Aut($\mathbb{R}^2$)

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1 What's in Aut($\mathbb{R}^2$)

Intuitively, we can see that every translation, reflection and rotation around the origin, and their composition is in Aut($\mathbb{R}^2$). For any $x, y \in \mathbb{R}^2$:

1. $T := \{T_y : \mathbb{R}^2 \to \mathbb{R}^2 | T_y(x) = x + y\}$
2. $D_\infty := \{\phi : \mathbb{R}^2 \to \mathbb{R}^2 | \langle \phi(x), \phi(y) \rangle = \langle x, y \rangle\}$

The informal notation D_∞ is inspired by D_{2n} when the polygon approaches a circle, but is not defined in the same way as D_{2n} was. Moreover, they make a semidirect product, which we will prove later.

We claim that the set of functions preserving the inner product, which is D_∞, is exactly the set of all reflections and rotations around the origin. Let's denote the latter with S. We first prove that $S \subseteq D_\infty$. For all $\phi \in D_\infty$, we have

$$||\phi(x) - \phi(y)|| = ||x - y||$$

Using the cosine rule, so

$$||\phi(x)|| - 2\langle \phi(x), \phi(y) \rangle + ||\phi(y)|| = ||x|| - 2\langle x, y \rangle + ||y||$$

Notice that the origin should be a fixed point according to inner product preservation, and we have $||\phi(x)|| = ||x||$ for all x. So

$$\langle \phi(x), \phi(y) \rangle = \langle x, y \rangle$$

The above shows that $D_\infty = \{\phi \in \text{Aut}(\mathbb{R}^2) | \phi(0) = 0\}$, which implies $S \subseteq D_\infty$.

Next, we go from the other direction. Obviously, $D_\infty \subset \mathcal{L}(\mathbb{R}^2)$. Consider the standard orthonormal basis (e_1, e_2), and for any $\phi \in D_\infty$, $(\phi(e_1), \phi(e_2))$ must be an orthonormal basis as well, as it preserves inner product.

Let's denote $v_1 = \phi(e_1)$ and $v_2 = \phi(e_2)$. They must stay on the unit circle. We then have two cases:

1. If the right-hand orientation is preserved, $v_1 = (\cos\theta, \sin\theta)$ and $v_2 = (-\sin\theta, \cos\theta)$. Now

$$\phi(x) = (x_1(\cos\theta - \sin\theta), x_2(\sin\theta + \cos\theta))$$

which is exactly the rotation by θ.

2. If it becomes a left-hand orientation, $v_1(\cos\theta, \sin\theta)$ and $v_2 = (\sin\theta, -\cos\theta)$. And

$$\phi(x) = (x_1(\cos\theta + \sin\theta), x_2(\sin\theta - \cos\theta))$$

which is exactly reflection over the line with angle $\frac{\theta}{2}$.

The consideration of compositions of rotations and reflections is omitted, as a composition of a rotation and a reflection produces a new reflection, and a composition of two rotations or reflections produces a rotation. Now we have proven $D_\infty \subseteq S$, thus $D_\infty = S$, and the definitions we gave at first are exactly translation, rotation, and reflections.

2 Nothing more in it

Say an arbitrary member $\phi \in \mathrm{Aut}(\mathbb{R}^2)$. Let $\phi(0) = a$, then $T_{-a} \cdot \phi \in \mathrm{Aut}(\mathbb{R}^2)$, and $T_{-a} \cdot \phi(0) = 0$, which is exactly in D_∞. This means the product of T and D_∞ sufficiently covers $\mathrm{Aut}(\mathbb{R}^2)$.

3 They make a semidirect product

A semidirect product basically means every element can be uniquely expressed as a product of elements of some subgroups. It's semidirect instead of direct because it's not abelian.

We have proven that every element can be expressed as a product of T and D_∞. To show that it's a unique product, it's equivalent to show that $T \cap D_\infty = \{I\}$. Assume a non-identity element ϕ in $T \cup D_\infty$. As it's in T, $\phi(0) = y \neq 0$, but we have shown that D_∞ is also the set of all functions fixing the origin. Contradict, therefore $T \oplus D_\infty = \mathrm{Aut}(\mathbb{R}^2)$. This also means there is no redundancy in our definition of $\mathrm{Aut}(\mathbb{R}^2)$.

4 Changing the metric/inner product/norm

Obviously, $\mathrm{Aut}(\mathbb{R}^2)$ changes a lot if any of the metric/inner product/norm is changed. For example, if the norm is defined as $||x|| = ||x_1|| + ||x_2||$, the unit circle will be a square connecting (1,0), (0,1), (-1,0), and (0,-1), and $\mathrm{Aut}(\mathbb{R}^2) = T \oplus D_8$. However, notice that our definition and reasoning on T and D_∞ are never dependent on a specific definition of metric/inner product/norm, so one can always get $\mathrm{Aut}(\mathbb{R})$ using the definition of T and D_∞ under any metric/inner product/norm, and one should realize that T is always good, and D_∞ sometimes becomes finite. Probably I should use another notation for it at first.