Naive Lie Theory 2.6 Exercises

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Notes

This chapter is about the definition of (external) direct product of groups.

1. Technically, the definition given in this chapter is called external direct product. There is another different concept called internal direct product. If C is an internal product of two subgroups A and B, for any $c \in C$ can be uniquely expressed as $ab = ba$ where $a \in A$ and $b \in B$. It is equivalent to say $AB = C$ and $A \cap B = C$ make C an internal direct product of A and B.

2. Back to the external direct product, here is an important proposition: if $C = A \times B$, and A' and B' are normal in A and B, then $A' \times B'$ is normal in C. Take any $(a, b) \in C$ and $(a', b') \in A' \times B'$, we have $(a, b)(a', b')(a, b)^{-1} = (aa'a^{-1}, bb'b^{-1}) = (a', b')$.

3. This is a useful method to prove or disprove simplicity of groups as an external direct product. To that end, the proposition can be weakened to that C is not simple if one of A and B is not simple. Say A' is a non-trivial normal subgroup in A, then $A' \times \{e_B\}$ is always normal in C.

2.6.1

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.6.2

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

2.6.3

Say the groups of matrices from 2.6.1 and 2.6.2 are H_1 and H_2, and we have $H_1 \cong H_2 \cong \mathbb{S}^1$. Consider the group $H = \{h_1 h_2 | h_1 \in H_1, h_2 \in H_2\}$ (in fact, this is an internal direct product of H_1 and H_2). H is a subgroup of SO(4) by definition. Now we prove that $H \cong H_1 \times H_2$. $\varphi : h_1 h_2 \mapsto (h_1, h_2)$ is obviously surjective. It's also injective because any matrix in H has a unique block-diagonal (upper left for H_1 the bottom right for H_2) structure that uniquely determines its H_1 and H_2 components. Putting it all together, we have $H \cong H_1 \times H_2 \cong \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$, so $\mathbb{T}^2$ is isomorphic to H, which is a subgroup of SO(4).

2.6.4

(x, r, θ) is uniquely represented by a point on an infinite cylinder's surface.

2.6.5

$\mathbb{S}^3$ is not abelian but $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ is becasue each $\mathbb{S}^1$ is abelian.