TD6

(1)

Augmented Matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 2 & 2 & 1 \\ 2 & 4 & 2 & 3 & 4 & 2 \\ 3 & 5 & 2 & 1 & 5 & 1 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -5 & -1 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 2 & 2 & 1 \\ 0 & -1 & -1 & -5 & -1 & -2 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right)$$

The last row shows us that the matrix does not a solution, thus (B_1 & B_2) do not generate the same VS.

~~Addition~~

(2)

(a) $h_3 : \mathbb{Q}^3 \rightarrow \mathbb{Q}^2, (x, y, z)^t \longmapsto (x+y+z, x-y-z)^t$

$$h_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-y-z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & w_1 \\ 1 & -1 & -1 & 0 & w_2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & w_1 \\ 0 & -2 & -2 & 0 & w_2 - w_1 \end{array}\right)$$

h_3 is surjective, because rank of matrix in upper triangle form is equal to $\dim(\mathbb{Q}^2)$, which is 2.

Using the ~~rank~~ rank-nullity theorem, we get:

$$\dim(\text{range}(h_3)) + \dim(\text{Ker}(h_3)) = \dim(\mathbb{Q}^3) \Leftrightarrow$$

$$\Leftrightarrow \dim(\text{Ker}(h_3)) = \dim(\mathbb{Q}^3) - \dim(\text{range}(h_3)) \Leftrightarrow$$

$$\Leftrightarrow \dim(\text{Ker}(h_3)) = \cancel{2}\ 3 - (\text{at most } 2), \text{ thus}$$

$\dim(\text{Ker}(h_3))$ will never be 0, hence h_3 is not injective.

(b) $h_4 : \mathbb{Q}^3 \to \mathbb{Q}^3, (x, y, z)^t \to (x+y+z, x-y+z, x+y-z)^t$

$$h_4\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-y+z \\ x+y-z \end{pmatrix} = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & w_1 \\ 1 & -1 & 1 & 0 & w_2 \\ 1 & 1 & -1 & 0 & w_3 \end{array}\right) \to$$

$$\to \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & w_1 \\ 0 & -2 & 0 & 0 & w_2 - w_1 \\ 0 & 0 & -2 & 0 & w_3 - w_1 \end{array}\right)$$ the rank of the matrix is $3 = \dim(\mathbb{Q}^3)$, thus h_4 is surjective.

Applying the rank-nullity theorem we have:

$$\dim(\text{Ker}(h_4)) = \dim(\mathbb{Q}^3) - \dim(\text{range}(h_4))$$

$$\dim(\text{Ker}(h_4)) = 3 - (\text{at most } 3), \text{ thus}$$

$$\dim(\text{Ker}(h_4)) \geq 0, \text{ hence } h_4 \text{ is also}$$

injective.

R: h_4 is bijective.