Naive Lie Theory 1.1 Exercises

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1.1.1

$z_{\theta+\varphi}$ can be interpreted as

1. a point on the unit circle with argument $\theta + \varphi$.
2. the product of two rotations, z_θ and z_φ.

Therefore the equivalence above gives

$$\cos(\theta + \varphi) + i\sin(\theta + \varphi) = (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$$

Separating the real and imaginary part,

$$\begin{cases} \cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi \\ \sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi \end{cases}$$

1.1.2

Just plug in $\varphi = \theta$.

1.1.3

Using the fraction definition of tan,

$$\tan(\theta + \varphi) = \frac{\sin(\theta + \varphi)}{\cos(\theta + \varphi)} = \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\cos\theta\cos\varphi - \sin\theta\sin\varphi}$$

Then divide both the numerator and denominator by $\cos\theta\cos\varphi$

$$\implies \tan(\theta + \varphi) = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi}$$

1.1.3

Remember that $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -\varphi$ by the fact that cos is an even function and sin is an odd function. Similar to 1.1.1, we have the following equivalence:

$$\cos(\theta - \varphi) + i\sin(\theta - \varphi) = (\cos\theta + i\sin\theta)(\cos\varphi - i\sin\varphi)$$

Then separate the real and imaginary,

$$\begin{cases} \cos(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi \\ \sin(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi \end{cases}$$

Do the same as we did in 1.1.3,

$$\tan(\theta - \varphi) = \frac{\tan\theta - \tan\varphi}{1 + \tan\theta\tan\varphi}$$

$\theta - \varphi = \pm\frac{\pi}{2} \iff \tan(\theta - \varphi) = \infty \iff 1 + \tan\theta\tan\varphi = 0 \iff \tan\theta = -\frac{1}{\tan\varphi}$

1.1.5

Let's denote the rotation through θ by R_θ, which sends a complex number z to zz_θ. We have

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Now we consider the rotation through $-\theta$ on standard basis $(1, i)$ to find $R_{-\theta}$ as a matrix.

$$\begin{cases} 1z_{-\theta} = \cos\theta - i\sin\theta \\ iz_{-\theta} = \sin\theta + i\cos\theta \end{cases}$$

$$\implies R_{-\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Then we calculate the inverse of R_θ

$$(R_\theta)^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_{-\theta}$$