8.6: Routhian Reduction

Based on LibreTexts Article 8.6: Routhian Reduction

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(%i2)
```
info:build_info()$info@version;
```

(%o2)

5.38.1

(%i2)
```
reset()$kill(all)$
```

(%i1)
```
derivabbrev:true$
```

(%i2)
```
ratprint:false$
```

(%i3)
```
fpprintprec:5$
```

(%i4)
```
if get('draw,'version)=false then load(draw)$
```

(%i5)
```
wxplot_size:[1024,768]$
```

(%i6)
```
if get('optvar,'version)=false then load(optvar)$
```

(%i7)
```
if get('rkf45,'version)=false then load(rkf45)$
```

(%i8)
```
declare(t,mainvar)$
```

(%i9)
```
declare(trigsimp,evfun)$
```

1 Example 8.6.1: Spherical pendulum

Using Hamiltonian mechanics

(%i10) `kill(labels,m,g,b)$`

(%i1) `unorder()$`

(%i2) `orderless(m,g,b)$`

(%i3) `declare([m,g,b],constant)$`

(%i4) `assume(m>0,g>0,b>0)$`

(%i5) `params:[m=1,g=9.8,b=5]$`

(%i6) `τ:60$`

Generalized coordinates

(%i7) `ξ:[θ,ϕ]$`

(%i8) `depends(ξ,t)$`

(%i9) `dim:length(ξ)$`

Kinetic energy

(%i10) `ldisplay(T:`$\frac{1}{2}$`*m*b`^{2}`*diff(θ,t)`^{2}`+`$\frac{1}{2}$`*m*b`^{2}`*sin(θ)`^{2}`*diff(ϕ,t)`^{2}`)$`

$$T = \frac{m\,b^2 \sin(\theta)^2 \left(\dot{\phi}\right)^2}{2} + \frac{m\,b^2 \left(\dot{\theta}\right)^2}{2} \tag{\%t10}$$

Potential energy

(%i11) `ldisplay(U:-m*g*b*cos(θ))$`

$$U = -mgb\,\cos(\theta) \tag{\%t11}$$

Lagrangian

(%i12) `ldisplay(L:T-U)$`

$$L = \frac{m\,b^2 \sin(\theta)^2 \left(\dot{\phi}\right)^2}{2} + \frac{m\,b^2 \left(\dot{\theta}\right)^2}{2} + mgb\,\cos(\theta) \tag{\%t12}$$

Generalized momenta

(%i13) `ldisplay(P_θ:ev(diff(L,'diff(θ,t))))$`

$$P_\theta = m\,b^2 \left(\dot{\theta}\right) \tag{\%t13}$$

(%i14) `linsolve(p_θ=P_θ,diff(θ,t));`

$$\left[\dot{\theta} = \frac{p_\theta}{m\,b^2}\right] \tag{\%o14}$$

(%i15) `ldisplay(P_ϕ:ev(diff(L,'diff(ϕ,t))))$`

$$P_\phi = m\,b^2 \sin(\theta)^2 \left(\dot{\phi}\right) \tag{\%t15}$$

(%i16) `linsolve(p_ϕ=P_ϕ,diff(ϕ,t));`

$$\left[\dot{\phi} = \frac{p_\phi}{m\,b^2 \sin(\theta)^2}\right] \tag{\%o16}$$

Generalized Forces

(%i17) `ldisplay(F_θ:diff(L,θ))$`

$$F_\theta = m\,b^2\, \cos(\theta)\, \sin(\theta) \left(\dot{\phi}\right)^2 - mgb\, \sin(\theta) \tag{\%t17}$$

(%i18) `ldisplay(F_ϕ:diff(L,ϕ))$`

$$F_\phi = 0 \tag{\%t18}$$

Euler-Lagrange Equation

(%i19) `aa:el(L,ξ,t)$`

(%i23) `bb:ev(aa,eval,diff)$`

(%i24) `declare([E,J],constant)$`

(%i26)
```
bb[1]:subst([k[0]=-E],-bb[1])$
bb[4]:subst([k[2]=J],bb[4])$
```

(%i30)
```
bb[1]:rhs(bb[1])=lhs(bb[1])$
bb[2]:lhs(bb[2])-rhs(bb[2])=0$
bb[3]:lhs(bb[3])-rhs(bb[3])=0$
bb[4]:rhs(bb[4])=lhs(bb[4])$
```

Conservation Laws

(%i31) `map(ldisp,part(bb,[1,4])),expand$`

$$E = \frac{m\,b^2 \sin(\theta)^2 \left(\dot{\phi}\right)^2}{2} + \frac{m\,b^2 \left(\dot{\theta}\right)^2}{2} - mgb\, \cos(\theta) \tag{\%t31}$$

$$J = m\,b^2 \sin(\theta)^2 \left(\dot{\phi}\right) \tag{\%t32}$$

Energy as a function of Angular momentum

(%i33) `expand(solve(eliminate(part(bb,[1,4]),[diff(ϕ,t)]),E));`

$$\left[E = \frac{m\,b^2 \left(\dot{\theta}\right)^2}{2} + \frac{J^2}{2m\,b^2 \sin(\theta)^2} - mgb\, \cos(\theta)\right] \tag{\%o33}$$

Equations of Motion

(%i34) `map(ldisp,part(bb,[2,3])),trigsimp$`

$$-m\,b^2\,\cos(\theta)\,\sin(\theta)\,\left(\dot{\phi}\right)^2+m\,b^2\,\left(\ddot{\theta}\right)+mgb\,\sin(\theta)=0 \tag{\%t34}$$

$$m\,b^2\sin(\theta)^2\,\left(\ddot{\phi}\right)+2m\,b^2\,\cos(\theta)\,\sin(\theta)\,\left(\dot{\theta}\right)\,\left(\dot{\phi}\right)=0 \tag{\%t35}$$

Solve for second derivative of coordinates

(%i36) `linsol:linsolve(part(bb,[2,3]),diff(ξ,t,2))$`

(%i37) `map(ldisp,expand(linsol))$`

$$\ddot{\theta}=\cos(\theta)\,\sin(\theta)\,\left(\dot{\phi}\right)^2-\frac{g\,\sin(\theta)}{b} \tag{\%t37}$$

$$\ddot{\phi}=-\frac{2\cos(\theta)\,\left(\dot{\theta}\right)\,\left(\dot{\phi}\right)}{\sin(\theta)} \tag{\%t38}$$

Check Conservation of Energy

(%i39) `bb[1];`

$$E=\frac{m\,b^2\sin(\theta)^2\,\left(\dot{\phi}\right)^2}{2}+\frac{m\,b^2\,\left(\dot{\theta}\right)^2}{2}-mgb\,\cos(\theta) \tag{\%o39}$$

(%i40) `subst(linsol,diff(rhs(bb[1]),t)),fullratsimp;`

$$0 \tag{\%o40}$$

Check Conservation of Angular Momentum

(%i41) `bb[4];`

$$J=m\,b^2\sin(\theta)^2\,\left(\dot{\phi}\right) \tag{\%o41}$$

(%i42) `subst(linsol,diff(rhs(bb[4]),t)),fullratsimp;`

$$0 \tag{\%o42}$$

Legendre Transformation

(%i43) `kill(labels)$`

(%i1) `Legendre:linsolve([p_θ=P_θ,p_ϕ=P_ϕ],['diff(θ,t),'diff(ϕ,t)])$`

(%i2) `map(ldisp,Legendre)$`

$$\dot{\theta}=\frac{p_\theta}{m\,b^2} \tag{\%t2}$$

$$\dot{\phi} = \frac{p_\phi}{m\,b^2\sin(\theta)^2} \tag{\%t3}$$

Hamiltonian

(%i4) `ldisplay(H:ev(p_θ*'diff(θ,t)+p_φ*'diff(φ,t)-L,Legendre,expand))$`

$$H = \frac{{p_\phi}^2}{2m\,b^2\sin(\theta)^2} - mgb\,\cos(\theta) + \frac{{p_\theta}^2}{2m\,b^2} \tag{\%t4}$$

Equations of Motion

(%i5) `Hq:makelist(Hq[i],i,1,2*dim)$`

(%i9)
```
Hq[1]:'diff(θ,t)=diff(H,p_θ)$
Hq[2]:'diff(φ,t)=diff(H,p_φ)$
Hq[3]:'diff(p_θ,t)=-diff(H,θ)$
Hq[4]:'diff(p_φ,t)=-diff(H,φ)$
```

(%i10) `map(ldisp,Hq:expand(Hq))$`

$$\dot{\theta} = \frac{p_\theta}{m\,b^2} \tag{\%t10}$$

$$\dot{\phi} = \frac{p_\phi}{m\,b^2\sin(\theta)^2} \tag{\%t11}$$

$$\dot{p}_\theta = \frac{{p_\phi}^2\cos(\theta)}{m\,b^2\sin(\theta)^3} - mgb\,\sin(\theta) \tag{\%t12}$$

$$\dot{p}_\phi = 0 \tag{\%t13}$$

Analytical solution of ϕ, p_ϕ (Depends on θ)

(%i16)
```
atvalue(φ(t),[t=0],φ_0)$
atvalue(p_φ(t),[t=0],J)$
desol:desolve(convert(part(Hq,[2,4]),[φ,p_φ],t), convert([φ,p_φ],[φ,p_φ],t));
```

$$\left[\phi(t) = \frac{J\,t}{m\,b^2\sin(\theta)^2} + \phi_0, p_\phi(t) = J\right] \tag{desol}$$

Check Conservation of Energy

(%i17) `depends([p_θ,p_φ],t)$`

(%i18) `subst(Hq,diff(H,t)),fullratsimp;`

$$0 \tag{\%o18}$$

Routhian Transformation

(%i19) `ldisplay(Routhian:linsolve(bb[4],'diff(φ,t)))$`

$$Routhian = \left[\dot{\phi} = \frac{J}{m\,b^2\sin(\theta)^2}\right] \tag{\%t19}$$

(%i20) `ldisplay(R:ev(L-p_ϕ*'diff(ϕ,t),[J=p_ϕ],Routhian,expand))$`

$$R = \frac{m\,b^2 \left(\dot{\theta}\right)^2}{2} - \frac{{p_\phi}^2}{2m\,b^2 \sin\left(\theta\right)^2} + mgb\,\cos\left(\theta\right) \tag{\%t20}$$

Generalized momenta

(%i21) `ldisplay(P_θ:ev(diff(R,'diff(θ,t))))$`

$$P_\theta = m\,b^2 \left(\dot{\theta}\right) \tag{\%t21}$$

(%i22) `linsolve(p_θ=P_θ,diff(θ,t)),factor;`

$$\left[\dot{\theta} = \frac{p_\theta}{m\,b^2}\right] \tag{\%o22}$$

Generalized Forces

(%i23) `ldisplay(F_θ:expand(diff(R,θ)))$`

$$F_\theta = \frac{{p_\phi}^2 \cos\left(\theta\right)}{m\,b^2 \sin\left(\theta\right)^3} - mgb\,\sin\left(\theta\right) \tag{\%t23}$$

Euler-Lagrange Equations

(%i24) `aa:el(R,θ,t)$`

(%i26) `cc:ev(aa,eval,diff)$`

(%i27) `cc[1]:subst([k[0]=-E],-cc[1])$`

(%i29) `cc[1]:rhs(cc[1])=lhs(cc[1])$`
`cc[2]:lhs(cc[2])-rhs(cc[2])=0$`

Conservation Laws

(%i30) `cc[1];`

$$E = \frac{m\,b^2 \left(\dot{\theta}\right)^2}{2} + \frac{{p_\phi}^2}{2m\,b^2 \sin\left(\theta\right)^2} - mgb\,\cos\left(\theta\right) \tag{\%o30}$$

Equations of Motion

(%i31) `cc[2];`

$$m\,b^2 \left(\ddot{\theta}\right) + mgb\,\sin\left(\theta\right) - \frac{{p_\phi}^2 \cos\left(\theta\right)}{m\,b^2 \sin\left(\theta\right)^3} = 0 \tag{\%o31}$$

Solve for second derivative of coordinates

(%i32) `linsol:linsolve(cc[2],diff(θ,t,2))$`

(%i33) `map(ldisp,expand(linsol))$`

$$\ddot{\theta} = \frac{{p_\phi}^2 \cos(\theta)}{m^2\, b^4 \sin(\theta)^3} - \frac{g\,\sin(\theta)}{b} \tag{\%t33}$$

(%i34) `subst(bb[4],linsol),expand;`

$$\left[\ddot{\theta} = \frac{{p_\phi}^2 \cos(\theta)}{m^2\, b^4 \sin(\theta)^3} - \frac{g\,\sin(\theta)}{b}\right] \tag{\%o34}$$

Check Conservation of Energy

(%i35) `cc[1];`

$$E = \frac{m\, b^2 \left(\dot{\theta}\right)^2}{2} + \frac{{p_\phi}^2}{2m\, b^2 \sin(\theta)^2} - mgb\,\cos(\theta) \tag{\%o35}$$

(%i36) `subst(linsol,diff(rhs(cc[1]),t)),fullratsimp;`

$$\frac{p_\phi\, (\dot{p}_\phi)}{m\, b^2 \sin(\theta)^2} \tag{\%o36}$$

Equations of Motion (I don't know what this is!)

(%i37) `Bq:makelist(Bq[i],i,1,2)$`

(%i39) `Bq[1]:'diff(θ,t)=diff(-R,p_θ)$`
`Bq[2]:'diff(p_θ,t)=-diff(-R,θ)$`

(%i40) `map(ldisp,Bq:expand(Bq))$`

$$\dot{\theta} = 0 \tag{\%t40}$$

$$\dot{p}_\theta = \frac{{p_\phi}^2 \cos(\theta)}{m\, b^2 \sin(\theta)^3} - mgb\,\sin(\theta) \tag{\%t41}$$

Legendre Transformation

(%i42) `kill(labels)$`

(%i1) `Legendre:linsolve([p_θ=P_θ],['diff(θ,t)])$`

(%i2) `map(ldisp,Legendre)$`

$$\dot{\theta} = \frac{p_\theta}{m\, b^2} \tag{\%t2}$$

Hamiltonian

(%i3) `ldisplay(H:ev(p_θ*'diff(θ,t)-L,Legendre,expand))$`

$$H = -\frac{m\, b^2 \sin(\theta)^2 \left(\dot{\phi}\right)^2}{2} - mgb\,\cos(\theta) + \frac{{p_\theta}^2}{2m\, b^2} \tag{\%t3}$$

Equations of Motion

(%i4) `Rq:makelist(Rq[i],i,1,2)$`

(%i6) `Rq[1]:'diff(θ,t)=diff(H,p_θ)$`
`Rq[2]:'diff(p_θ,t)=-diff(H,θ)$`

(%i7) `map(ldisp,Rq:expand(Rq))$`

$$\dot{\theta} = \frac{p_\theta}{m\,b^2} \tag{\%t7}$$

$$\dot{p}_\theta = m\,b^2\,\cos(\theta)\,\sin(\theta)\left(\dot{\phi}\right)^2 - mgb\,\sin(\theta) \tag{\%t8}$$

Check Conservation of Energy

(%i9) `subst(Hq,diff(H,t)),fullratsimp;`

$$-p_\phi\left(\ddot{\phi}\right) \tag{\%o9}$$

(%i10) `subst(Rq,diff(H,t)),fullratsimp;`

$$-m\,b^2\sin(\theta)^2\left(\dot{\phi}\right)\left(\ddot{\phi}\right) \tag{\%o10}$$

Compare

θ, p_θ

(%i11) `part(Hq,[1,3]),expand;`

$$\left[\dot{\theta} = \frac{p_\theta}{m\,b^2}, \dot{p}_\theta = \frac{{p_\phi}^2\cos(\theta)}{m\,b^2\sin(\theta)^3} - mgb\,\sin(\theta)\right] \tag{\%o11}$$

(%i12) `part(Rq,[1,2]),expand;`

$$\left[\dot{\theta} = \frac{p_\theta}{m\,b^2}, \dot{p}_\theta = m\,b^2\,\cos(\theta)\,\sin(\theta)\left(\dot{\phi}\right)^2 - mgb\,\sin(\theta)\right] \tag{\%o12}$$

(%i13) `subst(part(Hq,[2,4]),%),expand;`

$$\left[\dot{\theta} = \frac{p_\theta}{m\,b^2}, \dot{p}_\theta = \frac{{p_\phi}^2\cos(\theta)}{m\,b^2\sin(\theta)^3} - mgb\,\sin(\theta)\right] \tag{\%o13}$$

(%i14) `is(%=%th(3));`

$$\text{true} \tag{\%o14}$$

2 Example 8.6.4: Single particle

moving in a vertical plane under the influence of an inverse-square central force

(%i15) `kill(labels,m,K)$`

(%i1) `unorder()$`

(%i2) `orderless(m,K)$`

(%i3) `declare([m,K],constant)$`

(%i4) `assume(m>0,K>0)$`

(%i5) `params:[m=1,K=0.5]$`

(%i6) `τ:60$`

Generalized coordinates

(%i7) `ξ:[r,θ]$`

(%i8) `depends(ξ,t)$`

(%i9) `dim:length(ξ)$`

Lagrangian

(%i10) `ldisplay(L:`$\frac{1}{2}$`*m*(diff(r,t)`^{2}`+r`^{2}`*diff(θ,t)`^{2}`)+K/r)$`

$$L = \frac{m\left(r^2\left(\dot{\theta}\right)^2 + (\dot{r})^2\right)}{2} + \frac{K}{r} \qquad (\%t10)$$

Generalized momenta

(%i11) `ldisplay(P_r:ev(diff(L,'diff(r,t))))$`

$$P_r = m\,(\dot{r}) \qquad (\%t11)$$

(%i12) `linsolve(p_r=P_r,diff(r,t));`

$$\left[\dot{r} = \frac{p_r}{m}\right] \qquad (\%o12)$$

(%i13) `ldisplay(P_θ:ev(diff(L,'diff(θ,t))))$`

$$P_\theta = m\,r^2\left(\dot{\theta}\right) \qquad (\%t13)$$

(%i14) `linsolve(p_θ=P_θ,diff(θ,t));`

$$\left[\dot{\theta} = \frac{p_\theta}{m\,r^2}\right] \qquad (\%o14)$$

Generalized Forces

(%i15) `ldisplay(F_r:diff(L,r))$`

$$F_r = mr\left(\dot{\theta}\right)^2 - \frac{K}{r^2} \tag{\%t15}$$

(%i16) `ldisplay(F_θ:diff(L,θ))$`

$$F_\theta = 0 \tag{\%t16}$$

Euler-Lagrange Equation

(%i17) `aa:el(L,ξ,t)$`

(%i21) `bb:ev(aa,eval,diff)$`

(%i22) `declare([E,J],constant)$`

(%i24)
```
bb[1]:subst([k[0]=-E],-bb[1])$
bb[4]:subst([k[2]=J],bb[4])$
```

(%i28)
```
bb[1]:rhs(bb[1])=lhs(bb[1])$
bb[2]:lhs(bb[2])-rhs(bb[2])=0$
bb[3]:lhs(bb[3])-rhs(bb[3])=0
bb[4]:rhs(bb[4])=lhs(bb[4])$
```

Conservation Laws

(%i29) `map(ldisp,part(bb,[1,4])),expand$`

$$E = \frac{m\,r^2\left(\dot{\theta}\right)^2}{2} + \frac{m\left(\dot{r}\right)^2}{2} - \frac{K}{r} \tag{\%t29}$$

$$J = m\,r^2\left(\dot{\theta}\right) \tag{\%t30}$$

Energy as a function of Angular momentum

(%i31) `expand(solve(eliminate(part(bb,[1,4]),[diff(θ,t)]),E));`

$$\left[E = \frac{m\left(\dot{r}\right)^2}{2} - \frac{K}{r} + \frac{J^2}{2m\,r^2}\right] \tag{\%o31}$$

Equations of Motion

(%i32) `map(ldisp,part(bb,[2,3])),expand$`

$$-mr\left(\dot{\theta}\right)^2 + m\left(\ddot{r}\right) + \frac{K}{r^2} = 0 \tag{\%t32}$$

$$m\,r^2\left(\ddot{\theta}\right) + 2mr\left(\dot{r}\right)\left(\dot{\theta}\right) = 0 \tag{\%t33}$$

Solve for second derivative of coordinates

```
(%i34) linsol:linsolve(part(bb,[2,3]),diff(ξ,t,2))$
(%i35) map(ldisp,expand(linsol))$
```

$$\ddot{r} = r\left(\dot{\theta}\right)^2 - \frac{K}{m\,r^2} \tag{\%t35}$$

$$\ddot{\theta} = -\frac{2\left(\dot{r}\right)\left(\dot{\theta}\right)}{r} \tag{\%t36}$$

Check Conservation of Energy

```
(%i37) bb[1];
```

$$E = -\frac{m\left(r^2\left(\dot{\theta}\right)^2 + \left(\dot{r}\right)^2\right)}{2} + m\,r^2\left(\dot{\theta}\right)^2 + m\left(\dot{r}\right)^2 - \frac{K}{r} \tag{\%o37}$$

```
(%i38) subst(linsol,diff(rhs(bb[1]),t)),fullratsimp;
```

$$0 \tag{\%o38}$$

Check Conservation of Angular Momentum

```
(%i39) bb[4];
```

$$J = m\,r^2\left(\dot{\theta}\right) \tag{\%o39}$$

```
(%i40) subst(linsol,diff(rhs(bb[4]),t)),fullratsimp;
```

$$0 \tag{\%o40}$$

Legendre Transformation

```
(%i41) kill(labels)$
(%i1)  Legendre:linsolve([p_r=P_r,p_θ=P_θ],['diff(r,t),'diff(θ,t)])$
(%i2)  map(ldisp,Legendre)$
```

$$\dot{r} = \frac{p_r}{m} \tag{\%t2}$$

$$\dot{\theta} = \frac{p_\theta}{m\,r^2} \tag{\%t3}$$

Hamiltonian

```
(%i4)  ldisplay(H:ev(p_r*'diff(r,t)+p_θ*'diff(θ,t)-L,Legendre,expand))$
```

$$H = -\frac{K}{r} + \frac{{p_\theta}^2}{2m\,r^2} + \frac{{p_r}^2}{2m} \tag{\%t4}$$

Equations of Motion

```
(%i5) Hq:makelist(Hq[i],i,1,2*dim)$
(%i9) Hq[1]:'diff(r,t)=diff(H,p_r)$
      Hq[2]:'diff(θ,t)=diff(H,p_θ)$
      Hq[3]:'diff(p_r,t)=-diff(H,r)$
      Hq[4]:'diff(p_θ,t)=-diff(H,θ)$
(%i10) map(ldisp,Hq:expand(Hq))$
```

$$\dot{r} = \frac{p_r}{m} \quad (\%t10)$$

$$\dot{\theta} = \frac{p_\theta}{m\, r^2} \quad (\%t11)$$

$$\dot{p}_r = \frac{{p_\theta}^2}{m\, r^3} - \frac{K}{r^2} \quad (\%t12)$$

$$\dot{p}_\theta = 0 \quad (\%t13)$$

Analytical solution of ϕ, p_ϕ (Depends on r)

```
(%i16) atvalue(θ(t),[t=0],θ_0)$
       atvalue(p_θ(t),[t=0],J)$
       desol:desolve(convert(part(Hq,[2,4]),[θ,p_θ],t), convert([θ,p_θ],[θ,p_θ],t));
```

$$\left[\theta(t) = \frac{J t}{m\, r^2} + \theta_0, p_\theta(t) = J\right] \quad (\text{desol})$$

Check Conservation of Energy

```
(%i17) depends([p_r,p_θ],t)$
(%i18) subst(Hq,diff(H,t)),fullratsimp;
```

$$0 \quad (\%o18)$$

Routhian Transformation

```
(%i19) ldisplay(Routhian:linsolve(bb[4],'diff(θ,t)))$
```

$$Routhian = \left[\dot{\theta} = \frac{J}{m\, r^2}\right] \quad (\%t19)$$

```
(%i20) ldisplay(R:ev(L-p_θ*'diff(θ,t),[p_θ=J],Routhian,expand))$
```

$$R = \frac{m\,(\dot{r})^2}{2} + \frac{K}{r} - \frac{J^2}{2m\, r^2} \quad (\%t20)$$

Generalized momenta

```
(%i21) ldisplay(P_r:ev(diff(R,'diff(r,t))))$
```

$$P_r = m\,(\dot{r}) \quad (\%t21)$$

(%i22) `linsolve(p_r=P_r,diff(r,t)),factor;`

$$\left[\dot{r}=\frac{p_r}{m}\right] \quad (\%o22)$$

Generalized Forces

(%i23) `ldisplay(F_r:expand(diff(R,r)))$`

$$F_r=\frac{J^2}{m\,r^3}-\frac{K}{r^2} \quad (\%t23)$$

Euler-Lagrange Equations

(%i24) `aa:el(R,r,t)$`

(%i26) `cc:ev(aa,eval,diff)$`

(%i27) `cc[1]:subst([k[0]=-E],-cc[1])$`

(%i29) `cc[1]:rhs(cc[1])=lhs(cc[1])$`
`cc[2]:lhs(cc[2])-rhs(cc[2])=0$`

Conservation Laws

(%i30) `cc[1];`

$$E=\frac{m\,(\dot{r})^2}{2}-\frac{K}{r}+\frac{J^2}{2m\,r^2} \quad (\%o30)$$

Equations of Motion

(%i31) `cc[2];`

$$m\,(\ddot{r})+\frac{K}{r^2}-\frac{J^2}{m\,r^3}=0 \quad (\%o31)$$

Solve for second derivative of coordinates

(%i32) `linsol:linsolve(cc[2],diff(r,t,2))$`

(%i33) `map(ldisp,expand(linsol))$`

$$\ddot{r}=\frac{J^2}{m^2\,r^3}-\frac{K}{m\,r^2} \quad (\%t33)$$

(%i34) `subst(bb[4],linsol),expand;`

$$\left[\ddot{r}=r\left(\dot{\theta}\right)^2-\frac{K}{m\,r^2}\right] \quad (\%o34)$$

Check Conservation of Energy

(%i35) `cc[1];`

$$E = \frac{m\,(\dot{r})^2}{2} - \frac{K}{r} + \frac{J^2}{2m\,r^2} \quad (\%o35)$$

(%i36) `subst(linsol,diff(rhs(cc[1]),t)),fullratsimp;`

$$0 \quad (\%o36)$$

Equations of Motion (I don't know what this is!)

(%i37) `Bq:makelist(Bq[i],i,1,2)$`

(%i39) `Bq[1]:'diff(r,t)=diff(-R,p_r)$`
`Bq[2]:'diff(p_r,t)=-diff(-R,r)$`

(%i40) `map(ldisp,Bq:expand(Bq))$`

$$\dot{r} = 0 \quad (\%t40)$$

$$\dot{p}_r = \frac{J^2}{m\,r^3} - \frac{K}{r^2} \quad (\%t41)$$

Legendre Transformation

(%i42) `kill(labels)$`

(%i1) `Legendre:linsolve([p_r=P_r],['diff(r,t)])$`

(%i2) `map(ldisp,Legendre)$`

$$\dot{r} = \frac{p_r}{m} \quad (\%t2)$$

Hamiltonian

(%i3) `ldisplay(H:ev(p_r*'diff(r,t)-L,Legendre,expand))$`

$$H = -\frac{m\,r^2\left(\dot{\theta}\right)^2}{2} - \frac{K}{r} + \frac{{p_r}^2}{2m} \quad (\%t3)$$

Equations of Motion

(%i4) `Rq:makelist(Rq[i],i,1,2)$`

(%i6) `Rq[1]:'diff(r,t)=diff(H,p_r)$`
`Rq[2]:'diff(p_r,t)=-diff(H,r)$`

(%i7) `map(ldisp,Rq:expand(Rq))$`

$$\dot{r} = \frac{p_r}{m} \quad (\%t7)$$

$$\dot{p}_r = mr\left(\dot{\theta}\right)^2 - \frac{K}{r^2} \tag{\%t8}$$

Check Conservation of Energy

(%i9) `subst(Hq,diff(H,t)),fullratsimp;`

$$-p_\theta\left(\ddot{\theta}\right) \tag{\%o9}$$

(%i10) `subst(Rq,diff(H,t)),fullratsimp;`

$$-m\,r^2\left(\dot{\theta}\right)\left(\ddot{\theta}\right) \tag{\%o10}$$

Compare

θ, p_θ

(%i11) `part(Hq,[1,3]),expand;`

$$\left[\dot{r} = \frac{p_r}{m}, \dot{p}_r = \frac{{p_\theta}^2}{m\,r^3} - \frac{K}{r^2}\right] \tag{\%o11}$$

(%i12) `part(Rq,[1,2]),expand;`

$$\left[\dot{r} = \frac{p_r}{m}, \dot{p}_r = mr\left(\dot{\theta}\right)^2 - \frac{K}{r^2}\right] \tag{\%o12}$$

(%i13) `subst(part(Hq,[2,4]),%),expand;`

$$\left[\dot{r} = \frac{p_r}{m}, \dot{p}_r = \frac{{p_\theta}^2}{m\,r^3} - \frac{K}{r^2}\right] \tag{\%o13}$$

(%i14) `is(%=%th(3));`

$$\text{true} \tag{\%o14}$$