No 3-Square Identity

Qige Wang

September 2025

We want to prove that there is no 3-square identity that makes a length-preserving 3D number system, which is characterized by the following equation

$$(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) = z_1^2 + z_2^2 + z_3^2$$

Let's consider x, y, z as column vectors.

We now make an important assumption: each component of z is bilinear in x and y. This is essentially making the multiplication distributive. Moreover, otherwise, one can just give some trivial 3-square identity with zero respect to the number system structure. The bilinear condition gives:

$$\begin{cases} z_1 = x^T A_1 y \\ z_2 = x^T A_2 y \\ z_3 = x^T A_3 y \end{cases}$$

for some 3×3 matrices A_i.

Write $A_x = \begin{pmatrix} x^T A_1 \\ x^T A_2 \\ x^T A_3 \end{pmatrix}$, and A_x is another 3×3 matrix. We have

$$z = A_x y$$

We assume that there exists a 3-square identity that preserves the bilinear condition, so

$$||x||^2 ||y||^2 = ||A_x y||^2 = y^T A_x^T A_x^T y$$

The left-hand side can be rewritten as $y^T(||x||^2 I)y$, thus

$$||x||^2 I = A_x^T A_x$$

Notice that $A_x = x_1 B_1 + x_2 B_2 + x_3 B_3$ for some matrices B_i. Specifically, $B_i = \begin{pmatrix} i\text{th row of } \mathrm{A}_1 \\ i\text{th row of } \mathrm{A}_2 \\ i\text{th row of } \mathrm{A}_3 \end{pmatrix}$. We then rewrite our previous equation as

$$(x_1 B_1^T + x_2 B_2^T + x_3 B_3^T)(x_1 B_1 + x_2 B_2 + x_3 B_3) = (x_1^2 + x_2^2 + x_3^2)I$$

For the quadratic terms, the coefficients are $B_i^T B_i = I$, therefore Bs are orthogonal thus $\det(B_i) = \pm 1$, and $\det(B_i) = \det(B_i^T)$.

On the left-hand side, the $x_i x_j$ terms give $B_i^T B_j = -B_i B_j^T$. It follows that

$$\det(B_i^T B_j) = \det(-B_i B_j^T) = -\det(B_i B_j^T)$$

and the left-hand side is $\det(B_i)\det(B_j)$ and the right-hand side is $-\det(B_i)\det(B_j)$, so they give

$$2\det(B_i)\det(B_j) = 0$$

which is a contradiction.