Differential Forms
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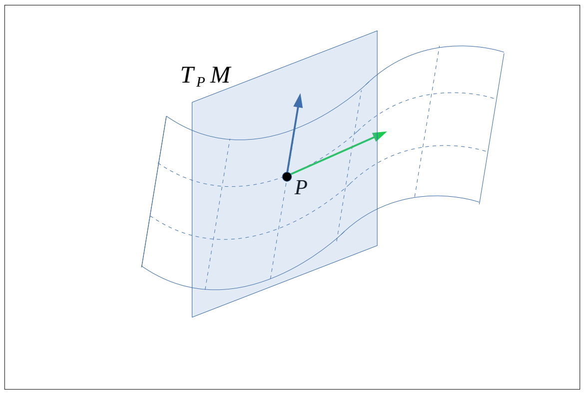
Introduction to Differential forms

Background material:

1. Manifolds and Vector spaces (functions and functionals)
2. Tangent space and Vector fields
3. Directional derivative

Manifolds: this is a “space” that is locally Euclidean



A Manifold can be n dimensional $\mathbb{R}^n$

From this lecture we will try to develop our ideas of Calculus on a Manifold and to do that we are going to learn these Differential forms and today we are going to review some background materials that will be necessary for discussing this Differential forms. Firstly we want to discuss about Manifolds and Vector spaces and they will be relevant to define functions and functionals that will play key role when we try to discuss Differential forms then we'll go on to Tangent space and Vector fields and finally at the end of the lecture we want to talk about something called a Directional derivative and from these places we will see the emergence of Differential forms.

Let's get started by doing the following: first we want to discuss Manifolds, so what is a Manifold? That's the first question so a very hand waving or non rigorous definition would be a Manifold is such a “space” that locally looks Euclidean so this is though it's not really appropriate to call it a space but for now, to visualize the thing we want to call it a space, this is a space that is locally Euclidean. What do we mean by this statement? Think of the following: let's say you have a rubber sheet like this and if you consider a point let's say the point P and you consider a very small neighborhood around P, if this neighborhood is flat then this is Euclidean that means there is no ups and downs no curvatures and all that. If this is the case for all the points and their neighborhood on this rubber sheet then this rubber sheet can be considered as a Manifold.

This locally Euclidean term is actually coming from the fact that we are considering a neighborhood an example would be if you take Earth and you consider a very small portion of it let's say your room doesn't look curved, locally it looks flat so this is the case for all the points on Earth and as a result the surface of the Earth can be considered as a Manifold and these Manifolds can be of arbitrary dimension so what do I mean by dimension? If you consider your room which can be idealized as a cubical box any point on that room or inside that room can be totally described by three points and those are your very familiar Cartesian coordinates that is x, y, z , even though we did not define this yet but we will define it soon so when I say that a Manifold can be n-dimensional where n is some number, an integer. A Manifold can be n-dimensional and by this we mean that every point in that Manifold, let's say P_1 it will be completely characterized by some n numbers and let's call those numbers as $x_1, x_2, \dots, x_n$ that means if you want to pinpoint a point on the Manifold you'll need to have n entries. That is what the dimension means and typically it is written as if it's n-dimensional and all the entries are real then it is written as $\mathbb{R}^n$, this is what an n dimensional Manifold would look like.

This is the informal definition of a Manifold and now we want to discuss another extreme point of view of mathematics Manifold is one of them another one is called a Vector space so for us knowing the definition would suffice so what is a Vector space you can think of Vector space as a space where all the vectors live in, that's an informal way to think of Vector space but formally what we can say is that if you have a collection of objects let's say the collection is denoted with some V , so collection of objects note the word that I'm using objects I'm not specifying what they truly are so this is a very abstract definition if you have a collection of objects let's say V which includes some objects labeled as $V = \{v_1, v_2, \dots, v_n\}$ and so on, if for any real number, if you have a something like this let's say you take two of them you add them together and the resulting object will also belong to V and the second property is that if there is some α where $\alpha \in \mathbb{R}$ is a real number if you multiply this thing with any v_i this will also belong to V . If you have a set or collection of objects that satisfy these things then that would be called a real Vector space so this definition will help us in defining vectors fields and later we will define something called an inner product. This definition will help us in constructing those objects as well so this is the very basic definition of Manifolds and Vector spaces.

Vector space: Collection of objects $V = \{v_1, v_2, \dots, v_n\}$ such that

1. $v_1 + v_2 \in V$
2. $\alpha v_i \in V$ where $\alpha \in \mathbb{R}$

Now what we want to do is that we want to define functions and functionals so you already know about functions but from the Manifold point of view a function is something let's call the function f and what it does is that it takes a point of the Manifold that's an n-dimensional Manifold to be precise and it maps that point which consists of n entries into a single real number.

Functions: $\mathrm{f} : \mathbb{R}^n \to \mathbb{R}$ where f is a function and $\mathbb{R}^n$ is the Manifold

The next thing that we are going to consider the definition of functionals so just like functions we can define functionals for vector spaces and just like Manifolds Vector space can be of n dimension as well it can even be infinite dimensional so let's call the functional as F and let's say it works on a Vector space let's call that V^n , I mean this is n dimensional vector space and a functional would take in a vector or a vector function in fact we can even call them a vector function as well and this will even this will ultimately map this vector thing into a real number as well note the fact that we are only considering real Vector spaces so you can also apply an r over here just to be sure that you are dealing with real Vector spaces and stuff so this mapping over the Vector space is known as a functional and you already know the different examples for function and for functional what you can recall from your high school that if you had two vectors like this

Functionals: $F: V_{\mathbb{R}}^{n} \to \mathbb{R}$ It takes in a function and spits out a number.

e.g. $\vec{A}=2\hat{x}+3\hat{y}$, $\vec{B}=\hat{x}-\hat{y}$ then $\vec{A}\cdot\vec{B}=2-3=-1$

This can be thought of as one example of functional but the very idea of functionals is that it takes in a function and spits out a number so in that sense a definite integral would would also be a functional a derivative at some point would also be a functional so you have already seen examples of functional but this is the definition that is not introduced in your general first year Calculus books.

Coordinate functions

Now what we want to do is that we are done with the very basic definitions now we want to define coordinates or coordinate functions so let's call it coordinate functions. Note the fact that I have only talked about Manifolds and points and points can exist on a Manifold regardless of the coordinate system you choose because this coordinate systems only help you to describe some certain physical phenomena. However the points itself they are not dependent on the coordinate functions only when you put a coordinate system on a Manifold then you can reload or use those points to describe some phenomena using the tools of mathematics so what does this coordinate functions do?

Similar to other functions will define coordinate functions are as maps that takes points on the Manifold and spits out a real number so we are going to start with a very familiar Cartesian coordinate so this will be the definition of Cartesian coordinates and I'll just go with only three coordinates that means this Cartesian coordinates are such functions that if there is some x at a point P that means this is a function and it's taking a number from the 3D Manifold $\mathbb{R}^3$ and it's mapping into a real number. Similarly y and z can be defined as the same way and this mapping for x, y, z are defined as follows:

Cartesian coordinates: x, y, z $\begin{cases} x:\mathbb{R}^3 \to \mathbb{R} \\ y:\mathbb{R}^3 \to \mathbb{R} \\ z:\mathbb{R}^3 \to \mathbb{R} \end{cases}$

If there is a point P on $\mathbb{R}^3$ such that $P=(x_1, x_2, x_3)$ then $x(P)=x(x_1, x_2, x_3)=x_1$, $y(P)=y(x_1, x_2, x_3)=x_2$ and $z(P)=z(x_1, x_2, x_3)=x_3$

This is how you can define your coordinate functions similarly if you go to Polar coordinates:

Polar Coordinates: (r, θ) $\begin{cases} r:\mathbb{R}^2 \to \mathbb{R} \\ \theta:\mathbb{R}^2 \to \mathbb{R} \end{cases}$

Valid at any point $P=(x_1, x_2)$ on $\mathbb{R}^2$

r is such a map that $r(x_1, x_2)=\sqrt{x_1^2+x_2^2}$ and $\theta=\tan^{-1}\left(\dfrac{x_2}{x_1}\right)$

These are examples of coordinate functions now let's move on obviously you can do all of this with numbers as well that is instead of x_1, x_2, x_3 you could take any real number you want and you would get some numerical values once you plug those x_1, x_2 values into this r, θ so now what we want to do is that since we are done with discussing manual force Vector space and coordinate functions the next important thing is to discuss about Tangent space and Vector fields so for that again we will make use of this picture so again take this point P and since this is a Manifold and we have already defined functions what I can do right now is that I can imagine that there might be curves on this Manifold. Since there are some curves from our knowledge of single variable Calculus we can imagine that we may be able to draw some Tangents, that means we can differentiate.

Obviously this curves have to be differentiable they have to be smooth so if we now try to draw a Tangent on P we can do it in many ways and let me show you how for example if you are going from one way to the other way that crosses the point P then you can have a Tangent vector. Now what you can do is that you can also go on another direction, also both ways also crossing the point P and similarly for other directions. You can also go in other directions as well. You can see that on this point you can draw very different Tangent vectors depending on which direction you want to travel. As a result what we have on point P is that we have a set of Tangent vectors, not just one Tangent vector.

To deal with that set we have to define the Tangent space. We have different Tangents and we can actually get generator set and this is what motivates us to define a Tangent space in the first place. Let's go down to the definition of a Tangent space

Tangent space definition: this Tangent space is a space on a particular point on a Manifold that contains all the possible Tangent Vectors that pass through that point. Graphically it would look something like this if this is our Manifold and this is the point of interest then a Tangent plane would look something like this and if this is our Manifold which is denoted with M then the Tangent space would be denoted as $T_P M$ where P is the point and this tells you that if you have $v \in T_P M$ that means this v or this Tangent Vector belongs to the Tangent space of a point P which residing in the Manifold.

Why do we bother even defining these things? Because it turns out that to define a Vector field we will need these ideas because you cannot define a Vector on a Manifold itself, to define the Vectors you need those Tangent spaces, these Tangent spaces are the Vector spaces in fact. There are Manifolds that exhibit both the behaviors of Manifold itself, there's general spaces which exhibits the behavior of Manifolds and Vector spaces but not all the Manifolds show this behavior that's why we need to make these two spaces distinct that when we deal with more general type of Manifolds in future we don't get confused and an example that there are spaces which exhibit both of the behaviors would be our familiar Cartesian coordinates. It's a Manifold and at the same time it's a Vector space but yet we want to keep these definitions distinct so that we don't get confused.

Since we can define Vectors, we can define Tangent spaces, now let's try to define a Vector field from very rough idea or very basic Vector Calculus you know that a Vector field points towards a distribution that tells you that if you have some space at every point there is a Vector. That means at every point you will have some value and some direction. Now let's think this idea in terms of this Manifold language. To define Vectors at every point on this Manifold that means on the Tangent spaces you will need all the copies of the Tangent spaces of the points of all the points in this Manifold. For example here we have only one point but there will be other points let's say P_1 , P_2 , P_3 , P_4 and all of these points will carry a Tangent space of their own. All these Tangent spaces with the Manifold itself is known as a Tangent bundle and you define a Vector field on a Tangent bundle because otherwise all the points will not be included. This is the definition of Vector fields from real variable Calculus or ordinary Calculus gets transformed into the Manifold language. Let me just write in short what is the Tangent bundle.

Tangent bundle just means the Manifold itself plus, I mean you are considering them together don't mistake it for the ordinary arithmetic operation, and Tangent spaces of all the points P . All this is a Tangent bundle and if you take a section of the Tangent bundle on that section you can define this Vector fields and a remark about the Vector field is that if your Vector fields, if they are to be called smooth then that would mean that on every point that is a Vector field is called smooth if it is possible to find curves on the Manifold such that all the Vectors in the Vector field are Tangent to the curves:

Smooth Vector fields: A Vector field is called smooth if it is possible to find curves on a Manifold such that all the Vectors in the Vector field are Tangent to those curves. In fact these are called Integral curves and what are these Integral curves, they are nothing but the solutions to differential equations.

This is the power of this Manifold language you can apparently link all those things that seem a little bit unrelated to each other and use this Manifold language thing and you can see that they're actually connected with each other this is one way to appreciate this new language and now we have had these ideas of Tangent spaces we know that there are things called Vectors, Manifolds all that stuff finally we want to jump into the thing called a Directional derivative.

What is the Directional derivative? Since we have defined what are functions we can try to define a function on it obviously this can be thought of as an n-dimensional Manifold for simplicity I am considering $\mathbb{R}^3$ that is a 3D Manifold $M=\mathbb{R}^3$ so what we can do right now we can define function on this Manifold and that function would take in three arguments and for simplicity again we can think of this Manifold as the 3D Euclidean space. We can define a function like $\mathrm{f}(x,y,z)$ for simplicity but this procedure would also work when there are n dimensional Manifolds and you have n variables

As you can see that in real variable Calculus we had only two ways to go that is we can go left or we can go right on the x axis but here on the Manifold there are three degrees of freedom so I can either move in x , I can either move in y or z or I can move in between x, y, z so there are more degrees of freedom so now what we do is that since we know that we can even define vectors as well on a Manifold so let's take a point let's say the point has this arguments x_0, y_0, z_0 the point itself is P and we try to find a derivative that means we want to see how this function changes. Let's say the derivative along this direction this is a vector so let's call this vector u and a new way to write our vectors on this Manifold language would be to write it as a column vector that means a column matrix and let's say it has elements $u=(a,b,c)^T$. If you write it in your high school language this would look like $u=a\hat{i}+b\hat{j}+c\hat{k}$.

This is our vector and we want to see what happens to this function that means how it changes if we go on to this direction if we move into this direction so this is the motivation behind defining this Directional derivative now the formal mathematical definition that people use usually see in their vector Calculus course is something like this:

Vector Calculus: Directional derivative

$$D_u \mathrm{f}(x,y,z)=\lim_{t\to 0}\frac{\mathrm{f}(x_0+at, y_0+bt, z_0+ct)-\mathrm{f}(x_0,y_0,z_0)}{t} \qquad (1)$$

Where $P=(x_0,y_0,z_0)$ and $P'=(x_0+at, y_0+bt, z_0+ct)$. This is the definition and this can be written in a more compact form using the vector notations and that compact form is nothing but:

$$D_u \mathrm{f}(x,y,z) = \lim_{t \to 0} \frac{\mathrm{f}(P+ut) - \mathrm{f}(P)}{t} \tag{2}$$

Note that P itself contains three arguments or you can say a vector as well on that point you can think of it both ways. If this limit exists then the value of this limit evaluated at point $P=(x_0, y_0, z_0)$ which corresponds to $t=0$ would be the Directional derivative. Working in this way it becomes a bit tough to evaluate these limits so what we do right now we have to do a little bit of algebra to simplify this definition so let's try to simplify this definition and we want to see that whether we can actually compute something that is easier to do than evaluating this limit.

(2) almost looks like a derivative that is analogous to a real variable function I mean a single variable function so what you can do right now is that since you are trying to evaluate the Directional derivative at a point P where $t=0$. You use this definition and you write it as:

$$D_u \mathrm{f}(x,y,z) = \frac{d}{dt}\left(\mathrm{f}(P+ut)\right)\bigg|_{t=0} \tag{3}$$

What is the advantage of this thing (3)? Well, recall the fact that we can write a total differential of a function or a multi valued function as follows since we are considering three variables:

$$d\,\mathrm{f}(x,y,z) = \frac{\partial \mathrm{f}}{\partial x} dx + \frac{\partial \mathrm{f}}{\partial y} dy + \frac{\partial \mathrm{f}}{\partial z} dz \tag{4}$$

That is an infinitesimal change in the Calculus perspective but we will later see that these are actually 1-forms which you will see in the next lecture. Finally you have this guy (4) so you can use this over here and use the fact that P and u they actually contain three entries so by doing that what you are actually doing is writing it as:

$$D_u \mathrm{f}(x,y,z) = \frac{d}{dt}\left\{\mathrm{f}(x_1, x_2, x_3)\right\}\bigg|_{t=0} \tag{5}$$

Where $x_1 = x_0 + at, x_2 = y_0 + bt, x_3 = z_0 + ct$. By doing this what I can do right now is that I can use some tools from multi variable Calculus to simplify my calculations. Now you can use the chain rule that means you will have:

$$D_u \mathrm{f}(x,y,z) = \frac{\partial \mathrm{f}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathrm{f}}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \mathrm{f}}{\partial x_3}\frac{dx_3}{dt}\bigg|_{t=0} \tag{6}$$

Since $x_1 = x_0 + at$ then $dx_1/dt = a$. Similarly you can show $dx_2/dt = b$, $dx_3/dt = c$. We finally have:

$$D_u \mathrm{f}(x, y, z) = \left(\frac{\partial \mathrm{f}}{\partial x_1}\right) a + \left(\frac{\partial \mathrm{f}}{\partial x_2}\right) b + \left(\frac{\partial \mathrm{f}}{\partial x_3}\right) c \tag{7}$$

If you can calculate partial derivatives you can use that power to calculate Directional derivatives as well, in fact Directional derivatives reduce to partial derivatives and you can easily see that if you consider the vector u as a unit vector that is u was written as it was written like $u=(a, b, c)^T$ now I am only interested in the directional Derivative segment where the direction is only along x_1 , let's say, so that would mean that I'm not looking what is happening over the other axes, that is, x_2 and x_3 that means I don't care about those variables or coordinates so the direction vector that is u will contain only one element and it will be one instead of a you will have $a=1$ and $b=0$ and if you do that, you will see that the only term that survives will be the first one in (7).

This is our Directional derivative that generalizes partial derivatives and as a special case you get the partial derivative back if you use the special conditions. There is another remark is that these vectors can be unit vectors as well and the reason is kind of arbitrary because if you keep these vectors as unit vectors that means the modulus of the those vectors are 1 then you can define the Directional derivative but at the same time you also retain the definition of Tangents so that's one motivation to keep those vectors as unit vectors. This would be our final formula for computing Directional derivatives:

$$D_u \mathrm{f}(x, y, z) = \left(\frac{\partial \mathrm{f}}{\partial x}\right) a + \left(\frac{\partial \mathrm{f}}{\partial y}\right) b + \left(\frac{\partial \mathrm{f}}{\partial z}\right) c \text{ where } u=(a, b, c)^T \tag{8}$$

This idea can be generalized into more variables with more more general Manifolds or higher dimensional Manifolds or even lower dimensional Manifolds as well. Now that we are almost done through our definitions there is a one last thing that we need to see and that will be very important for the emergent of Differential forms but let's now try to do a numerical example of this Directional derivatives. In our example our function is just a two variable function given by:

$$\mathrm{f}(x, y) = y^2 - 3xy + x^3 \tag{9}$$

We want to compute $D_u \mathrm{f}$ at $P=(1,2)$ with $u=(\cos\pi/6, \sin\pi/6)^T$ so all we have to do is that we need to just use the formula that we derived a few minutes ago (8) and we'll be good to go. First things first, we want to compute the partial derivatives so when you are differentiating with respect to x we you treat everything that is not x as a constant including the y :

$$\frac{\partial \mathrm{f}}{\partial x} = 3x^2 - 3y \quad , \quad \frac{\partial \mathrm{f}}{\partial y} = -3x + 2y \tag{10}$$

We can finally write that this is nothing but:

$$D_u \mathrm{f}(x, y) = \frac{\partial \mathrm{f}}{\partial x} a + \frac{\partial \mathrm{f}}{\partial y} b = (3x^2 - 3y)\cos\pi/6 + (-3x + 2y)\sin\pi/6 \tag{11}$$

Remember that we have to use the point because in our definition we were saying that $t=0$ but $t=0$ when these two points coincided so now we have to use the information that we are interested in this Directional derivative at point $P=(1,2)$ and since this is the Cartesian coordinate system $x=1$ and $y=2$. Now we have to use that as well so using this point $P=(1,2)$:

$$D_u \mathrm{f}(x,y)=(3-6)\cos\pi/6+(3-6)\sin\pi/6=-3\cos\pi/6-4\sin\pi/6 \tag{12}$$

This is your Directional derivative at that particular point. Now let's talk about the last thing that will help us in understanding the emergence of Differential forms is that we have seen the Directional derivatives definition from the vector Calculus point of view. What we can do right now is that we can define this Directional derivative in another way so this is the new version of the definition of the Directional derivative

New definition/Alternative definition: Let f be a function that works on a Manifold $\mathrm{f}:\mathbb{R}^n \to \mathbb{R}$. Let v_P be a Tangent vector at point P on that Manifold M that is $v_P \in T_P M$. You can also specify the dimension by writing $v_P \in T_P M(\mathbb{R}^n)$. If this is true then the number which is defined as:

$$v_P[\mathrm{f}] \equiv \frac{d}{dt}\left(\mathrm{f}(P+t v_P)\right)\Big|_{t=0} \tag{13}$$

This number is our Directional derivative now you can already see that on the right side we have the same thing that we saw from the definition of the vector Calculus itself so you might think, why are we doing this? If we already know that this definition is there so there is another reason for doing it and you will see why so from our previous result over here (8). You can see that not only this derivative contain this partial derivatives of the function but also it contain the components of the vector, that unit vector and now the difference is that instead of unit vector we are saying that there should be a Tangent vector. That is on that Manifold so similarly just like before we can write it as:

$$v_P[\mathrm{f}]=\frac{\partial \mathrm{f}}{\partial x_1}v_1+\frac{\partial \mathrm{f}}{\partial x_2}v_2+\cdots+\frac{\partial \mathrm{f}}{\partial x_n}v_n \tag{14}$$

I'm working on an n dimensional Manifold so v_P and it will have components $v_P=(v_1,v_2,\ldots,v_n)$ and I'm also assuming that my Vector space is also n dimensional even though that's not the case always, in fact it can be larger or it can be smaller but for now for simplicity I'm just saying that the dimension of the Manifold and the dimension of the Tangent vectors are the same though that is not the general case and even we will see that most of the time this is not the case. Then under a summation symbol I can write this thing in a very compact manner:

$$v_P[\mathrm{f}]=\sum_{i=1}^{n} v_i \frac{\partial \mathrm{f}}{\partial x_i} \tag{15}$$

Note that these these are just components that means they are some numbers so what does this tell us? A very interesting thing, let's take a special case $\mathbb{R}^3$ so when it's $\mathbb{R}^3$ you will write it as:

$$D_u f(x, y, z) = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y} + v_3 \frac{\partial f}{\partial z} \equiv v_P[f] \tag{16}$$

Now this was defined as the Tangent vector at the point P and it is acting on the function so this third bracket thing is also used to denote that this is indeed a function, this is another way of denoting stuff. Now what you can do is that instead of these v_P that means the Tangent vector at some points what if we take the unit vectors. by unit vector I mean the basis vectors of x, y, z that means $\hat{x}$, $\hat{y}$, $\hat{z}$ so here we are going to use a more general notation instead of $\hat{x}$ we are going to write it as $\boldsymbol{e}_1$, instead of $\hat{y}$ we are going to write it as $\boldsymbol{e}_2$ and instead of $\hat{z}$ we are going to write it as $\boldsymbol{e}_3$. Whenever you see $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ you should understand that as the basis vectors and what are basis vectors again?

These are the fundamental vectors that you need to make up any vector inside a Vector space. If you do that and these vectors have this components:

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ , \ \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ , \ \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{17}$$

If you do that you can easily see that:

$$\boldsymbol{e}_1[f] = v_1 \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[f] \ , \ \boldsymbol{e}_2[f] = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[f] \ , \ \boldsymbol{e}_3[f] = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[f] \tag{18}$$

The component is v_1 of $\boldsymbol{e}_1$ is actually 1 . From here you can see that you are taking a function then you are using some mapping on that function which is $\boldsymbol{e}_1$ and finally this mapping is giving your output of the derivative of that function with respect to that basis vector. Similarly $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. Now comes a remarkable conclusion: by judging with these things you can think this as an operator, it is acting on something and it's giving you something else and these are partial differential operators so you can finally conclude that these basis vectors are nothing but the differential operator themselves.

$$\boldsymbol{e}_1 = \frac{\partial}{\partial x} \ , \ \boldsymbol{e}_2 = \frac{\partial}{\partial y} \ , \ \boldsymbol{e}_3 = \frac{\partial}{\partial z} \tag{19}$$

At first this may look very weird and shocking but this is in fact how basis vectors are defined in multi-variable Calculus and there is no way to see how this comes in the first place but once you adopt the language of Manifolds, this is emergent, it comes very naturally. It will be a pinnacle in our upcoming discussions where we will discuss about 1-forms which will be actually dual to this differential operators and they are nothing but dx, dy, dz so this will be it for today and I'll see you guys on the next lecture, thank you.

Differential 1-forms

In this lecture we are going to finally define differential 1-forms. The definition for a differential 1-form would be the following: it says that a differential 1-form let's call it α on a Manifold $\mathbb{R}^n$ is a linear functional on the set of the Tangent vectors to the Manifold $\mathbb{R}^n$ on the set of the Tangent vectors on the Manifold $\mathbb{R}^n$ this means that on each point let us say P of Manifold $\mathbb{R}^n$, $\alpha : T_P(\mathbb{R}^n) \rightarrow \mathbb{R}$

This is the definition of a differential 1-form that we are denoting with α and we will see what α looks like actually in a few minutes but in addition to this definition α will also satisfy the following:

Is a linear property. let's say you have two elements v_P and w_P at some point P of the Tangent space so $\alpha(v_P + w_P) = \alpha v_P + \alpha w_P$ and then the second property will be the following that is similar to a scalar multiplication that is $\alpha(a v_p) = a \alpha(v_p)$ where $a \in \mathbb{R}$ and $v_p, w_P \in T_P M$, this is the definition. Now if you recall from our previous lectures that whenever we talk about a vector space we also have to discuss about its dual vector space as well so let me give a very brief review over what a Dual space is.

A dual vector space is something that also contains vectors as well but when you take an element from the Dual space and you act those vectors on the vectors that are on the Tangent space of it you should get a real number, this is how Dual space is defined that means let's say you have a Tangent space denoted by $T_P M$, this is our Tangent space then $T_P^* M$ is the Dual space of $T_P M$. A very simple example or a heuristic example will be something like this so let's say you have a vector $v_1 = (1, 2, 3)^T$ written as a column vector so this thing belongs to $T_P M$. What is $v^1 \in T_P^* M$?

Since we are only dealing with real numbers we won't have to go into the complications of doing complex conjugates but in general v subscript one will be written as a transposition and then complex conjugation of v^1 itself and this is denoted with a dagger: $v^1 = (v_1)^\dagger = (v_1^*)^T$ so this dagger means that all the elements of v_1 will first get complex conjugated and then you have to make a transpose of it. Since we are not dealing with complex numbers you can, for now, ignore this star symbol, what this star symbol means and since $v_1 = (1, 2, 3)^T$ then $v^1 = (1, 2, 3)$. This is a very simple example of a Dual space and if you now multiply this thing you should get $v_1 \cdot v^1 = 1^2 + 2^2 + 3^2 = 14 \in \mathbb{R}$.

Now from our previous lectures we saw that the Cartesian basis vectors were equal to this differential operators.

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\} = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\} \tag{20}$$

Since the left hand side are elements of vector spaces then the right hand side are also elements of vector spaces so how do you find the elements of these vectors onto the Dual space and note that the Dual space is sometimes also referred to as a Cotangent space. This is another remark Dual spaces are also known as Cotangent spaces and all this subscript and superscript things are related with some covariant and contravariant stuff but I won't go into that because that is a part of tensor analysis which is not our topic here so we'll stick to the notion of Dual spaces and Tangent spaces or Cotangent spaces

Now our goal is to find the dual elements, that means the dual basis elements of these operators (20) and we write the dual elements or dual version of this operators as following, this is the dual version:

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\} \Rightarrow \{d x_1, d x_2, d x_3\} \tag{21}$$

You might say these $d x_i$ are something you learn as differential in a Calculus course or this is the thing that you have seen in your first four weeks and we'll get to that that we are calling this thing a form, we will call it a 1-form but also we are calling it as a differential and how these two ideas relate we'll get into that. For the partial derivative with respect to x_1 coordinate it will have a dual element as $d x_1$ such that $d x_1$ when acting upon this operator $d x_1(\partial / \partial x_1)$ will give you 1 . However if you act $d x_1(\partial / \partial x_2)$ you should get a 0 so in a more compact notation you can write this multiplication that we defined at the very beginning of the lecture that in the definition of 1-form is that:

$$d x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij} \text{ where } \delta_{ij} \text{ is the Kronecker delta} \tag{22}$$

What does the Kronecker delta mean? Is just an identity matrix:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{23}$$

Finally we see that our differential 1-forms for a 3D Cartesian coordinate or a 3D Cartesian Vector space will be the following set:

$$\{d x_1, d x_2, d x_3\} \tag{24}$$

Another hand-waving way of knowing that this is indeed a form is to show is to count that how many d are there, that means you have some variables or coordinates and how many d are there so it has only one d so people call it a 1-form or a differential 1-form but there's just a hand-waving way of remembering things not the correct rigorous definition, keep that in mind. Now let's try to see an example that what are we going to do with this Differential forms that is what these Differential forms actually do so let's first take a very simple example and what we want to compute is the following:

$$v = \left(2\frac{\partial}{\partial x_1} + 3\frac{\partial}{\partial x_1}\right) \tag{25}$$

I want you to compute what is $d x_1(v)$. This is pretty easy, you just multiply them as we have seen before you take the 2 out according to the second property that was introduced in the definition and you have this $2+3=5$. If you look closely you will actually understand the fact that these differential 1-forms are just measuring the projections of these vectors on the Cotangent space axes, that's one way to think of it or you can also say in another way that this Differential forms are actually giving you the projection along the axes of this Tangent space in fact that is more correct way to think of it. Similarly you can also try to compute another quantity let's say this is our example 2:

$$\alpha = 2\, d x - 3\, d y + 5\, d z \tag{26}$$

I want you to compute the quantity:

$$\alpha\left(-\frac{\partial}{\partial x}+3\frac{\partial}{\partial y}-4\frac{\partial}{\partial z}\right) \tag{27}$$

If you understand that $d\,x$ will only produce a nonzero thing when it is acted upon $\partial/\partial x$ and all the other elements will be zero, this is just a one line computation just like the dot product that you see in your ordinary Vector analysis this is just $-2-9-20=-31$. Why are we even bothering with these things? Well because first of all we want to know what is the geometrical interpretation of a total differential, I mean we have defined 1-form and from there we can also go to 2-form, 3-forms and so on and we will define something new type of product that is called a wage product that will actually help us in achieving our goal of understanding how volumes areas these are all anti-symmetric products and all that but we'll come to that a bit later first we want to use this one from tool to gain some better understanding of what we have learned in our Calculus course so far.

I want to give you a slight review on the total differential because most often in Calculus courses we say differential there is a differential let's say $d\,x\,d\,y\,d\,z$ but we don't actually give a geometrical interpretation of what the differential measures and that's a very important thing to do and for that we need this 1-form language. What is the total differential? If you have a multi variable function $\mathrm{f}(x,y,z)$ then the total differential is just:

$$d\,\mathrm{f}=\frac{\partial \mathrm{f}}{\partial x}d\,x+\frac{\partial \mathrm{f}}{\partial y}d\,y+\frac{\partial \mathrm{f}}{\partial z}d\,z \tag{28}$$

Remember from Calculus point of view $d\,x$ is just a differential x , some infinitesimal change in stuff so this is given by this thing. Now what about the one from language? How does 1-form defines a differential, because we want to see what this one from language is actually good for. Now we define the differential or the total differential in terms of 1-form language so this is on the Calculus language Now we want to see the definition in the form language so the definition for a total differential or a differential if you want to call it will be the following: let there be some function $\mathrm{f}:\mathbb{R}^n\rightarrow\mathbb{R}$ on the Manifold then the differential $d\,\mathrm{f}$ of the function f is defined as the 1-form on the Manifold $\mathbb{R}^n$ such that for all vectors $v_P\in T_P M$ then the following is true:

$$d\,\mathrm{f}(v_P)=v_P[\mathrm{f}] \tag{29}$$

This means that $d\,\mathrm{f}$ acting upon the vector v_P on a Tangent space will give us the same output of this guy over here which is a linear functional that means you are taking a function you are doing some vector operation on it with v_p and you are getting a number and Differential form also does the same thing it takes a Tangent vector the 1-from itself is a member of the Cotangent space or the Dual space and it gives you a real number so this is what is actually happening over here and if you recall from the past lectures that this was nothing but the Directional derivative so this guy was just the Directional derivative on that Manifold or on that point of that Manifold so this was written as (13). In our past lecture we saw that this was nothing but (16) where $v_P=(v_1,v_2,v_3)^T$.

Before we proceed we want to see another thing that means we want to see how we can start from here $v_P[\mathrm{f}]$, the Directional derivative and we can end up over here $d\,\mathrm{f}(v_P)$. For that we want to write:

$$v_P = v_1 \boldsymbol{e}_1 + v_2 \boldsymbol{e}_2 + v_2 \boldsymbol{e}_2 \tag{30}$$

Where $\boldsymbol{e}_i$ are the basis vectors on that Cartesian coordinate but recall that $\boldsymbol{e}_i = \partial/\partial x_i$. We can write:

$$v_P = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_2 \frac{\partial}{\partial x_3} \tag{31}$$

Now we use this, that means, we use the fact that $\boldsymbol{e}_i = \partial/\partial x_i$ and we also use (22) and $d\,x_1(v_P) = v_1$ because it's measuring the projection along the $\boldsymbol{e}_1$ axis that's what differential 1-from does to a vector from a Tangent space. Similarly $d\,x_2(v_P) = v_2$ and $d\,x_3(v_P) = v_3$ so instead of this v_1, v_2, v_3, I can now substitute this fellas so let's try to do that, we can write:

$$d\,\mathrm{f}(v_P) = \frac{\partial \mathrm{f}}{\partial x_1}\, d\,x_1(v_P) + \frac{\partial \mathrm{f}}{\partial x_2}\, d\,x_2(v_P) + \frac{\partial \mathrm{f}}{\partial x_3}\, d\,x_3(v_P) \tag{32}$$

Please don't confuse this with ordinary multiplication even though the notation might be a little bit confusing this doesn't mean they are ordinarily multiplying with each other they are like operators and they are acting on those vectors. Since these are partial derivatives it will give you some type of function or if you want to evaluate it at a point it will give you some number and the only operator type of thing is these Differential forms and the operand that means the guy who's getting operated are these vectors so in that sense you can write this as:

$$d\,\mathrm{f}(v_P) = \left(\frac{\partial \mathrm{f}}{\partial x_1}\, d\,x_1 + \frac{\partial \mathrm{f}}{\partial x_2}\, d\,x_2 + \frac{\partial \mathrm{f}}{\partial x_3}\, d\,x_3 \right)(v_P) \tag{33}$$

Comparing the left hand side with the right hand side should convince you that one can conclude that the differential is given by this 1-form, again this differential is just a 1-form:

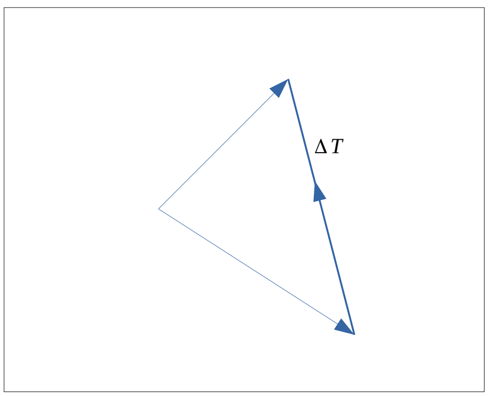
$$d\,\mathrm{f} = \frac{\partial \mathrm{f}}{\partial x_1}\, d\,x_1 + \frac{\partial \mathrm{f}}{\partial x_2}\, d\,x_2 + \frac{\partial \mathrm{f}}{\partial x_3}\, d\,x_3 \tag{34}$$

Now the question is what does this Differential form actually mean? What does it mean geometrically? Well let's try to look at the definition again, if you see this definition (29) you can see that on the left hand side we are talking about this differential 1-form acting on some vector but on the right hand side this is equivalent to a Directional derivative so from there we can conclude that this Differential form on a Manifold and I want to see how this function changes because on Calculus we say that how this function changes on multiple axes, let's say this is our coordinate axes x_1, x_2, x_3 and we want to measure the change in some direction because on the right hand side we had a Directional derivative definition so at this point you can draw a Tangent space and what this Differential form tells you is how that direction you want to move, that means the choice of the direction vector, this is the vector along which you want to measure the rate of change this gives out a number this Differential form, it actually gives you a number that is the rise of this plane, the Tangent plane, to the graph of the function as one moves from this initial starting point to the end point.

This Differential form $d\mathrm{f}$ actually tells you or gives you a number that measures the rise of the Tangent space along the direction of the vector t or so to say the vector chosen. This is the geometrical meaning of this total differential. There is another way to see it as well. What you can do is that you can call this point $S=\mathrm{f}(x_0, y_0)$. You can call this point as the point where you have this thing and then you can have some derivatives that actually gives you the projections along the lines of x and y or should I say x_1 and y_1 so instead of x_0, y_0 you could write, let's say x_1, x_2 and you could measure this slopes and finally what you want to do is that you want to see the difference of the quantity that will be given by:

$$\Delta T = T\left(x_0^1 + a, x_0^2 + b\right) - T\left(x_0^1 + x_0^2\right) \tag{35}$$

That means you are just ultimately measuring the difference between two Tangent vectors and you can think of this as, at this point, that is the point E (the tip of the vector) you can draw another Tangent vector then you can translate it back and a 2D picture would look something like this:



What you are doing is that you are only measuring this difference call it by ΔT and denote it with this (35). This is a more vector type or draw based approached but in differential 1-form language these things are immanent. To sum up what we have learned is that this differential 1-forms are something that comes up with one d that is related with the infinite decimal changes as we saw in the total differential discussion and what it does is that it takes something from the Tangent space and it spits out a number.

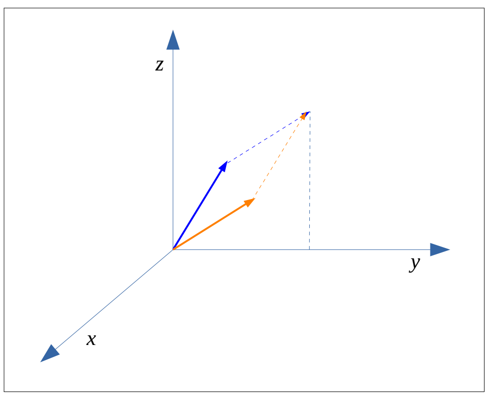
In the next in the next lectures we are going to see more Differential forms like 2-forms and 3-forms and we will define a new type of product that is called a Wedge product and from there we will see how the Jacobian arrives very naturally when we do a coordinate transformation of area or the volume and how areas and volume they will even have signs that are pretty much neglected in any Calculus courses but when we use this Differential form we have to take them under consideration and all that. This will be it for today and I'll see you guys on the next lecture thank you.

Wedge Product and Differential n-forms

In this lecture we are going to talk about Wedge product and Differential forms. In our previous lectures we saw that a differential 1-form is such an object that it takes a vector in and spits out a number in other words if you want to see geometrically what it does is that it takes a vector and it finds out the projection of the different basis along some certain basis what is the projection of that vector. If we recall that that means what 1-forms did is that 1-forms or differential 1-forms found projections of vectors along a basis or an axis whatever you want to call it.

What we want to do right now is that we want to define a differential 2-form. How this 2-form will look like? I will come to that later but from the fact that a 1-from find out projections gives us an idea that if you consider a vector let's say in a 3D space and you take the projections onto the different axis so this 1-forms take this vector and finds out this projections and if you just look at one of these axis that means that the projections are equal to some certain length and this length can be thought of as a 1D volume so this projections can be thought of as one deep volume because you can only move left or right so the only dimension you get is the length and you can think of it as a one-dimensional volume.

What we want to do right now is that we want the differential 2-forms to show some certain property like differential 1-forms that means what it will do is that it will give us some information about a 2D volume of some pair of vectors. Why I'm using a pair of vectors? Because if I use only one vector I cannot generate a 2D volume that means an area I have to take at least two vectors and I can even do it with more vectors but then I will have to just slice I'll have to take a slice of that higher dimensional volume to get a 2D volume so we want the differential 2-forms to show some volume and invariant properties this will be a 2D volume invariant properties just like 1-forms.



With that in mind we write our first differential 2-form as follows. Let's say you have a vector like this and another vector like this and obviously you can think that there is a parallel of these vectors will span a parallelpiped and let's say this is your x, this is your y, this is your z and if you want to show or if you want to find out the projection of this area along the xy plane you can do that with the differential 2-form because here you have to find out the projection onto the x and y axes you also have to multiply them in the Euclidean geometric sense.

This differential 2-form is written as let's say $dx \wedge dy$ and there is this symbol $\wedge$ which represents a Wedge product of 1-forms. You might ask what is this Wedge product? You can think of it as a generalization of the cross product that you see in your 3D space. Cross product doesn't exist for higher dimensional spaces, however, the cross product itself is a special case of this Wedge product and how do you work with Wedge product? Since we will be considering ourselves with areas and volumes and we already know the determinant and cross products are related with the area and the volume because if you take a cross product at the end of the day you have to take some determinant to get the scalar volume. This motivates us to define this differential 2-form as follows so this differential 2-form will act on this pair of vectors, if they act on this pair of vectors it will give you:

$$d x \wedge d y(v, w) = \begin{vmatrix} d x(v) & d x(w) \\ d y(v) & d y(w) \end{vmatrix} \tag{36}$$

If you consider v and w to be 3D vectors let's say $v=(v_1, v_2, v_3)^T$ and $w=(w_1, w_2, w_3)^T$ you can immediately say that the element of the matrix in (36) are just one 1-forms that you saw in your previous lectures and immediately you can write that this is just:

$$d x \wedge d y(v, w) = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = v_1 w_2 - v_2 w_1 \tag{37}$$

This is how you deal with a differential 2-form and one thing to note is that this differential 2-form shows anti-symmetry that means that:

$$d x \wedge d y = - d y \wedge d x \tag{38}$$

This is an anti-symmetric property and this anti-symmetry has to do with the idea of the sign of the area or the volume because if you consider a surface like this you can always define a positive because the surface has two directions if you take this as a surface there is one opening and there is another opening so if you consider one of them to be positive the other is negative. The Wedge product itself is defined as an anti-symmetric product. What about differential 3-forms? Similarly just like differential 2-forms you can define differential 3-forms as follows, let's say instead of x, y, z we turn our notations into x_1, x_2, x_3 , it will be something like:

$$d x_1 \wedge d x_2 \wedge d x_3(x, y, z) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \tag{39}$$

Which will act on three vectors x, y, z . You will have some determinant. What about differential 4-forms, 5-forms, 6-forms or let's say differential n-forms how do you write those things? Just like this and now comes the interpretation of this Wedge product. Just like differential 1-forms let's say you take $d x$ and what it does it takes the vector and it measures the projection along this basis:

$$d x \xrightarrow{v} \frac{\partial}{\partial x} \tag{40}$$

Similarly differential 2-forms measure the projection of this volume or area if you consider 2D you can call it an area but for the general case we are regarding this as a volume so what this differential 2-form does is that it takes some pairs of vectors and when it does it's already considering the space or the object that is spanned by those vectors and it finds out the projection of the volume along these basis that means if it's $d x \wedge d y$, if you think of it as the Cotangent space then the component of $d x \wedge d y$ will be your projection of the volume in the Tangent space because this $d x \wedge d y$ are living in the Cotangent space so it's a bit hard to visualize because usually we visualize the bases as a line, a 1D thing but here what is happening is that this $d x \wedge d y$ a 2D thing it becomes a basis.

This is the geometrical interpretation of these differential 2-forms, now we want to move 2 differential n-forms, we just want to see how to write them and obviously before we move on to that we want to state another thing that let's say you want to have some projections over differential 3-form that means you are taking three pairs of vectors and you want to find out those projections and at the end of the day you want to express that projections into a compact way using these differential 2 forms. Any differential 2-form can be written as follows so this differential to form let's say:

$$a_{12}\, d x_1 \wedge d x_2 + a_{23}\, d x_2 \wedge d x_3 + a_{31}\, d x_3 \wedge d x_1 \tag{41}$$

You might ask what is happening with the other bases like $d x_2 \wedge d x_1$, $d x_3 \wedge d x_1$ or $d x_2 \wedge d x_2$. It's a very easy exercise to show that those things with the anti-symmetric property can be absorbed into these coefficients, it's as simple as that so that's why we try to maintain a cyclic order so that we don't have to come up with a minus sign in between and since $d x_2 \wedge d x_1 = -d x_1 \wedge d x_2$, $d x_3 \wedge d x_1 = -d x_1 \wedge d x_3$ and $d x_2 \wedge d x_2 = 0$ we want to observe and simplify the coefficients as much as possible so this is one way to see it and if you are working with let's say n-forms that's the thing we wanted to see if you are working with differential n-forms any n-form can be written in a general n-form can be written as the following summation of terms:

$$a_{12\cdots n}\, d x_1 \wedge d x_2 \wedge \cdots \wedge d x_n + \text{cyclic terms} \tag{42}$$

This is how we can write down any differential n-form as well but for now we will mostly focus on differential 2-forms and differential 3-forms so let me get to another page. These differential n-forms will also give you some determinant. Let's take an easy example to see whether we have actually understood the idea behind these Differential forms or not so let's consider an example where I have two vectors given by $v=(1,2,3)^T$ and $w=(4,5,6)^T$ and we want to find out what is $d x_1 \wedge d x_1 (v, w)$ If we recall the discussion we just had a few minutes ago this is just:

$$d x_1 \wedge d x_2 (v, w) = \begin{vmatrix} d x_1(v) & d x_1(w) \\ d x_2(v) & d x_2(w) \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = -3 \tag{43}$$

What if I tell you to calculate $d x_1 \wedge d x_1$? Would you go with this method? Obviously you can and you will see that that result is 0 but there is a very straightforward way of seeing that recall from the anti-symmetric property that $d x_1 \wedge d x_2 = -d x_2 \wedge d x_1$. Similarly using this property I can write $d x_1 \wedge d x_1 = -d x_1 \wedge d x_1$ and these are the same thing and they are equal and opposite to each other so that means it must be 0. This is another useful property that will help us a lot in the future and similarly you can do things for differential 3-forms and all that.

Since we are almost done we want to show some properties of differential n-forms some algebraic properties we will not go into the proofs but we'll just state them here for the completeness of this lecture. Algebraic properties properties of Differential forms and this will hold for any Differential form that means 1-from, 2-forms, n-forms whatever. Let's say $\omega, \omega_1, \omega_2$, k-forms and η, η_1, η_2 era l-forms and $a \in \mathbb{R}$ then the following properties result, the first one is:

$$(a\omega)\wedge\eta=a(\omega\wedge\eta)=\omega\wedge(a\eta) \tag{44}$$

The second one is a linear property:

$$(\omega_1+\omega_2)\wedge\eta=\omega_1\wedge\eta+\omega_2\wedge\eta \tag{45}$$

The third property is that you have:

$$\omega\wedge(\eta_1+\eta_2)=\omega\wedge\eta_1+\omega\wedge\eta_2 \tag{46}$$

The fourth one is that:

$$\omega\wedge\eta=(-1)^{kl}\eta\wedge\omega \tag{47}$$

This is a very interesting one, all of these are but this 4th property, the proof of this one is a little bit interesting but we are not going to prove it this will help us in the long run when we are dealing on algebra that involves manipulations of different Differential forms. We will try to see that whether we can actually make use of these properties and in fact they will help us in the in computing different quantities that comes with Differential forms. Now what we want to do is that let's say you are taking let's say two Differential forms one is α a k-form another is β an l-form.

$$\alpha=\sum_I a_I\, dx^I \text{ where } dx^I=dx^1\wedge dx^2\wedge\cdots\wedge dx^k \tag{48}$$

Then the coefficients are summed over with this differential k-forms so this is one way to simplifying your notation:

$$\beta=\sum_J b_J\, dx^J \text{ where } dx^J=dx^1\wedge dx^2\wedge\cdots\wedge dx^l \tag{49}$$

$$\alpha\wedge\beta=\sum_{IJ} a_I b_J\, dx^I\wedge dx^J \tag{50}$$

The reason we're calling it simplifying because all of these dx^I and dx^J will contain actually k and l forms respectively that means there are more dx inside those things. The last topic for today is the interior product of our Differential forms. Before we go into that we can try to talk about the general formula of the Wedge products but we won't go into the proof because there is another related product that is called a tensor product and from that point of view the proof becomes easier so I'll just write this formula for now that the general formula for Wedge product for different forms:

$$(\alpha\wedge\beta)(v_1,v_2,\ldots,v_{k+l})=\frac{1}{k!l!}\sum_{\sigma\in k+l}\text{sign}(\sigma)\alpha(v_1,v_2,\ldots,v_{k+l})\beta(v_{\sigma(k+1)},v_{\sigma(k+2)},\ldots,v_{\sigma(k+l)}) \tag{51}$$

That actually comes from the totally anti-symmetric tensor ε , I'm not using this though this is the most straightforward thing because we are not discussing tensor so that's why I'm not going to use it and the σ will belongs to $k+l$ for different case in l you have to see what is the sign then you have this α that is the Differential form which will act upon these vectors. This formula is totally optional as you can see it gets very messy however if you can use this Levi-Civita symbol it becomes pretty straightforward and now to move on to the interior product this is the last thing for today.

The interior product or the inner product of Differential forms you can already guess from the name that it's kind of similar to the dot product that because we have been saying this interior product is kind of similar to a dot product where you know you take some stuff you do something and you get some number right but it turns out that when you take different interior products of Differential form it's not necessarily the same thing so that's why I would like to call it the interior product so that we don't get confused with the inner product of the things that we see in the vector space.

Let's say we have a k-form α so let's say α and sometimes we can even denote a k-form like this $\alpha \in \Lambda^k(\mathbb{R}^n)$, that would mean this is a k-form on $\mathbb{R}^n$. This is another way to way to express it. Given a vector v so the interior product is denoted with this ι_v according to the textbook we are following in fact you can choose any other notation you want so this is the symbol for denoting this interior product and this interior product is defined as:

$$\iota_v \alpha(v_1, v_2, \ldots, v_{k-l}) = \alpha(v, v_1, v_2, \ldots, v_{k-l}) \tag{52}$$

This is the interior product between a vector v and a Differential form α and this $v_1, v_2, \ldots, v_{k-1}$ are just $k-1$ vector and what this interior product does is it just puts the vector v into α first slot. What does that mean, because we are considering the internal product between this α and this v and we are only interested in a product between in between them. Notice what happens in this definition. When we are saying that I want to internal product with respect to v which I denoted with ι_v then what it does is that it gets rid of this v_1 that means the dummy vector that was there and it puts that v on the first slot.

You can think of it as it just creates a new slot and puts v inside it so whatever is happen happening over here it really doesn't matter all this $v_1, v_2, \ldots$, they don't really matter and we'll see why this is so in an example. Let's take a very simple example. Let's say our Differential form is $\alpha = dx \wedge dz$ and $v = (v_1, v_2, v_3)^T$. We want to compute this quantity $\iota_v \alpha$. By the definition it should do the following it will put v and then will be w_1, w_2 like this:

$$\iota_v dx\, dz(w_1, w_2) = dx \wedge dz(v, w_1, w_2) \tag{53}$$

We should face the fact we don't have any clue about w_1 and w_2 we only know what is v so how we write this is the following: o instead of this w_1 and w_2 we just put put a dot over here that means these slots are empty we don't know about them so this is written as:

$$\iota_v dx\, dz(w_1, w_2) = dx \wedge dz(v, \cdot) \tag{54}$$

If you consider how these 2-forms were defined when this they act upon vectors they give you a determinant like this:

$$\iota_v dx\, dz(w_1, w_2) = \begin{vmatrix} dx(v) & dx(\cdot) \\ dz(v) & dz(\cdot) \end{vmatrix} \tag{55}$$

Since there is nothing to act upon these operators that means the Differential forms remain intact so $dx(\cdot)=dx$ and $dz(\cdot)=dz$ and then:

$$\iota_v dx dz(w_1, w_2)=\begin{vmatrix} v_1 & dx \\ v_3 & dz \end{vmatrix}=v_1 dz-v_3 dx \tag{56}$$

As you can see that this interior product actually turned a 2-form into a 1-form and we'll have to say something more about this type of things where you can make a 2-form into a 3-form and let's say 3-form into a 4-form or you can convert a 4-form into a 0-form for a 4D space. We'll have to say more about these things in the future lectures where we will discuss Hodge operators and exterior differentiation and stuff so this is how you compute it, pretty simple.

Similarly you can go for, if you want to take interior products of 3-forms you can similarly flow this way but before I finish off I want to again show you some algebraic properties that comes from the definition of the interior product. Consider $\alpha, \beta \in \Lambda^k(\mathbb{R}^n)$. The first one is:

$$\iota_v(\alpha+\beta)=\iota_v \alpha+\iota_v \beta \tag{57}$$

You can show this easily using the definition. Similarly if you take two vectors v and w instead of just v and you add them up:

$$\iota_{(v+w)}\alpha=\iota_v \alpha+\iota_w \alpha \tag{58}$$

There is another property that is if you want to compute the interior product to a Differential form where $\alpha \in \Lambda^k(\mathbb{R}^n)$ and $\beta \in \Lambda^q(\mathbb{R}^n)$. If you want to compute a quantity like this:

$$\iota_v(\alpha \wedge \beta)=(\iota_v \alpha)\wedge \beta+(-1)^k \alpha \wedge(\iota_v \beta) \tag{59}$$

These are some algebraic properties and a final one is a little bit interesting which kind of looks like an anti-commutator if you are familiar with Linear Algebra and if you are not just take this property and try to prove it is that:

$$(\iota_u \iota_v+\iota_v \iota_u)\alpha=0 \tag{60}$$

This thing kind of looks like an anti-commutator. This will be it for today even though you might think that these are not pretty big deals but when we try to do this Exterior derivatives and Hodge star operations you will begin to understand why we want to bother with this algebraic properties and all these stuffs that we have done in this lecture so this is it and thank you.

Exterior Derivative

In this lecture we are going to discuss about Exterior derivatives that means we want to take derivatives over forms on a Manifold. In calculus what we do is that we take some function and then we can define a derivative that will measure the rate of change of that function but when we do calculus on Manifolds We have seen that the objects that we concern ourselves with is forms whether it's a 1-form or a 2-form or a 3-form and the goal of this Exterior derivative will be to measure in some sense it will measure how this forms change so the idea is to define a similar analog of derivative for forms.

This is our goal and we'll try to see a little bit of a geometrical picture again. This picture is not entirely accurate but it gives you some idea what this Exterior derivative does. First things first let us define a 1-form so let's say we have a 1-form like this, a 1-form that is written as α which is defined:

$$\alpha=\sum_i \mathrm{f}_i d x_i \tag{61}$$

Where $d x_i$ is a 1-form and f_i can be a function of our coordinates, is a scalar, that means it's a 0-form. Now the Exterior derivative which is written as $d\alpha$ in this case because it will try to measure the rate of change of this α will be defined as follows:

$$d\alpha=\sum_i d\,\mathrm{f}_i \wedge d x_i \tag{62}$$

Since you know about how to take the total derivative of a 0-form, that means a scalar function or a vector function, you already know how to evaluate this $d\alpha$ so this will just be:

$$d\alpha=\sum_{ij} \frac{\partial \mathrm{f}_i}{\partial x_{ij}}\, d x_j \wedge d x_i \tag{63}$$

Don't mind this indices, they're just here to keep tracks of stuff. This is how an Exterior derivative is defined. How do we know that this is the way to measure some changes? That will be a long discussion and we'll try to provide a very short geometrical picture in this lecture. As you can see from here that we are explicitly introducing the coordinates however you can write the Exterior derivative in such a way that it doesn't depend on the coordinate itself so this is the coordinate dependent version. The end result would be the same we should emphasize that as well that whatever you do the end result will be same but there are just two ways of writing the Exterior derivative so this is the form where the coordinates appear explicitly. It's also known as the local formula. This is the coordinate free version and this is written as:

$$d\alpha(v,w)=v[\alpha(w)]-w[\alpha(v)]-\alpha[v,w] \tag{64}$$

The last term is α acting on the commutator of v and w . We already know how a form acts on a vector field. Now we want to discuss these things. The motivation behind this Exterior derivative is that it should take an n-form and it will return a (n+1)-form.

$$d:\Lambda^n(M)\rightarrow\Lambda^{n+1}(M) \tag{65}$$

Why because if you look at the ordinary derivatives in Calculus your function let's say $\mathrm{f}(x)$ was a 0-form, when you take the derivative of this function it gives you a 1-form like this:

$$d\,\mathrm{f} \rightarrow \frac{\partial \mathrm{f}}{\partial x} d\,x \tag{66}$$

This is where the motivation comes from this is one way to look at it and obviously you can if you are dealing with multi variable you just can put a summation symbol over here and you are good to go. Now let's say you have an n-form and you want to apply an Exterior derivative on that n-form so just like this definition over here let's say our n-form is something like this and we want to apply the External derivative operator this n-form is written as:

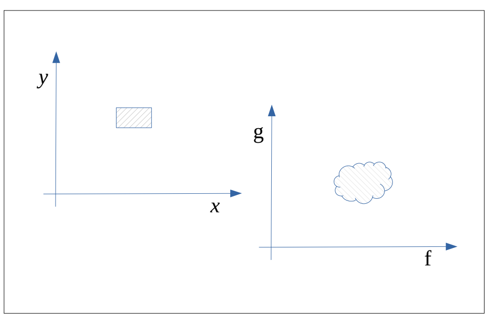
$$d\left(\sum \alpha_{i_1 i_2 \ldots i_n} d\,x_{i_1} \wedge d\,x_{i_2} \wedge \cdots \wedge d\,x_{i_n}\right) \tag{67}$$

This α is a coefficient that is coming with some indexes. Obviously there will be a Wedge product otherwise it won't be considered as an n-form. Once you apply this Exterior derivative on to this thing:

$$\frac{\partial\left(\alpha_{i_1 i_2 \ldots i_n}\right)}{\partial\,x_{i\,j}} d\,x_j \wedge d\,x_{i_1} \wedge d\,x_{i_2} \wedge \cdots \wedge d\,x_{i_n} \tag{68}$$

These are just indexes there is no chance that this is a two variable thing, it's a single variable thing but this index will actually give you what is this $i\,j$ component of the coordinate that you are considering. Now it will put another $d\,x$ and this will come as index of j in front of this whole thing and then you will have the thing that you started with initially. Let's try to see an example right now.

Let's say you have two function $\mathrm{f}(x,y)$ and $\mathrm{g}(x,y)$ and if you notice a little bit carefully what it does is that it's just a mapping from x,y there is a mapping to this f and there is a mapping to this g . Let's consider a very simple thing: What is $d\,\mathrm{f} \wedge d\,\mathrm{g}$? We learned in our previous lecture is a projection of the area, we want to compute this quantity. Let's say this is your $x\,y$ plane, and this is your $\mathrm{f}\,\mathrm{g}$ plane:



This is f and this is g so you have some infinitesimal area over here and you want to see how this area looks when you go to this $\mathrm{f}\,\mathrm{g}$ plane and let's see, it looks something like this and we want to see how they are related so that's why we want to carry out this computation. What is $d\,\mathrm{f}$? This is easy since this is a two variable functions and we are computing a 2-form this is a 2-form thing $d\,\mathrm{f}$ is nothing but, by the definition this is:

$$d\,\mathrm{f} = \frac{\partial \mathrm{f}}{\partial x} d\,x + \frac{\partial \mathrm{f}}{\partial y} d\,y \tag{69}$$

Now you can consider this:

$$d\,\mathrm{g}=\frac{\partial \mathrm{g}}{\partial x}\,d\,x+\frac{\partial \mathrm{g}}{\partial y}\,d\,y \tag{70}$$

Then you can easily compute the Wedge product:

$$d\,\mathrm{f}\wedge d\,\mathrm{g}=\left(\frac{\partial \mathrm{f}}{\partial x}\,d\,x+\frac{\partial \mathrm{f}}{\partial y}\,d\,y\right)\wedge\left(\frac{\partial \mathrm{g}}{\partial x}\,d\,x+\frac{\partial \mathrm{g}}{\partial y}\,d\,y\right) \tag{71}$$

Notice that even though we are considering a Wedge product you can see from the fact that this f,g are 0-forms so we are taking 1-forms using this Exterior derivative over here which you already know as a total derivative thing so if you compute it this is nothing but, now I'm going to show a new notation which will just make our life a little bit easier is that instead of writing this $\partial\,\mathrm{f}/\partial\,x$ we can write it as $\partial_x \mathrm{f}$. The index will indicate with respect to which variable we are taking the derivative or the partial derivative so if I write in that way this becomes very simple to write:

$$\begin{aligned} d\,\mathrm{f}\wedge d\,\mathrm{g}&=\partial_x \mathrm{f}\,\partial_x \mathrm{g}\,\cancel{d\,x\wedge d\,x}+\partial_x \mathrm{f}\,\partial_y \mathrm{g}\,d\,x\wedge d\,y\\ &\quad+\partial_y \mathrm{f}\,\partial_x \mathrm{g}\,d\,y\wedge d\,x+\partial_y \mathrm{f}\,\partial_y \mathrm{g}\,\cancel{d\,y\wedge d\,y}\\ &=\left(\partial_x \mathrm{f}\,\partial_y \mathrm{g}-\partial_y \mathrm{f}\,\partial_x \mathrm{g}\right)d\,x\wedge d\,y\\ &=\begin{vmatrix}\partial_x \mathrm{f} & \partial_y \mathrm{f}\\ \partial_x \mathrm{g} & \partial_y \mathrm{g}\end{vmatrix}d\,x\wedge d\,y \end{aligned} \tag{72}$$

$d\,\mathrm{f}\wedge d\,\mathrm{g}$ is the so called area in $\mathrm{f}\,\mathrm{g}$ plane. $d\,x\wedge d\,y$ can be thought of as the area in the $x\,y$ plane and the determinant is our scaling factor which is also known as the Jacobian or the determinant of the Jacobian matrix. In multi-variable calculus these things are derived in a very tedious way but once you adapt to this notation of calculation Manifolds and Differential forms you can see that this arises very naturally so this is one example and let's consider another example.

let's say we have some function $\mathrm{f}(x,y,z)=x^3 y^2 z^4$ and we want to compute what is this guy when you take a form of it that means f is already a 0-form so you can obviously make a 1-form out of it by using $d\,\mathrm{f}$ but what we want to see is that how to do this calculations when you are actually going to compute a quantity like this $d(d\,\mathrm{f})$. That's what we want to consider because that was the motivation for introducing this Exterior derivative so that we can take derivatives of forms:

$$\begin{aligned} d\,\mathrm{f}&=\partial_x \mathrm{f}\,d\,x+\partial_y \mathrm{f}\,d\,y+\partial_z \mathrm{f}\,d\,z\\ &=3\,x^2 y^2 z^4\,d\,x+2\,x^3 y\,z^4\,d\,y+4\,x^3 y^2 z^3\,d\,z \end{aligned} \tag{73}$$

Let's say this is just $d\,\mathrm{f}=\alpha$ which is equals to some 1-form and we want to compute what is $d\,\alpha$. How do you compute $d\,\alpha$? Well, use the definition that introduced above and you have to first take partial derivatives of the coefficients and then Wedge this the 1-forms. If you recall if you had a 0-form this is how you would write the 1-form $d\,\mathrm{f}=d\,\mathrm{f}_i\wedge d\,x^i$ so using this we can write it as:

$$d\alpha = d\left(3x^2y^2z^4\right)\wedge dx + d\left(2x^3yz^4\right)\wedge dy + d\left(4x^3y^2z^3\right)\wedge dz \tag{74}$$

Now we have to evaluate three coefficients. One thing you will notice is that there is already a dx over here, a dy and a dz. You know how to do it already since this is a function which is a 0-form:

$$d\left(3x^2y^2z^4\right) = 6xy^2z^4\ dx + 6x^2yz^4\ dy + 12x^2y^2z^3\ dz \tag{75}$$

You are just calculating the total differential. There is a Wedge like this $\wedge dx$ and you already know that $dx\wedge dx = 0$ so the first term will contribute nothing and these two terms will contribute some non-zero stuff so this will yield the following results:

$$d\left(3x^2y^2z^4\right)\wedge dx = 6x^2yz^4\ dy\wedge dx + 12x^2y^2z^3\ dz\wedge dx \tag{76}$$

$$d\left(2x^3yz^4\right)\wedge dy = 6x^2yz^4\ dx\wedge dy + 8x^3yz^3\ dz\wedge dy \tag{77}$$

$$d\left(4x^3y^2z^3\right)\wedge dz = 12x^2y^2z^3\ dx\wedge dz + 8x^3yz^3\ dy\wedge dz \tag{78}$$

Now according to this line (75) = (76) + (77) + (78) will give me the $d\alpha$ thing. Let's try to collect all the terms and simplify directly so first we want to write the terms with $dx\wedge dy$. If you write $dy\wedge dx$ then you have to switch the order over here which will give you a minus sign so if you see carefully that these two terms are equal to each other and when you switch this you are getting a minus sign so in total you are actually getting a 0. Does this happen for the other terms as well? Let's see what about $dy\wedge dz$? There are two terms, there are no other Wedge product that involves dy and dz and again these two terms are similar, if you switch this order over here or another order over here one of the coefficients will pick up a minus sign and these terms will actually cancel each other out. Similarly for these two terms that contain $dx\wedge dz$.

In total what you are getting is a 0 so what is happening over here? We are calculating this $d(d\mathrm{f})$ and we are getting a 0. This is a very famous property that Exterior derivatives show is that for any n-form γ then $d(d\gamma)=0$ always. This is sometimes also written as $d^2=0$ as an operator: it takes something and it puts it 0 and it has a famous interpretation that the boundary of a boundary is 0, this might not make sense right now but we'll get to that later. This will be more clear when we do integration with forms then we will realize why these are called boundary of a boundary terms.

Now since we have seen some examples we now want to write down a few algebraic properties of this Exterior derivatives and after this I'll try to show you a very simple geometrical picture and then we'll be done with discussing this Exterior derivative stuff. Consider $\alpha,\beta\in\Lambda^n(M)$ and $\omega\in\Lambda^m(M)$

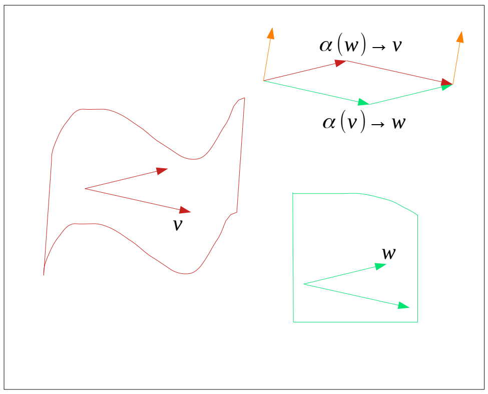
$$d(\alpha+\beta) = d\alpha + d\beta \tag{79}$$

$$d(\alpha\wedge\omega) = d\alpha\wedge\omega + (-1)^n\alpha\wedge d\omega \tag{80}$$

The third one is the boundary of a boundary thing that means:

$$d(d\alpha)=0 \tag{81}$$

In fact you can think these three things as axioms as well. If you take these axioms then these axioms will lead you down to this definition of an Exterior product where you have this d that takes some n-form and returns an (n+1)-form like this $d:\Lambda^n \to \Lambda^{n+1}$. This might seem a bit circular but that's how things are and now we want to show you a very simple geometrical picture of this Exterior derivatives so if you recall that we said that there is a coordinate free way that means a coordinate independent way of writing the Differential forms and that was written as follows is that if you recall that we wrote it a bit while ago so let me get back to that this one this is the coordinate free way of writing an Exterior derivative (64). What does it mean?



It means that when you have a Manifold and you have some Differential form α and you have two vectors given by this v and w you want to measure the change in α so what you do is that you introduce two vectors and let's say this is the Manifold and this is the first case and let's say this is the Manifold again and this is the second case. In the first case what happens is that you can take α and you want to get a weighted sum along the direction or you want to see what happens to this α when you take this v vector and make a weighted sum of it.

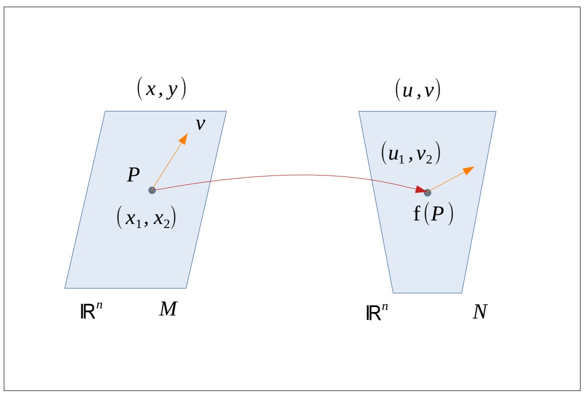
Obviously v can be a constant or a non-constant vector and then what you do is that you move in the w direction and see how things change and after that you do the converse that means now you take your α and make a weighted sum with the w and then you want to move in the v direction and see how the form changes because there is no direct way of measuring a change in the form so what you do you associate some vector fields with the form and then you try to measure the changes so if this gives you Tangent plane like this on the Manifold and this gives you another Tangent plane, from a geometric point of view what you are doing is that you are actually going towards this way first and then this way so if this is your $\alpha(v)\to w$. The other like this would be say $\alpha(w)\to v$.

The orange vector is your test vector and you are going towards this way, this is one way and this is another way and you want to finally measure the difference so this is one way to see this geometrical picture of the Exterior derivative but it will become clearer when we see the integration of Differential forms and we will also see that this idea of this Exterior derivative is extremely powerful because you can write all of Vector calculus all of multi variable calculus within a few lines. Theorems like curl, divergence and Stoke theorem can be written in a very compact way and they can be interpreted as such with this Exterior derivatives. This will be it for today and I'll see you guys on the next lecture thank you.

Push forward of vectors

In this lecture we are going to first discuss the Push forward of vectors from one Manifold to another so let's say you have a Manifold like this let's call this Manifold M and you have another Manifold called N and for the time being let us assume that both of them have the same dimension, that means if this is $\mathbb{R}^n$ this is also $\mathbb{R}^n$ so now what you can do is that you can equip this Manifolds both of them with some coordinate system so let's say we have the Cartesian coordinate system on the Manifold M and we have some other coordinate system denoted with u and v on Manifold N and there is some function that takes this x and y to this u and v : $\mathrm{f}(x,y) \rightarrow (u,v)$.

What happens is that points on Manifold M gets mapped to points on the Manifold N so let's say you have some point over here that is denoted by (x_1, y_1) it gets mapped to some point on Manifold N as (u_1, v_1) like this. Now this seems pretty simple that is just a mapping between two Manifolds but what we want to do is that we want to see what happens when I change my points from (x_1, y_1) to some other value let's say (x_1+a, y_1+b) and this can be thought of as a fact that you are moving along a vector that is given by $v=(a,b)^T$ which has two components a and b like this.



How does that change on the picture that we have over N ? This is where the concept of the push forward will come. Now let's define this Push forward of vectors formally so let's say we have some function that takes points from the (x, y) Manifold, a 2D Manifold to some other Manifold which is also 2D Manifold:

$$\mathrm{f}:\mathbb{R}^2_{xy} \rightarrow \mathbb{R}^2_{uv} \tag{82}$$

Each point gets mapped with this function $P \rightarrow \mathrm{f}(P)$ then because of this mapping we can get an induced mapping on the Tangent bundle as well, and it will be clear in a moment why we are getting an induced mapping on the Tangent bundle. This is given by:

$$T_P\mathrm{f}: T_P\mathbb{R}^2_{xy} \rightarrow T_{\mathrm{f}(P)}\mathbb{R}^2_{uv} \tag{83}$$

All the points are getting mapped under the function where the vectors at some point on the Manifold M is getting mapped with the following:

$$v_P \rightarrow T_P\mathrm{f}.v_P \tag{84}$$

This is actually the Push forward mapping which is an induced mapping that we initially defined, it's coming from the induced mapping itself so how does this come well think of this as the following: on Manifold M at point P which is given by this (x_1, y_1) you can think of there as a Tangent space and on that Tangent space you can have some vector so what will happen is this Push forward mapping is that this vector will get mapped into another vector into this point which is denoted by $\mathrm{f}(P)$ this is the mapped vector or so to say the pushed forward vector.

If you want to determine how a vector like v gets mapped into a Push forward vector first we have to consider how this function itself is taking the points from M and mapping it to N so now first we want to consider that let's say there is some vector $v=(a,b)^T$ with components a and b and let us say our function is such that:

$$\mathrm{f}=(x+y, x-y) \text{ where } \begin{cases} u=x+y \\ v=x-y \end{cases} \tag{85}$$

What it does is it takes some point on the x and y Manifold or the Manifold M and it maps those x and y into $x+y$ and $x-y$ respectively. When we take Manifold M and we go from x, y to a point let's say $(x+a, y+b)$ how do we do that? We can think of it as a Directional derivative, this change, why? Because when you have these two points you can have a Tangent space at any of those points (x, y) and you can think of a vector $v=(a,b)^T$ and you can move along that vector on Manifold M and you want to see what is the consequence when we consider the mapping from M to N so that's why we can think of it as a Directional derivative so since we already know that the Directional derivative is given by this guy:

$$D\mathrm{f}.v=\frac{\partial \mathrm{f}}{\partial x}a+\frac{\partial \mathrm{f}}{\partial y}b \tag{86}$$

Note the fact that it happens for both the $x+y$ mapping which let's say you can denote with $\mathrm{f}_1(x,y)$ and $\mathrm{f}_2(x,y)$. You have to consider the changes that are coming from f_1 and f_2 .

$$D\mathrm{f}_1.v=\frac{\partial \mathrm{f}_1}{\partial x}a+\frac{\partial \mathrm{f}_1}{\partial y}b \text{ and } D\mathrm{f}_2.v=\frac{\partial \mathrm{f}_2}{\partial x}a+\frac{\partial \mathrm{f}_2}{\partial y}b \tag{87}$$

You can write this whole thing in a compact form because this f_1 and f_2 they are acting like the components of the total mapping f and we want to find out how f itself is changing so that's why we have to consider how f_1 and f_2 is changing so if you encode them together this will look like this:

$$D\mathrm{f}.v=\begin{pmatrix} \frac{\partial \mathrm{f}_1}{\partial x} & \frac{\partial \mathrm{f}_1}{\partial y} \\ \frac{\partial \mathrm{f}_2}{\partial x} & \frac{\partial \mathrm{f}_2}{\partial y} \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} \tag{88}$$

Recall $v=(a,b)^T$ and from there you can see that this is just the Jacobian matrix so you can say that the change of the mapping itself is given by the Jacobian matrix:

$$D\mathrm{f}=\begin{pmatrix} \frac{\partial \mathrm{f}_1}{\partial x} & \frac{\partial \mathrm{f}_1}{\partial y} \\ \frac{\partial \mathrm{f}_2}{\partial x} & \frac{\partial \mathrm{f}_2}{\partial y} \end{pmatrix} \tag{89}$$

In a sense the Jacobian matrix encodes the information about how the mapping if contains information about the changes in the Tangent space or Tangent bundle if you want to consider every point so this will also be denoted as sometimes called a $T\mathrm{f}$, this in fact this will be the notation that we will be carrying throughout the lectures and now you can take any arbitrary vector on Manifold M so let's say you take any arbitrary vector on Manifold M let's call that vector $v_1=(a_1, b_1)^T$ then the Push forward vector on Manifold N will be given by $T\mathrm{f}.v_1$ that means that you can find out the new vector let's call that w_1 if you just multiply this:

$$w_1=\begin{pmatrix} \frac{\partial \mathrm{f}_1}{\partial x} & \frac{\partial \mathrm{f}_1}{\partial y} \\ \frac{\partial \mathrm{f}_2}{\partial x} & \frac{\partial \mathrm{f}_2}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \tag{90}$$

Obviously this vector v_1 will be at a point P however this w vector will be at a point $\mathrm{f}(P)$ and now we will see a very simple example and we will consider this mapping that we talked about earlier (85) We just introduced it but we kind of went ahead with the more general notation so if you consider this mapping then you can easily find out that the Jacobian is nothing but the following:

$$D\mathrm{f}=\begin{pmatrix} \frac{\partial \mathrm{f}_1}{\partial x} & \frac{\partial \mathrm{f}_1}{\partial y} \\ \frac{\partial \mathrm{f}_2}{\partial x} & \frac{\partial \mathrm{f}_2}{\partial y} \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{91}$$

Let's consider any vector let's say $v_1=(1,3)^T$ so w_1 will be just obviously you can take the base points to be $(0,0)$ or you can take the base point to be whatever you want let's say the base point this vector $(1,3)^T$ is defined on a point $(1,1)$ like this and w_1 will be a vector which is given by:

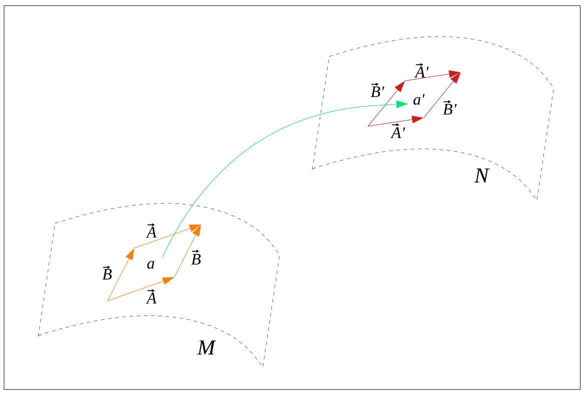
$$v_1=\begin{pmatrix} 1 \\ 3 \end{pmatrix}_{(1,1)} \text{ then } w_1=\left[\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]_{(1+1,1-1)}=\begin{pmatrix} 4 \\ -2 \end{pmatrix}_{(2,0)} \tag{92}$$

Because the points get mapped by the function itself which is given by $(x+y, x-y)$, the u coordinate will be 2 and the v coordinate will be 0 so the vector $v_1=(1,3)$ at point $(1,1)$ of Manifold M gets mapped to a vector $w_1=(4,-2)$ at point $(2,0)$ in the Manifold N . You might ask why we are bothering with this Push forward thing? The idea is that when you talk about coordinate changes what you are doing essentially is that you are going from one Manifold to another and the reason we are considering the Push forward of vectors is because when we try to define areas or volumes we have to deal with this vectors and that's why we need to see how these vectors get mapped from one Manifold to another because the vectors themselves don't live on the Manifold, they live on the Tangent spaces and that's why we are interested to see how this Tangent space mapping is working.

When you consider all the points then it becomes a mapping on the Tangent bundle so that's why we have to formally define this Push forward of vectors and if you consider the area spanned on the Manifold, we are again dealing with the same kind of coordinate changes as in this example so if you consider an area form like this $dx \wedge dy$ you will see that if you want to know how this area form changes when you change your coordinates from (x, y) to (u, v) you will need to consider this Push forward of vectors so that is another reason why we are actually bothering and even if you do any elementary calculation that is you write $u=x+y$ and $v=x-y$ and you solve for x and y you will find that $x=(u+v)/2$ and $y=(u-v)/2$ so $dx=½(du+dv)$ and $dy=½(du-dv)$ and then if you want to evaluate $dx\wedge dy$ or find the area form or the volume form, whatever you want to call it this is just:

$$dx\wedge dy=¼(du+dv)\wedge(du-dv)=-½du\wedge dv \tag{93}$$

This is one way, an elementary way, to see it that how your area form changes and note that this minus sign is actually telling you the orientation of the area because if you consider $dx\wedge dy$ to be a positive thing that means this kind of orientation where you go from x to y in a counter clockwise direction, if this is positive then when you go from du to dv in the counter clockwise direction that will give you a negative sign so this anti-clockwise rotation on the (x, y) plane can be thought of as a clockwise rotation on the (u, v) plane so that's where this minus sign is coming from. It's because of this orientation so this is an indicator of orientation.



Let's come back to the idea of the Push forward and we'll see that the result is the same. You see that to get any kind of area if you think of it like this way that you have one vector like this $\vec{A}$, you have another vector like this $\vec{B}$ and what you can do you can make a parallelogram with these two vectors. You can get some area and if this is on the Manifold M so let's say this is our Manifold M on the background denoted with this dashed line.

Since these two are vectors we should be able to find the pushed forward versions of these vectors and it will get mapped to this Manifold N and let's say this is the pushed forward version of this area with $\vec{A}'$ and $\vec{B}'$ and we'll see that they will yield the same result. Notice that these $dx\wedge dy$ are area forms and what do forms do they eat up vectors that's what we have learned so far that if you have any kind of form it will just eat your vectors up that's it simple as that so if we use our idea of the Push forward of vectors then we can map these vectors from $\vec{A}$ to $\vec{A}'$ and from $\vec{B}$ to $\vec{B}'$ and we should be able to map this area, let's call this area a and let's call this as a'. We should be able to map this a to a' and let's see how that is done. Note the fact that they are forms.

$$dx\wedge dy\left(\begin{bmatrix}1\\0\end{bmatrix}_{(0,0)}, \begin{bmatrix}0\\1\end{bmatrix}_{(0,0)}\right)=\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}=1 \tag{94}$$

They should eat up some vectors on the (x, y) plane and for simplicity we can just take the basis vectors, that means the x vector and the y vector or the unit vectors, whatever you want to call them. Let's denote those vectors by $(1,0)^T$ and this is on the point $(0,0)$ and vector $(0,1)^T$ also at the point $(0,0)$. If you use the ideas that we discussed on the previous lectures you can easily show that this is nothing but the determinant of the identity matrix which gives you a result of 1.

Now let's consider the idea of push forwards, that is, these two vectors will get pushed forward so the Push forward version of this vector $(1,0)^T$ will be this:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Bigg|_{xy(0,0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Bigg|_{uv(0,0)} \tag{95}$$

Note the fact that this is coming from the (x, y) plane or the Manifold M where you are equipped with the (x, y) coordinates on point $(0,0)$ and the result is on the (u,v) plane on Manifold N on point $(0,0)$ of the (u,v) plane. Then similarly:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Bigg|_{xy(0,0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Bigg|_{uv(0,0)} \tag{96}$$

Now what do you do? You simply want to find out $du \wedge dv$ but now these forms will eat the vectors that are the pushed forward versions of these two guys, that means, The results of (95) and (96):

$$du \wedge dv \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{(0,0)}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{(0,0)} \right) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 = -2\, dx \wedge dy \tag{97}$$

Finally you have the relationship that we computed earlier (93). This very idea of Push forward is actually helpful to understand how the projection of the area on the Manifold M gets mapped into Manifold N and this will also enable us to discuss about Pull backs in the next lecture.

Pullback of Differential forms

In this lecture we are going to discuss about Pullback of Differential forms. In the last lecture we discussed about the push forward vectors and we said that let's say you have these vectors v_1, v_2, v_3 on some Manifold M and you are inducing some coordinate change function or a coordinate transformation that induces a Tangent mapping or a mapping on the Tangent bundle that takes vectors v_1, v_2, v_3 as $T v_1, T v_2, T v_3$ where T itself was the Jacobian. Now what we ask is something like this: let's say you have some differential n-form on Manifold N called $\omega(T v_1, T v_2, T v_3)$ which eats up the vectors on this Manifold, that means these vectors denoted with $T v_1, T v_2, T v_3$, let's call this as let's say $T v_1 = v'_1, T v_2 = v'_2, T v_3 = v'_3$ so it will eat this vectors up and produce a real number. We ask can we do something like this: Let's say this v'_i are pushed forward vectors of its corresponding v_i that are situated on the Tangent bundle of M and we ask can we define some ω' ?

This is the question that we are asking: can we define ω' such that it will produce the same number, doesn't have to be real necessarily but since we're dealing with only functions of real variables or Manifolds where the points only give you a real number so that's why we're calling it a real number but it can be complex as well if you're dealing with a complex Manifold. This is the question that we are asking that you have some form ω on the Manifold N which its vector from the Tangent spaces of N produces a number and if you consider the vectors of N to be the pushed forward versions of some vector that were in the Tangent bundle of M can you define some ω' that will produce the same number as well, that's why it's called a Pullback.

What it's doing is that it's pulling you back from the Manifold N to Manifold M so that's why it's called a Pullback. Push forward was defined as you are pushing your vectors forward from Manifold M to N and in Pullback you first do something on Manifold N and then you try to induce a map by pulling the form defined on N onto the Manifold M so that's why it's called a Pullback. Formally we write the Pullback of a Differential form form as follows:

$$(T^* \mathrm{f}.\omega)(v_1, v_2, \ldots, v_n) = \omega(T\mathrm{f}.v_1, T\mathrm{f}.v_2, \ldots, T\mathrm{f}.v_n) \tag{98}$$

This whole thing is the Pullback and it's also known as the Pullback of ω this whole quantity, which eats vectors of Manifold M $v_1, v_2, \ldots, v_k$. These $T\mathrm{f}.v_i$ over the right hand side of (98) are the pushed forward versions of the vectors v_i over the left hand side of (98). This is our definition of the Pullback of Differential forms and now we will try to find out what is this $T^*\mathrm{f}$. We will just do things in general over here so let's consider a mapping, also we are assuming that these two Manifolds have the same dimension so what happens when they are not of the same dimension we'll get back to that through an example but first we want to find out what is the form of this $T^*\mathrm{f}$ using the fact that the Manifolds under consideration will have the same dimension. Let's consider a mapping denoted by:

$$\phi = (\phi_1(x_1, x_2, \ldots x_n), \phi_2(x_1, x_2, \ldots x_n), \ldots, \phi_n(x_1, x_2, \ldots x_n)) \tag{99}$$

By the definition of the Pullback of Differential forms or Pullback of forms we see that both sides of (98) are numbers and ω is a, let's say, differential n-form. If ω is an n-form and this guy produces a number that means there should be n vectors over here. Similarly if there are n vectors over here it should tell us that this is also, this Pullback guy, is also an n-form. That was pretty easy to do, now what we will do is we will say that we want to consider an n-form ω that is coming from the definition of ω as follows: this ω is defined on Manifold N where this mapping ϕ takes the points of N and it projects those points into M so that's why the ω form will be given by:

$$\omega = d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n \tag{100}$$

You can also say that $\omega \in \mathbb{R}^n_{(\phi_1, \phi_2, \ldots, \phi_n)}$. Since we want to find out what is this $T^* \phi$, because we are using a ϕ over here (99) so you can call it a $T^* \phi$ as well we want to find this guy. The first thing we note that this $T^* \mathrm{f}.\omega$ which is given over here (98) and it will translate into this $T^* \phi.\omega$ this is also an n-form. Why? Because we said a few minutes earlier if ω is an n-form and it eats n-vectors produces a number where on the left hand side you also have n vectors and you want to see what kind of a form this $T^* \mathrm{f}.\omega$ is you can easily conclude that this is also an n-form because if it's not an n-form then it won't be able to eat up n vectors and produce a number so the equality would be violated.

We already know that this $T^* \phi.\omega$ is an n-form so we can write this $T^* \phi.\omega$ as a general n-form where f is a 0-form or a function whatever you want to call it because 0-forms are just functions given by:

$$T^* \phi.\omega = \mathrm{f}\ dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \tag{101}$$

Now we proceed as follows: we will consider some vectors that are the basis vectors because they're easiest to deal with so we consider the following vectors:

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ , \ \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ , \ \ldots \ , \ \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{102}$$

We also know that this $T\phi$ which is just the Jacobian matrix is given by the following:

$$T\phi = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{pmatrix} \tag{103}$$

This is $n \times n$ matrix. Now let's consider the following: our 0-form f can be written as:

$$\mathrm{f} = \mathrm{f}\ \left(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n\right)\left(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\right) \tag{104}$$

What this does? It eats out these vectors this is by definition. That's why we consider this basis vector so that we can use this trick over here and note the fact that this f along with this $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ This is equals to our Pullback $T^* \phi.\omega$ by the definition because this Pullback was also an n-form and we said let the n-form take the following phase where you have some function and then you have this

Wedge products so you write this as $T^* \phi . \omega$ and then you have $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Now use the definition of the Pullback again that is the main definition (98). We use this definition again and you can write:

$$\mathrm{f} = \left(T^* \phi . \omega\right)\left(\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\right) = \omega\left(T\phi . \boldsymbol{e}_1, T\phi . \boldsymbol{e}_2, \ldots, T\phi . \boldsymbol{e}_n\right) \tag{105}$$

So ω was what? ω was an n-form on the Manifold where you have the coordinates $\phi_1, \phi_2, \ldots, \phi_n$. You write ω as (100). We use that now over here so using this identity we can write down this line:

$$\mathrm{f} = d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n \left(T\phi . \boldsymbol{e}_1, T\phi . \boldsymbol{e}_2, \ldots, T\phi . \boldsymbol{e}_n\right) \tag{106}$$

This is what? This is just another matrix given by:

$$\mathrm{f} = \begin{pmatrix} d\phi_1(T\phi . \boldsymbol{e}_1) & d\phi_1(T\phi . \boldsymbol{e}_2) & \cdots & d\phi_1(T\phi . \boldsymbol{e}_n) \\ d\phi_2(T\phi . \boldsymbol{e}_1) & d\phi_2(T\phi . \boldsymbol{e}_2) & \cdots & d\phi_2(T\phi . \boldsymbol{e}_n) \\ \vdots & \vdots & \ddots & \vdots \\ d\phi_n(T\phi . \boldsymbol{e}_1) & d\phi_n(T\phi . \boldsymbol{e}_2) & \cdots & d\phi_n(T\phi . \boldsymbol{e}_n) \end{pmatrix} \tag{107}$$

Now note the fact that $T\phi$ was our Jacobian matrix (103) and all these vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ these were the basis vector so if you consider any quantity that is given by let's say the first:

$$T\phi . \boldsymbol{e}_1 = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} \\ \frac{\partial\phi_2}{\partial x_1} \\ \vdots \\ \frac{\partial\phi_n}{\partial x_1} \end{pmatrix} \tag{108}$$

The first is an $n \times n$ matrix, the second is a column vector where you have n rows and one column only it should only spit out the first component because all the other components would be 0 so this is just this guy over here (108), similarly for the other components you can find that this $T\phi . \boldsymbol{e}_n$ will just give you the partial derivative of ϕ_2 with respect to x_n so finally we can write (107) as:

$$\mathrm{f} = \begin{vmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{vmatrix} \tag{109}$$

Which is nothing but the determinant of the Jacobian matrix itself so finally we have found f and our Pullback formula becomes:

$$T^*\phi\left(d\phi_1\wedge d\phi_2\wedge\cdots\wedge d\phi_n\right)=\begin{vmatrix}\frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_n}\\ \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_n}\end{vmatrix} d x_1\wedge d x_2\wedge\cdots\wedge d x_n \tag{110}$$

We can also write it as the very familiar formula that we see in multi variable calculus:

$$T^*\phi\left(d\phi_1\wedge d\phi_2\wedge\cdots\wedge d\phi_n\right)=|J|\ d x_1\wedge d x_2\wedge\cdots\wedge d x_n \tag{111}$$

This is what your Pullback does. What we do in multi variable calculus is that we introduce the change of coordinates but we never mention about the Manifolds because when you change your coordinates you are also changing the Manifold as well. We finally found the Pullback, that is, this $T^*\omega$ where ω is given by (100) is just the determinant of the Jacobian and this form itself:

$$T^*\omega=|J|\ d x_1\wedge d x_2\wedge\cdots\wedge d x_n \tag{112}$$

If you recall the example from the last lecture where our mappings were given by:

$$\phi=\left(\phi_1=x+y,\phi_2=x-y\right) \tag{113}$$

Then the determinant of the Jacobian was what? The Jacobian was just:

$$|J|=\begin{vmatrix}\frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_1}{\partial y}\\ \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_2}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-2 \tag{114}$$

If you want to consider that the area on the (u,v) plane like this $d u\wedge d v$ then it would be related with the area form of the (x,y) as:

$$d u\wedge d v=-2\ d x\wedge d y \tag{115}$$

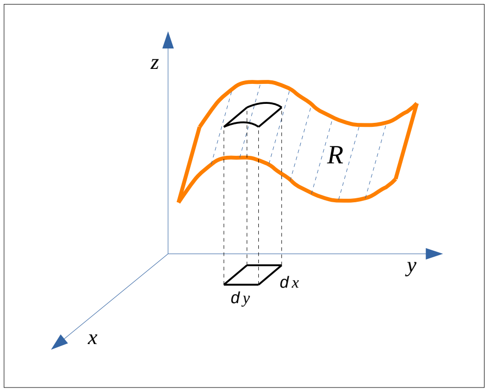
When we do this kind of things we actually calculate the Pullback that's what the Pullback of Differential forms tells us that what you are doing here is that the Pullback of $d u\wedge d v$ is just this whole thing $-2\ d x\wedge d y$ so if this is the initial $\omega=d u\wedge d v$ that means the initial form on the Manifold N or the second Manifold then the Pullback of this ω would be $-2\ d x\wedge d y$ and in fact this type of Pullback can also be defined for mapping such that:

$$\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ where } m \neq n \tag{116}$$

Maybe we'll get back to that later but this will be it for today and I'll see you guys on the next lecture. Thank you

Integration of Forms

This will be our last topic in this course that we're going to discuss and today's topic is integration of forms what do we mean by integration of Differential forms. Let's say we have some function like this $f(x,y)$ which is a 0-form and if you want to draw a graph on the x, y, z axes it should give you a surface area like this a region R .



Now this region is denoted with this function $f(x,y)$ and our interest is to compute a quantity that is defined with this double integral over this region R is this something like this:

$$\iint_R f(x,y)\, dx\wedge dy \tag{117}$$

We want to compute integrals that involve Differential forms.

How do we proceed? Well just like an ordinary single variable or multiple variable calculus we can use the idea of Riemann summations and we can go along with it however if you look at this form over here (117) it doesn't tell you much on how to compute or how to handle this Wedge products over here because so far we have seen the Differential forms eat up vectors and we don't see any vectors over here so we should look into another tool that will enable us to compute this type of quantities.

There is a relation, the relation is something like this let me first state the relation and then I'll explain what this relation means and what we are essentially doing so instead of writing double or triple integrals I will just use a single integral sign that can represent an n dimensional integral and instead of writing $f(x,y,z)$, I will just write f which can be a function of n variables and then we have our Differential forms like this:

$$\int_R f\, dx_1\wedge dx_2\wedge\cdots\wedge dx_n=\int_{\phi(R)} f\circ\phi^{-1}\, T^*\phi^{-1}\left(dx_1\wedge dx_2\wedge\cdots\wedge dx_n\right) \tag{118}$$

Where $\phi(R)$ means this region R is getting mapped with some mapping that is denoted with ϕ and f will get mapped with ϕ^{-1} . This $\circ$ like thing over here is also known as a binary operation and then there will be a Pullback mapping on the Differential forms. In fact this is the tool that will enable us to compute integrals involving Differential forms. How does this come about? Well the idea is as follows: let's say you have two copies of the same Manifold, let's say this is copy #1 and let's say this is copy #2.

They're the same Manifold and what you are doing essentially is that let's say this Manifold is endowed with the coordinate system x_i that means $x_1, x_2, \ldots, x_n$. When you are trying to introduce an integral that involves Differential forms on Manifold #1 what you do is that you want to introduce some vectors. Why?

Because those vectors will make up a lattice and on those lattice points you can define a Riemann sum and then take the continuous limit to find this relation (118) and when you are considering this mapping that is going from Manifold #1 to Manifold #2 even though they are the same they are just endowed with two different coordinate systems. These x_i coordinates will get mapped to $\phi(x_i)=\phi_i$ and when you do that you also have to take into account the fact that in Manifold #1 there are vectors that make up your lattice points or the lattice structure and those vectors will be pushed forward on Manifold #2.

That's why you need to have a Pullback on the Differential forms that you introduce in your Manifold #1. This is one way to think of it the proof is not very hard but we will skip that for now and we will try to see how we can use this quantity in equation (118) and we want to see how we can use this tool or quantity to help us evaluate different kind of integrals that involve Differential forms. Before we go on oftentimes people regard a double integral let's say:

$$\iint_R \mathrm{f}(x,y)\, dx\, dy \equiv \iint_R \mathrm{f}(x,y)\, dx \wedge dy \tag{119}$$

Those are equivalent but not equal. The reason is the following: if you are familiar with Fubini's theorem, it says that if you have a nice sort of function then the order of the integration over here won't matter. The result will be the same as if you interchange to $dy\,dx$ but for that, this function has to have some nice sort of properties and most of the functions will deal within this course will have that property so from an ordinary calculus point of view there is no difference but in here (119), on the right hand side we have this Wedge product and we know that Wedge products have this anti-symmetry built inside of them. There is a minus sign thing in $dx \wedge dy = -dy \wedge dx$ so that's why when you work with Differential forms you are actually taking into account the orientation of the volume because this quantity is actually a volume.

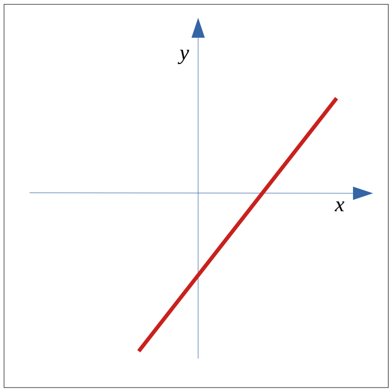
If you want to see it just draw the graph this is your region $\mathrm{f}(x,y)$ and essentially what you are doing is that this is like a height let's say $z=\mathrm{f}(x,y)$ and this is the area that is endowed with $dx\,dy$ and you are just computing this $z\, dx\, dy$ so that's why we said this gives us a volume under this region $\mathrm{f}(x,y)$ and when you are working with forms you are also accounting for the orientation of that volume that means whether you are working with a left-handed coordinate system or writing that coordinate system and so on. Now let's try to see how we can actually use this formula to our advantage and let's start with a very simple example.

Example 1: Let's say we want to compute our integral with some function $\mathrm{f}(x,y)=x$ over a region in the Cartesian Manifold $\mathbb{R}^n_{xy}$ bounded by the lines, the reason I'm keep referring to the Cartesian Manifold or $\mathbb{R}^n$ or $\mathbb{R}^2$ whatever, is because things are much easier when you work with them as the Manifold itself is isomorphic to the Tangent space of this Manifold so when you introduce those Differential forms and vectors you really don't have to be worried about what you are working with, whether you are working on the Manifold, whether you are working on the Tangent space and so on.

Continuing with the example, it will be bounded by the lines $x=0$ and $y=0$ and let's say $y=x-2$ something like that. Let's try to integrate this thing, this will look something like this:

$$\iint x\, dx\, dy \tag{120}$$

We have to find out our limits so let's say we want to integrate with respect to x first and then we want to integrate with respect to y . In this region you can see that if you want to integrate in terms of x first then you have to solve for x from this equation $y=x-2$, this will give you $x=y+2$. The question is what is the lower limit and what is the upper limit? To visualize that you have to draw the region. If we draw the region it will look something like this:



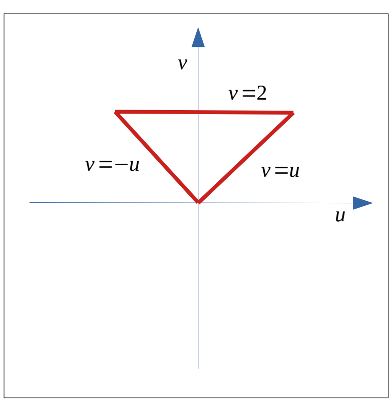
From here you can see that our lower limit where this red line is given by $y=x-2$ and from this drawing or this graph over here you can see that the lower limit would be $y+2$ and the upper limit would be 0 and then you integrate with respect to y . What is the lowest limit of y ? It's -2 over here and the upper limit is 0 . If you compute that:

$$\int_{-2}^{0}\int_{y+2}^{0} x\, d x\, d y=-\frac{4}{3} \tag{121}$$

This is an example of an ordinary double integral. Now what we want to do is that we want to use this idea, this one in equation (118). What does it tell us? It tells us that you have something like this $d x_1 \wedge d x_2 \wedge \cdots \wedge d x_n$ and now what you're doing is that you're changing your coordinate system why we want to do that well let's say in the coordinates $x_1, x_2, \ldots, x_n$ the integral is very difficult to compute so that's why we want to change the coordinates and we want to see what happens. Now what we can do is that we can take this integral but now we want to change the coordinate systems.

Let's change the coordinate systems as follow: our function was what function $f(x, y)=x$ and let's say I want to compute that integral with the help of Differential forms in a new coordinate and the new coordinate is given by (85) $u=x+y$ and $v=x-y$, these will be our new coordinates. If you do that you can immediately solve for $x=(u+v)/2$ and $y=(u-v)/2$. Now we have to use the information given in the problem is that our region R was bounded by this $x=0$, $y=0$ and $y=x-2$.

We need to translate those things in terms of the new coordinates so $x=0$ will give us $u=-v$. $y=0$ would give us $u=v$ and the final end which is $y=x-2$ will gives us $v=2$. If you draw this region in the (u, v) coordinates it will look something like the following:



This is now our new region. Now what we want to do we want to use the ideas of the Differential forms of the tools that are coming from the Differential forms and note that this change of coordinates is just that map this $\phi(x, y)$ this ϕ is taking your x and y coordinates and it's returning the new values as:

$$\phi(x, y)=(x+y, x-y) \tag{122}$$

Which are written as u and v so this is our mapping.

The ϕ^{-1} will induce the Pullback on $d x \wedge d y$ that means this mapping $T^{*} \phi^{-1}(d x \wedge d y)$.

This guy now we have to compute the Pullback and this is pretty easy if you just compute:

$$dx = \frac{du + dv}{2} \quad , \quad dy = \frac{du - dv}{2} \tag{123}$$

$$dx \wedge dy = -½\, du \wedge dv \tag{124}$$

We computed the Pullback on the forms, we computed how x and y transfers into u and v all we got to do right now is to just use this tool on equation (118) to find out our integration so let's try to do that and if you write it like this let's say this is $\tilde{R}$ which is the new region and what we can do is that we can now use this equivalence that we saw earlier:

$$I = \iint_{\tilde{R}} \mathrm{f}(u,v)(-½\, du \wedge dv) = \iint_{\tilde{R}} \mathrm{f}(u,v)(-½\, du\, dv) \tag{125}$$

Now all you have to do is to just decide on which variable you are going to integrate first, let's say we want to integrate over u first we have the lower and upper limits of u which are just $-v$ and $+v$ and then we have to integrate over v the values vary from 0 to 2 and you can take the constant out and you have to put $\mathrm{f}(u,v)$ as $(u+v)/2$:

$$I = -\frac{1}{2} \int_0^2 \int_{-v}^{v} \frac{(u+v)}{2}\, du\, dv = -\frac{4}{3} \tag{126}$$

Which is identical to (121). Note the fact that we took the orientation $du \wedge dv$ and if we took the opposite thing this minus sign in (126) it wouldn't be here so what would happen is that you would get a $4/3$ and this is a pretty weird thing when you see it at first because we told that this type of integrals should give you a volume and you are having a negative sign on the volume again the volume is not negative it's just the orientation that has been changed when you went from the (x, y) coordinates to the (u,v) coordinates and this is what happens with the minus sign so when you are invoking these Differential forms you are inherently considering the orientation of your coordinates as well.

Now that we have some idea on how this integrals work let us try to go for a general kind of problems that we will see and that is how we can actually formally do this things let's say we want to now go for a parameterized surfaces. We want to parameterize our regions or curves whatever we have so the first example is actually associated with the Line integrals so this is a parameterized example where the surface or curves are just parameters with respect to some parameters. The first thing is an example of a Line integral. The integration of Differential forms had this kind of form:

$$\int \mathrm{f}\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = \int \alpha = \int_{\phi(C)} T^* \phi^{-1} \cdot \alpha \tag{127}$$

In short we can write it like with α this is some form and using the right hand side of equation (118) that we discussed a little bit earlier would be written with the Pullback. In this example we are going to integrate a 1-form along some curve so let's say we want to integrate $y^2\, dx + x\, dy$ this is our 1-form and we want to integrate it on a parameterized curve so let's say the curve is parameterized as follows:

$$\phi^{-1}(t) = (x(t), y(t)) = (5t-5, 5t-3) \tag{128}$$

Since this is a curve on a 2D space and that's why it will have two components x and y . You can also induce more components like say w and whatever but the procedure is the same. Now first things first since we are dealing with a parameterized curve now our goal should be to change the coordinates and bring the whole integral that means the integral over this guy over here $y^2\, dx + x\, dy$ in terms of t . That should be our goal. What we want to do is first compute the Pullback, the Pullback mapping:

$$T^* \phi^{-1} = \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{129}$$

You are dealing with the form and forms will eventually act on vectors so that's why you need to have this Pullback mapping. Now what you do? Since this is done we note that we have an $\mathrm{f}(x, y)$ but we want to go for $\mathrm{f}(t)$ because we want to get rid of this x and y and we want to express the whole thing as a function of t that is the function of the parameter so this $\mathrm{f}(t)$ can be written as the following:

$$\mathrm{f}(t) = \mathrm{f}(t)\, dt[1] = T^* \phi^{-1}(y^2\, dx + x\, dy)[1] \tag{130}$$

This is a 1-form and it will let's say take a vector which is a 1×1 , 1 row and 1 column vector. This $dt[1]$ can be written with the help of Pullback. That means the initial function that we were working with times one. Then from here what we can do is that we can write it as:

$$\mathrm{f}(t) = (y^2\, dx + x\, dy)(T^* \phi^{-1}[1]) = (y^2\, dx + x\, dy)\begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{131}$$

Which should give me the following column vector and this is a linear multiplication so you just multiply the column vectors:

$$\mathrm{f}(t) = y^2\, dx \begin{pmatrix} 5 \\ 5 \end{pmatrix} + x\, dy \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5y^2 + 5x \tag{132}$$

Now what we can do is that we can use the parameterizations that were given in the first place (128). If you do that the final expression will be something like this:

$$\mathrm{f}(t) = 125(t^2 - t) + 20 \tag{133}$$

Now we finally have to integrate this thing. Let's say our parameter varies from $0 \le t \le 1$ this is how the limit is being set and we want to compute:

$$\int_{\phi(C)} T^* \phi^{-1}.\alpha = \int_0^1 \left(125(t^2 - t) + 20\right) dt = -\frac{5}{6} \tag{134}$$

This is a simple integral and you can evaluate to know that the answer is just $-5/6$. Now you might ask: I get everything but what are we doing over here (131)? Why is this $T^* \phi^{-1}[1]$ over here and why is this giving me a $(5,5)^T$ or a column matrix? Let's try to justify this as the following way, we already computed this $T^* \phi^{-1}[1]$ is just a $(5,5)^T$ and by doing this we have induced the Tangent and Cotangent mapping. This is a justification, when you do (129), you write it as $(5,5)^T$ using our parameterization. By doing this we introduced the Tangent and Cotangent mappings as follows:

$$T_P(\mathbb{R}) \xrightarrow{T\phi^{-1}} T_{\phi^{-1}(P)}(\mathbb{R}^2) \tag{135}$$

This $[1]$ which belong to the Tangent space of the Manifold will become something like this:

$$[1] \in T_P(\mathbb{R}) \to T\phi^{-1}.[1] = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \cdot [1] = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \in T_{\phi^{-1}(P)}(\mathbb{R}^2) \tag{136}$$

Because now you are just using the value of this $T\phi^{-1}$ and this is just $(5,5)^T$ which belongs to $T_{\phi^{-1}(P)}(\mathbb{R}^2)$. This is the justification. What is the summary? How do you approach it? The summary for this is that at least for parametric surfaces first you find whatever is given inside the integral sign, if this is your quantity of interest (127), 1st you find this f in terms of the parameter, 2nd you compute the Pullback mapping of the forms with respect to the parameter because that's how Pullback works, 3rd you see the limits or find out the limits of the parameter then you just integrate it.

In fact if you can just define your whole problem using a single parameter, your n-dimensional problem can be reduced to just 1D problem and so on. Let us end by seeing another example and this time we are going to not deal with Line integrals but rather what we are going to do is that we are going to take an example of a Surface integral so that you can get a better feel of how to compute this Pullback maps We're going to integrate $z^2\, dx \wedge dy$ over the top half of the unit sphere, this is our region.

The best way to do is to just go to the parameterization of (r,θ) for this case. I'm giving you the parameterization over here and this parameterization is given by:

$$\phi^{-1}(r,\theta) = \left(x(r,\theta), y(r,\theta), z(r,\theta)\right) \tag{137}$$

Notice that I'm using this ϕ^{-1} one because the (x,y,z) goes inside ϕ and then pops out this (r,θ) and whatever and so that's why I use this ϕ^{-1} on (r,θ) . Those are the parameters and I get (x,y,z) :

$$\phi^{-1}(r,\theta) = \left(r\cos\theta, r\sin\theta, \sqrt{1-r^2}\right) \tag{138}$$

How does this z come? Well the unit sphere will have an equation like this $x^2+y^2+z^2=1$ and since I'm talking about the top half I will only take the positive square root offset, the negative square root would give you the bottom half of the unit sphere so that's why I wrote it as $z=\sqrt{1-r^2}$ and we are done. Now we have to compute the Pullback:

$$T\phi^{-1}=\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{bmatrix}=\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ -\frac{r}{\sqrt{1-r^2}} & 0 \end{bmatrix} \tag{139}$$

All of this is coming from the definition of the Pullback that we started earlier so if you're confused I suggest to look at it once more. Now what you want to do is put the final piece of the puzzle, that is, let's say this is our $g(x,y)=z^2\, dx\wedge dy$ for example and we want to compute it in terms of (r,θ) because we are parameterizing our problem in that way and we want to finally use the fact that:

$$g(r,\theta)\, dr\wedge d\theta\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=g(r,\theta) \tag{140}$$

If you introduce two column vectors which will give you $g(r,\theta)$. This is pretty easy to see because these forms with this column vectors will give you a determinant and on the next line you want to introduce that map:

$$g(r,\theta)=T^*\phi^{-1}(z^2\, dx\wedge dy)\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)=(z^2\, dx\wedge dy)\left(T^*\phi^{-1}\begin{bmatrix}1\\0\end{bmatrix},T^*\phi^{-1}\begin{bmatrix}0\\1\end{bmatrix}\right) \tag{141}$$

Now you put the values of $T^*\phi^{-1}$ that you found earlier and you multiply them with this column vectors. Note that the reason we are introducing two column vectors is that we have a 2-form and on the previous problem we had a 1-from so we just introduced something like this a box that contains a $[1]$ that means here it's just a 1×1 with a single entry and this time we have 2-form (r,θ) so we need two vectors, that's why we are doing this whole thing. If you use the value of $T^*\phi^{-1}$ that you computed here and you multiply them with this column vectors this will become:

$$g(r,\theta)=z^2\, dx\wedge dy\left(\begin{bmatrix}\cos\theta\\ \sin\theta\\ -r/\sqrt{1-r^2}\end{bmatrix},\begin{bmatrix}-r\sin\theta\\ r\cos\theta\\ 0\end{bmatrix}\right) \tag{142}$$

Now it's pretty simple you have two vectors you have two forms, you just take dx will just eat the first quantity and dy will take the second quantity over here, that's what you think but this is actually a determinant so the determinant will only consist of these two first parts because there is no third form over here like dz . If it had a dz it would be 3×3 matrix.

This $dx \wedge dy$ will act on these two vectors and it should give you something like this determinant:

$$g(r,\theta) = z^2 \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = z^2 r = r(1-r^2) \tag{143}$$

Then what you do you express z in terms of r so $z=\sqrt{1-r^2}$. Finally what we can write is:

$$T^* \phi^{-1}(z^2\, dx \wedge dy) = (r - r^3)\, dr \wedge d\theta \tag{144}$$

Now we can use that equivalence that we used earlier this will be equivalent to:

$$\int_R (r-r^3)\, dr \wedge d\theta \equiv \int_R (r-r^3)\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\, dr = \frac{\pi}{2} \tag{145}$$

Now we have to find the limits well this is the unit square so $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. This is your answer. A seemingly difficult integral can be put to much simpler notation even though a little bit of long computation with the help of Differential forms and this is how you actually work around with Differential forms and their integration and this is the last lecture, thank you