Naive Lie Theory 3.3 Exercises

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October 2025

Notes

A notation used to check orthonormal basis is the Kronecker delta function:

$$\delta_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$$

So, for a list of vectors $u = (u_1, \cdots, u_n)$ to be orthonormal, it has to satisfy that $\langle u_i, u_j \rangle = \delta_{ij}$ for any i, j.

3.3.1

For a 1×1 matrix the determinant is just the only entry itself, so $SU(1)$ has only one element 1. It's the trivial group.

3.3.2

It's easy to verify that $\langle \overline{u}, \overline{v} \rangle = \Sigma_{i=1}^{n} \overline{u_i} v_i = \langle v, u \rangle = \overline{\langle u, v \rangle}$. Assume $(u_1, \cdots, u_n)$ is orthogonal, so $\langle u_i, u_j \rangle = \delta_{ij}$. We also have $\langle \overline{u_i}, \overline{u_j} \rangle = \overline{\langle u, v \rangle} = \overline{\delta_{ij}} = \delta_{ij}$, thus the conjugate is also orthogonal. The other direction is trivial.

3.3.3

Say the row vectors of A are $u_1, \cdots, u_n$. Notice that the ijth entry of $A\overline{A}^T$ is just $\langle u_i, u_j \rangle = \delta_{ij}$. $A\overline{A}^T$ is 1 on the diagonal and 0 everywhere else, which is just I.

3.3.4

(I think the problem statement is a bit weird. It should be "If $A\overline{A}^T = I$, deduce...")
Assume that $A\overline{A}^T = I$. In a group, inverses are commutative, so $A\overline{A}^T = \overline{A}^T A =$

I. Let $B = \overline{A}^T$, so $B\overline{B}^T = I$. From 3.3.2 row vectors of B are orthogonal, and row vectors of B are just column vectors of A.

The inverse, i.e., if the column vectors are orthogonal $A\overline{A}^T = I$, is also true. One can assume A's column vectors are orthogonal then take $B = \overline{A}^T$. B's row vectors are orthogonal. By 3.3.3, $B\overline{B}^T = \overline{B}^T B = A\overline{A}^T = I$.

3.3.5

As we have shown that $A\overline{A}^T = I$ is equivalent to orthogonal column vectors in 3.3.4, here we only need to prove that the inner product preservation, namely, $\langle u, v\rangle = \langle Au, Av\rangle$, implies $A\overline{A}^T = I$. Assume $\langle u, v\rangle = \langle Au, Av\rangle$. We have:

$$\langle Au, Av\rangle = \Sigma_{i=1}^n (Au)_i \overline{(Av)_i} = \Sigma_{i=1}^n ((\Sigma_{j=1}^n a_{ij} u_j)(\Sigma_{k=1}^n \overline{a_{ik} v_k}))$$

Rearranging the terms we get $\langle Au, Av\rangle = \Sigma_{j=1}^n \Sigma_{k=1}^n u_j \overline{v_k} (\Sigma_i^n a_{ij} \overline{a_{ik}})$. If $\langle u, v\rangle = \langle Au, Av\rangle$, $\Sigma_i^n a_{ij} \overline{a_{ik}} = \delta_{jk}$, and this is precisely $A\overline{A}^T = I$.

3.3.6

Every implication in 3.3.5 is invertible.