A0232935 A

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1. $M_X(r) = E[e^{rX}] = \int_0^\infty e^{rX} \cdot f_X(x)\,dx$

$$= \int_0^\infty e^{rX} \cdot \lambda e^{-\lambda X}\,dx$$

$$= \frac{\lambda}{r-\lambda} e^{(r-\lambda)x} \Big|_0^\infty$$

$$= \frac{\lambda}{\lambda - r}$$

$M_X(r)_{S_n} = E[e^{rS_n}] = E[e^{r\cdot(X_1+X_2+\dots+X_n)}] = E[e^{rX_1} \cdot e^{rX_2} \cdot e^{rX_3} \cdots e^{rX_n}]$

Because $X_1, X_2 \dots X_n$ are iid.

$M_X(r)_{S_n} = E[e^{rX_1}] \cdot E[e^{rX_2}] \cdots E[e^{rX_n}] = \frac{\lambda^n}{(\lambda - r)^n}$

Invert:

$\mathcal{L}^{-1}[M_X(r)_{S_n}] = \mathcal{L}^{-1}[\frac{\lambda^n}{(\lambda-r)^n}] = \frac{\lambda^n}{(n-1)!} \cdot \mathcal{L}^{-1}[\frac{(n-1)!}{(\lambda-r)^n}]$

$$= \frac{\lambda^n}{(n-1)!} \cdot S_n^{n-1} \cdot e^{-\lambda S_n}$$

2.

(a)

Mean:

$$\cancel{E[N(t)] = \int_0^\infty n \cdot P_{N(t)}(n)\, dt}$$

$$\cancel{= \int_0^\infty n \cdot \frac{(\lambda t)^n e^{-\lambda t}}{n!}\, dt}$$

$$=$$

$$E[N(t)] = \sum_{n=1}^{\infty} n \cdot P_{N(t)}(n)$$

$$= \sum_{n=1}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{(n-1)!}$$

$$= \lambda t \cdot e^{-\lambda t} \cdot \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t \cdot e^{-\lambda t} \cdot e^{\lambda t}$$

$$= \lambda \cdot t$$

$$Var(N(t)) = E[N(t)^2] - (E[N(t)])^2$$

$$= \sum_{n=1}^{\infty} n^2 \cdot P_{N(t)}(n) - (\lambda t)^2$$

$$= \lambda t \cdot e^{-\lambda t} \cdot \sum_{n=1}^{\infty} \frac{n (\lambda t)^{n-1}}{(n-1)!} - (\lambda t)^2$$

$$= \lambda t \cdot e^{-\lambda t} \sum_{n=1}^{\infty} \left[\frac{(\lambda t)^{n-1}}{(n-2)!} + \frac{(\lambda t)^{n-1}}{(n-1)!} \right] - (\lambda t)^2$$

$$= (\lambda t)^2 \cdot e^{-\lambda t} \cdot \sum_{n=2}^{\infty} \frac{(\lambda t)^{n-2}}{(n-2)!} + (\lambda t) \sum \frac{(\lambda t)^{n-1}}{(n-1)!} - (\lambda t)^2$$

$$= (\lambda t)^2 + \lambda t - (\lambda t)^2$$

$$= \lambda t$$

MGF:

$$E[e^{rN(t)}] = \sum_{n=0}^{\infty} e^{rn} \cdot \frac{(\lambda t)^n \cdot e^{-\lambda t}}{n!}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda \cdot t \cdot e^r)^n}{n!}$$

$$= e^{-\lambda t} \cdot e^{\lambda t e^r}$$

$$= e^{\lambda t (e^r - 1)}$$

2.(b)

~~$Pr\{Y = X_1+X_2\} = \sum_{k=0}^{Y} Pr\{X_1=k\} \cdot Pr\{X_2 = Y-k\}$.~~

~~$= \sum_{k=0}$~~

Let $X_1+X_2=n$,

$$Pr(X_1+X_2=n) = \sum_{i=0}^{n} Pr\{X_1=i\} Pr\{X_2=n-i\}$$

$$= \sum_{i=0}^{n} \frac{\lambda_1^i e^{-\lambda_1}}{i!} \cdot \frac{\lambda_2^{n-i} e^{-\lambda_2}}{(n-i)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \sum_{i=0}^{n} \frac{\lambda_1^i \cdot \lambda_2^{n-i}}{i!(n-i)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{i=0}^{n} \binom{n}{i} \cdot \lambda_1^i \lambda_2^{n-i}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot (\lambda_1+\lambda_2)^n.$$

$X_1+X_2 \sim Poi(\lambda_1+\lambda_2)$

3. (a)

$$P_{N(t)}(n) = \frac{(\lambda t)^n \cdot e^{-\lambda t}}{n!}$$

$$P_r\{X_1 > x\} = P_r\{N(t)=0\} = e^{-\lambda t}.$$

(b) Because $N(t)$ has IIP and SIP.

$N(\tau, \tau+x)$ and $N(\tau)$ are independent.

$$P_r(X_n > x \mid S_n = \tau) = P_r\{\tilde{N}(\tau, \tau+x) = n-1 \mid N(\tau) = n-1\}.$$

$$= P_r\{N(t)=0\} = e^{-\lambda x}.$$

(c) $P_r\{X_n > x\} = P_r\{\tilde{N}(\tau, \tau+x) = n-1\} = P_r\{N(t)=0\} = e^{-\lambda x}$

Because $P_r\{X_n > x\} = P_r\{X_n > x \mid S_n = \tau\}$, X_n is independent of S_{n-1}

(d). Let $X_1 = t_1$, $X_2 = t_2 \ldots$, $X_{n-1} = t_{n-1}$,

$$P_r\{X_n > x \mid X_1 = t_1, X_2 = t_2, \ldots X_{n-1} = t_{n-1}\}$$

$$= P_r\{X_n > x \mid S_{n-1} = (t_1 + \cdots + t_{n-1})\}$$

$$= P_r(X_n > x).$$

So X_n is independent of $X_1, X_2, \ldots X_{n-1}$.

4.

(a) $\frac{dF_0(\tau)}{d\tau} = \lim_{\delta \to 0} \frac{F_0(\tau+\delta) - F_0(\tau)}{\delta}$

$= \lim_{\delta \to 0} \frac{Pr(N(\tau+\delta)=0) - Pr(N(\tau)=0)}{\delta}$

$= \lim_{\delta \to 0} \frac{Pr\{N(\tau)=0\} \cdot Pr\{\tilde{N}(\tau,\tau+\delta)=0\} - Pr\{N(\tau)=0\}}{\delta}$

$= \lim_{\delta \to 0} \frac{Pr\{N(\tau)=0\} \cdot \{Pr\{\tilde{N}(\tau,\tau+\delta)=0\} - 1\}}{\delta}$

$= Pr\{N(\tau)=0\} \cdot \lim_{\delta \to 0} \frac{1-\lambda\delta+o(\delta)-1}{\delta}$

$= -\lambda \cdot F_0(\tau)$

(b) $Pr\{X_1 > x\} = Pr\{N(t)=0\}$

$\frac{dF^c(X_1)}{dx} = -\lambda F^c(X_1) = -\lambda F_0(\tau)$

$\Rightarrow F^c(X_1) = e^{-\lambda x} \quad x > 0$

$\therefore F_x(X_1) = 1 - e^{-\lambda x}$

$\therefore f_{X_1}(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & \text{otherwise.} \end{cases}$

(c) ~~$F_n^c(\tau) = Pr\{N(\tau)=0\}$~~. $F_n^c(t) = \{\tilde{N}(t, t+\tau)=0 \mid S_{n-1}=t\}$

$= \{\tilde{N}(t, t+\tau)=0 \mid N(t)=n-1\}$

Because $N(t)$ is IIP & SIP.

$LHS = Pr\{\tilde{N}(t, t+\tau)=0\}$

$= Pr\{N(\tau)=0\}.$

From 4.(a), we can get $\frac{dF_n^c(\tau)}{d\tau} = -\lambda F_n^c(\tau)$

4.(d)

From 4.(c).

$Pr\{\tilde{N}(t, t+\tau)=0 \mid S_{n-1}=t\}$

$= Pr\{X_n > \tau \mid S_{n-1}=t\}$

$= Pr\{X_n \geqslant \tau \mid X_1=\tau_1, X_2=\tau_2, \ldots X_{n-1}=\tau_{n-1}\}$

$= Pr(X_n > \tau) = e^{-\lambda t} = F^c(X_n)$ [4.(b) and 4.(c)]

$\Rightarrow X_n$ is indepent of $X_1, X_2 \ldots X_{n-1}$

5.

$W = A + B$. Because A, B are Poisson process, W is also a Poisson Process

(2.(b)) $W \sim Poi(\lambda_A + \lambda_B)$

(a) $P_r\{N(t)=9\} = \frac{(\lambda_A+\lambda_B)^9}{9!} \cdot e^{-(\lambda_A+\lambda_B) \cdot t}$

(b) $E[N] = E[E[N \mid N(t)=n]]$

For 1 message, the expected of words is:

$E[N \mid N(t)=1] = \frac{2}{6} \times 1 + \frac{3}{6} \times 2 + \frac{1}{6} \times 3 = \frac{11}{6}$

Because the transimitte process is indepenet

$E[N \mid N(t)=n] = \frac{11}{6} n$.

$E[N] = E[\frac{11}{6} n] = \frac{11}{6} E[N(t)=n] = \frac{11}{6} \cdot (\lambda_A+\lambda_B) \cdot t$

5.

(c) $Pr\{W=3 \mid N(t)=8\} = (\frac{1}{6})^8$

$$Pr\{W=3, N(t)=8\} = Pr\{W=3 \mid N(t)=8\} \cdot Pr\{N(t)=8\}$$

$$= (\frac{1}{6})^8 \cdot \frac{\lambda_A^8 \cdot e^{-\lambda_A t}}{8!}$$

5.

(d) $\cancel{Pr\{A=8 \mid A+B=12\}}$ $Pr\{A=8, A+B=12\}$

$\cancel{= Pr\{A=8 \mid B=4\}}$

$$= Pr\{A=8, B=4\}$$

$$= Pr(A=8) \cdot Pr(B=4)$$ (A and B are indepent)

$$= \frac{\lambda_A^8 e^{-\lambda_A t}}{8!} \cdot \frac{\lambda_B^4 \cdot e^{-\lambda_B t}}{4!}$$

6.

(i) Because Poisson Process have both the stationary increment and independent increment properties.

(ii) exponential distribution.

(iii) $\lambda \cdot X$

(iv) ~~$Pr\{t^*-U \geq x\} = Pr\{N(t^*) - \ldots\}$~~

$P\{t^*-U \leq x\} = e^{-\lambda x}$.

So, t^*-U is a exponential distribution

$$f_{(t^*-U)}(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & o.w. \end{cases}$$

(v) $Pr\{L \leq l\}$

$= Pr\{t^*-U + L-(t^*-U) \leq l\}$.

$= Pr\{t^*-U \leq l_1\} \cdot Pr\{L - (t^*-U) \leq l-l_1\}$.

$= e^{-\lambda l_1} \cdot e^{-\lambda(l-l_1)}$

$= e^{-\lambda l}$

$\therefore f_L(l) = \begin{cases} \lambda e^{-\lambda l}, & l>0 \\ 0, & o.w \end{cases}$

(vi) It's a exponetial distribution with rate λ.