TD 4 CALCULUS – SOLUTIONS

(1) **Consider the recurrent sequence defined for $n > 0$ by $u_n = u_{n-1} + \frac{1}{u_{n-1}}$ and $u_0 = 1$. Examine whether this sequence converges to a finite limit or not.**

Assume that the series converges to some finite limit $u = \lim_{n\to\infty} u_n$. It is clear that $u_n > 0$ and so $u \geq 0$, since u_0 is positive. For $u > 0$, we get the equality

$$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(u_{n-1} + \tfrac{1}{u_{n-1}}\right) = u + \tfrac{1}{u} .$$

A simple transformation shows that $\frac{1}{u} = 0$, but this is a contradiction. Similarly we get a contradiction for $u = 0$. So the series can not be convergent.

(2) **Do the following series converge?**

(a) $\sum_{n\geq 0} \frac{(n+2)^2}{2^{n+2}}$,

For this series, the terms of the series are given as quotients, such that we should try the quotient criterion first. We determine

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+3)^2 2^{n+2}}{2^{n+3}(n+2)^2} = \frac{(n+3)^2}{2(n+2)^2} .$$

We divide both numerator and denominator by the most dominant term n^2, and we see that $\lim_{n\to\infty} |a_{n+1}/a_n|$ equals $\frac{1}{2}$. Therefore, for sufficiently large n, we can guarantee that $|a_{n+1}/a_n| < \frac{3}{4} < 1$, and the quotient criterion proves convergence.

(b) $\sum_{n\geq 0} 0.9999^n$.

One might be tempted to also apply the quotient criterion for this series, but we can do better. A closer look at the series shows that this is indeed a geometric sum, and we can simply prove convergence by calculating the value if the series, which is $1/(1 - 0.9999) = 10000$. So the series is convergent.

(c) $\sum_{n\geq 2} (-1)^{n-1} \frac{n^2-n}{n^2+n}$, It is easy to see that the given series is alternating, so we guess that we should try the Leibniz criterion. First we observe that $\sum_{n\geq 2}(-1)^{n-1}\frac{n^2-n}{n^2+n} = -\sum_{n\geq 2}(-1)^n\frac{n^2-n}{n^2+n}$. If we define $a_n = \frac{n^2-n}{n^2+n}$, we easily see that $\lim_{n\to\infty} a_n = 1$, such that Leibniz can NOT be used, and we do not get any information about convergence.

The next idea is based on the transformation

$$a_n = \frac{n^2-n}{n^2+n} = \frac{n^2+n}{n^2+n} - \frac{2n}{n^2+n} = 1 - \frac{2n}{n^2+n} .$$

Therefore, the original alternating series is equal to

$$\sum_{n\geq 2}(-1)^{n-1}\left(1-\tfrac{2n}{n^2+n}\right) = \sum_{n\geq 2}(-1)^{n-1} + \sum_{n\geq 2}(-1)^n \underbrace{\tfrac{2n}{n^2+n}}_{:=b_n} .$$

We first consider the second sum in the latter equality. We again try to apply Leibniz: clearly $b_n > 0$, $\lim_{n\to\infty} b_n = 0$, and

$$b_n = \tfrac{2n}{n^2+n} > \tfrac{2n+2}{n^2+3n+2} = b_{n+1} \text{ for all } n \geq 2 .$$

Therefore, the second sum is convergent as proven by the Leibniz criterion. Now let us assume that the input series is convergent. Then clearly (as sum of two convergent series) also

$$\sum_{n\geq 2}(-1)^{n-1}$$

must be convergent, say with limit λ. Therefore, for every $\epsilon > 0$ there exists an integer m_0 such that for <u>all</u> $m > m_0$ we have

$$\left|\sum_{n=2}^{m}(-1)^{n-1} - \lambda\right| < \epsilon .$$

But this clearly is impossible if $\epsilon < \frac{1}{2}$, since the value of the partial sum up to m and the value of the partial sum up to $m+1$ differ by 1 or -1. We get a contradiction to our assumption, and in conclusion, the input series can not be convergent.

(3) **Nature of functions.** Are the functions given below injective from their domain to $\mathbb{R}$? Surjective? Continuous on their respective domain?

(a) $f(x) = x^3$,

Function is injective (it is strictly monotonous increasing), surjective and continuous everywhere in $\mathbb{R}$.

(b) $g(x) = x^3 - x$,

The function $g(x)$ can not be injective since $g(0) = g(-1) = g(1) = 0$. It is continuous since both x^3 and x are continuous. Note that

$$x^3 - x \;=\; x \cdot (x^2 - 1) \;\geq\; x \quad \text{for} \quad |x| \geq 1 \,.$$

Therefore, we can conclude that $\lim_{x\to-\infty} g(x) = -\infty$ and $\lim_{x\to\infty} g(x) = \infty$. In class, we have seen the intermediate theorem which proves that for every $y \in \mathbb{R}$ the equation $g(x) = y$ has at least one real solution (since g is continuous). Thus, $g(x)$ is surjective.

(c) $h(x) = \lfloor x \rfloor$ (floor of x is defined as the largest integer at most equal to x),

The function $h(x)$ is not injective ($h(1) = 1 = h(1.5)$), not surjective (no x exists with $h(x) = \frac{1}{2}$), and not continuous for all $x \in \mathbb{Z}$.

(d) $j(x) = |x|$,

The function $j(x)$ can not be injective since $j(1) = 1 = j(-1)$. It is also not surjective since there does not exist $x \in \mathbb{R}$ with $j(x) = -1$. However, the function is continuous everywhere in its domain.

(e) $k(x) = \frac{1}{x}$.
The function $k(x)$ is injective, but not surjective. There is a single real number which is not in the image of $k(x)$, namely 0, since $k(x) \neq 0$ for all $x \in \mathbb{R}$. The function is continuous for all numbers in its domain.