Naive Lie Theory 2.3 Exercises

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Notes on this section

This chapter formalizes what we proved in 1.5 with group languages: each antipodal pair in $\mathbb{S}^3$ corresponds to a rotation of $\mathbb{R}^3$, or $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.

1. Sometimes the book says $\mathbb{R}^3$ and sometimes $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. These two are the same space. It's intuitive that two spaces with the same (finite) dimension are the same, but it's hard to prove this theorem from scratch at this stage.

2. The formal definition of SO(3) is not given at this point, because it's easier to understand the group homomorphism from an intuitive and geometric perspective at this point. It's formal definition as real matrices will be discussed in Chapter 3.

3. The proof of simplicity of SO(3) is a bit confusing. The idea is to assume a non-trivial normal subgroup, then "generate"-as the normal subgroup has to be closed-the whole SO(3). SO(3) has no nontrivial normal subgroup besides itself, thus is simple.

2.3.1

A tetrahedron has 4 faces. Any one of these four faces can be chosen to be in the position of the "front face" shown in the diagram. Once a face has been selected as the front face, that triangular face has 3 edges. Any of these three edges can be rotated into the "bottom" position. We have $4 \times 3 = 12$ positions.

2.3.2

A cube has 6 faces. Any one of these faces can be rotated to be on top. This gives us 6 initial choices. Once you've fixed which face is on top, the cube can still be rotated around the vertical axis passing through the top and bottom faces. There are 4 faces around the side. We have $6 \times 4 = 24$.

2.3.3

For each vertex of the tetrahedron, there are 2 rotations ($\frac{2\pi}{3}$ and $\frac{4\pi}{3}$) about the axis perpendicular to corresponding face (which is the face doesn't contain the vertex). There are 4 vertices, so 8 distinct rotations in total. For each pair of disjoint edges, there is 1 rotations (π) about the axis passing their midpoints. There are 4 such pairs, so 3 distinct rotations in total. The 12th rotation is the identity (not rotating essentially).

2.3.4

As mentioned in 1.5, each pair of unitary quaternions represents a rotation of $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. The real axis is invariant, so ± 1 is the identity. You can also plugin $\theta = 0$. For non-trivial rotations, $\theta = \pi$, so $q = u$. Notice that the other three non-trivial rotations are indeed about three orthogonal axis. We can align these axes with the standard coordinate axes of $\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$, so they correspond to $\pm i$, $\pm j$, and $\pm k$.

2.3.5

Notice that the real part of q decides whether the rotational angle is $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. That said, the imaginary part decides the rotational axis. $\pm(i + j + k)$, $\pm(-i + j + k)$, $\pm(i - j + k)$, and $\pm(i + j - k)$ correspond to the four axes.

One can verify the geometrical relationship between any of the two imaginary quaternions above. Say $u = i + j + k$ and $v = -i + j + k$. Let's try their inner product. $u \cdot v = -1$, meanwhile $u \cdot v = |u||v|\cos\alpha$. It gives $-1 = 3\cos\alpha$. $\alpha \approx 109°$, which is the tetrahedral angle. However, the rigorous definition of tetrahedral angle might be given by quaternion, and this verification might be circular.