Defintion of O(n)

Qige Wang

October 2025

We want to prove that the following definitions are equivalent:

- $O(n)$ is all $n \times n$ matrices A where $AA^T = I$.
- For a n-dimensional vector space V over a field F, and a commutative and positive-definite bilinear $\langle , \rangle$ form $V \times V \to F$, $O(n) := \{A \in \mathrm{End}(V) | \forall u, v \in V,\ \langle Au, Av\rangle = \langle u, v\rangle\}$.

We first prove the direction from the matrix definition to the bilinear form one. Assume that $AA^T = I$. Then we have $\langle Au, Av\rangle = (Au)^T Av = u^T A^T Av = u^T v = \langle u, v\rangle$. This shows that $O(n)$ by the matrix definition is a subset of $O(n)$ by the bilinear form definition.

Now we prove the other way around. The bilinear condition gives $\langle u, v\rangle = u^T Mv$. One can eliminate M to I by choosing the basis orthogonal to this bilinear form, which is always plausible. Say $(e_1, \cdots, e_n)$ is an orthonormal basis with respect to $B = \langle , \rangle$, then

$$\langle e_i, e_j\rangle = \begin{cases} 0,\ i \neq j \\ 1,\ i = j \end{cases}$$

Note that here 0 and 1 are additive and multiplicative identities in F. The bilinear form is reduced to the conventional inner product.

$$\langle u, v\rangle = \langle \Sigma_{i=1}^n a_i e_i, \Sigma_{j=1}^n b_j e_j\rangle = \Sigma_{i=1}^n a_i \Sigma_{j=1}^n b_j \langle e_i, e_j\rangle = \Sigma_{i=1}^n a_i b_i$$

Now we can say that $\langle u, v\rangle = a^T b$. The map sends u to a and v to b is a bijection, as it's essentially no more than changing the basis. For the sake of symbolic simplicity, we will just write $\langle u, v\rangle = u^T v$.

Using the bilinear form preservation condition, we have$(Au)^T Av = u^T v$ and thus $u^T(AA^T - I)v = 0$. For the sake of contradiction, assume that $AA^T - I \neq 0$. Let $u = e_i$ and $v = e_j$. $u^T(AA^T - I)v = (AA^T - I)_{ij} = 0$, so it must be the zero matrix. Now $O(n)$ defined by the bilinear form is a subset of $O(n)$ defined by the matrix.