Naive Lie Theory 1.3 Exercises

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September 2025

1.3.1

Omitted. Pure matrix calculation.

1.3.2

Say we have two quaternions in the form

$$q_1 = \begin{pmatrix} \alpha_1 & -\beta_1 \\ \overline{\beta_1} & \overline{\alpha_1} \end{pmatrix},\ q_2 = \begin{pmatrix} \alpha_2 & -\beta_2 \\ \overline{\beta_2} & \overline{\alpha_2} \end{pmatrix}$$

Therefore their product is

$$q_1 q_2 = \begin{pmatrix} \alpha_1\alpha_2 - \beta_1\overline{\beta_2} & -\alpha_1\beta_2 + \overline{\alpha_2}\beta_1 \\ \alpha_2\overline{\beta_1} + \overline{\alpha_1\beta_2} & \overline{\alpha_1\alpha_2} - \beta_1\beta_2 \end{pmatrix}$$

Now we verify that $q_1 q_2$ is in the form given by the hint.

$$\begin{cases} \overline{\alpha_1\alpha_2 - \beta_1\overline{\beta_2}} = \overline{\alpha_1\alpha_2} - \beta_1\beta_2 \\ \overline{-(-\alpha_1\beta_2 - \overline{\alpha_2}\beta_1)} = \alpha_2\overline{\beta_1} + \overline{\alpha_1\beta_2} \end{cases}$$

1.3.3

Let $q = a + ib + jc + kd$, so $\overline{q} = a - ib - jc - kd$, represented by the matrix

$$\begin{pmatrix} a - id & b + ic \\ -b + ic & a + id \end{pmatrix}$$

where

$$q = \begin{pmatrix} a + id & -b - ic \\ b = ic & a - id \end{pmatrix} = \begin{pmatrix} \overline{a - id} & \overline{b + ic} \\ \overline{-b + ic} & \overline{a + id} \end{pmatrix}$$

We then verify this rule preserves multiplication. Remember the expressions we got from 1.3.2. Let q_1, q_2 be two arbitrary quaternions, then

$$\overline{q_2}\,\overline{q_1} = \begin{pmatrix} \overline{\alpha_2} & \beta_2 \\ \overline{-\beta_2} & \alpha_2 \end{pmatrix} \begin{pmatrix} \overline{\alpha_1} & \beta_1 \\ \overline{-\beta_1} & \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 - \beta_1\overline{\beta_2} & -\alpha_1\beta_2 + \overline{\alpha_2}\beta_1 \\ \alpha_2\overline{\beta_1} + \overline{\alpha_1\beta_2} & \overline{\alpha_1\alpha_2} - \beta_1\beta_2 \end{pmatrix} = \overline{q_1 q_2}$$

1.3.4

$q\overline{q} = (a+(ib+jc+kd))(a-(ib+jc+kd)) = a^2-(ib+jc+kd)^2 = a^2-(-b^2-c^2-d^2) = |q|^2$

1.3.5

From 1.3.4, we have $\det(q_1) = |a_1|^2+|\beta_1|^2$ and same for q_2. Using the expression from 1.3.2, we have $\det(q_1q_2) = |\alpha_1\alpha_2-\beta_1\overline{\beta_2}|^2+|a_1\beta_2+\overline{\alpha_2}\beta_1|^2$. Then the relation $\det(q_1)\det(q_2) = \det(q_1q_2)$ gives the desired identity.

1.3.6

The left-hand side is $|q_1|^2|q_2|^2$. One can fully expand q_1q_2 to get the right-hand side as $|q_1q_2|$, which includes a lot of boring work.

1.3.7

All you need is attention, as you can't use the four-square identity yet to two numbers in the vacuum. Check if $97 = 2^2+2^2+5^2+8^2$ and $99 = 1^2+1^2+4^2+9^2$. The answer is not unique.

1.3.8

Now one should apply the four-square identity.

$$97 \times 99 = (2^2 + 2^2 + 5^2 + 8^2)(1^2 + 1^2 + 4^2 + 9^2) = ...$$

A lot of boring work again. Check if it's $7^2 + 13^2 + 13^2 + 96^2 = 9603$.