Naive Lie Theory 2.4 Exercises

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Notes

This chapter is basically about proving the theorem of reflection representations of isometries.

1. I think the main confusion in the proof given by the book is why f reduces to a reflection plus an isometry of $\mathbb{R}^{n-1}$. Here is my explanation of this. $\mathbb{R}u$ is invariant under fr_u, as f and r_u both send $\mathbb{R}u$ onto itself. As f is a bijection, the remaining $n-1$-dimensional space is invariant. It follows that $f(\mathbb{R}^{n-1}) = g(\mathbb{R}^{n-1})$, where g is an isometry of $\mathbb{R}^{n-1}$, so $f = r_u g$. According to our inductive assumption, proof is closed.

2.4.1

Say w is on the line segment connecting u and v. We have $||u-w|| + ||w-v|| = ||u-v||$. As f preserves distances,

$$||f(u) - f(w)|| + ||f(w) - f(v)|| = ||f(u) - f(v)||$$

which suggests that $f(u)$, $f(w)$, and $f(v)$ are collinear (as the triangular inequality reduces to an equality).

Now let w be the midpoint of line segment connecting u and v. Similarly, as f preserves distances, we have $||f(u)-f(w)|| = ||f(v)-f(w)||$ and $||f(u)-f(w)|| = \frac{1}{2}||f(u) - f(v)||$, so $f(w)$ is the midpoint of $f(u)$ and $f(v)$.

2.4.2

Consider the midpoint between 0 and v. As shown in 2.4.1, f preserves midpoints, so $f(\frac{u+0}{2}) = \frac{1}{2}(f(u)+f(0))$. As f fixes 0, $f(\frac{u}{2}) = \frac{1}{2}f(u)$, which is equivalent to $2f(u) = f(2u)$. Then consider $u+v$ as the midpoint between $2u$ and $2v$, and we have $f(u+v) = \frac{1}{2}(f(2u)+f(2v)) = \frac{1}{2}(2f(u)+2f(v)) = f(u)+f(v)$.

2.4.3

We first prove for natural numbers by induction on $f((n+1)u) = f(nu)+f(u) = (n+1)f(u)$.

We then extend it to integers. $0 = f(nu - nu) = f(nu) + f(-nu)$, so $f(-nu) = -f(nu) = -nf(u)$. This proves the statement for negative integers, and 0 and $\mathbb{Z}^+$ are already shown previously. The statement is proved for all integers.

We then extend it to rationals. We have $f(u) = f(n\frac{u}{n}) = nf(\frac{u}{n})$, so $f(\frac{u}{n}) = \frac{1}{n}f(u)$. For a rational number q, $f(qu) = f(\frac{m}{n}u) = mf(\frac{u}{n}) = \frac{m}{n}f(u) = qf(u)$.

Now we extend to the real numbers. An arbitrary real number is defined as the limit of a Cauchy sequence of rational numbers. Distance preservation is a way stronger condition than continuity, as $\forall u, v \in \mathbb{R}^n$, $\forall \epsilon > 0$, $\exists \delta = \epsilon$, $||u - v|| = ||f(u) - f(v)|| < \epsilon = \delta$. We can now move the limit out, $f(ru) = f(\lim_{n\to\infty} q_n u) = \lim_{n\to\infty} f(q_n u) = \lim_{n\to\infty} q_n f(u) = rf(u)$.

2.4.4

I will just give the generalized result. Say r_k is the reflection in hyperplane orthogonal to the unit vector e_k. One can generate a basis $(e_1, \cdots, e_k, \cdots, e_n)$ for any e_k using Gram-Schmidt process. For any $i \neq k$, $r_k(e_i) = e_i$, and $r_k(e_k) = -e_k$.