Naive Lie Theory 2.7 Exercises

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Notes

Some important propositions in group theory is not emphasized much in this textbook due to limited space. You may refer to an algebra textbook alongside. Don't simply rely on sections 2.1 and 2.2. They are really a "crash".

1. The proof of SO(4) is not simple, requires the proof of that image of a normal subgroup is normal, which is not included in 2.1 nor 2.2. Now let's prove it. Let φ be a homomorphism and H be a subgroup of G. If H is normal, let $\varphi(h) \in \varphi(H)$ and $\varphi(g) \in \varphi(G)$, then $\varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(ghg^{-1}) \in \varphi(H)$, so $\varphi(H)$ is normal.

2.7.1

Let $\rho(p) = u^{-1}pu$. Quaternions have both left and right distributive law, so

$$\rho(p+q) = u^{-1}(p+q)u = (u^{-1}p + u^{-1}q)u = u^{-1}pu + u^{-1}qu = \rho(p) + \rho(q)$$

And

$$\rho(pq) = u^{-1}p(1)qu = u^{-1}puu^{-1}qu = \rho(p)\rho(q)$$

2.7.2

Now ρ is an arbitrary automorphism. Firstly, we have $\rho(0+p) = \rho(p) = \rho(0) + \rho(p)$, so $\rho(0) = 0$. Then $\rho(q-q) = \rho(q) + \rho(-q) = 0$, so $\rho(-q) = -\rho(q)$ and $\rho(p-q) = \rho(p) - \rho(q)$.

2.7.3

Similar to 2.7.2. We have $\rho(1 \cdot p) = \rho(p) = \rho(1)\rho(p)$, so $\rho(1) = 1$. And $\rho(\frac{p}{p}) = \rho(p)\rho(\frac{1}{p}) = 1$, so $\rho(\frac{1}{p}) = \frac{1}{\rho(p)}$, and $\rho(\frac{p}{q}) = \frac{\rho(p)}{\rho(q)}$.

2.7.4

To rigorously prove this, we need to construct integers then rational numbers then real numbers. Many details are omitted, as they belong to an analysis textbook.

By induction on $\rho(1+1) = 2$, one can get $\rho(n) = n$ for all natural numbers. Then use fact that ρ preserves additive inverse, $\rho(-n) = -\rho(n) = -n$, so we are done for $\mathbb{Z}$. Then according to its quotient preservation,

$$\rho(\frac{m}{n}) = \frac{\rho(m)}{\rho(n)} = \frac{m}{n}$$

for any $m, n \in \mathbb{Z}$. Now we are done for $\mathbb{Q}$. Now we define real numbers to limits of all the Cauchy sequences,

$$\mathbb{R} = \{\lim_{n\to\infty} q_n | \forall n \in \mathbb{N},\, q_n \in \mathbb{Q}\}$$

Say $r \in \mathbb{R}$. Given that ρ is continuous, so we can pull the limit out,

$$\rho(r) = \rho(\lim_{n\to\infty} q_n) = \lim_{n\to\infty} \rho(q_n) = \lim_{n\to\infty} q_n = r$$

Now we are done for all real numbers.

ρ is a linear map of $\mathbb{R}$ by definition because it has additivity $\rho(p+q) = \rho(p) + \rho(q)$ and homogeneity $\rho(rp) = r\rho(p)$ for all real r and p. One can then extend this it easily for all $p \in \mathbb{R}^4$ using Cartesian products.

2.7.5

Say $q = \mathrm{Re}(q) + \mathrm{Im}(\mathrm{q})$, where $\mathrm{Re}(q) = a$ and $\mathrm{Im}(q) = ib + jc + kd$, and $\overline{q} = \mathrm{Re}(q) - \mathrm{Im}(q)$. As shown before, $\rho(\mathrm{Re}(q)) = \mathrm{Re}(q)$. Then we write

$$\rho(\overline{q}) = \rho(\mathrm{Re}(q) - \mathrm{Im}(q)) = \mathrm{Re}(q) - \rho(\mathrm{Im}(q))$$

On the other hand, consider the imaginary part of $\rho(q)$, and one can easily show that $\mathrm{Im}(p+q) = \mathrm{Im}(p) + (\mathrm{q})$.

$$\mathrm{Im}(\rho(q)) = \mathrm{Im}(\mathrm{Re}(\rho(q))) + \mathrm{Im}(\mathrm{Im}(\rho(q))) = \mathrm{Im}(\rho(q))$$

Thus ρ preserves imaginary parts as well.

Then our previous equation becomes

$$\rho(\overline{q}) = \mathrm{Re}(q) - \rho(\mathrm{Im}(q)) = \mathrm{Re}(q) - \mathrm{Im}(\rho(q)) = \overline{\rho(q)}$$

thus ρ preserves conjugates.

We have $|\rho(q)|^2 = \rho(q)\overline{\rho(q)} = \rho(q)\rho(\overline{q}) = \rho(q\overline{q}) = \rho(|q|^2)$. Notice that $|q^2|$ is a real number, so $|\rho(q)| = |q|$ for any $q \in \mathbb{H}$. It follows that $|\rho(u) - \rho(v)| = |u - v|$ by defining $u - v = q$.

The inner product of two imaginary quaternions is $\langle u, v\rangle = -\mathrm{Re}(uv)$. We have $\langle \rho(u), \rho(v)\rangle = -\mathrm{Re}(\rho(u)\rho(v)) = -\mathrm{Re}(\rho(uv))$. As ρ preserves real part, the right-hand side reduces to $-\mathrm{Re}(uv) = \langle u, v\rangle$.

The cross product is in turn $u \times v = \mathrm{Im}(uv)$. We have $\rho(u) \times \rho(v) = \mathrm{Im}(\rho(uv))$. As ρ preserves imaginary part as well, $\mathrm{Im}(\rho(uv)) = \rho(\mathrm{Im}(uv)) = \rho(u \times v)$.

A map that preserves both the inner and cross product must preserve orientation. This is not trivial, but the proof is long. In short, a map preserving the inner product must have a determinant of either +1 or -1. The cross product directly tests orientation (i.e., the "right-hand rule"). By preserving the cross product, the map must maintain the right-hand rule for any basis vectors. This forces the determinant to be +1, meaning the map must preserve orientation. We will have a closer look of this in 3.2.