TD 1 CALCULUS – SOLUTIONS

(1) **Prove that for all integers $n \geq 1$ we have $8 \,|\, 9^n - 1$.**

We use mathematical induction. For $n = 1$, we clearly see that 8 divides $9^1 - 1 = 8$. Therefore, assume that the statement is true for some fixed integer $n \geq 1$. We test the statement for $n + 1$:

$$9^{n+1} - 1 \;=\; (9^n - 1) \cdot 9 + 8 \,.$$

Since by assumption 8 divides $9^n - 1$, we can conclude that the same is true for $9^{n+1} - 1$.

(2) **Prove that for $n \geq 1$ we have**

$$\sum_{i=1}^{n} 2^i \quad = \quad 2^{n+1} - 2 \,.$$

Again we use induction. For $n = 1$, we have $\sum_{i=1}^{1} 2^i = 2^1 = 2^2 - 2$. Then assume that the statement is true for some fixed index $n \geq 1$. We test the statement for $n + 1$:

$$\sum_{i=1}^{n+1} 2^i \;=\; \left(\sum_{i=1}^{n} 2^i\right) + 2^{n+1} \;=\; 2^{n+1} - 2 + 2^{n+1} \;=\; 2^{n+2} - 2 \,.$$

(3) **Prove by induction that for $n \geq 4$ the inequality $n! > 2^n$ is true.**

The minimal value for n in this case is $n = 4$. We calculate $4! = 24$ and $2^4 = 16$, so the statement is true. Therefore, assume that the statement is true for an $n \geq 4$. Then we test the inequality for $n + 1$:

$$(n+1)! \;=\; (n+1) \cdot n! \;>\; (n+1) \cdot 2^n \;>\; 2 \cdot 2^n \;=\; 2^{n+1} \,.$$

(4) **(Binomial Theorem) Determine the value of the following sum (without induction)**

$$\sum_{k=0}^{n} \binom{n}{k} \quad = \quad ??? \,.$$

If you check the formula for the Binomial Theorem given in the handouts for last week's class, then you can find the formula

$$(a+b)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} a^k \, b^{n-k} \, .$$

If you compare the right hand side with the given problem, then you see that the a and b terms are "missing" in the exercise, or equivalently they are equal to 1. Therefore, a^k and b^{n-k} must be equal to 1 for every value of k, so $a = b = 1$. So we conclude that the answer is equal to $(1+1)^n = 2^n$.