Simple SO(n)

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The Basic Idea

We claim that $SO(n)$ is simple if and only if n is odd. It's easy to tell that for all even n there is a non-trivial normal subgroup $\{I, -I\}$. We therefore focus on proving simplicity of all $SO(n)$ when n is odd. For the sake of contradiction, assume that there is a non-trivial normal subgroup N that contains a non-identity element A. The basic idea of the proof is as follows:

1. Use A's conjugation to generate a plane rotation B that must be in N.
2. Use B's conjugations to generate all plane rotations.
3. Use the above condition to generate the whole $SO(n)$.

1 N Contains A Plane Rotation

Say V is the space that A actually acts on, where $\forall v \in V : Av \neq v$, and W is the space that A fixes, where $\forall w \in W : Aw = w$. It follows that $V \oplus W = \mathbb{R}^n$ and $\dim V + \dim W = n$. As A is not the identity, $\dim V \neq 0$. $\dim V$ must be even as if it's odd, A fixes a vector in V, in other words, has an eigenvalue of 1 (5.26 of *Linear Algebra Done Right*).

Let $B = RAR^{-1}A^{-1} \in N$. We want to carefully choose R so that B is a plane rotation. Let R be the π rotation on the plane spanned by (e_1, e_2), where we specifically choose $e_1 \in V$ and $e_2 \in W$. W is not empty, as $\dim V$ is even and n is odd.

As R is a π plane rotation, $R = R^{-1}$. $B = RARA^{-1}$. Let $R' = ARA^{-1}$, which is a conjugation of R. R' therefore is a π rotation on the plane spanned by $(Ae_1, Ae_2) = (Ae_1, e_2)$. Therefore, $B = RR'$ is compounded by two plane rotations, and the two plane intersect at the line spanned by e_2. Notice that $R, R' \in SO(3)$, so $B \in SO(3)$ and B is therefore a plane rotation.

2 N Contains All Plane Rotations

Now we have a plane rotation $B \in N$. Without loss of generality, say B is a θ rotation on the plane spanned by (e_1, e_2) (as the indices of basis does not matter).

One can easily prove that any θ rotation is in N. Say we want a θ rotation on the arbitrary plane spanned by (e_3, e_4). By choosing $S \in SO(n)$ such that $Se_1 = e_3$ and $Se_2 = e_4$, we have $SBS^{-1} \in N$, where SBS^{-1} is a θ rotation on the desired plane.

Now we want to generate $SO(2)$ on the plane spanned by (e_1, e_2) then we are done. Given $SO(3)$ is simple, B generates $SO(3)$, which covers all angles from 0 to 2π on the plane spanned by (e_1, e_2). Combining with the previous argument, all plane rotations in $\mathbb{R}^n$ are generated in N.

3 $N = SO(n)$

According to the Cartan–Dieudonne theorem (2.4 of *Naive Lie Theory*), any member of $SO(n)$ can be expressed as an even number of reflections. The product of two reflections is a rotation on the plane which the normal vectors of the two hyperplanes span. For an arbitrary $R \in SO(n)$,

$$R = r_1 r_2 \cdots r_k$$

where $k < n$ is an even number. One can simply pair r_i up, and each pair is a plane rotation. $N = SO(n)$.