Naive Lie Theory 1.2 Exercises

OblivionIsTheName

September 2025

1.2.1

Say we have two complex numbers, $a_1 + ib_1$ and $a_2 + ib_2$, represented as two matrices:

$$A_1 = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix},\ A_2 = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}$$

And the matrix representation of the product of them:

$$A_1 A_2 = \begin{pmatrix} a_1a_2 - b_1b_2 & -a_1b_2 - a_2b_1 \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{pmatrix}$$

Then the left hand side of the two-square identity is transformed to the right hand side as follows:

$$(a_1^2+b_1^2)(a_2^2+b_2^2) = \det(A_1)\det(A_2) = \det(A_1A_2) = (a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2$$

1.2.2

We have

$$\begin{cases} 5 = 1^2 + 2^2 \\ 13 = 2^2 + 3^2 \end{cases}$$

Therefore applying the two-square identity

$$65 = (1^2 + 2^2)(2^2 + 3^2) = (2 - 6)^2 + (3 + 4)^2 = 4^2 + 7^2$$

1.2.3

Use the two-square identity twice, check if $37^4 = 1081^2 + 840^2$

1.2.4

Directly from the distributive law and multiplicative absolute value, we have

$$|uv - uw| = |u(v - w)| = |u||v - w|$$

Now consider the function $f_u(x) = ux$ from $\mathbb{C}$ to $\mathbb{C}$. For any $x_1, x_2 \in \mathbb{C}$, we have

$$|f_u(x_1) - f_u(x_2)| = |ux_1 - ux_2| = |u||x_1 - x_2|$$

therefore all distances on the complex plane is multiplied by $|u|$.

1.2.5

If u is in the form of $\cos\theta + i\sin\theta$, $|u| = 1$, therefore for any $x_1.x_2 \in \mathbb{C}$, we have

$$|f_u(x_1) - f_u(x_2)| = |x_1 - x_2|$$

so all distances are preserved.