# TD 2 – SOLUTIONS

(1) **Solve in $\mathbb{R}$:**

(a) $|x-2| \leq 4$ ,

We distinguish the two cases $x-2 \geq 0$ and $x-2 < 0$. In the first case, we have $x \geq 2$ and $|x-2| = x-2 \leq 4$. In total, we get $2 \leq x \leq 6$. For the second case, we get $x < 2$ and $|x-2| = -(x-2) \leq 4$, or equivalent$-2 \leq x < 2$. So in total we have $-2 \leq x \leq 6$.

(b) $|x^2 - x - 1| = 1$ .

First we have to use the ABC rule to find the roots of the quadratic polynomial as $x_{1,2} = \frac{1}{2} \mp \frac{1}{2}\sqrt{5} \approx -0.618, 1.618$. Therefore, we examine two situations: for all $x \leq x_1$ or $x \geq x_2$, we have $|x^2 - x - 1| = x^2 - x - 1$, and we have to solve $x^2 - x - 2 = 0$. The ABC rule again gives us the two solutions 2 and $-1$, which also satisfy the condition $x \leq x_1$ or $x \geq x_2$. For the second case $x_1 \leq x \leq x_2$, we have $|x^2 - x - 1| = -(x^2 - x - 1)$, and we must find the roots of $x^2 - x$ which are directly determined as $0, 1$. In total, we get four solutions for the initial equation, namely $-1, 0, 1, 2$.

(2) **Are the following sequences convergent or not if the index $n$ goes to infinity (give an argument for your answer)? For convergent sequences, also determine their limit:**

(a) $a_n = \frac{n}{n^2+1}$

We determine the most dominant term of numerator and denominator as $n^2$. Therefore, we deduce

$$a_n = \frac{n}{n^2+1} = \frac{n^2(1/n)}{n^2(1+1/n^2)} = \frac{1/n}{1+1/n^2} .$$

Taking the limit for all individual terms and using the addition / division rule, we get $\lim_{n\to\infty} a_n = 0$.

(b) $b_n = \frac{3n^2+1/n}{2n^2+n}$

We proceed as in the last exercise and determine the most dominant term of numerator and denominator as $n^2$. Therefore, we deduce

$$\lim_{n\to\infty} b_n \;=\; \lim_{n\to\infty} \tfrac{3n^2+1/n}{2n^2+n} \;=\; \lim_{n\to\infty} \tfrac{3+1/n^3}{2+1/n} \;=\; \tfrac{3}{2}\,.$$

(c) $c_n = 2^{-n}$

In this example we can not use one of the basic composition rules. Therefore, we try to find suitable lower and upper bounds such that the corresponding sequences converge. Trivially we know that $2^{-n} \geq 0$. We guess the upper bound as $2^{-n} \leq \frac{1}{n}$. We can prove this bound with mathematical induction: $n = 1$ is trivial, and

$$2^{-n-1} \;=\; \tfrac{1}{2}2^{-n} \;\leq\; \tfrac{1}{2}\cdot\tfrac{1}{n} \;\leq\; \tfrac{1}{n+1}\,.$$

Since both the lower and the upper bound sequences converge to zero, we have proven that $\lim_{n\to\infty} c_n = 0$.

(3) **(Recurrent sequences) The general term of a recurrent sequences is defined by a function of one or more previous terms. Consider for $n \geq 0$ the sequence**

$$u_{n+1} \;=\; \tfrac{1}{2}\left(u_n + \tfrac{a}{u_n}\right) \qquad \text{where } a \in \mathbb{R}^+, u_0 = a\,.$$

Prove that $\lim_{n\to\infty} u_n = \sqrt{a}$.

Assume first that we know that this recursive sequence converges, e.g. assume $\lim_{n\to\infty} u_n = u$. Then clearly also $\lim_{n\to\infty} u_{n+1} = u$. It is obvious that for $u_0 > 0$ all subsequent terms of the sequence will also be positive. Applying the addition and division rules, we get

$$u \;=\; \lim_{n\to\infty} u_{n+1} \;=\; \frac{1}{2}\left((\lim_{n\to\infty} u_n) + \frac{a}{(\lim_{n\to\infty} u_n)}\right) \;=\; \frac{1}{2}\left(u + \frac{a}{u}\right)\,.$$

Easily we can transform this equality to $u^2 = a$ which proves the statement on the limit value.

Proving the convergence of this sequence is more complicated. Assume that $\epsilon_n \geq 0$ is the multiplicative error for the $n$-th term of the sequence, i.e. $u_n =$

$\sqrt{a} \cdot (1 + \epsilon_n)$. Then

$$u_{n+1} \;=\; \tfrac{1}{2}\left(\sqrt{a}(1+\epsilon_n) + \frac{a}{\sqrt{a}(1+\epsilon_n)}\right) \;=\; \sqrt{a} \cdot \left(\frac{1+\epsilon_n}{2} + \frac{1}{2(1+\epsilon_n)}\right)$$

Now

$$\frac{1+\epsilon_n}{2} + \frac{1}{2(1+\epsilon_n)} \;=\; \frac{(1+\epsilon_n)^2}{2(1+\epsilon_n)} + \frac{1}{2(1+\epsilon_n)} \;=\; 1 + \frac{\epsilon_n^2}{2(1+\epsilon_n)}\ .$$

Therefore, we have determined $\epsilon_{n+1} = \frac{\epsilon_n^2}{2(1+\epsilon_n)}$. We distinguish two cases:

- For $\epsilon_n < 1$, we get $\epsilon_{n+1} \leq \frac{\epsilon_n^2}{2}$.
- For $\epsilon_n \geq 1$, we have $\epsilon_{n+1} = \frac{\epsilon_n}{2(1/\epsilon_n + 1)} \leq \frac{\epsilon_n}{2}$.

For both situations we see that $0 \leq \epsilon_{n+1}$ is smaller than $\epsilon_n$, more precisely, the relative error $\epsilon_n$ itself forms a sequence with limit 0. Therefore, the distance between $u_n$ and $\sqrt{a}$ becomes smaller and smaller, and the definition of convergence can be applied to prove convergence.