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Q.1

a) Calculus of variations

The input signal is:

$$u = -\frac{2}{a}y + \frac{1}{a}\dot{y}$$

The function $J(y, u)$ changes to:

$$\begin{aligned} J(y) &= \int_0^\infty [y^2 + w(-\frac{2}{a}y + \frac{1}{a}\dot{y})^2]dt \\ &= \int_0^\infty [(\frac{4w}{a^2} + 1)y^2 + \frac{w}{a^2}\dot{y}^2]dt - \frac{4w}{a^2}\int_0^\infty y\dot{y}dt \end{aligned}$$

Because

$$\begin{aligned} \int_0^\infty y\dot{y}dt &= y^2|_0^\infty - \int_0^\infty y\dot{y}dt \\ \int_0^\infty y\dot{y}dt &= \frac{1}{2}[y^2(\infty) - y^2(0)] \end{aligned}$$

So:

$$J(y) = \int_0^\infty [(\frac{4w}{a^2} + 1)y^2 + \frac{w}{a^2}\dot{y}^2]dt - \frac{2w}{a^2}y^2(\infty) + \frac{2w}{a^2}c^2$$

Let $z(t)$ denote any function of t with the property that $J(z)$ exists. Take ε to be a scalar parameter.

$$J(y_0 + \varepsilon z) = \int_0^\infty [(\frac{4w}{a^2} + 1)(y_0 + \varepsilon z)^2 + \frac{w}{a^2}(\dot{y}_0 + \varepsilon \dot{z})^2]dt - \frac{2w}{a^2}y^2(\infty) + \frac{2w}{a^2}c^2$$

The $J(y_0 + \varepsilon z)$ must have an absolute minimum at $\varepsilon = 0$

$$\frac{d}{d\varepsilon}J(y_0 + \varepsilon z)|_{\varepsilon=0} = 0$$

$$\begin{aligned} J(y_0 + \varepsilon z) = &\int_0^\infty [(\frac{4w}{a^2} + 1)y_0^2 + \frac{w}{a^2}\dot{y}_0^2]dt + 2\varepsilon \int_0^\infty [(\frac{4w}{a^2} + 1)(y_0 z) + \frac{w}{a^2}\dot{y}_0\dot{z}]dt \\ &+ \varepsilon^2 \int_0^\infty [(\frac{4w}{a^2} + 1)z^2 + \frac{w}{a^2}\dot{z}^2]dt - \frac{2w}{a^2}y^2(\infty) + \frac{2w}{a^2}c^2 \end{aligned}$$

We see then that the variational condition derived is

$$\begin{aligned} \int_0^\infty [(\frac{4w}{a^2} + 1)(y_0 z) + \frac{w}{a^2}\dot{y}_0\dot{z}]dt = 0 \\ \dot{y}_0 z|_0^\infty + \int_0^\infty z[(\frac{4w}{a^2} + 1)y_0 - \frac{w}{a^2}\ddot{y}_0]dt = 0 \\ \dot{y}_0(\infty)z(\infty) - \dot{y}_0(0)z(0) + \int_0^\infty z[(\frac{4w}{a^2} + 1)y_0 - \frac{w}{a^2}\ddot{y}_0]dt = 0 \end{aligned}$$

Since $y_0 + \varepsilon z$ is an admissible function satisfies the initial condition

$$y_0(0) + \varepsilon z(0) = c$$

We see that $z(0) = 0$.

Since the left-hand side must be zero for all admissible z, we suspect that

$$(\frac{4w}{a^2} + 1)y_0 - \frac{w}{a^2}\ddot{y}_0 = 0$$

First, we use T to replace ∞, $\dot{y}_0(T) = 0$. And we obtain no condition on $\dot{y}_0(0)$. We can get:

$$y_0(0) = c, \ \dot{y}_o(T) = 0$$

The general solution of the differential equation is:

$$y = c_1 e^{\sqrt{4+a^2/w}\cdot t} + c_2 e^{-\sqrt{4+a^2/w}\cdot t}$$

Using the boundary conditions, we have the two equations to determine the coefficients c_1 and c_2.

$$c = c_1 + c_2$$
$$0 = c_1 e^{\sqrt{4+a^2/w}\cdot T} - c_2 e^{-\sqrt{4+a^2/w}\cdot T}$$

Solving, we obtain the expression:

$$y_o(t) = c(\frac{e^{\sqrt{4+a^2/w}(t-T)} + e^{-\sqrt{4+a^2/w}(t-T)}}{e^{-\sqrt{4+a^2/w}\cdot T} + e^{\sqrt{4+a^2/w}\cdot T}}) = c\frac{cosh(\sqrt{4+a^2/w}(t-T))}{cosh(\sqrt{4+a^2/w}\cdot T)}$$

Let $T \to \infty$, We have

$$y_o(t) = c(\frac{e^{\sqrt{4+a^2/w}(t-T)} + e^{-\sqrt{4+a^2/w}(t-T)}}{e^{-\sqrt{4+a^2/w}\cdot T} + e^{\sqrt{4+a^2/w}\cdot T}}) = c(\frac{e^{\sqrt{4+a^2/w}(t-2T)} + e^{-\sqrt{4+a^2/w}\cdot t}}{e^{-2\sqrt{4+a^2/w}\cdot T} + 1}) \to ce^{-\sqrt{4+a^2/w}\cdot t}$$
$$\dot{y}_o(t) = c\sqrt{4+a^2/w}\cdot(\frac{e^{\sqrt{4+a^2/w}(t-T)} - e^{-\sqrt{4+a^2/w}(t-T)}}{e^{-\sqrt{4+a^2/w}\cdot T} + e^{\sqrt{4+a^2/w}\cdot T}}) = c\sqrt{4+a^2/w}\cdot(\frac{e^{\sqrt{4+a^2/w}(t-2T)} - e^{-\sqrt{4+a^2/w}\cdot t}}{e^{-2\sqrt{4+a^2/w}\cdot T} + 1}) \to -c\sqrt{4+\frac{a^2}{w}}\cdot e^{-\sqrt{4+a^2/w}\cdot t}$$
$$\dot{y}_o = -\sqrt{4+\frac{a^2}{w}}\cdot y_o(t)$$

So we have the control laws

$$u(t) = -\frac{2}{a}y(t) + \frac{1}{a}\dot{y}(t) = -\frac{1}{a}(2+\sqrt{4+\frac{a^2}{w}})y(t)$$

b) Dynamic programming

Optimal Value function:

$$V(c,T) = \min_y J(y)$$
$$J(y) = \int_0^\Delta + \int_\Delta^T = (c^2 + wu^2)\Delta + V(c + (ac+au)\Delta, T-\Delta) + O(\Delta^2)$$

We can use Taylor series to relate $V(c+(2c+au)\Delta, T-\Delta)$ with $V(c,T)$, $J(y)$ will change to

$$V(c,T) = \min_u[(c^2+wu^2)\Delta + V(c,T) + \frac{\partial V}{\partial c}(2c+au)\Delta - \frac{\partial V}{\partial T}\Delta + O(\Delta^2)]$$

Ignoring the higher order terms of Δ, we have

$$\frac{\partial V}{\partial T} = \min_u[(c^2+wu^2) + \frac{\partial V}{\partial c}(2c+au)]$$

When $T \to \infty$, $V(c,T)$ becomes $V(c)$,

$$V(c) = \min_u[(c^2+wu^2)\Delta + V(c+(2c+au)\Delta)] + O(\Delta^2)$$
$$0 = \min_u[(c^2+wu^2) + \dot{V}(c)(2c+au)]$$

Take the derivative respect to u gives $2wu + \dot{V}(c)a = 0$, so

$$u = -\frac{a}{2w}\dot{V}(c)$$
$$0 = (c^2 + w(-\frac{a}{2w}\dot{V}(c))^2) + \dot{V}(c)(2c + a(-\frac{a}{2w}\dot{V}(c)))$$
$$\dot{V}^2(c) - \frac{8wc}{a^2}\dot{V}(c) - \frac{4wc^2}{a^2} = 0$$
$$\dot{V}(c) = \frac{4wc}{a^2} \pm \frac{2c}{a^2}\sqrt{4w^2+wa^2}$$

So we have two possibilities, with the condition $V(0) = 0$, we can obtain two possible solutions:

$$V(c) = (\frac{2w}{a^2} + \frac{1}{a^2}\sqrt{4w^2+wa^2})c^2$$
$$V(c) = (\frac{2w}{a^2} - \frac{1}{a^2}\sqrt{4w^2+wa^2})c^2$$

Since $V(c) \geqslant 0$, we see that $V(c) = (\frac{2w}{a^2} + \frac{1}{a^2}\sqrt{4w^2+wa^2})c^2$, the optimal value can be easily obtained as

$$u = -\frac{a}{2w}\dot{V}(c) = -\frac{1}{a}(2+\sqrt{4+\frac{a^2}{w}})c$$

Since $y(0) = c$, so we have $u(0) = -\frac{1}{a}(2+\sqrt{4+\frac{a^2}{w}})y(0)$. At any time t, we will have the control law:

$$u(t) = -\frac{1}{a}(2+\sqrt{4+\frac{a^2}{w}})y(t)$$

As we can see, the results from two method are same

c) Weight factor

If $w \to \infty$, the $u(t)$ and $y(t)$ will change to

$$u(t) = -\frac{4}{a}y(t)$$
$$y(t) = ce^{-2t}$$

If $w \to 0$, the $u(t)$ and $y(t)$ will change to

$$u(t) \to -\infty$$
$$y(t) = 0$$

As we can see, if weight is very big, the input signal will very small. If weight is small, the control signal will very big, and the output will change to 0 rapidly

Q.2

We write the optimal value function as

$$V_N(c) = \min_{u_n} J_N(y,u)$$

After $u(0)$ is chosen, the new state of the system is $y(1) = 2c + au(0)$, The cost function takes the form

$$c^2 + wu^2(0) + \sum_{n=1}^{N}(y^2(n) + wu^2(n))$$

The long term cost can be expressed as optimal value starting from $2c + au(0)$ withe $N-1$ steps left

$$\sum_{n=1}^{N}(y^2(n) + wu^2(n)) = V_{N-1}(2c + au(0))$$

Then

$$V_N(c) = \min_{u(0)}[c^2 + wu^2(0) + V_{N-1}(2c + au(0))]$$

For the continuous case we have $V(c,T) = c^2 r(T)$

It is reasonable to guess that

$$V_N(c) = c^2 r_N$$
$$c^2 r_N = \min_{u(0)}[c^2 + wu^2(0) + (2c + au(0))^2 r_{N-1}]$$

The value of $u(0)$ that minimizes is readily obtained by differentiation

$$2wu(0) + 2a(2c + au(0))r_{N-1} = 0$$
$$u(0) = -\frac{2acr_{N-1}}{w + a^2 r_{N-1}}$$

Using this value, we obtain the recurrence relation

$$r_N = 1 + \frac{4wr_{N-1}}{w + a^2 r_{N-1}}$$

At each time $t = k$, the input control is

$$u(k) = -\frac{2ar_{N-k-1}y(k)}{w + a^2 r_{N-k-1}}$$

When $N \to \infty$, let $r = \lim_{N\to\infty} r_N$, then r is the positive root of the quadratic equation

$$r = 1 + \frac{4wr}{w + a^2 r}$$
$$r = \frac{(a^2 + 3w) + \sqrt{(a^2 + w)(a^2 + 9w)}}{2a^2}$$

The control signal will change to:

$$\lim_{N\to\infty} u(0) = -\frac{2acr}{w + a^2 r}$$

We see that

$$V(c) = \min_{u(n)} \sum_{n=0}^{\infty}(y^2(n) + wu^2(n))$$
$$V(c) = \min_{u(0)}[c^2 + wu^2(0) + V(2c + au(0))]$$
$$V(c) = rc^2$$

Therefore, for the infinite time process, the optimal feedback controller is:

$$u(k) = -\frac{2ary(k)}{w + a^2 r}$$

Q.3

Assume that the lifeguard will run to $(a, 0)$, and then swim to the swimmer. The parameter we can get from question: $v_1 = 8m/s,\ v_2 = 2m/s$. The optimal function can be expressed as:

$$J(a) = \min_a[(\frac{\sqrt{a^2+10^2}}{v_1})^2 + (\frac{\sqrt{(20-a)^2+(-20)^2}}{v_2})^2] = \min_a[\frac{a^2+10^2}{v_1^2} + \frac{(20-a)^2+20^2}{v_2^2}]$$

Take the derivative respect to a, we get

$$\frac{2a}{v_1^2} - \frac{2(20-a)}{v_2^2} = 0$$

$$a = \frac{20v_1^2}{v_1^2+v_2^2} = 18.823$$

So, the shortest time path is that lifeguard run to $(18.823, 0)$ and then swim to the swimmer. The shortest time is:

$$t_{min} = \frac{\sqrt{a^2+10^2}}{v_1} + \frac{\sqrt{(20-a)^2+(-20)^2}}{v_2} = 12.68s$$

Q.4

First, we can put all attractions and hotel in the x-y plane and sort them by x coordinate from small to large $p_1, p_2, \ldots, p_n$. Assume that $V_{i,j}$ $(i \leqslant j)$ is the shortest closed curve which contain $p_1, p_2, \ldots, p_j$. This path goes from p_i left to p_1, and then goes from p_1 right to p_j. So, $V_{n,n}$ is what we want in this topic.

Assume that the length of $V_{i,j}$ is $l(i,j)$, the distance between p_i and p_j is $dist(i,j) = \sqrt{(x_i - xj)^2 + (y_i - y_j)^2}$

In the path $V_{i,j}$, p_i is in the path $p_i \to p_1$, p_j is in the path $p_1 \to p_j$. Now, let's talk about the position of p_{j-1}

(1) $i < j - 1$

Because p_{j-1} is on the right side of p_i, so p_{j-1} is in the path $p_1 \to p_j$. Besides, p_{j-1} is the rightmost point except p_j, so it connect to p_j directly. We can get

$$l(i,j) = l(i,j-1) + dist(j-1,j)$$

(2) $i = j - 1$

In this case, p_{j-1} is p_i, so p_{j-1} is in the path $p_i \to p_1$. Any point from $p_1, p_2, \ldots, p_{j-2}$ can connect to $p + j$. Assume that point is $p_k (1 \leqslant k \leqslant j-2)$. We need to chose an appropriate point p_k so that we can get the shortest $l(i,j)$

$$l(i,j) = \min_{1 \leqslant k \leqslant j-2} [l(k,j-1) + dist(k,j)]$$

(3)$i = j$

This only happens when $i = j = n$. In this case, p_{n-1} connect to p_n, we can get:

$$l(n,n) = l(n-1,n) + dist(n-1,n)$$

In conclusion the optimal function is:

$$l(i,j) = \begin{cases} l(i,j-1) + dist(j-1,j), & i < j-1 \\ \min\limits_{1 \leqslant k \leqslant j-2} [l(k,j-1) + dist(k,j)], & i = j-1 \\ l(n-1,n) + dist(n-1,n), & i = j = n \end{cases}$$

This function is what we want.