Chapter 3 Probability

Experiment
- ⇨ An experiment is a process when performed will give one result only

Outcomes
- ⇨ The possible results of an experiment

Sample space
- ⇨ The set of all possible outcomes of an experiment
- ⇨ Is denoted by S

Sample point
- ⇨ The element of a sample space

Example 1:

Experiment	Outcomes	Sample Space
Toss a coin once		
Roll a die once		
Toss a coin twice		
Birth of a baby		
Take a test		
Select a student		

Event
- ⇨ An event is a collection of one or more of the outcomes of an experiment
- ⇨ Consists of 2 types: simple event or compound event

Simple event (Elementary event)
- ⇨ An event with only one outcome for an experiment
- ⇨ Usually is denoted by E_i

Compound event (Composite event)

⇨ An event with more than one outcome for an experiment

Example 2:

Experiment: Toss a coin twice

Outcomes = HH, HT, TH, TT

Let E_1 be the event that both 'head' are obtained, then E_1 = {HH}

Let E_2 be the event that a 'head' followed by a 'tail', then E_2 = {HT}

Let E_3 be the event that a 'tail' followed by a 'head', then E_3 = {TH}

Let E_4 be the event that both 'tail' are obtained, then E_4 = {TT}

⇨ Simple event: E_1, E_2, E_3 and E_4

Let Q be the event that at least a 'head' is obtained, then Q = {HH, HT, TH}

Let R be the event that both toss show the same outcome, then R = {HH, TT}

⇨ Compound event: Q and R

Probability

⇨ A numerical measure of the likelihood that a specific event will occur

⇨ Is denoted by P

Notation:

P(E_i) ≡ The probability that a simple event E_i will occur

P(A) ≡ The probability that a compound event A will occur

Two properties of probability

1. The probability of an event always lies in the range 0 to 1
   - ⇨ $0 \le P(E_i) \le 1$ and $0 \le P(A) \le 1$
   - ⇨ B ≡ impossible event if and only if P(B) = 0
   - ⇨ C ≡ sure event if and only if P(C) = 1

2. The sum of the probabilities of all simple events (assume *n* simple events) for an experiment is always 1
   - ⇨ $\sum P(E_i) = P(E_1) + P(E_2) + \cdots + P(E_n) = 1$

Example 3:

Two tosses of a coin ⇨ P(HH) + P(HT) + P(TH) + P(TT) = 1
Football game ⇨ P(Win) + P(Loss) + P(Tie) = 1

Equal likely outcomes/events
⇨ 2 or more outcomes/events that have the same probability of occurrence

Let *n*(A) ≡ total number of outcomes belong to event A
n(S) ≡ total number of outcomes for the experiment
Then

The probability of an event A , $P(A) = \dfrac{n(A)}{n(S)}$

Complement event
⇨ The complement of event A, denoted by $\overline{A}$ is the event that includes all the outcomes for an experiment that are not in A
⇨ $\overline{A}$ denote the event "A does not occur"
⇨ P($\overline{A}$) = 1 – P(A)

Example 4:

A ball is chosen at random from a box containing 6 red balls, 4 white balls and 5 blue balls. Find the probability that the ball is

a) Red b) white c) blue d) not red

Solution:

a)

b)

c)

d)

Example 5:

A die is thrown. State the sample space. Let E be the event “the number is odd” and F be the event “the number is greater than 4”. Find P(E) and P(F).

Solution:

The Addition Rule

Let A and B to be two events defined in a sample space S.
The Addition Rule state that

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example 6:
In a group of 100 people, 40 own a cat, 25 own a dog and 15 own a cat and a dog. Find the probability that a person chosen at random will

a) Owns a cat or a dog
b) owns a cat or a dog, but not both

Solution:

Example 7:
In a group of 20 adults, 4 out of the 7 women and 2 out of the 13 men wear glasses. What is the probability that a person chosen at random from the group is a woman or someone who wear glasses?
Solution:

Example 8:
One white die and one black die are rolled. Find the probability that

a) the white die shows ‘1’ or ‘2’ or the sum of the dice is greater than 9

b) the white die shows ‘1’ or ‘2’ or both dice show the same number

Solution:

Mutually exclusive events

- ⇨ Events that cannot occur together
- ⇨ A and B are mutually exclusive events $\Rightarrow$ $P(A \cap B) = 0$
- ⇨ $P(A \cup B) = P(A) + P(B)$

Example 9:

A die is thrown once. E is the event of getting even number and O is the event of getting odd number. Are the two events mutually exclusive?

Solution:

Independent events

- ⇨ Two events A and B are said to be independent if the occurrence of event A does not affect the probability of the occurrence of event B
- ⇨ $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$ or $P(A \cap B) = P(A) \times P(B)$

Dependent events

- ⇨ Two events A and B are said to be dependent if the occurrence of event A affect the probability of the occurrence of event B
- ⇨ $P(A \mid B) \neq P(A)$ or $P(B \mid A) \neq P(B)$ or $P(A \cap B) \neq P(A) \times P(B)$

Example 10:

A fair die is thrown twice. Let A be the event where a '3' is obtained on the first throw and B be the event where an odd number on the second throw. Are the two events A and B independent or dependent?

Solution:

Example 11:

Students take two independent tests. 30% of the students pass the test A and 60% pass the test B. Find the probability that a student selected at random will pass

a) both tests b) only test A c) only one test

Solution:

Note:

1. Two events A and B with P(A) $\neq$ 0 and P(B) $\neq$ 0 are mutually exclusive
   ⇨ A and B are dependent events
2. Two events A and B with P(A) $\neq$ 0 and P(B) $\neq$ 0 are independent
   ⇨ A and B cannot be mutually exclusive
3. Dependent events may or may not be mutually exclusive

Conditional probability

⇨ probability an event A will occur given that event B has already occurred

⇨ denoted by $P(A \mid B)$

⇨ If P(B) $\neq$ 0 then $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Example 12:
When a die is thrown, an odd number is obtained. What is the probability that the number is a prime number?
Solution:

Example 13:
The probabilities that a student will fail accounting, mathematics, or both are $P(A) = 0.20$, $P(M) = 0.15$, and $P(A \cap M) = 0.03$ respectively. What is the probability that he will fail accounting given that he fails mathematics?
Solution:

<u>The Multiplication Rule</u>

Let A and B to be two events defined in a sample space S.

If A and B are independent, then $P(A \cap B) = P(A) \times P(B)$

If A and B are dependent, then $P(A \cap B) = P(A) \times P(B \mid A) = P(B) \times P(A \mid B)$

Example 14:

When drawing two cards from a shuffled deck, find the probability that the first card is an ace and the second card is king. Assume that the first card is not replaced before the second card is drawn.

Solution:

Example 15:

Swee Ling and Chee Seng played ten games of chess of which six are won by Swee Ling, three are won by Chee Seng and one game ended in a tie. Now they agree to play another three games. Find the probability that

(i) Swee Ling and Chee Seng win alternatively,

(ii) Chee Seng wins at least one game,

(iii) two games will end in a tie.

Solution:

Probability tree

⇨ A useful way of tackling many probability problems

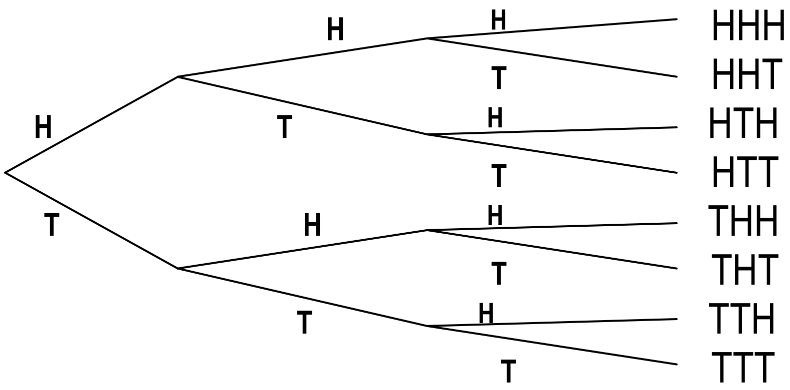
⇨ Each outcome is represented by a branch of the tree

Example 16:

Draw the probability tree for the experiment of tossing a coin 3 times. Find the sample space for this experiment.

Solution:

First toss Second toss Third toss Final outcomes



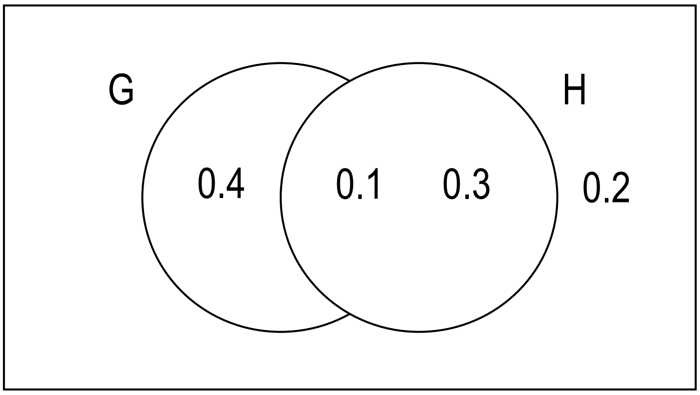
Sample space, S = { HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

Example 17:

A box contains 20 cassettes, 4 of which are defective. If 2 cassettes are selected at random without replacement from this box, what is the probability that both are defective?

Solution:

Example 18:
Given P(G) = 0.5, P(H) = 0.4 and P(G and H) = 0.1. (Refer Venn diagram)



a) Calculate
   i) P(G | H)   ii) P(H | G)   iii) P($\overline{H}$)
   iv) P(G or H)   v) P(G or $\overline{H}$)

b) Are the events G and H mutually exclusive? Independent?

Solution:

Example 19:
A shipment of grapefruit arrived containing 10% pink seedless, 20% white seedless, 30% pink with seeds and 40% white with seeds. A grapefruit is selected at random from the shipment. Find the probability that it is

a) seedless
b) pink
c) Pink given that it is seedless
d) seedless given that it is pink

Solution:

Example 20:
A sample of 500 respondents was selected from a large city to determine various information concerning consumers' behaviour.

Enjoy shopping for clothing	Gender		Total
	Male	Female	
Yes	138	222	360
No	106	34	140
Total	244	256	500

(a) Suppose that a respondent is selected at random. Find the probability that the respondent is

(i) a female,

(ii) a female and she enjoys shopping for clothing,

(iii) a female or he/she enjoys shopping for clothing.

(b) Suppose the respondent chosen is a female. What is the probability that she enjoys shopping for clothing?

(c) Suppose the respondent chosen does not enjoy shopping for clothing. What is the probability that the individual is a male?

Exhaustive event

⇨ $A_1, A_2, \cdots, A_n$ are exhaustive event if $A_1 \cup A_2 \cup \cdots \cup A_n = S$

Bayes theorem

Suppose $A_1, A_2, \cdots, A_n$ are *n* mutually exclusive and exhaustive events.
Let B be an arbitrary event of S. Then, for $i = 1, 2, \cdots, n$

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)}$$

where

$$\begin{aligned} P(B) &= \sum_{i=1}^{n} P(B \mid A_i)P(A_i) \\ &= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_n)P(A_n) \end{aligned}$$

⇨ P(B) is called the total probability of event B

Example 21:
Alice, Barbara and Cathy pack biscuits in a factory. From the batch allotted to them, Alice packs 55%, Barbara 30% and Cathy 15%. The probability that Alice, Barbara and Cathy break some biscuits in a packet is 0.7, 0.2 and 0.1 respectively. What is the probability that a packet with broken biscuits was packed by Alice?

Solution:

Example 22:

In a certain college, 25% of the boys and 10% of the girls are studying Statistics. The girls constitute 60% of the student population. If a student is selected at random and he/she is studying Statistics, determine the probability that the student is a girl.

Solution:

Arrangements

Counting Rule

If one operation consists of n steps in which the first step can be done in k_1 ways, a second steps in k_2 ways, ..., the n-th step in k_n ways, then, the number of ways performing the operation are $k_1 \times k_2 \times \cdots \times k_n$

Example 1:

If a travel agency offers special weekend trips to 7 different cities by air, rail, or bus, in how many different ways can such a trip be arranged?

Solution:

Example 2:

How many 3 digit numbers can be made from the integers 2, 3, 4, 5 and 6 if

a) each integer is used only once?

b) there is no restriction on the number of times each integer can be used?

Solution:

Result 1

The number of ways of arranging n unlike (different) objects in a line is n!

Note:
1. $n! = n(n-1)(n-2)\cdots(3)(2)(1)$
2. 0 ! = 1

Example 3:
In how many ways we can arrange the letters A, B, C and D in a line?
Solution:

The arrangements are

ABCD	ABDC	ACBD	ACDB	ADCB	ADBC	BCDA	BCAD
BDAC	BDCA	BACD	BADC	CDBA	CDAB	CABD	CADB
CBAD	CBDA	DABC	DACB	DBCA	DBAC	DCAB	DCBA

Result 2
The number of ways of arranging in a line n objects of which p of first type are alike, q of a second type are alike, r of a third type are alike, and so on, is $\frac{n!}{p!q!r!\cdots}$

Example 4:
In how many ways can the letters of the word STATISTICS be arranged?
Solution:

Example 5:
Five red balls, two white balls and three blue balls are arranged in a row. How many ways the balls can be arranged?

Solution:

Permutation

- ⇨ If r objects are selected from n different objects with attention given to the order of arrangement, then, the selection is called a permutation
- ⇨ The number of permutations of r objects chosen from n unlike objects is ^nP_r where $^nP_r = \dfrac{n!}{(n-r)!}$

Note: $^nP_n = n!$

Example 6:
How many ways the two letters from 'A', 'B', 'C' and 'D' can be arranged?

Solution:

The arrangements are
AB BA AC CA AD DA BC CB BD DB CD DC

Example 7:
If 20 students are entered in a contest, in how many different ways can a judge award a first prize and a second prize to the students?
Solution:

Combination

- ⇨ If r objects are selected from n different objects with no attention given to the order of arrangement, then, the selection is called a combination
- ⇨ The number of combinations of r objects chosen from n unlike objects is ^nC_r where $^nC_r = \dfrac{^nP_r}{r!} = \dfrac{n!}{r!(n-r)!}$

Example 8:
In how many ways, four letters can be chosen from the word RANDOMLY?
Solution:

Example 9:
A committee consists of 2 ladies and 3 men are chosen from 5 ladies and 7 men. In how many ways this can be done if

a) any lady and any man can be included?
b) one particular man must be in the committee?
c) two particular ladies cannot be in the committee?

Solution:

Example 10:
Find the number of ways in which 10 students can be divided into 2 groups of

a) six and four students
b) five students each

Solution:

Example 11:

Three letters are selected at random from the word BIOLOGY. Find the number of selection that will contain

a) no letter O

b) only one letter O

c) both of the letters O

d) any letters

Solution:

Check answer:

Selection with	Possible selections
no letter O	BIL, BIG, BIY, BLG, BLY, BGY, ILG, ILY, IGY, LGY
one letter O	OBI, OBL, OBG, OBY, OIL, OIG, OIY, OLG, OLY, OGY
two letter O	OOB, OOI, OOL, OOG, OOY