to vary it.

Hamiltonian formulation of the dynamics

History

- This was formulated by William Rowan Hamilton in 1833. An Irish Mathematician. It contributed to the formulation of
  - Statistical mechanics
  - Quantum mechanics
- It is equivalent to the Newton i's laws of motion in the frame of classical mechanics
- However, the time evolution is obtained by computing the *Hamiltonian* of the system in the generalized coordinates and inserting it into Hamilton's equations
- Hamilton's equations consist of 2n first-order differential equations, while Lagrange's equations consist of n second-order equations.

Time evolution

- It is described by a set of canonical of coordinates $\boldsymbol{r} = (\boldsymbol{q}, \boldsymbol{p})$, where q_i, p_i are compoeenents of coordinates indexed to the frame of reference of the system.
  - q_i generalized coordinates
  - p_i conjugate momenta
- The time evolution of the system is uniquely defined by the Hamilton's equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q} \tag{36}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = +\frac{\partial \mathcal{H}}{\partial p} \tag{37}$$

- where $\mathcal{H} = \mathcal{H}(q, p, t)$ is the Hamiltonian which often corresponds to the total energy of the system. For a closed system, it corresponds to the sum of the Kinetic and potential energy int he system.

$$\mathcal{H} = T + V\,, \qquad T = \frac{p^2}{2m}\,, \qquad V = (q) \tag{38}$$

Relationship between Lagrangian and Hamiltonian formulation

- Hamilton's equations can be derived from Lagrangian by total differentia wrt time

$$\mathrm{d}\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q^i} \mathrm{d}q^i + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \mathrm{d}\dot{q}^i \right) + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \tag{39}$$

- Lets define the generalized momenta were defined as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \tag{40}$$

- Substituting, re-writing and making use of the product rule gives

$$\mathrm{d}\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q^i} \mathrm{d}q^i + \mathrm{d}\left(p_i \dot{q}^i \right) - \dot{q}^i \mathrm{d}p_i \right) + \frac{\partial \mathcal{L}}{\partial t} \mathrm{d}t \tag{41}$$

Relationship between Lagrangian and Hamiltonian formulation

- Move the second term in the first bubble to the LHS gives

$$\mathrm{d}\left(\sum_i p_i \dot{q}^i - \mathcal{L}\right) = \sum_i \left(-\frac{\partial \mathcal{L}}{\partial q^i}\mathrm{d}q^i + \dot{q}^i \mathrm{d}p_i\right) - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \tag{42}$$

- The term on the LHS is just the Hamiltonian

$$\mathcal{H} = \sum_i \dot{q}^i p_i - \mathcal{L} \tag{43}$$

- This is also known as Legendre transformation of the Lagrangian.
- Putting equation (43) in equation (42)

$$\mathrm{d}\mathcal{H} = \sum_i \left(-\frac{\partial \mathcal{L}}{\partial q^i}\mathrm{d}q^i + \dot{q}^i \mathrm{d}p_i\right) - \frac{\partial \mathcal{L}}{\partial t}\mathrm{d}t \tag{44}$$

Relationship between Lagrangian and Hamiltonian formulation

- Noting that $\mathcal{H} = \mathcal{H}(q, \dot{q}, t)$ we can take its total differential

$$\mathrm{d}\mathcal{H} = \sum_i \left(\frac{\partial \mathcal{H}}{\partial q^i} \mathrm{d}q^i + \frac{\partial \mathcal{H}}{\partial p_i} \mathrm{d}p_i \right) + \frac{\partial \mathcal{H}}{\partial t} \mathrm{d}t \tag{45}$$

- Comparing equation (44) to equation (45) gives

$$\frac{\partial \mathcal{H}}{\partial q^i} = -\frac{\partial \mathcal{L}}{\partial q^i} \quad , \quad \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}^i \quad , \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{46}$$

- Using the Euler-Lagrangian equation to define $\dot{p}_i$ gives the Hamiltons equations

$$\frac{\partial \mathcal{H}}{\partial q^j} = -\dot{p}_j \quad , \quad \frac{\partial \mathcal{H}}{\partial p_j} = \dot{q}^j \quad , \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{47}$$

Lessions from the Hamiltons formulation

- Hamilton's equations consist of 2n first-order differential equations, while Lagrange's equations consist of n second-order equations.
- The coordinates and momenta are independent variables with nearly symmetric roles.
- Links for conservation equation and symmetry: if a system has a symmetry, such that a coordinate does not occur in the Hamiltonian, the corresponding momentum is conserved.
- In a conserved system, the the problem reduces from n coordinates to $(n-1)$ coordinates.

Examples

- Example 1: Consider a spherical pendulum of mass m, moving without friction on the surface of a sphere. The only surface acting on the mass are the reaction from the sphere and the gravity. The Lagrangian of the system is given gy

$$\mathcal{L} = \frac{1}{2} ml^2 \left(\dot{\theta}^2 + \sin^2\theta \; \dot{\phi}^2 \right) + mgl\cos\theta. \tag{48}$$

- Solution: We work in spherical coordinates $\boldsymbol{r} = r(r, \theta, \phi) = r(l, \theta, \phi)$, where l is fixed. The Hamiltonian of the system maybe obtained from the Lagrangian

$$\mathcal{H} = p_\theta \dot{\theta} + p_\phi \dot{\phi} - \mathcal{L} \tag{49}$$

Using $p_i = \partial\mathcal{L}/\partial\dot{q}^i$, we compute the components

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} \tag{50}$$

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = ml^2\sin^2\theta\,\dot{\phi} \tag{51}$$

Solution

- The Hamiltionain becomes

$$H = \underbrace{\left[\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi}^2\right]}_{T} + \underbrace{\left[-\,mgl\cos\theta\right]}_{V} \tag{52}$$

$$= \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta \tag{53}$$

- Now compute the Hamilton equations

$$\frac{\partial\mathcal{H}}{\partial q^j} = -\dot{p}_j \quad , \quad \frac{\partial\mathcal{H}}{\partial p_j} = \dot{q}^i \tag{54}$$

- In components

Solution

- Hamilton equations in components

$$\begin{aligned}
\dot{\theta} &= \frac{p_\theta}{ml^2} && (55)\\
\dot{\phi} &= \frac{p_\phi}{ml^2\sin^2\theta} && (56)\\
\dot{p}_\theta &= \frac{p_\phi^2}{ml^2\sin^3\theta}\cos\theta - mgl\sin\theta && (57)\\
\dot{p}_\phi &= 0 && (58)
\end{aligned}$$

- where Momentum p_ϕ, which corresponds to the vertical component of angular momentum, $L_z = l\sin\theta \times ml\sin\theta\,\dot{\phi}$ is a constant of motion.
- That is a consequence of the rotational symmetry of the system around the vertical axis. Being absent from the Hamiltonian, azimuth ϕ is a cyclic coordinate , which implies conservation of its conjugate momentum.

Noether Theorem and Conservation laws

Noether theorem: First integral

- When f is of the form $f(y,y')$ i.e has no explicit dependence on x,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \cancelto{0}{\frac{\partial f}{\partial x}} \tag{59}$$

We see that

$$\frac{d}{dx}\left(f - y'\frac{\mathrm{d}f}{dy'}\right) = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} - y''\frac{\partial f}{\partial y'} - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) \tag{60}$$

$$= y'\left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] \tag{61}$$

This is zero if EL equation is satisfied.

$$I \equiv f - y'\frac{\partial f}{\partial y'} = f - \sum_i y'_i \frac{\partial f}{\partial y'_i} \tag{62}$$

Noether theorem:Symmetry and Conservation law

- The time-independence of the first integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right] = 0 = \frac{\mathrm{d}I}{\mathrm{d}t} \tag{63}$$

- Noether showed that they are related to the underlying symmetry of of the system.
- Stated differently, for each symmetry of the Lagrangian, there is a conserved quantity

$$\text{Symmetry} \Leftrightarrow \text{Conservation law} \tag{64}$$

- where I is the constant of motion

$$I = L - \dot{q}\frac{\partial L}{\partial \dot{q}} \tag{65}$$

Examples: Angular momentum conservation

The action integral for the central force problem is given by

$$S = \int_0^T \left[\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\Theta}^2\right) - V(r)\right] \mathrm{d}t \tag{66}$$

The integral is left unchange, if we make the variation

$$\Theta(t) \to \Theta(t) + \varepsilon\alpha \tag{67}$$

Noether observed that S is stationary under the specific variation

$$\Theta(t) \to \Theta(t) + \varepsilon(t)\alpha \tag{68}$$

where $\varepsilon(t)$ is now allowed to be time-dependent. Action variation gives

$$\delta S = \alpha \int_0^T \left[mr^2\dot{\Theta}\right] \dot{\varepsilon}\mathrm{d}t \tag{69}$$

Since $\delta S = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\Theta}\right) = 0 \tag{70}$$

rotationally invariant implies angular momentum conservation.

Examples: Energy conservation

A system is invariant under time-translation:

$$t \to t + \varepsilon\,, \qquad q(t) \to q(t+\varepsilon) = q(t) + \varepsilon\dot{q} \tag{71}$$

The variation of the action leads to

$$\delta S = \int_0^T \left[\frac{\partial L}{\partial q}\dot{q}\varepsilon + \frac{\partial L}{\partial \dot{q}}\left(\ddot{q}\varepsilon + \dot{q}\dot{\varepsilon}\right)\right] \tag{72}$$

Absence of any explicit time dependence in L and Integration by parts and applying fixed point criterion leads to leads to

$$\delta S = \int_0^T \varepsilon(t)\frac{\mathrm{d}}{\mathrm{d}t}\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right]\mathrm{d}t \tag{73}$$

Now assume that $q(t)$ obeys the EOM,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right] = 0\,, \quad \text{where} \quad E = L - \frac{\partial L}{\partial \dot{q}}\dot{q} \tag{74}$$

The time-translation invariant implies conservation of energy.

Noether Theorem:Harmonic oscillator

- The total energy of a a simple Harmonic osciallator is given by

$$E(x,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 \tag{75}$$

Where $\dot{x} = p$ and $\dot{p} = -\omega^2 x$

$$\frac{dE(x(t),p(t))}{dt} = p\dot{p} + \omega^2 \dot{x} \tag{76}$$

$$= p(-\omega^2 x) + \omega^2 x p = 0 \tag{77}$$

- the energy of a simple harmonic oscillator is a first integral.

The energy-momentum tensor I

If we consider an action of the general form

$$S = \int \mathcal{L}(\Psi, \Phi_{\mu}) \mathrm{d}^{d+1}x \tag{78}$$

where $\mathcal{L}$ does not depend explcitly on any of the coordinates x^{μ}

$$\Psi(x^{\mu}) \to \Psi\left(x^{\mu} + \varepsilon^{\mu}(x)\right) = \Psi(x^{\mu}) + \varepsilon^{\mu}(x)\partial_{\mu}\Psi \tag{79}$$

The resulting variation leads to

$$\delta S = \int \varepsilon^{\mu}(x) \frac{\partial}{\partial x^{\nu}} \left[\mathcal{L}\delta^{\nu}{}_{\mu} - \frac{\partial \mathcal{L}}{\partial \Psi_{\nu}} \partial_{\mu}\Psi\right] \mathrm{d}^{d+1}x \tag{80}$$

where Ψ satisfies the equation of motion

$$\frac{\partial}{\partial x^{\nu}} \left[\mathcal{L}\delta^{\nu}{}_{\mu} - \frac{\partial \mathcal{L}}{\partial \Psi_{\nu}} \partial_{\mu}\Psi\right] = 0 \tag{81}$$

The energy-momentum tensor II

Thus we define the canonical energy-momentum tensor as

$$T^{\nu}{}_{\mu} \;=\; \mathcal{L}\delta^{\nu}{}_{\mu} - \frac{\partial \mathcal{L}}{\partial \Psi_{\nu}} \partial_{\mu}\Psi \tag{82}$$

And the conservation law

$$\nabla_{\nu} T^{\nu}{}_{\mu} = 0 \tag{83}$$

Conclusion and further reading

- Watch this programme on why the conservation of energy breaks down in an expanding spacetime :http://backreaction.blogspot.com/2020/10/what-is-energy-is-energy-conserved.html
- Read up on variational calculus: http://galileoandeinstein.physics.virginia.edu/7010/CM-02-CalculusVariations.html