A Primer to Qualitative Methods in Inverse Scattering Theory

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Contents

Abstract

These notes are based off reading courses I take with Peijun Li on inverse scattering theory problems. I summarize basic ideas necessary for research into the field. I also review important theorems and necessary PDE theory. This primer summarizes Chapters 3 and 4 of *Inverse Acoustic and Electromagnetic Scattering Theory* by Colton and Kress.

1 Overview

Inverse scattering tries to recover the perturbation: the potential, the obstacle, etc., from scattering data. The main questions are uniqueness, stability and reconstruction. Another important question is range characterization. Scattering theory, in the time dependent formulation, is the study of the long time behaviour of solutions of an evolution equation that move out to infinity. The evolution equation might be the time dependent Schrodinger equation, (quantum scattering), the scalar wave equation (acoustical scattering), Maxwell's equations (electromagnetic scattering) or even a non-linear evolution equation. The underlying space might be Euclidean space or a Riemannian manifold. In each problem there is a localized scattering target. Moving in space away from the target to infinity, the equations, or the geometry, become simpler. The idea is that in the distant past and in the far future, the scattered wave will be located in the region where the equation or geometry is simple. It then becomes possible to compare distant past input to the far future output.

The task of inverse scattering theory is to determine properties of the target, given sufficiently many input output pairs. In fact, most often scattering problems are stated in a time-harmonic formulation that results after taking a Fourier transform in the time variable. Although the immediate connection to the original scattering experiments is obscured, it can be easier to state and study scattering problems in the this formulation.

2 Ill-Posed Problems

We want to make Hadamard's concept of well-posedness precise.

Definition 1. *Let $A : U \subset X \to V \subset Y$ be a linear operator between subsets of normed spaces. The equation*

$$A(\varphi) = f \tag{1}$$

is **well-posed** *if $A : U \to V$ is bijective and the inverse operator $A^{-1} : V \to U$ is continuous. Otherwise the equation is* **ill-posed**.

Idea 1. *Note that we may distinguish three types of ill-posedness.*

1. *If A is not surjective, then (1) is not solvable for all $f \in V$ (nonexistence)*
2. *If A is not injective, then (1) may have more than one solution (nonuniqueness)*
3. *If $A^{-1} : V \to U$ exists but is not continuous then the solution φ of (1) does not depend continuously on the data f (instability)*

Example 1. *Let $A : U \subset X \to Y$ be a completely continuous operator from a subset U of a normed space X into a subset V of a normed space Y. Then (1) is ill-posed if U is not of finite dimension.*

Proof. Assume A^{-1} exists and is continuous. Then we have $I = A^{-1}A$ and so the identity operator on U is compact since the product of a continuous and compact operator is compact. So U must be finite-dimensional. $\square$

Example 2. Differentiation: *Finding $u(x)$ given*

$$\int_0^x u(y)\,dy$$

is ill-posed.

Example 3. Inverse Heat Equation: *Finding $u(x,0)$ for given*

$$u(x,T) = \int_0^\pi k(x,y,T)f(y)\,dy$$
$$k(x,y,T) = \frac{2}{\pi}\sum_{n=1}^{\infty} \exp(-n^2T)\sin(nx)\sin(ny)$$

is ill-posed.

Example 4. Deconvolution: *Finding $u(x)$ for given*

$$\int_\Omega k(x-y)u(y)\,dy$$

is ill-posed.

2.1 Regularization Methods

Methods for constructing a stable approximate solution of an ill-posed problem are called **regularization methods**. We want to approximate the solution φ to the equation (1) from a knowledge of a perturbed RHS f^δ with a known error level

$$||f^\delta - f|| \leq \delta. \tag{2}$$

When $f \in A(X) := \{A\varphi : \varphi \in X\}$, then there is a unique solution φ of (1). RHS is perturbed, so need to approximate exact solution of unperturbed eqution. This approximation needs to be stable.

Definition 2. *Let X, Y be normed spaces and let $A \in \hom(X,Y)$ be an injective bounded linear operator. Then a family of bounded linear operators $R_\alpha : Y \to X, \alpha > 0$, with the property of pointwise convergence*

$$\lim_{\alpha\to 0} R_\alpha A\varphi = \varphi \tag{3}$$

for all $\varphi \in X$ is a **regularization scheme** *for operator A. The parameter α is a* **regularization parameter**.

Theorem 1. *Let X, Y be normed spaces, let $A \in \hom(X, Y)$ be a compact, linear operator, and supposed $\dim X = \infty$. Then for a regularization scheme the operators $R_\alpha A$ cannot be uniformly bounded w.r.t. α and the operators $R_\alpha A$ cannot be norm convergent.*

Idea 2. $\varphi_\alpha^\delta := R_\alpha f^\delta$. *Then for the approximation error, we obtain following estimate*

$$||\varphi_\alpha^\delta - \varphi|| \leq \delta ||R_\alpha|| + ||R_\alpha A\varphi - \varphi||. \tag{4}$$

Definition 3. *A strategy for a regularization scheme $R_\alpha, \alpha > 0$, that is, the choice of the regularization parameter $\alpha = \alpha(\delta, f^\delta)$ depending on the error level δ and on f^δ is called* **regular** *if for all $f \in A(X)$ and all $f^\delta \in Y$ with $||f^\delta - f|| \leq \delta$ we have*

$$R_\alpha f^\delta \to A^{-1} f, \quad \delta \to 0.$$

2.2 Singular Value Decomposition

Here we describe a regularization scheme in a Hilbert space setting. One such approach is via singular value decomposition (SVD) for compact operators which generalizes the spectral decomposition for compact self-adjoint operators.

Theorem 2 (Eigendecomposition)**.** *Let $A \in \mathcal{L}(X, X)$ be self-adjoint and compact. Then there exist at most countably many nonzero eigenvalues $\{\lambda_n\}$ of A. All eigenvalues are real and for a set of O.N. eigenvectors $\{\varphi_n\}$ with $||\varphi_n|| = 1$ one has*

$$A\varphi = \sum_{n \in \mathcal{I}} \lambda_n (\varphi, \varphi_n) \varphi_n.$$

Also known as spectral decomposition.

Theorem 3 (SVD). *Any compact operator $A \in \mathcal{L}(X, Y)$ has a representation*

$$A\varphi = \sum_{n\in\mathcal{I}} \mu_n(\varphi, \varphi_n) g_n$$
$$A^* g = \sum_{n\in\mathcal{I}} \mu_n(g, g_n) \varphi_n$$

with the O.N. singular vectors φ_n and g_n and singular values μ_n. The singular values are ordered so that

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots$$

Theorem 4 (Picard). *Let $A \in \mathcal{L}(X, Y)$ be a compact linear operator with singular system (μ_n, φ_n, g_n). The equation of the first kind (1) is solvable iff f belongs to the orthogonal complement $N(A^*)^\perp$ and satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} |(f, g_n)|^2 < \infty.$$

Then a solution is given by

$$\varphi = A^\dagger f = \sum_{n=1}^{\infty} \frac{1}{\mu_n}(f, g_n)\varphi_n. \tag{5}$$

Idea 3. *From (5), we can see that*

1. *errors corresponding to g_n are amplified by $\frac{1}{\mu_n}$.*
2. *errors corresponding to large n (high frequencies) are amplified much stronger.*
3. *The faster μ_n decays, the more severe the error amplification.*
4. *Inversion of compact $A \in \mathcal{L}(X, Y), \dim(A(X)) = \infty$ is ill-posed!*
5. *The decay of μ_n causes the ill-posedness.*

Idea 4 (Classification of Ill-Posedness)**.** *A problem $A\varphi = f$ with a compact linear operator $A \in \mathcal{L}(X, Y)$ with infinite-dimensional range is called*

- **Mildly ill-posed** *if there exist a $\gamma \leq 1$ and $C > 0$ satisfying $\mu_n \geq Cn^{-\gamma}$ for all n.*
- **Moderately ill-posed** *if it is not mildly ill-posed but there is a $\gamma > 1$ and $C > 0$ such that $\mu_n \geq Cn^{-\gamma}$.*
- **Severely ill-posed** *if the singular values decay faster than with polynomial speed.*

Theorem 5 (SVD based Regularizations)**.** *For $R_\alpha : Y \to X$ defined by*

$$R_\alpha g = \sum_{n=1}^{\infty} g_\alpha(\mu_n)(g_n, g)\varphi_n$$

and a map $g_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ with

- *$g_\alpha(\mu) \to 1/\mu$ fro $\mu > 0$ as $\alpha \to 0$.*
- *$g_\alpha(\mu) \leq C_\alpha < \infty$ for all $\mu > 0$.*
- *$\mu g_\alpha < \infty$ for all α and $\mu > 0$.*

R_α is a regularization for $A^\dagger$.

Theorem 6. *Let $A \in \mathcal{L}(X, Y)$ be an injective compact linear operator with singular system (μ_n, φ_n, g_n). Then the spectral cut-off*

$$R_m f := \sum_{\mu_n \geq \mu_m} \frac{1}{\mu_n}(f, g_n)\varphi_n$$

describes a regularization scheme with regularization parameter $m \to \infty$ and $||R_m|| = 1/\mu_m$.

Proof Sketch. Apply Bessel's inequality to obtain estimate

$$||R_m f||^2 = \sum_{\mu_n \geq \mu_m} \frac{1}{\mu_n^2}|(f, g_n)|^2 \leq \frac{1}{\mu_m^2} \sum_{\mu_n \geq \mu_m} |(f, g_n)|^2 \leq \frac{1}{\mu_m^2}||f||^2,$$

whence $||R_m|| \leq 1/\mu_m$. Then apply $R_m g_m = \varphi_m/\mu_m$. □

2.3 Tikhonov Regularization

Idea 5. *Tikhonov regularization is defined by*

$$g_\alpha(\mu) = \frac{\mu}{\mu^2 + \alpha}. \tag{6}$$

The $x_\alpha^\delta = R_\alpha y^\delta$ *generated by Tikhonov regularization are the unique solutions of*

$$(A^* A + \alpha I)\varphi = A^* g.$$

The $x_\alpha^\delta = R_\alpha y^\delta$ *generated by Tikhonov regularization are the unique elements*

$$x_\alpha^\delta = \arg\min_\varphi ||A\varphi - g^\delta||^2 + \alpha||\varphi||^2. \tag{7}$$

3 Direct Acoustic Obstacle Scattering

The starting point of any discussion of classical acoustic scattering theory is the Helmholtz equation. We note in particular spherical Bessel functions and spherical harmonics which arise when separation of variables is implemented. To be more precise, we consider solutions of the Helmholtz equation in $\mathbb{R}^3$

$$\nabla^2 + k^2 u = 0 \tag{8}$$

for $k > 0$ in the form

$$u(x) = f(k|x|)Y_n^m(\hat{x}),$$

where $x \in \mathbb{R}^3, \hat{x} := \frac{x}{|x|}$, and $Y_n^m(\hat{x})$ is a *spherical harmonic* defined by

$$Y_n^m(\theta, \varphi) := \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}} P_n^m(\cos\theta)e^{im\varphi}, \tag{9}$$

with P_n^m being the associated Legendre polynomial. We note that $\{Y_n^m\}$ is a complete O.N. system in $L^2(S^2)$ where $S^2 := \{x : |x| = 1\}$ and $Y_0^0 = \frac{1}{\sqrt{4\pi}}$. Then f is the solution of the *spherical Bessel equation*

$$t^2 \frac{d^2 f}{dt^2} + 2t\frac{df}{dt} + [t^2 - n(n+1)]f(t) = 0 \tag{10}$$

with two linearly independent solutions

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)} \tag{11}$$

$$y_n(t) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p! (-2n+1)(-2n+3)\cdots(-2n+2p+1)}. \tag{12}$$

The functions

$$h_n^{(1,2)} := j_n \pm i y_n \tag{13}$$

are known as *spherical Hankel functions* of the first and second kind of order n. From the series representations (11) and (12), by equating powers of t, it is verified that both $f_n = j_n$ and $f_n = y_n$ satisfy the recurrence relation

$$f_{n+1}(t) + f_{n-1}(t) = \frac{2n+1}{t} f_n(t), \quad n = 1, 2, \dots$$

and satisfy differentiation formulas

$$\begin{aligned} f_{n+1}(t) &= -t^n \frac{d}{dt}\{t^{-n} f_n(t)\}, \quad n = 0, 1, 2, \dots, \\ t^{n+1} f_{n-1}(t) &= \frac{d}{dt}\{t^{n+1} f_n(t)\}, \quad n = 1, 2, \dots \end{aligned}$$

From this, we see that the spherical Hankel functions have the asymptotic behavior

$$\begin{aligned} h_n^{(1)}(t) &= \frac{1}{t} e^{i(t - \frac{n\pi}{2} - \frac{\pi}{2})} \left\{1 + O\left(\frac{1}{t}\right)\right\} \\ h_n^{(2)}(t) &= \frac{1}{t} e^{-i(t - \frac{n\pi}{2} - \frac{\pi}{2})} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \end{aligned}$$

as $t \to \infty$.

Idea 6. *Note in particular that $h_n^{(1)}(kr)$ satisfies the Sommerfield radiation condition (SRC)*

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0,$$

i.e., if $u(x) = h_n^{(1)}(kr) Y_n^m(\hat{x})$ then $u(x)\exp(-i\omega t)$ is an outgoing wave. Solutions of the Helmholtz equation satisfying SRC uniformly in $\hat{x}$ are radiating.

Theorem 7. *Let Y_n be a spherical harmonic of order n. Then*

$$u_n(x) = j_n(k|x|) Y_n\left(\frac{x}{|x|}\right)$$

is an entire solution to the Helmholtz equation and

$$v - n(x) = h_n^{(1)}(k|x|) Y_n\left(\frac{x}{|x|}\right)$$

is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 - \{0\}$.

Theorem 8. *Let Y_n^m, $m = -n, \dots, n, n = 0, 1, 2, \dots,$ be a set of O.N. spherical harmonics. Then for $|x| > |y|$ we have*

$$\frac{e^{ik|x-y|}}{4\pi|x-y|} = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k|x|) Y_n^m(\hat{x}) j_n(k|y|) \overline{Y_n^m(\hat{y})}.$$

The series and its term by term first derivatives w.r.t. $|x|$ and $|y|$ are absolutely and uniformly convergent on compact subsets of $|x| > |y|$.

3.1 The Far Field Pattern

Let D be a bounded domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and assume that ∂D is of class C^2 with outward unit normal ν directed into the exterior of D. Let

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y$$

be the radiating fundamental solution to the Helmholtz equation in $\mathbb{R}^3$. Then we can apply Green's second identity

$$\int_D u\Delta v - v\Delta u \, dx = \int_{\partial D} (u\partial_\nu v - v\partial_\nu u) \, dS \tag{14}$$

we can deduce Green's representation formula $u \in C^2(D) \cap C^1(\overline{D})$

$$\begin{aligned} u(x) = &\int_D \left\{ \frac{\partial u}{\partial \nu} \Phi(x, y) - u \frac{\partial}{\partial \nu(y)} \phi(x, y) \right\} dS(y) \\ &- \int_D \{(\Delta u + k^2 u)\phi(x, y)\} \, dy, \quad x \in D \end{aligned}$$

Lemma 1. *Let $u \in C^2(D)$ be a solution of the Helmholtz equation in D. Then u is analytic in D.*

Proof. Let $x \in D$ and choose a ball contained in D with center x. Apply Green's representation formula to the ball. ☐

Theorem 9 (Holmgren's Theorem)**.** *Let $u \in C^2(D) \cap C^1(\overline{D})$ be a solution to the Helmholtz equation in D so that*

$$u = 0 = \frac{\partial u}{\partial \nu} \quad \text{on } \Gamma$$

for some open subset $\Gamma \subset \partial D$. Then $u \equiv 0$.

Proof. Can extend u by setting

$$u(x) = \int_{\partial D \setminus \Gamma} \left\{ \frac{\partial u}{\partial \nu} \Phi(x,y) - u \frac{\partial}{\partial \nu(y)} \phi(x,y) \right\} dS(y)$$

for $x \in (\mathbb{R}^3 \setminus \overline{D}) \cup \Gamma$. Apply Green's second identity to u and $\Phi(x,\cdot)$ and observe $u = 0$ on $\mathbb{R}^3 \setminus \overline{D}$. Apply analyticity of u to get $u = 0$ in D. ☐

Theorem 10. *Let $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C^1(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Helmholtz equation. Then we have Green's formula*

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu} \Phi(x,y) - u \frac{\partial}{\partial \nu(y)} \phi(x,y) \right\} dS(y)$$

where $x \in \mathbb{R}^3 \setminus \overline{D}$.

Proof is a bit involved but still relatively straightforward. (cf. Colton and Kress). From this, we get two corollaries:

Theorem 11 (Corollary 1)**.** *An entire solution to the Helmholtz equation satisfying SRC must vanish identically.*

Proof is immediate consequence of last theorem.

Theorem 12 (Corollary 2)**.** *Every radiating solution u to the Helmholtz equation has asymptotic behavior of an outgoing spherical wave*

$$u(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty, \tag{15}$$

uniformly in all directions $\hat{x}$. The function $u_\infty(\hat{x})$ defined on the unit sphere is called the **far field pattern** *of u.*

Proof. Note

$$|x-y| = \sqrt{|x|^2 - 2x \cdot y + |y|^2} = |x| - \hat{x} \cdot y + O\left(\frac{1}{r}\right)$$

so we get

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ikr}}{r} \left\{ e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{r}\right) \right\}$$

and

$$\frac{\partial}{\partial\nu(y)}\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ikr}}{r}\left\{\frac{\partial}{\partial\nu(y)}e^{-ik\hat{x}\cdot y} + O\left(\frac{1}{r}\right)\right\}$$

as $r \to \infty$ uniformly for all $y \in \partial D$. Now substitute Green's formula. □

Lemma 2 (Rellich's Lemma)**.** *Assume the bounded set D is the open complement of an unbounded domain and let $u \in C^2(\mathbb{R}^3 \setminus \overline{D}$ be a solution to the Helmholtz equation satisfying*

$$\lim_{r\to\infty}\int_{|x|=r}|u(x)|^2\,ds = 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Rellich's lemma is very important: it establishes a one-to-one correspondence between radiating waves and their far field patterns.

Theorem 13 (Corollary 3)**.** *Let D be as in Rellich's lemma and ∂D be of class C^2 with unit normal ν directed into the exterior of D and assume $u \in C^2(\mathbb{R}^3 \setminus \overline{D}) \cap C(\mathbb{R}^3 \setminus D)$ is a radiating solution to the Helmholtz equation with wave number $k > 0$ which has a normal derivative in the sense of uniform convergence and for which*

$$\Im\int_{\partial D} u\frac{\partial\bar{u}}{\partial\nu}\,ds \geq 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

So Rellich's lemma ensures uniqueness for solutions to exterior boundary value problems.

3.2 Far Field Operator and Direct Scattering Problem

Assume the incident field $u^i = e^{ikx\cdot d}$ where $|d| = 1$. Then the solution of the scattering problem

$$\Delta u + k^2 n(x)u = 0 \tag{16}$$

$$e^{ikx\cdot d} + u^s(x) = u(x) \tag{17}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{18}$$

will depend on incident direction d and the far field pattern $u_\infty(\hat{x}) = u_\infty(\hat{x}, d)$ will also depend on d.

Theorem 14 (Reciprocity Principle)**.** *Let $u_\infty(\hat{x}, d)$ be the far field pattern corresponding to direct scattering problem (DSP). Then $u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x})$.*

Definition 4. *Define the* **far field operator** $F : L^2(S^2) \to L^2(S^2)$ *by*

$$(Fg)(\hat{x}) := \int_{S^2} u_\infty(\hat{x}, d) g(d) \, dS(d).$$

F is clearly compact since far field pattern is smooth w.r.t. each of its variables. The corresponding **scattering operator** $S : L^2(S^2) \to L^2(S^2)$ is defined by

$$S := I + \frac{ik}{4\pi} F.$$

Definition 5 (Herglotz wave function)**.** *The* **Herglotz wave function** *is a function of the form*

$$v(x) = \int_{S^2} e^{ikx\cdot d} g(d) dS(d), \quad x \in \mathbb{R}^3$$

where $g \in L^2(S^2)$. The function g is the **Herglotz kernel of** v*. Herglotz wave functions are entire solutions to the Helmholtz equation.*

Theorem 15. *Let $g, h \in L^2(S^2)$ and let v^i and w^i be the Herglotz wave functions with kernels g and h respectively. Then if both v and w are solutions of the DSP corresponding to the incident field $e^{ikx\cdot d}$ being replaced by the incident fields v^i, w^i respectively, we have*

$$0 = 2\pi(Fg, h) - 2\pi(g, Fh) - ik(Fg, Fh).$$

Theorem 16 (Corollary 4)**.** *The far field operator F is normal and the scattering operator S is unitary.*

We introduce the **transmission eigenvalue problem:** determine $k > 0$ and $x, w \in L^2(D)$, $v - w \in H_0^2(D)$, such that $v \neq 0, w \neq 0$ and

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 \quad &\text{in } D \\ \Delta v + k^2 v = 0 \quad &\text{in } D \\ v = w \quad &\text{on } \partial D \\ \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} &\text{on } \partial D. \end{aligned}$$

Such values of $k > 0$ are **transmission eigenvalues**. $D := \{x : n(x) \neq 0\}$ and D is bounded with Lipschitz boundary such that $\mathbb{R}^3 \setminus \overline{D}$ is connected.

Theorem 17. *If F is the far field operator corresponding to DSP, then it is injective if k is not a transmission eigenvalue.*

Proof. Suppose $Fg = 0$. We need to show $g = 0$. Note that the far field pattern w_∞ of the scattered field w^s corresponding to incident field

$$w^i = \int_{S^2} e^{ikx\cdot d} g(d)\, ds(d)$$

vanishes. Applying Rellich's lemma, $w^s = w - w^i$ vanishes outside D. Then $w = w^i + w^s$ satisfies $\Delta w + k^2 n(x) w = 0$ in $\mathbb{R}^3$ and $w - w^i = 0$ on boundary of D and $\frac{\partial}{\partial \nu}(w - w^i) = 0$ on boundary. With k not being a transmission eigenvalue, $w^i = w = 0$ and hence $g = 0$. $\square$

Theorem 18 (Corollary 5)**.** *Let F be the far field operator corresponding to DSP. Then F has dense range if k is not a transmission eigenvalue.*

Proof. $(\mathcal{R}(F))^\perp = \mathcal{N}(F^*)$, hence it suffices to show $F^*h = 0$ implies $h = 0$. If $F^*h = 0$, then

$$\int_{S^2} \overline{u_\infty(d, \hat{x})} h(d)\, dS(d) = 0.$$

Then

$$\int_{S^2} u_\infty(-\hat{x}, d)\overline{h(d)}\, dS(d) = 0,$$

hence, applying reciprocity

$$\int_{S^2} u_\infty(\bar{x}, d)\overline{h(-d)}\, dS(d) = 0.$$

F is injective, hence $h = 0$, as desired. $\square$

3.3 Uniqueness of Direct Scattering Problem

Consider the DSP of determining u such that

$$\begin{aligned} \Delta u + k^2 n(x) u &= 0 \\ e^{ikx\cdot d} + u^s(x) &= u(x) \\ \lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) &= 0. \end{aligned}$$

where $n(x) = 1$ outside of the inhomogeneous medium. We now assume $n \in L^\infty(\mathbb{R}^3)$ with nonnegative imaginary part, set $m := 1 - n$ and let D be a bounded domain with Lipschitz boundary such that $\mathbb{R}^3 \setminus \overline{D}$ is connected and $m(x) = 0$ in $\mathbb{R}^3 \setminus \overline{D}$.

Theorem 19. *Given two bounded domains D and G, the volume potential*

$$(V\varphi)(x) := \int_D \Phi(x,y)\varphi(y)\,dy, \quad x \in \mathbb{R}^3$$

defines a bounded operator $V : L^2(D) \to H^2(G)$.

A classical approach to solving the scattering problem is based on reformulating the problem as a volume integral equation known as the **Lippmann-Schwinger integral equation**. Solving DSP is equivalent to solving

$$u(x) := u^i(x) - k^2 \int_{\mathbb{R}^{\nvDash}} \Phi(x,y)m(y)u(y)\,dy, \quad x \in \mathbb{R}^3, \tag{19}$$

where $u \in L^2(D)$ with $supp(m) = \hat{D}$.

Theorem 20. *If $u \in H^2_{loc}(\mathbb{R}^3)$ is a solution to DSP, then u is solution to (19) in $L^2(D)$. Conversely, if u is a solution to (19), then $u \in H^2_{loc}(\mathbb{R}^3)$ and u solves DSP.*

When $k > 0$ is sufficiently small, can show uniqueness of solution u to (19) and therefore DSP using method of successive approximations. The proof of th existence of unique solution to (19) for arbitrary $k > 0$ is more delicate and is based on the *unique continuation principle*.

Theorem 21 (Unique Continuation Principle)**.** *Let G be a domain of $\mathbb{R}^3$ and suppose $u \in H^2(G)$ is a solution to $\Delta u + k^2 n(x)u = 0$ in G for $n \in L^2(G)$. Then if u vanishes in a neighborhood of some point in G, $u \equiv 0$ in G.*

Theorem 22 (Corollary 6)**.** *For each $k > 0$ $\exists! u \in H^2_{loc}(\mathbb{R}^3)$ to DSP.*

Figure 1: Plane wave $e^{ik\cdot d}$ on $(-1,1)^2$, $k=30$, angle $=0.524$

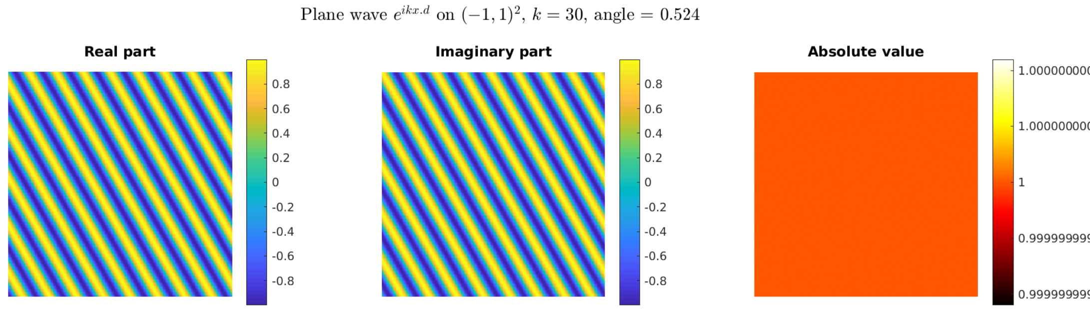


Figure 2: Scattered Field Solution of a plane wave of direction **d**$=(\frac{\sqrt{3}}{2},\frac{1}{2})$ by a sound-soft disc with radius 0.25 at $k=30$.

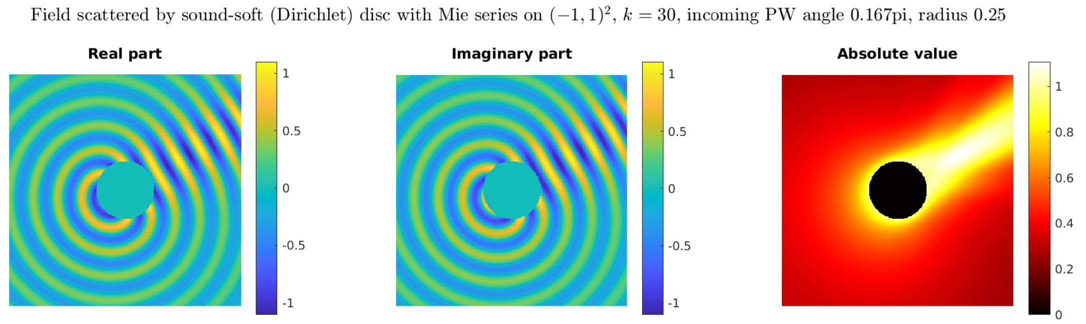
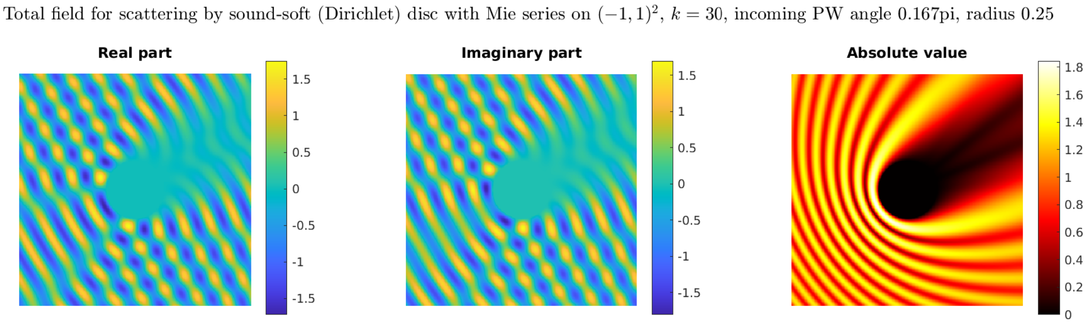


Figure 3: Total Field of a plane wave of direction **d**$= (\frac{\sqrt{3}}{2}, \frac{1}{2})$ by a sound-soft disc with radius 0.25 at $k = 30$.



4 Inverse Scattering: Factorization Method

In the inverse medium scattering problem with far field data one seeks to determine properties of the scatterer from the knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all directions $\hat{x}$ in a given set of measurement directions and all d in a given set of directions of incidence.

The **factorization method** is a *qualitative approach* to inverse scattering theory. Namely the method helps in determining the support $\hat{D}$ of $m := 1 - n$ which is not based on nonlinear optimization techniques or weak scattering approximation. Consider the following theorem from functional analysis:

Theorem 23. *Let X and H be Hilbert spaces and assume $F : H \to H$, $B : X \to H$, and $T : X \to X$ are bounded linear operators such that*

$$F = BTB^*$$

where B^* is the adjoint of B,

$$\Im(Tf, f) \neq 0$$

for all $f \in \overline{B^*H}$ with $f \neq 0$ and $T = T_0 + C$ where C is compact such that

$$(T_0 f, f) \in \mathbb{R},$$
$$(T_0 f, f) \geq c||f||^2$$

for all $f \in \overline{B^*H}$ and some $c > 0$. Then for any $g \in H$ with $g \neq 0$ we have that $g \in \mathcal{B}(X)$ iff

$$\inf\{|(F\psi, \psi)| : \psi \in H, (g, \psi) = 1\} > 0.$$

The factorization method solves the following inverse scattering problem:

(ISP) Given $u_\infty(\hat{x}, d)$ for all $\hat{x} \in S^2$ and all $d \in S^2$, find the support of D of the scattering object.

We recall that the far field pattern u_∞ defines the far field operator $F : L^2(S^2) \to L^2(S^2)$ by

$$(Fg)(\hat{x}) = \int_{S^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in S^2.$$

For an inverse problem, this operator contains the known data. The aim of the inverse problem is to give explicit characterizations of unknown data D by unknown domain D by this data operator F.

We introduce factorization of operator F in the form

$$F = GTG^*,$$

where G is a compact operator and T is an isomorphism between suitable spaces which depend on D. There is a simple relationship between the range of the operator G and the shape of D.

Definition 6. *Let the* **data-to-pattern operator** $G : H^{1/2}(\Gamma) \to L^2(S^2)$ *be defined by* $Gf = v_\infty$*, where* $v_\infty \in L^2(S^2)$ *is the far field pattern of the solution* v *to the exterior Dirichlet problem with boundary data* $f \in H^{1/2}(\Gamma)$*:*

$$\begin{aligned} \Delta v + k^2 v &= 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ v &= f \quad \text{on } \Gamma, \\ \frac{\partial v}{\partial \nu} - ikv &= O(r^{-2}) \text{ for } r = |x| \to \infty \end{aligned}$$

uniformly w.r.t. $\hat{x}$*.*

Theorem 24. *Let* G *be the data-to-pattern operator. For* $z \in \mathbb{R}^3$ *define the map* $\phi_z \in L^2(S^2)$ *by*

$$\phi_z(\hat{x}) = e^{ikx\cdot z}, \quad \hat{x} \in S^2.$$

Then ϕ_z *belongs to* $\mathcal{R}(G)$ *of* G *iff* $z \in D$*.*

Proof. Let $z \in D$ and define

$$v(x) := \Phi(x, z) = \frac{\exp(ik|x - z|)}{|x - z|}, \quad x \neq z$$

with $f := v|_\Gamma$. Then $f \in H^{1/2}(\Gamma)$ and the corresponding far field pattern of v is given by

$$v^\infty(\hat{x}) = \exp(-ik\hat{x} \cdot z), \quad \hat{x} \in S^2,$$

which coincides with phi_z, i.e., $Gf = v^\infty = \phi_z$. Hence, $\phi_z \in \mathcal{R}(G)$.
Let $z \notin D$ and argue by contradiction: assume there is a $f \in H^{1/2}(\Gamma)$ with $Gf = \phi_z$. Let v be solution to the EDP with boundary data f and $v^\infty = Gf$ be the far field pattern. We know ϕ_z is the far field pattern of $\Phi(\cdot, z)$ hence by Rellich's Lemma we have $v(x) = \Phi(x, z)$ for all x outside any sphere containing D and z. Finally, by analytic continuation we have that v and $\Phi(\cdot, z)$ coincide on $\mathbb{R}^3 \setminus (\overline{D} \cup \{z\})$.
If $z \notin \overline{D}$ we contradict analyticity of v outside D with $\Phi(x, z)$ being singular at $x = z$.
If $z \in \Gamma$ we have $\Phi(x, z) = f(x)$, hence $x \mapsto \Phi(x, z)$ is in $H^{1/2}(\Gamma)$. But this map is not $H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})$ since $\nabla\Phi(x, z) = \mathcal{O}(1/|x - z|^2)$ as $x \to z$. $\square$

Theorem 25. *$G : H^{1/2}(\Gamma) \to L^2(S^2)$ is compact, 1:1, and has dense range in $L^2(S^2)$.*

The operator T is the adjoint of the single-layer operator $S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$. Note that the single layer boundary operator $S : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is defined by

$$(S\varphi)(x) := \int_\Gamma \Phi(x, y)\varphi(y)\, ds(y),$$

where $\varphi(y)$ is the surface density of the boundary integral. By $\langle H^{-1/2}(\Gamma), H^{1/2}(\Gamma)\rangle$ we denote the dual form which is the extension of the inner product of $L^2(\Gamma)$. This dual form is sesquilinear, i.e., a mapping $\varphi \mapsto \langle\varphi, \psi\rangle$ and $\psi \mapsto \overline{\langle\varphi, \psi\rangle}$ are linear where the bar denotes complex conjugation.

Theorem 26 (Lemma)**.** *Assume k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the following holds:*

1. *S is an isomorphism from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$.*
2. *$\Im\langle\varphi, S\varphi\rangle < 0$ for all $\varphi \in H^{-1/2}(\Gamma)$ with $\varphi \neq 0$.*
3. *Let S_i be the single layer boundary operator corresponding to the wavenumber $k = i$. Then S_i is self-adjoint and coercive as an operator from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$, i.e., there is a $c_0 > 0$ with*
$$\langle\varphi, S_i\varphi\rangle \geq c_0||\varphi||^2_{-\frac{1}{2},\Gamma} \quad \forall\varphi \in H^{-1/2}(\Gamma).$$
4. *Difference $S - S_i$ is a compact operator.*

The next theorem displays the fundamental factorization of far field operator F:

Theorem 27. *The following relation holds between F, G, S:*

$$F = -GS^*G.$$

Proof. As an auxiliary operator, we define $H : L^2(S^2) \to H^{1/2}(\Gamma)$ by

$$(Hg)(x) = \int_{S^2} g(d) \exp(ikx \cdot d)\, ds(d), \quad x \in \Gamma.$$

Now Hg is the trace on Γ of the Herglotz wave function with density g. Its adjoint $H^* : H^{-1/2}(\Gamma) \to L^2(S^2)$ is given by

$$(H^*\varphi)(\hat{x}) = \int_{S^2} \varphi(y) \exp(-ikx \cdot y)\, ds(y), \quad \hat{x} \in S^2.$$

By the asymptotic behavior of the fundamental solution we know that $H^*\varphi$ is the far field pattern of the single layer potential. So $H^*\varphi = GS\varphi$, i.e.,

$$H^* = GS \quad \text{and} \quad H = S^*G^*.$$

Finally, Fg is the far field pattern of the exterior Dirichlet problem with boundary data

$$-\int_\Gamma g(d) \exp(ikx \cdot d)\, ds(d) = -(Hg)(x), \quad x \in \Gamma.$$

Hence, $Fg = -GHg$. Finally, substitute $H = S^*G^*$. □

In short, we have the following commuative diagram:

$$\begin{array}{ccc} L^2(S^2) & \xrightarrow{F} & L^2(S^2) \\ {\scriptstyle G^*}\big\downarrow & & \big\uparrow{\scriptstyle G} \\ H^{-1/2}(\Gamma) & \xrightarrow{S^*} & H^{1/2}(\Gamma) \end{array}$$

Theorem 28. *Let X and H be Hilbert spaces and let the operators F, T, and B satisfy the assumptions of Theorem 23. In addition, let the operator $F : H \to H$ be compact, injective and assume that $I + i\gamma F$ is unitary for some $\gamma > 0$. Then the ranges $\mathcal{B}(X)$ and $(F^*F)^{1/4}(H)$ coincide.*

Theorem 29 (Corollary)**.** *Let F be the far field operator and assume k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D. Then $z \in D$ iff*

$$(F^*F)^{1/4} g_z = \Phi_\infty(\cdot, z)$$

is solvable in $L^2(S^2)$.

This characterization of the scatterer in terms of the range $(F^*F)^{1/4}$ can be used for a reconstruction with the aid of a singular system $(|\lambda_n|, \psi_n, \psi_n)$ of the operator F.

5 Inverse Scattering: MUSIC Algorithm

Factorization method has a discrete analogue known as the MUSIC algorithm (**MU**ltiple-**SI**gnal-**C**lassification). This algorithm originated with signal-processing applications, but it can also be applied to imaging. It provides a method to determine one or more unknown objects (point scatterers) from the so-called multistatic response matrix $\mathbf{F}$.

5.1 The MUSIC Algorithm

Consider an array of M point scatterers at locations $y_1, \dots, y_M \in \mathbb{R}^3$ in the homogeneous, isotropic space $\mathbb{R}^3$. Incident plane waves of the form

$$u^i(x, d) = \exp(ikx \cdot d), \quad x \in \mathbb{R}^3,$$

scattered at the targets at y_m. We neglect all multiple scattering between the scatterers. Then u^s is given by

$$u^s(x, d) := \sum_{m=1}^{M} \tau_m u^i(y_m, d)\Phi(x, y_m).$$

Here $\tau_m \in \mathbb{C} \setminus \{0\}$ is the scattering strength of the $m-$th target, $m = 1, \dots, M$, and Φ is again the fundamental solution of the Helmholtz equation in $\mathbb{R}^3$. Can obtain the far field pattern from the asymptotic behavior of $\Phi(x, y)$ we conclude

$$u^s(\hat{x}, d) = \frac{\exp(ik|x|)}{4\pi|x|} \sum_{m=1}^{M} \tau_m u^i(y_m, d) e^{-ik\hat{x}\cdot y_m} + \mathcal{O}(|x|^{-2}), \quad |x| \to \infty$$

yields

$$u^\infty(\hat{x}, d) = \sum_{m=1}^{M} \tau_m u^i(y_m, d) e^{-ik\hat{x}\cdot y_m}, \quad \hat{x} \in S^2.$$

We can now state the *inverse scattering problem* as follows:

Definition 7 (Inverse Scattering Problem)**.** *Determine the locations $y_1, \dots, y_M$ from the response $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in S^2$ or from a finite subset $\{\theta_j : j = 1, \dots, N\} \subset S^2$. In the finite case we assume $N \geq M$ and define the multistatic response matrix $\mathbf{F} \in \mathbb{C}^{N\times N}$ by*

$$F_{jl} := u^\infty(\hat{x}_j, d_l),$$

for $j, l = 1, \dots, N$.

We can define the matrices $\mathbf{H} \in \mathbb{C}^{M\times N}$ and $\mathbf{T} \in \mathbb{C}^{M\times M}$ by

$$\begin{aligned} H_{ml} &= \sqrt{|\tau_m|}e^{ikd_l\cdot y_m}, \quad l = 1, \dots, N,\ m = 1, \dots, M \\ \mathbf{T} &= \operatorname{diag}(\operatorname{sign}\tau_m). \end{aligned}$$

We observe $\mathbf{F}$ has a factorization in the form

$$\mathbf{F} = \mathbf{H}^*\mathbf{T}\mathbf{H}$$

with adjoint $\mathbf{H}^*$. If $N \geq M$ and if the locations y_m are such that $\mathbf{H}$ has maximal rank M then, by a standard result in linear algebra, the ranges $\mathcal{R}(\mathbf{H}^*)$ and $\mathcal{R}(\mathbf{F})$ of $\mathbf{H}^*$ and $\mathbf{F}$ respectively, coincide.

For any point $z \in \mathbb{R}^3$ we define the vector $\phi_z \in \mathbb{C}^N$ by

$$\phi_z = (e^{-ikd_1\cdot z}, \dots, e^{-ikd_n\cdot z})^T.$$

Then we obtain the following theorem

Theorem 30. *Let $\{d_n : n \in \mathbb{N}\} \subset S^2$ be a countable set of directions such that any analytic function on S^2 that vanishes in d_n for all $n \in \mathbb{N}$ vanishes identically. Let K be a compact subset of $\mathbb{R}^3$ containing all y_m. Then there is a N_0 such that for any $N \geq N_0$ the following characterization holds for every $z \in K$:*

$$z \in \{y_1, \dots, y_M\} \iff \phi_z \in \mathcal{R}(\mathbf{H}^*).$$

Furthermore, the ranges of $\mathbf{H}^*$ and $\mathbf{F}$ coincide and so

$$z \in \{y_1, \dots, y_M\} \iff \phi_z \in \mathcal{R}(\mathbf{F}) \iff \mathbf{P}\phi_z = 0,$$

where $\mathbf{P} : \mathbb{C}^N \to \mathcal{R}(\mathbf{F})^\perp = \mathcal{N}(\mathbf{F}^*)$ is the orthogonal projection onto left null space of $\mathbf{F}$.

Hence the key to implementing MUSIC is plotting the function

$$W(z) = \frac{1}{|\mathbf{P}\phi_z|}, \quad z \in \mathbb{R}^3,$$

which should result in sharp peaks at $y_1, \dots, y_M$.

6 Inverse Scattering: Linear Sampling Method

The linear sampling method (LSM) is the most frequently used qualitative inversion method, and has been numerically proven to be a fast and reliable method in many situations. Roughly speaking, the idea of the linear sampling method (LSM) is to consider approximate solutions to the range of the far field operator $F : L^2(S^2) \to L^2(S^2)$ defined by

$$(Fg)(\hat{x}) := \int_{S^2} u^\infty(\hat{x}, d) g(d)\, ds(d),$$

i.e., $g_z \in L^2(S^2)$ satisfying

$$Fg_z \simeq \Phi(\cdot, z), \tag{20}$$

with $\Phi(x, z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}$ being the far field pattern associated to the fundamental solution $\Phi(\cdot, z)$. Then we use $z \mapsto 1/||g_z||_{L^2(S^2)}$ as an indicator function for the domain D. We mainly give a presentation of the method in the special case where $u^\infty(\cdot, d)$ is the far field pattern associated with the scattered field solution $u^s(\cdot, d) \in H^1_{loc}(\mathbb{R}^3)$. The index of refraction $n \in L^\infty(\mathbb{R}^3)$ is such that $\Re n > 0, \Im n \geq 0, n = 1$ outside the support $\overline{D}$ of $m := 1 - n$, and assume D contains the origin , has Lipschitz boundary ∂D and connected component in $\mathbb{R}^3$.

Let us define for $u_0 \in L^2(D)$ the unique function $w \in H^1_{loc}(\mathbb{R}^3)$ satisfying

$$\Delta w + k^2 n w = k^2 m u_0 \quad \text{in } \mathbb{R}^3$$

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial w}{\partial r} - ikw \right|^2 ds = 0.$$

If $u_0(x) = e^{ikx\cdot d}$ then $w = u^s(\cdot, d)$, hence the far field pattern w^∞ coincides with u^∞. Let us consider the compact (Hilbert-Schmidt) operator $\mathcal{H} : L^2(S^2) \to L^2(D)$ defined by

$$\mathcal{H}g := v_g|_D,$$

where $v_g(x) := \displaystyle\int_D e^{ikx\cdot d} g(d)\, ds(d)$ is the Herglotz wave kernel. Let us denote by $\mathcal{H}_{inc} = \overline{\mathcal{R}(\mathcal{H})}$ in $L^2(D)$. We then consider the compact operator $G : \mathcal{H}_{inc} \to L^2(S^2)$ defined by

$$G(u_0) := w^\infty.$$

Then we can factorize F as

$$F = G\mathcal{H}.$$

The justification of the LSM is then based on the characterization of D in terms of the range of the operator G (i.e., LSM is again a special case of the factorization method mentioned earlier). This characterization uses the solvability of the *interior transmission eigenvalue problem* outlined earlier. We discussed well-posedness assumptions for this problem. For the sake of simplicity, we assume only real transmission eigenvalues. A first step towards the justification of LSM is the characterization of the closure of the range of $\mathcal{H}$.

Theorem 31 (Lemma)**.** *The operator $\mathcal{H}$ is compact and injective. Let $\mathcal{H}_{inc} = \overline{\mathcal{R}(\mathcal{H})}$. Then*

$$\mathcal{H}_{inc}(D) = \{v \in L^2(D) \,:\, \Delta v + k^2 v = 0 \text{ in } D\}.$$

Proof. $\mathcal{H}$ is a Hilbert-Schmidt operator and therefore compact. Assume $\mathcal{H}g = 0$ in D. Since

$$\Delta \mathcal{H}g + k^2 \mathcal{H}g = 0 \quad \text{in } \mathbb{R}^3,$$

by the unique continuation principle. $\mathcal{H}g = 0$ in all of $\mathbb{R}^3$. Applying the Jacobi-Anger expansion, we obtain $g = 0$.

Set $\widetilde{\mathcal{H}_{inc}} = \{v \in L^2(D) \,:\, \Delta v + k^2 v = 0 \text{ in } D\}$. Then clearly $\mathcal{H}_{inc} \subseteq \widetilde{\mathcal{H}_{inc}}$. To verify the other inclusion, it is sufficient to prove that the adjoint $\mathcal{H}^* : L^2(D) \to L^2(S^2)$ given by

$$\mathcal{H}^* = \int_D e^{-ik\hat{x}\cdot y} \varphi(y)\, dy, \quad \varphi \in L^2(D), \quad \hat{x} \in S^2,$$

is injective on $\widetilde{\mathcal{H}_{inc}}$. Let $u_0 \in \widetilde{\mathcal{H}_{inc}}$ and set u to be the volume potential, i.e.,

$$u(x) := \int_D \Phi(x, y) u_0(y)\, dy, \quad x \in \mathbb{R}^3.$$

From the regularity properties of the volume potential, we infer $u \in H^2_{loc}(\mathbb{R}^3)$ and satisfies

- $\Delta u + k^2 u = -u_0 \quad \text{in } D,$
- $\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}.$

By construction $4\pi u^\infty = \mathcal{H}^*(u_0)$, so $\mathcal{H}^*(u_0) = 0$ implies $u^\infty = 0$ which implies $u = 0$ in $\mathbb{R}^3 \setminus \overline{D}$ by Rellich's Lemma. The regularity of $u \in H^2_{loc}(\mathbb{R}^3)$ implies $u \in H^2_0(D)$. Taking the $L^2(D)$ scalar product with u_0 and applying $u \in H^2_0(D)$ yields $||u_0||_{L^2(D)} = 0$, as desired. ☐

We now have a familiar theorem discussed in factorization section.

Theorem 32. *We assume that the refractive index n and the real wave number k are such that the interior transmission eigenvalue problem defines a well posed problem. Then the operator $G : \mathcal{H}_{inc} \to L^2(S^2)$ is injective with dense range. Moreover,*

$$\Phi^\infty(\cdot, z) \in \mathcal{R}(G) \iff z \in D.$$

Proof. We will mainly prove the last part of the theorem. (Injectivity of G follows by Rellich's Lemma and dense range follows from Rellich's Lemma and the unique continuation principle). First observe that $\Phi^\infty(\cdot, z)$ is the far field pattern of $u_e = \Phi(\cdot, z)$ satisfying $\Delta u_e + k^2 u_e = \delta_z$ in $\mathbb{R}^3$ and the Sommerfield radiation condition. Let $z \in D$. We consider $(u, u_0) \in L^2(D) \times L^2(D)$ as being the solution to the interior transmission eigenvalue problem with

$$f(x) = u_e(x; z); \quad \text{and } h(x) = \frac{\partial u_e(x; z)}{\partial \nu(x)} \text{ for } x \in \partial D.$$

We then define w by

$$\begin{aligned} w(x) &= u(x) - u_0(x) \quad \text{in } D, \\ w(x) &= u_e(x; z) \quad \text{outside } D. \end{aligned}$$

By the prior lemma, $w \in H^2_{loc}(\mathbb{R}^3)$ and hence $Gu_0 = \Phi^\infty(\cdot, z)$. Now let $z \in \mathbb{R}^3 \setminus \overline{D}$. Assume there exists $u_0 \in \mathcal{H}_{inc}$ such that $Gu_0 = \Phi^\infty(\cdot, z)$. By Rellich's lemma we deduce $w = u_e(\cdot; z)$ in $\mathbb{R}^3 \setminus \overline{D}$. But this gives a contradiction: $u_e(\cdot; z) \notin H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})$ but $w \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{D})$. $\square$

Since the operator $\mathcal{H}$ is compact, the characterization of D in terms of the range of G does not imply a similar characterization in terms of the range of F. However, we can deduce the following:

Theorem 33. *The operator F is injective and has dense range. Moreover,*

- *If $z \in D$ then there exists a sequence $g_z^\alpha \in L^2(S^2)$ such that the*

$$\lim_{\alpha \to 0} ||Fg_z^\alpha - \Phi^\infty(\cdot, z)||_{L^2(S^2)} = 0,$$
$$\lim_{\alpha \to 0} ||\mathcal{H} g_z^\alpha||_{L^2(S^2)} < \infty.$$

- *If $z \notin D$ then for all $g_z^\alpha \in L^2(S^2)$ such that $\lim_{\alpha \to 0} ||Fg_z^\alpha - \Phi^\infty(\cdot, z)||_{L^2(S^2)} = 0$:*

$$\lim_{\alpha \to 0} ||\mathcal{H} g_z^\alpha||_{L^2(S^2)} = \infty.$$

The main weak point in this theorem is that it does not indicate how to construct the sequence g_z^α when $z \in D$. In practice one relies on the use of Tikhonov regularization and considers $\widetilde{g_z^\alpha} \in L^2(S^2)$ satisfying

$$(\alpha + F^* F)\widetilde{g_z^\alpha} = F^*(\Phi^\infty(\cdot, z)). \tag{21}$$

Since F has dense range, $\lim_{\alpha \to 0} ||F\widetilde{g_z^\alpha} - \Phi^\infty(\cdot, z)||_{L^2(S^2)} = 0$. However, one cannot guarantee in general that $\lim_{\alpha \to 0} ||\mathcal{H}\widetilde{g_z^\alpha}||_{L^2(S^2)} < \infty$. In the case of $\Im n = 0$, the latter was proved to hold based on the $(F^* F)^{1/4}$ method. [0]

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