TD 3 CALCULUS – SOLUTIONS

(1) **Prove the formula for a geometric series:**

$$\forall n \geq 0, \forall x \neq 1 : \qquad 1 + x + x^2 + \ldots + x^n \;=\; \frac{x^{n+1} - 1}{x - 1}\,.$$

The formula is easy to prove if we simply multiply both its sides with $x - 1$. Then we get

$$\begin{aligned}&(x-1)\cdot(1 + x + x^2 + \ldots + x^n)\\ =\;& x^{n+1} + x^n + \ldots + x - (x^n + \ldots + 1) \;=\; x^{n+1} - 1\end{aligned}$$

and almost all x^i terms are both added and subtracted (remember this trick!!).

(2) **Find a general formula for $\sum_{i=n_0}^{\infty} x^i$ and $\sum_{i=n_0}^{m} x^i$ for $n_0, m \in \mathbb{N}$ and $|x| < 1$.**

We can use the formula for the finite geometric series given in class. Then

$$\sum_{i=n_0}^{\infty} x^i \;=\; \sum_{i=0}^{\infty} x^i \;-\; \sum_{i=0}^{n_0-1} x^i \;=\; \frac{1}{1-x} - \frac{1 - x^{n_0}}{1-x} \;=\; \frac{x^{n_0}}{1-x}\,.$$

Similarly, we obtain for the second sum the value

$$\sum_{i=n_0}^{m} x^i \;=\; \sum_{i=0}^{m} x^i \;-\; \sum_{i=0}^{n_0-1} x^i \;=\; \frac{1 - x^{m+1}}{1-x} - \frac{1 - x^{n_0}}{1-x} \;=\; \frac{x^{n_0} - x^{m+1}}{1-x}\,.$$

(3) **Compute the values of the following series:**

(a) $10 + 20 + 40 + 80 + \ldots + 10240$

We note that this sequence can be written as $10 \cdot (1 + 2 + 4 + \ldots + 1024)$. Therefore, we can use the formula for a finite geometric series, and we obtain the value of the series as

$$10 \cdot (1 + 2 + 4 + \ldots + 1024) \;=\; 10 \cdot \sum_{i=0}^{10} 2^i \;=\; 10 \cdot \frac{1 - 2^{11}}{1 - 2} \;=\; 20470\,.$$

(b) $12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \dots$

We note that this series can be rewritten as an infinite geometric series with some "added noise":

$$12 + 4 \cdot \left(1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \dots\right) \;=\; 12 + 4 \cdot \sum_{i=0}^{\infty} \left(\tfrac{1}{3}\right)^i \;=\; 12 + 4 \cdot \frac{1}{1 - \frac{1}{3}} \;=\; 18 \; .$$

(c) $S = 2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$

This series looks similar to an alternating geometric series, but the minus terms are shifted by one position. To correct this shift, we have to multiply the series by -1. After doing this we get

$$-S = -2 + 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - \dots \;=\; -2 + \sum_{i=0}^{\infty} \left(\tfrac{-1}{2}\right)^i \;=\; -2 + \frac{1}{1 + \frac{1}{2}} \;=\; -\tfrac{4}{3} \; .$$

From this observation we get the value of the infinite series as $\frac{4}{3}$.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2^n}{3^{n-1}}$

Again the series looks like an infinite geometric series with the small problem that the starting index does not equal 0. So we must correct the starting index:

$$\sum_{n=1}^{\infty} \tfrac{(-1)^n \cdot 2^n}{3^{n-1}} \;=\; \sum_{n=0}^{\infty} \tfrac{(-1)^{n+1} \cdot 2^{n+1}}{3^n} \;=\; -2 \cdot \sum_{n=0}^{\infty} \tfrac{(-2)^n}{3^n} \;=\; -2 \cdot \frac{1}{1 + \frac{2}{3}} \;=\; -\tfrac{6}{5} \; .$$

(4) **Assume a series whose general term a_i is given by $a_i = 2 - \frac{1}{3^{2i}}$. What is the value of the partial sum $\sum_{i=0}^{n} a_i$? Does the series converge for $n \to \infty$?**

We start with finding an expression for the partial sum up to index n. A simple calculation gives us

$$\begin{aligned} \sum_{i=0}^{n} \left(2 - \tfrac{1}{3^{2i}}\right) &= \sum_{i=0}^{n} 2 - \sum_{i=0}^{n} \frac{1}{3^{2i}} \;=\; 2(n+1) - \sum_{i=0}^{n} \frac{1}{9^i} \\ &= 2(n+1) - \frac{1 - (1/9)^{n+1}}{1 - (1/9)} \;=\; 2(n+1) - \frac{9 - (1/9)^n}{8} \; . \end{aligned}$$

To answer the question whether the series converges for $n \to \infty$, we have to examine convergence of this partial sum. Since $\lim_{n\to\infty} (1/9)^n = 0$, we see that the second part of the partial sum expression converges to some constant, whereas $2(n+1)$ clearly is divergent. In total, the sequence defined by the partial sums can not be convergent, and the infinite series therefore also does not converge for $n \to \infty$.