Chapter 4 Probability Distribution

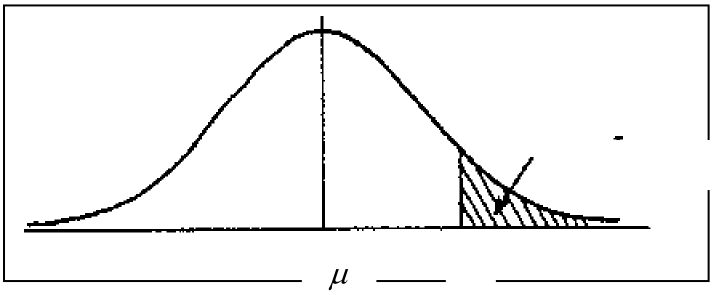
Consider only
1. Continuous distribution: Normal
2. Discrete distribution: Binomial and Poisson

Normal distribution

A random variable X has a Normal distribution with mean μ and variance σ^2 if and only if its probability function is given by

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), \qquad x \in R, \quad \mu \in R, \quad \sigma > 0$$

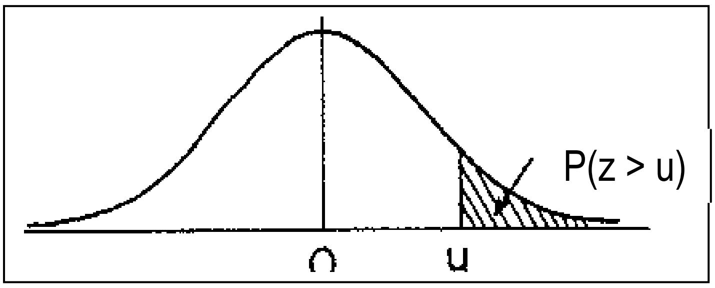
and it is denoted by $X \sim N(\mu,\sigma^2)$.



If $X \sim N(\mu,\sigma^2)$, then the random variable $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. The random variable Z is called the standard Normal distribution with probability function

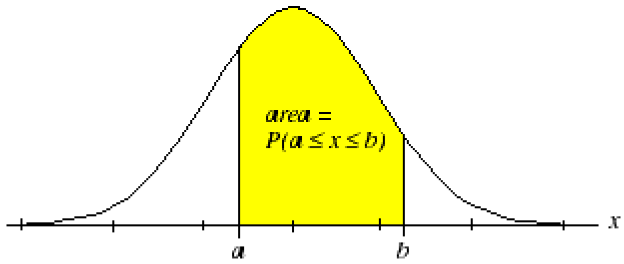
$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right), \qquad z \in R$$

and the curve is symmetry about z = 0



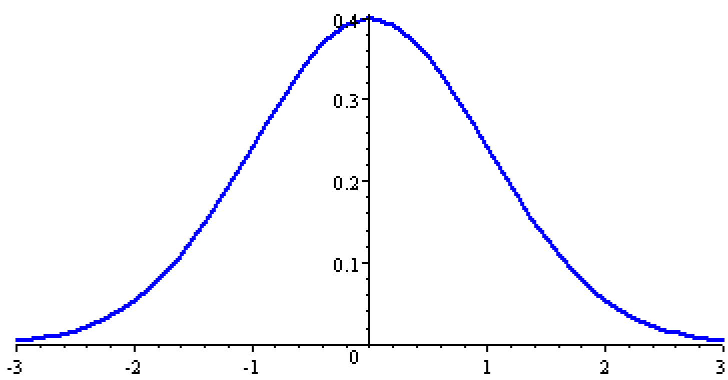
Note:
A normal probability distribution, when plotted, gives a bell-shaped curve such that

1. The total area under the curve is 1
2. The curve is symmetric about the mean
3. The two tails of the curve extend indefinitely



Hence, the area under the curve between two ordinates X = *a* and X = *b* where *a* < *b*, represents the probability that X lies between *a* and *b*, denoted by P(*a* < X < *b*)

Note:
In the Normal distribution tables, the partial areas under the Normal curve are tabulated against the standardised variable Z.



Example 1:

Find the following probabilities for the standard Normal curve.

a) $P(Z > 1.5)$

b) $P(Z > -2)$

c) $P(Z < 0.8)$

d) $P(Z < -1.5)$

e) $P(-1 < Z < -0.5)$

f) $P(-2 < Z < 1)$

g) $P(1.19 < Z < 2.12)$

Solution:

Example 2:

The mean weight of 200 people is 67 kg and the standard deviation is 7 kg. Assuming that the weights are Normally distributed, determine how many people have a weight

a) between 60 and 74 kg   b) more than 81kg

c) between 53 and 88kg

Solution:

Example 3:

If $Z \sim N(0, 1)$, find the value of a if

a) $P(Z > a) = 0.3783$

b) $P(Z > a) = 0.7823$

c) $P(Z < a) = 0.0793$

d) $P(Z < a) = 0.9693$

Solution:

Example 4:
The score on a final examination was normally distributed with mean 72 and the standard deviation 9. The top 10% of the students are receive A's. What is the minimum score that a student must get in order to receive an A?

Solution:

Example 5:
If X ~ N (100, σ^2) and P(X < 106) = 0.8849, find the standard deviation σ .

Solution:

Binomial Distribution

A Binomial experiment must satisfy the following 4 conditions

1. There are *n* identical trials
   - ⇨ an experiment is repeated *n* times under identical conditions
2. Each trial has only two possible outcomes
   - ⇨ success or failure
3. The probabilities of the two outcomes remain constant
   - ⇨ The probability of success is denoted by *p*
   - ⇨ The probability of failure is denoted by *q*
   - ⇨ $p + q = 1$
4. The trials are independent
   - ⇨ Outcome of one trial does not affect the outcome of another trial

Let random variable X ≡ the total number of success in the *n* trials

⇨ X is a Binomial distribution with parameter *n* and *p*

⇨ denoted by $X \sim B(n, p)$ and $x = 0, 1, ..., n$

For a Binomial distribution,

⇨ mean = np and variance = $np(1-p)$

The probability of exactly *x* successes in *n* trials is given by

$$P(X = x) = {}^{n}C_{x} p^{x}(1-p)^{n-x}; \quad x = 0,1,\cdots,n; \quad p \in [0,1]$$

where

n = total number of trials

p = probability of success

x = number of successes in *n* trials

Example 6:

If the probability of a defective bottle is 0.1, find the mean and standard deviation for the distribution of defective bottles in a total of 400.

Solution:

Example 7:

If 20% of the bottles produced by a machine are defective, determine the probability that, out of 4 bottles chosen at random,

a) 1, b) 0, c) at most 2 bottles will be defective

Solution:

Example 8:

The probability that an entering college student will graduate is 0.4. Determine the probability that out of 5 students

a) none, b) 1, c) at least 1, d) all will graduate

Solution:

Note:

Inequality	**Sign**
At most/ not more than	$X \leq$
At least/ not less than	$X \geq$
Exceed / more than	$X >$
Less than/ fewer than	$X <$

<u>Normal approximation to the Binomial distribution</u>

⇨ In a Binomial situation when *np* > 5 and *nq* > 5, the Normal distribution with mean = μ = *np* and standard deviation = σ = $\sqrt{np(1-p)}$ can be used to approximate the Binomial distribution

⇨ $X \sim B(n, p) \approx X \sim N(\mu, \sigma^2)$ when *np* > 5 and *nq* > 5

Continuity correction factor

⇨ The addition and/or subtraction of 0.5 from the value(s) of *x* when the Normal distribution is used as an approximation to the Binomial distribution, where *x* is the number of successes in *n* trials

Example 9:

Binomial	⇨ Normal
P(X = 2)	P(1.5 < X < 2.5)
P$(3 \leq X \leq 5)$	P(2.5 < X < 5.5)
P$(3 < X \leq 5)$	P(3.5 < X < 5.5)
P$(3 \leq X < 5)$	P(2.5 < X < 4.5)
P$(3 < X < 5)$	P(3.5 < X < 4.5)
P(X < 4)	P(X < 3.5)
P$(X \leq 4)$	P(X < 4.5)
P$(X \geq 4)$	P(X > 3.5)
P(X > 4)	P(X > 4.5)
P$(X \geq 0)$	P(X > – 0.5)
P(X > 0)	P(X > 0.5)
P(X = 0)	P(– 0.5 < X < 0.5)

Example 10:
Find the probability that 200 tosses of a bias coin will result in less than 51 heads in which the probability of getting a head is 0.2 for each toss.
Solution:

Example 11:
Find the probability of getting between 3 and 6 heads inclusive in 10 tosses of a fair coin by using the Normal approximation to the Binomial distribution

Solution:

Poisson distribution

A discrete random variable having probability function of the form

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0,1,2,\cdots$$

where λ can take any positive value, is said to follow the Poisson distribution

Note: If λ is not given, then $\lambda = np$

If an event is randomly scatted in time (or space) and has mean number of occurrence λ in a given interval of time (or space) and X is the random variable “the number of occurrences in the given interval”

⇨ X is a Poisson distribution with parameter λ

⇨ denoted by X ~ $P_o(\lambda)$ and x = 0, 1, 2, ...

For a Poisson distribution with parameter λ,

⇨ mean = λ and variance = λ

Example 12:

The mean number of bacteria per ml of a liquid is known to be 4. Assuming that the number of bacteria follows a Poisson distribution, find the probability that, in 1 ml of liquid, there will be

a) No b) 4 c) less than 3 bacteria

Solution:

Example 13:
Based on information obtained in previous example, find the probability that
a) in 3 ml of liquid , there will be less than 2 bacteria
b) in 0.5 ml of liquid, there will be more than 2 bacteria
Solution:

Poisson approximation to the Binomial distribution

⇨ A Binomial distribution with parameters *n* and *p* can be approximated by a Poisson distribution with parameter $\lambda = np$ if *n* is large and *p* is small.

⇨ The approximation gets better as $n \to \infty$ and $p \to 0$

⇨ $X \sim B(n, p) \approx X \sim P_o(\lambda = np)$ when $n \to \infty$ and $p \to 0$

Example 14:
Eggs are packed in boxes of 500. On average, 0.8% of the eggs are found to be broken when the eggs are unpacked. Find the probability that in a box of 500 eggs

i) exactly 3 ii) less than 2 iii) more than 2 will be broken

Solution:

Normal approximation to the Poisson distribution

⇨ If X ~ P_o(λ), then mean = μ = λ and variance = σ^2 = λ

⇨ For large λ, X ~ N(λ, λ) approximately

⇨ Generally, we require λ > 20 for a good approximation

⇨ $X \sim P_o(\lambda) \approx X \sim N(\lambda, \lambda)$ when λ is large

Example 15:

The number of calls received by an office switchboard per hour follows a Poisson distribution with parameter 30. Using the Normal approximation to the Poisson distribution, find the probability that in one hour, there are

a) more than 23 calls b) between 25 and 28 calls inclusive

Solution:

Let X ≡ the number of calls in one hour. Then X ~ P_o(30)

Using the Normal approximation, X ~ N(30, 30)

a) $P(X > 23) \approx$

b) $P(25 \le X \le 28) \approx$