Mathematical Methods for Graduate Students in Physics

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Linear first- order ordinary differential equations

- The general form of the linear first order ODE is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = q(x) \tag{1}$$

- It has a unique result via the integrating factor

$$\alpha(x) = \exp\left[\int^{x} P(x)\mathrm{d}x\right] \tag{2}$$

- The general solution is given by

$$\begin{aligned} y(x) &= \frac{1}{\alpha(x)}\int \alpha(x)q(x)\mathrm{d}x + C & (3)\\ &= \exp\left[-\int^{x} p(t)\mathrm{d}t\right]\int^{x} \exp\left[\int p(t)\mathrm{d}t\right] q(s)\mathrm{d}s + C & (4) \end{aligned}$$

A system Linear first- order ordinary differential equations

Consider a system of first order DE

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = X^2\left(x^i, x^2, \cdot x^n, t\right) \quad (5)$$

$$\frac{\mathrm{d}x^2}{\mathrm{d}t} = X^2\left(x^i, x^2, \cdot x^n, t\right) \quad (6)$$

$$. \quad (7)$$

$$. \quad (8)$$

$$. \quad (9)$$

$$\frac{\mathrm{d}x^n}{\mathrm{d}t} = X^n\left(x^i, x^2, \cdot x^n, t\right) \quad (10)$$

Now introduce a single function $y(t)$, and set
$x^1 = y, x^2 = \dot{y}, x^3 = \ddot{y} \cdots x^n = y^{(n-1)}$

From a system of Ist order ODE to n-th order ODE

Given a set of smooth functions $\{p_0(x), p_1(x), \cdots p_n(x)\}$ which are no where vanishing and consider a particular system

$$
\begin{aligned}
\frac{dx^1}{\mathrm{d}t} &= x^2 && (11)\\
\frac{\mathrm{d}x^2}{\mathrm{d}t} &= x^3 && (12)\\
&. && (13)\\
&. && (14)\\
&. && (15)\\
\frac{\mathrm{d}x^n}{\mathrm{d}t} &= -\frac{1}{p_0}\left[p_1 x^n + p_2 x^{n-1} + \cdots p_n x^1\right] && (16)
\end{aligned}
$$

n-th order ODE

1. This may be expressed as an n-th order ODE

$$P_0(t)\frac{d^2y}{dt^2} + P_1(t)\frac{\mathrm{d}^{n-1}}{dt^{n-1}} + \cdots P_{n-1}\frac{dy}{\mathrm{d}t} + P_n(t)y(t) = 0 \qquad (17)$$

2. This n-th order ODE can be written as a first order in n-dimensions.
3. Provided P_0 never vansishes, the differentials equation $Ly = 0$ has a unique solution $y(t)$, for each of the initial data $(y(0), \dot{y}(0), \ddot{y}(0), \cdots y^{(n-1)}(0)$.

Normal form

- In elementary algebra a polynomial equation

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{18}$$

- with $a_0 \neq 0$ is said to be in the normal form if $a_1 = 0$.
- If it wasn't originally in the normal form, it can always be put in the normal form by defining a variable $\bar{x}$ with

$$x = \bar{x} - a_1 (n a_0)^{-1} \tag{19}$$

- By analogy an n-th order linear ODE with no $y^{(n-1)}$ terms is also said to be in a normal form.
- Putting an ODE in a normal form is done by substitution

$$y = \omega \tilde{y} \tag{20}$$

for a suitable function $\omega(x)$.

- Let

$$P_0(x) y^{(n)} + P_1(x) y^{(n-1)} + \cdots + P_n(x) y = 0 \tag{21}$$

- Set $y = \omega \tilde{y}$, using Leibniz rule

$$(\omega \tilde{y})^{(n)} = \omega \tilde{y}^{(n)} + n\omega' \tilde{y}^{(n-1)} + \frac{n(n-1)}{2!} \omega'' \tilde{y}^{(n-2)} + \cdots + \omega^{(n)} \tilde{y} \tag{22}$$

The differential equation becomes

$$(\omega p_0) \tilde{y}^{(n)} + \left(p_1 \omega + p_0 n \omega' \right) \tilde{y}^{(n-1)} + \cdots = 0 \tag{23}$$

- If we choose ω to be a solution of

$$p_1\omega + p_0 n\omega' = 0 \tag{24}$$

- we have

$$w(x) = \exp\left[-\frac{1}{n}\int_0^x \left(\frac{p_1(\xi)}{p_0(\xi)}\right) \mathrm{d}\xi\right] \tag{25}$$

With no second highest derivative.

Example

Given

$$y'' + p_1 y' + p_2 y = 0 \tag{26}$$

Set

$$y(x) = V(x) \exp\left[-\frac{1}{2}\int^{x} p_1(\xi)\mathrm{d}\xi\right] \tag{27}$$

We find that V obeys

$$V'' + \Omega V = 0 \tag{28}$$

where

$$\Omega = p_2 - \frac{1}{2}p_2' - \frac{1}{4}p_1^2 \tag{29}$$

Reducing an equation to it normal form gives us the best chance of solving it by inspection

- `Physicsts:` Schrodinger equation

$$-\frac{d^2\Psi}{dx^2} + (V(x) - E)\,\Psi = 0 \tag{30}$$

- `Ex:` `Inflation`: The classical Klien-Gordon equation for the inflation field on an expanding spacetime is given by

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \frac{k^2}{a^2}\chi_k = 0 \tag{31}$$

where χ is a scalar field promoted to the status of an operator, H is the Hubble paprameter $(H = da/(a d\tau))$. Show that it may be put in the normal form as

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0 \tag{32}$$

where $'$ denotes the conformal time $d\tau = a d\eta$.

Linear Independence

$$P_0(t)\frac{d^2y}{dt^2} + P_1(t)\frac{\mathrm{d}^{n-1}}{dt^{n-1}} + \cdots P_{n-1}\frac{dy}{\mathrm{d}t} + P_n(t)y(t) = 0 \tag{33}$$

Consider a of n continuous functions $y_i(x)]\,[i = 1, 2, 3, \cdots, n]$, each of which is differentiable at least n times. Then if there exist a set of constants λ_i that are not all zero such that

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n = 0 \tag{34}$$

then we say that the set of functions $\{y_i(x)\}$ is linear dependent.
If the only solution to (33) is $\lambda_i = 0$ for all i, then the set of functions $\{y_i(x)\}$ are linearly independent.

Consider the second ODE

$$y'' + a(x)y' + b(x)y = 0 \tag{35}$$

Let y_1 and y_2 be the solution

$$W = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2 \neq 0 \tag{36}$$

Taking the derivative

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(y_1 y_2' - y_1' y_2\right) = y_1 y_2'' - y_1'' y_2 \tag{37}$$

Since the terms proportional to $y_1' y_2'$ exactly cancel. Using the fact that y_1 and y_2 are solutions to eqution (35)

$$\begin{aligned} y_1'' + a(x)y_1' + b(x)y_1 &= 0 && (38) \\ y_2'' + a(x)y_2' + b(x)y_2 &= 0 && (39) \end{aligned}$$

Multiply equation (38) by y_2 and multiply y_1 (39)

$$y_1 y_2'' - y_1'' y_2 + a(x)\left[y_1 y_2' - y_1' y_2\right] = 0 \tag{40}$$

or

$$\frac{\mathrm{d}W}{\mathrm{d}x} + a(x)W(x) = 0 \tag{41}$$

Finally, the Wronskian is given by

$$W(x) = C\exp\left[-\int^{x} a(x')\mathrm{d}x'\right] \tag{42}$$

Applications of Wronskian