Q.1 (a) Consider a dynamic system described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \alpha f(x_2(t), z(t)) + \beta u \end{aligned}$$

where $\beta > 0$ is unknown constant, the state variables, $x_1(t)$ and $x_2(t)$, are measurable, $f(x_2(t), z(t))$ is a known function of $x_2(t)$ and an external measurable signal, $z(t)$, with an unknown constant multiplier, α, and u is the input signal.

Based on the Lyapunov synthesis method, design an adaptive controller which ensures that

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

tracks the state of a reference model with $r(t)$ as its reference input. The reference model is to have a characteristic polynomial, $(s^2 + 10s + 25)$, where s is the Laplace Transform variable.

Show clearly the structure of the control law that you use, the state variable description of the reference model, and the error system that is the basis of your adaptive laws. Discuss the asymptotic behaviour of the overall system and state any necessary assumptions on $f(x_2(t), z(t))$ to ensure asymptotic tracking.

(15 Marks)

(b) Consider a dynamic system described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + f_1(x_1(t)) \\ \dot{x}_2(t) &= \alpha f(x_2(t), z(t)) + \beta u \end{aligned}$$

where all the variables, parameters and functions are the same as in Question Q.1(a), except for the introduction of the known function, $f_1(x_1(t))$.

Based on the Lyapunov synthesis method, design an adaptive controller which ensures that the output, $y(t) = x_1(t)$, can be regulated to zero.

(10 Marks)

Q.2 Consider a class of second order system described by

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^2 + a_1 s + a_2}$$

where a_1, a_2 and $b_0 \neq 0$ are unknown constant parameters of the plant.

Under the assumption that only the input, $u(t)$, and output, $y(t)$, are available for feedback, consider the commonly used controller structure shown in Figure Q.2.

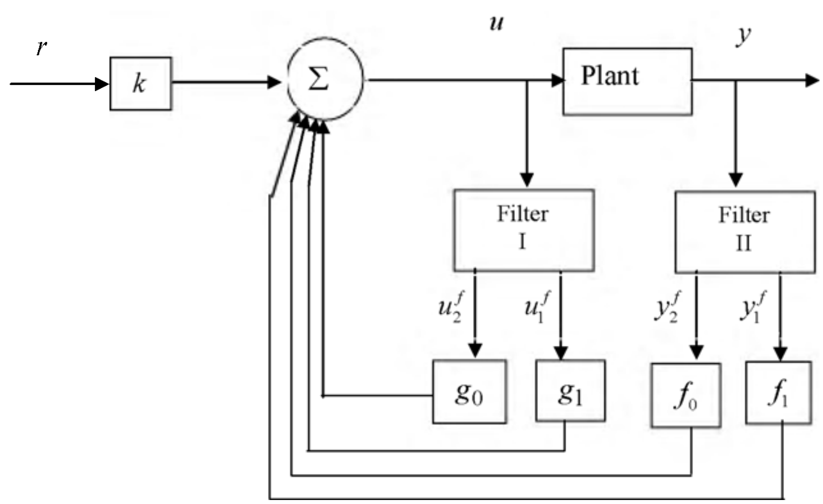

Figure Q.2

In Figure Q.2, g_0, g_1, f_0 and f_1 are control gains to be adaptively updated. The two state filters are identical and are described in the general form as

$$\dot{z} = Az + bv, \qquad A = \begin{bmatrix} 0 & 1 \\ -t_2 & -t_1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with z being the state vector and v the input, u_1^f and u_2^f are the states of Filter I, and y_1^f and y_2^f are the states of Filter II.

(a) Provide an analysis to show that there exists a set of controller gains, $\{f_0, f_1, g_0, g_1, k\}$, that will achieve a suitable desired closed-loop response. You need not be concerned about the issue of boundedness in the adaptation procedure.

(15 marks)

(b) Discuss the class of reference models that can be matched using the given structure. If the adaptive controller is to be implemented digitally, discuss the factors that affect the choice of the sampling interval.

(10 marks)

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Solution To Question No. : Q.1(a) **Marks 15**

Assume the transfer function of the reference model be

$$\frac{Y_m}{R} = G_m(s) = \frac{k_m}{s^2 + a_{2m}s + a_{1m}}$$

with $a_{1m} = 25, a_{2m} = 10$, and known constant $k_m > 0$.

Let states be $x_{1m} = y_m, x_{1m} = \dot{y}_m$. Then the state space description of the reference model is given by

$$\dot{x}_{1m} = x_{2m}(t)$$
$$\dot{x}_{2m} = -a_{1m}x_{1m} - a_{2m}x_{2m} + k_m r$$

i.e.,

$$\dot{x}_m = A_m x_m + k_m b r, \quad \text{where} \quad A_m = \begin{bmatrix} 0 & 1 \\ -a_{1m} & -a_{2m} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $\hat{\alpha}, \hat{\beta}$ be the estimate of the unknown constants, α, β, and consider the control of the form

$$u = \frac{1}{\hat{\beta}}[-\hat{\alpha} f(x_2, z) - a_{1m}x_1 - a_{2m}x_2 + k_m r]$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = \alpha f(x_2, z) + \beta u$

Then, we have

$$\hat{\beta}u = -\hat{\alpha} f(x_2, z) - a_{1m}x_1 - a_{2m}x_2 + k_m r$$

Substituting into the dynamics of the system leads to

$\dot{x}_1 = x_2$;

$$\dot{x}_2(t) = \alpha f(x_2(t), z(t)) + \hat{\beta}u + \beta u - \hat{\beta}u$$
$$= -a_{1m}x_1 - a_{2m}x_2 + \tilde{\alpha} f(x_2, z) + \tilde{\beta}u + k_m r$$
$$= -a_{1m}x_1 - a_{2m}x_2 + \tilde{\theta}^T \omega + k_m r$$

where

$$\tilde{\theta} = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \omega = \begin{bmatrix} f(x_2, z) \\ u \end{bmatrix}, \ \tilde{\alpha} = \alpha - \hat{\alpha}, \tilde{\beta} = \beta - \hat{\beta}.$$

Thus, we have the standard closed loop dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A_m x + b\tilde{\theta}^T \omega + bk_m r$$

Compared with $\dot{x}_m = A_m x_m + k_m br$, we have the closed-loop error equation

$$\dot{e} = A_m e + b\tilde{\theta}^T \omega$$

$A_m^T P + P A_m = -Q$; must have then $\dot{\tilde{\theta}} = -\Gamma e^T P b w$

As the reference model matrix, A_m, is a stable matrix, it satisfies the Lyapunov equation $A_m^T P + PA_m = -Q$, i.e., for any symmetric positive definite matrix Q, there exists a symmetric positive definite P satisfying the above equation.

Consider a Lyapunov function candidate $V = e^T Pe + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$, where Γ is a symmetric positive definite (s.p.d.) matrix.

Evaluate $\dot{V}$ along the trajectory of the system

$$\dot{V} = 2e^T P\dot{e} + 2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \quad = 2e^T P\left\{A_m e + b\tilde{\theta}^T \omega\right\} + 2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$

$$= e^T \left(A_m^T P^T + PA_m\right)e + 2e^T Pb\tilde{\theta}^T \omega + 2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$

$$= -e^T Qe + 2e^T Pb\tilde{\theta}^T \omega + 2\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$

Letting $\dot{\tilde{\theta}} = \begin{bmatrix} \dot{\tilde{\alpha}} \\ \dot{\tilde{\beta}} \end{bmatrix} = -\Gamma \omega e^T Pb$, $i.e., \begin{bmatrix} \dot{\hat{\alpha}} \\ \dot{\hat{\beta}} \end{bmatrix} = \Gamma \omega e^T Pb$, we have

$$\dot{V} = -e^T Qe \le 0$$

Note that

(i) $V(t)$ is positive definite, (ii) $V(t)$ is decrescent, and (iii) $V(t)$ is radically unbounded.

Accordingly, we have the following conclusion

- $V(t)$ is positive definite and $\dot{V}(t) \le 0 \Rightarrow V(t)$ is bounded, implies that $\|e\|, \|\tilde{\theta}\| \quad \left(hence \quad \|\hat{\alpha}\|, \|\hat{\beta}\|\right)$ are uniformly bounded
- Furthermore, $\int_0^\infty e^T Qe d\tau \le V(0) - V(\infty) \le V(0)$, i.e., $\lambda_{\min}(Q) \int_0^\infty e^T e d\tau \le V(0)$, which implies that $\|e\|^2$ is square integrable.
- To conclude asymptotic convergence of e, we need $\dot{e}$ be bounded. Examine the error equation,

  $$\dot{e} = A_m e + b\tilde{\theta}^T \omega$$

  where $\tilde{\theta} = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \omega = \begin{bmatrix} f(x_2, z) \\ u \end{bmatrix}$

In addition to the proved bounded of e, and $\tilde{\theta}$, we also need the boundedness of ω, accordingly $f(x_2, z)$, and

$$u = \frac{1}{\hat{\beta}}[-\hat{\alpha} f(x_2, z) - a_{1m} x_1 - a_{2m} x_2 + k_m r], \hat{\beta} \ne 0 .$$

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		10 marks

The system is in the so called strict feedback form

$$\dot{x}_1(t) = x_2(t) + f_1(x_1(t))$$
$$\dot{x}_2(t) = \alpha f(x_2(t), z(t)) + \beta u$$

where all the variables, parameters and functions are the same as in Question Q.1(a), except for the introduction of the known function, $f_1(x_1(t))$. The standard backstepping technique can be used for Lyapunov control design.

Step 1. Let $z_1 = x_1, z_2 = x_2 - \alpha_1$, where the virtual control, α_1, acts as the control for the first equation in place of x_2 under the assumption of $z_2 = 0$. Then, $\dot{z}_1 = \dot{x}_1 = z_2 + \alpha_1 + f_1(x_1(t))$.
Consider the following virtual control, $\alpha_1 = f_1(x_1(t)) - c_1 x_1$, we have the closed loop error equation as

$$\dot{z}_1 = z_2 - c_1 x_1$$

For the z_1 dynamics, consider the Lyapunov candidate, $V_1 = \frac{1}{2} z_1^2$. Its derivative is

$$\dot{V}_1 = z_1 \dot{z}_1 = -c_1 z_1^2 + z_1 z_2.$$

which leads to the conclusion of asymptotic stabilization of z_1 if $z_2 = 0$.
According to the standard backstepping procedure, as x_2 is not the physical control, backstepping design has to proceed and the coupling term $z_1 z_2$ will be cancelled in the next step for global asymptotic stability of the whole system.

$z_1 \triangleq x_1$

$z_2 \triangleq x_2 - \alpha_1$

$\alpha_1 \triangleq -f_1(x_1) - c_1 x_1$

Step 2. The derivative of z_2 is given by

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = \alpha f(x_2(t), z(t)) + \beta u + \frac{\partial f_1(x_1)}{\partial x_1}\dot{x}_1 + c_1 \dot{x}_1$$
$$= \alpha f(x_2(t), z(t)) + \left[\frac{\partial f_1(x_1)}{\partial x_1} + c_1\right]\underbrace{[x_2 + f_1(x_1)]}_{\dot{x}_1} + \beta u$$
$$= \alpha f(x_2(t), z(t)) - f_{known}(x) + \beta u$$

where $f_{known}(x) = -\left\{\left[\frac{\partial f_1(x_1)}{\partial x_1} + c_1\right][x_2 + f_1(x_1)]\right\}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\dot{z}_1 = \dot{x}_1$
$= x_2 + f_1(x_1)$
$= (z_2 + \alpha_1) + f_1(x_1)$
$= z_2 - c_1 z_1$

To stabilize z_2, consider the following equivalent control,

$$u = \frac{1}{\hat{\beta}}[-\hat{\alpha} f(x_2, z) + f_{known}(x) - c_2 z_2]$$

Then, we have

$$\hat{\beta} u = -\hat{\alpha} f(x_2, z) + f_{known}(x) - c_2 z_2 - z_1$$

Substituting into the dynamics of the system leads to

$\dot{z}_1 = -c_1 z_1 + z_2$

$$\dot{z}_2(t) = \alpha f(x_2(t), z(t)) - f_{known}(x) + \hat{\beta}u + \beta u - \hat{\beta}u$$
$$= -z_1 - c_2 z_2 + \tilde{\alpha} f(x_2, z) + \tilde{\beta}u = -z_1 - c_2 z_2 + \tilde{\theta}^T \omega$$

where

$$\tilde{\theta} = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \omega = \begin{bmatrix} f(x_2, z) \\ u \end{bmatrix}, \ \tilde{\alpha} = \alpha - \hat{\alpha}, \tilde{\beta} = \beta - \hat{\beta}.$$

which apparently includes the physical control u.

To design the physical control u to stabilize the whole system, the (z_1 , z_2) system, consider the augmented Lyapunov function candidate

$$V_2 = V_1 + V_2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

$\dot{\hat{\theta}} = -\Gamma \omega z_2$

Its derivative is given by

$$\dot{V}_2 = -c_1 z_1^2 + z_1 z_2 + z_2 \dot{z}_2 + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$
$$= -c_1 z_1^2 + z_1 z_2 + z_2[-z_1 - c_2 z_2 + \tilde{\theta}^T \omega] + \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$
$$= -c_1 z_1^2 - c_2 z_2^2 + \cancel{z_2 \tilde{\theta}^T \omega} + \cancel{\tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}}$$

Letting $\dot{\tilde{\theta}} = \begin{bmatrix} \dot{\tilde{\alpha}} \\ \dot{\tilde{\beta}} \end{bmatrix} = -\Gamma \omega z_2, \quad i.e., \begin{bmatrix} \dot{\hat{\alpha}} \\ \dot{\hat{\beta}} \end{bmatrix} = \Gamma \omega z_2$, we have

$$\dot{V} = -c_1 z_1^2 - c_2 z_2^2 \le 0, \qquad c_1 \text{ and } c_2 > 0$$

Note that

(i) $V(t)$ is positive definite, (ii) $V(t)$ is decrescent, and (iii) $V(t)$ is radically unbounded.

Similarly, we have the following conclusion

- $V(t)$ is positive definite and $\dot{V}(t) \le 0 \Rightarrow V(t)$ is bounded, implies that $\|z\|, \|\tilde{\theta}\| \ \left(hence \quad \|\hat{\alpha}\|, \|\hat{\beta}\| \right)$ are uniformly bounded

- Furthermore, $\int_0^\infty z^T Q z d\tau \le V(0) - V(\infty) \le V(0)$, i.e., $\lambda_{\min}(Q) \int_0^\infty z^T z d\tau \le V(0)$, which implies that $\|z\|^2$ is square integrable.

- To conclude asymptotic convergence of e, we need $\dot{e}$ be bounded. Examine the error equation,

$$\dot{z}_1 = z_2 - c_1 z_1$$
$$\dot{z}_2 = -z_1 - c_2 z_2 + \tilde{\theta}^T \omega$$

where $\tilde{\theta} = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}, \omega = \begin{bmatrix} f(x_2, z) \\ u \end{bmatrix}$.

In addition to the proved bounded of z, and $\tilde{\theta}$, we also need the boundedness of ω, accordingly $f(x_2, z)$, and

$$u = \frac{1}{\hat{\beta}} [-\hat{\alpha} f(x_2, z) + f_{known}(x) - c_2 z_2], \hat{\beta} \ne 0.$$

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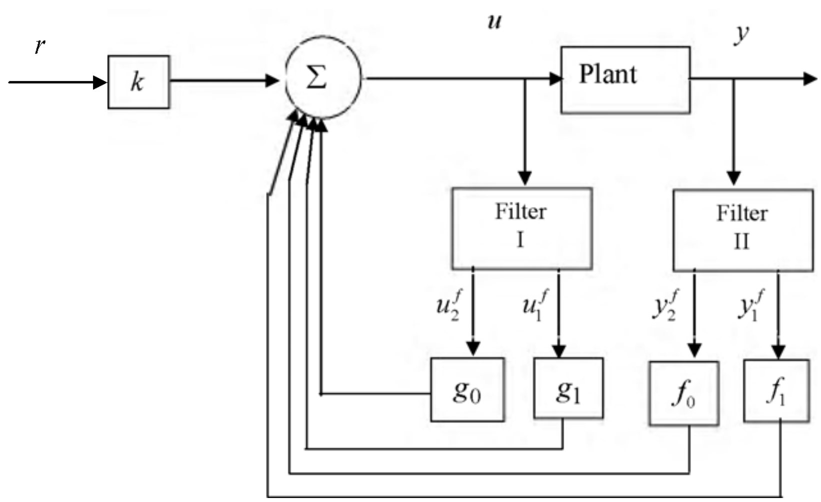
Solution To Question No. :	Q2(a)	Marks 15



For the structure given, we first note that for

$$\begin{bmatrix} \dot{y}_1^f \\ \dot{y}_2^f \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -t_2 & -t_1 \end{bmatrix} \begin{bmatrix} y_1^f \\ y_2^f \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y$$

we have

$$y_1^f(p) = \frac{1}{T(p)} y(p)$$

$$y_2^f(p) = \frac{p}{T(p)} y(p)$$

where $T(p) = p^2 + t_1 p + t_2$ which is Hurwitz in s.

Similarly, for

$$\begin{bmatrix} \dot{u}_1^f \\ \dot{u}_2^f \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -t_2 & -t_1 \end{bmatrix} \begin{bmatrix} u_1^f \\ u_2^f \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

we have

$$u_1^f(p) = \frac{1}{T(p)} u(p)$$

$$u_2^f(p) = \frac{p}{T(p)} u(p)$$

Accordingly, the fixed gains given, we have

$$u(s) = kr(p) + f_0 y_2^f(p) + f_1 y_1^f(p) + g_0 u_2^f(p) + g_1 u_1^f(p) \qquad (1)$$

We need to show that the control law (1), with appropriate gains, can match a suitable reference model.

The plant is

$$R(p)y(p) = k_p Z(p)u(p)$$

where

$$R(p) = p^2 + a_1 p + a_2$$
$$Z(p) = 1$$
$$k_p = b_0$$

Consider the "Diophantine" identity

$$T(p)R_m(p) = R(p)E(p) + F(p) \tag{2}$$

with $\deg(T) = \deg(R) = n = 2, \deg(R_m) = \deg(E) = n^* = 2$, and $\deg(F) = n - 1 = 1$.

From the plant, we have $R(p)E(p)y(p) = k_p E(p)Z(p)u(p)$

i.e.,

$$T(p)R_m(p)y = F(p)y + k_p E(p)Z(p)u = F(p)y + k_p \overline{G}(p)u,$$

where $\overline{G}(p) = E(p)Z(p)$.

Note that (a) Z_p monic, and E monic, $\therefore$ $\overline{G}$ monic;

(b) $\deg(\overline{G}) = \deg(Z_p) + \deg(E) = n$

Thus, we have

$$\begin{aligned}
R_m(p)y &= \frac{k_p \bar{G}(p)}{T(p)}u + \frac{F(p)}{T(p)}y \\
&= k_p \frac{T(p) - [T(p) - \bar{G}(p)]}{T(p)}u + \frac{F(p)}{T(p)}y \\
&= k_p u - \frac{\tilde{g}_0 p + \tilde{g}_1}{T(p)}u + \frac{f_0 p + f_1}{T(p)}y \\
&= k_p u - \tilde{g}_0 u_1^f(p) - \tilde{g}_1 u_2^f(p) + f_0 y_1^f(p) + f_1 y_2^f(p)
\end{aligned}$$

If we set the R.H.S. to be equal to $k_m r$, *i.e.,*

$$k_p u - \tilde{g}_0 u_1^f(p) - \tilde{g}_1 u_2^f(p) + f_0 y_1^f(p) + f_1 y_2^f(p) = k_m r \tag{3}$$

Thus, we have

$$R_m(s)Y(s) = k_m R(s)$$

$$\text{or } \frac{Y(s)}{R(s)} = \frac{k_m}{R_m(s)} = \frac{k_m}{s^2 + a_{2m}s + a_{1m}} \text{ in this case} \tag{4}$$

But (2) is in fact equivalent to

$$u = \frac{1}{k_p}\left[k_m r - f_0 w_2 - f_1 w_1 + \tilde{g}_0 w_4 + \tilde{g}_1 w_3\right]$$

which is of the same structure as (1), i.e., as in the given diagram.

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It can be seen that the class of reference model that can matched are of the form with transfer function

$$\frac{Y(s)}{R(s)} = \frac{k_m}{R_m(s)} = \frac{k_m}{s^2 + a_{2m}s + a_{1m}}$$

If the adaptive controller is to be implemented digitally, then the time constants to be considered are

(i) the factors of the $T(s) = s^2 + t_1 s + t_2$, the states of the filter, and

(ii) the closed-loop dynamics desired, i.e., factor of the $R_m(s)$.

The states filters act as observers. As such, it should be chosen with dynamics tow to five times faster than the feedback dynamics (of $R_m(s)$). Thus assuming this is the case, sampling should be chosen to be approximately

$$h = \frac{1}{10}\frac{1}{\lambda_{\max}}$$

where $\lambda_{\max}$ is the factor of $T(s)$ with largest magnitude.