Naive Lie Theory 2.2 Exercises

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Notes

Again, this is a crash course. Here are some important supplementary details:

1. As cosets are disjoint, there is the bijection $gh \mapsto h$ between gH and H. $|gH| = |H|$ for any g.
2. As the fundamental theorem of homomorphism states, a homomorphism gives $G/\ker\varphi \cong H$. There is a bijection between the cosets of $\ker\varphi$ and H. The $|\ker\varphi|$ elements of each coset is mapped to one element in H, so φ is $|\ker\varphi|$-to-1. I am going to use this proposition in 2.2.2.
3. Each homomorphism indicates a normal subgroup as its kernel. This is an important method to prove that a group is not simple: if you can find a non-bijective (so the normal subgroup is non-trivial) and non-trivial (so the normal subgroup is not the whole group) homomorphism, the group is not simple.

2.2.1

Let $\varphi(z) = z^2$, $\ker\varphi = \left\{z \in \mathbb{S}^1 \mid z^2 = 1\right\} = \{1, -1\}$, and any left coset by z is $\{z, -z\}$.

2.2.2

The identity will be ± 1; the inverse is given by $\{\pm z\}^{-1} = \left\{\pm z^{-1}\right\}$; the closure is given by the absolute-value-preservation of multiplications in $\mathbb{S}^1$. We first show that φ is a homomorphism:

$$\varphi\left(z_1 z_2\right) = \{\pm z_1 z_2\} = \{z_1\}\{z_2\} = \varphi\left(z_1\right)\varphi\left(z_2\right).$$

If $\varphi(z) = \{\pm 1\}$, then $z \in \{1, -1\}$, so $\ker\varphi = \{1, -1\}$. $|\ker\varphi| = 2$, and φ is therefore 2-to-1.

2.2.3

(Note: I think the problem statement is a bit confusing, as G is the group of $\{\pm z\}$, but the map sends z to z^2. This is a 2-to-1 homomorphism, as we proved in 2.2.2. I will just pretend that it's sending $\{\pm z\}$ to z^2.)

We have $\varphi(\{\pm z\}) = z^2$, and $(z)^2 = (-z)^2$. φ is well-defined. To prove that φ is an isomorphism, it's sufficient and necessary to prove operation-preservation and bijectivity. We have

$$\varphi\left(\{\pm z_1\}\{\pm z_2\}\right) = \left(z_1 z_2\right)^2 = z_1^2 z_2^2 = \varphi\left(\{\pm z_1\}\right)\varphi\left(\{\pm z_2\}\right),$$

so φ is operation-preserving.

Assume that $\varphi\left(\{\pm z_1\}\right) = \varphi\left(\{\pm z_2\}\right)$, so $z_1^2 - z_2^2 = 0$ and $\left(z_1 + z_2\right)\left(z_1 - z_2\right) = 0$. It follows that $z_1 = z_2$ or $z_1 = -z_2$, so $\{\pm z_1\} = \{\pm z_2\}$. Thus φ is injective. Assume $w = e^{i\theta} \in \mathbb{S}^1$. We can always find $z = e^{i\frac{\theta}{2}} \in \mathbb{S}^1$ such that $\varphi(\{\pm z\}) = w$. φ is surjective, thus bijective.

2.2.4

We want to show that for any $q \in \mathbb{S}^3$ and $h \in \{1, -1\}$, $qhq^{-1} = h$. If $h = 1$, $q1q^{-1} = qq^{-1} = 1$; else $q(-1)q^{-1} = (-1)qq^{-1} = -1$. Note that all real numbers are commutative in quaternions.

2.2.5

Now we want to find counterexamples $q \in \mathbb{S}^3$ and $z \in \mathbb{S}^1$ such that $qzq^{-1} \neq z$. Consider $j \in \mathbb{S}^3$ and $i \in \mathbb{S}^1$. $jij^{-1} = ji(-j) = -j(ij) = -jk = -i \neq i$.