A symbolic solver for linear ODE of order 2

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We introduce an algorithm for a symbolic solver for linear homogeneous ordinary differential equations of order 2. It tries to detect the singular points of the equation and it computes the transformation of the coefficients of the equation under a change of variable. We give some applications.

1 Singular points of a linear ordinary differential equation

Consider a linear ordinary differential equation of order n of the form

$$\sum_{i=0}^{n} p_i(x) \frac{d^i f}{dx^i}(x) = 0 \tag{1}$$

for an unknown function f, where the p_i are meromorphic functions on $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and $p_n(x) = 1$ for all $x \in \widehat{\mathbb{C}}$.

Definition 1.1. 1. A point $x_0 \in \widehat{\mathbb{C}}$ is called an *ordinary point* of the equation (1) if all the functions p_i are analytic at x_0. Otherwise x_0 is called a *singular point* of the equation (1).
2. A singular point x_0 of the equation (1) is called a *regular singular point* of (1) if for every i the function p_{n-i} has a pole of order at most i at x_0. Otherwise x_0 is called an *irregular singular point* of (1).

The reader can find more information on these notions in many textbooks on ordinary differential equations, e. g. in [3] or [2].

Example 1.2. Let $n = 2$, i.e. consider

$$\frac{d^2 f}{dx^2}(x) + p_1(x) \frac{df}{dx}(x) + p_0(x) f(x) = 0. \tag{2}$$

In order to examine the point $x_0 = \infty$ we use the Möbius transformation $w = \frac{1}{x}$. We compute

$$\begin{aligned} \frac{df}{dx}(x) &= \frac{dw}{dx}(x) \frac{df}{dw}(w) = -w^2 \frac{df}{dw}(w) \\ \frac{d^2 f}{dx^2}(x) &= \frac{dw}{dx}(x) \frac{d}{dw}\Big(-w^2 \frac{df}{dw}(w) \Big) = w^4 \frac{d^2 f}{dw^2}(w) + 2w^3 \frac{df}{dw}(w) \end{aligned}$$

and thus
$$\frac{d^2f}{dw^2}(w)+\Big(\frac{2}{w}-\frac{1}{w^2}p_1\Big(\frac{1}{w}\Big)\Big)\frac{df}{dw}(w)+\frac{1}{w^4}p_0\Big(\frac{1}{w}\Big)f(w)=0.$$
It follows that $x_0=\infty$ is an ordinary point of (2) if the functions $2x-x^2p_1(x)$ and $x^4p_0(x)$ are bounded as $|x|\to\infty$. Moreover, we see that $x_0=\infty$ is a regular singular point of (2) if $x_0=\infty$ is not an ordinary point of (2) and the functions $xp_1(x)$ and $x^2p_0(x)$ are bounded as $|x|\to\infty$.

2 Q-form of a linear second order ordinary differential equation

We consider a linear second order ordinary differential equation of the form
$$\frac{d^2f}{dx^2}(x)+p_1(x)\frac{df}{dx}(x)+p_0(x)f(x)=0, \tag{3}$$
where p_1,p_0 are meromorphic functions. We make the ansatz $f(x)=h(x)g(x)$ and we obtain
$$\begin{aligned}\frac{df}{dx}(x)&=\frac{dh}{dx}(x)g(x)+h(x)\frac{dg}{dx}(x),\\ \frac{d^2f}{dx^2}(x)&=\frac{d^2h}{dx^2}g(x)+2\frac{dh}{dx}(x)\frac{dg}{dx}(x)+h(x)\frac{d^2g}{dx^2}(x)\end{aligned}$$
and therefore
$$h(x)\frac{d^2g}{dx^2}(x)+\Big(2\frac{dh}{dx}(x)+p_1(x)h(x)\Big)\frac{dg}{dx}(x)+\Big(\frac{d^2h}{dx^2}(x)+p_1(x)\frac{dh}{dx}(x)+p_0(x)h(x)\Big)g(x)=0.$$
Now, we put
$$h(x):=\exp\Big(-\frac{1}{2}\int_{x_0}^{x}p_1(s)\,ds\Big), \tag{4}$$
where $x\in\mathbb{R}$ is such that p_1 is not singular at x and x_0 is such that the real interval with endpoints x,x_0 does not contain any singularities of p_1. As a consequence, the coefficient of $\frac{dg}{dx}(x)$ vanishes and g satisfies the equation $g''(x)+q_0(x)g(x)=0$ where
$$q_0(x)=-\frac{1}{2}\frac{dp_1}{dx}(x)-\frac{1}{4}p_1(x)^2+p_0(x).$$

Definition 2.1. We say that the function g satisfies a second order linear *ordinary differential equation in Q-form* if
$$\frac{d^2g}{dx^2}(x)+q_0(x)g(x)=0$$
where q_0 is a meromorphic function.

Remark 2.2. For an equation in Q-form the point $x_0=\infty$ is always a singular point by Example 1.2 and it is a regular singular point if $x^2q_0(x)$ is bounded as $|x|\to\infty$.

Remark 2.3. Let $P_1, P_0\colon I \to \mathbb{C}$ and let $F\colon I \to \mathbb{C}$ be a solution to

$$F''(x) + P_1(x)F'(x) + P_0(x)F(x) = 0$$

and write $F(x) = H(x)G(x)$ where G satisfies an equation in Q-form $G''(x)+Q_0(x)G(x) = 0$. Let $\varphi\colon J \to I$ be a diffeomorphism and put $f := F \circ \varphi$. We compute

$$f'(y) = F'(\varphi(y))\varphi'(y), \quad f''(y) = F''(\varphi(y))\varphi'(y)^2 + F'(\varphi(y))\varphi''(y).$$

Therefore, f satisfies the equation

$$f''(y) + p_1(y)f'(y) + p_0(y)f(y) = 0$$

where

$$p_1(y) = P_1(\varphi(y))\varphi'(y) - \frac{\varphi''(y)}{\varphi'(y)}, \tag{5}$$

$$p_0(y) = P_0(\varphi(y))\varphi'(y)^2. \tag{6}$$

We write $f(y) = h(y)g(y)$ where

$$\begin{aligned} h(y) &= \exp\Big(-\frac{1}{2}\int_{y_0}^{y} p_1(t)\,dt\Big) = \exp\Big(-\frac{1}{2}\int_{\varphi(y_0)}^{\varphi(y)} P_1(s)\,ds\Big)\frac{\varphi'(y)^{1/2}}{\varphi'(y_0)^{1/2}} \\ &= C \cdot H(\varphi(y))\varphi'(y)^{1/2} \end{aligned} \tag{7}$$

with a constant $C \in \mathbb{R}$ and g satisfies the equation $g(y)'' + q_0(y)g(y) = 0$ where

$$q_0(y) = Q_0(\varphi(y))\varphi'(y)^2 + \frac{\varphi'''(y)}{2\varphi'(y)} - \frac{3\varphi''(y)^2}{4\varphi'(y)^2}. \tag{8}$$

From $H(\varphi(y))G(\varphi(y)) = f(y) = h(y)g(y)$ we get for some $C \in \mathbb{R}$

$$(G \circ \varphi)(y) = C\varphi'(y)^{1/2}g(y).$$

and therefore

$$g(y) = \frac{(G \circ \varphi)(y)}{C\varphi'(y)^{1/2}} = \frac{1}{C\varphi'(y)^{1/2}} \cdot \frac{(F \circ \varphi)(y)}{(H \circ \varphi)(y)}. \tag{9}$$

We note that the map

$$T_\varphi : \quad C^\infty(I, \mathbb{C})^3 \to C^\infty(J, \mathbb{C})^3, \quad T_\varphi(P_1, P_0, Q_0) := (p_1, p_0, q_0) \tag{10}$$

is bijective.

Example 2.4. Let $F\colon I \to \mathbb{C}$ be a solution to (3) and write $F(x) = H(x)G(x)$ where G satisfies an equation in Q-form $G''(x)+Q_0(x)G(x) = 0$. Let $\varphi\colon J \to I$ be a diffeomorphism, let $f := F \circ \varphi$ and write $f = \widehat{h}\widehat{g}$ such that $\widehat{g}$ satisfies an equation in Q-form. If there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\delta - \beta\gamma \neq 0$ and $\varphi(y) = \frac{\alpha y+\beta}{\gamma y+\delta}$ for all y then we have $g''(y) + q_0(y)g(y) = 0$ where $q_0(y) = Q_0(\varphi(y))\frac{(\alpha\delta-\beta\gamma)^2}{(\gamma y+\delta)^4}$.

Example 2.5. 1. Let p_0, p_1 be constant. We get $f(x) = h(x)g(x)$ where

$$h(x) = \exp\Big(-\frac{1}{2}p_1 x\Big)$$

and g satisfies

$$\frac{d^2 g}{dx^2}(x) + \Big(p_0 - \frac{1}{4}p_1^2\Big)g(x) = 0.$$

If $p_0 - \frac{1}{4}p_1^2 = 0$ we get $g(x) = Ax + B$ with $A, B \in \mathbb{C}$. Otherwise we get

$$g(x) = A\exp\Big(\frac{1}{2}\sqrt{p_1^2 - 4p_0}x\Big) + B\exp\Big(-\frac{1}{2}\sqrt{p_1^2 - 4p_0}x\Big), \quad A, B \in \mathbb{C}.$$

2a) Let $a, b, c \in \mathbb{C}$ and let

$$p_1(x) = \frac{c - (a+b+1)x}{x(1-x)}, \quad p_0(x) = -\frac{ab}{x(1-x)}.$$

Then the equation is called a hypergeometric equation (notation as in [1]). We get

$$h(x) = \exp\Big(-\frac{c\log(x) + (a+b+1-c)\log(x-1)}{2}\Big),$$
$$q_0(x) = \frac{(1-(a-b)^2)x^2 - ((1-a-b)2c + 4ab)x - c^2 + 2c}{4x^2(x-1)^2}.$$

The equation for f and the equation for g both have the regular singular points $0, 1, \infty$ and no irregular singular points.
2b) Let $\nu, \mu \in \mathbb{C}$ and let

$$p_1(x) = \frac{2x}{x^2 - 1}, \quad p_0(x) = \frac{-\nu(\nu+1)x^2 + \nu(\nu+1) - \mu^2}{(x^2-1)^2}.$$

Then the equation is called Legendre's equation. We get

$$h(x) = \begin{cases} \frac{1}{\sqrt{1-x^2}}, & x \in (-1, 1) \\ \frac{1}{\sqrt{x^2-1}}, & |x| > 1 \end{cases}, \quad q_0(x) = \frac{-\nu(\nu+1)x^2 + \nu(\nu+1) + 1 - \mu^2}{(x^2-1)^2}.$$

The equation for f and the equation for g both have the regular singular points $-1, 1, \infty$ and no irregular singular points.
3a) Let $a \in \mathbb{C}$ and let

$$p_1(x) = \frac{1}{x}, \quad p_0(x) = 1 - \frac{a^2}{x^2}.$$

Then the equation is called Bessel's equation. We get for $x > 0$

$$h(x) = x^{-1/2}, \quad q_0(x) = \frac{4x^2 + 1 - 4a^2}{4x^2}.$$

We denote by J_a, Y_a the Bessel functions of the first kind and second kind respectively. If $a \notin \mathbb{Z}$ then $\{J_a, J_{-a}\}$ is a fundamental system for Bessel's equation. Otherwise $\{J_a, Y_a\}$ is a fundamental system.

3b) Let $a, b \in \mathbb{C}$ and let

$$p_1(x) = \frac{b}{x} - 1, \quad p_0(x) = -\frac{a}{x}.$$

Then the equation is called Kummer's equation. We get

$$h(x) = x^{-b/2} \exp\left(\frac{x}{2}\right), \quad q_0(x) = \frac{-x^2 + (2b - 4a)x + 2b - b^2}{4x^2}.$$

In the special case that $a = -n \in \{0, -1, -2, ...\}$ is a non-positive integer and $b = \alpha + 1$ the equation is called Laguerre's equation with parameters n, α.
In examples 3a), b) the equations for f and g both have a regular singular point at $x_0 = 0$ and an irregular singular point at $x_0 = \infty$. We denote by M_{ab}, U_{ab} the confluent hypergeometric functions of the first and second kind respectively. By definition we have

$$M_{ab}(x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(b)_k k!}, \quad U_{ab}(x) = \frac{\pi}{\sin(\pi b)} \left(\frac{M_{ab}(x)}{\Gamma(1 + a - b)\Gamma(b)} - x^{1-b} \frac{M_{1+a-b,2-b}(x)}{\Gamma(a)\Gamma(2 - b)} \right)$$

where

$$(a)_0 := 1, \quad (a)_k := a(a + 1) \cdots (a + k - 1), \quad k \geq 1$$

and Γ denotes the Gamma function. The function M_{ab} is not defined if b is a non-positive integer. The function U_{ab} is defined for all complex numbers a, b. Namely, if b is an integer then U_{ab} is defined as the limit U_{ab_n} as $b_n \to b$, $b_n \notin \mathbb{Z}$ (see [1], p. 504). If $a \in \{0, -1, -2, -3, ...\}$ and $b \notin \{0, -1, -2, -3, ...\}$ then M_{ab} is a polynomial of degree $-a$ and U_{ab} is a multiple of M_{ab}.
Moreover, if $n \in \{0, 1, 2, ...\}$ and $\alpha \in \mathbb{R}$ we define the generalized Laguerre polynomial with parameters n, α by (see [1], p. 509-510)

$$L_n^{(\alpha)}(x) := \begin{cases} \frac{(\alpha+1)_n}{n!} M_{-n,\alpha+1}(x) & \text{if } \alpha \notin \{-1, -2, -3, ...\} \\ \frac{(-1)^n}{n!} U_{-n,\alpha+1}(x) & \text{if } \alpha \in \{-1, -2, -3, ...\} \end{cases}$$

From the computation of Wronskians in [1], p. 505 we get:
Case 1: $a \in \{0, -1, -2, -3, ...\}$:

- If $b \notin \mathbb{Z}$ then $\{M_{ab}, x \mapsto x^{1-b} M_{1+a-b,2-b}(x)\}$ is a fundamental system and moreover $\{L_{-a}^{(b-1)}, x \mapsto x^{1-b} M_{1+a-b,2-b}(x)\}$ is a fundamental system
- If $b \in \{1, 2, 3, ...\}$ then $\{M_{ab}, x \mapsto e^x U_{b-a,b}(-x)\}$ is a fundamental system and moreover $\{L_{-a}^{(b-1)}, x \mapsto e^x U_{b-a,b}(-x)\}$ is a fundamental system
- If $b \in \{0, -1, -2, -3, ...\}$ then $\{U_{ab}, x \mapsto e^x U_{b-a,b}(-x)\}$ is a fundamental system and moreover $\{L_{-a}^{(b-1)}, x \mapsto e^x U_{b-a,b}(-x)\}$ is a fundamental system

Case 2: $a \notin \{0, -1, -2, -3, ...\}$:

- If $b \notin \{0, -1, -2, -3, ...\}$ then $\{M_{ab}, U_{ab}\}$ is a fundamental system. In the special case $a = b$ we have $M_{ab} = \exp$ and thus $\{\exp, U_{ab}\}$ is a fundamental system.

- If $b \in \{0, -1, -2, -3, ...\}$ then $\{x \mapsto x^{1-b} M_{1+a-b,2-b}(x), U_{ab}\}$ is a fundamental system

4. Let $b, c, q \in \mathbb{R}$ with $b > -1$ and let

$$p_1(x) := -\frac{2(b+1)x}{1-x^2}, \quad p_0(x) := \frac{c - 4qx^2}{1-x^2}$$

Then the equation is called a spheroidal wave equation (notation as in [1], p. 722). We get

$$h(x) = \begin{cases} (1-x^2)^{-(b+1)/2}, & \text{if } |x| < 1 \\ (x^2-1)^{-(b+1)/2}, & \text{if } |x| \geq 1 \end{cases}, \quad q_0(x) = \frac{4qx^4 - (4q + c + b^2 + b)x^2 + c + b + 1}{(1-x)^2(1+x)^2}.$$

The equation for f and the equation for g both have regular singular points at -1, 1 and an irregular singular point at ∞. In the special case $b = -\frac{1}{2}$, $c = a + 2q$ the equation is called the algebraic form of Mathieu's equation because it is obtained from Mathieu's equation

$$f''(t) + (a - 2q\cos(2t))f(t) = 0$$

by using the transformation $t = \varphi(x) := \arccos(x)$. In this case we get

$$h(x) = \begin{cases} (1-x^2)^{-1/4}, & \text{if } |x| < 1 \\ (x^2-1)^{-1/4}, & \text{if } |x| \geq 1 \end{cases}, \quad q_0(x) = \frac{16qx^4 + (1 - 24q - 4a)x^2 + 8q + 4a + 2}{4(1-x)^2(1+x)^2}.$$

Example 2.6. In the following we use the formulas (5), (6), (7) and (8).

1. Let $a \in \mathbb{C}$, $k \in \mathbb{R}$, $x_0 \in \mathbb{R}$, $\alpha \in \mathbb{C}$ and let B_a be a solution to Bessel's equation with parameter a. We put $f(x) := B_a(\alpha(x-x_0)^k)$. Then f satisfies the equation

$$\frac{d^2 f}{dx^2}(x) + \frac{1}{x - x_0}\frac{df}{dx}(x) + \frac{\alpha^2 k^2 (x-x_0)^{2k} - a^2 k^2}{(x-x_0)^2} f(x) = 0.$$

We get

$$h(x) = (x-x_0)^{-1/2}, \quad q_0(x) = \frac{4\alpha^2 k^2 (x-x_0)^{2k} + 1 - 4a^2k^2}{4(x-x_0)^2}.$$

If $1 - 4a^2k^2 = 0$ we get $q_0(x) = \alpha^2 k^2 (x-x_0)^{2k-2}$. In this case, if $k \geq 1$ the only singular point of the equation for g is an irregular singular point at ∞.
2. Let $a, b \in \mathbb{C}$ and let K_{ab} be a solution to Kummer's equation with parameters a, b. For $k, x_0 \in \mathbb{R}, \alpha \in \mathbb{C}$ we put $f(x) = K_{ab}(\alpha(x-x_0)^k)$. Then f satisfies the equation

$$\frac{d^2 f}{dx^2}(x) + \Big(\frac{kb - k + 1}{x - x_0} - \alpha k (x-x_0)^{k-1}\Big)\frac{df}{dx}(x) - a\alpha k^2 (x-x_0)^{k-2} f(x) = 0.$$

We get

$$h(x) = \exp\Big(\frac{\alpha}{2}(x-x_0)^k\Big)(x-x_0)^{(k-1-bk)/2}$$

and

$$q_0(x) = \frac{-\alpha^2 k^2 (x-x_0)^{2k} + (2b-4a)\alpha k^2 (x-x_0)^k + 1 - (b-1)^2 k^2}{4(x-x_0)^2}$$

If $1-(b-1)^2k^2 = 0$ then we get $q_0(x) = -\frac{\alpha^2 k^2}{4}(x-x_0)^{2k-2} + \frac{2b-4a}{4}\alpha k^2 (x-x_0)^{k-2}$. In this case, if $k \geq 2$ then the only singular point of the equation for g is an irregular singular point at ∞.

3. Let $\lambda \in \mathbb{C}$ and let

$$p_1(x) = -2x, \quad p_0(x) = 2\lambda.$$

Then the equation is called Hermite's equation. We get

$$h(x) = \exp\left(\frac{x^2}{2}\right), \quad q_0(x) = -x^2 + 1 + 2\lambda.$$

The equations for f and g both have no regular singular point and an irregular singular point at $x_0 = \infty$. By Example 2 with $x_0 = 0$, $k = 2$, $\alpha = 1$ and $b = \frac{1}{2}$ or $b = \frac{3}{2}$ we see that the solutions can be expressed in terms of Kummer functions. We consider the following special cases:

- If $\lambda \in \{0, 2, 4, ...\}$ is an even non-negative integer then

  $$H_\lambda(x) := M_{-\lambda/2,1/2}(x^2)$$

  is a polynomial function and $\{H_\lambda, x \mapsto x M_{(1-\lambda)/2,3/2}(x^2)\}$ is a fundamental system.

- If $\lambda \in \{1, 3, 5, ...\}$ is an odd positive integer then

  $$H_\lambda(x) := x M_{(1-\lambda)/2,3/2}(x^2)$$

  is a polynomial function and $\{H_\lambda, x \mapsto M_{-\lambda/2,1/2}(x^2)\}$ is a fundamental system.

4. Let $a \in \mathbb{C}$ and let B_a be a solution to Bessel's equation. For $\alpha, \beta, \gamma, \in \mathbb{C}$ we consider $\varphi(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$. We have already discussed the case $\alpha = 0$ or $\gamma = 0$. Thus we may assume that $\alpha \neq 0$ and $\gamma \neq 0$ and since we can divide out common factors we may assume that $\gamma = 1$. We therefore put $f(x) := B_a(\frac{\alpha(x-\beta)}{x-\delta})$ with $\alpha, \beta, \delta \in \mathbb{C}$, $\alpha \neq 0$, $\beta \neq \delta$. Then f satisfies the equation

$$f''(x) + \frac{2(x-\beta) + (\beta-\delta)}{(x-\beta)(x-\delta)} f'(x) + \frac{(\beta-\delta)^2(\alpha^2(x-\beta)^2 - a^2(x-\delta)^2)}{(x-\beta)^2(x-\delta)^4} f(x) = 0.$$

We get

$$h(x) = \frac{1}{\sqrt{(x-\beta)(x-\delta)}}$$

and

$$q_0(x) = \frac{4\alpha^2(x-\beta)^2 + (1-4a^2)(x-\delta)^2}{4(x-\beta)^2} \cdot \frac{(\beta-\delta)^2}{(x-\delta)^4}$$

The equation for g has regular singular points at β, ∞ and an irregular singular point at δ.
5. Similarly, for $k > 0$ and $f(x) := B_a(\alpha(\frac{x-\beta}{x-\delta})^k)$ we get using (7), (8)

$$h(x) = \frac{1}{\sqrt{(x-\beta)(x-\delta)}}$$

and

$$q_0(x) = (\beta-\delta)^2 \frac{4\alpha^2k^2(x-\beta)^{2k} + (1-4a^2k^2)(x-\delta)^{2k}}{(x-\beta)^2(x-\delta)^{2k+2}}.$$

The equation for g has regular singular points at β, ∞ and an irregular singular point at δ.
6. Let $a, b \in \mathbb{C}$ and let K_{ab} be a solution to Kummer's equation. For $\alpha, \beta, \delta \in \mathbb{C}$, $\alpha \neq 0$, we put $f(x) := K_{ab}(\frac{\alpha(x-\beta)}{x-\delta})$. Then f satisfies the equation

$$f''(x) + \Big(\frac{b(\beta-\delta)+2}{x-\delta} - \frac{\alpha(\beta-\delta)}{(x-\delta)^2}\Big) f'(x) - \frac{\alpha a(\beta-\delta)^2}{(x-\beta)(x-\delta)^3} f(x) = 0.$$

We get

$$h(x) = (x-\beta)^{-b/2}(x-\delta)^{(b-2)/2} \exp\Big(\frac{\alpha(x-\beta)}{2(x-\delta)}\Big)$$

and

$$q_0(x) = \frac{-\alpha^2(x-\beta)^2 + (2b-4a)\alpha(x-\beta)(x-\delta) + (2b-b^2)(x-\delta)^2}{4(x-\beta)^2(x-\delta)^4}(\beta-\delta)^2$$

The equation for g has regular singular points at β, ∞ and an irregular singular point at δ.
7. Similarly, for $k \in \mathbb{R} \setminus \{0\}$ and $f(x) := K_{ab}(\alpha(\frac{x-\beta}{x-\delta})^k)$ we get using (7), (8)

$$h(x) = (x-\beta)^{(k-1-bk)/2}(x-\delta)^{(bk-1-k)/2} \exp\Big(\frac{\alpha(x-\beta)^k}{2(x-\delta)^k}\Big)$$

and

$$q_0(x) = \frac{-\alpha^2k^2(x-\beta)^{2k} + (2b-4a)\alpha k^2(x-\beta)^k(x-\delta)^k + (1-(b-1)^2k^2)(x-\delta)^{2k}}{4(x-\beta)^2(x-\delta)^{2k+2}}(\beta-\delta)^2.$$

The equation for g has regular singular points at β, ∞ and an irregular singular point at δ.

3 An algorithm for a symbolic ODE solver

Assume we want to solve an ordinary differential equation

$$f''(x) + p_1(x)f'(x) + p_0(x)f(x) = 0.$$

By defining h as in Formula (4) and writing $f = h \cdot g$ we get an equation of the form

$$g''(x) + q_0(x)g(x) = 0$$

for g.
Assume that we can find a diffeomorphism φ: $J \to I$ such that $f = F \circ \varphi$ where F is a known function satisfying an equation of the form

$$F''(x) + P_1(x)F'(x) + P_0(x)F(x) = 0$$

with known P_1, P_0. Then the solutions g can be computed from the formula (9). In order to find such a diffeomorphism φ we consider $\varphi = \varphi_N \circ ... \circ \varphi_1$ with suitable diffeomorphisms φ_k, $k = 1, ..., N$. The algorithm is then the following:

Step 0:
We put $m_0(x) := 1$, $p_{0,0}(x) := p_0(x)$, $p_{1,0}(x) := p_1(x)$, $q_{0,0}(x) := q_0(x)$ and $\arg_0(x) := x$.

Step k: $(k \geq 1)$
We take a suitably chosen diffeomorphism φ_k and we compute

$$\begin{aligned} (p_{1,k}, p_{0,k}, q_{0,k}) &:= T_{\varphi_k}^{-1}(p_{1,k-1}, p_{0,k-1}, q_{0,k-1}) \\ \arg_k(x) &:= \varphi_k(\arg_{k-1}(x)) \end{aligned}$$

where T_φ^{-1} is the inverse of the map in (10). If we choose the φ_k suitably then after N finitely many repetitions of Step k the function q_N describes the Q-form of a known equation for the function F, i. e. $F = H \cdot G$ for some H and G satisfies $G''(x) + q_N(x)G(x) = 0$ and F, H are known functions. In this case we put $m(x) := (\frac{d}{dx} \arg_N(x))^{1/2}$ and

$$f(x) := \frac{h(x)}{m(x)} \cdot G(\arg_N(x)) = \frac{h(x) \cdot F(\arg_N(x))}{m(x) \cdot H(\arg_N(x))} \tag{11}$$

and then f is a solution to our problem by (9).
In the following we assume that q_0 is a rational function. Then we distinguish several cases.

3.1 Three regular singular points

a) Assume that the equation for g has three regular singular points at $\alpha, \beta \in \mathbb{C}$ and at ∞ and no irregular singular point. Then the denominator of the rational function q_0 has exactly the zeros α, β with multiplicity at most 2 each. Moreover, for the degrees of the numerator n and of the denominator d of q_0 we have $\deg(d) \geq \deg(n) + 2$. It follows that $q_0(x) = \frac{Ax^2+Bx+C}{4(x-\alpha)^2(x-\beta)^2}$ where $\alpha, \beta \in \mathbb{C}$ are distinct and $A, B, C \in \mathbb{C} \setminus \{(0, 0, 0)\}$. Using the coordinate transformation

$$\varphi_1(x) := \frac{x - \alpha}{\beta - \alpha}, \quad \varphi_1^{-1}(x) = (\beta - \alpha)x + \alpha$$

we get

$$
\begin{aligned}
q_{0,1}(x) &= \frac{A(\beta-\alpha)^2x^2 + (\beta-\alpha)(2A\alpha+B)x + A\alpha^2 + B\alpha + C}{4x^2(x-1)^2(\beta-\alpha)^2},\\
p_{1,1}(x) &= (\beta-\alpha)p_1\big(\varphi_1^{-1}(x)\big),\\
p_{0,1}(x) &= (\beta-\alpha)^2 p_0\big(\varphi_1^{-1}(x)\big),\\
\arg_1(x) &= \frac{x-\alpha}{\beta-\alpha}.
\end{aligned}
$$

We distinguish two cases:
Case 1: We have $p_{1,1}(x) = \frac{C_1+C_2x}{x(1-x)}$ and $p_{0,1}(x) = \frac{C_3}{x(1-x)}$ where $C_1, C_2, C_3 \in \mathbb{C}$ are constant. We now put $c := C_1$ and we want to solve

$$
ab = -C_3, \quad a+b+1 = -C_2
$$

This is satisfied with

$$
a = \frac{-1-C_2-\sqrt{(1+C_2)^2+4C_3}}{2}, \quad b = \frac{-1-C_2+\sqrt{(1+C_2)^2+4C_3}}{2}.
$$

Then we get

$$
f(x) = F_{abc}(\varphi_1(x)) = F_{abc}\Big(\frac{x-\alpha}{\beta-\alpha}\Big)
$$

where F_{abc} is a solution to the hypergeometric equation with parameters a, b, c.
Case 2: $p_{0,1}$ and $p_{1,1}$ are not of the form as in Case 1. In this case we write

$$
\widehat{A} := A, \quad \widehat{B} := \frac{2A\alpha+B}{\beta-\alpha}, \quad \widehat{C} := \frac{A\alpha^2+B\alpha+C}{(\beta-\alpha)^2}
$$

and we have $q_{0,1}(x) = \frac{\widehat{A}x^2+\widehat{B}x+\widehat{C}}{4x^2(x-1)^2}$. The goal is now to find $a, b, c \in \mathbb{C}$ such that

$$
1-(a-b)^2 = \widehat{A}, \tag{12}
$$

$$
(1-(a+b))2c + 4ab = -\widehat{B}, \tag{13}
$$

$$
c^2 - 2c = -\widehat{C}. \tag{14}
$$

From (14) we get

$$
c = 1 \pm \sqrt{1-\widehat{C}}.
$$

We rewrite (13) as

$$
(1-(a+b))2c + (a+b)^2 - (a-b)^2 = -\widehat{B}
$$

and by (12) we get

$$
(a+b)^2 - 2c(a+b) + \widehat{A} - 1 + \widehat{B} + 2c = 0
$$

and thus using (14)

$$a+b=\frac{2c\pm\sqrt{4c^2-4(\widehat{A}-1+\widehat{B}+2c)}}{2}=c\pm\sqrt{1-\widehat{A}-\widehat{B}-\widehat{C}}.$$

By (12) we get

$$a-b=\pm\sqrt{1-\widehat{A}}.$$

Altogether, we thus have

$$\begin{aligned}
c&=1+\varepsilon_1\sqrt{1-\widehat{C}},\\
a&=\frac{1}{2}\big(c+\varepsilon_2\sqrt{1-\widehat{A}-\widehat{B}-\widehat{C}}+\varepsilon_3\sqrt{1-\widehat{A}}\big),\\
b&=a-\varepsilon_3\sqrt{1-\widehat{A}}
\end{aligned}$$

with $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\{\pm1\}$. In case that $1-\widehat{A}$ is not positive real we choose the same square root of $1-\widehat{A}$ in the formulas for a,b. We then see that a,b,c solve (12)-(14). Therefore, if we define

$$G(y):=\exp\Big(\frac{c\log(y)+(a+b+1-c)\log(y-1)}{2}\Big)F_{abc}(y)$$

where F_{abc} is a solution to the hypergeometric equation with parameters a,b,c then we get

$$f(x)=h(x)G(\varphi_1(x))$$

by (11) with $m(x)$ constant.
b) Assume that $q_0(x)=\frac{Ax^2+Bx+C}{4(x-\alpha)^2(x-\beta)^2}$ as in 1. In the following we check whether the equation for g can be transformed to a Legendre equation. Using the coordinate transformation

$$\varphi_1(x):=\frac{2x-\alpha-\beta}{\beta-\alpha},\quad \varphi_1^{-1}(x):=\frac{\beta-\alpha}{2}x+\frac{\beta+\alpha}{2}$$

we get

$$\begin{aligned}
q_{0,1}(x)&=\frac{A(\beta-\alpha)^2x^2+2(\beta-\alpha)(A(\beta+\alpha)+B)x+A(\beta+\alpha)^2+2B(\beta+\alpha)+4C}{4(x-1)^2(x+1)^2(\beta-\alpha)^2}\\
p_{1,1}(x)&=\frac{\beta-\alpha}{2}p_1(\varphi_1^{-1}(x))\\
p_{0,1}(x)&=\frac{(\beta-\alpha)^2}{4}p_0(\varphi_1^{-1}(x))\\
\arg_1(x)&=\frac{2x-\alpha-\beta}{\beta-\alpha}
\end{aligned}$$

We distinguish two cases:
Case 1: We have $p_{0,1}(x)=\frac{C_2x^2+C_0}{(x^2-1)^2}$ where $C_2,C_0\in\mathbb{C}$ are constant and $p_{1,1}(x)=\frac{2x}{x^2-1}$. Then we put $\nu=\frac{-1+\sqrt{1-4C_2}}{2}$ and $\mu=\sqrt{-C_2-C_0}$. Then we get

$$f(x)=L_{\mu\nu}(\varphi_1(x))=L_{\mu\nu}\Big(\frac{2x-\alpha-\beta}{\beta-\alpha}\Big)$$

and $L_{\mu\nu}$ is a solution to the Legendre equation with parameters ν, μ.
Case 2: $p_{0,1}$ or $p_{1,1}$ is not as in Case 1. Then we write

$$\widehat{A} := A, \quad \widehat{B} := \frac{2(A(\beta+\alpha)+B)}{\beta-\alpha}, \quad \widehat{C} := \frac{A(\beta+\alpha)^2+2B(\beta+\alpha)+4C}{(\beta-\alpha)^2}$$

and we have $q_{0,1}(x) = \frac{\widehat{A}x^2+\widehat{B}x+\widehat{C}}{4(x-1)^2(x+1)^2}$. If $\widehat{B} \neq 0$ then the equation cannot be transformed to a Legendre equation. If $\widehat{B} = 0$ then it can be transformed to a Legendre equation. We determine ν, μ such that

$$-\nu(\nu+1) = \frac{1}{4}\widehat{A}, \quad \nu(\nu+1)+1-\mu^2 = \frac{1}{4}\widehat{C},$$

i. e. we put $\nu = \frac{-1+\sqrt{1-\widehat{A}}}{2}$ and $\mu = \sqrt{1-\frac{1}{4}\widehat{A}-\frac{1}{4}\widehat{C}}$. Then, if we define

$$G(y) = \sqrt{1-y^2}L_{\mu\nu}(y)$$

where $L_{\mu\nu}$ is a solution to Legendre's equation with parameters ν, μ then we get

$$f(x) = h(x)G(\varphi_1(x))$$

by (11) with $m(x)$ constant.

3.2 One regular and one irregular singular point

Assume that the equation for g has a regular singular point at $\beta \in \mathbb{C}$ and an irregular singular point at ∞. Then the denominator of the rational function q_0 has exactly the zero β with multiplicity at most 2 and therefore $q_0(x) = \frac{N(x)}{4(x-\beta)^2}$, where $N(x)$ is a non-constant polynomial function. Similarly, if the equation for g has a regular singular point at ∞ and an irregular singular point at $\beta \in \mathbb{C}$ then the rational function q_0 takes the form $q_0(x) = \frac{N(x)}{4(x-\beta)^k}$, where $k > 2$ and N is a polynomial function of order at most $k-2$. In the following we assume that $q_0(x) = \frac{A(x-\beta)^{2k}+B(x-\beta)^k+C}{4(x-\beta)^2}$ with $A, B, C \in \mathbb{C}$ and A, B not both zero and $k \in \mathbb{R} \setminus \{0\}$. For integer k, we get some special cases of the function $N(x)$ and for non-integer k we get further special cases in which the function q_0 is possibly not rational. Using the coordinate transformation

$$\varphi_1(x) = \alpha(x-\beta)^k, \quad \varphi_1^{-1}(x) = \alpha^{-1/k}x^{1/k}+\beta, \quad \alpha \in \mathbb{C} \setminus \{0\},$$

we get

$$\begin{aligned} q_{0,1}(x) &= \frac{A\alpha^{-2}x^2+B\alpha^{-1}x+C+k^2-1}{4x^2k^2} \\ m(x) &= (\alpha k)^{1/2}(x-\beta)^{(k-1)/2} \end{aligned}$$

Case 1: $A \neq 0$ and $B = 0$. Then g can be expressed in terms of Bessel functions as follows. We choose $\alpha \in \mathbb{C}$ such that $\alpha^2 = \frac{A}{4k^2}$ and we choose $a \in \mathbb{C}$ such that $4a^2 = \frac{1-C}{k^2}$. Then by (11) we get using $H(x) = x^{-1/2}$ with some constant C:

$$f(x) = \frac{h(x)}{m(x)} \cdot \frac{B_a(\varphi_1(x))}{H(\varphi_1(x))} = C \cdot h(x)(x-\beta)^{1/2}B_a\big(\alpha(x-\beta)^k\big)$$

where B_a is a solution to the Bessel equation with parameter a.
Case 2: $A = 0$ and $B \neq 0$. This is just Case 1 with $2k$ replaced by k.
Case 3: $A \neq 0$ and $B \neq 0$. Then g can be expressed in terms of Kummer functions as follows. We choose $\alpha \in \mathbb{C}$ such that $\alpha^2 = -\frac{A}{k^2}$ and we solve

$$\frac{B}{\alpha k^2} = 2b - 4a, \qquad \frac{C + k^2 - 1}{k^2} = 2b - b^2$$

for a and b by putting

$$b := 1 + \frac{\sqrt{1-C}}{k}, \quad a := \frac{b}{2} - \frac{B}{4\alpha k^2}.$$

Then by (11) we get using $H(x) = x^{-b/2}\exp(\frac{x}{2})$ that

$$\begin{aligned} f(x) &= \frac{h(x)}{m(x)} \cdot \frac{K_{ab}(\varphi_1(x))}{H(\varphi_1(x))} \\ &= C(x-\beta)^{(1-k)/2}(x-\beta)^{kb/2}\exp\Big(-\frac{\alpha(x-\beta)^k}{2}\Big)K_{ab}\big(\alpha(x-\beta)^k\big) \\ &= C(x-\beta)^{(1+kb-k)/2}\exp\Big(-\frac{\alpha(x-\beta)^k}{2}\Big)K_{ab}\big(\alpha(x-\beta)^k\big) \end{aligned}$$

where K_{ab} is a solution to Kummer's equation with parameters a, b.

3.3 Two regular singular points

Assume that the equation for g has two regular singular points at $x_0 = \beta \in \mathbb{C}$ and $x_0 = \infty$ and no irregular singular point. Similarly as in 3.1 we conclude that $q_{0,1}(x) = \frac{C}{(x-\beta)^2}$ for some constant $C \in \mathbb{C} \setminus \{0\}$. If $C = \frac{1}{4}$ we get

$$g(x) = A\sqrt{x-\beta} + B\log(x-\beta)\sqrt{x-\beta}, \quad A, B \in \mathbb{C}.$$

If $C \neq \frac{1}{4}$ we make the ansatz $g(x) = (x-\beta)^k = \exp(k\log(x-\beta))$, $k \in \mathbb{C}$ and we get

$$g(x) = A\exp\Big(\frac{1+\sqrt{1-4C}}{2}\log(x-\beta)\Big) + B\exp\Big(\frac{1-\sqrt{1-4C}}{2}\log(x-\beta)\Big), \quad A, B \in \mathbb{C}.$$

3.4 One irregular singular point

Assume that the equation for g has an irregular singular point at $x_0 = \infty$ and no regular singular point. Then the rational function q_0 is a polynomial function.
In the following we assume that $q_0(x) = A(x-x_0)^{2k-2} + B(x-x_0)^{k-2}$ with $k \geq 2$, $x_0 \in \mathbb{R}$ and $A, B \in \mathbb{C}$ and not both zero. We first consider the case $k = 2$. If f is a solution to Hermite's equation

$$f''(x) - 2xf'(x) + 2\lambda f(x) = 0, \quad \lambda \in \mathbb{C}$$

and $\widetilde{f}(x) := f(\alpha(x-x_0))$ for some α, x_0 then writing $\widetilde{f} = \widetilde{h}\widetilde{g}$ with $\widetilde{h}(x) := \exp(x^2/2)$ we get $\widetilde{g}''(x) + \widetilde{q}_0(x)\widetilde{g}(x) = 0$ with $\widetilde{q}_0(x) = A(x-x_0)^2 + B$ with $A = -\alpha^4$ and $B =$

$(2\lambda+1)\alpha^2$. Assume that our computer algebra system can determine A and B then it can also determine $\lambda = \frac{1}{2}(\frac{B}{(-A)^{1/2}} - 1)$. In case that λ is an integer we can express the solutions in terms of Hermite polynomials and Kummer functions. In the general case we distinguish again two cases:
Case 1: $A = 0$ and $B \neq 0$. Then the solution can be expressed in terms of Bessel functions as follows. We put

$$\alpha := \frac{\sqrt{A}}{k}, \quad a := \frac{1}{2k}$$

and we put

$$\varphi_1(x) := \alpha(x - x_0)^k, \quad \varphi_1^{-1}(x) = \alpha^{-1/k}x^{1/k} + x_0.$$

Then we get by (11)

$$f(x) = \frac{h(x)}{m(x)} \cdot \frac{B_a(\varphi_1(x))}{H(\varphi_1(x))} = C \cdot h(x)(x - x_0)^{1/2} B_a\big(\alpha(x - x_0)^k\big)$$

where B_a is a solution to the Bessel equation with parameter a.
Case 2: $A \neq 0$. We put

$$b := 1 + \frac{1}{k}, \quad \alpha := \frac{2\sqrt{-A}}{k}, \quad a := \frac{b}{2} - \frac{B}{\alpha k^2}$$

and we put

$$\varphi_1(x) := \alpha(x - x_0)^k, \quad \varphi_1^{-1}(x) = \alpha^{-1/k}x^{1/k} + x_0.$$

Then we get by (11)

$$f(x) = \frac{h(x)}{m(x)} \cdot \frac{K_{ab}(\varphi_1(x))}{H(\varphi_1(x))} = C \cdot (x - x_0)^{(1+kb-k)/2} \exp\Big(-\frac{\alpha(x - x_0)^k}{2}\Big) K_{ab}\big(\alpha(x - x_0)^k\big)$$

where K_{ab} is a solution to the Kummer equation with parameters a, b.

3.5 Four regular singular points

Assume that $q_0(x) = \frac{Ax^2+Bx+C}{(x-x_1)^2(x-x_2)^2(x-x_3)^2}$ with $x_1, x_2, x_3 \in \mathbb{C}$ distinct, $A, B, C \in \mathbb{C}$. We put

$$\delta := x_1, \quad \beta := x_2, \quad \alpha := \frac{x_1 - x_3}{x_2 - x_3}$$

and we put

$$\varphi_1(x) := \frac{\alpha(x - \beta)}{x - \delta}, \quad \varphi_1^{-1}(x) = \frac{\delta x - \alpha\beta}{x - \alpha}.$$

Then we get

$$q_{0,1}(x) = \frac{\widehat{A}x^2 + \widehat{B}x + \widehat{C}}{4x^2(x-1)^2}$$

for some $\widehat{A}, \widehat{B}, \widehat{C} \in \mathbb{C}$ and thus the equation for $f \circ \varphi_1^{-1}$ has three regular singular points and can be treated as described above.

3.6 A method for the determination of coefficients

a) If we implement the above method in a computer algebra system it happens that we need to determine whether a function g specified by the user is of the form $g(x) = Bx^k + C$ or of the form $g(x) = Ax^{2k} + Bx^k + C$ with real constants A, B, C, k and if so we need to determine these constants. We assume that our computer algebra system can compute symbolic derivatives of functions. For solving this problem the following remarks are useful.
Assume that g is a non-constant smooth function. We define

$$h(x) := \frac{1}{\frac{dg}{dx}(x)} \frac{d}{dx}\Big(x \frac{dg}{dx}(x)\Big).$$

If $h = 0$ is constant then $g(x) = B\log(x) + C$ with constants B, C.
If $h \neq 0$ is constant then $g(x) = Ax^{2k} + C$ with constants A, C and $k = \frac{h}{2}$.
If h is not constant then g is neither of the form $g(x) = Ax^{2k} + C$ nor of the form $g(x) = B\log(x) + C$.
In the following we assume the latter and we want to determine whether g is of the form $g(x) = Ax^{2k} + Bx^k + C$ with constants A, B, C, k and $k \neq 0$. If $g(x) = Ax^{2k} + Bx^k + C$ then we have

$$\frac{d}{dx}\Big(x \frac{d}{dx}\Big(x \frac{dg}{dx}\Big)\Big)(x) - 3k \frac{d}{dx}\Big(x \frac{dg}{dx}\Big)(x) + 2k^2 \frac{dg}{dx}(x) = 0$$

Assuming that g is not constant we get

$$\frac{d}{dx}\Big(\frac{1}{\frac{dg}{dx}(x)} \frac{d}{dx}\Big(x \frac{d}{dx}\Big(x \frac{dg}{dx}\Big)\Big)\Big)(x) - 3k \frac{dh}{dx}(x) = 0$$

where h is as defined above. Thus if

$$k(x) := \frac{1}{3\frac{dh}{dx}(x)} \frac{d}{dx}\Big(\frac{1}{\frac{dg}{dx}(x)} \frac{d}{dx}\Big(x \frac{d}{dx}\Big(x \frac{dg}{dx}\Big)\Big)\Big)(x)$$

is not constant or is zero then g is not of the desired form. On the other hand, if k is a non-zero constant we still need to check whether g is of the desired form. We therefore compute

$$A(x) := \frac{x^{1-k}}{2k^2} \frac{d}{dx}\Big(x^{1-k} \frac{dg}{dx}\Big).$$

If A is constant, then we define

$$B(x) := \frac{x^{1-k}}{k} \frac{d}{dx}(g(x) - Ax^{2k}).$$

If B is constant then we check whether

$$C(x) := g(x) - Ax^{2k} - Bx^k$$

is constant.

b) Also we want to determine whether a function g specified by the user is of the form $g(x) = A(x - x_0)^k$ for unknown A, x_0, k. Assume that g is of this form. Then we get $h(x) := \frac{g'(x)}{g(x)} = \frac{k}{x - x_0}$ and therefore $\frac{h'(x)}{h(x)^2}$ is independent of x. On the other hand, if $\frac{h'(x)}{h(x)^2}$ is a non-zero constant we know that $h(x) = \frac{k}{x - x_0}$ for some constants $k \neq 0, x_0$ and thus $g(x) = A(x - x_0)^k$ for some constant A.
c) In order to determine whether a function g specified by the user is of the form $g(x) = A(x - x_0)^k + C$ for some constants $A, C, k \neq 0, x_0$ we check whether $g'(x)$ is of the form $\widetilde{A}(x - x_0)^r$ for some $r \neq -1, \widetilde{A}, x_0$.
d) In order to determine whether a function g specified by the user is of the form $g(x) = A(x - x_0)^{2k-2} + C(x - x_0)^{k-2}$ for some constants $A, C, k \neq 0, x_0$ we compute $h(x) := \frac{g'(x)}{g(x)}$ and we check whether the numerator of h is of the form $\widetilde{A}(x - x_0)^k + \widetilde{C}$ for some constants $\widetilde{A}, \widetilde{C}, x_0, k \neq 0$.

4 Discussion

The algorithm introduced in this note tries to detect the singular points of an ordinary differential equation. Furthermore, it computes the transformation of the coefficients of the equation under a change of variable. Therefore, one may implement this algorithm in such a way that the user of the program has the possibility to propose a change of variable if an equation cannot be solved.
It seems difficult to give an algorithm that finds a suitable change of variable to reduce a general equation to a known equation. Still, it might be possible to give an algorithm that finds under certain conditions a change of variable that decreases the number of singular points of the equation.

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