EE5137 Stochastic Processes: Problem Set 7
Assigned: 04/03/22, Due: 11/03/22

There are four (4) non-optional problems in this problem set to give you time to prepare for quiz 2.

1. Exercise 4.1 (Gallager's book) Let $[P]$ be the transmission matrix for a finite state Markov chain and let state i be recurrent. Prove that i is aperiodic if $P_{ii} > 0$.

   Solution: Since $P_{11} > 0$, there is a walk of length 2 that starts in state 1, goes to state 1 at epoch 1 and goes from there to state 1 at epoch 2. In the same way, for each $n > 2$, a walk of length n exists, going through state 1 at each step. Thus $P_{11}^n > 0$, so the greatest common denominator of $\{n : P_{11}^n > 0\}$ is 1.

2. Exercise 4.2 (Gallager's book) Show that every Markov chain with $M < \infty$ states contains at least one recurrent set of states. Explaining each of the following statements is sufficient.

   (a) If state i_1 is transient, then there is some other states i_2 such that $i_1 \to i_2$ and $i_2 \nrightarrow i_1$.
   Solution: If there is no such state i_2, then i_1 is recurrent by definition. That state is distinct from i_1 since otherwise $i_1 \to i_2$ would imply $i_2 \to i_1$.

   (b) If the i_2 of (a) is also transient, there is a third state i_3 such that $i_2 \to i_3$, $i_3 \nrightarrow i_2$; that state must satisfy $i_3 \neq i_2, i_3 \neq i_1$.
   Solution: The argument why i_3 exists with $i_2 \to i_3, i_3 \nrightarrow i_2$ and with $i_3 \neq i_2$ is the same as (a). Since $i_1 \to i_2$ and $i_2 \to i_3$, we have $i_1 \to i_3$. We must also have $i_3 \nrightarrow i_1$, since otherwise $i_3 \to i_1$ and $i_1 \to i_2$ would imply the contradiction $i_3 \to i_2$. Since $i_1 \to i_3$ and $i_3 \nrightarrow i_1$, it follows as before that $i_3 \neq i_1$.

   (c) Continue iteratively to repeat (b) for successive states, $i_1, i_2, \ldots$. That is, if $i_1, i_2, \ldots, i_k$ are generated as above and are all transient, generate i_{k+1} such that $i_k \to i_{k+1}$ and $i_{k+1} \nrightarrow i_k$. Then $i_{k+1} \neq i_j$ for $1 \leq j \leq k$.
   Solution: The argument why i_{k+1} exists with $i_k \to i_{k+1}$, $i_{k+1} \nrightarrow i_k$ and with $i_{k+1} \neq i_k$ is the same as before. The show that $i_{k+1} \neq i_j$ for each $j < k$, we use contradiction, noting that if $i_{k+1} = i_j$, then $i_{k+1} \to i_{j+1} \to i_k$.

   (d) Show that for some $k \leq M$, k is not transient, i.e., it is recurrent, so a recurrent class exists.
   Solution: For transient states $i_1, i_2, \ldots, i_k$ generated in (c), state i_{k+1} found in (c) must be distinct from distinct states $\{i_j; j \leq k\}$. Since there are only M states, there can not be M transient states, since then, with $k = M$, a new distinct state i_{M+1} would be generated, which is impossible. Thus, there must be some $k < M$ for which the extension to i_{k+1} leads to recurrent state.

3. A spider and a fly move along a straight line in unit increments. At any point in time, two events happen. First, the spider always moves towards the fly by one unit. The fly then moves towards the initial position of the spider by one unit with probability 0.3, move away from the spider by one unit with probability 0.3 and stays in place with probability 0.4. The initial distance between the spider and the fly is integer. When the spider and the fly land in the same position, the spider captures the fly.

(a) Construct a Markov chain that describes the relative location of the spider and fly.

Solution: We introduce a Markov chain with state equal to the distance between spider and fly. Let n be the initial distance. Then, the states are $0, 1, 2, \ldots, n$, and we have

$$p_{00} = 1, p_{0i} = 0, \quad \text{for} \quad i \neq 0, p_{10} = 0.4, p_{11} = 0.6, p_{1i} = 0, \quad \text{for} \quad i \neq 0, 1, \tag{1}$$

and for all $i \neq 0, 1$,

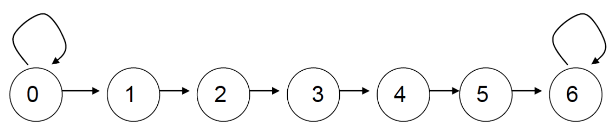
$$p_{i(i-2)} = 0.3, p_{i(i-1)} = 0.4, p_{ii} = 0.3, p_{ij} = 0, \quad \text{for} \quad j \neq i-2, i-1, i. \tag{2}$$

(b) Identify the transient and recurrent states.

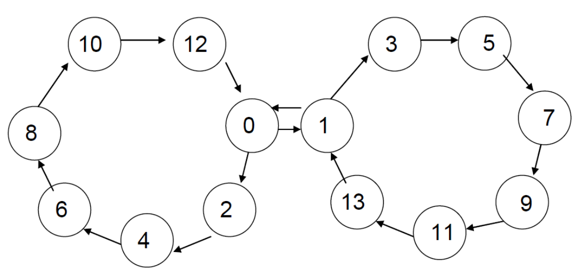
Solution: All states are transient except for state 0 which forms a recurrent class.

4. For the following finite-state Markov chains, each transition is marked with $\longleftarrow$ or $\longrightarrow$, the transition probability is nonzero. For each chain, identify all classes, determine the period of each class, and specify whether each class is recurrent or transient. Explain your answer carefully.

- (5 points) Chain 1:



- (5 points) Chain 2:



Solutions:

- Chain 1: Recall that two states i and j in a Markov chain communicate if each is accessible from the other, i.e., if there is a walk from each to the other. Since all transitions move from left to right, each state is accessible only from those to the left, and therefore no state communicates with any other state. Thus each state is in a class by itself. States 0 to 5 (and thus the classes $\{0\}, \ldots, \{5\}$ are each transient since each is inaccessible from an accessible state (i.e., there is a path away from each from which there is no return). State 6 is recurrent. States 1 and 6 (and thus class $\{1\}$ and $\{6\}$) are each aperiodic since $P^1_{00} > 0$ and $P^1_{66} > 0$. The periods of classes $\{1\}$ to $\{5\}$ are defined to be infinity since we adopt the convention that the gcd of an empty set is ∞. But if you say that the periods of classes $\{1\}$ to $\{5\}$ are undefined, this is also fine.
- Chain 2: Each state on the circle on the left communicates with all other states on the left and similarly for the circle on the right. Since there is a transition from left to right, and also from right to left, the entire set of states communicate, so there is single class containing all states.

State 0 has a cycle of length 2 through state 1 and of length 7 via the left circle. The greatest common divisor of 2 and 7 is 1, so state 1 has period 1. The chain is then aperiodic since all states in a class have the same period.

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5. (Optional) Exercise 4.3 (Gallager's book) Consider a finite-state Markov chain in which some given state, say state 1, is accessible from every other state. Show that the chain has exactly one recurrent class $\mathcal{R}$ of states and state $1 \in \mathcal{R}$.

   Solution: Since $j \to 1$ for each j, there can be no state j for which $1 \to j$ and $j \nrightarrow 1$. Thus state 1 is recurrent. Next, for any given j, if $1 \nrightarrow j$, then j must be transient since $j \to 1$. On the other hand, if $1 \to j$, then 1 and j communicate and j must be in the same recurrent class as 1. Thus each state is either transient or in the same recurrent class as 1.

6. (Optional) Exercise 4.8 (Gallager's book) A transition probability matrix $[P]$ is said to be doubly stochastic if

$$\sum_j P_{ij} = 1 \quad \text{for all} \;\; i; \qquad \sum_i P_{ij} = 1 \quad \text{for all} \;\; j. \tag{3}$$

   That is, each row sum and each column sum equals 1. If a doubly stochastic chain has M states and is ergodic (i.e., has a single class of states and is aperiodic), calculate its steady-state probabilities.

   Solution: It is easy to see that if the row sums are all equal to 1, then $P[\mathbf{e}] = \mathbf{e}$. If the column sums are also equal to 1, then $\mathbf{e}^T[P] = \mathbf{e}^T$. Thus $\mathbf{e}^T$ is a left eigenvector of $[P]$ with eigenvalue 1, and it is unique within a scale factor since the chain is ergodic. Scaling $\mathbf{e}^T$ to be probabilities, $\pi = (1/M, 1/M, \ldots, 1/M)$.

7. (Optional) Consider a Markov chain with states $1, 2, \ldots, 9$ and the following transition probabilities.

$$P_{12} = P_{17} = 1/2, \qquad P_{i,i+1} = 1, \quad i \neq 1, 6, 9 \qquad P_{61} = P_{91} = 1.$$

   Is the recurrent class of the chain periodic?

   Solution: It is periodic with period 2. The two corresponding subsets are $\{2, 4, 6, 7, 9\}$ and $\{1, 3, 5, 8\}$.