TD 8 CALCULUS – SOLUTIONS

(1) **Determine an anti-derivative for the following functions:**

(a) $(2x+1)\cdot\exp(x)$,

We notice that the given function is a product of two functions. Therefore, we use the integration by parts rule with $f = 2x+1$ and $g' = \exp(x)$ (note that the described thumb rule was applied to choose f as the polynomial factor). Then we get a result as

$$\int (2x+1)\cdot\exp(x)\,dx \;=\; (2x+1)\cdot\exp(x) \;-\; \int 2\exp(x)\;dx \;=\; (2x-1)\cdot\exp(x)\;.$$

(b) $(5x^2+1)\cdot\cos(x)$,

Again we note that the function is a product of two functions, and we choose $f = 5x^2+1$ and $g' = \cos(x)$ in the integration by parts rule. Then we determine an antiderivative as (note that we have to apply integration by parts twice, since the first application does not yet result in a basic function)

$$\begin{aligned}\int (5x^2+1)\cdot\cos(x)\,dx \;&=\; (5x^2+1)\cdot\sin(x) \;-\; \int 10\,x\cdot\sin(x)\,dx\\ &=\; (5x^2+1)\cdot\sin(x) \;-\; \Big(-10\,x\cdot\cos(x) + \int 10\,\cos(x)\,dx\Big)\\ &=\; (5x^2+1)\cdot\sin(x) \;+\; 10\,x\cdot\cos(x) \;-\; 10\sin(x)\\ &=\; (5x^2-9)\cdot\sin(x) \;+\; 10\,x\cdot\cos(x)\;.\end{aligned}$$

(c) $(3x^2+2x+1)\cdot\exp(ax)$ for some constant $a\neq 0$,

This is another example of the integration by parts, with $f = (3x^2+2x+1)$ and $g' = \exp(ax)$. With the substitution rule, it is easy to see that an antiderivative of g' is then given as $g = \frac{1}{a}\exp(ax)$. Therefore, we determine the result as

$$\begin{aligned}\int (3x^2+2x+1)\cdot \exp(ax) &= \frac{(3x^2+2x+1)}{a}\cdot \exp(ax) - \int \frac{6x+2}{a}\exp(ax)\,dx \\ &= \frac{(3x^2+2x+1)}{a}\cdot \exp(ax) - \left(\frac{6x+2}{a^2}\exp(ax) - \int \frac{6}{a^2}\exp(ax)\,dx\right) \\ &= \left(\frac{3x^2+2x+1}{a} - \frac{6x+2}{a^2} + \frac{6}{a^3}\right)\cdot \exp(ax)\ .\end{aligned}$$

(2) **Find the antiderivative of the function x^2e^x which has a zero at a (for some fixed constant a).**

We again use integration by parts with $f = x^2$ and $g' = e^x$. Then we get an antiderivative as

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - \int 2xe^x\,dx \\ &= x^2e^x - 2\left(xe^x - \int e^x\,dx\right) \\ &= \left(x^2-2x+2\right)\cdot e^x \ =: \ T(x)\ .\end{aligned}$$

It is clear from class that infinitely many other antiderivatives can be determined by adding arbitrary constants to $T(x)$. We are requested to find the unique antiderivative $T(x)+c$ (for some suitable constant c), which has a zero at a. It is straightforward that c can be chosen as $-T(a) = -(a^2-2a+2)\cdot e^a$.

(3) **Determine the value of the following integrals:**

(a) $\int_0^{\pi/2}(-x^2+x+1)\cdot\sin(x)\,dx$,

We choose the same strategy as in Exercise (1) and apply the integration by parts rule to find an antiderivative, then we use that antiderivative to determine the value of the integral. We get

$$\begin{aligned}\int(-x^2+x+1)\cdot\sin(x)\,dx &= (x^2-x-1)\cos(x) - \int(2x-1)\cos(x)\,dx \\ &= (x^2-x-1)\cos(x) - \left((2x-1)\sin(x) - \int 2\sin(x)\,dx\right) \\ &= (x^2-x-3)\cos(x) - (2x-1)\sin(x)\ .\end{aligned}$$

Now we note that $\sin(0) = 0 = \cos(\pi/2), \sin(\pi/2) = 1 = \cos(0)$, and we get

$$\int_0^{\pi/2} (-x^2 + x + 1) \cdot \sin(x)\, dx \;=\; -\pi + 1 - (-3) \;=\; 4 - \pi\ .$$

(b) $\int_0^x t \cdot e^t\, dt$.

Finally, we use integration by parts again to determine

$$\int t \cdot e^t\, dt \;=\; t \cdot e^t - \int e^t\, dt \;=\; (t-1) \cdot e^t\ .$$

So, the result of the exercise is given as

$$(x-1) \cdot e^x + 1\ .$$