Chapter 7 Pole Placement

§7.1 State Feedback

Consider a multiple-input and multiple-output (MIMO) n-dimensional linear plant with m-inputs and p-outputs described by

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx. \tag{1b}$$

with constant matrices A, B, and C of appropriate dimensions,

$$x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T,$$
$$u = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}^T,$$
$$y = \begin{bmatrix} y_1 & \cdots & y_p \end{bmatrix}^T.$$

A control law:

$$u = -Kx + Fr, \tag{2}$$

with a **non-singular *F***, is called the state feedback, where *m*-dimensional *r* is the reference, *K and F* are constant matrices to be designed.

The resulting state feedback system is

$$\dot{x} = (A - BK)x + BFr, \tag{3a}$$

$$y = Cx. \tag{3b}$$

Thus, by the control in (2), the plant triple (*A*, *B*, *C*) is changed to the closed-loop one (*A* – *BK*, *BF*, *C*). The system block diagram is depicted in Figure 1 below.

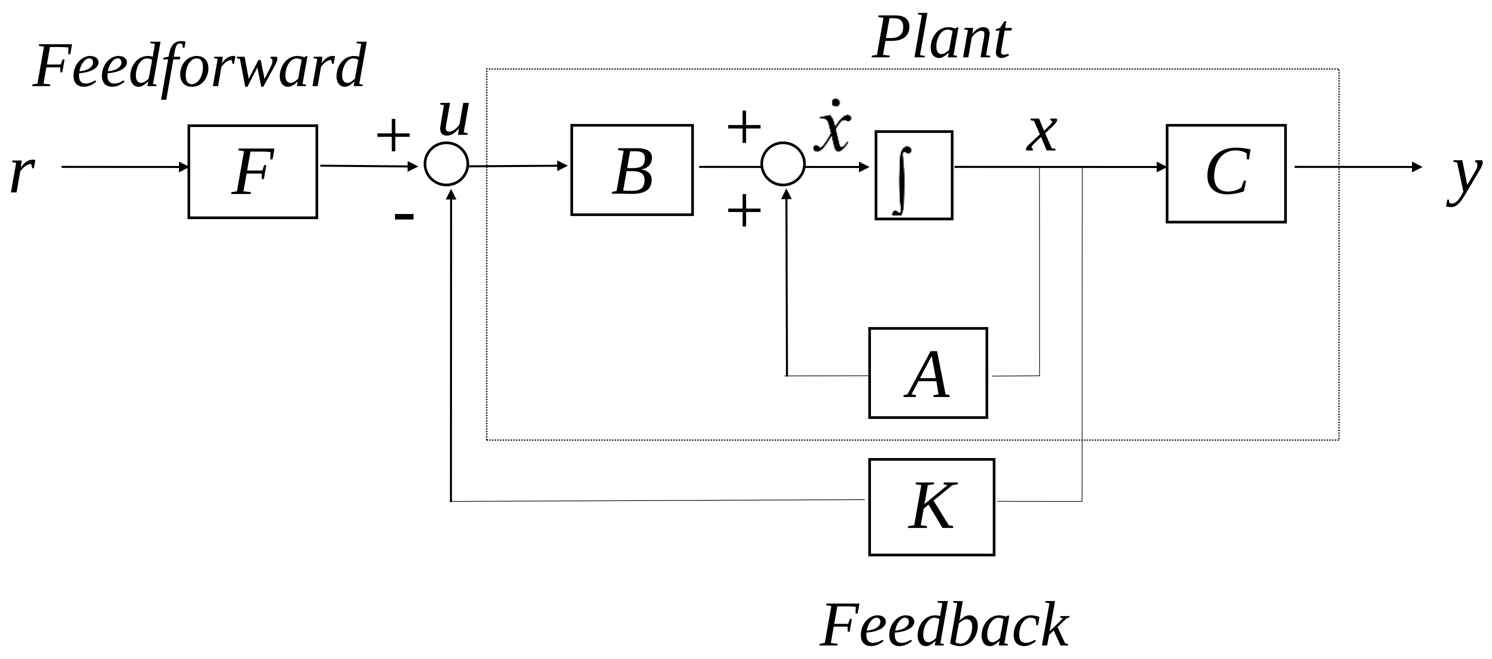


Figure 1 State feedback control system.

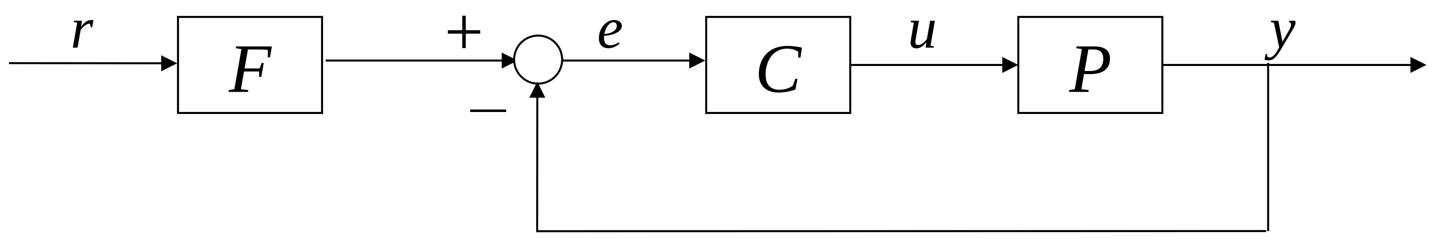


Figure 2 Unity output feedback system.

What are changes of system properties?

What properties are changed and what remain unchanged under state-feedback control?

Lemma 1. *The controllability of the system is unaltered by state feedback, that is, if* (*A*,*B*) *is controllable* (*uncontrollable*), *so is* (*A*-*BK*,*BF*) *for any K.*

Proof 1: For any *u*,

$$u^* = -Kx + Fr,$$

there is always an *r*

$$r^* = F^{-1}(u^* + Kx),$$

and vice versa.

Proof 2: (A, B) is controllable iff Rank[$sI - A$, B] = n, $\forall s \in \mathbb{C}$.

$$Rank\begin{bmatrix} sI-(A-BK) & BF \end{bmatrix}$$

$$= Rank\left\{\begin{bmatrix} sI-A & B \end{bmatrix}\begin{bmatrix} I & 0 \\ K & F \end{bmatrix}\right\}$$

$$= Rank\begin{bmatrix} sI-A & B \end{bmatrix}.$$

Remark. Rank[$sI - A$, B] = n for all complex s is equivalent to Rank[$sI - A$, B] = n for every eigenvalue of A. One sees the statement that Rank[$sI - A$, B] = n where s is any complex number, may break into two conditions:

1) Rank[$sI - A$, B] = n for s with det($sI - A$) = 0, that is, Rank[$sI - A$, B] = n for every eigenvalue of A;
2) Rank[$sI - A$, B] = n for other s, i.e. det($sI - A$) is nonzero. This is always true as the nonzero det($sI - A$) makes Rank[$sI - A$, B] = n.

Example 1. The system:

$$\dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

has the controllability matrix:

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix},$$

with rank of 1, so it is uncontrollable. When the control is applied:

$$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x + fr, \qquad f \neq 0,$$

the feedback system becomes

$$\dot{x} = \underbrace{\begin{bmatrix} -k_1 & -k_2 - 2 \\ -k_1 + 1 & -k_2 - 3 \end{bmatrix}}_{A-BK} x + \underbrace{f \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{Bf} r.$$

Its controllability matrix is

$$\begin{bmatrix} 1 & -k_1 - k_2 - 2 \\ 1 & -k_1 - k_2 - 2 \end{bmatrix} f,$$

with rank of 1, too, and the feedback system is still uncontrollable.

Lemma 2. *The closed-loop poles are affected only by the feedback gain matrix K but independent of F.*

Proof : The closed-loop characteristic polynomial:

$$\phi_f(s) = \det(sI - A + BK),$$

is independent of F.

Why state feedback?

Improve performance and robustness such as

(i) pole placement control
(ii) optimal control
(iii) decoupling control
(iv) servo control
(v) …

How to do?

- See this and coming chapters, and
- *Take other modules to learn robust, adaptive and nonlinear control*

§7.2 Why Pole Placement?

Time-domain Specifications

Specifications for a control system design often involve certain requirements associated with the time response of the system. The requirements for a step response of the stable system output are often expressed in terms of

The **rise time:** t_r is the time for the system output to reach 90% of its final value from its 10% value;

The **settling time:** t_s is the time for the system transients to decay to and stay in a small percentage of its final value, say 2%;

The **overshoot:** M_p is the maximum amount by which the system output overshoots its final value divided by its final value (and often expressed as a percentage),

as illustrated in Figure 3.

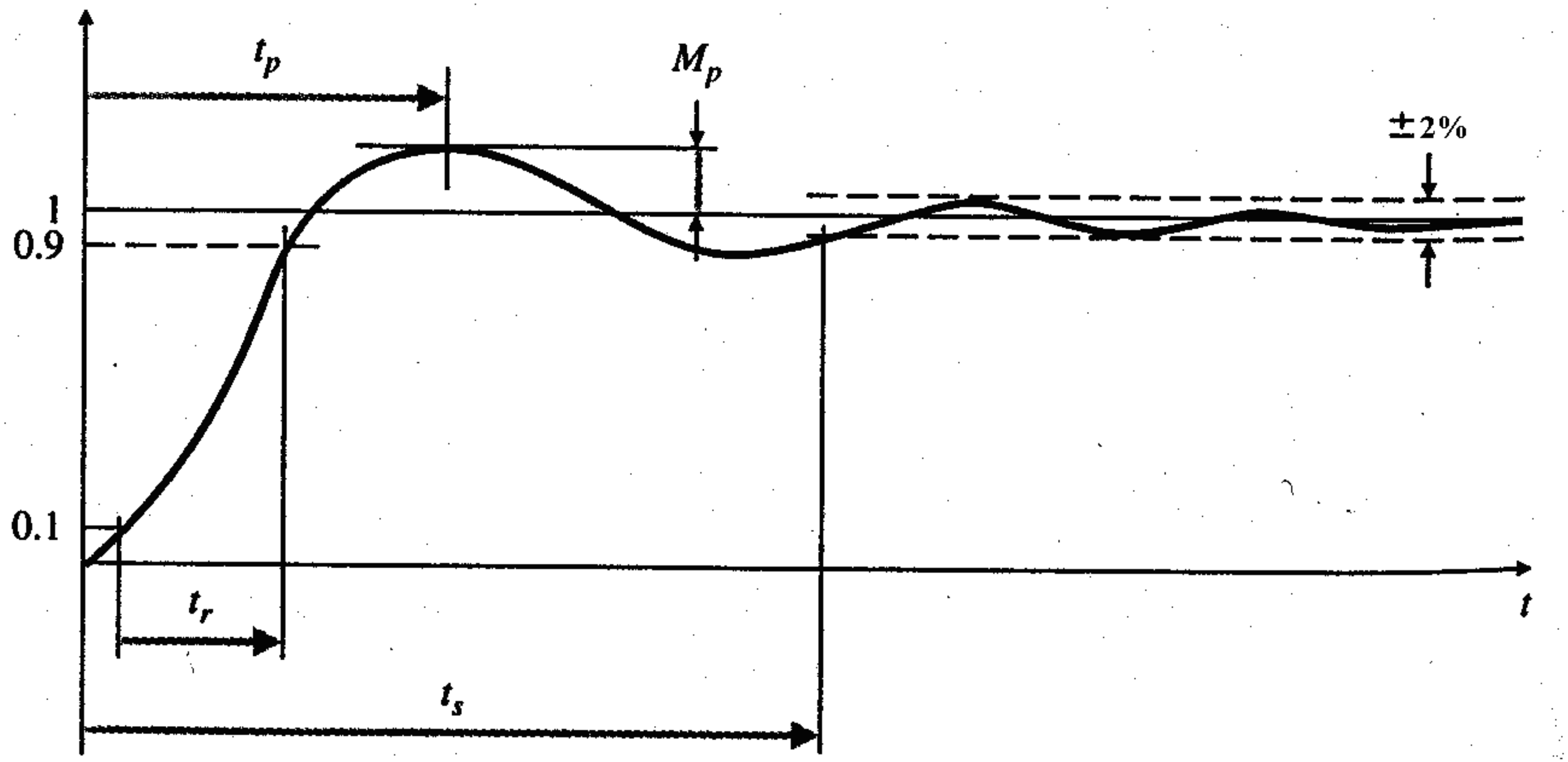


Figure 3 Control specifications.

Consider a standard second-order system:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Its poles are at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ (with $0 < \zeta < 1$). Two parameters, or the poles determine performance completely:

(a) Rise time, $t_r = \dfrac{1.8}{\omega_n}$;

(b) Peak overshoot, $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$;

(c) 2% settling time, $t_s = \dfrac{4}{\zeta\omega_n}$.

Example 2. Consider $G(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. Sketch and shade the allowable region in the s-plane for the system poles if the step response requirements are

$$t_r < 0.9\text{sec},\quad t_s < 3\text{sec},\quad M_p < 10\%.$$

Solution. Rise time t_r is given by $1.8/\omega_n$, hence,

$$\frac{1.8}{\omega_n} < 0.9 \quad \Rightarrow \omega_n > 2.$$

2% settling time meets $t_s = 4/\zeta\omega_n$ so that

$$\frac{4}{\zeta\omega_n} < 3 \quad \Rightarrow \zeta\omega_n > \frac{4}{3}.$$

Finally, the maximum overshoot should be less than 10%, i.e.

$$e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.1 \quad \Rightarrow \zeta > 0.59.$$

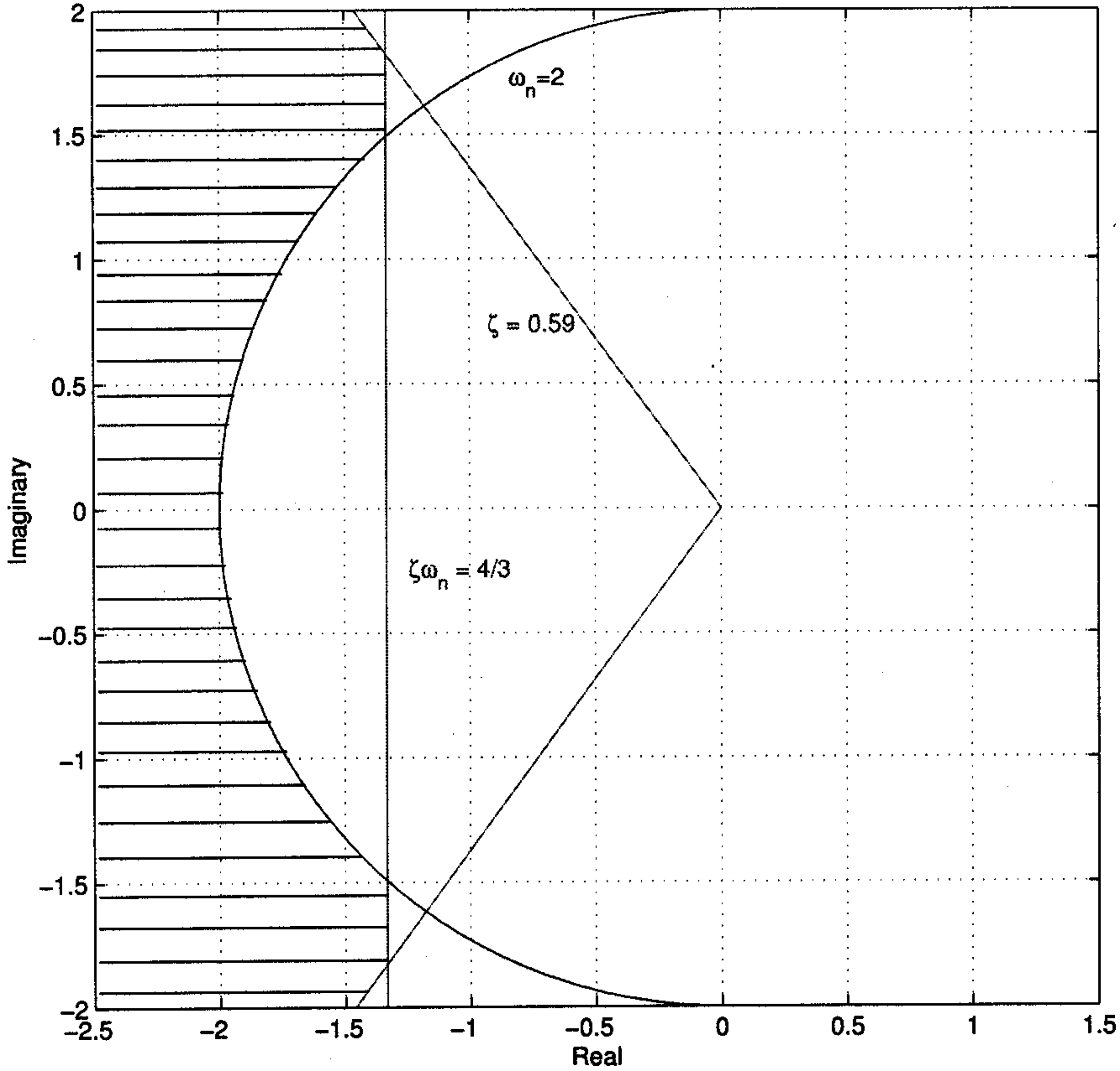


Figure 1: Allowable region in the s-plane for the system poles

Consider now a general state feedback system:

$$\dot{x} = (A - BK)x + Br,$$
$$y = Cx.$$

If all its closed-loop poles, λ_i , are distinct, then the output impulse response has a form of

$$y(t) = \sum \alpha_i e^{\lambda_i t}.$$

Conclusion: The pole locations of a system determine its stability and significantly impact its dynamic performance.

Applications?

An Industrial Motivation: DC Motor Position Control

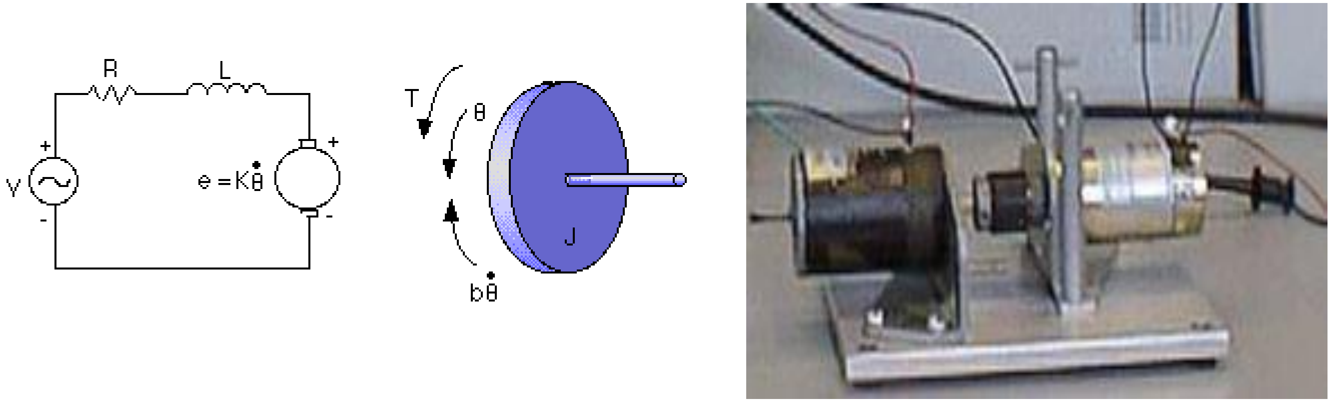


Figure 5 DC Motor.

1) Modeling. Suppose for this example:

- moment of inertia of the rotor, J = 3.2284×10^{-6} kg·m^2/s^2
- damping ratio of the mechanical system, b = 3.5077×10^{-6} N·m·s
- electromotive force constant, K=K_e=K_t = 0.0274 N·m/Amp
- electric resistance, R = 4 ohm
- electric inductance, L = 2.75×10^{-6} H

- input, V : Source Voltage
- output, θ : position of shaft

From the figure above we can write the following equations based on Newton's law combined with Kirchhoff's law:

$$J\ddot{\theta} + b\dot{\theta} = T,$$

$$L\frac{di}{dt} + Ri = V - e,$$

where the motor torque, T, is related to the armature current, i, by a constant factor K_t, and the back emf, e, is related to the rotational velocity by the following equations:

$$T = K_t i,$$

$$e = K_e \dot{\theta}.$$

In SI units (which we will use), K_t (armature constant) is equal to K_e (motor constant).The plant can be modeled as follows,

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & -\frac{b}{J} & \frac{K_t}{J}\\ 0 & -\frac{K_e}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}V,$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix},$$

where the motor position, motor speed, and armature current are three state variables.

2) Control. Requirements on the closed-loop step response are that

- the settling time be less than 40 milliseconds, and
- the overshoot be less than 16%.

The plant has a pole at s=0, and is unstable. Indeed, check the plant output step response in Figure 6, which goes with no bound and does not satisfy the design criteria at all.

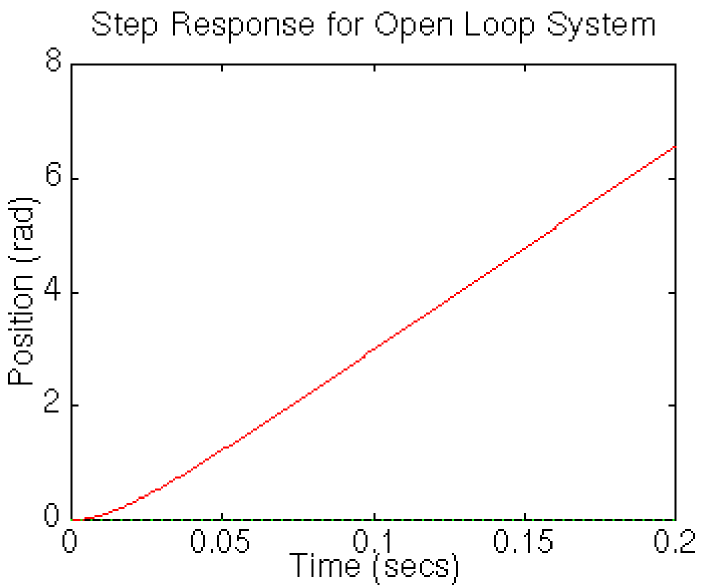


Figure 6 Output step response.

3) Conclusion: good control (good closed-loop poles) is demanded.

Summing up:

Rationale of Pole Placement

Reasonable to represent system performance by the poles

Time domain ↘

Specifications $\Rightarrow$ Desired pole locations

Frequency domain ↗

But questions arise:

Q1: Is pole placement possible?

Q2: How can pole placement be done?

§7.3 Single-input Case

Consider a *single input* system:

$$\dot{x} = Ax + bu, \tag{4}$$

where *u is a scalar*. Use the state feedback:

$$u = r - k^T x. \tag{5}$$

Note $k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$ is a column vector, $k^T = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$ is a row vector,

$$k^T x = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} k_i x_i,$$

Pole placement problem. For the plant (4) , let $\lambda_1, \lambda_2, \cdots, \lambda_n$, be the desired stable closed-loop poles, where for each complex pole, its conjugate must be also included in. Define

$$\phi_d(s) = \prod_{i=1}^{n}(s-\lambda_i) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \gamma_0$$

as the desired closed-loop characteristic polynomial with real coefficients. We want to determine the control law (5) such that the closed-loop system:

$$\dot{x} = (A - bk^T)x + br$$

meets

$$\det(sI - A + bk^T) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_0 . \qquad (6)$$

Theorem 1. *The closed-loop poles of the system with* (4) *and* (5) *can be arbitrarily assigned if and only if* (A, b) *is controllable.*

Proof of sufficiency: Let (A,b) be controllable. Use the state transformation, $\bar{x} = Tx$, such that

$$\dot{\bar{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}}_{\bar{A}} \bar{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\bar{b}} u,$$

where $(\bar{A}, \bar{b})$ is in the controllable canonical form. The characteristic polynomial of $\bar{A}$ is

$$\phi_0(s) = \det\left(sI - \bar{A}\right) = s^n + \sum_{i=0}^{n-1} \alpha_i s^i ,$$

where α_i are from the last row of $\bar{A}$, the only non-trivial elements in $\bar{A}$. For

$$u = r - \bar{k}^T \bar{x} = r - \sum_{i=1}^{n} \bar{k}_i \bar{x}_i ,$$

one sees that

$$\overline{bk}^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \bar{k}^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{k}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_n \end{bmatrix}.$$

$(\bar{A} - \overline{bk}^T)$ will differ from $\bar{A}$ only in the last row, and the controllable canonical structure of $\bar{A}$ is retained in $(\bar{A} - \overline{bk}^T)$. Indeed, one has

$$\dot{\bar{x}} = (\bar{A} - \bar{b}\bar{k}^T)\bar{x} + \bar{b}r$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 - \bar{k}_1 & -\alpha_1 - \bar{k}_2 & -\alpha_2 - \bar{k}_3 & \cdots & \left(-\alpha_{n-1} - \bar{k}_n\right) \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} r.$$

The *closed-loop* characteristic polynomial is

$$\phi_f(s) = s^n + \sum_{i=0}^{n-1} (\alpha_i + \bar{k}_{i+1}) s^i,$$

which can be made by a proper choice of $\bar{k}_i$ equal to any desired characteristic polynomial:

$$\phi_d(s) = \prod_{i=1}^{n}(s-\lambda_i) = s^n + \sum_{i=0}^{n-1}\gamma_i s^i,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$, be the desired stable closed-loop poles, that is

$$\phi_f(s) = \phi_d(s) \Leftrightarrow \alpha_i + \bar{k}_{i+1} = \gamma_i$$

which always has a solution for $\bar{k}$ as well as k since

$$u = r - \bar{k}^T \bar{x} = r - \bar{k}^T T x = r - k^T x \rightarrow k^T = \bar{k}^T T \; .$$

Proof of Necessity. Conversely, suppose (A, b) is uncontrollable. There is a transformation $\tilde{x} = Tx$ such that

$$\tilde{x} = \begin{bmatrix} \tilde{x}_c \\ \tilde{x}_{\bar{c}} \end{bmatrix}, \quad \dot{\tilde{x}} = \begin{bmatrix} A_c & A_{c\bar{c}} \\ 0 & A_{\bar{c}} \end{bmatrix} \tilde{x} + \begin{bmatrix} b_c \\ 0 \end{bmatrix} u,$$

$$\dot{\tilde{x}}_c = A_c \tilde{x}_c + A_{c\bar{c}} \tilde{x}_{\bar{c}} + b_c u, \quad \text{controllable } \tilde{x}_c,$$

$$\dot{\tilde{x}}_{\bar{c}} = A_{\bar{c}} \tilde{x}_{\bar{c}}, \qquad \text{uncontrollable } \tilde{x}_{\bar{c}},$$

$$u = r - \tilde{K}^T \tilde{x} = r - \begin{bmatrix} \tilde{k}_c & \tilde{k}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \tilde{x}_c \\ \tilde{x}_{\bar{c}} \end{bmatrix}.$$

Note

$$\tilde{b}\tilde{K}^T = \begin{bmatrix} b_c \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{k}_c^T & \tilde{k}_{\bar{c}}^T \end{bmatrix} = \begin{bmatrix} b_c \tilde{k}_c^T & b_c \tilde{k}_{\bar{c}}^T \\ 0 & 0 \end{bmatrix},$$

so that $(\tilde{A}-\tilde{b}\tilde{K}^T)$ has the same block-triangular structure as $\tilde{A}$. The feedback system is

$$\dot{\tilde{x}} = (\tilde{A}-\tilde{b}\tilde{K}^T)\tilde{x}+\tilde{b}r$$

$$= \begin{bmatrix} A_c - b_c\tilde{k}_c^T & A_{c\bar{c}} - b_c\tilde{k}_{\bar{c}}^T \\ 0 & A_{\bar{c}} \end{bmatrix}\tilde{x} + \begin{bmatrix} b_c \\ 0 \end{bmatrix} r,$$

with

$$\det(sI-\tilde{A}+\tilde{b}\tilde{K}^T) = \det(sI-A_c+b_c\tilde{k}_c^T)\det(sI-A_{\bar{c}}),$$

where $\det\left(sI-A_{\bar{c}}\right)$ is independent of $\tilde{K}$, and can NOT be changed. Hence, the closed-loop poles cannot be assigned arbitrarily.

Theorem 1 answers Q1 on when pole placement has a solution. Now:

Q2: "how to get K for pole placement?"

That is, ***given a controllable pare of (A, b) and a stable ϕ_d, find k to meet***

$$\det(sI - A + bk^T) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_0 .$$

The simplest method is to directly solve (6) by coefficient comparison.

Example 3. For

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

assign the desired closed-loop poles at $\lambda_1 = -1$ and $\lambda_2 = -2$.

Solution. Let $u = -k^T x = -[k_1\ k_2]x$. Then the closed-loop system becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$$
$$= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x.$$

The closed-loop characteristic polynomial is

$$\det\left(sI - A + bk\right) = s^2 + k_2 s + k_1.$$

On the other hand, we have

$$\phi_d\left(s\right) = \left(s - \lambda_1\right)\left(s - \lambda_2\right) = s^2 - \left(\lambda_1 + \lambda_2\right)s + \lambda_1\lambda_2.$$

Comparing the corresponding coefficients of the above two gives

$$k_1 = \lambda_1\lambda_2 = 2,$$
$$k_2 = -\left(\lambda_1 + \lambda_2\right) = 3.$$

But the direct comparison will be difficult to solve if the system order is 3 or higher.

We need systematic method for pole placement.
Ackermann formula is given by

$$k^T = [0,\ 0,\ \cdots,\ 0,\ 1]\mathrm{C}^{-1}\phi_d(A),$$

Where C is the controllability matrix

$$C = [b \quad Ab \quad \cdots \quad A^{n-1}b]$$

$$\phi_d(A) = \phi_d(s)\big|_{s=A} = A^n + \gamma_{n-1}A^{n-1} + \ldots + \gamma_0 I_n.$$

Note $A^0 = I_n$.

Example 4. Let the desired poles be $\lambda_1 = -3$, $\lambda_2 = -4$, and the plant be

$$\dot{x} = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Then, it follows that

$$\phi_d(s) = (s+4)(s+3) = s^2 + 7s + 12,$$

$$\begin{aligned}
\phi_d(A) &= A^2 + 7A + 12I_2 \\
&= \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} + 7\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} + 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 13 & 3 \\ 12 & 4 \end{bmatrix} + \begin{bmatrix} 21 & 7 \\ 28 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} \\
&= \begin{bmatrix} 46 & 10 \\ 40 & 16 \end{bmatrix}.
\end{aligned}$$

$$\mathsf{C} = [b \quad Ab] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$[0 \quad 1]\mathsf{C}^{-1} = [1 \quad 0].$$

Hence, Ackermann's formula yields

$$k^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 46 & 10 \\ 40 & 16 \end{bmatrix} = \begin{bmatrix} 46 & 10 \end{bmatrix}.$$

Example 5. Consider the system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Design the state feedback control $u = -k^T x$ such that the desired closed-loop poles are located at $\lambda_1 = -2 + j4$, $\lambda_2 = -2 - j4$, $\lambda_3 = -10$.

Solution. Since

$$\phi_d(s) = (s-\lambda_1)(s-\lambda_2)(s-\lambda_3) = s^3 + 14s^2 + 60s + 200.$$

$$\phi_d\left(A\right) = A^3 + 14A^2 + 60A + 200I$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2$$

$$+60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix},$$

and

$$\mathsf{C} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix},$$

we obtain

$$k^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \end{bmatrix}.$$

It is the same as before.

For single-input systems, the solution to pole placement is unique.

What happens to the zeros?

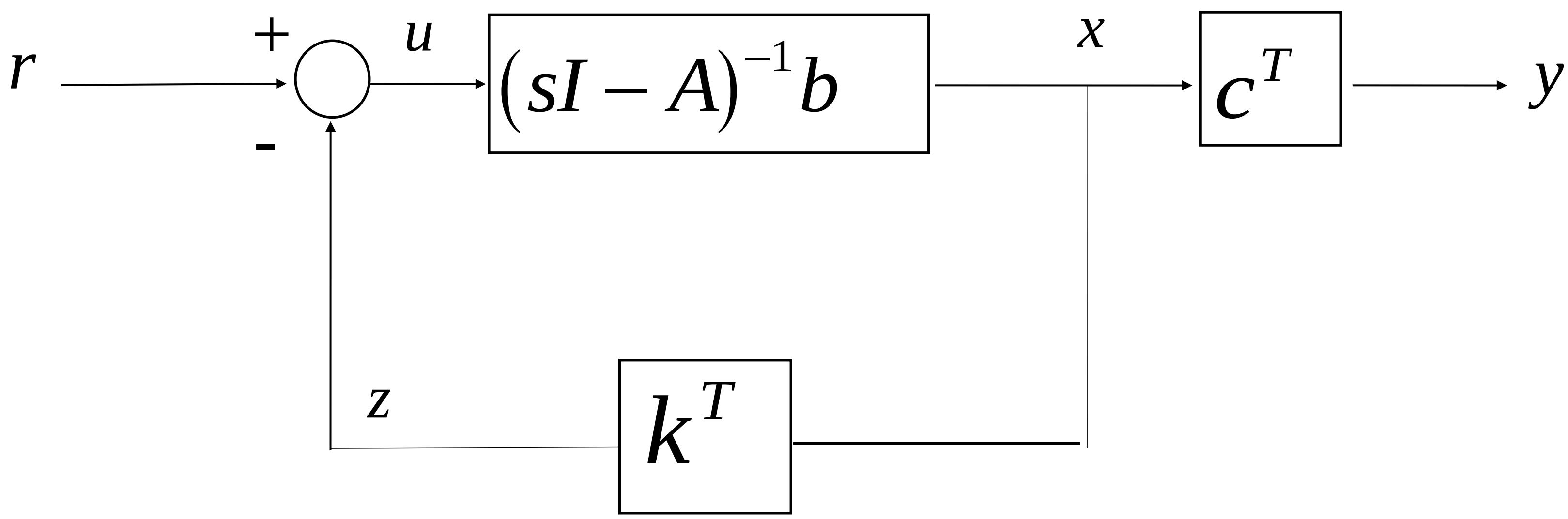


Figure 7 State feedback control system.

Let

$$\frac{Y(s)}{U(s)} = c^T(sI - A)^{-1}b \;:= \frac{\beta(s)}{\phi_o(s)}\,,$$

with

$$\phi_o(s) = \det(sI - A)\,,$$

and

$$\frac{Z(s)}{U(s)} = k^T(sI - A)^{-1}b = \frac{g(s)}{\phi_o(s)}\,.$$

Then,

$$\frac{Y(s)}{R(s)} = \frac{c^T(sI - A)^{-1}b}{1 + k^T(sI - A)^{-1}b} = \frac{\dfrac{\beta}{\phi_0}}{1 + \dfrac{g}{\phi_0}} = \frac{\beta(s)}{\phi_0(s) + g(s)} = \frac{\beta(s)}{\phi_f(s)}$$

Conclusion: No change on zeros.

Example 6. Consider a plant,

$$\dot{x} = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u ,$$

$$y = [5 \quad 1] x .$$

It follows that

$$(sI - A)^{-1} = \begin{bmatrix} s-3 & -1 \\ -4 & s \end{bmatrix}^{-1}$$

$$= \frac{1}{s(s-3)-4} \begin{bmatrix} s & 1 \\ 4 & s-3 \end{bmatrix}$$

$$G(s) = c(sI - A)^{-1} b$$

$$= c \begin{bmatrix} s & 1 \\ 4 & s-3 \end{bmatrix} \frac{1}{s^2 - 3s - 4} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 - 3s - 4} \begin{bmatrix} 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s - 3 \end{bmatrix}$$

$$= \frac{s + 2}{s^2 - 3s - 4},$$

which has a zero at $s = -2$. Suppose that we design

$$u = -k^T x + fr,$$

with k^T to achieve pole placement and f to make the unity static gain, that is,

$$H_{y,r}(s) = c\left(sI - A + Bk^T\right)^{-1} bf,$$

making $H_{y,r}(0) = 1$ yields f as

$$f = \left[c\left(-A + bk^T\right)^{-1} b\right]^{-1}.$$

(i) If the desired closed-loop poles are (-5,-8) and the steady state error to a unit step is zero, the solution is

$$k^T = [92 \quad 16], \quad f = 20.$$

The corresponding closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{20(s+2)}{(s^2+13s+40)},$$

for which s = –2 remains as a zero.

(ii) If the desired closed-loop poles are (-2,-8) and the steady state error to a unit step is zero, the solution is

$$k^T = [58.8 \quad 13], \qquad f = 8, \qquad \frac{Y(s)}{R(s)} = \frac{8}{(s+8)}.$$

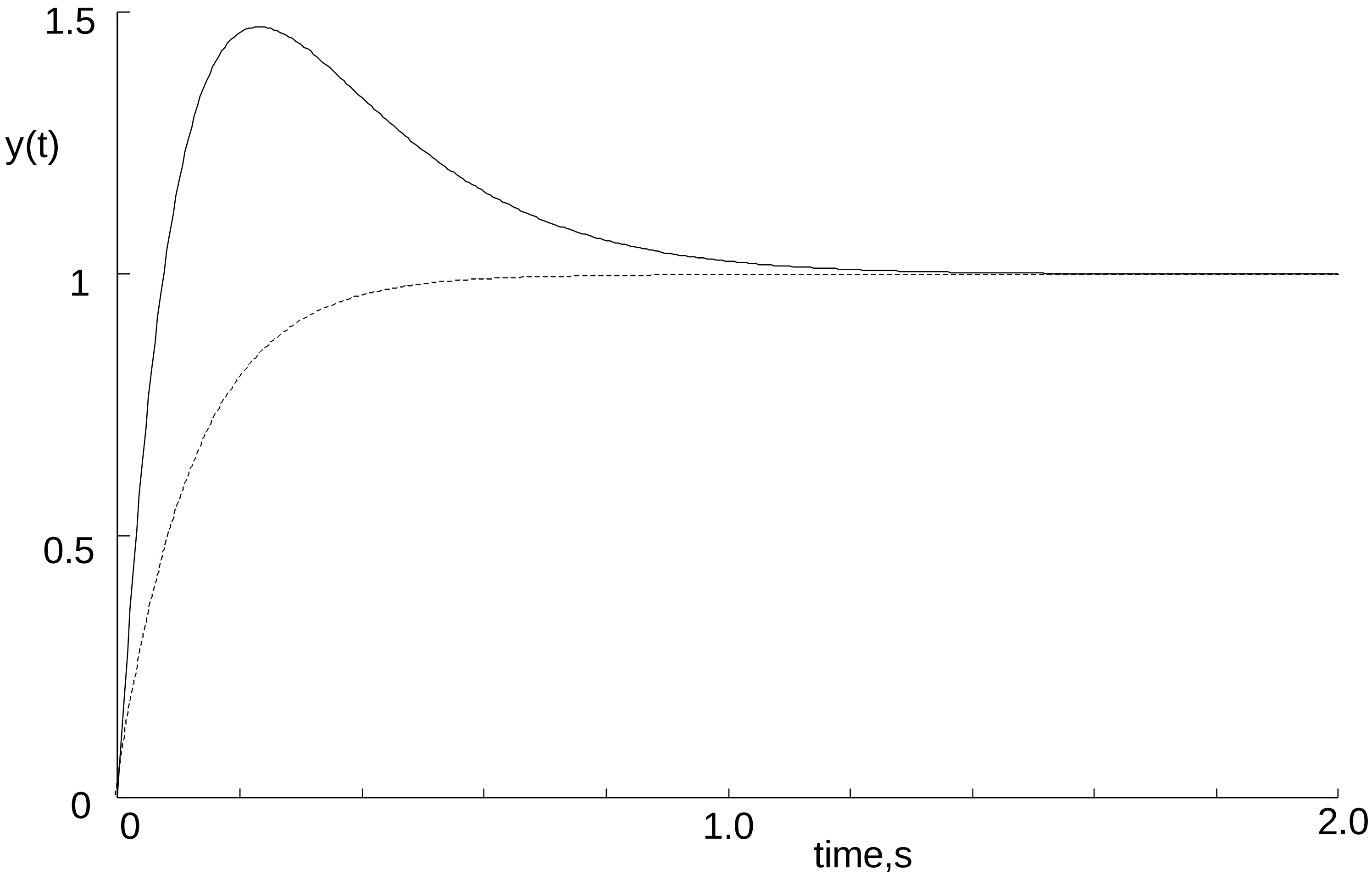


Figure 8 Unit step response: case (i) ———, case (ii) ········ .

Summary on pole placement for the single-input case

Solvability $\leftrightarrow$ Controllability

Solutions: Unique

Solution procedures:

- Direct comparison using $\det\left[sI - A + bK\right] = \phi_d(s)$
- Ackermann's formula

Selection of procedures

- Direct comparison recommended for systems of order 1 or 2
- Ackermann's formula recommended for systems of order 3 or higher

No zeros change under pole-placement and state-feedback in general.

Selection of closed-loop poles

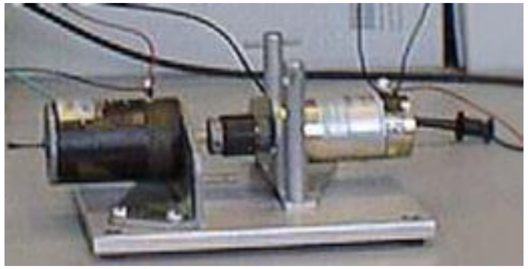
- Use performance specifications to determine a pair of dominant poles (i.e. slowest among all): λ_1, λ_2 with $\mathrm{Re}\,\lambda_1 = \mathrm{Re}\,\lambda_2 < 0.$ Say, $\lambda_{1,2} = -2 \pm j$, with the corresponding decay function of e^{-2t}.

- Locate extra poles other than the dominant ones to be 2-5 time faster than the dominant ones: $\lambda_i = (2 \sim 5)\,\mathrm{Re}\,\lambda_1, i = 3, 4, \ldots, n.$ Say, e^{-10t}, which decays much faster than e^{-2t}.

- Fine-tune pole locations/performance by taking into account plant zeros and doing simulation.

An Industrial Application: DC Motor Position Control revisited

1) Model:



$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K_t}{J} \\ 0 & -\frac{K_e}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \frac{1}{L}\end{bmatrix} V,$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix}.$$

with

$J = 3.2284\times10^{-6}$, moment of inertia of the rotor

$b = 3.5077\times10^{-6}$, damping ratio of the mechanical system

$K = 0.0274$, electromotive force constant

$R = 4$, electric resistance

$L = 2.75\times10^{-6}$ electric inductance

2) Control Requirements

- Settling time less than 40 milliseconds
- Overshoot less than 16%

3) Design Solution:

Since the system is of order 3, there will be 3 poles to be placed for the closed-loop system. We may decide two dominant poles from design requirements, and then let the third pole be 2 to 5 times left to these two dominant poles. From

$$t_s = \frac{4}{\zeta\omega_n} < 0.04 \quad \Rightarrow \quad \zeta\omega_n > 100,$$

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.16 \quad \Rightarrow \quad \zeta > 0.47,$$

choose $\zeta = 0.707$ and $\zeta\omega_n = 100$. Then two dominant poles are

$$\lambda_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -100 \pm j100.$$

The third pole is chosen as $\lambda_3 = -200$. By Ackermann's formula, we obtain

$$K = \begin{bmatrix} 0.0013 & -0.0274 & -3.9989 \end{bmatrix}.$$

The step response of the closed-loop system is shown in Figure 9.

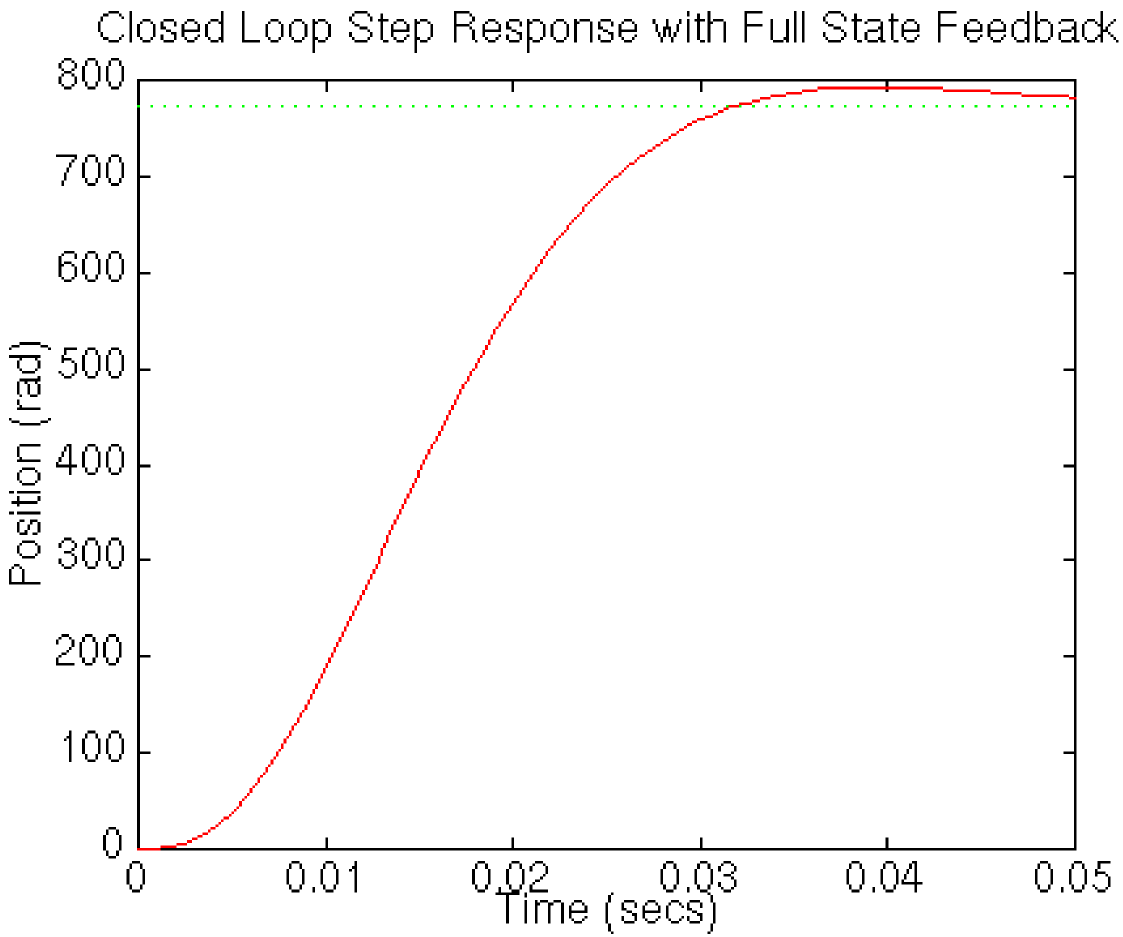


Figure 9 Unit step response.

§7.4 Multi-input Case

Consider a multi-input plant:

$$\dot{x} = Ax + Bu, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

$$y = Cx.$$

where u has 2 or more elements, or B has 2 or more columns. The control law is

$$u = -Kx + Fr.$$

The closed-loop system becomes

$$\dot{x} = (A - BK)x + BFr,$$

$$y = Cx.$$

In the following let's first try to use direct comparison method and see what happens to the multi-input case.

Example 7. Consider the system:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u$$

The closed-loop poles are to be at $\lambda_{1,2} = -1 \pm j.$

$$\varphi_d(s) = (s+1+j)(s+1-j) = s^2 + 2s + 2.$$

Check:

$$[B \quad AB] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \text{controllable } (A, B).$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u \qquad u = -Kx + Fr, \qquad K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$\begin{aligned}
&\det(sI - (A - BK)) \\
&= \det(sI - (\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix})) = \det(sI - \begin{bmatrix} -k_{11} & 1 - k_{12} \\ -k_{21} & -k_{22} \end{bmatrix}) \\
&= \det(\begin{bmatrix} s + k_{11} & -1 + k_{12} \\ k_{21} & s + k_{22} \end{bmatrix}) = (s + k_{11})(s + k_{22}) - (k_{12} - 1)k_{21} \\
&= s^2 + (k_{11} + k_{22})s + k_{11}k_{22} - k_{12}k_{21} + k_{21}
\end{aligned}$$

Compare this with the desired C.P. $\varphi_d(s) = (s + 1 + j)(s + 1 - j) = s^2 + 2s + 2.$

We have
$$\begin{cases} k_{11} + k_{22} = 2 \\ k_{11}k_{22} - k_{12}k_{21} + k_{21} = 2 \end{cases}$$

which results in infinity number of solutions.

In general, by direct comparison, let

$$\det(sI - (A - BK)) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_0$$

The number of design parameters in K is $m \times n$, while the number of equations is only n. Hence the number of solutions is infinity.
In the following, we are going to show two systematic ways to obtain the solutions such that we gain more insights.

7.4.1 Unity Rank K

Since we already know how to place the poles for single-input system, the first way is to convert the multi-input to single input. But is that possible?

Multi-input means that you can choose multiple input signals at the same time. But can we limit our choice to only one input signal?

The answer simply is YES because you are free to design whatever input you like!

For example, we can design all the input signals as the same such that

$$u = \begin{bmatrix} v \\ v \\ \vdots \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} v.$$

where v is a scalar.
In general, we can have different weights for different input such that

$$u = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} v = qv,$$

One sees that the system:

$$\begin{aligned} \dot{x} &= Ax - Bu \\ &= Ax - Bqv \\ &= Ax + \underbrace{(Bq)}_{b} v, \end{aligned} \tag{10}$$

may be viewed as a single-input system with *(A, Bq)* and *v* as the single input.

Then we can apply the state feedback control law

$$v = -k^T x, \tag{11}$$

Overall, we have,

$$u = qv = -qk^T x = -Kx,$$

where *K* has unity rank:

$$K = qk^T = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix},$$

************Revision on Matrix Rank***********

$$\text{Rank}\left(A\cdot B\right)\le \min\left(\text{Rank } A, \text{Rank } B\right)$$

Let

$$A=\begin{bmatrix}1\\2\end{bmatrix},\quad B=\begin{bmatrix}3 & 4\end{bmatrix},\quad \text{Rank } A=\text{Rank } B=1$$

$$AB=\begin{bmatrix}3 & 4\\6 & 8\end{bmatrix},\quad \det AB=0,\quad \text{Rank}\left(AB\right)=1.$$

If A and B are non-zero column vectors, then one sees

$$A=\begin{bmatrix}a_1\\a_2\\\vdots\\a_m\end{bmatrix},\quad B^T=\begin{bmatrix}b_1 & b_2 & \cdots & b_m\end{bmatrix},$$

$$AB^T=\begin{bmatrix}Ab_1 & Ab_2 & \cdots & Ab_m\end{bmatrix},\quad \text{columns proportional to each other}$$

$$\text{Rank}\left(AB\right)=1,\quad \text{if } A\ne 0, B^T\ne 0.$$

************* End of Revision *************

Key: If *the single-input pair* {*A,Bq*} can be made *controllable, the single input (SI) solutions can be applied.*

It can be shown that if {A,B} is controllable, then for almost any vector q, the single-input pair {A,Bq}is also controllable.

Example 8. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The system is controllable. The possible SI system is

$$\tilde{A} = A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\tilde{B} = Bq = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_1 + q_2 \end{bmatrix} = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix}.$$

Look at controllability matrix

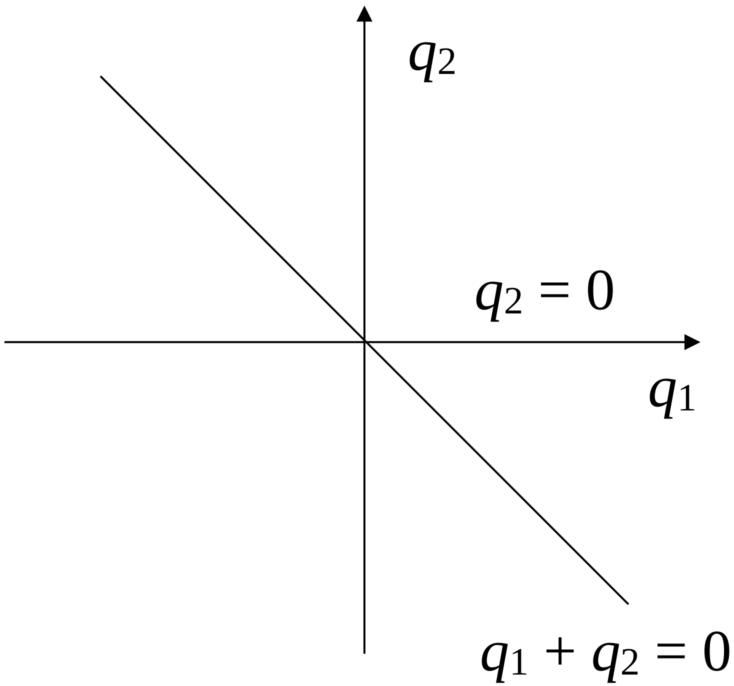
$$\tilde{Q}_C = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} \end{bmatrix} = \begin{bmatrix} q_1 & 2q_1 + q_2 \\ q_1 + q_2 & 2(q_1 + q_2) \end{bmatrix}$$

$$\det(\tilde{Q}_C) = q_2(q_1 + q_2).$$

$\tilde{Q}_c$ is nonsingular except

(i) $q_1 + q_2 = 0$, or

(ii) $q_2 = 0$.



Algorithm for unity rank pole placement

Given a controllable pair $\{A, B\}$. Proceed with

*Step 1***.** Choose the weight vector q such that the *pair* {A, Bq} is controllable

*Step 2***.** Use any single-input pole placement algorithm for the *pair* $\{A, Bq\}$ to determine k^T such that

$$A - Bqk^T$$

has the desired eigenvalues (closed-loop poles).

Then the required state feedback matrix is

$$K = qk^T. \tag{12}$$

Example 9. Consider the system:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u \quad .$$

The closed-loop poles are to be at $\lambda_{1,2} = -1 \pm j$.

$$\phi_d(s) = (s+1+j)(s+1-j) = s^2 + 2s + 2.$$

Check:

$$[B \quad AB] = \begin{bmatrix} 1 & 0 & \times & \times \\ 0 & 1 & \times & \times \end{bmatrix} \quad \Rightarrow \quad \text{controllable}\left(A, B\right).$$

Step 1. Choose

$$q=\begin{bmatrix}0\\1\end{bmatrix},$$

Then

$$Bq=\begin{bmatrix}0\\1\end{bmatrix}$$

and

$$\begin{bmatrix}Bq & ABq\end{bmatrix}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

So (A,Bq) is controllable.

Step 2. Let $k^T = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, thus

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{Bq} k^T = \underbrace{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}}_{A_1},$$

$$\det\left(sI - A_1\right) = s^2 + k_2 s + k_1,$$

$$\phi_d\left(s\right) = s^2 + 2s + 2,$$

and directly comparing coefficients yields

$$k^T = [2 \quad 2].$$

Hence, we have the state feedback gain as

$$K = qk^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [2 \quad 2] = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}.$$

One may check that

$$\det(sI - A + BK) = \det\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}\right\}$$

$$= \det\begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}$$

$$= s^2 + 2s + 2$$

which is indeed the same as $\phi_d(s)$.

Is the solution unique?

If we choose another weight vector

$$q = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\text{then } Bq = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and $\begin{bmatrix} Bq & ABq \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has full rank. So (A, Bq) is thus controllable. Let $k^T = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. One has

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{Bq} k^T = \underbrace{\begin{bmatrix} -k_1 & 1-k_2 \\ -k_1 & -k_2 \end{bmatrix}}_{A_1},$$

$$\det\left(sI - A_1\right) = s^2 + \left(k_1 + k_2\right)s + k_1,$$

$$\phi_d\left(s\right) = s^2 + 2s + 2,$$

and directly comparing coefficients yields

$$k^T = [2 \quad 0].$$

This leads to the state feedback gain as

$$K = qk^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [2 \quad 0] = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}.$$

One may check that

$$\begin{aligned} \det(sI - A + BK) &= \det\left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \right\} \\ &= \det \begin{bmatrix} s+2 & -1 \\ 2 & s \end{bmatrix} \\ &= s^2 + 2s + 2 \end{aligned}.$$

which is also the same as $\phi_d(s)$.

This case shows that K is not unique. Why?

There are infinite number of ways to choose the weight vector q such that {A, Bq} is controllable!

Key: If *the single-input pair {A,Bq}* can be made *controllable, the single input (SI) solutions can be applied.*

But there are exceptions like

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u$$

You can easily verify that the system is controllable since B itself is already full rank!

However, for any vector q, the single-input pair {A,Bq}= {A,q} is uncontrollable!

Is there any other systematic way?

For the single-input case, we showed that the controllable canonical form is the key for the solution.

Can we also rely on the controllable canonical form for multi-input?

7.4.2 Pole Placement with Controllable Canonical Form – Full Rank Method

Example 10. Consider the system:

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} x \,.$$

Solution: (i) Obtain the controllable canonical form of A:

How to Obtain the controllable canonical for MIMO system?

First compute the controllability matrix

$$W_c = \{B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\}$$

Check whether it is full rank or not. If it is controllable, then move on to the next step.

For single input system, the controllability matrix is a square matrix, all of the vectors are independent.

For MIMO system, we need to select the n independent vectors from the controllability matrix in the strict order from left to right.

And group them in a square matrix C in the following form

$$C = \{b_1 \quad Ab_1 \quad \cdots \quad A^{d_1-1}b_1 \quad b_2 \quad Ab_2 \quad \cdots \quad A^{d_2-1}b_2 \cdots \quad b_m \quad Ab_m \quad \cdots \quad A^{d_m-1}b_m \quad \}$$

Where the indices di imply the number of vectors in C related to the i-th input, ui

The controllable canonical form can then be computed from this matrix!

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$W_c = \{B \quad AB \quad A^2B\} = \{b_1 \quad b_2 \quad Ab_1 \quad Ab_2 \quad A^2b_1 \quad A^2b_2\}$$
$$= \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 & 4 \end{bmatrix}$$

Select 3 independent vectors from the controllability matrix in the order from left to right and group them in a matrix C in the following form

$$C = \{b_1 \quad b_2 \quad Ab_2\} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Once we have the matrix C, we need to compute the inverse of C,

$$C^{-1} = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

How to construct T for the single input case? Which row to take out?

For single input case, just take the last row of C^{-1}, q_n^T, and construct T as

$$T = \begin{bmatrix} q_n^T \\ q_n^T A \\ \vdots \\ q_n^T A^{n-1} \end{bmatrix} \Rightarrow \bar{b} = Tb = \begin{bmatrix} q_n^T b \\ q_n^T Ab \\ \vdots \\ q_n^T A^{n-1} b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For multi-input case, we need to take out m rows from C^{-1} corresponding to the m inputs selected in the matrix C, and form T as

$$T = \begin{bmatrix} q_{d_1}^T \\ q_{d_1}^T A \\ \vdots \\ q_{d_1}^T A^{d_1-1} \\ q_{d_1+d_2}^T \\ q_{d_1+d_2}^T A \\ \vdots \\ q_{d_1+d_2}^T A^{d_2-1} \\ \vdots \end{bmatrix} \Rightarrow \overline{B} = TB = \begin{array}{cl} \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & \times & \cdots & \times \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{array}\right] & \begin{array}{l} \\ \\ \\ d_1^{th} row \\ \\ \\ \\ (d_1 + d_2)^{th} row \\ \\ \\ \\ n^{th} row \end{array} \end{array}$$

Where the indices d_i imply the number of vectors in C related the i-th input, u_i

$$C = \{ b_1 \quad Ab_1 \quad \cdots \quad A^{d_1-1} b_1 \quad b_2 \quad Ab_2 \quad \cdots \quad A^{d_2-1} b_2 \cdots \quad b_m \quad Ab_m \quad \cdots \quad A^{d_m-1} b_m \quad \}$$

$$C = \{b_1 \quad b_2 \quad Ab_2\} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

What is d_1 and d_2? $d_1 = 1$ and $d_2 = 2$

Which rows should we take out to form T?

The first (d_1)and third ($d_1 + d_2$)rows!

$$T = \begin{bmatrix} q_1^T \\ q_3^T \\ q_3^T A \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The key is to form the transformation matrix T. The rest is simple.

$$\bar{A} = TAT^{-1} = \left[\begin{array}{c|cc} -1 & 7 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -2 & 3 \end{array}\right], \quad \bar{B} = TB = \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right],$$

Design the feedback gain matrix $\bar{K} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} \end{bmatrix}$

Form the closed loop matrix:

$$\bar{A} - \bar{B}\bar{K} = \left[\begin{array}{c|cc} -\bar{k}_{11} - 1 & -\bar{k}_{12} + 7 & -\bar{k}_{13} \\ \hline 0 & 0 & 1 \\ -\bar{k}_{21} & -\bar{k}_{22} - 2 & -\bar{k}_{23} + 3 \end{array}\right].$$

(iii). Let the desired eigenvalues be -1, -2, -3. Then a possible desired closed-loop matrix is

$$A_d = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -6 & -5 \end{array}\right] \qquad \tilde{A}_d = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ -6 & -11 & -6 \end{array}\right]$$

since

$$\begin{aligned}\det(sI - A_d) &= (s+1)(s+2)(s+3) \\ &= (s+1)(s^2+5s+6) \\ &= s^3+6s^2+11s+6.\end{aligned}$$

(iv) Equating $\bar{A} - \bar{B}\bar{K}$ and A_d, we get

$$\bar{k}_{11} = \bar{k}_{13} = \bar{k}_{21} = 0,\ \bar{k}_{12} = 7,\ \bar{k}_{22} = 4,\ \bar{k}_{23} = 8.$$

(v) Finally, the original feedback gain is obtained as

$$K = \overline{K}T = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 12 & 8 \end{bmatrix}.$$

Algorithm For Full rank pole placement

Given controllable $\{A, B\}$ and the desired $\varphi_d(s)$.

(i) Obtain the controllable canonical form in the state $\overline{x}$ via

$$\overline{x} = Tx.$$

Such that

$$\dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u$$

is in the controllable canonical form

(ii) Specify a desired closed-loop matrix A_d as

$$A_d = \text{block diag}\{A_{d_1}, A_{d_2}, \ldots., A_{d_m}\},$$

such that $\det(sI - A_d) = \varphi_d(s)$ and A_d is the same as $\bar{A}$ except non-trivial rows.

(iii) Compare the non-trivial rows of the canonical form with the desired one, and compute $\bar{K}$

(iv) Compute the original feedback gain K

$$K = \bar{K}T.$$

Example 11. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 0 & 4 & 1 \\ 10 & 13 & 2 & 8 \\ -3 & -3 & 0 & -2 \\ -10 & -14 & -5 & -9 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 4 & -3 \\ -1 & 1 \\ -3 & 3 \end{bmatrix} u.$$

It is desired to have the closed-loop poles at -2, -3, $(-1 \pm j\sqrt{3})/2$.

Case* (*i*): Full rank *Method. Construct

$$\mathsf{C} = (b_1 \quad Ab_1 \quad b_2 \quad Ab_2) = \begin{bmatrix} -2 & -7 & 0 & 7 \\ 4 & 6 & -3 & -13 \\ -1 & 0 & 1 & 3 \\ -3 & -4 & 3 & 10 \end{bmatrix}.$$

Then

$$C^{-1} = \begin{bmatrix} 10 & 21 & 21 & 14 \\ 1 & 2 & 3 & 1 \\ -2 & -3 & -5 & -1 \\ 4 & 8 & 9 & 5 \end{bmatrix} \begin{matrix} \\ \leftarrow q_2^T \\ \\ \leftarrow q_4^T \end{matrix},$$

$$T = (q_2 \quad A^T q_2 \quad q_4 \quad A^T q_4)^T = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 4 & 8 & 9 & 5 \\ 3 & 7 & 7 & 5 \end{bmatrix},$$

so that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & -2 & 1 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

in controllable canonical form.

Design the feedback gain matrix

$$\bar{K} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix}$$

Compute

$$\bar{B}\bar{K} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ 0 & 0 & 0 & 0 \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix}$$

Form the closed loop matrix:

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5-\bar{k}_{11} & 3-\bar{k}_{12} & 3-\bar{k}_{13} & -3-\bar{k}_{14} \\ 0 & 0 & 0 & 1 \\ 3-\bar{k}_{21} & 1-\bar{k}_{22} & -2-\bar{k}_{23} & 1-\bar{k}_{24} \end{bmatrix}$$

The desired characteristic polynomial is

$$\phi_f(s) = (s+2)(s+3)(s+\frac{1+j\sqrt{3}}{2})(s+\frac{1-j\sqrt{3}}{2})$$

$$= s^4 + 6s^3 + 12s^2 + 11s + 6.$$

Choose

$$A_d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -11 & -12 & -6 \end{bmatrix},$$

which meets requirements. Compare it with $\bar{A}-\bar{B}\bar{K}$, we have

$$\bar{K} = \begin{bmatrix} -5 & 3 & 2 & -3 \\ 9 & 12 & 10 & 7 \end{bmatrix},$$

and

$$K = \overline{K}T = \begin{bmatrix} -3 & -6 & -9 & -4 \\ 82 & 183 & 202 & 118 \end{bmatrix}.$$

Case **(*ii*)*:* Full rank *K*, different from *case (i)*.** Choose the desired canonical form as

$$A_d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix},$$

which leads to

$$K = \begin{bmatrix} 12 & 29 & 33 & 17 \\ 6 & 15 & 17 & 10 \end{bmatrix}.$$

Case **(*iii*)*:* Unity rank *K*.** Let $K = qk^T$ where $q^T = (1 \quad 1)$.

the required K is

$$K = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -0.245 & 9.509 & 20.358 & 7.415 \end{bmatrix}.$$

***Case* (*iv*): Unity rank *K*.** Choose another $q^T = (1 \quad 3)$

$$K = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 4.792 & 10.328 & 12.307 & 7.768 \end{bmatrix}.$$

We shall compare the four designs for the case when $r = 0$ and $x^T(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$.

(i) Although all the four designs resulted in the same closed-loop poles the nature of K was different in each case.

(ii) The transient performance was substantially different.

(iii) This example shows that K is not unique.

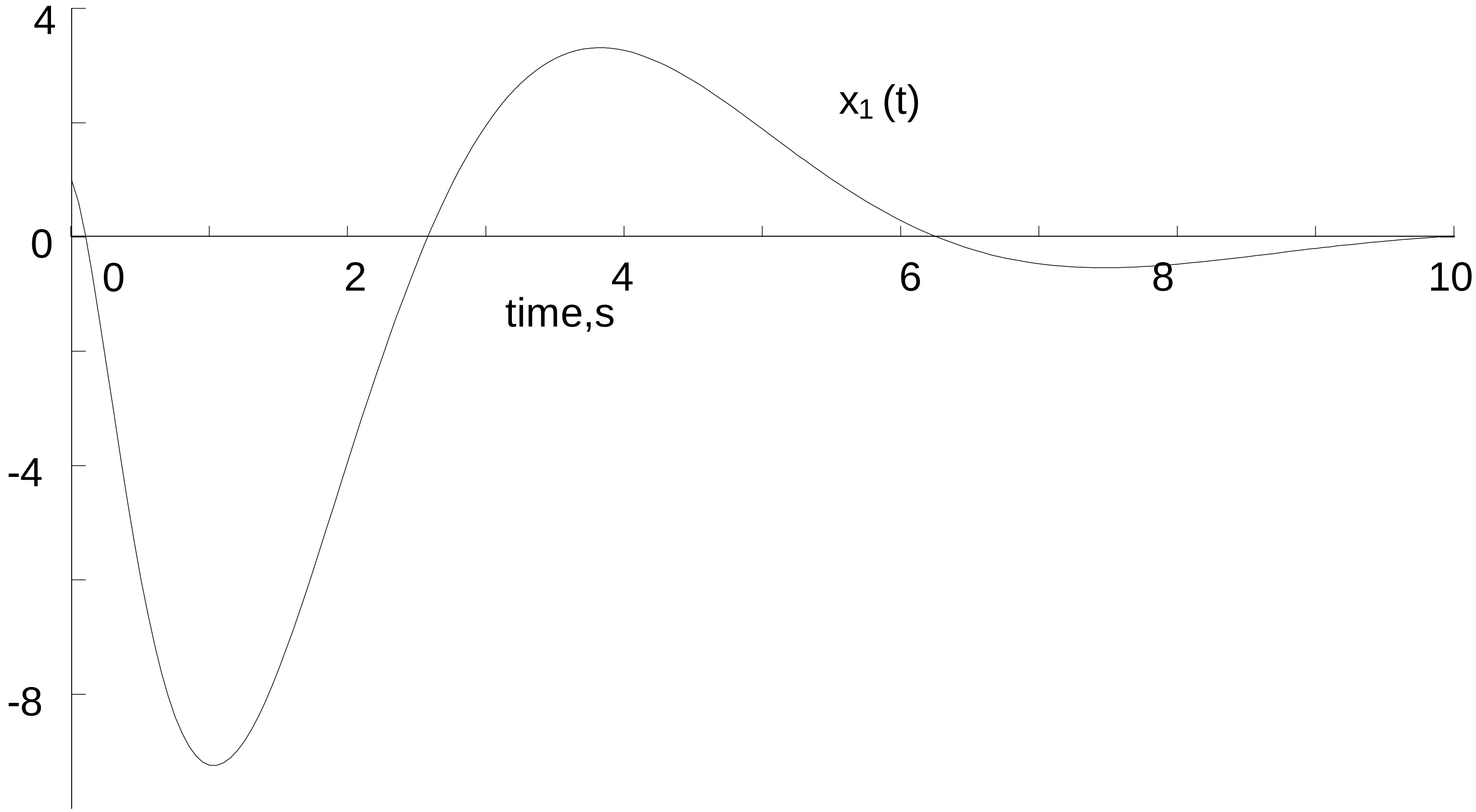


Figure 10. Performance of the pole placement controller: *case* (*i*)

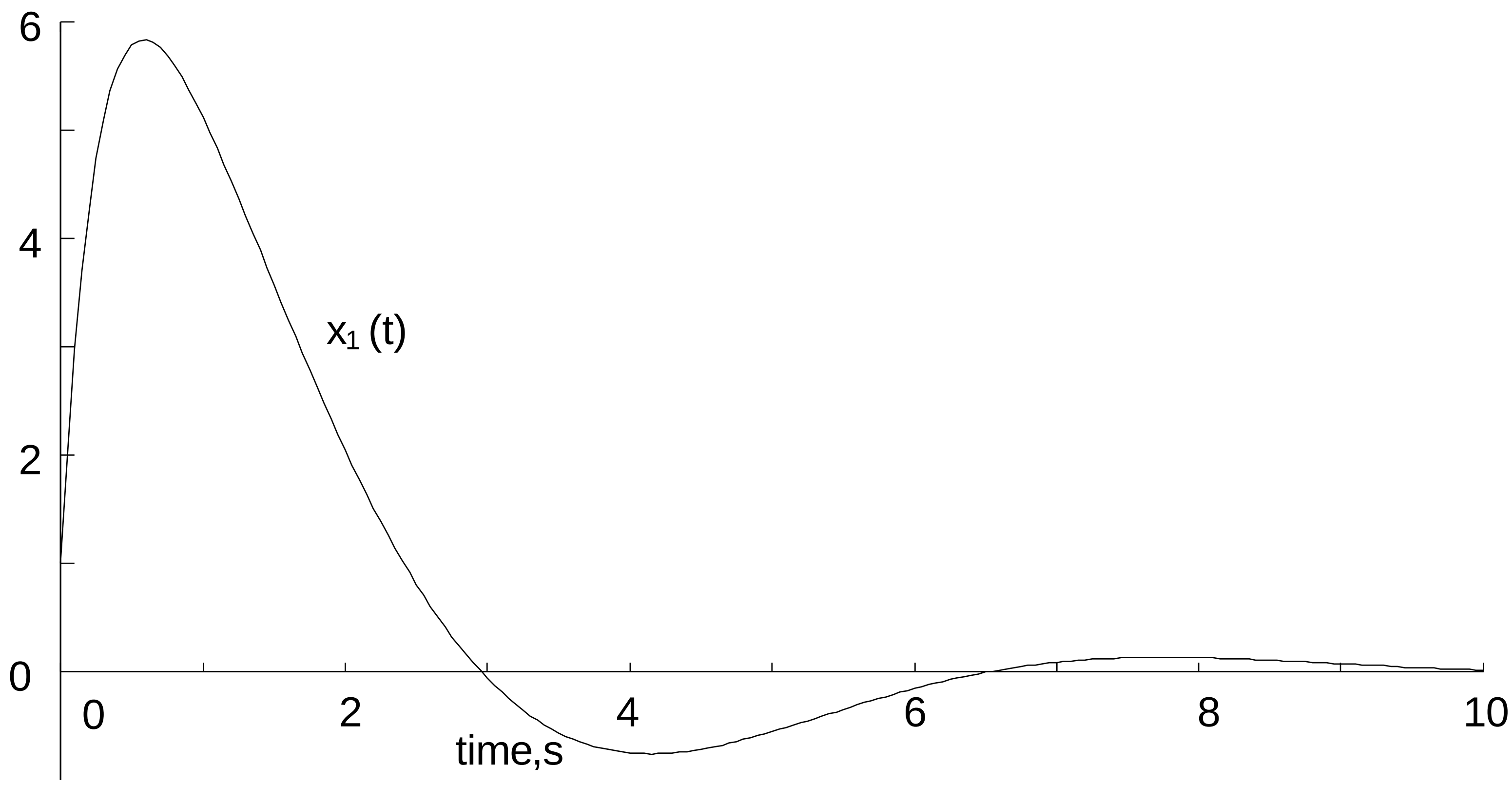


Figure 11. Performance of the pole placement controller: *case* (*ii*)

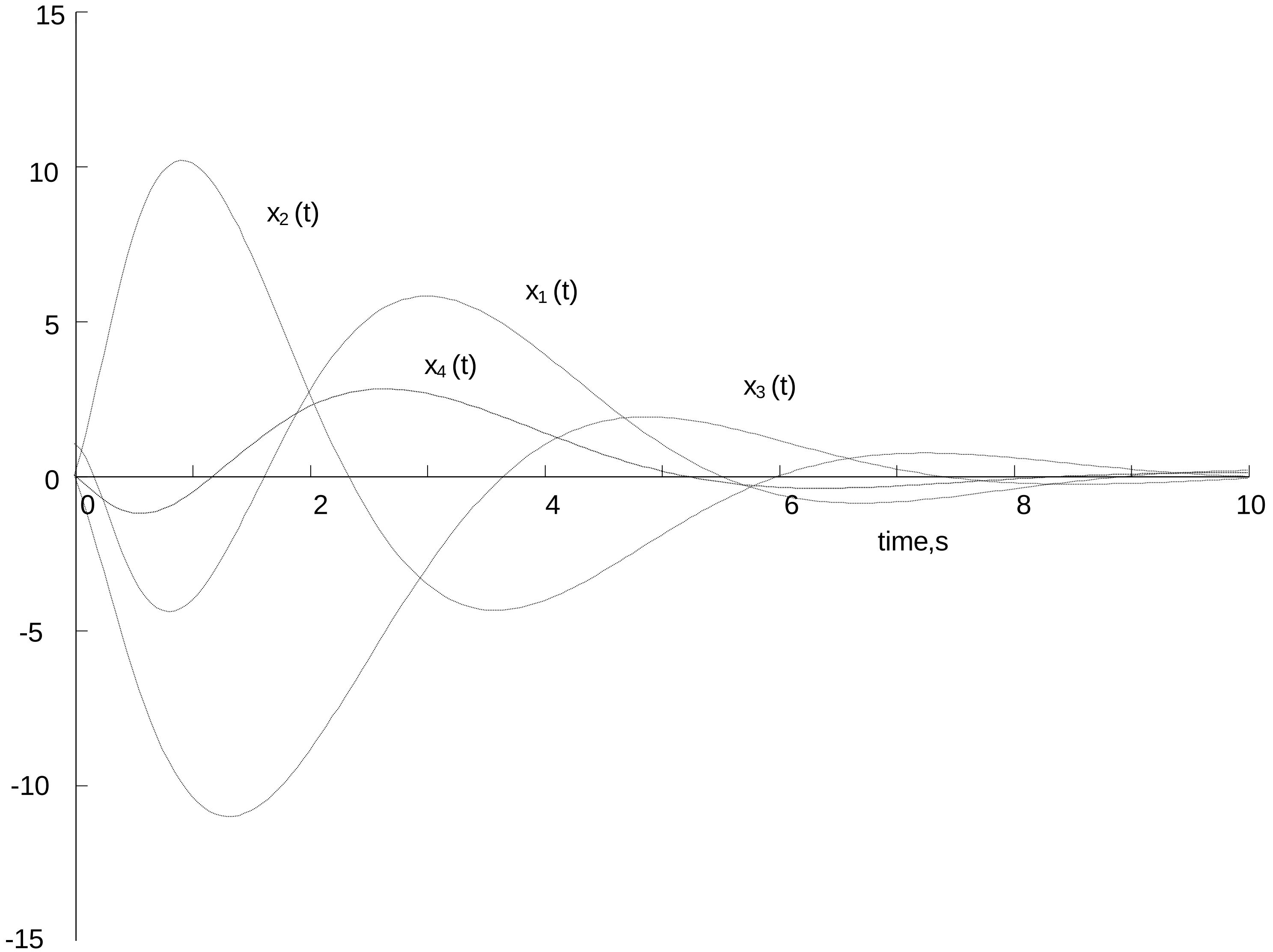


Figure 12. Performance of the pole placement controller: *case* (iii)

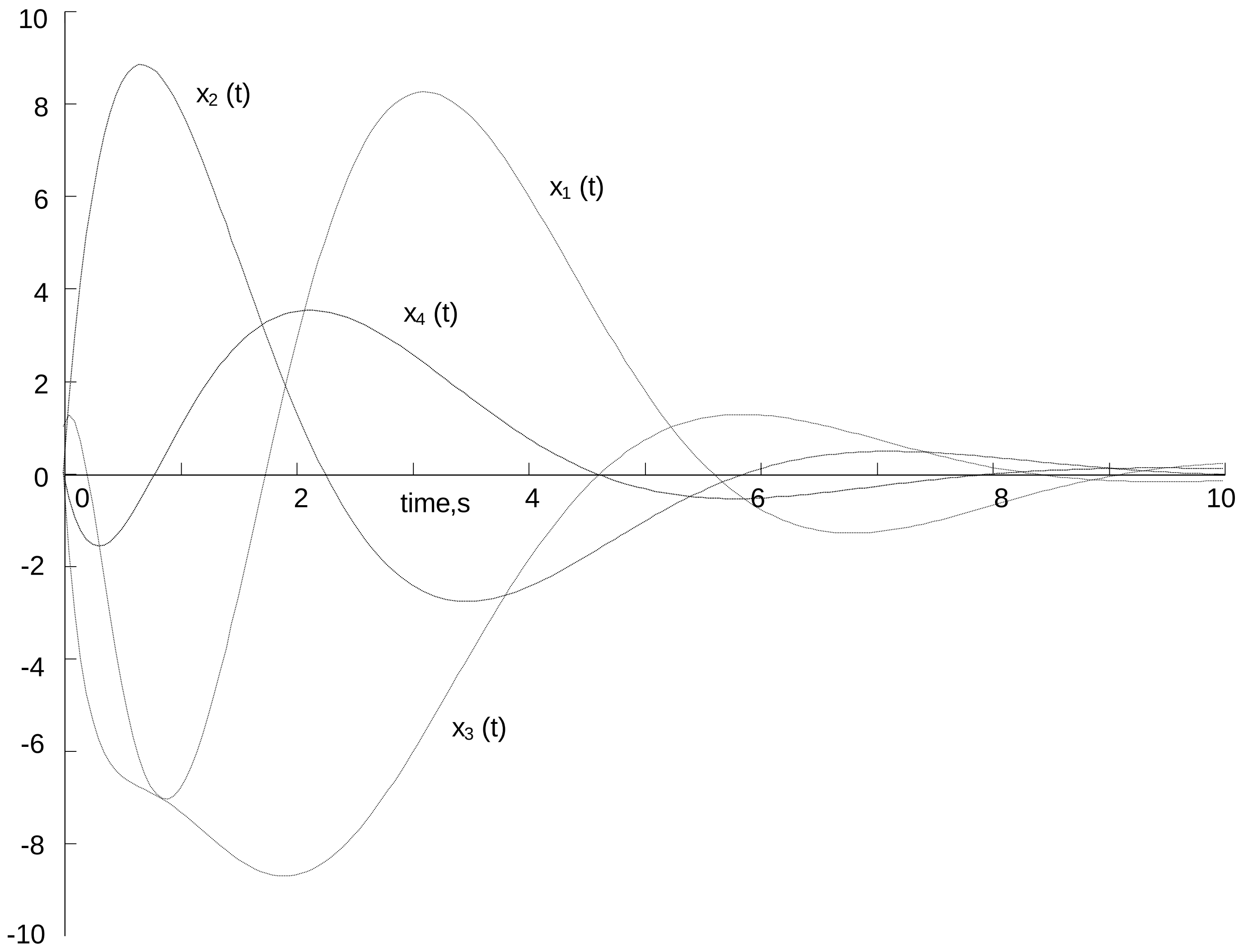


Figure 13. Performance of the pole placement controller: *case* (iv)

§7.5 Summary

In this Chapter, we have discussed general properties of state feedback and its use in pole placement for both the single-input case and the multi-input case.

1. Properties of state feedback:

- the controllability of the system is unaltered by state feedback
- the closed-loop poles are unaffected by the feedforward gain

2. Pole placement:

- Reasons
- Solvability
- Methods

In particular, we focused on the single-input case and showed that

- the conditions for arbitrary assignment of closed-loop poles (*Theorem 1*);
- two methods for the determination of feedback gain : Direct comparison and Ackermann's formula ; and
- the zeros are unchanged by state feedback.

We also extended the theorems to the multi-input case and discussed:

- two methods for the determination of feedback gain :
  Unity Rank and Full-Rank Methods.