Shaikot Differential Forms

Based on Shaikot Jahan Shuvo Playlist: Differential Forms

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(%i2) `info:build_info()$info@version;`

(%o2)

5.38.1

(%i2) `reset()$kill(all)$`

(%i1) `derivabbrev:true$`

(%i2) `load(linearalgebra)$`

(%i3) `load(draw)$`

(%i4) `wxplot_size:[1024,768]$`

(%i5) `load(drawdf)$`

(%i6) `if get('vect,'version)=false then load(vect)$`

(%i7) `norm(u):=block(trigsimp(radcan(√(u.u))))$`

(%i8) `normalize(v):=block(v/norm(v))$`

(%i9) `if get('cartan,'version)=false then load(cartan)$`

(%i10) `declare(trigsimp,evfun)$`

Define the space

(%i11) `r:[x,y,z]$`

(%i12) `scalefactors(r)$`

(%i13) `sf;`

$$[1,1,1] \tag{\%o13}$$

(%i14) `sfprod;`

$$1 \tag{\%o14}$$

(%i15) `init_cartan(r)$`

(%i16) `cartan_coords;`

$$[x,y,z] \tag{\%o16}$$

(%i17) `cartan_basis;`

$$[dx,dy,dz] \tag{\%o17}$$

(%i18) `cartan_dim;`

$$3 \tag{\%o18}$$

(%i19) `extdim;`

$$3 \tag{\%o19}$$

1 Introduction to Differential forms

(%i21)
```
A:[2,3]$
B:[1,-1]$
```
(%i22)
```
A.B;
```
$$-1 \tag{\%o22}$$

Vector $\vec{u}$

(%i23)
```
kill(a,b,c)$
```
(%i24)
```
u:[a,b,c]$
```
Scalar field f

(%i25)
```
depends(f,r)$
```
∇f

(%i26)
```
gradf:ev(express(grad(f)),diff);
```
$$[f_x, f_y, f_z] \tag{gradf}$$

$u \cdot \nabla f$

(%i27)
```
u.gradf;
```
$$c\ (f_z) + b\ (f_y) + a\ (f_x) \tag{\%o27}$$

Define the space

(%i28)
```
r:[x,y]$
```
(%i29)
```
scalefactors(r)$
```
(%i30)
```
init_cartan(r)$
```
Vector $\vec{u}$

(%i31)
```
u:['cos(π/6),'sin(π/6)]$
```
Scalar field f

(%i32)
```
ldisplay(f:y^2-3*x*y+x^3)$
```
$$f = y^2 - 3xy + x^3 \tag{\%t32}$$

∇f

(%i33)
```
gradf:ev(express(grad(f)),diff);
```
$$[3x^2 - 3y, 2y - 3x] \tag{gradf}$$

Potential

(%i34) `potential(gradf);`

$$y^2 - 3xy + x^3 \tag{\%o34}$$

$u \cdot \nabla f$

(%i35) `u.gradf,ratsimp;`

$$-\frac{\left(3^{\frac{3}{2}} - 2\right) y - 3^{\frac{3}{2}}\, x^2 + 3x}{2} \tag{\%o35}$$

Lie derivative

$\mathcal{L}_u f$

(%i36) `lie_diff(u,f),ratsimp;`

$$-\frac{\left(3^{\frac{3}{2}} - 2\right) y - 3^{\frac{3}{2}}\, x^2 + 3x}{2} \tag{\%o36}$$

Check

(%i37) `is(%th(1)=%th(2));`

$$\text{true} \tag{\%o37}$$

$\mathrm{d}f(u)$

(%i38) `u|ext_diff(f),ratsimp;`

$$-\frac{\left(3^{\frac{3}{2}} - 2\right) y - 3^{\frac{3}{2}}\, x^2 + 3x}{2} \tag{\%o38}$$

Check

(%i39) `is(%th(1)=%th(3));`

$$\text{true} \tag{\%o39}$$

Standard basis

(%i41)
```
e_1:[1,0]$
e_2:[0,1]$
```

$\mathrm{d}f(e_1)$

(%i42) `e_1|ext_diff(f);`

$$3x^2 - 3y \tag{\%o42}$$

$\mathrm{d}f(e_2)$

(%i43) `e_2|ext_diff(f);`

$$2y - 3x \tag{\%o43}$$

2 Differential 1-forms

(%i44) `kill(labels,x,y,z,α,v)$`

Define the space

(%i1) `r:[x,y,z]$`

(%i2) `scalefactors(r)$`

(%i3) `init_cartan(r)$`

$\alpha \in \mathcal{A}^1(\mathbb{R}^3), \quad \alpha = 2\,\mathrm{d}x - 3\,\mathrm{d}y + 5\,\mathrm{d}z$

(%i4) `ldisplay(α:2*dx-3*dy+5*dz)$`

$$\alpha = 5dz - 3dy + 2dx \qquad (\%t4)$$

Vector $\vec{v}$

(%i5) `v:[-1,3,-4]$`

$v[\alpha] = \alpha(v)$

(%i6) `v|α;`

$$-31 \qquad (\%o6)$$

3 Wedge product

(%i7) `kill(labels,x_1,x_2,x_3)$`

Define the space

(%i1) `r:[x_1,x_2,x_3]$`

(%i2) `init_cartan(r)$`

Vectors $\vec{v}, \vec{w}$

(%i4)
```
v:[1,2,3]$
w:[4,5,6]$
```

$\alpha \in \mathcal{A}^2(\mathbb{R}^3), \quad \alpha = \mathrm{d}x_1 \wedge \mathrm{d}x_2$

(%i5) `ldisplay(α:dx_1~dx_2)$`

$$\alpha = dx\_1\, dx\_2 \tag{\%t5}$$

$\alpha(v, w)$

(%i6) `w|(v|α);`

$$-3 \tag{\%o6}$$

$\alpha(w, v)$

(%i7) `v|(w|α);`

$$3 \tag{\%o7}$$

4 Exterior derivative

(%i8) `kill(labels,x,y,f,g)$`

Define the space

(%i1) `r:[x,y]$`

(%i2) `init_cartan(r)$`

Scalar fields f, g

(%i3) `depends([f,g],r)$`

$\mathrm{d}f \wedge \mathrm{d}g$

(%i4) `edit(ext_diff(f)~ext_diff(g));`

$$((f_x)\ (g_y) - (f_y)\ (g_x))\ dx\ dy \tag{\%o4}$$

(%i5) `determinant(jacobian([f,g],r));`

$$(f_x)\ (g_y) - (f_y)\ (g_x) \tag{\%o5}$$

Define the space

(%i6) `r:[x,y,z]$`

(%i7) `init_cartan(r)$`

Scalar field f

(%i8) `ldisplay(f:x`^{3}`*y`^{2}`*z^4)$`

$$f = x^3\, y^2\, z^4 \tag{\%t8}$$

$\alpha \in \mathcal{A}^1(\mathbb{R}^3), \quad \alpha = \mathrm{d}f$

(%i9) `ldisplay(`α`:edit(ext_diff(f)))$`

$$\alpha = 4x^3\, y^2\, z^3\ dz + 2x^3 y\, z^4\ dy + 3x^2\, y^2\, z^4\ dx \tag{\%t9}$$

$\mathrm{d}(\mathrm{d}f)$

(%i10) `edit(ext_diff(`α`));`

$$0 \tag{\%o10}$$

(%i11) `kill(x,y,z,`α`,`β`,`γ`)$`

(%i12) `depends([`α`,`β`],r)$`

Define the space

```
(%i13) r:[x,y,z]$
(%i14) init_cartan(r)$
```

$\gamma \in \mathcal{A}^1(\mathbb{R}^3), \quad \gamma = \mathrm{d}(\alpha \wedge \beta)$

```
(%i15) ldisplay(γ:edit(ext_diff(α~β)))$
```

$$\gamma = dz\ (\alpha\ (\beta_z) + (\alpha_z)\,\beta) + dy\ (\alpha\ (\beta_y) + (\alpha_y)\,\beta) + dx\ (\alpha\ (\beta_x) + (\alpha_x)\,\beta) \tag{\%t15}$$

$\gamma \in \mathcal{A}^1(\mathbb{R}^3), \quad \gamma = \mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta$

```
(%i16) ldisplay(γ:edit(ext_diff(α)~β+α~ext_diff(β)))$
```

$$\gamma = dz\ (\alpha\ (\beta_z) + (\alpha_z)\,\beta) + dy\ (\alpha\ (\beta_y) + (\alpha_y)\,\beta) + dx\ (\alpha\ (\beta_x) + (\alpha_x)\,\beta) \tag{\%t16}$$

Define the space

```
(%i17) r:[x,y]$
(%i18) init_cartan(r)$
```

$\gamma \in \mathcal{A}^1(\mathbb{R}^2), \quad \gamma = \mathrm{d}(\alpha \wedge \beta)$

```
(%i19) ldisplay(γ:edit(ext_diff(α~β)))$
```

$$\gamma = dy\ (\alpha\ (\beta_y) + (\alpha_y)\,\beta) + dx\ (\alpha\ (\beta_x) + (\alpha_x)\,\beta) \tag{\%t19}$$

$\gamma \in \mathcal{A}^1(\mathbb{R}^2), \quad \gamma = \mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta$

```
(%i20) ldisplay(γ:edit(ext_diff(α)~β+α~ext_diff(β)))$
```

$$\gamma = dy\ (\alpha\ (\beta_y) + (\alpha_y)\,\beta) + dx\ (\alpha\ (\beta_x) + (\alpha_x)\,\beta) \tag{\%t20}$$

5 Push forward of vectors

(%i21) `kill(labels,x,y,u,v,a,b,f,ϕ,ω)$`

Define the space

(%i1) `ζ:[x,y]$`

(%i2) `init_cartan(ζ)$`

Change of coordinates φ

(%i3) `ldisplay(φ:[x+y,x-y])$`

$$\varphi = [y + x, x - y] \tag{\%t3}$$

Inverse change of coordinates $i\varphi$

(%i4) `ξ:[u,v]$`

(%i5) `ldisplay(iφ:map('rhs,linsolve(map("=",ξ,φ),ζ)))$`

$$i\varphi = \left[\frac{v+u}{2}, \frac{u-v}{2}\right] \tag{\%t5}$$

Vector $\vec{w}$

(%i6) `w:[a,b]$`

Vector field $\vec{f}$

(%i7) `f:[f_1,f_2]$`

(%i8) `depends(f,ζ)$`

df

(%i9) `ext_diff(f);`

$$\left[\left(f_{1y}\right) dy + \left(f_{1x}\right) dx, \left(f_{2y}\right) dy + \left(f_{2x}\right) dx\right] \tag{\%o9}$$

d$f(\vec{w})$

(%i10) `w|ext_diff(f);`

$$\left[b\ \left(f_{1y}\right) + a\ \left(f_{1x}\right), b\ \left(f_{2y}\right) + a\ \left(f_{2x}\right)\right] \tag{\%o10}$$

Jacobian

(%i11) `J:jacobian(f,ζ);`

$$\begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} \tag{J}$$

$J \cdot \vec{w}$

(%i12) `list_matrix_entries(J.w);`

$$\left[b\,\left(f_{1y}\right)+a\,\left(f_{1x}\right),b\,\left(f_{2y}\right)+a\,\left(f_{2x}\right)\right] \tag{\%o12}$$

Vector $\vec{v}_1 = (1,3)^T$

(%i13) `ldisplay(v_1:[1,3])$`

$$v_1 = [1,3] \tag{\%t13}$$

Jacobian of φ

(%i14) `J:jacobian(φ,ζ);`

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{J}$$

(%i15) `ldisplay(w_1:list_matrix_entries(J.v_1))$`

$$w_1 = [4,-2] \tag{\%t15}$$

$\omega \in \mathcal{A}^2(\mathbb{R}^2), \quad \omega = \mathrm{d}x \wedge \mathrm{d}y$

(%i16) `ldisplay(ω:dx~dy)$`

$$\omega = dx\,dy \tag{\%t16}$$

Pullback $i\varphi^*\omega \in \mathcal{A}^2(\mathbb{R}^2)$

(%i17) `ldisplay(Pb:diff(iφ,v)|(diff(iφ,u)|ev(ω,map("=",ζ,iφ))))$`

$$Pb = -\frac{1}{2} \tag{\%t17}$$

Define the new space

(%i18) `init_cartan(ξ)$`

$\psi \in \mathcal{A}^2(\mathbb{R}^2), \quad \psi = i\varphi^*\omega\,\mathrm{d}u \wedge \mathrm{d}v$

(%i19) `ldisplay(ψ:Pb*(du~dv))$`

$$\psi = -\frac{du\,dv}{2} \tag{\%t19}$$

$\omega \in \mathcal{A}^2(\mathbb{R}^2), \quad \omega = \mathrm{d}u \wedge \mathrm{d}v$

(%i20) `ldisplay(ω:du~dv)$`

$$\omega = du\,dv \tag{\%t20}$$

Pullback $\varphi^*\omega \in \mathcal{A}^2(\mathbb{R}^2)$

(%i21) `ldisplay(Pb:diff(φ,y)|(diff(φ,x)|ev(ω,map("=",ξ,φ))))$`

$$Pb = -2 \qquad (\%t21)$$

Define the old space

(%i22) `init_cartan(ζ)$`

$\psi \in \mathcal{A}^2(\mathbb{R}^2), \quad \psi = \varphi^*\omega \, \mathrm{d}x \wedge \mathrm{d}y$

(%i23) `ldisplay(ψ:Pb*(dx~dy))$`

$$\psi = -2dx\, dy \qquad (\%t23)$$

$\omega \in \mathcal{A}^2(\mathbb{R}^2), \quad \omega = \mathrm{d}x \wedge \mathrm{d}y$

(%i24) `ldisplay(ω:dx~dy)$`

$$\omega = dx\, dy \qquad (\%t24)$$

Standard basis

(%i26)
```
e_1:[1,0]$
e_2:[0,1]$
```

$(\mathrm{d}x \wedge \mathrm{d}y)(\vec{e}_1, \vec{e}_2)$

(%i27) `e_2|(e_1|ω);`

$$1 \qquad (\%o27)$$

$P = J \cdot \vec{e}_1$

(%i28) `P:list_matrix_entries(J.e_1);`

$$[1, 1] \qquad (P)$$

$Q = J \cdot \vec{e}_2$

(%i29) `Q:list_matrix_entries(J.e_2);`

$$[1, -1] \qquad (Q)$$

Define the new space

(%i30) `init_cartan(ξ)$`

$\psi \in \mathcal{A}^2(\mathbb{R}^2), \quad \psi = \mathrm{d}u \wedge \mathrm{d}v$

(%i31) `ldisplay(ψ:du~dv)$`

$$\psi = du\, dv \qquad (\%t31)$$

(%i32) `Q|(P|ψ);`

$$-2 \qquad (\%o32)$$

6 Pullback of Differential forms

(%i33) `kill(labels)$`

7 Integration of forms

(%i1) `kill(labels,x,y,u,v,f)$`

Define the space

(%i1) `ζ:[x,y]$`

(%i2) `init_cartan(ζ)$`

Scalar field $f(x,y)$

(%i3) `f(x,y):=x$`

(%i4) `I:'integrate('integrate(f(x,y),x,y+2,0),y,-2,0)$`

(%i5) `ldisplay(I=ev(I,integrate))$`

$$\int_{-2}^{0}\int_{y+2}^{0} x\,\mathrm{d}x\,\mathrm{d}y = -\frac{4}{3} \tag{\%t5}$$

Change of coordinates $\varphi:\mathbb{R}^2\to\mathbb{R}^2$

(%i6) `ldisplay(φ:[x+y,x-y])$`

$$\varphi = [y+x, x-y] \tag{\%t6}$$

Inverse change of coordinates $i\varphi:\mathbb{R}^2\to\mathbb{R}^2$

(%i7) `ξ:[u,v]$`

(%i8) `ldisplay(iφ:map('rhs,linsolve(map("=",ξ,φ),ζ)))$`

$$i\varphi = \left[\frac{v+u}{2}, \frac{u-v}{2}\right] \tag{\%t8}$$

$\omega \in \mathcal{A}^2(\mathbb{R}^2), \quad \omega = f(x,y)\,\mathrm{d}x\wedge\mathrm{d}y$

(%i9) `ldisplay(ω:f(x,y)*(dx~dy))$`

$$\omega = x\ dx\ dy \tag{\%t9}$$

Pullback $i\varphi^*\omega \in \mathcal{A}^2(\mathbb{R}^2)$

(%i10) `ldisplay(Pb:diff(iφ,v)|(diff(iφ,u)|ev(ω,map("=",ζ,iφ))))$`

$$Pb = -\frac{v}{4} - \frac{u}{4} \tag{\%t10}$$

(%i11) `I:'integrate('integrate(Pb,u,-v,v),v,0,2)$`

(%i12) `ldisplay(I=ev(I,integrate))$`

$$\int_{0}^{2}\int_{-v}^{v} -\frac{v}{4} - \frac{u}{4}\,\mathrm{d}u\,\mathrm{d}v = -\frac{4}{3} \tag{\%t12}$$

$\omega \in \mathcal{A}^2(\mathbb{R}^2), \quad \omega = y^2\,\mathrm{d}x + x\,\mathrm{d}y$

(%i13) `ldisplay(ω:y^2*dx+x*dy)$`

$$\omega = x\ dy + y^2\ dx \tag{\%t13}$$

Parametrization $i\varphi : \mathbb{R}^2 \to \mathbb{R}$

(%i14) `ldisplay(iφ:[5*t-5,5*t-3])$`

$$i\varphi = [5t-5, 5t-3] \tag{\%t14}$$

$T^*\varphi^{-1}$

(%i15) `ldisplay(iφ\':diff(iφ,t))$`

$$i\varphi' = [5, 5] \tag{\%t15}$$

Pullback $i\varphi^*\omega \in \mathcal{A}^1(\mathbb{R}^2)$

(%i16) `ldisplay(Pb:iφ\'|ev(ω,map("=",ζ,iφ)))$`

$$Pb = 125t^2 - 125t + 20 \tag{\%t16}$$

(%i17) `I:'integrate(Pb,t,0,1)$`

(%i18) `ldisplay(I=ev(I,integrate))$`

$$\int_0^1 125t^2 - 125t + 20dt = -\frac{5}{6} \tag{\%t18}$$

Define the space

(%i19) `ζ:[x,y,z]$`

(%i20) `init_cartan(ζ)$`

$\omega \in \mathcal{A}^2(\mathbb{R}^3), \quad \omega = z^2\,\mathrm{d}x \wedge \mathrm{d}y$

(%i21) `ldisplay(ω:z^2*(dx~dy))$`

$$\omega = z^2\ dx\ dy \tag{\%t21}$$

Parametrization $i\varphi : \mathbb{R}^3 \to \mathbb{R}^2$

(%i22) `kill(r,θ,ξ)$`

(%i23) `ξ:[r,θ]$`

(%i24) `ldisplay(iφ:[r*cos(θ),r*sin(θ),√(1-r^2)])$`

$$i\varphi = [r\ \cos(\theta), r\ \sin(\theta), \sqrt{1-r^2}] \tag{\%t24}$$

$T^*\varphi^{-1}$

(%i25) `J:jacobian(i`φ`,`ξ`);`

$$\begin{pmatrix} \cos(\theta) & -r\,\sin(\theta) \\ \sin(\theta) & r\,\cos(\theta) \\ -\frac{r}{\sqrt{1-r^2}} & 0 \end{pmatrix} \tag{J}$$

Pullback $i\varphi^*\omega \in \mathcal{A}^2(\mathbb{R}^3)$

(%i26) `ldisplay(Pb:trigsimp(diff(i`φ`,`θ`)|(diff(i`φ`,r)|ev(`ω`,map("=",`ζ`,i`φ`)))))$`

$$Pb = r - r^3 \tag{\%t26}$$

(%i27) `I:'integrate('integrate(Pb,r,0,1),`θ`,0,2*`π`)$`

(%i28) `ldisplay(I=ev(I,integrate))$`

$$2\pi \int_0^1 r - r^3\,\mathrm{d}r = \frac{\pi}{2} \tag{\%t28}$$