Mathematical Methods for Graduate Students in Physics

Obinna Umeh

Institute of Cosmology & Gravitation,
University of Portsmouth,
Portsmouth PO1 3FX, United Kingdom

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Recall previous lecture

- Given a Lagrangian. $L = L(x, y, y', y'', y''' \cdots)$

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right) - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial L}{\partial y'''}\right) + \cdots = 0 \tag{1}$$

- Leads to n-order differential equation.
- The Hamiltonian formulation: Legendre transformation

$$\mathcal{H} = \sum_i \dot{q}^i p_i - L \tag{2}$$

- The Hamiltons equations: For a close system $H = E$

$$\frac{\partial \mathcal{H}}{\partial q^j} = -\dot{p}_j \quad , \quad \frac{\partial \mathcal{H}}{\partial p_j} = \dot{q}^j \quad , \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{3}$$

- Reduces the (n-1)-order differential equation.

Symmetry and Conservation law (Noether theorem)

- If L has no explicit dependence on ($L \neq L(x)$; , you can derive

$$\frac{d}{dx}\overbrace{\left[L - y'\frac{\mathrm{d}L}{dy'}\right]}^{\text{CoM}} = y'\left[\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)\right] , \tag{4}$$

- Say $x = x(t, r, \theta, \phi)$
  - t: time translation invariance, conservation of energy
  - r: all locations are equivalent. linear momentum conservation.
  - θ: rotational symmetry, angular momentum conservation.
  - ϕ: azimuthal symmetry, angular momentum conservation.
  - Energy conservation in FLRW space time? $E \sim 1/a$
  - Symmetric under spatial inversion, time inversion, and particle inversion?
- In terms of classical fields $y = \Phi, A_\mu, g_{\mu\nu}$

$$T^{\nu}{}_{\mu} \;=\; L\delta^{\nu}{}_{\mu} - \frac{\partial L}{\partial_\nu \Phi}\partial_\mu \Phi \tag{5}$$

It leads to energy conservation law $\nabla_\nu T^{\nu}{}_{\mu} = 0$

Euler-Lagrange equations

- Product of varying an action

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial L}{\partial y'''}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial L}{\partial y''}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right)+\frac{\partial L}{\partial y}=0 \tag{6}$$

- How do you find a set of solutions to this equation.

Differential equations: Linear differential equations

- Linear differential equations

$$P_n(x)\frac{d^n y(x)}{dx^n} + \quad P_{n-1}(x)\frac{\mathrm{d}^{n-1}}{dx^{n-1}}y(x) \\ + \cdots P_1(x)\frac{dy(x)}{\mathrm{d}x} + P_0(t) = g(x) \tag{7}$$

- The solution is a linear combination of derivatives of y.

$$y^{(n)} = \sum_{i=0}^{n-1} a_i(x) y^{(i)} + r(x) \tag{8}$$

- $r(x)$ is the part of the solution activated by the source term $g(x)$
  - Homogeneous equation if $r(x) = 0$
  - Inhomogeneous equation if $r(x) \neq 0$

- Nonlinear first order DE (Abel equation)

$$\frac{dy}{dx} = f_o(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 \tag{9}$$

- Nonlinear second order DE (Rayleigh-Plesset equation)

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + \frac{4\nu}{R}\frac{dR}{dt} + \frac{2\gamma}{\rho R} + \frac{\Delta P(t)}{\rho} = 0 \tag{10}$$

- Very few methods of solving nonlinear differential equations exactly; those that are known typically depend on the equation having particular symmetries.
- If the differential equation is a correctly formulated representation of a meaningful physical process, then one expects it to have a solution

Autonomous differential equation

- Autonomous differential equation

$$\frac{d^n y(x(t))}{dt^n} = g(y^{n-1}(x(t)) \cdots y(t)) \tag{11}$$

- It does not explicitly depend on the independent variable. In this case, the variable is time, they are also called time-invariant systems.
- Very important in dynamical system analysis in cosmology.

Partial differential equation

- Black-Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{12}$$

- Atimes it is easier to solve by change of variables

$$V(S,t) = Kv(x,\tau)\,, \qquad x = \ln\left(\tfrac{S}{K}\right) \tag{13}$$

$$\tau = \tfrac{1}{2}\sigma^2(T-t)\,, \qquad v(x,\tau) = e^{-\alpha x - \beta\tau}u(x,\tau) \tag{14}$$

- to obtain a simpler form (Heat equation)

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{15}$$

Linear first-order ordinary differential equations

- The general form of the linear first order ODE is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = q(x) \tag{16}$$

  where P and q to be continuous functions.

- Separable limit $P = 0$ (Linear separable equations)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = q(x) \tag{17}$$

- Solution in the Separable limit

$$y = \int q(x) + c \tag{18}$$

Linear first- order ordinary differential equations

- The general form of the linear first order ODE is given by

$$\frac{dy}{dx} + P(x)y = q(x) \tag{19}$$

where P and q are continuous functions.

- The integrating factor

$$\alpha(x) = \exp\left[\int^x P(x)dx\right] \tag{20}$$

- The general solution is given by

$$\begin{aligned} y(x) &= \frac{1}{\alpha(x)}\int \alpha(x)q(x)dx + C \qquad (21)\\ &= \exp\left[-\int^x p(t)dt\right]\int^x \exp\left[\int p(t)dt\right] q(s)ds + C \qquad (22)\end{aligned}$$

Solution to differential equation

- Exercise: Find the solution to the following IVP

$$2y' - y = 4\sin(3t) \qquad y(0) = y_0 \tag{23}$$

- Soln: Divide by 2: $y' - \frac{1}{2}y = 2\sin(3t)$
- Find IF: $\alpha(t) = e^{-\int \frac{1}{2}\mathrm{d}t} = e^{-t/2}$
- The solution becomes

$$\begin{aligned} y(t) &= e^{\frac{t}{2}} \int 2e^{-\frac{t}{2}} \sin(3t)\mathrm{d}t + c & (24) \\ &= -\frac{24}{37}e^{-\frac{t}{2}}\cos(3t) - \frac{4}{37}e^{-\frac{t}{2}}\sin(3t) + ce^{\frac{t}{2}} & (25) \end{aligned}$$

- Apply initial conditions $y_0 = y(0) = -24/37 + c \to y = y_0 + 24/37$
- Final solution

$$y(t) = -\frac{24}{37}e^{-\frac{t}{2}}\cos(3t) - \frac{4}{37}e^{-\frac{t}{2}}\sin(3t) + \left(y_0 + \frac{24}{37}\right)e^{\frac{t}{2}} \tag{26}$$

Exercise

1. Solve for initial value problem v

$$\frac{dv}{dt} = 9.8 - 0.196v\,, \qquad v(0) = 48 \tag{27}$$

2. Solve for initial value problem y

$$ty' - 2y = t^5 \sin(2t) - t^3 + 4t^4\,, \qquad y(\pi) = \frac{3}{2}\pi^4 \tag{28}$$

n-th order ODE to n-number of first order ODE

1. This may be expressed as an n-th order ODE

$$P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{\mathrm{d}^{n-1}y}{dt^{n-1}} + \cdots P_1\frac{dy}{\mathrm{d}t} + P_0(t)y(t) = g(t) \tag{29}$$

2. This n-th order ODE can be written as a first order in n-dimensions.
3. Provided P_n never vansishes, the differentials equation $\mathcal{L}y = g(t)$ has a unique solution $y(t)$, for each of the initial data $(y(0), \dot{y}(0), \ddot{y}(0), \cdots y^{(n-1)}(0)$.
4. where $\mathcal{L}$ is the differential operator

$$\mathcal{L} = P_n(t)\frac{d^n}{dt^n} + P_{n-1}(t)\frac{\mathrm{d}^{n-1}}{dt^{n-1}} + \cdots P_1\frac{d}{\mathrm{d}t} + P_0(t) \tag{30}$$

Example and Exercise

- Given $y''' - 3y'' - 2y = t$ with I.C $y(o) = 1, y(0) = 2, y(0) = 3$: Soln

$$\begin{aligned}
x_1 &= y \qquad & x_1' = x_2 \qquad (31)\\
x_2 &= y' \qquad & x_2' = x_3 \qquad (32)\\
x_3 &= y'' \qquad & x_3' = t + 2y + 3y'' = t + 2x_1 + 3x_3 \qquad (33)
\end{aligned}$$

- With $y(0) = x_1(0) = 1, y'(0) = x_2(0) = 2, y''(0) = x_3(0) = 3$
- Exercise: Given $4y'''' + ty'' - 3' = 0$ with IC $y(1) = 2, y'(0) = 3, y''(0) = 6, y'''(0) = 4$
- Soln

$$\begin{aligned}
x_1 &= y \qquad & x_1' = x_2 \qquad (34)\\
x_2 &= y' \qquad & x_2' = x_3 \qquad (35)\\
x_3 &= y'' \qquad & x_3' = x_4 \qquad (36)\\
x_4 &= y''' \qquad & x_4' = \frac{3}{2}y' - \frac{t}{4}y'' = \frac{3}{4}x_2 - \frac{t}{4}x_3 \qquad (37)
\end{aligned}$$

Going beyond first order differential equation

Homogeneous 2nd order ODE with constant coefficients

Starting with a linear homogeneous differential equation

$$p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = 0 \tag{38}$$

where $p_n, p_{n-1}, \ldots, p_1, p_0$ are constant coefficients

- if $y(x) = e^{rx}$, each term would be a constant multiple of e^{rx}:

$$p_n r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0 = 0 \tag{39}$$

- 2nd order ODE leads to quadratic equation

$$p_2 r^2 + p_1 r + p_0 = 0 \tag{40}$$

- it is straight forward to find the roots of the equation

1. `Real and distinct Roots`: If the characteristic equation has distinct real number roots $r_1, \ldots, r_n$, then the complementary solution will be of the form

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x} \tag{41}$$

2. Real and repeated Roots:

$$y_c(x) = e^{r_1 x}(c_1 + c_2 x + \cdots + c_k x^{k-1}) \tag{42}$$

where k is the degree of the polynomial.

3. Complex roots If the roots are of the form $r_1 = a + bi$ and $r_2 = a - bi$, then the general solution is accordingly

$$y(x) = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x} \tag{43}$$

may be simplified further using (De Moivre's formula)

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{44}$$

this solution can be rewritten as

$$y(x) \quad = \quad (c_1 + c_2)e^{ax}\cos bx + i(c_1 - c_2)e^{ax}\sin bx \tag{45}$$

1. Find the general solution to

$$y''' + 4y'' - 7y' - 10y = 0 \tag{46}$$

solution Solve the characteristic equation

$$\lambda^3 + 4\lambda^2 - 7\lambda - 10 = 0 \tag{47}$$

with the solution $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = -5$, thus the general solution becomes

$$y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{-5t} \tag{48}$$

2. **Ex** Find the general solution to

$$y^{(4)} + 8y'' + 16y = 0 \tag{49}$$

Inhomogeneous ODE:Particular solution

1. `Method of undetermined coefficients:` It is involves making a guess of the form of the particular solution by looking at the form of the forcing function. For example

$$y'' - 4y' - 12y = 3\exp^{5t} \tag{50}$$

2. `Find the full solution` Solution:
    - $y_c = c_1 \exp^{-2t} + c_2 \exp^{6t}$
    - For the particular solution, try $y_p = A\exp^{5t}$
    - sub this in the original ODE $(25 - 20 - 12)A = 3$
    - Solve for A(ans $A = -3/7$)
    - particular solution $y_p = -3\exp^{5t}/7$.
    - The general solution becomes
      $y = y_c + y_p = c_1 \exp^{-2t} + c_2 \exp^{6t} - 3\exp^{5t}/7$

Example and exercise

- Example Find the full solution to the following ODE

$$y'' - 2y' - 3y = 3t^2 + 4t - 5 \tag{51}$$

Here the forcing function is a Quadratic polynomial and it is smooth. Hence for the y_p should have the following form

$$y_p = At^2 + Bt + C \tag{52}$$

- Exercise
Use this solution to show a possible break down of the rules and how to fix it

$$y'' - 2y' - 3y = 5\exp^{3t} \tag{53}$$

If you try $y_p = A\exp^{3t}$, the LHS leads to 0, remedy multiply by t: $y_p = At\exp^{3t}$. See table below for further guidelines

Trial solutions for the method of undetermined coefficients

	Form of $g(x)$	Guess for particular solution
1.	1 (any constant)	A
2.	$5x+7$	$Ax+B$
3.	$3x^2-2$	Ax^2+Bx+C
4.	$\sin 4x$	$A\cos 4x + B\sin 4x$
5.	$\cos 4x$	$A\cos 4x + B\sin 4x$
6.	e^{5x}	Ae^{5x}
7.	$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
8.	x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
9.	$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
10.	$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x + (E^2+Fx+G)\sin 4x$
11.	$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$
12.	$(5x+7)+\sin 4x$	$(Ax+B)+(C\cos 4x + D\sin 4x)$

Figure 1: Guideline for the method of undetermined coefficients[http://huzpibcz.ddns.net/undetermined-coefficients-212787.html]

Variation of parameters

Given a second order ODE of the form

$$y'' + a(x)y' + b(x)y = f(x) \tag{54}$$

The general solution is a sum of the complementary solution and the particular solution

$$y = c_1 y_1 + c_2 y_2 + \overbrace{v_1(x)y_1(x) + v_2(x)y_2(x)}^{\text{Particular solution}} \tag{55}$$

We solve for v_1 and v_2 using Lagrange method;take first derivative of the particular solution

$$y_p' = v_1 y_1' + v_2 y_2' + \left[v_1' y_1 + v_2' y_2\right] \tag{56}$$

Set the square bracket to zero for convenient

$$\left[v_1' y_1 + v_2' y_2\right] = 0 \tag{57}$$

Take another derivative of it and plug the result in equation (54). Since y_1 and y_2 satisfies equation equation (54), we will be left with

$$v_1' y_1' + v_2' y_2' = f(x) \tag{58}$$

Solve equation (57) and equation (58) simultaneously leads to

$$v_1' = -\frac{y_2(x) f(x)}{W(x)} \qquad v_2' = \frac{y_1(x) f(x)}{W(x)} \tag{59}$$

where W is the Wronskian

$$W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2 \neq 0 \tag{60}$$

Finally, you the general solution becomes

$$y = c_1 y_1(x) + c_2 y_2(x) - y_1 \int \frac{y_2(x) f(x)}{W(x)} \mathrm{d}x + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} \mathrm{d}x \tag{61}$$

The same procedure is easily extended to the n-th order linear differential equation .

1. Show that the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{1+x^2} \tag{62}$$

is Repeated roots $r = 1, 1$: $y_1 = e^x$ and $y_2 = xe^x$

$$y = Ae^x + Vxe^x - \frac{1}{2}e^x \ln\left(1+x^2\right) + xe^x \tan^{-1} x \tag{63}$$

2. Wronskian

$$W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \tag{64}$$

Going beyond linear ODE with constant coefficients

Normal form

- Given an ODE of the form

$$p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \tag{65}$$

- Putting an ODE in a normal form is done by substitution

$$y = \omega\tilde{y} \tag{66}$$

for a suitable function $\omega(x)$.

- With $y = \omega\tilde{y}$ and using Leibniz rule gives

$$(\omega\tilde{y})^{(n)} = \omega\tilde{y}^{(n)} + n\omega'\tilde{y}^{(n-1)} + \frac{n(n-1)}{2!}\omega''\tilde{y}^{(n-2)} + \cdots + \omega^{(n)}\tilde{y} \tag{67}$$

- Substituting in the differential equation gives

$$(\omega p_0)\tilde{y}^{(n)} + \left(p_1\omega + p_0 n\omega'\right)\tilde{y}^{(n-1)} + \cdots = 0 \tag{68}$$

- We require the coeficient of $\tilde{y}^{(n-1)}$ to vanish

$$(\omega p_0)\tilde{y}^{(n)} + \left(p_1\omega + p_0 n\omega'\right)\tilde{y}^{(n-1)} + \cdots = 0 \tag{69}$$

- If we choose ω to be a solution of

$$p_1\omega + p_0 n\omega' = 0 \tag{70}$$

- we have

$$w(x) = \exp\left[-\frac{1}{n}\int_0^x \left(\frac{p_1(\xi)}{p_0(\xi)}\right) \mathrm{d}\xi\right] \tag{71}$$

- The resulting ODE has no second highest derivative

$$(\omega p_0)\tilde{y}^{(n)} + \cdots = 0 \tag{72}$$

Example

- Given '$y'' + p_1 y' + p_2 y = 0$
- Set and differentiate

$$y(x) = V(x) \exp\left[-\frac{1}{2}\int^{x} p_1(\xi)\mathrm{d}\xi\right] \tag{73}$$

- We find that V obeys $V'' + \Omega V = 0$, where

$$\Omega = p_2 - \frac{1}{2}p_2' - \frac{1}{4}p_1^2 \tag{74}$$

- The equation now looks like the Schrodinger equation

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + (V(x) - E)\,\Psi = 0 \tag{75}$$

- Then use WKB approximation to solve or any well known approximation scheme.

Exercise

- `Ex 1: Inflation`: The classical Klien-Gordon equation for the inflation field on an expanding spacetime is given by

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \frac{k^2}{a^2}\chi_k = 0 \qquad (76)$$

where χ is a scalar field promoted to the status of an operator, H is the Hubble paprameter $(H = \mathrm{d}a/(a\mathrm{d}\tau))$. Show that it may be put in the normal form as

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0 \qquad (77)$$

where $'$ denotes the conformal time $\mathrm{d}\tau = a\mathrm{d}\eta$.

Linear Independence

- Consider an n-th order homogenous ODE

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{\mathrm{d}^{n-1}}{dt^{n-1}} + \cdots P_{n-1}\frac{dy}{\mathrm{d}t} + P_n(t)y(t) = 0 \tag{78}$$

- If there exist a set of constants λ_i that are not all zero such that

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n = 0 \tag{79}$$

- then we say that the set of functions $\{y_i(x)\}$ is linear dependent.
- If the only solution to (77) is $\lambda_i = 0$ for all i, then the set of functions $\{y_i(x)\}$ are linearly independent.

Example

Consider the second ODE

$$y'' + a(x)y' + b(x)y = 0 \tag{80}$$

Let y_1 and y_2 be the solution, $\exists$ a Wronskian

$$W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2 \neq 0 \tag{81}$$

Taking the derivative

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(y_1 y_2' - y_1' y_2\right) = y_1 y_2'' - y_1'' y_2 \tag{82}$$

Since the terms proportional to $y_1' y_2'$ exactly cancel. Using the fact that y_1 and y_2 are solutions to eqution (79)

$$\begin{aligned} y_1'' + a(x)y_1' + b(x)y_1 &= 0 & (83) \\ y_2'' + a(x)y_2' + b(x)y_2 &= 0 & (84) \end{aligned}$$

Multiply equation (82) by y_2 and multiply y_1 (83)

$$y_1 y_2'' - y_1'' y_2 + a(x)\left[y_1 y_2' - y_1' y_2\right] = 0 \tag{85}$$

or

$$\frac{\mathrm{d}W}{\mathrm{d}x} + a(x)W(x) = 0 \tag{86}$$

Finally, the Wronskian is given by

$$W(x) = C\exp\left[-\int^{x} a(x')\mathrm{d}x'\right] \tag{87}$$

From the definition of Wronskian, one can work out an algebraic identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2} \tag{88}$$

we we have made use of the definition of Wronskian. Integrating with respect to x yields

$$\frac{y_2}{y_1} = \int^x \frac{W(x)}{[y_1(x)]^2}\mathrm{d}x \tag{89}$$

The other solution is given by

$$y_2(x) = y_1(x)\left[\int^x \frac{W(x)}{[y_1(x)]^2}\mathrm{d}x + C\right] \tag{90}$$

Since y_1 is a solution, y_2 is also a solution so is $y_2 + c_1 y_1(x)$

Excercise

1. One solution to an ODE of the form
$$y'' - 2xy' = 0 \tag{91}$$
is given by $y_1 = 1$, find the other solution.
2. $W \sim C_1 e^{-2\int x \mathrm{d}x}$ and $y_2 = y_1 \left[\int^x W[x] \mathrm{d}x + C_2 \right]$
3. $y_1(x) = e^x$ solves the ODE below
$$\left(x^2 - 2x\right) y'' - (x^2 - 2)y' + (2x - 2)y = 0 \tag{92}$$
Find the other solution.

Boundary value problem

Green function approach

- Green's function is a solution to inhomogeneous differential equations

$$\mathcal{L}y(x) = f(x) \tag{93}$$

where $\mathcal{L}$ is the differential operator

$$\mathcal{L} = P_n(x)\frac{d^n}{dx^n} + P_{n-1}(x)\frac{\mathrm{d}^{n-1}}{dx^{n-1}} + \cdots P_1\frac{d}{\mathrm{d}x} + P_0(x) \tag{94}$$

- Green's function method is one of the most effective ways of solving the ODE given some **boundary conditions** $(y(\xi), y'(\xi))$.
- The approach involves first understanding how the ODE responds to a **unit impulse**

$$\mathcal{L}g(x) = \delta(x-\xi) \tag{95}$$

- where ξ is an arbitrary point of excitation of the unit impulse.
- The solution normally appears as an integral involving the Green's function $g(x, \xi)$ (or $g(x|\xi)$)

Green function approach

- The Green's function is so powerful because given the solution for $g(x|\xi)$, we can immediately solve the general problem (equation (92)) for an arbitrary $f(x)$ by writing

$$y(x) = \int_a^b g(x|\xi) f(\xi) \mathrm{d}\xi \tag{96}$$

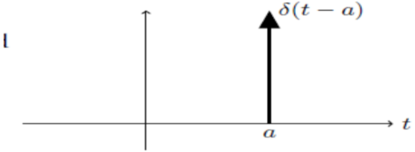
- You can see that it solves equation (92) by plugging in equation (95) in equation (92)

$$\begin{aligned} \mathcal{L}y &= \mathcal{L}\left[\int_a^b g(x|\xi) f(\xi) \mathrm{d}\xi\right] && (97) \\ &= \int_a^b \left[\mathcal{L}g(x|\xi)\right] f(\xi) \mathrm{d}\xi && (98) \\ &= \int_a^b \delta(x-\xi) f(\xi) \mathrm{d}\xi = f(x) && (99) \end{aligned}$$

- Its total area is 1

$$\int_c^d \delta(x)\mathrm{d}x = \begin{cases} 1 & \text{if} \quad c \le 0 \le d \\ 0 & 0 \quad \text{otherwise} \end{cases} \tag{100}$$

- when shifted $\delta(x-a)$ its spike shifts to $x = a$



- Shift delta function

$$\int_c^d f(x)\delta(x-a)\mathrm{d}x = \begin{cases} \mathrm{f(a)} & \text{if} \quad c \le 0 \le d \\ 0 & 0 \quad \text{otherwise} \end{cases} \tag{101}$$

Steps

1. Ensure $g(x|\xi)$ satisfies the homogeneous equation $f(x) = 0$ except at $x = \xi$.
2. $g(x|\xi)$ satisfies certain homogeneous boundary conditions
3. Check continuity of $g(x|\xi)$ at $x = \xi$.
4. Ensure derivative of $g(x|\xi)$ satisfies the jump condition in the neighbourhood of $x = \xi$

General example 1

- Consider a general linear second order differential equation

$$\mathcal{L}y(x) = \alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x}y + \gamma(x)y = f(x) \tag{102}$$

- α, β, γ are continuous functions of x within the range [a,b] and α is non zero.
- Firstly for $x \neq \xi$, we can solve $\mathcal{L}g(x|\xi) = 0$ for $x < \xi$ and $x > \xi$.
- We suppose that $\{y_1, y_2\}$ constitute the basis for the linearly independent solutions to the homogeneous equation. Thus, we can write

$$g(x|\xi) = \begin{cases} A(\xi)y_1(x) + B(\xi)y_2(x) & a \leq x < \xi \\ C(\xi)y_1(x) + D(\xi)y_2(x) & \xi < x \leq b \end{cases} \tag{103}$$

General example 1

- Impose stationary boundary conditions $y_1(a) = y_2(b) = 0$

$$\begin{aligned} g(a|\xi) &= A(\xi)y_1(a) + B(\xi)y_2(x) && (104)\\ &= A(\xi).0 + B(\xi)y_2(x) = B(\xi)y_2(x) && (105)\\ g(b|\xi) &= C(\xi)y_1(x) + D(\xi)y_2(b) && (106)\\ &= C(\xi)y_1(x) + D(\xi).0 = C(\xi)y_1(x) && (107)\end{aligned}$$

- Our solution now reduce to

$$g(x|\xi) = \begin{cases} B(\xi)y_2(x) & a \leq x < \xi \\ C(\xi)y_1(x) & \xi < x \leq b \end{cases} \qquad (108)$$

- Boundary conditions for the derivatives

$$\lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\left[\alpha(x)\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}+\beta(x)\frac{\mathrm{d}g}{\mathrm{d}x}+\gamma(x)g\right]\mathrm{d}x \tag{109}$$

$$=\int_{\xi-\epsilon}^{\xi+\epsilon}\delta(x-\xi)\mathrm{d}x = 1 \tag{110}$$

- Note we require that $\alpha(x), \beta(x)$ and $\gamma(x)$ are continuous function.
- Also we require that g is continuous at $x=\xi$

$$\lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\gamma(x)g(x)\mathrm{d}x \approx 0 \tag{111}$$

- For infinitesimally small region. For the first derivative

$$\lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\beta(x)\frac{\mathrm{d}g}{\mathrm{d}x}\mathrm{d}x \sim \beta(\xi)\left[g(x)\Big|_{x=\xi+\epsilon} - g(x)\Big|_{x=\xi-\epsilon}\right]\approx 0 \tag{112}$$

- For the second derivative

$$\lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\alpha(x)\frac{\mathrm{d}^2 g}{\mathrm{d}x^2}\mathrm{d}x = \alpha(\xi)\left[\left.\frac{\mathrm{d}g}{\mathrm{d}x}\right|_{x=\xi+\epsilon} - \left.\frac{\mathrm{d}g}{\mathrm{d}x}\right|_{x=\xi-\epsilon}\right] \tag{113}$$

- Now we need to impose continuity condition at $x=\xi$ and and jump in derivative conditions as we approach to ξ from lefft and right

$$g(\xi_+|\xi) = g(\xi_-|\xi) \tag{114}$$

$$\frac{\partial g(\xi_+|\xi)}{\partial x} - \frac{\partial g(\xi_-|\xi)}{\partial x} = \frac{1}{\alpha(\xi)} \tag{115}$$

- Applying these will lead to

$$B(\xi)y_2(\xi) = C(\xi)y_1(\xi) \tag{116}$$

$$B(\xi)y_2'(\xi) - C(\xi)y_1'(\xi) = \frac{1}{\alpha(\xi)} \tag{117}$$

- These are two linear system of equations for B and C

$$C(\xi) = \frac{y_2(\xi)}{\alpha(\xi)W(\xi)} \qquad \text{and} \qquad B(\xi) = \frac{y_1(\xi)}{\alpha(\xi)W(\xi)} \tag{118}$$

- where $W(x) = y_1 y_2' - y_2 y_1'$ is the Wronskian.
- It would be evaluate at $x = \xi$ in order to determine constants in equation (117).

$$\begin{aligned} g(x|\xi) &= \begin{cases} \dfrac{y_1(\xi)y_2(x)}{\alpha(\xi)W(\xi)} & a \leq x < \xi \\ \dfrac{y_1(x)y_2(\xi)}{\alpha(\xi)W(\xi)} & \xi < x \leq b \end{cases} \\ &= \frac{1}{\alpha(\xi)W(\xi)}\left[\Theta(\xi - x)y_1(\xi)y_2(x) + \Theta(x-\xi)y_1(x)y_2(\xi)\right] \end{aligned} \tag{119}$$

- where Θ is a step function The final solution is given by

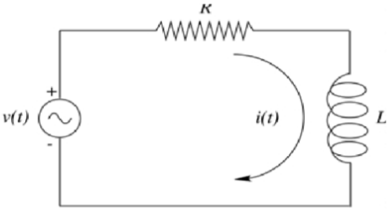
$$\begin{aligned} y(x) &= \int_a^b g(x|\xi) f(\xi) \mathrm{d}\xi \qquad (120) \\ &= y_2(x) \int_a^x \frac{y_1(\xi)}{\alpha(\xi) W(\xi)} f(\xi) \mathrm{d}\xi + y_1(x) \int_x^b \frac{y_2(\xi)}{\alpha(\xi) W(\xi)} f(\xi) \mathrm{d}\xi \end{aligned}$$

Electric Circuit problem

- In electrical engineering, the equation describing current flow within a circuit is given by

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = v(t) \tag{121}$$

where $v(t)$ is the voltage source, R is the resistance, L is the inductance, i is the current. The circuit was initially dead, allow the voltage to become $V_0/\Delta\tau$ during a very short duration $\Delta\tau$ starting at $t = \tau$, calculate the current flow in the system at $t > \tau + \Delta\tau$.



- How does it respond to a unit impulse?

$$L\frac{\mathrm{d}g(t|\xi)}{\mathrm{d}t} + Rg(t|\xi) = \delta^D(t-\xi) \tag{122}$$

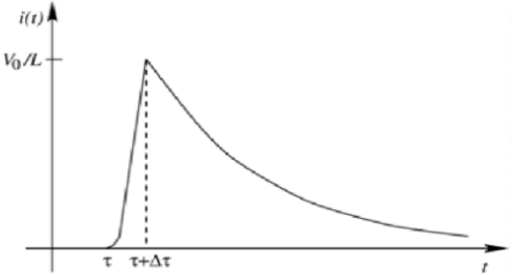


Figure 3: Current after a voltage surge

Find the homogeneous solution

- Homogeneous solution

$$g(t) = Ae^{-Rt/L}, \qquad t > \tau + \Delta\tau \tag{123}$$

- Now we need to fix the constant A
- From the boundary conditions: voltage during a very short duration $\Delta\tau$ starting at $t = \tau$

$$\lim_{\Delta\tau\to 0} L\int_{\tau}^{\tau+\Delta\tau} \mathrm{d}g + R\int_{\tau}^{\tau+\Delta\tau} g\mathrm{d}t = \int_{\tau}^{\tau+\Delta\tau} \delta^D(t-\xi)\mathrm{d}t \tag{124}$$

$$\lim_{\Delta\tau\to 0} L\left[g(\tau+\Delta\tau) - g(\tau)\right] = 1 \tag{125}$$

Full solution

- then for $\Delta\tau$ small

$$\lim_{\Delta\tau\to 0}[g(\tau+\Delta\tau)-g(\tau)] = A\left[e^{-R(\tau+\Delta\tau)/L} - e^{-R\tau/L}\right] \tag{126}$$
$$= Ae^{-R\tau/L} = 1 \tag{127}$$

- which then implies $A = (1/L)e^{R\tau/L}$, we can finally write

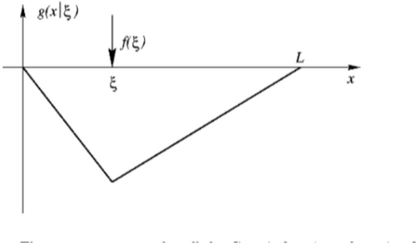
$$g(t|\tau) = \begin{cases} 0 & t < \tau \\ \frac{1}{L}e^{-R(t-\tau)/L} & \tau \leq t \end{cases} \tag{128}$$

- For a continuous voltage surges, the current flow becomes

$$i(t) = \int_\tau^t v(t)g(t|\tau)\mathrm{d}\tau = \int_\tau^t \frac{v(t)}{L}e^{-R(t-\tau)/L}\mathrm{d}\tau \tag{129}$$

Load on a String problem

- Consider a string of length L connected at both ends of the support and subjected to an external force acting downwards:



- it satisfies the equation

$$T\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f(x) \tag{130}$$

- where T denotes the uniform tensile force of the string.

Boundary conditions

- The string is stationary at both ends.

$$u(0) = u(L) = 0 \tag{131}$$

- Solution: Instead of solving for $f(x)$ directly, we solve for its response to a unit load at a point $x = \xi$.

$$T\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \delta(x-\xi) \tag{132}$$

- subject to the boundary conditions

$$g(0|\xi) = g(L|\xi) = 0 \tag{133}$$

  $g(x|\xi)$ denotes the displacement of the string when it is subjected to a unit load at $x = \xi$.

- As soon as $g(x|\xi)$ is known, the displacement at any other location is found by convolving $f(x)$ with $g(x|\xi)$.

Homogeneous solution

- At $x \neq \xi$, we have the homogeneous equation

$$T\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \tag{134}$$

- $T \neq 0$ hence the solution becomes

$$g(x|\xi) = \begin{cases} ax + b & 0 \leq x < \xi \\ cx + d & \xi < x \leq \tau \end{cases} \tag{135}$$

- There four arbitrary constants to fix.

Apply the boundary conditions

- Apply the stationary boundary conditions to $g(x|\xi)$

$$
\begin{aligned}
g(0|\xi) &= a.0 + b = b = 0\,, && (136)\\
g(L|\xi) &= c.L + d = 0\,, \quad \text{or} \quad d = -cL && (137)
\end{aligned}
$$

- Applying these to equation (134) gives

$$
g(x|\xi) = \begin{cases} ax & 0 \le x < \xi \\ c(x-L) & \xi < x \le \tau \end{cases} \tag{138}
$$

- where a and c are yet to be determined.
- Impose continuity at $x = \xi$ for $g(x)$ since the string is not broken

Full solution

- Continuity of $u(x)$ implies continuity of both solutions of $g(x|\xi)$

$$a\xi = c(\xi - L) \longrightarrow c = \frac{a\xi}{\xi - L} \tag{139}$$

- still have to determine a.
- There is a discontinuity for first derivatives at $x = \xi$

$$T \int_{\xi-\epsilon}^{\xi+\epsilon} \frac{\mathrm{d}^2 g(x|\xi)}{\mathrm{d}x^2} \mathrm{d}x = \int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x - \xi) \mathrm{d}x \tag{140}$$

- As we approach $x = \xi$ from left and right, equation (139) becomes

$$\lim_{\epsilon \to 0} \left[\frac{\mathrm{d}g(\xi + \epsilon|\xi)}{\mathrm{d}x} - \frac{g(\xi - \epsilon|\xi)}{\mathrm{d}x} \right] = \frac{1}{T} = \frac{\mathrm{d}g(\xi_+|\xi)}{\mathrm{d}x} - \frac{g(\xi_-|\xi)}{\mathrm{d}x} = \frac{1}{T} \tag{141}$$

where ξ_+ and ξ_- are nearest point to right and left of ξ.

- Using equation (137), we find

$$\frac{g(\xi_-|\xi)}{\mathrm{d}x} = a\,, \quad \text{and} \quad \frac{\mathrm{d}g(\xi_+|\xi)}{\mathrm{d}x} = c = \frac{a\xi}{\xi - L} \tag{142}$$

- Plugging this in the jump condition (equation (140)) gives

$$\frac{a\xi}{\xi - L} - a = \frac{1}{T} \Rightarrow \frac{aL}{\xi - T} = \frac{1}{T} \tag{143}$$

- Hence

$$g(x|\xi) = \begin{cases} \dfrac{x(\xi - L)}{LT} & 0 \leq x < \xi \\ \dfrac{x(x - L)}{LT} & \xi < x \leq \tau \end{cases} \tag{144}$$

- Finally

$$u(x) = \frac{(x - L)}{LT}\int_0^x f(\xi)\xi \mathrm{d}\xi + \frac{x}{LT}\int_x^L f(\xi)(\xi - L)\mathrm{d}\xi \tag{145}$$

Exercise

1. Find the Green's function for the system
$$y'' - 3y' + 2y = f(t) \qquad \text{with} \qquad y(0) = y'(0) = 0 \tag{146}$$
Ans:
$$g(t|\tau) = \left[e^{2(t-\tau)} - e^{(t-\tau)}\right] H(t-\tau) \tag{147}$$
where $H(t-\tau)$ is Heavside function.
2. Damped harmonic oscillator
$$my'' + cy' = ky = f(t) \tag{148}$$
where k is the spring constant, m is the mass attached to the string and c is the damping coefficient. Find the Green's function for $g(0|\tau) = g'(0|\tau) = 0$ Ans
$$mg(t|\tau) = \frac{e^{-\gamma(t-\tau)}}{\sqrt{\omega_0^2 - \gamma^2}} \sin\left[(t-\tau)\sqrt{\omega_0^2 - \gamma^2}\right] \tag{149}$$
$\omega_0 = k/m$ and $\gamma = c/(2m)$