RATIONAL PARAMETRIZATION OF LEMNISCATES

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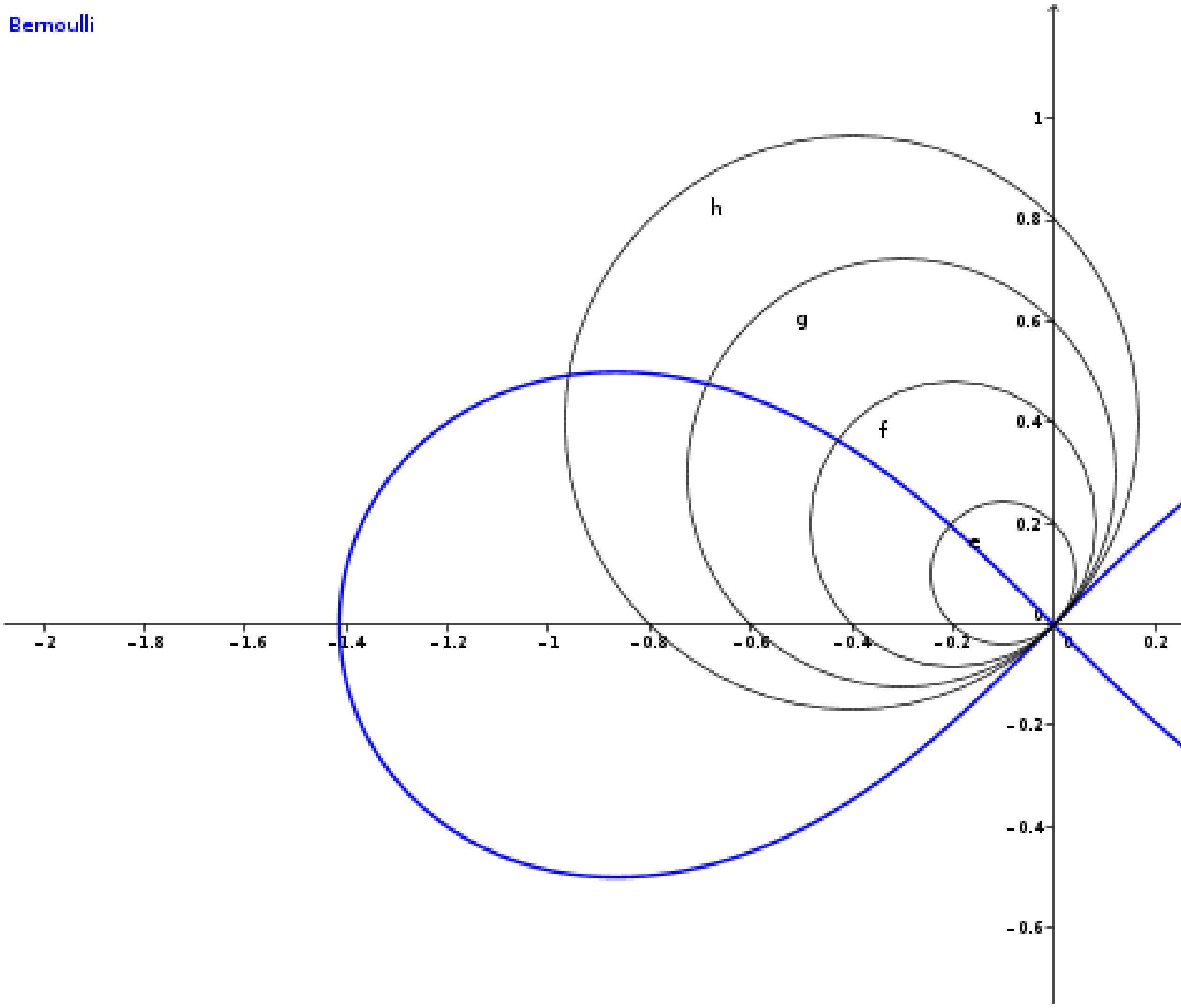
1 Lemniscate of Bernoulli

The implicit equation for the Lemniscate of Bernoulli is given by:

$$\left(x^2+y^2\right)^2-2\left(x^2-y^2\right)=0 \tag{1}$$



Parametrize with a substitution using a pencil of circles through (0,0):

$$x^2 + y^2 = t\,(x - y) \tag{2}$$

Substituting 2 into 1 and expanding the difference of two squares:

$$t^2\,(x-y)^2 - 2\,(x-y)\,(x+y) = 0$$

$$(x-y)\left(t^2\,(x-y) - 2\,(x+y)\right) = 0$$

Solving for the second term as it contains t:

$$t^2x - 2x - t^2y - 2y = x\left(t^2-2\right) - y\left(t^2+2\right) = 0$$

$$y = x\frac{t^2-2}{t^2+2}$$

Substitute $y = x\frac{t^2-2}{t^2+2}$ in 2:

$$x^2 + x^2\frac{\left(t^2-2\right)^2}{\left(t^2+2\right)^2} - xt + x\frac{t\left(t^2-2\right)}{t^2+2} = 0$$

$$x\left(x + x\frac{\left(t^2-2\right)^2}{\left(t^2+2\right)^2} - t + \frac{t\left(t^2-2\right)}{t^2+2}\right) = 0$$

Which has solutions $x = 0$ and x as a function of t:

$$x\left(1 + \frac{\left(t^2-2\right)^2}{\left(t^2+2\right)^2}\right) = t - \frac{t\left(t^2-2\right)}{t^2+2}$$

$$x\left(\frac{\left(t^2+2\right)^2 + \left(t^2-2\right)^2}{\left(t^2+2\right)^2}\right) = \frac{t\left(t^2+2\right) - t\left(t^2-2\right)}{t^2+2}$$

$$x = \frac{\left(t^2+2\right)^2}{t^4+4t^2+4+t^4-4t^2+4} \bullet \frac{t^3+2t-t^3+2t}{t^2+2}$$

$$x = \frac{4t\left(t^2+2\right)}{2t^4+8}$$

$$y = \frac{2t\left(t^2+4\right)}{t^4+4} \bullet \frac{t^2-2}{t^2+2}$$

A rational parametrization of the Lemniscate of Bernoulli is:

$$x\left(t\right) = \frac{2t\left(t^2+2\right)}{t^4+4}$$

$$y\left(t\right) = \frac{2t\left(t^2-2\right)}{t^4+4}$$

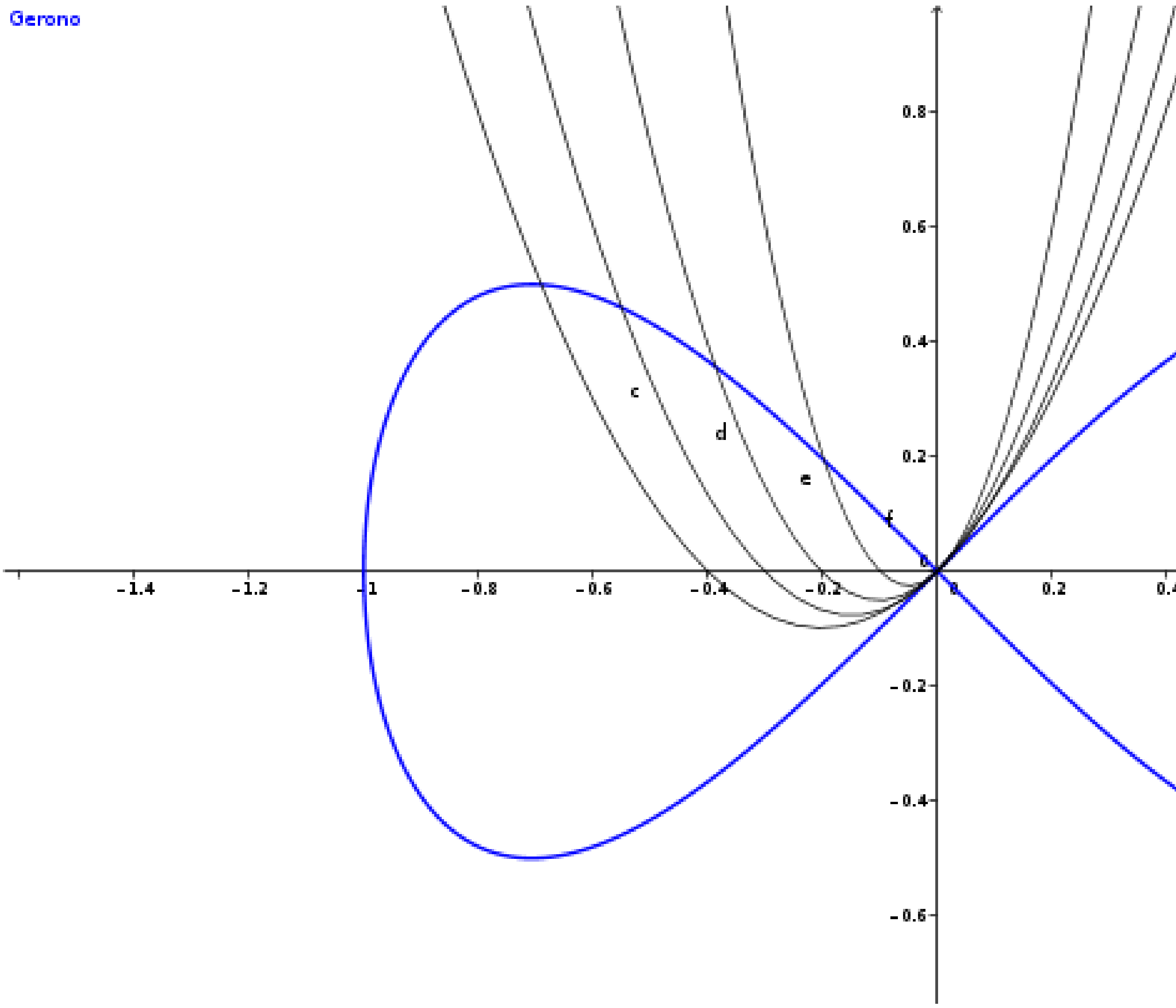
2 Lemniscate of Gerono

The lemniscate of Gerono is given by the implict equation:

$$x^4 - a^2\left(x^2 - y^2\right) = 0 \tag{3}$$

Parametrize with a substitution using a pencil of parabolas through (0,0):

$$x^2 = t\left(x - y\right) \tag{4}$$



Substituting 4 into 3 and expanding the difference of two squares:

$$t^2\left(x-y\right)^2-a^2\left(x-y\right)\left(x+y\right)=0$$

$$\left(x-y\right)\left(t^2\left(x-y\right)-a^2\left(x-y\right)\right)=0$$

Solving for the second term as it contains t:

$$x\left(t^2-a^2\right)-y(t^2+a^2)=0$$

So we can substitute $y=x\frac{t^2-a^2}{t^2+a^2}$ in 4:

$$x^2-t\left(x-x\frac{t^2-a^2}{t^2+a^2}\right)=0$$

$$x^2 - x\left(\frac{t\left(t^2+a^2\right) - t\left(t^2-a^2\right)}{t^2+a}\right) = 0$$

$$x\left(x - \frac{2a^2t}{t^2+a^2}\right) = 0$$

Which has solutions $x = 0$ and $x = \frac{2a^2t}{t^2+a^2}$. Then:

$$y = \frac{2a^2t}{t^2+a^2} \bullet \frac{t^2-a^2}{t^2+a^2}$$

A rational parametrization for the Lemniscate of Gerono is:

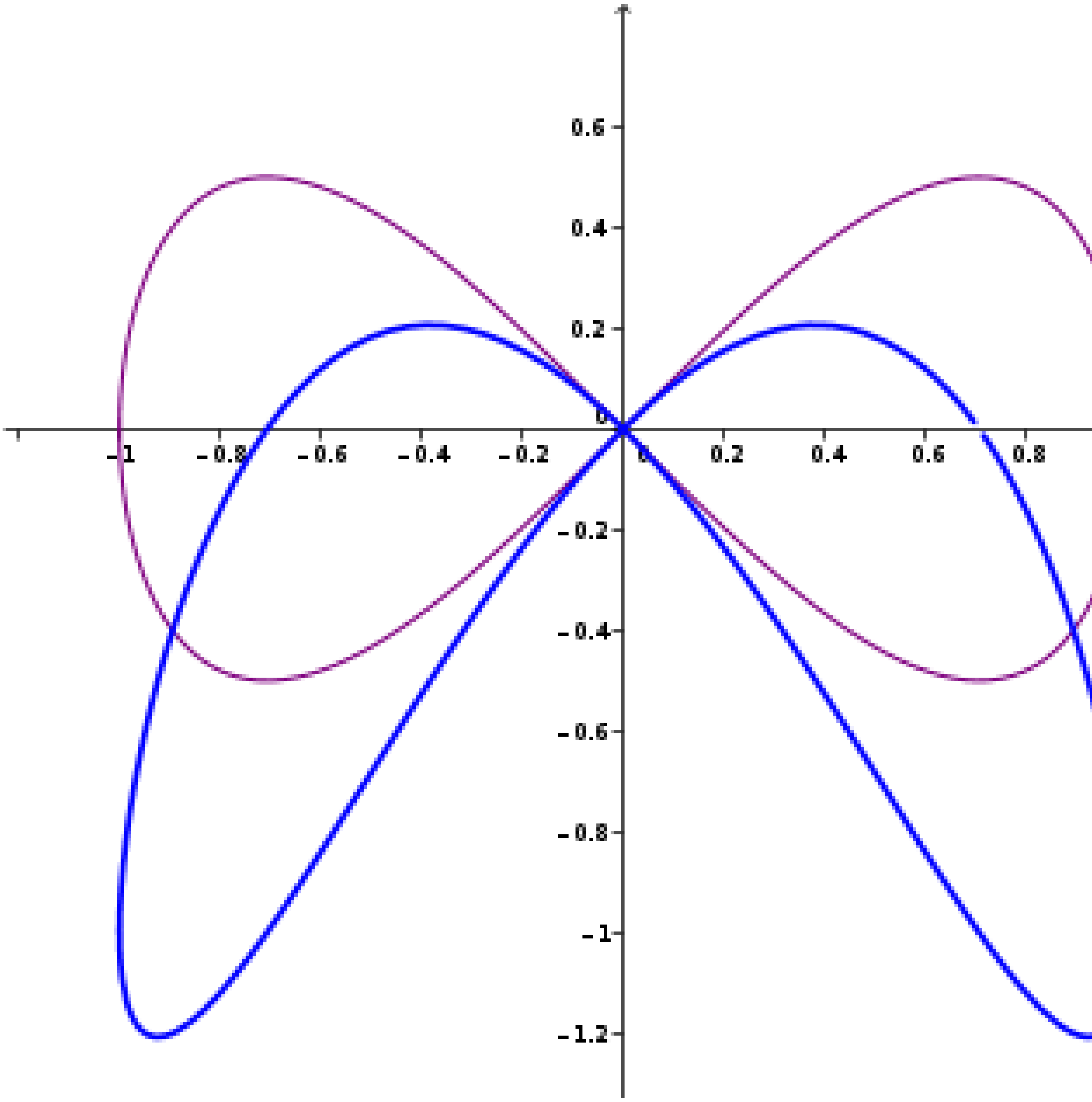
$$x\left(t\right) = \frac{2a^2t}{t^2+a^2}$$

$$y\left(t\right) = \frac{2a^2t\left(t^2-a^2\right)}{\left(t^2+a^2\right)^2}$$

3 Cramer's Besace

The Besace is a lemniscate with coefficient a in a position that makes the parametrization method less obvious.

$$x^4 - x^2 + \left(y^2 + ax^2\right)^2 = 0 \tag{5}$$



Note that when $a = 0$, we have the lemniscate of Gerono of 3 with $a = 1$. However, using the same initial substitution doesn't work well and another approach is required.

Rearrange 4 to collect terms of the same order:

$$f(x, y) = \left(a^2 + 1\right) x^4 + 2ax^2y + \left(y^2 - x^2\right)$$

$$f = f_d + f_{d-1} + f_{d-2}$$

Parametrization is a two step process, first in terms of τ and then the more specific τ_0 in terms of a polynomial in t such that the polynomial ensures a discriminant becomes a perfect square.

First, define $y = \tau x$ and solve

$$F(1,\tau) = x^2 f_4(1,\tau) + x f_3(1,\tau) + f_2(1,\tau)$$

$$F(1,\tau) = \left(a^2+1\right) x^2 + 2a\tau x + \tau^2 - 1$$

$$D(\tau) = (2a\tau)^2 - 4\left(a^2+1\right)^2 \left(\tau^2-1\right)$$

$$D(\tau) = 4\left(a^2+1-\tau^2\right)$$

$D(\tau) = P^2.Q(\tau)$ where $Q(\tau) = \alpha + \beta\tau + \gamma\tau^2$, $\alpha = \left(a^2+1\right), \beta = 0, \gamma = -1$allows the rationalizer to be set with parameter t:

$$\tau_0 = \frac{4\left(a^2+1\right)t}{t^2+4\left(a^2+1\right)}$$

$D(\tau_0) = P^2.Q(\tau_0) = P^2.R^2(\tau_0)$, then

$$x = \frac{\frac{-2a.4\left(a^2+1\right)t}{t^2+4\left(a^2+1\right)} + P.R(\tau_0)}{2\left(a^2+1\right)}$$

$$x = \frac{\frac{-2a.4\left(a^2+1\right)t}{t^2+4\left(a^2+1\right)} + 2\frac{\sqrt{(a^2+1)}\left(t^2-4\left(a^2+1\right)\right)}{t^2+4\left(a^2+1\right)}}{2\left(a^2+1\right)}$$

A parametrization for the Besace is

$$x(t) = \frac{\frac{1}{\sqrt{(a^2+1)}}t^2 - 4at - 4\sqrt{a^2+1}}{t^2+4\left(a^2+1\right)}$$

$$y(t) = \frac{4\left(a^2+1\right)t}{t^2+4\left(a^2+1\right)} \bullet \frac{\frac{1}{\sqrt{(a^2+1)}}t^2 - 4at - 4\sqrt{a^2+1}}{t^2+4\left(a^2+1\right)}$$