Mathematical Methods for Graduate Students in Physics

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Non-constant coefficient ODE

- Missing a dependent function in the ODE
- it can be reduced to an ODE of one-order less. FOr example

$$ty'' + 4y' = t^2 \tag{1}$$

- Set $y' = z$ which implies that $y'' = z'$,
- It leads to a linear first order ODE.

$$z' + \frac{4}{t}z = t \tag{2}$$

Non-constant coefficient ODE:Exact Equations

- If the ODE can be written as a derivative of another ODE, we say it is exact

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x) \tag{3}$$

- is exact if the LHS can be written as

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_0(x)y = \frac{\mathrm{d}}{dx}\left[b_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{dx^{n-1}} + \cdots + b_0(x)y\right] \tag{4}$$

- This holds if

$$a_0(x) - a_1'(x) + a_2''(x) - \cdots + (-1)^n a_n^{(n)}(x) = 0 \tag{5}$$

Example

- Find the general solution to the ODE

$$(1-x^2)\frac{\mathrm{d}^2y}{dx^2} - 3x\frac{\mathrm{d}x}{\mathrm{d}x} - y = 1 \tag{6}$$

- **solution**
- Comparing to equation (3), we find $a_2 = 1 - x^2$, $a_1 = -3x$, $a_0 = 1$.
- Check that $a_0 - a_1' + a_2'' = 0$, hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[b_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b_0(x)y\right] = 0 \tag{7}$$

- Matching

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[b_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b_0(x)y\right] = b_1\frac{\mathrm{d}^2y}{dx^2} + (b_1' + b_0)\frac{\mathrm{d}x}{\mathrm{d}x} + b_1'y \tag{8}$$

- We can then compute b_1 and b_0 by comparing with the original equation

$$b_1 = 1 - x^2\,, \quad b_1' + b_0 = -3x\,, \qquad b_0' = -1 \tag{9}$$

Solution

- This implies that $b_1 = 1 - x^2$ ad $b_0 = -x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy\right] = 1 \tag{10}$$

- Integrating give

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{x}{1 - x^2}\right)y = \frac{x + c_1}{(1 - x^2)} \tag{11}$$

- Solving further gives

$$y = \frac{c_1 \sin^{-1} x + c_2}{\sqrt{1 - x^2}} - 1 \tag{12}$$

Solution to Laplace equation[Spherical Harmonics]

- In spherical coordinates, the Laplacian is given by

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = 0 \quad (13)$$

- Uusing the method of separation of variables

$$f(r,\theta,\phi). = .R(r)\Theta(\theta)\Phi(\phi) \quad (14)$$

- Inserting this decomposing in equation (13)

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = 0 \quad (15)$$

Laplace equation

- The Φ evolves independently

$$+\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\partial r}\right) - \frac{\sin^2\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = -m^2 \quad (16)$$

- We can write

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2\,, \quad (17)$$

- where m^2 is the separation constant, which is chosen to be negative so that the solution for $\Phi(\phi)$ are periodic in ϕ

$$\Phi(\phi) = \begin{cases} e^{im\phi} \\ \\ e^{-im\phi} \end{cases} \quad \text{for} \quad m = 0, 1, 2, \cdots \quad (18)$$

Euler equation

- The radial compoent becomes

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda, \tag{19}$$

- Imposing requlairyt, $\lambda = \ell(\ell+1)$Leading to a well-known Euler equation

$$r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}R}{\mathrm{d}r} - \ell(\ell+1)R = 0 \tag{20}$$

- This implies that the solution has the following form $R = r^s$. Putting the solution back into equation (20)

$$R(r) = \begin{cases} r^{\ell} \\ r^{-\ell-1} \end{cases} \tag{21}$$

Associated Legendre equation

- Now consider the angular component

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0 \tag{22}$$

- Charging variables to $x=\cos\theta$ and $y=\Theta(\theta)$, the above equation becomes

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]y=0 \tag{23}$$

- The solution this equation leads to associated Legendre polynomials [details later]

$$y(x)=P_\ell^m(x)\,,\qquad \ell=0,1,2,3and-\ell\le m\le\ell \tag{24}$$

- Combining the Θ and Φ solutions leads to the Spherical Harmonics.

$$Y_\ell^m(\theta,\varphi)=\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi} \tag{25}$$

Legendre equations: Series solution to ODE

- Legendre's equation has the form

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0 \tag{26}$$

where ℓ is a constant. Show that $x = 0$ is an ordinary point and $x = \pm 1$ are regular singular points of this equations

Solution

- Since $x = 0$ is an ordinary point in the Legendre equation, we expect to find two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (27)$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (28)$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad (29)$$

- This implies

$$2xy' = \sum_{n=1} 2xna_n x^{n-1} = \sum_{n=0} 2na_n x^n \quad (30)$$

$$(1-x^2)y'' = \sum_{n=0} \left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n \quad (31)$$

Solution

- Substituting all in equation (26) gives

$$\sum_{n=0}^{\infty}\left[n(n-1)a_n x^{n-2} - n(n-1)a_n x^n - 2na_n x^n + \ell(\ell+1)a_n x^n\right] = 0 \quad (32)$$

- Shifting the index of the first term and collecting like terms

$$\sum_{n=0}^{\infty}\left\{(n+2)(n+1)a_{n+2} - \left[n(n+1) - \ell(\ell+1)\right]a_n\right\}x^n = 0 \quad (33)$$

- The recurrence relation

$$a_{n+2} = \frac{\left[n(n+1) - \ell(\ell+1)\right]a_n}{(n+2)(n+1)} \quad (34)$$

Solution

- For the first few integer values $n = 0, 1, 2, \cdots$ for the even

$$
\begin{aligned}
a_2 &= -\frac{\ell(\ell+1)}{2.1}a_0 \\
a_4 &= -\frac{(\ell-2)(\ell-3)}{3.4} = (-1)^2\frac{\ell(\ell-2)(\ell+1)(\ell+3)}{4!}a_0 \\
a_n &= (-1)^n\frac{\ell(\ell-2)\cdots(\ell-2n+2)\cdot(\ell+1)(\ell+3)\cdots(\ell+2+2\ell-}{(2n)!}
\end{aligned}
$$

Similarly for the odd number sin terms of a_1

$$
\begin{aligned}
a_3 &= -\frac{(-1)(\ell+2)}{2\cdot 3}a_1 \\
a_5 &= -\frac{(\ell-3)(\ell-4)}{4\cdot 5}a_3 = (-1)^2\frac{(\ell-1)(\ell-3)(\ell+2)(\ell+4)}{5!}a_1 \\
a_{2n+1} &= (-1)^n\frac{(\ell-1)(\ell-3)\cdots(\ell-2n+1)(\ell+2)\cdots(\ell+2n)}{(2n+1)!}a_1
\end{aligned}
$$

Legendre polynomial

- If we choose $a_0 = 1$ and $a_1 = 0$, we obtain

$$\begin{aligned} y_1(x) &= 1 - \ell(\ell+1)\frac{x^2}{2!} + (\ell-2)\ell(\ell+1)(\ell+3)\frac{x^4}{4!} - \cdots \\ y_2(x) &= x - (\ell-1)(\ell+2)\frac{x^3}{3!} + (\ell-3)(\ell-1)(\ell+2)(\ell+4)\frac{x^5}{5!} - \cdots \end{aligned}$$

The two solutions are linearly independent

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{35}$$

In general

$$\begin{aligned} y_1(x) &= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{\ell(\ell-2)\cdots(\ell-2n+2)\cdot(\ell+1)(\ell+3)\cdots(\ell +}{(2n)!} \\ y_2(x) &= x + \sum_{n=1}^{\infty}(-1)^n \frac{(\ell-1)(\ell-3)\cdots(\ell-2n+1)(\ell+2)\cdots(\ell+2}{(2n+1)!} \end{aligned}$$

Legendre polynomial

- These solution consititue the Legendre polynomials

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}, \tag{36}$$

And for the first few terms int he series

$$P_0 = 1 \tag{37}$$
$$P_1 = x \tag{38}$$
$$P_2 = \frac{1}{2}\left(3x^2-1\right) \tag{39}$$
$$P_3 = \frac{1}{2}\left(5x^3-3x\right) \tag{40}$$
$$P_4 = \frac{1}{8}\left(35x^4-30x^2+3\right) \tag{41}$$
$$P_5 = \frac{1}{8}\left(63x^5-70x^3+15x\right) \tag{42}$$
$$P_6 = \frac{1}{16}\left(231x^6-315x^4+105x^2-5\right) \tag{43}$$

Excercises

Find the solution to the following

1. Bessel

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0 \tag{44}$$

2. Associated Legendre

$$(1 - z^2)y'' - 2zy' + \left[\ell(\ell + 1) - \frac{m^2}{1 - z^2}\right] y = 0 \tag{45}$$

3. Hypergeometric

$$z(1 - z)y'' + [c - (a + b + 1)z]\, y' - aby = 0 \tag{46}$$