Inverted Pendulum

Based on Prof. Soumitro Banerjee Using the Lagrangian Equation to Obtain Differential Equations

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(%i2)
```
info:build_info()$info@version;
```

(%o2)

5.38.1

(%i2)
```
reset()$kill(all)$
```
(%i1)
```
derivabbrev:true$
```
(%i2)
```
ratprint:false$
```
(%i3)
```
fpprintprec:5$
```
(%i4)
```
if get('draw,'version)=false then load(draw)$
```
(%i5)
```
wxplot_size:[1024,768]$
```
(%i6)
```
if get('optvar,'version)=false then load(optvar)$
```
(%i7)
```
if get('rkf45,'version)=false then load(rkf45)$
```
(%i8)
```
declare(trigsimp,evfun)$
```
(%i9)
```
declare(t,mainvar)$
```

1 Settings

(%i10) `orderless(m,g,l)$`

(%i11) `declare([m_1,m_2,g,l],constant)$`

(%i12) `assume(m_1>0,m_2>0,g>0,l>0)$`

(%i13) `params:[m_1=1,m_2=1,g=9.8,l=1]$`

(%i14) `τ:1$`

Generalized coordinates

(%i15) `ζ:[x,θ]$`

(%i16) `depends(ζ,t)$`

(%i17) `dim:length(ζ)$`

2 Lagrangian Formalism

Kinetic Energy

(%i18) `ldisplay(T:`$\frac{1}{2}$`*m_1*diff(x,t)`^{2}`+`$\frac{1}{2}$`*m_2*(l*diff(`θ`,t)*cos(`θ`)+diff(x,t))`^{2}`+`$\frac{1}{2}$`*m_2*(l*diff(`θ`,t)*sin(`θ`))`^{2}`)$`

$$T = \frac{m_2\left(l\,\cos(\theta)\,\left(\dot{\theta}\right)+\dot{x}\right)^2}{2} + \frac{l^2\,m_2\sin(\theta)^2\,\left(\dot{\theta}\right)^2}{2} + \frac{m_1(\dot{x})^2}{2} \qquad (\%t18)$$

Potential Energy

(%i19) `ldisplay(V:m_2*g*l*cos(`θ`)-F*x)$;`

$$V = g l\, m_2 \cos(\theta) - F x \qquad (\%t19)$$

Lagrangian

(%i20) `ldisplay(L:expand(trigsimp(T-V)))$`

$$L = \frac{l^2\,m_2\left(\dot{\theta}\right)^2}{2} + l\,m_2\,(\dot{x})\,\cos(\theta)\,\left(\dot{\theta}\right) - g l\,m_2\cos(\theta) + \frac{m_2(\dot{x})^2}{2} + \frac{m_1(\dot{x})^2}{2} + F x \qquad (\%t20)$$

Momentum Conjugate

(%i21) `ldisplay(P_x:diff(L,'diff(x,t)))$`

$$P_x = l\,m_2\cos(\theta)\,\left(\dot{\theta}\right) + m_2\,(\dot{x}) + m_1\,(\dot{x}) \qquad (\%t21)$$

(%i22) `linsolve(p_x=P_x,diff(x,t));`

$$\left[\dot{x} = -\frac{l\,m_2\cos(\theta)\,\left(\dot{\theta}\right) - p_x}{m_2+m_1}\right] \qquad (\%o22)$$

(%i23) `ldisplay(P_`θ`:diff(L,'diff(`θ`,t)))$`

$$P_\theta = l^2\,m_2\,\left(\dot{\theta}\right) + l\,m_2\,(\dot{x})\,\cos(\theta) \qquad (\%t23)$$

(%i24) `linsolve(p_`θ`=P_`θ`,diff(`θ`,t));`

$$\left[\dot{\theta} = -\frac{l\,m_2\,(\dot{x})\,\cos(\theta) - p_\theta}{l^2\,m_2}\right] \qquad (\%o24)$$

Generalized Forces

(%i25) `ldisplay(F_x:diff(L,x))$`

$$F_x = F \qquad (\%t25)$$

(%i26) `ldisplay(F_θ:factor(diff(L,θ)))$`

$$F_\theta = l\, m_2 \sin(\theta) \left(g - (\dot{x}) \left(\dot{\theta}\right)\right) \tag{\%t26}$$

Euler-Lagrange Equation

(%i27) `aa:el(L,ζ,t)$`

(%i30) `bb:ev(aa,eval,diff)$`

(%i31) `bb[1]:subst([k[0]=-E],-bb[1])$`

Conservation Laws

(%i32) `expand(trigsimp(bb[1]));`

$$\frac{l^2\, m_2 \left(\dot{\theta}\right)^2}{2} + l\, m_2\, (\dot{x})\, \cos(\theta)\, \left(\dot{\theta}\right) + g l\, m_2 \cos(\theta) + \frac{m_2 (\dot{x})^2}{2} + \frac{m_1 (\dot{x})^2}{2} - F x = E \tag{\%o32}$$

Equations of Motion

(%i33) `map(ldisp,part(bb,[2,3]))$`

$$l\, m_2 \cos(\theta)\, \left(\ddot{\theta}\right) - l\, m_2 \sin(\theta)\, \left(\dot{\theta}\right)^2 + m_2\, (\ddot{x}) + m_1\, (\ddot{x}) = F \tag{\%t33}$$

$$l^2\, m_2\, \left(\ddot{\theta}\right) - l\, m_2\, (\dot{x})\, \sin(\theta)\, \left(\dot{\theta}\right) + l\, m_2\, (\ddot{x})\, \cos(\theta) = g l\, m_2 \sin(\theta) - l\, m_2\, (\dot{x})\, \sin(\theta)\, \left(\dot{\theta}\right) \tag{\%t34}$$

Solve for second derivative of coordinates

(%i35) `linsol:linsolve(part(bb,[2,3]),diff(ζ,t,2))$`

(%i36) `map(ldisp,linsol)$`

$$\ddot{x} = -\frac{l\, m_2 \sin(\theta)\, \left(\dot{\theta}\right)^2 - g\, m_2 \cos(\theta)\, \sin(\theta) + F}{m_2 \cos(\theta)^2 - m_2 - m_1} \tag{\%t36}$$

$$\ddot{\theta} = \frac{l\, m_2 \cos(\theta)\, \sin(\theta)\, \left(\dot{\theta}\right)^2 + (-g\, m_2 - g\, m_1) \sin(\theta) + F\, \cos(\theta)}{l\, m_2 \cos(\theta)^2 - l\, m_2 - l\, m_1} \tag{\%t37}$$

3 Hamiltonian Formalism

Legendre Transformation

(%i38) `Legendre:linsolve([p_x=P_x,p_θ=P_θ],['diff(x,t),'diff(θ,t)])$`

(%i39) `map(ldisp,Legendre)$`

$$\dot{x} = \frac{p_\theta \cos(\theta) - l\, p_x}{l\, m_2 \cos(\theta)^2 - l\, m_2 - l\, m_1} \tag{\%t39}$$

$$\dot{\theta} = \frac{l\, m_2 p_x \cos(\theta) - m_2 p_\theta - m_1 p_\theta}{l^2\, m_2^2 \cos(\theta)^2 - l^2\, m_2^2 - l^2\, m_1 m_2} \tag{\%t40}$$

Hamiltonian

(%i41) `ldisplay(H:ev(p_x*'diff(x,t)+p_θ*'diff(θ,t)-L,Legendre,expand,factor))$`

$H = (2g\, l^3\, m_2^3 \cos(\theta)^3 - 2l^2\, m_2^2 F x \cos(\theta)^2 + 2l\, m_2 p_x p_\theta \cos(\theta) - 2g\, l^3\, m_2^3 \cos(\theta) - 2g\, l^3\, m_1 m_2^2 \cos(\theta) +$
$2l^2\, m_2^2 F x + 2l^2\, m_1 m_2 F x - m_2 p_\theta^2 - m_1 p_\theta^2 - l^2\, m_2 p_x^2)/(2l^2\, m_2 \left(m_2 \cos(\theta)^2 - m_2 - m_1\right))$

Equations of Motion

(%i42) `Hq:makelist(Hq[i],i,1,4)$`

(%i46)
```
Hq[1]:'diff(x,t)=diff(H,p_x)$
Hq[2]:'diff(θ,t)=diff(H,p_θ)$
Hq[3]:'diff(p_x,t)=-diff(H,x)$
Hq[4]:'diff(p_θ,t)=-diff(H,θ)$
```

(%i47) `map(ldisp,Hq)$`

$$\dot{x} = \frac{2l\, m_2 p_\theta \cos(\theta) - 2l^2\, m_2 p_x}{2l^2\, m_2 \left(m_2 \cos(\theta)^2 - m_2 - m_1\right)} \tag{\%t47}$$

$$\dot{\theta} = \frac{2l\, m_2 p_x \cos(\theta) - 2m_2 p_\theta - 2m_1 p_\theta}{2l^2\, m_2 \left(m_2 \cos(\theta)^2 - m_2 - m_1\right)} \tag{\%t48}$$

$$\dot{p}_x = -\frac{-2l^2\, m_2^2 F \cos(\theta)^2 + 2l^2\, m_2^2 F + 2l^2\, m_1 m_2 F}{2l^2\, m_2 \left(m_2 \cos(\theta)^2 - m_2 - m_1\right)} \tag{\%t49}$$

$\dot{p}_\theta = -(-6g\, l^3\, m_2^3 \cos(\theta)^2 \sin(\theta) + 4l^2\, m_2^2 F x \cos(\theta) \sin(\theta) - 2l\, m_2 p_x p_\theta \sin(\theta) + 2g\, l^3\, m_2^3 \sin(\theta) + 2g\, l^3\, m_1 m_2^2 \sin(\theta))/(2l^2\, m_2$
$(\cos(\theta)(2g\, l^3\, m_2^3 \cos(\theta)^3 - 2l^2\, m_2^2 F x \cos(\theta)^2 + 2l\, m_2 p_x p_\theta \cos(\theta) - 2g\, l^3\, m_2^3 \cos(\theta) - 2g\, l^3\, m_1 m_2^2 \cos(\theta) +$
$2l^2\, m_2^2 F x + 2l^2\, m_1 m_2 F x - m_2 p_\theta^2 - m_1 p_\theta^2 - l^2\, m_2 p_x^2) \sin(\theta))/(l^2 \left(m_2 \cos(\theta)^2 - m_2 - m_1\right)^2)$

4 Reduce Order

(%i52) `ξ:[X,Θ]$`
`depends(ξ,t)$`

(%i54) `gradef(x,t,X)$`
`gradef(θ,t,Θ)$`

Euler-Lagrange Equations

(%i55) `aa:el(L,ζ,t)$`

(%i58) `bb:ev(aa,eval,diff)$`

(%i59) `bb[1]:subst([k[0]=-E],-bb[1])$`

Conservation Laws

(%i60) `bb[1];`

$X\,\left(l\,m_2\,\Theta\,\cos\left(\theta\right)+m_2\,X+m_1\,X\right)+\Theta\,\left(l\,m_2\,X\,\cos\left(\theta\right)+l^2\,m_2\,\Theta\right)-l\,m_2\,X\,\Theta\,\cos\left(\theta\right)+g\,l\,m_2\,\cos\left(\theta\right)-\frac{l^2\,m_2\,\Theta^2}{2}-F\,x-\frac{m_2\,X^2}{2}-\frac{m_1\,X^2}{2}=E$

Equations of Motion

(%i61) `map(ldisp,part(bb,[2,3]))$`

$$-l\,m_2\,\Theta^2\,\sin\left(\theta\right)+l\,m_2\,\left(\dot{\Theta}\right)\,\cos\left(\theta\right)+m_2\,\left(\dot{X}\right)+m_1\,\left(\dot{X}\right)=F \tag{\%t61}$$

$$-l\,m_2\,X\,\Theta\,\sin\left(\theta\right)+l\,m_2\,\left(\dot{X}\right)\,\cos\left(\theta\right)+l^2\,m_2\,\left(\dot{\Theta}\right)=g\,l\,m_2\,\sin\left(\theta\right)-l\,m_2\,X\,\Theta\,\sin\left(\theta\right) \tag{\%t62}$$

Solve for second derivative of coordinates

(%i63) `linsol:linsolve(part(bb,[2,3]),diff(ζ,t,2))$`

(%i64) `map(ldisp,linsol)$`

$$\dot{X}=\frac{\left(g\,m_2\,\cos\left(\theta\right)-l\,m_2\,\Theta^2\right)\,\sin\left(\theta\right)-F}{m_2\,\cos\left(\theta\right)^2-m_2-m_1} \tag{\%t64}$$

$$\dot{\Theta}=\frac{\left(l\,m_2\,\Theta^2\,\cos\left(\theta\right)-g\,m_2-g\,m_1\right)\,\sin\left(\theta\right)+F\,\cos\left(\theta\right)}{l\,m_2\,\cos\left(\theta\right)^2-l\,m_2-l\,m_1} \tag{\%t65}$$