RATIONAL PARAMETRIZATION OF THE CIRCLE

LOGICMONKEY

The unit circle $\mathcal{C}$ centred at the origin is given implicitly by

$$x^2 + y^2 = 1 \tag{0.1}$$

The general line through point $[x_0, y_0]$ with gradient m is given by

$$y - y_0 = m\,(x - x_0)$$

We fix a point $[0, 1]$ at the top of the circle $\mathcal{C}$ and have line ℓ pass through it with gradient t and equation

$$y - 1 = t\,(x - 0)$$

$$y = tx + 1 \tag{0.2}$$

By varying only the gradient t as a parameter, ℓ is a secant through $[0, 1]$ and any other point $[x, y]$ on $\mathcal{C}$ as in figure 0.1.

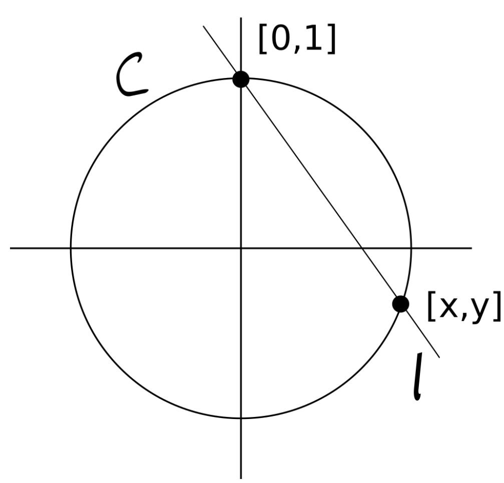

FIGURE 0.1. Circle $\mathcal{C}$ and secant ℓ

Solve for x and y, substitute 0.2 into 0.1:

$$x^2 + (tx + 1)^2 - 1 = 0$$

$$\left(1 + t^2\right) x^2 + 2tx = 0$$

One solution is $x = 0$, which we already knew since ℓ passes through $[0, 1]$. The other is:

$$x = \frac{-2t}{1+t^2}$$

Substitute $x = \frac{-2t}{1+t^2}$ into 0.2:

$$y = \frac{-2t^2}{1+t^2} + 1 = \frac{1-t^2}{1+t^2}$$

Arbitrarily dropping the sign of t, the rational parametrization is

$$\mathcal{C} : \left[\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right]$$

These rational coordinates are the lengths of the sides of a right triangle with $\mathcal{C}$'s radius as its hypoteneuse. So

$$\left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 = 1 \tag{0.3}$$

$$(2t)^2 + \left(1-t^2\right)^2 = \left(1+t^2\right)^2 \tag{0.4}$$

The terms of 0.4 can be used as a generator of Pythagorean triples. The RHS is a perfect square and could alternatively be expressed as

$$\left(a^2+t^2\right)^2 = a^4 + 2a^2t^2 + t^4 = (2at)^2 + (a^2-t^2)^2$$

$$(2at)^2 + (a^2-t^2)^2 = \left(a^2+t^2\right)^2$$

$$\left(\frac{2at}{a^2+t^2}\right)^2 + \left(\frac{a^2-t^2}{a^2+t^2}\right)^2 = 1 \tag{0.5}$$

Equating terms in the LHS of 0.5 with those of 0.3 yields the more general parametrization:

$$\mathcal{C} : \left[\frac{2at}{a^2+t^2}, \frac{a^2-t^2}{a^2+t^2}\right]$$