National University of Singapore
Department of Electrical & Computer Engineering

Examination for

EE5137 Stochastic Processes
(Semester I, 2017/18)
November/December 2017

Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- This paper contains **FOUR (4)** questions, printed on **FIVE (5)** pages.
- The total number of marks is 100.
- Answer all questions.
- Programmable calculators are NOT allowed.
- Electronic communicating devices MUST be turned off and inaccessible throughout the examination. They CANNOT be used as calculators, timers or clocks.
- You are allowed to bring ONE (A4) size help sheet.
- No other material is allowed.

Question 1

We toss a biased coin n times. The probability of heads, denoted by q, is the value of a random variable Q with a given mean μ and variance σ^2. Let X_i be a Bernoulli random variable that models the outcome of the i-th toss (i.e., $X_i = 1$ if the i-th toss is a head). In other words, for each $1 \leq i \leq n$,

$$X_i = \begin{cases} 1 & \text{w.p. } Q \\ 0 & \text{w.p. } 1-Q \end{cases},$$

where $Q \in [0,1]$ is a random variable with

$$\mathbb{E}[Q] = \mu, \quad \text{and} \quad \mathrm{Var}(Q) = \sigma^2.$$

We assume that $X_1, X_2, \ldots, X_n$ are conditionally independent given $\{Q = q\}$. let

$$S_n = X_1 + X_2 + \ldots + X_n$$

be the total number of heads in the n tosses.

1(a) (5 points) Use the law of iterated expectations to find $\mathbb{E}[X_i]$ and $\mathbb{E}[S_n]$.

1(b) (3 points) Using the fact that $X_i^2 = X_i$, show that $\mathrm{Var}(X_i) = \mu - \mu^2$.

1(c) (5 points) Using the law of iterated expectations, find

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j], \quad \text{for} \quad i \neq j.$$

Are X_i and X_j independent?

1(d) (7 points) By writing $\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$, show that

$$\mathrm{Var}(S_n) = \mathbb{E}[\mathrm{Var}(S_n|Q)] + \mathrm{Var}(\mathbb{E}[S_n|Q]), \tag{1}$$

where $\mathrm{Var}(S_n|Q)$ is the random variable that takes on the value $\mathrm{Var}(S_n|Q = q)$ with probability $\Pr(Q = q)$.

1(e) (5 points) Calculate the variance of S_n by using the formula (1) in part 1(d).

Question 2

2(a) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda = \ln 2 > 0$.

- (6 points) Find the probability that there are *no arrivals* in $(3, 5]$.
  Express your answer as a rational number (fraction).
- (7 points) Find the probability that there is *exactly one arrival* in each of the intervals $(0, 1]$, $(2, 3]$, and $(99, 100]$.
  Express your answer in terms of $\ln 2$.

2(b) (12 points) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda > 0$. Find the covariance function of this process, i.e.,

$$C_N(t_1, t_2) := \mathrm{Cov}\big(N(t_1), N(t_2)\big), \quad \text{for } t_1, t_2 \in [0, \infty)$$

Hints: (i) First assume that $t_1 > t_2$. (ii) Write

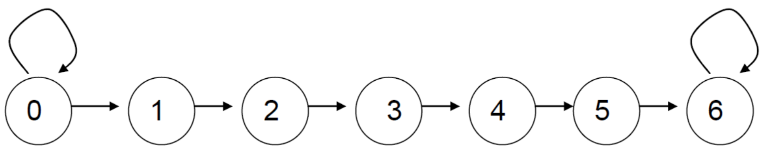
$$N(t_1) = \big[N(t_1) - N(t_2)\big] + \big[N(t_2)\big]$$

in some formula. (iii) You may use the fact that the variance of a Poisson random variable is the same as its mean.

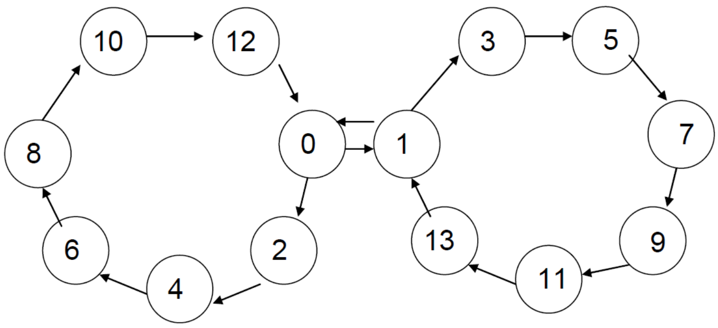
Question 3

3(a) For the following finite-state Markov chains, each transition is marked with $\longleftarrow$ or $\longrightarrow$, the transition probability is nonzero. For each chain, identify all classes, determine the period of each class, and specify whether each class is recurrent or transient. Explain your answer carefully.

- (5 points) Chain 1:



- (5 points) Chain 2:



3(b) An auto insurance company classifies its customers in three categories: bad, satisfactory and preferred. No one moves from bad to preferred or from preferred to bad in one year. 40% of the customers in the bad category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become bad; 20% of those in the preferred category are downgraded to satisfactory.

- (5 points) Write the state transition matrix for the model.
- (10 points) What is the limiting fraction of customers in each of these categories, i.e., the fraction of bad, satisfactory, and preferred customers after many years?

Question 4

Binary frequency shift keying (FSK) on a Rayleigh fading channel can be modeled in terms of a 4-dimensional observation vector

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix}$$

which is given by $\mathbf{Y} = \mathbf{X}+\mathbf{Z}$ where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$ and $\mathbf{Z}$ is independent of $\mathbf{X}$. Under the two hypotheses, $\mathbf{X}$ takes the following two values

$$\mathbf{X} = \begin{cases} \begin{bmatrix} X_1 \\ X_2 \\ 0 \\ 0 \end{bmatrix} & \text{if } H = H_0 \\ \begin{bmatrix} 0 \\ 0 \\ X_3 \\ X_4 \end{bmatrix} & \text{if } H = H_1 \end{cases}$$

The X_i's are i.i.d. $\mathcal{N}(0, \alpha^2)$ random variables. Furthermore, the two hypotheses are equally likely. Assume that $\alpha, \sigma \neq 0$.

4(a) (10 points) Show that the maximum likelihood receiver calculates $V_0 = Y_1^2 + Y_2^2$ and $V_1 = Y_3^3 + Y_4^2$ and chooses $\hat{H} = H_0$ if $V_0 \geq V_1$ and chooses $\hat{H} = H_1$ otherwise.

4(b) (2 points) It is known that if $A \sim \mathcal{N}(0, \nu)$ and $B \sim \mathcal{N}(0, \nu)$ are independent Gaussians, then the distribution (pdf) of $R = A^2 + B^2$ is exponential

$$f_R(t) = \frac{1}{2\nu} e^{-t/(2\nu)}, \quad \forall\, t \geq 0.$$

Using this fact, write down the distributions (pdfs) $f_{V_0|H}(v_0|H_0)$ and $f_{V_1|H}(v_1|H_0)$.

4(c) (6 points) Let $U = V_0 - V_1$. Using convolutions, show that $f_{U|H}(u|H_0)$ is

$$f_{U|H}(u|H_0) = \begin{cases} \frac{ab}{a+b} e^{bu} & u < 0 \\ \frac{ab}{a+b} e^{-au} & u \geq 0 \end{cases}$$

and identify the constants a and b in terms of σ^2 and α^2.

4(d) (7 points) Define the error event

$$\mathcal{E} := \{\hat{H} \neq H\}.$$

Find an expression in terms of a and b (and hence σ^2 and α^2 if you manage to get part 4(c)) for the conditional probability of error $\Pr(\mathcal{E}|H = H_0)$. Also find $\Pr(\mathcal{E}|H = H_1)$ and hence the unconditional probability of error $\Pr(\mathcal{E})$.

END OF PAPER

Solutions to EE5137 Exam (Semester 1 2017/8)

November 30, 2017

1 Problem 1

(a) Using the law of iterated expectations,

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i|Q]] = \mathbb{E}[Q] = \mu$$

where the second equality is because $\mathbb{E}[X_i|Q = q] = q$. Now, the expectation of the sum is

$$\mathbb{E}[S_n] = \mathbb{E}[X_1 + \ldots + X_n] = n\mu.$$

(b) The variance is

$$\mathrm{Var}(X_i) = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = \mathbb{E}[X_i] - (\mathbb{E}[X_i])^2 = \mu - \mu^2.$$

(c) The covariance can be computed as

$$\begin{aligned}
\mathrm{Cov}(X_i, X_j) &= \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \\
&= \mathbb{E}[\mathbb{E}[X_i X_j|Q]] - \mu^2 \\
&= \mathbb{E}[\mathbb{E}[X_i|Q]\mathbb{E}[X_j|Q]] - \mu^2 \\
&= \mathbb{E}[Q^2] - \mu^2 \\
&= \sigma^2 + \mu^2 - \mu^2 \\
&= \sigma^2 > 0.
\end{aligned}$$

Hence, the random variables X_i and X_j are not independent.

(d) We now derive $\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$ using the law of iterated expectations. We have

$$\begin{aligned}
\mathbb{E}[S_n^2] - \mathbb{E}[S_n]^2 &= \mathbb{E}[\mathbb{E}[S_n^2|Q]] - (\mathbb{E}[\mathbb{E}[S_n|Q]])^2 \\
&= \mathbb{E}[\mathbb{E}[S_n^2|Q] - \mathbb{E}[S_n|Q]^2] + \mathbb{E}[\mathbb{E}[S_n|Q]^2] - (\mathbb{E}[\mathbb{E}[S_n|Q]])^2 \\
&= \mathbb{E}[\mathrm{Var}(S_n|Q)] + \mathrm{Var}(\mathbb{E}[S_n|Q])
\end{aligned}$$

(e) We calculate $\mathrm{Var}(S_n|Q)$ first. By conditional independence, we have

$$\begin{aligned}
\mathrm{Var}(S_n|Q) &= \mathrm{Var}(X_1 + \ldots + X_n|Q) \\
&= nQ(1 - Q)
\end{aligned}$$

Next we recall that $\mathbb{E}[S_n|Q] = nQ$. Thus,

$$\begin{aligned}
\mathrm{Var}(S_n) &= \mathbb{E}[nQ(1 - Q)] + \mathrm{Var}(nQ) \\
&= n[\mu - \sigma^2 - \mu^2] + n^2\sigma^2 \\
&= n[\mu - \mu^2] + n(n - 1)\sigma^2.
\end{aligned}$$

2 Problem 2

(a) The Poisson process has rate $\lambda = \ln 2$. The probability that there are zero arrivals in $(3, 5]$ is

$$\Pr(N(2) = 0) = e^{-2\lambda} = e^{-2\ln 2} = \frac{1}{4}.$$

Here we used stationary increment property.

Probability that there is exactly one arrival in the three non-overlapping intervals of length 1 each is

$$\Pr(N(1) = 1)^3 = (e^{-\lambda}\lambda)^3 = (\frac{1}{2}\ln 2)^3 = \frac{1}{8}\ln^3 2.$$

(b) Assume $t_1 \geq t_2$. Consider

$$\begin{aligned}
C_N(t_1, t_2) &= \mathbb{E}[N(t_1)N(t_2)] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] \\
&= \mathbb{E}[(N(t_1) - N(t_2) + N(t_2))N(t_2)] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] \\
&\overset{(a)}{=} \mathbb{E}[N(t_1) - N(t_2)]\mathbb{E}[N(t_2)] + \mathbb{E}[N(t_2)^2] - \mathbb{E}[N(t_1)]\mathbb{E}[N(t_2)] \\
&= \lambda(t_1 - t_2) \cdot \lambda t_2 + \mathbb{E}[N(t_2)^2] - \lambda t_1 \cdot \lambda t_2 \\
&= \mathbb{E}[N(t_2)^2] - \lambda^2 t_2^2 \\
&= \mathbb{E}[N(t_2)^2] - \mathbb{E}[N(t_2)]^2 \\
&= \mathrm{Var}(N(t_2)) \\
&= \lambda t_2
\end{aligned}$$

where (a) follows from the independent increment property (i.e., that $N(t_1) - N(t_2)$ is independent of $N(t_2)$). By symmetry, if $t_1 \leq t_2$,

$$C_N(t_1, t_2) = \lambda t_1.$$

Thus,

$$C_N(t_1, t_2) = \lambda \min\{t_1, t_2\}.$$

3 Problem 3

(a) Chain 1: Recall that two states i and j in a Markov chain communicate if each is accessible from the other, i.e., if there is a walk from each to the other. Since all transitions move from left to right, each state is accessible only from those to the left, and therefore no state communicates with any other state. Thus each state is in a class by itself. States 0 to 5 (and thus the classes $\{0\}, \ldots, \{5\}$ are each transient since each is inaccessible from an accessible state (i.e., there is a path away from each from which there is no return). State 6 is recurrent. States 1 and 6 (and thus class $\{1\}$ and $\{6\}$) are each aperiodic since $P_{00}^1 \neq 0$ and $P_{66}^1 \neq 0$ The periods of classes $\{1\}$ to $\{5\}$ are undefined but no points will be taken off if some other answer was given for these periods.

Chain 2: Each state on the circle on the left communicates with all other states on the left and similarly for the circle on the right. Since there is a transition from left to right, and also from right to left, the entire set of states communicate, so there is single class containing all states. State 0 has a cycle of length 2 through state 1 and of length 7 via the left circle. The greatest common divisor of 2 and 7 is 1, so state 1 has period 1. The chain is then aperiodic since all states in a class have the same period.

(b) The transition matrix is

$$[P] = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

We will find the limiting fraction of drivers in each of these categories from the components of the stationary distribution vector π, which satisfies the following equation:

$$\pi = \pi[P].$$

This is equivalent to the following system of equations:

$$\begin{aligned} \pi_1 &= 0.6\pi_1 + 0.1\pi_2 \\ \pi_2 &= 0.4\pi_1 + 0.6\pi_2 + 0.2\pi_3 \\ \pi_3 &= 0.3\pi_2 + 0.8\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

This has the following solution

$$\pi = \frac{1}{11}(1, 4, 6)$$

Thus, the limiting fraction of drivers in the bad category is $1/11$, in the satisfactory category $4/11$ and in the preferred category $6/11$.

4 Problem 4

(a) The ML decision rule decides in favor of H_1 if

$$f_{\mathbf{Y}|H}(\mathbf{y}|H_1) \geq f_{\mathbf{Y}|H}(\mathbf{y}|H_0).$$

The conditional densities are jointly Gaussian, with zero mean and the following covariance matrices

$$\Sigma_{\mathbf{Y}|H_0} = \begin{bmatrix} (\alpha^2+\sigma^2)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{I} \end{bmatrix}, \quad \Sigma_{\mathbf{Y}|H_1} = \begin{bmatrix} \sigma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\alpha^2+\sigma^2)\mathbf{I} \end{bmatrix}$$

We plug these matrices into the Gaussian density functions, take logarithms and cancel constants, obtaining that we decide in favor of H_1 if

$$\frac{Y_1^2+Y_2^2}{\alpha^2+\sigma^2} + \frac{Y_3^2+Y_4^2}{\sigma^2} \geq \frac{Y_1^2+Y_2^2}{\sigma^2} + \frac{Y_3^2+Y_4^2}{\alpha^2+\sigma^2}$$

Now, we can cancel terms to get the desired result:

$$Y_3^2+Y_4^2 \geq Y_1^2+Y_2^2 \quad \Leftrightarrow \quad V_1 \geq V_0$$

to decide in favor of H_1.

(b) Simply use the hint to obtain

$$f_{V_0|H}(v_0|H_0) = \frac{1}{2(\alpha^2+\sigma^2)} \exp\left(-\frac{v_0}{2(\alpha^2+\sigma^2)}\right), \quad v_0 \geq 0$$

and

$$f_{V_1|H}(v_1|H_0) = \frac{1}{2\sigma^2} \exp\left(-\frac{v_1}{2\sigma^2}\right), \quad v_1 \geq 0$$

(c) Now given $H = H_0$, V_0 and V_1 are independent, so we can find the density of $U = V_0 - V_1$ by convolution. Let

$$a = \frac{1}{2(\alpha^2+\sigma^2)} \quad \text{and} \quad b = \frac{1}{2\sigma^2}.$$

Then

$$f_{U|H}(u|H_0) = f_{V_0|H}(u|H_0) * f_{V_1|H}(-u|H_0)$$

After convolving, we obtain

$$f_{U|H}(u|H_0) = \begin{cases} \frac{ab}{a+b}e^{bu} & u < 0 \\ \frac{ab}{a+b}e^{-au} & u \geq 0 \end{cases}.$$

(d) Given that $H = H_0$, an error occurs if $V_1 > V_0$ or equivalently $U < 0$. We compute

$$\begin{aligned} \Pr(\mathcal{E}|H=H_0) &= \int_{-\infty}^{0} f_{U|H}(u|H_0)\,\mathrm{d}u \\ &= \int_{-\infty}^{0} \frac{ab}{a+b}e^{bu}\,\mathrm{d}u \\ &= \frac{a}{a+b} = \frac{1}{2+\alpha^2/\sigma^2} \end{aligned}$$

Note that if $H = H_1$, then V_1 and V_0 switch roles in the problem. That is, they will both still be exponentially distributed, but their variances will be swapped. If we define $W = V_1 - V_0$, note that all of the above results will hold with a slight change in the subscripts.

In particular,

$$\Pr(\mathcal{E}|H=H_1) = \frac{1}{2+\alpha^2/\sigma^2} = \Pr(\mathcal{E}|H=H_0).$$

Since the two hypotheses are equally likely, this must also be the unconditional probability of error.

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Question 1

1(a) A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel, The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours of travel. If we assume that the miner is at all times equally likely to choose any one of the doors, let us find the expected time for him to get to safety X using the law of iterated expectations. Let $Y \in \{1, 2, 3\}$ be the identity of the door that he initially chooses.

(i) (3 points) It is known that

$$\mathbb{E}[X|Y = 1] = a_1\mathbb{E}[X] + b_1, \quad \mathbb{E}[X|Y = 2] = a_2\mathbb{E}[X] + b_2 \quad \mathbb{E}[X|Y = 3] = a_3\mathbb{E}[X] + b_3$$

Find the constants $a_1, b_1, a_2, b_2, a_3, b_3$.

(ii) (8 points) Use the law of iterated expectations and the above part to find $\mathbb{E}[X]$.

1(b) Here we will show that convergence in probability does not imply convergence in mean, i.e., that $X_n \xrightarrow{\text{p}} b$ as $n \to \infty$ does not imply that $\mathbb{E}[X_n] \longrightarrow b$ as $n \to \infty$. Consider the sequence of random variables $\{X_n\}_{n=1}^{\infty}$, each with probability mass function

$$\Pr(X_n = n^2) = \frac{1}{n}, \qquad \Pr(X_n = 0) = 1 - \frac{1}{n}.$$

Please provide detailed justifications to each of the following problems.

(i) (7 points) It is known that $X_n \xrightarrow{\text{p}} b$ for $n \to \infty$ or some $b \in \mathbb{R} \cup \{\pm\infty\}$. Find b.

(ii) (7 points) It is known that $\mathbb{E}[X_n] \longrightarrow c$ for $n \to \infty$ for some $c \in \mathbb{R} \cup \{\pm\infty\}$. Find c.

Question 2

2(a) (6 points) Consider the hypothesis test

$$\mathsf{H}_0 : X \sim \mathcal{N}(x; 0, \sigma_0^2) \qquad \mathsf{H}_1 : X \sim \mathcal{N}(x; 0, \sigma_1^2)$$

where $\sigma_1 > \sigma_0$. Show that whatever the prior probabilities of H_0 and H_1, we will decide in favor of H_1 if and only if x belongs to

$$\mathcal{Z} := \{x : x < -\gamma\} \cup \{x : x > +\gamma\}$$

for some $\gamma > 0$.

Hint: Diagrams of the two pdfs would help. You can use the fact that the normal pdf takes the form

$$\mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

2(b) Let X_1, X_2 and X_3 be three IID Bernoulli random variables with $\Pr(X_i = 1) = p$ for $i \in \{1, 2, 3\}$. This means that $\Pr(X_i = x) = p^x(1-p)^{1-x}$ for $x \in \{0, 1\}$. It is known that p can take on two values 1/2 or 2/3. In this problem, we consider the hypothesis test

$$\mathsf{H}_0 : p = 1/2, \qquad \mathsf{H}_1 : p = 2/3$$

based on $(X_1, X_2, X_3) \in \{0, 1\}^3$.

(i) (5 points) Let $T = X_1 + X_2 + X_3$ be the number of ones in the random vector (X_1, X_2, X_3). Let P_0 and P_1 be the distributions of X_1, X_2, and X_3 under hypothesis H_0 and H_1 respectively. Write down the likelihood ratio

$$L(X_1, X_2, X_3) := \frac{P_0(X_1, X_2, X_3)}{P_1(X_1, X_2, X_3)}$$

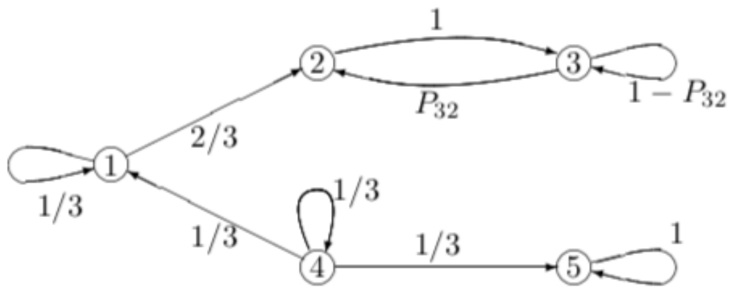
in terms of T. Hence, argue that T is a sufficient statistic for deciding between H_0 and H_1.

(ii) (4 points) Clearly $T \in \{0, 1, 2, 3\}$. Evaluate the values of the likelihood ratio in terms of T.

(iii) (3 points) What is the best probability of missed detection $P_1(\text{declare } \mathsf{H}_0)$ if we allow the probability of false alarm $P_0(\text{declare } \mathsf{H}_1)$ to be 1/8? What is the corresponding test in terms of T?

(iv) (7 points) What is the best probability of missed detection $P_1(\text{declare } \mathsf{H}_0)$ if we allow the probability of false alarm $P_0(\text{declare } \mathsf{H}_1)$ to be 1/4? What is the corresponding test in terms of T?

Hint: You need to consider randomized tests here.



Question 3

3(a) (3 points) Refer to the state transition diagram above. Identify the transient states and identify each class of recurrent states.

3(b) (5 points) For each recurrent class, find the steady-state probability vector $\boldsymbol{\pi} := (\pi_1, \pi_2, \dots, \pi_5)$ for that class.

3(c) (12 points) Find the following n-step transition probabilities, $P_{ij}^n = \Pr\{X = j \,|\, X = i\}$ as a function of n. Give a brief explanation of each.

- (i) P_{44}^n;
- (ii) P_{45}^n;
- (iii) P_{41}^n;
- (iv) $P_{43}^n + P_{42}^n$;
- (v) $\lim_{n\to\infty} P_{43}^n$.

3(d) (5 points) This part is not related to the above parts. We have a 2-state Markov chain whose probability transition matrix is

$$[P] = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$$

The stationary distribution is $\boldsymbol{\pi} = (\pi_1, \pi_2)$. It is known that $[P^n]_{11} - \pi_1$ decays to zero exponentially fast. This means that

$$[P^n]_{11} - \pi_1 = c\phi^n$$

for some constant $c \in \mathbb{R}$ and $\phi \in (-1, 1)$. Find ϕ.

Question 4

4(a) Consider a Poisson process of rate $\lambda > 0$. Let t^* be a fixed time instant and consider the length of the interarrival interval $[U, V]$ that contains t^*. In this question, we would like to determine the distribution of

$$L = (t^* - U) + (V - t^*).$$

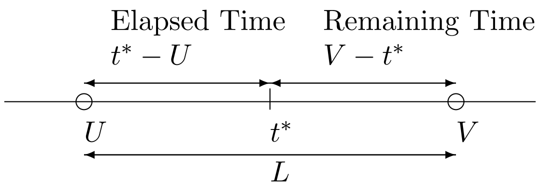


Figure 1: Here U and V are successive arrival epochs and t^* is a fixed time instance between U and V.

(i) (2 point) Give a one sentence answer as to why $V - t^*$ is independent of $t^* - U$.

(ii) (2 point) In class, we determined the distribution of $V - t^*$. What is this distribution?

(iii) (2 points) Consider the event

$$\{t^* - U > x\}.$$

This event is the same as

$$\{\text{there are } k \text{ arrivals in the interval } [t^* - x, t^*]\}.$$

Find the integer k. No explanation is needed.

(iv) (3 points) Hence, find the distribution of $t^* - U$.

(v) (3 points) By using the preceding parts, find the distribution of L.

(vi) (3 points) What is the distribution of an interarrival time of a Poisson process? Why is this the same or different from that of L in part (v)?

4(b) (10 points) Let $X_1, X_2, \ldots$ be a sequence of IID inter-renewal random variables. Let

$$S_n = X_1 + \ldots + X_n$$

be the corresponding renewal epochs for each $n \geq 1$. Assume that each X_i has a finite expectation $\mu > 0$. For any given $t > 0$, show (e.g., using Chebyshev's inequality) that

$$\lim_{n\to\infty} \Pr\{S_n \leq t\} = 0.$$

END OF PAPER

Solutions to EE5137 Exam (Semester 1 2018/9)

December 3, 2018

1 Problem 1

(a) (i) By the question, we know that

$$\mathbb{E}[X|Y=1]=3, \qquad \mathbb{E}[X|Y=2]=\mathbb{E}[X]+5, \qquad \mathbb{E}[X|Y=3]=\mathbb{E}[X]+7,$$

Hence, $a_1=0, b_1=3, a_2=1, b_2=5, a_3=1, b_3=7$.

(ii) By the law of iterated expectations,

$$\begin{aligned}\mathbb{E}[X]&=\mathbb{E}[\mathbb{E}[X|Y]]\\&=\frac{1}{3}(3)+\frac{1}{3}(\mathbb{E}[X]+5)+\frac{1}{3}(\mathbb{E}[X]+7)\end{aligned}$$

This yields after solving the equation

$$\frac{1}{3}\mathbb{E}[X]=1+\frac{5}{3}+\frac{7}{3}=5, \quad \Longrightarrow \quad \mathbb{E}[X]=15.$$

(a) (i) We claim that $b=0$. Let's prove it. For fixed $\epsilon>0$

$$\mathbb{P}(|X_n-0|>\epsilon)=\mathbb{P}(X_n=n^2)=\frac{1}{n}\to 0$$

Hence, X_n converges to 0 in probaiblity and so $b=0$.

(ii) We claim that $c=\infty$. Let's prove it. Consider

$$\mathbb{E}[X_n]=\Pr(X_n=n^2)\cdot n^2+\Pr(X_n=0)\cdot 0=\frac{1}{n}\cdot n^2=n$$

Clearly, $\mathbb{E}[X_n]$ diverges to ∞ and so $c=\infty$.

2 Problem 2

(a) Consider the likelihood ratio test

$$\frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} \overset{\text{decide } \mathsf{H}_0}{\gtrless} \frac{p_1}{p_0} =: \eta,$$

where p_1 and p_0 are the prior probabilities of H_0 and H_1 resp. Then we have

$$\frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} = \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp(-\frac{x^2}{2\sigma_0^2})}{\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp(-\frac{x^2}{2\sigma_1^2})} = \frac{\sigma_1}{\sigma_0}\exp\left(-\frac{x^2}{2}(\sigma_0^{-2} - \sigma_1^{-2})\right).$$

Comparing this to the threshold η, we obtain

$$-\frac{x^2}{2}(\sigma_0^{-2} - \sigma_1^{-2}) \overset{\text{decide } \mathsf{H}_0}{\gtrless} \ln\left(\eta\frac{\sigma_0}{\sigma_1}\right)$$

Hence, by using the fact that $\sigma_0 < \sigma_1$

$$x^2 \overset{\text{decide } \mathsf{H}_1}{\gtrless} 2\frac{1}{\sigma_1^{-2} - \sigma_0^{-2}}\ln\left(\eta\frac{\sigma_0}{\sigma_1}\right) =: \gamma^2$$

So we decide in favor of H_1 iff

$$|x| > \gamma.$$

(b) (i) We have

$$L(X_1, X_2, X_3) = \frac{P_0(X_1)P_0(X_2)P_0(X_3)}{P_1(X_1)P_1(X_2)P_1(X_3)} = \frac{\prod_{i=1}^3 (\frac{1}{2})^{X_i}(\frac{1}{2})^{1-X_i}}{\prod_{i=1}^3 p^{X_i}(1-p)^{1-X_i}} = \frac{1/8}{(2/3)^T(1/3)^{3-T}}$$

Since $L(X_1, X_2, X_3)$ depends only on T, T is a sufficient statistic.

(ii) Note that $T \in \{0, 1, 2, 3\}$. Evaluating the likelihood ratio,

$$L(X_1, X_2, X_3) = \begin{cases} 27/8 & T = 0 \\ 27/16 & T = 1 \\ 27/32 & T = 2 \\ 27/64 & T = 3 \end{cases}$$

(iii) For probability of false alarm to be $1/8$, we need to put the threshold at $(27/64, 27/32)$ and declare that if $T > 2$, then H_1 is declared. This is because $P_0(T > 2) = P_0(T = 3) = 1/8$. Hence, the best probability of detection is $P_1(T > 2) = P_1(T = 3) = (2/3)^3 = 8/27$.

(iv) For probability of false alarm to be $1/4$, we consider that $P_0(T > 1) = 1/2$ and the corresponding probability of detection is $P_1(T > 1) = (2/3)^3 + 3(2/3)^2(1/3) = 20/27$. Hence, we need to randomize between the strategy that places the threshold at $T > 2$ and $T > 1$. Now we find $\alpha \in [0, 1]$ such that

$$\alpha\frac{1}{8} + (1-\alpha)\frac{1}{2} = \frac{1}{4}, \qquad \Longrightarrow \qquad \alpha = \frac{2}{3}$$

Thus, the best probability of detection is

$$\alpha\frac{8}{27} + (1-\alpha)\frac{20}{27} = \frac{12}{27}.$$

The best test in terms of T would be to randomize between $T > 2$ and $T > 1$ where the former has probability $2/3$.

3 Problem 3

(a) States 1 and 4 are transient. States 2 and 3 constitute a class of recurrent states and state 5 constitues another class (a singleton class) of recurrent states.

(b) Let $\boldsymbol{\pi}^{(1)}$ be the be the steady-state vector for class $\{2,3\}$ and let $\boldsymbol{\pi}^{(2)}$ be the steady state vector for class $\{5\}$. For the class $\{5\}$, $\boldsymbol{\pi}^{(2)} = (0,0,0,0,1)$. For the class $\{2,3\}$, the steady-state equations (written just for the recurrent class) are

$$\pi_3^{(1)} P_{32} = \pi_2^{(1)}; \quad \pi_2^{(1)} P_{23} + \pi_3^{(1)} P_{33} = \pi_3^{(1)}; \quad \pi_2^{(1)} + \pi_3^{(1)} = 1$$

Solving these, we obtain

$$\pi_3^{(1)} = \frac{1}{P_{32}+1}, \qquad \pi_2^{(1)} = \frac{P_{32}}{P_{32}+1}.$$

(c) (i) $P_{44}^n = (1/3)^n$ since n successive self-loop transitions, each of probability 1/3, are required.

(ii) $P_{45}^n = (1/3)^1 + (1/3)^2 + \ldots + (1/3)^n = \frac{1}{2}(1 - 3^{-n})$. The reason for this is that there are n walks going from 4 to 5 in n steps; each such walk contains the 4 to 5 transition at a different time. If it occurs at time i then there are $i-1$ self-transitions from 4 to 4, so the probability of that walk is $(1/3)^i$.

(iii) $P_{41}^n = n(1/3)^n$ since there are n walks that go from 4 to 1 in n steps, one for each step in which the $4 \to 1$ transition can be made. Each walk has probability $(1/3)^n$.

(iv) $P_{43}^n + P_{42}^n = 1 - P_{44}^n - P_{45}^n - P_{41}^n = \frac{1}{2} - \frac{2n+1}{2}3^{-n}$ since $\sum_j P_{4j}^n = 1$.

(v) $\lim_{n\to\infty} P_{43}^n$: From (iv) note that $\lim_{n\to\infty} P_{43}^n + P_{42}^n = 1/2$. Given that $X_n \in \{2,3\}$,

$$\lim_{m\to\infty} \Pr(X_m = 3 | X_n \in \{2,3\}) = \pi_3^{(1)} = \frac{1}{P_{32}+1}$$

Since $\lim_{n\to\infty} \Pr(X_n \in \{2,3\} | X_0 = 4) = 1/2$, we see that

$$\lim_{m\to\infty} P_{43}^m = \frac{1}{2(P_{32}+1)}.$$

(d) We only have the find the eigenvalue with the second largest magnitude. For this purpose, consider

$$\det(P - \lambda I) = 0 \qquad \Rightarrow \qquad \det\left(\begin{bmatrix} 0.2-\lambda & 0.8 \\ 0.5 & 0.5-\lambda \end{bmatrix}\right) = 0.$$

Multiplying out, we see that

$$(0.2-\lambda)(0.5-\lambda) - 0.4 = 0.$$

This quadratic equation has two roots, $\lambda_1 = 1$ and $\lambda_2 = -0.3$. Hence, $\phi = -0.3$.

4 Problem 4

(a) (i) Independent increments property of the Poisson process.

(ii) This distribution is exponential with rate λ.

(iii) $k = 0$.

(iv) We need to evaluate

$$\Pr(\text{no arrivals in interval } [t^* - x, t^*]) = \Pr(N(x) = 0) = e^{-\lambda x}.$$

We have shown that

$$\Pr(t^* - U > x) = e^{-\lambda x}, \qquad \Pr(t^* - U \leq x) = 1 - e^{-\lambda x}$$

so $t^* - U$ is also exponential with rate λ.

(v) Since $t^* - U$ and $V - t^*$ are independent exponentials with rate λ, their sum L is Erlang of order 2 with rate λ.

(vi) The interarrival time of a Poisson process is an exponential with rate λ. It is more likely that t^*, being observed, is in a longer interarrival interval.

(b) We have

$$\Pr(S_n \leq t) = \Pr((X_1 - \mu) + \ldots + (X_n - \mu) \leq t - n\mu) = \Pr\left(\frac{1}{n}((X_1 - \mu) + \ldots + (X_n - \mu)) \leq \frac{t}{n} - \mu\right)$$

For n large enough $\frac{t}{n} - \mu \leq -\frac{\mu}{2}$. Hence,

$$\Pr(S_n \leq t) \leq \Pr\left(\frac{1}{n}((X_1 - \mu) + \ldots + (X_n - \mu)) \leq -\frac{\mu}{2}\right)$$

Since each of the summands on the right-hand-side have zero mean, by Chebyshev's inequality or the weak law of large numbers, the right-hand-side probability converges to zero so $\Pr(S_n \leq t) \to 0$.

National University of Singapore
Department of Electrical & Computer Engineering

Examination for

EE5137 Stochastic Processes
(Semester II, 2019/20)
April/May 2020

Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- Use A4 size paper and pen (blue or black ink) to write your answers.
- Write down your student number clearly on the top left of every page of the answers.
- Write the question number and page number on the top right corner of each page (e.g. Q1(a), Q1(b), . . ., Q2(a), . . .).
- This paper contains **FOUR (4)** questions, printed on **FIVE (5)** pages. Answer **ALL** questions.
- The total number of marks is **ONE HUNDRED (**100).
- This exam is **OPEN BOOK**.
- You may use any calculator. However, you should lay out systematically the various steps in the calculations.
- Join the Zoom conference and turn on the video setting at all time during the exam. Adjust your camera such that your face and upper body including your hands are captured on Zoom.
- You may go for a short toilet break (not more than 5 minutes) during the exam.
- At the end of the exam,
  - scan or take pictures of your work (make sure the images can be read clearly);
  - merge all your images into one pdf file (arrange them in the order: Q1 to Q4 in their page sequence);
  - name the pdf file by Matric No_Module Code (e.g. A123456R_EE5137);
  - upload your pdf into the LumiNUS folder "Exam Submission".
- The Exam Submission folder will close at 11.50am. After the folder is closed, exam answers that are not submitted will not be accepted, unless there is a valid reason.

Question 1 (Total 20 Marks)

Answer True or False to the following questions. No justification is required.

Two (2) marks to be awarded to each correct answer.
One (1) mark to be deducted for each wrong answer.
Zero (0) marks for each question that is not answered.
So do not guess.

(a) Let $g(y) := \mathbb{E}[X|Y=y]$. Then $\mathbb{E}[g(Y)] = \mathbb{E}[X]$.

(b) Let $f(y) := \mathrm{Var}(X|Y=y) = \mathbb{E}[X^2|Y=y] - (\mathbb{E}[X|Y=y])^2$. Then $\mathbb{E}[f(Y)] = \mathrm{Var}(X)$.

(c) Let $\gamma_X(r) = \ln \mathbb{E}[e^{rX}]$ be the cumulant generating function of the rv X with $\mathbb{E}X \neq 0$. Then

$$\gamma_X''(r)\Big|_{r=0} = \frac{\mathrm{d}^2}{\mathrm{d}r^2}\gamma_X(r)\Big|_{r=0} = \mathbb{E}[X^2].$$

(d) Let $X_1, \ldots, X_n$ be i.i.d. rvs with $\mathbb{E}[X] = 0$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Then the sequence of rvs

$$\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i, \qquad n = 1, 2, \ldots$$

converges in probability to 0.

(e) The geometric distribution $P_X(k) = (1-p)^{k-1}p$ for $k = 1, 2, \ldots$ has the memoryless property.

(f) Let $\{N_1(t) : t > 0\}$ and $\{N_2(t) : t > 0\}$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Given that there is an arrival of the combined or merged process $\{N(t) = N_1(t) + N_2(t) : t > 0\}$, the probability that it is from the first Poisson process is $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

(g) Let $\{N(t) : t > 0\}$ be a counting process. If for every $t > 0$, $N(t)$ is a Poisson rv with mean λt for some $\lambda > 0$, $\{N(t) : t > 0\}$ is a Poisson process.

(h) Consider the Markov chain with three states $\mathcal{S} = \{1, 2, 3\}$ that has the following transition matrix

$$[P] = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

State 1 is transient.

(i) Again consider the Markov chain as in Part (h) above. If $\Pr(X_1 = 1) = \Pr(X_1 = 2) = 1/4$, then $\Pr(X_1 = 3, X_2 = 2, X_3 = 1) = 1/12$.

(j) Consider a Poisson process for which the arrival rate λ is either λ_0 or λ_1 in which $\lambda_0 \neq \lambda_1$. Suppose we observe the first n interarrival times $Y_1, \ldots, Y_n$ and we would like to make a MAP decision about the arrival rate, i.e., whether it is λ_0 or λ_1. Then

$$\frac{1}{n-1} \sum_{i=1}^{n} Y_i$$

is a sufficient statistic for this binary hypothesis testing problem.

Question 2 (20 marks)

All parts can be done independently.

2(a) (10 marks) Let $\{N_1(t) : t > 0\}$ and $\{N_2(t) : t > 0\}$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. Find the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$ Write your answer in the form

$$\sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k},$$

by identifying the numbers m, n, and p.

2(b) (10 marks) Suppose that each arrival of a Poisson counting process $\{N(t) : t > 0\}$ with rate λ is classified as being a type-A or type-B arrival. Suppose that the probability of an arrival being classified as type-A or B depends on the time at which it occurs. If an arrival occurs at time s, then independently of all else, it is classified as being type-A with probability $Q(s) \in [0, 1]$ and being type-B with probability $1 - Q(s)$.

Show that if $N_A(t)$ and $N_B(t)$ respectively represent the number of type-A and type-B arrivals that occur by time t, then $N_A(t)$ and $N_B(t)$ are independent Poisson random variable having respective means $\lambda t p$ and $\lambda t(1 - p)$ where

$$p = \frac{1}{t} \int_0^t Q(s)\, \mathrm{d}s.$$

Question 3 (Total 30 Marks)

All parts can be done independently.

A Markov chain $\{X_n : n \geq 0\}$ on the state space $\mathcal{S} = \{0, 1, 2, 3, 4, 5\}$ has the transition matrix

$$[P] = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.6 & 0 & 0.3 & 0.1 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{bmatrix} \end{array}$$

Drawing a state transition diagram based on $[P]$ would help in the questions below.

3(a) (5 marks) List all the classes of the Markov chain. Briefly explain your answer.

3(b) (5 marks) For each state, determine whether it is transient or recurrent.

3(c) (3 marks) Suppose the Markov chain starts from state 5. What is the long run proportion of time that the Markov chain is at state 5?

3(d) (10 marks) Suppose the Markov chain starts from any state. Find the steady-state probabilities of the Markov chain $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_5)$ as $n \to \infty$.

3(e) (7 marks) Suppose the Markov chain starts from any state. It is known that $[P^n]_{00} - \pi_0$ decays to zero exponentially fast. This means that

$$\big|[P^n]_{00} - \pi_0\big| \leq c\,\phi^n$$

for some constant $c \in \mathbb{R}$ and $\phi \in [0, 1)$. Find the smallest possible ϕ.

Question 4 (30 marks)

All parts can be done independently.

Consider the following two state Markov chain $\{X_n : n \geq 0\}$ with state space $\mathcal{S} = \{0, 1\}$ and probability transition matrix

$$[P] = \begin{bmatrix} 1-\theta & \theta \\ \theta & 1-\theta \end{bmatrix}.$$

This means that $P_{ij} = 1 - \theta$ if $i = j$ and $P_{ij} = \theta$ if $i \neq j$. It is known that θ can take on two values

$$H_0 : \theta = \theta_0 \quad \text{and} \quad H_1 : \theta = \theta_1,$$

where $\theta_0 \neq \theta_1$. Hence,

$$\Pr(X_2 = j \mid X_1 = i, H_0) = (1 - \theta_0)^{\mathbb{1}\{i=j\}} \theta_0^{\mathbb{1}\{i \neq j\}}.$$

where $\mathbb{1}\{\text{clause}\}$ equals 1 if the clause is true and 0 otherwise. We also know that the chain (under both hypotheses) starts from state 0, i.e.,

$$\Pr(X_0 = 0) = 1.$$

4(a) (7 marks) Define the random variables

$$Y_i = \mathbb{1}\{X_i = X_{i-1}\}, \qquad \forall\, i = 1, 2, \ldots.$$

Suppose we observe X_0, X_1, X_2, and X_3. Show by considering the likelihood ratio

$$L(X_0, X_1, X_2, X_3) = \frac{\Pr(X_0, X_1, X_2, X_3 \mid H_1)}{\Pr(X_0, X_1, X_2, X_3 \mid H_0)}$$

and the likelihood ratio test that part (a) that

$$T = Y_1 + Y_2 + Y_3$$

is a <u>sufficient statistic</u> for deciding between hypotheses H_0 and H_1.

4(b) (5 marks) In this and the following parts, assume that

$$\theta_0 = 1/2 \quad \text{and} \quad \theta_1 = 1/4.$$

Calculate the values of the likelihood ratio $L(X_0, X_1, X_2, X_3)$ when T, defined in part (b), takes on values $\{0, 1, 2, 3\}$.

4(c) (5 marks) Suppose that the prior probability of H_0 is 1/2. What is the minimum Bayesian probability of error?

4(d) (5 marks) What is the smallest probability of missed detection Pr(declare $H_0 \mid H_1$) if we allow the probability of false alarm Pr(declare $H_1 \mid H_0$) to be at most 1/8? What is the corresponding test in terms of T?

4(e) (8 marks) What is the smallest probability of missed detection Pr(declare $H_0 \mid H_1$) if we allow the probability of false alarm Pr(declare $H_1 \mid H_0$) to be at most 1/6? What is the corresponding test in terms of T?

Hint: Here a randomization strategy is required.

END OF PAPER

Solutions to EE5137 Exam (Semester 2 2019/20)

April 28, 2020

1 Problem 1

(a) True. Law of iterated expectations.

(b) False. $\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y])$. This was proved in Quiz 2.

(c) False. $\gamma_X''(r)\Big|_{r=0} = \mathrm{Var}(X) \neq \mathbb{E}[X^2]$ if $\mathbb{E}X \neq 0$.

(d) True. By Chebyshev's inequality.

(e) True. Can directly verify that $\Pr(X > k + l \mid X > l) = \Pr(X > k)$ for any $k, l \in \mathbb{N}$.

(f) True. By splitting of Poisson process.

(g) False. We additionally need the independent increments property and the stationary increments property.

(h) False. It is recurrent.

(i) True. $\Pr(X_1 = 3, X_2 = 2, X_3 = 1) = \Pr(X_3 = 1 \mid X_2 = 2) \Pr(X_2 = 2 \mid X_1 = 3) \Pr(X_1 = 3) = 1/3 \times 1/2 \times (1 - 1/4 - 1/4) = 1/12$.

(j) True. Example 8.2.7 in the book says that $S_n = \sum_{i=1}^n Y_i$ is a sufficient statistic. Any one-to-one function of a sufficient statistic such as $\frac{1}{n-1} \sum_{i=1}^n Y_i$ is a sufficient statistic.

2 Problem 2

(a) Let $N(t)$ be a Poisson process with rate $\lambda = \lambda_1 + \lambda_2 = 3$. We split $N(t)$ into two processes $N_1(t)$ and $N_2(t)$ in the following way. For each arrival, a coin with $\Pr(H) = p = 1/3$ is tossed. If the coin lands heads up, the arrival is sent to the first process $N_1(t)$, otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then

- $N_1(t)$ is a Poisson process with rate $\lambda p = 1$.
- $N_2(t)$ is a Poisson process with rate $\lambda(1 - p) = 2$.
- $N_1(t)$ and $N_2(t)$ are independent.

Thus, $N_1(t)$ and $N_2(t)$ have the same probabilistic properties as the ones stated in the problem. We can now restate the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$ as the probability of observing at least two heads in four coin tosses, which is

$$\sum_{k=2}^{4} \binom{4}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{4-k}.$$

Hence, $m = 2, n = 4$ and $p = 1/3$.

(b) We compute the joint distribution of $N_A(t)$ and $N_B(t)$ by conditioning on $N(t)$:

$$\begin{aligned}
&\Pr(N_A(t) = n, N_B(t) = m) \\
&= \sum_{k=0}^{\infty} \Pr(N_A(t) = n, N_B(t) = m \mid N(t) = k) \Pr(N(t) = k) \\
&= \Pr(N_A(t) = n, N_B(t) = m \mid N(t) = n + m) \Pr(N(t) = n + m).
\end{aligned}$$

Now consider an arbitrary event that occurred during the interval $[0, t]$. If it had occurred at time s, the probability that it is of type-A is $Q(s)$. Hence, by the fact that the event occurred uniformly at random in the interval $[0, t]$, it follows that it is a type-A event with probability

$$p = \frac{1}{t} \int_0^t Q(s) \, \mathrm{d}s$$

independently of all the other events. Hence, $\Pr(N_A(t) = n, N_B(t) = m \mid N(t) = n + m)$ will just equal the probability of n successes and m failures in a total of $n + m$ independent trials when p is the probability of success of each trial. That is,

$$\Pr(N_A(t) = n, N_B(t) = m \mid N(t) = n + m) = \binom{n+m}{n} p^n (1 - p)^m.$$

Using the fact that $N(t)$ is a Poisson with rate λt, we have

$$\begin{aligned}
\Pr(N_A(t) = n, N_B(t) = m) &= \frac{(n+m)!}{n!m!} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!} \\
&= e^{-\lambda t p} \frac{(\lambda t p)^n}{n!} e^{-\lambda t (1-p)} \frac{(\lambda t (1-p))^m}{m!}
\end{aligned}$$

which completes the proof.

3 Problem 3

(a) The transition graph is

$$\begin{array}{ccccc}
0 & \leftarrow & 3 & & 2 \\
\updownarrow & \swarrow & \updownarrow & \searrow & \updownarrow \\
1 & \leftarrow & 5 & \leftrightarrow & 4
\end{array}$$

Clearly, $\{0, 1\}$ is a class and $\{2, 3, 4, 5\}$ is also a class.

(b) The class $\{0, 1\}$ is recurrent. The class $\{2, 3, 4, 5\}$ is transient.

(c) Since state 5 is transient, it will only be visited finite number of times. So the long run proportion of time that state 5 is visited is 0.

(d) Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_5)$ be the stationary distribution when we start anywhere for this ergodic unichain. Clearly, $\pi_2 = \pi_3 = \pi_4 = \pi_5 = 0$. To determine $\bar{\boldsymbol{\pi}} = (\pi_0, \pi_1)$, we solve the equation

$$\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}}[P],$$

or

$$\pi_0 = 0.5\pi_0 + 0.3\pi_1 \qquad \pi_1 = 0.5\pi_0 + 0.7\pi_1,$$

which is the same as $0.5\pi_0 = 0.3\pi_1$. Since $\pi_0 + \pi_1 = 1$, we have

$$\pi_0 + \frac{0.5}{0.3}\pi_0 = 1$$

or

$$\pi_0 = \frac{3}{8} \quad \text{and} \quad \pi_1 = \frac{5}{8}.$$

Thus, the stationary distribution is

$$\boldsymbol{\pi} = \left(\frac{3}{8}, \frac{5}{8}, 0, 0, 0, 0\right).$$

(e) We have to find the eigenvalue with the second largest magnitude for the transition matrix

$$[Q] = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$$

For this purpose

$$\det([Q] - \lambda[I]) = 0, \quad \implies \quad (0.5 - \lambda)(0.7 - \lambda) - 0.15 = 0.$$

Solving this yields $\lambda_1 = 1$ and $\lambda_2 = 0.2$. Thus $\phi = 0.2$.

4 Problem 4

(a) Let $\mathbf{X} = (X_0, X_1, X_2, X_3)$. The likelihood under H_0 is

$$\begin{aligned}
P_{\mathbf{X}|H_0}(\mathbf{x}) &= \Pr(X_0 = x_0) \prod_{i=1}^{3} \Pr(X_i = x_i \mid X_{i-1} = x_{i-1}) \\
&= 1 \cdot \prod_{i=1}^{3} (1-\theta_0)^{\mathbb{1}\{x_{i-1} = x_i\}} \theta_0^{\mathbb{1}\{x_{i-1} \neq x_i\}} \\
&= \prod_{i=1}^{3} (1-\theta_0)^{y_i} \theta_0^{1-y_i} \\
&= \theta_0^3 \left(\frac{1-\theta_0}{\theta_0}\right)^{\sum_{i=1}^3 y_i}
\end{aligned}$$

Similarly,

$$P_{\mathbf{X}|H_1}(\mathbf{x}) = \theta_1^3 \left(\frac{1-\theta_1}{\theta_1}\right)^{\sum_{i=1}^3 y_i}$$

The LRT states that we decide in favor of H_1 if

$$\log L(x_0, x_1, x_2, x_3) = \log \frac{P_{\mathbf{X}|H_1}(\mathbf{x})}{P_{\mathbf{X}|H_0}(\mathbf{x})} \geq \eta$$

for some $\eta \in \mathbb{R}$. This is equivalent to

$$3 \log \frac{\theta_1}{\theta_0} + \left(\sum_{i=1}^{3} y_i\right) \left(\log \frac{1-\theta_1}{\theta_1} - \log \frac{1-\theta_0}{\theta_0}\right) \geq \eta.$$

Since the test is in terms of $T = \sum_{i=1}^3 y_i$ and nothing else, T is a sufficient statistic for discriminating between the two hypotheses.

(b) Since $\theta_0 = 1/2$ and $\theta_1 = 1/4$, the log-likelihood ratio is

$$\log L(X_0, X_1, X_2, X_3) = 3 \log \frac{1}{2} + T \log 3.$$

Hence, the likelihood ratio is

$$L(X_0, X_1, X_2, X_3) = \frac{1}{8} \cdot 3^T$$

This means that

$$L(X_0, X_1, X_2, X_3) = \begin{cases} 1/8 & \text{if } T = 0 \\ 3/8 & \text{if } T = 1 \\ 9/8 & \text{if } T = 2 \\ 27/8 & \text{if } T = 3 \end{cases}.$$

(c) If $P(H_0) = 1/2$, this means we employ ML decoding, i.e., the threshold on the likelihood ratio is 1. Thus, we declare H_0 is true if $T \le 1$ and H_1 is true if $T > 1$. We have the probability of false alarm being

$$P_0(\text{declare } H_1) = P_0(T > 1) = P_0(T = 2) + P_0(T = 3) = 3(\frac{1}{2})^3 + (\frac{1}{2})^3 = \frac{1}{2}.$$

We also have the probability of mis-detection being

$$P_1(\text{declare } H_1) = P_1(T \le 1) = P_1(T = 0) + P_1(T = 1) = (\frac{1}{4})^3 + 3(\frac{3}{4})(\frac{1}{4})^2 = \frac{10}{64}.$$

Thus, the Bayesian error probability is

$$P_{\text{err}} = \frac{1}{2} P_0(T > 1) + \frac{1}{2} P_1(T \le 1) = \frac{21}{64}.$$

(d) Note that the likelihood ratio test simplifies to deciding in favor of H_1 if

$$L(X_0, X_1, X_2, X_3) = \frac{1}{8} \cdot 3^T \ge \eta'.$$

We want $P_0(\text{declare } H_1) \le 1/8$. To do so, we need to put the threshold $\eta' \in (9/8, 27/8)$ and declare that if $T > 2$ (i.e., $T = 3$), then H_1 is declared. This results in $P_0(T > 2) = P_0(T = 3) = (1/2)^3 = 1/8$.

The corresponding largest probability of detection is $P_1(\text{declare } H_1) = P_1(T > 2) = P_1(T = 3) = (3/4)^3 = 27/64$, hence the smallest probability of missed detection is $1 - 27/64 = 37/64$.

(e) We now want $P_0(\text{declare } H_1) \le 1/6$. We consider that $P_0(T > 1) = P_0(T = 2 \text{ or } T = 3) = 1/2$ and the corresponding probability of detection is $P_1(T > 1) = (3/4)^3 + 3(3/4)^2(1/4) = 27/32$. Hence, we need to randomize between the strategy that places the threshold at $T > 2$ and $T > 1$. Now we find an $\alpha \in [0, 1]$ such that

$$\alpha \frac{1}{8} + (1 - \alpha) \frac{1}{2} = \frac{1}{6} \quad \Longrightarrow \quad \alpha = \frac{8}{9}$$

Thus, we use the decision rule corresponding to declaring H_1 is true if $T > 2$ for a fraction of 8/9. Thus, the best probability of detection is

$$\alpha \frac{27}{64} + (1 - \alpha) \frac{27}{32} = \frac{15}{32}$$

The best probability of mis-detection is 17/32 and the optimal test in terms of T would be to randomize between $T > 2$ and $T > 1$ where the former has probability 8/9.

National University of Singapore
Department of Electrical & Computer Engineering

Examination for

EE5137 Stochastic Processes
(Semester II, 2020/21)
April/May 2021

Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- This paper contains **FOUR (4)** questions, printed on **FIVE (5)** pages.
- The total number of marks is 100.
- Answer all questions.
- Programmable calculators are NOT allowed.
- Electronic communicating devices MUST be turned off and inaccessible throughout the examination. They CANNOT be used as calculators, timers or clocks.
- You are allowed to bring ONE (A4) size help sheet.
- No other material is allowed.

Question 1 (Total 25 Marks)

1(a) (12 points) Let X be a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} e^{-2x} + \frac{1}{2}e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Write down the moment generating function $g_X(r)$ of X and use it to find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

1(b) (13 points) There is an ATM (Automated Teller Machine) at University Town which has to be loaded with enough money to cover the day's business. The number of students who withdraw money from the ATM per day is a random number N. Student i withdraws a random amount X_i. The total amount withdrawn during the day is the following random sum

$$T_N = X_1 + X_2 + \ldots + X_N.$$

If $\{X_i\}_{i=1}^{\infty}$ are independent random variables that are also independent of N and

$$\mathbb{E}[X_1] = \mu_X, \quad \mathrm{Var}(X_1) = \sigma_X^2, \quad \mathbb{E}[N] = \mu_N, \quad \mathrm{Var}(N) = \sigma_N^2,$$

find

$$\mathbb{E}[T_N] \quad \text{and} \quad \mathrm{Var}(T_N).$$

You might find it helpful to first find $\mathbb{E}[T_N \mid N = n]$, $\mathbb{E}[T_N^2 \mid N = n]$, *or* $\mathrm{Var}(T_N \mid N = n)$.

Question 2 (Total 25 Marks)

2(a) Men and women arrive at a store according to independent Poisson processes with parameters 3 and 4 respectively. Men shop for a time that is uniformly distributed on $[0, 1]$. The time that men shop in the store is independent of the two Poisson processes.

(i) (3 points) What is the expected number of people who arrive at the store during the interval $[0, 5]$?

(ii) (4 points) Given that 6 people arrive during the interval $[2, 3]$, what is the probability that exactly 2 of them are men?
Leave your answer as $\binom{n}{k}p^k(1-p)^{n-k}$ *for some numbers* p, n *and* k.

(iii) (6 points) Given that Andrew (a man) arrived during the interval $[4, 5]$, what is the probability that he is still in the store at time 5?

2(b) Let $\{N(t) : t > 0\}$ be a Poisson process with rate $\lambda > 0$. For each $t > 0$

$$Y(t) = (-1)^{N(t)}.$$

The process $\{Y(t) : t > 0\}$ is known as the *random telegraph signal.*

(i) (3 points) Does $\{Y(t) : t > 0\}$ have the stationary increment property? Explain.

(ii) (3 points) Does $\{Y(t) : t > 0\}$ have the independent increment property? Explain.

(iii) (6 points) Prove that $\mathbb{E}[Y(t)] = \exp(-a\lambda t)$ for some constant $a > 0$. Find a.
Hint: You may use the fact that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \forall x \in \mathbb{R}$$

without proof.

Question 3 (Total 25 Marks)

A rat runs through the maze shown below. At each step it leaves the room it is in by choosing uniformly at random one of the doors out of the room. Thus if it is at room 4 now, it is in rooms 3 and 6 with probability 1/2 each the next time step.

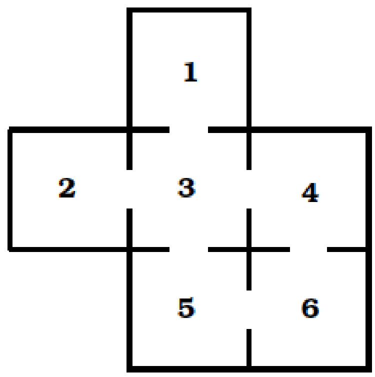


Figure 1: The rat's maze

3(a) (2 points) Write down the probability transition matrix $[P]$ for this Markov chain.

3(b) (3 points) Write down the class(es) of the Markov chain.

3(c) (2 points) For each state of the Markov chain, write down whether it is transient or recurrent.

3(d) (4 points) For each state of the Markov chain, write down its period. Does $[P^n]$ converge to a rank 1 matrix?

3(e) (7 points) Find the stationary distribution of the Markov chain.

3(f) (7 points) Now suppose that a piece of cheese is placed on a deadly trap in Room 5. The mouse starts in Room 1. Find the expected number of steps before reaching Room 5 for the first time and hence dying, starting in Room 1.

Parts (e) and (f) may be a little tedious. But you will get (a lot of) partial credit for (correctly) writing down systems of equations that you need to solve to get the answer even if you cannot get the exact numbers.

Question 4 (25 marks)

4(a) A sales executive suspects that one of his salespeople is routing half of his incoming sales to a competitor. Arriving sales are known to be Poisson at rate 1 per hour. If the sales executive's suspicion is false ($X = 0$), the interarrival times are i.i.d. Exp(1), i.e.,

$$f_{Y|X}(y \mid 0) = \exp(-y)\mathbb{1}\{y \geq 0\}.$$

If his suspicion is true ($X = 1$), the interarrival times are i.i.d. according to an Erlang distribution of order 2 and rate 1, i.e.,

$$f_{Y|X}(y \mid 1) = y\exp(-y)\mathbb{1}\{y \geq 0\}.$$

Assume that *a priori* the hypotheses are such that $\Pr(X = 0) = 3/4$ and $\Pr(X = 1) = 1/4$.

(i) (5 points) Starting at time 0, let S_i be the arrival epoch of the i^{th} subsequent successful sale. The executive observes $S_1, S_2, \ldots, S_n$ and chooses the maximum *a posteriori* (MAP) probability hypothesis given this observation. Find the joint probability densities

$$f_{S_1,\ldots,S_n|X}(s_1, \ldots, s_n \mid 0) \quad \text{and} \quad f_{S_1,\ldots,S_n|X}(s_1, \ldots, s_n \mid 1).$$

(ii) (4 points) Write down the the MAP rule. Simplify it as much as possible.

(iii) (3 points) Suppose the executive can only observe S_n (and not $S_1, \ldots, S_{n-1}$). Is S_n a sufficient statistic? Explain.

4(b) Let $x > 0$ and let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. random variables sampled from the density

$$f_Y(y; x) = \begin{cases} \dfrac{3y^2}{x^3} & \text{if } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

(i) (4 marks) Show that the estimator

$$\hat{X} = \frac{4}{3n}\sum_{i=1}^{n} Y_i$$

is an unbiased estimator of x.

(ii) (4 marks) By finding $\mathrm{Var}(Y_1)$, find $\mathrm{Var}(\hat{X})$.

(iii) (5 marks) Using the formula for Fisher information involving the first derivative of the score function, write down the Cramér-Rao bound for any unbiased estimator $\hat{X}_{\text{any}}$ used for estimating x given $Y_1, \ldots, Y_n$ each sampled i.i.d. from $f_Y(y; x)$.

(iv) (5 bonus marks) Does the Cramér-Rao theorem hold for the estimator in stated in part (a)? If not, why is it violated for this estimator?

END OF PAPER

Solutions to EE5137 Exam (Semester 2 2020/21)

May 2, 2021

1 Problem 1

(a) The moment generating function is

$$g_X(r) = \mathbb{E}[e^{-rX}] = \int_0^\infty e^{rx}\left(e^{-2x} + \frac{1}{2}e^{-x}\right) \mathrm{d}x.$$

Evaluating this yields

$$g_X(r) = \frac{1}{2-r} + \frac{1}{2(1-r)}.$$

The mean is

$$\mathbb{E}[X] = g_X'(0) = \frac{1}{(2-r)^2} + \frac{1}{2(1-r)^2}\bigg|_{r=0} = \frac{3}{4}$$

The second moment is

$$\mathbb{E}[X^2] = g_X''(0) = \frac{2}{(2-r)^3} + \frac{1}{(1-r)^3}\bigg|_{r=0} = \frac{5}{4}.$$

Hence, the variance is

$$\mathrm{Var}(X) = \frac{5}{4} - \left(\frac{3}{4}\right)^2 = \frac{11}{16}$$

(b) We have

$$\mathbb{E}[T_N \mid N = n] = n\mathbb{E}[X_1] = n\mu_X$$

and so by the law of total expectation,

$$\mathbb{E}[T_N] = \mathbb{E}[\mathbb{E}[T_N|N]] = \mathbb{E}[N\mu_X] = \mu_N\mu_X.$$

We also have

$$\mathrm{Var}[T_N \mid N = n] = \mathrm{Var}[X_1 + \ldots + X_n \mid N = n] = n\sigma_X^2.$$

Hence, $\mathbb{E}[T_N^2 \mid N = n] = \mathrm{Var}[T_N \mid N = n] + (\mathbb{E}[T_N \mid N = n])^2 = n\sigma_X^2 + n^2\mu_X^2$. By the law of total expectation,

$$\mathbb{E}[T_N^2] = \mathbb{E}[N\sigma_X^2 + N^2\mu_X^2] = \mu_N\sigma_X^2 + (\mu_N^2 + \sigma_N^2)\mu_X^2.$$

Finally,

$$\mathrm{Var}(T_N) = \mathbb{E}[T_N^2] - (\mathbb{E}[T_N])^2 = \mu_N\sigma_X^2 + \sigma_N^2\mu_X^2.$$

One can also get this using the law of total variance,

$$\mathrm{Var}(T_N) = \mathbb{E}[\mathrm{Var}(T_N \mid N)] + \mathrm{Var}(\mathbb{E}[T_N \mid N]) = \mathbb{E}[N\sigma_X^2] + \mathrm{Var}(N\mu_X) = \mu_N\sigma_X^2 + \mu_X^2\sigma_N^2.$$

2 Problem 2

(a) (i) The expected number of people who go to the store is $5(3+4) = 35$.

(ii) This is a Binomial distribution

$$\binom{6}{2}\left(\frac{3}{3+4}\right)^2\left(\frac{4}{3+4}\right)^4 = \binom{6}{2}\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)^4.$$

(iii) Let the exact time that Andrew arrived be S. Let W be the time that Andrew spends in the store (after he arrives). Note that $f_W(w) = 1$ for $w \in [0,1]$. Since the Poisson process is memoryless, S follows an exponential distribution with rate 3. Thus, the required probability is

$$\begin{aligned}
\Pr(S+W > 1 \mid S \le 1) &= \int_0^1 f_W(w)\Pr(S+W > 1 \mid S \le 1, W = w)\,\mathrm{d}w \\
&= \int_0^1 f_W(w)\Pr(S+w > 1 \mid S \le 1, W = w)\,\mathrm{d}w \\
&\overset{(a)}{=} \int_0^1 f_W(w)\Pr(S+w > 1 \mid S \le 1)\,\mathrm{d}w \\
&= \int_0^1 f_W(w)\Pr(S > 1-w \mid S \le 1)\,\mathrm{d}w \\
&\overset{(b)}{=} \int_0^1 1\cdot w\,\mathrm{d}w \\
&= 1/2,
\end{aligned}$$

where (a) follows from the fact S is independent of W and (b) follows from the fact that that given $S \le 1$, the arrival is uniformly distributed on $[0,1]$.

(a) (i) No. If $\{Y(t) : t > 0\}$ has the SIP, then $Y(t') - Y(t)$ should have the same distribution as $Y(t'-t)$ for $0 < t < t'$. But the set of values that $Y(t') - Y(t)$ can take on is $\{-2, 0, 2\}$ but the values that $Y(t'-t)$ is $\{-1, 1\}$. So the two rvs cannot have the same distribution.

(ii) No. If $\{Y(t) : t > 0\}$ has the IIP, then for any $0 < t_1 < t_2 < t_3$, $Y(t_2) - Y(t_1)$ should be independent of $Y(t_3) - Y(t_2)$. However, if we know that $Y(t_2) - Y(t_1) = 2$, this means that $N(t_2)$ is even and $N(t_1)$ is odd. Thus, $Y(t_3) - Y(t_2) = Y(t_3) - 1 \in \{0, -2\}$, i.e., $Y(t_3) - Y(t_2)$ cannot be 2. Hence, $Y(t_2) - Y(t_1)$ is not independent of $Y(t_3) - Y(t_2)$.

(iii) Note that $Y(t)$ can take on values -1 and 1. We have

$$\Pr(Y(t) = 1) = \Pr(N(t) \text{ is even}) = \sum_{n=0,\, n \text{ even}}^{\infty} \Pr(N(t) = n) = \sum_{n=0,\, n \text{ even}}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

and

$$\Pr(Y(t) = -1) = \Pr(N(t) \text{ is odd}) = \sum_{n=0,\, n \text{ odd}}^{\infty} \Pr(N(t) = n) = \sum_{n=0,\, n \text{ odd}}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

Thus, the expectation is

$$\begin{aligned}
\mathbb{E}[Y(t)] &= \Pr(Y(t)=1) - \Pr(Y(t)=-1) \\
&= \sum_{n=0,\, n \text{ even}}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} - \sum_{n=0,\, n \text{ odd}}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \sum_{n=0,\, n \text{ even}}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} + \sum_{n=0,\, n \text{ odd}}^{\infty} \frac{e^{-\lambda t}(-\lambda t)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(-\lambda t)^n}{n!} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!} \\
&= e^{-\lambda t} \cdot e^{-\lambda t} = e^{-2\lambda t}.
\end{aligned}$$

Thus $a = 2$.

3 Problem 3

(a) The state transition matrix is

$$[P] = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 1/2 \\
0 & 0 & 1/2 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1/2 & 1/2 & 0
\end{bmatrix}$$

(b) There is only 1 class $\{1,2,3,4,5,6\}$.

(c) All states are recurrent.

(d) Since $P_{11}^2 > 0$, $P_{11}^4 > 0, \ldots$, the period of state 1 is $d(1) = 2$. Similarly $d(i) = 2$ for all i. Since the chain consists of a single recurrent class which is periodic, $[P^n]$ does not converge to a rank 1 matrix.

(e) Writing out the steady state equations, we have

$$\begin{aligned}
\pi_3/4 &= \pi_1 \\
\pi_3/4 &= \pi_2 \\
\pi_1 + \pi_2 + \pi_4/2 + \pi_5/2 &= \pi_3 \\
\pi_3/4 + \pi_6/2 &= \pi_4 \\
\pi_3/4 + \pi_6/2 &= \pi_5 \\
\pi_4/2 + \pi_5/2 &= \pi_6
\end{aligned}$$

Furthermore,

$$\sum_{i=1}^{6} \pi_i = 1.$$

Solving this system yields

$$\pi_1 = \pi_2 = 1/12 \quad \pi_3 = 1/3 \quad \pi_4 = \pi_5 = \pi_6 = 1/6.$$

(f) Writing out the expected first-passage time equations, we have

$$\begin{aligned} v_1 &= 1 + v_3 \\ v_2 &= 1 + v_3 \\ v_3 &= 1 + v_1/4 + v_2/4 + v_4/4 \\ v_4 &= 1 + v_3/2 + v_6/2 \\ v_6 &= 1 + v_4/2 \end{aligned}$$

Solving this system of equations yields $v_1 = 6$ so the expected time for the rat to die is $1 + v_1 = 7$.

4 Problem 4

(a) (i) We let $s_0 = 0$. If $X = 0$, we have

$$f_{S_1,\ldots,S_n|X}(s_1,\ldots,s_n \mid 0) = \prod_{i=1}^{n} e^{-(s_i - s_{i-1})} = e^{-s_n}$$

Otherwise, if $X = 1$, we have

$$f_{S_1,\ldots,S_n|X}(s_1,\ldots,s_n \mid 1) = \prod_{i=1}^{n}(s_i - s_{i-1})e^{-(s_i - s_{i-1})} = e^{-s_n}\prod_{i=1}^{n}(s_i - s_{i-1})$$

(ii) The MAP rule stipulates that we declare $X = 1$ if

$$\frac{f_{S_1,\ldots,S_n|X}(s_1,\ldots,s_n \mid 1)}{f_{S_1,\ldots,S_n|X}(s_1,\ldots,s_n \mid 0)} \geq \frac{p_0}{p_1}.$$

This is equivalent to declaring $X = 1$ if

$$\prod_{i=1}^{n}(s_i - s_{i-1}) \geq 3$$

or

$$\log(s_1) + \sum_{i=2}^{n}\log(s_i - s_{i-1}) \geq \log 3.$$

(iii) No. S_n is not a sufficient statistic because the MAP test involves *all* $(S_1,\ldots,S_n)$.

(b) (i) The expectation of Y_1 is

$$\mathbb{E}Y_1 = \int_0^x y f_Y(y;x)\,\mathrm{d}y = \int_0^x \frac{3y^3}{x^3}\,\mathrm{d}y = \frac{3x}{4}.$$

Thus,

$$\mathbb{E}\hat{X} = \frac{4}{3n}\sum_{i=1}^{n}\mathbb{E}Y_i = \frac{4}{3n}\sum_{i=1}^{n}\frac{3x}{4} = \frac{4}{3n}\cdot\frac{3nx}{4} = x.$$

Since $\mathbb{E}\hat{X} = x$, the estimator is unbiased.

(ii) The second moment Y_1 is

$$\mathbb{E}[Y_1^2] = \int_0^x y^2 f_Y(y;x)\,\mathrm{d}y = \int_0^x \frac{3y^4}{x^3}\,\mathrm{d}y = \frac{3x^2}{5}.$$

Thus, the variance

$$\mathrm{Var}(Y_1) = \mathbb{E}[Y_1^2] - (\mathbb{E}Y_1)^2 = \left(\frac{3}{5} - \left(\frac{3}{4}\right)^2\right) x^2 = \frac{3x^2}{80}.$$

Thus, the variance of the estimator is

$$\mathrm{Var}(\hat{X}) = \left(\frac{4}{3n}\right)^2 \cdot n \cdot \mathrm{Var}(Y_1) = \left(\frac{4}{3n}\right)^2 \cdot n \cdot \frac{3x^2}{80} = \frac{x^2}{15n}.$$

(iii) The Fisher information of Y_1 is

$$J_{Y_1}(x) = \mathbb{E}\left[\left(\frac{\partial}{\partial x} \log\left(\frac{3y^2}{x^3}\right)\right)^2\right] = \frac{9}{x^2}$$

Thus, the Fisher information for $\mathbf{Y} = (Y_1, \dots, Y_n)$ is

$$J_{\mathbf{Y}}(x) = \frac{9n}{x^2}.$$

The Cramér-Rao bound reads

$$\mathrm{Var}(\hat{X}_{\text{any}}) \geq \frac{1}{J_{\mathbf{Y}}(x)} = \frac{x^2}{9n}.$$

(iv) We see that the Cramér-Rao theorem does not hold because

$$\mathrm{Var}(\hat{X}) = \frac{x^2}{15n} \not\geq \frac{1}{J_{\mathbf{Y}}(x)} = \frac{x^2}{9n}.$$

The reason for this is that the regularity condition

$$\mathbb{E}\left[\frac{\partial}{\partial x} \log f_Y(Y; x)\right] = 0, \qquad \forall\, x$$

does not hold. Let us check

$$\mathbb{E}\left[\frac{\partial}{\partial x} \log\left(\frac{3Y^2}{x^3}\right)\right] = \mathbb{E}\left[\frac{\partial}{\partial x}\big[\log(3Y^2) - \log(x^3)\big]\right] = -\frac{3}{x}$$

which is never 0. Since the regularity condition does not hold, the Cramér-Rao bound need not hold as is the case here.