Naive Lie Theory 2.1 Exercises

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Notes on this section

This section is a really "crash" course on groups, but I guess most of the readers have some knowledge of groups.

Subgroup is nothing more than a subset-who is also a group-within a group. The simplest way to verify if a subset is a subgroup is just checking closeness, as the other group properties are naturally inherited.

This section covers the proof of that cosets are either disjoint or the same. It follows that if you pick an element g' outside the coset gH and multiply it on the left, it gives a new disjoint coset $g'gH$. This fills an implicit gap in the proof of the Hopf fibration: you can always inductively choose a point outside all existing circles and make a new disjoint coset, which is still a circle. You can ultimately fill everywhere on the 3-sphere by disjoint congruent circles.

However, why an isometry necessarily preserves circles? This sounds trivial but it's not. You might initially try to prove it by using definition of circles and saying that every radius is preserved; therefore the image of an isometry is a congruent circle. Unfortunately, the isometry is not necessarily defined at the center, and we should only manipulate points on the circle. I have proved this using Thales's Theorem, but it's so nasty and irrelevant. Let me think if there's a better proof.

2.1.1

We have $ax + b = a_2(a_1x + b_1) + b_2$, so

$$\begin{cases} a = a_1a_2 \\ b = b_2 + a_2b_1 \end{cases}$$

For the matrices, we have

$$\begin{pmatrix} a_1 & b_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_2b_1 + b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

2.1.2

We can first manually find $f_{a,b}^{-1} = \frac{x}{a} - \frac{b}{a}$. Meanwhile, we have

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{a}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} = f_{a,b}^{-1}$$

2.1.3

$\{1, -1\}$.

2.1.4

Think about this geometrically. We want to find the n unitary complex numbers that evenly distributed over the, so that they are a closed subgroup. This is equivalent to roots of $z^n - 1 = 0$. We can also write the subgroup explicitly:

$$\{e^{i\frac{2k\pi}{n}} | k = 0, 1, \cdots, n-1\}$$

2.1.5

We are essentially proving that union of all subgroups with the form in 2.1.4 is a subgroup. The union can be written as

$$\{e^{i\frac{2p\pi}{q}} | p, q \in \mathbb{Z}\}$$

One can easily verify that the identity is 1, the inverse is $(e^{i\frac{2p\pi}{q}})^{-1} = e^{-i\frac{2p\pi}{q}}$, and closeness given by $e^{i\frac{2p_1\pi}{q_1}} e^{i\frac{2p_2\pi}{q_2}} = e^{i\frac{2\pi(p_1q_2+p_2q_1)}{q_1q_2}}$.

Now we have shown that the rotations of all rational multiples of 2π is a subgroup.

2.1.6

This is essentially filling the gaps between rational numbers. For the sake of contradiction, say there exists $n, m \in \mathbb{Z}$, $n \neq m$, $z^n = z^m$. It follows that $z^{n-m} = 1$, but $n - m \neq 0$, so we have a contradiction. The group proof is trivial.