Chapter 8 Quadratic Optimal Control

§8.1 Problem formulation and preliminaries

Daily life optimization examples:

- Minimum time for a trip, or shortest distance for a trip
- Lowest cost for a goods or service; or best quality within a cost

Optimal Control Problem: Given a plant to be controlled and the specifications to be met by control system, the specifications are first cast into a specific index or cost function, and the control is sought to minimize the cost function.

Examples include

- Minimum time
- Minimum energy
- Linear Quadratic Regulator (LQR)-minimum mixed error/energy

We focus on LQR.

What is LQR?

The system is described by the standard *linear* state space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \mathbf{x}(0) \neq 0, \qquad (1)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

The objective is to bring the state from *non-zero initial value to zero*. This is the *regulation* problem. The problem is cast into **the following** *quadratic* **cost function,**

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt. \qquad (2)$$

The LQR optimal control is to find the control law that minimizes (2). The optimal control law that minimizes (2) turns out to be in the form of *linear* state feedback:

$$\mathbf{u} = -\mathbf{K}\mathbf{x}, \qquad (3)$$

where **K** is a constant matrix if **A**, **B**, **C**, **Q**, and **R** are constant. We restrict ourselves to this time-invariant case.

Because J is a scalar, it is easy to show that the weighting matrices **Q** and **R** can be made symmetric; that is, $\mathbf{Q}^T = \mathbf{Q}$ and $\mathbf{R}^T = \mathbf{R}$. The matrices **Q** and **R** appear most often in diagonal form. Though there is no inherent restriction to such a form. In addition, it is assumed (will be explained shortly) that ***Q*** should be semi-positive, i.e.,

$$Q = \begin{bmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & \ddots & \\ 0 & & & q_n \end{bmatrix}, \; q_i \geq 0, \; i = 1, 2, \ldots, n,$$

whereas ***R*** be positive, i.e.,

$$R = \begin{bmatrix} r_1 & & & 0 \\ & r_2 & & \\ & & \ddots & \\ 0 & & & r_m \end{bmatrix}, \; r_i > 0, \; i = 1, 2, \ldots, m.$$

Note from

$$x^T Qx = q_1 x_1^2 + q_2 x_2^2 + \cdots + q_n x_n^2 ,$$

- q_i are relative weightings among x_i.
- *if* q_1 is bigger than q_2, there is higher penalty/price on error x_1 than x_2, and control will try to make smaller x_1^2 than x_2^2, vice versa.

The same can be said on $u^T Ru = r_1 u_1^2 + r_2 u_2^2 + \cdots + r_m u_m^2$.

For later use, let $Q = H^T H$. It is easy to see

$$Q = \begin{bmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & \ddots & \\ 0 & & & q_n \end{bmatrix}, \qquad H = \begin{bmatrix} \sqrt{q_1} & & & 0 \\ & \sqrt{q_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{q_n} \end{bmatrix}.$$

************ Revisions on Positive Definite Matrices *********

Positive Definite Matrices:

A symmetric matrix A is positive definite if for all vector x, $x^TAx \geq 0$, and $x^TAx = 0 \Rightarrow x = 0$ or for all $x \neq 0$, $x^TAx > 0$.

Note: $x = 0 \Rightarrow x_1 = x_2 = \ldots x_n = 0$.

Test 1: A is positive definite iff

$$a_{11} > 0,\ \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0,\ \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} > 0,\ \cdots$$

Test 2: The eigenvalues of A are all (strictly) positive.

Semi-Positive Definite Matrices

A symmetric matrix A is semi-positive definite iff for all vector x, $x^T A x \geq 0$.

Test 1: A is semi-positive definite iff

$$a_{11} \geq 0, \quad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \geq 0, \quad \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \geq 0, \quad \cdots$$

Test 2: The eigenvalues of A are all non-negative (positive or zero).

*********** End of Revisions *********

If the output **y**, but not the state **x**, is to be controlled to approach **0**, then

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{y}^T \mathbf{Q}_o \mathbf{y} + \mathbf{u}^T \mathbf{R}\mathbf{u}\right) dt,$$

may be used, where $\mathbf{Q}_o$ is the output weighting matrix. Substituting $\mathbf{y} = \mathbf{Cx}$, we see that $\mathbf{y}^T\mathbf{Q}_o\mathbf{y} = \mathbf{x}^T\mathbf{C}^T\mathbf{Q}_o\mathbf{Cx}$, yielding the equivalent $\mathbf{Q} = \mathbf{C}^T\mathbf{Q}_o\mathbf{C}$ in **x**. Thus, given $\mathbf{Q}_o$ and **C**, we can use the results based on the form (2).

Why LQR?

A scalar illustration

For the plant:

$$\dot{x} = x + u, \quad x(0) \neq 0, \tag{4}$$

we want to regulate the state to $x = 0$. Assume that we do not wish to apply any more control effort than is necessary. For example, we might wish to avoid saturation of the control elements or to use as little power as possible. Thus we should keep u as well as x near zero. The following extension of the integral squares error (ISE) index expresses this mathematically:

$$J = \frac{1}{2}\int_0^\infty \left(qx^2 + ru^2\right)dt, \tag{5}$$

- the factor of 1/2 is introduced for numerical convenience.
- The weighting factors q and r express the relative importance of keeping x and u near zero. If we place more importance on x, then we select q to be large relative to r, and so forth.
- Although we are interested in minimizing J, the actual value of J that results is usually not of interest. This also means that we can set either q or r to unity for convenience because it is their relative weight that is important.
- This step also reduces the number of weighting factors to be selected.

The optimal control is to find the control law that minimizes this J. *Hopefully, the control law is of feedback one:* $u = f(x)$. In fact, the feedback-control law that minimizes J is a linear law. In the current example, this means that u is related to x by

$$u = -Kx. \tag{6}$$

Later we will show how a feedback-gain matrix K can be computed for a general problem, but for now let us substitute (6) into (5) with r = 1. This results in

$$J = \frac{1}{2}\int_0^\infty \left(qx^2 + u^2\right)dt = \frac{1}{2}\left(q + K^2\right)\int_0^\infty x^2 dt. \tag{7}$$

It follows from (4) and (6) that

$$\dot{x} = x - Kx = -\left(K-1\right)x, \tag{8}$$

and its solution for constant K is

$$x\left(t\right) = x\left(0\right)e^{-\left(K-1\right)t}.$$

The system is stable for K>1. In this case, substituting $x(t)$ into (7) gives

$$J = \frac{1}{2}\left(q + K^2\right)x^2(0)\int_0^\infty e^{-2(k-1)t}dt = \frac{(q+K^2)}{4(K-1)}x^2(0).$$

To minimize J for fixed q and $x(0)$, we compute $\partial J/\partial K$ and set it to zero. This gives

$$K^2 - 2K - q = 0.$$

Its roots are

$$K_1 = 1 + \sqrt{1+q}, \quad K_2 = 1 - \sqrt{1+q}.$$

For a minimum we require that $\partial^2 J/\partial K^2 > 0$. Because $q > 0$, this implies that

$$K - 1 > 0. \tag{9}$$

This last condition is of interest. It says that the value of K that minimizes J must be such that the closed-loop system (8) will be stable.

- One root, K_1, will satisfy this condition. In this case, $x(t) \to 0$ and J has a minimum value of $J_{\min} = \frac{1}{2}\left(1+\sqrt{1+q}\right)x^2(0)$.
- For K_2, $K_2 - 1 < 0$ and $x(t) \to \infty$, $u(t) = \left(\sqrt{1+q} - 1\right)x(0)e^{\sqrt{1+q}t}$, $u(t) \to \infty$; Therefore, $J \to \infty$, and thus does not have a minimum.

The requirement (9) now seems obvious from the stability condition for system (8), but the lessons of this example will be useful later.

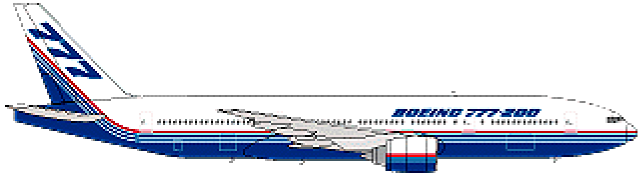
Finally, we note that the design problem has been transformed into one of selecting a value for q.

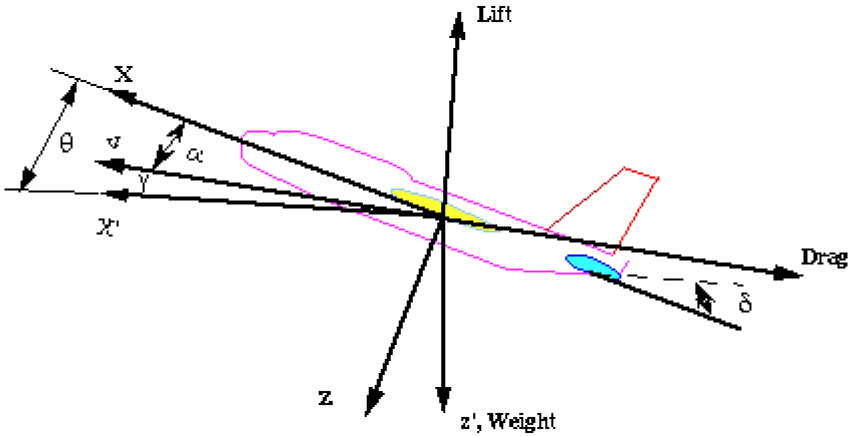
- The larger q is, the larger will be the gain K, and the faster will $x(t)$ approach zero.
- However, the peak magnitude of u will be larger.
- The parameter q is selected to achieve a compromise between these effects. We will indicate a general procedure for doing this.

One sees answers to why LQR:

- LQR represents a class of control problems well: balance/tradeoff between minimization of system errors and minimization of control efforts
- LQR has a neat solution: linear state feedback
- LQR is about easiest to solve among all optimal control problems
- LQR provides means for tuning.

- **An Industrial Motivation: A pitch controller design for an aircraft**





α – Angle of attack

q – Pitch rate

θ – Pitch angle

δ_e – Elevator deflection angle

Model: The basic coordinate axes and forces acting on an aircraft are shown in the figure above. After some simplifications, the longitudinal equations of motion of an aircraft can be written as:

$$\dot{\alpha} = -0.313\alpha + 56.7q + 0.232\delta_e,$$

$$\dot{q} = -0.0139\alpha - 0.426q + 0.0203\delta_e,$$

$$\dot{\theta} = 56.7q.$$

Its state-space model is

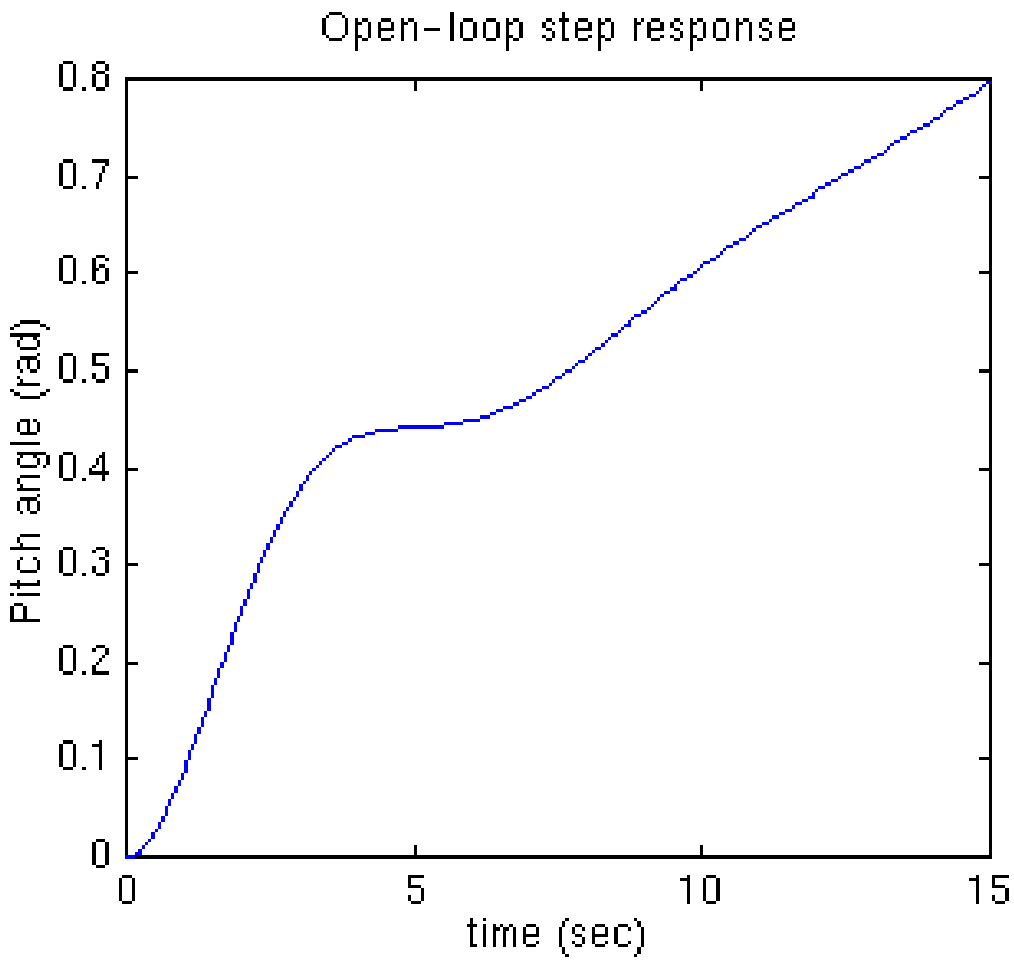
$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} \delta_e ,$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} .$$

Control design requirements

- Rise time: Less than 2 seconds;
- Overshoot: Less than 10%;
- 2% settling time: Less than 10 seconds.

Its open-loop response to a step input δ_e = 0.2 rad (11°) is shown in the following figure.



From the plot, we see that the open-loop response does not satisfy the design criteria at all. In fact the open-loop is unstable.

Is LQR problem solvable?

We need some assumptions before we can develop a procedure for computing **K**.

Consider a state space plant:

$$\begin{aligned}\dot{x}_1 &= x_1, \\ \dot{x}_2 &= x_2 + u.\end{aligned} \tag{10}$$

Let the performance index be

$$J = \frac{1}{2}\int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + u^2\right) dt. \tag{11}$$

Suppose that we are primarily interested in driving x_1 to zero. Therefore, we select $q_2 = 0$. It is easy to see from (10) that no minimum for J will exist because the state variable x_1 is uncontrollable and of an unstable mode. Thus the response of x_1 will be $x_1(t) = x_1(0)e^t$ regardless of what u and x_2 do. Therefore $x_1(t) \to \infty$ and no matter what $u(t)$ does,

$$J = \frac{1}{2}\int_0^\infty \left[q_1 x_1^2(0) e^{2t} + u^2 \right] dt \to \infty, \text{ if } q_1 \neq 0.$$

Any control system of practical usefulness must be stable. Any unstable plant must be stabilized by feedback control.

We can conclude that no minimum exists for the index J in this problem because of the following.

[1] The state variable x_1 is uncontrollable.

[2] The uncontrollable state variable x_1 is also unstable.

A minimum for J will exist if none of these two situations occur. We make the following assumption to avoid possibility of [1] above and enable a solution.

Assumption 1. The system, (A, B), is controllable.

The manner in which the state variables appear (or do not appear) in the performance index is also important. Consider another simple example:

$$\dot{x} = x + u,$$

$$J = \frac{1}{2}\int_0^\infty u^2 dt.$$

The system is controllable but unstable. It is obvious that the choice of u that minimizes J is $u(t) = 0$ for all t. Putting $u = 0$ in the state equation gives the solution $x(t) = x(0)e^t$. This simple example serves to point out that the optimal control u will not stabilize a controllable but unstable state variable that does not appear in the performance index. This situation is avoided if the following assumption is met. Once again, we want control system stability.

Assumption 2. The pair ($\mathbf{A}$, $\mathbf{H}$) is completely observable, where $\mathbf{H}$ is any matrix such that $\mathbf{H}\mathbf{H}^T = \mathbf{Q}$. The partitioning of the matrix $\mathbf{H}$ does not matter.

Assumption 2 says that *all* the state variables will be "observed" by the performance index. It happens that Assumption 2 is stronger than necessary because ***stable*** state variables that do not appear in the index cause no difficulty.

The algorithm to be developed later is capable of stabilizing any mode that is controllable (Assumption 1) and that appears in the performance index (Assumption 2).

But will the optimal control remain finite?

Assumption 3. The weighting matrices $\mathbf{Q}$ *and* $\mathbf{R}$ *are symmetric. The matrix* $\mathbf{R}$ *is positive definite, while the matrix* $\mathbf{Q}$ *is semi-positive definite.*

Assumption 3 is required for a minimum of J to exist with finite control. For example, if one takes

$$J = \frac{1}{2}\int_0^\infty \left(x^2 - u^2\right)dt, R < 0,$$

$$J = \frac{1}{2}\int_0^\infty \left(-x^2 + u^2\right)dt, Q < 0.$$

J may be made as negative as one wishes (without a minimum) when u or $x \to \infty$. These situations are avoided by requiring Q and R to be positive or semi-positive definite, which implies $J \geq 0$.

The different requirements on **Q** and **R** can be explained as follows. A semi-positive **Q** is acceptable as long as it meets Assumption 2 while a semi-positive **R** could yield infinite control signal, which is not acceptable. For example, the formulation

$$J = \frac{1}{2}\int_0^\infty x^2 dt,$$

$$\dot{x} = -x + u,$$

has a positive Q=1 and a semi-positive but not positive R=0. It is unacceptable because it results in $u \rightarrow \infty$. *u(t)* must be finite due to physical limitations.

In summary, Assumptions 1 and 2 guarantee that the optimal control system $\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{BK}\right)\mathbf{x}$ is stable, while Assumption 3 ensures that the optimal control *u(t)* is bounded for physical realization.

§8.2 General Solution

The LQR problem consists of the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{C}\mathbf{x},$$

and the index:

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt.$$

We now describe a solution for the optimal K due to Stear. Suppose that the optimal control law stabilizes the plant. Let an $n \times n$ symmetric matrix P satisfy the following *Algebraic Riccati Equation (ARE)*,

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}.$$

Differentiate $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ with respect to time, use the open-loop system and Riccati equation, and collect terms to obtain

$$
\begin{aligned}
\frac{d}{dt}\left(\mathbf{x}^T\mathbf{P}\mathbf{x}\right) &= \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} \\
&= \mathbf{x}^T\left(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A}\right)\mathbf{x} + \mathbf{u}^T\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{u} \\
&= \left(\mathbf{x}^T\mathbf{P}\mathbf{B} + \mathbf{u}^T\mathbf{R}\right)\mathbf{R}^{-1}\left(\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{R}\mathbf{u}\right) - \mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{u}^T\mathbf{R}\mathbf{u}. \qquad (12)
\end{aligned}
$$

Then, we integrate both sides. The LHS becomes

$$
\begin{aligned}
\int_0^\infty \frac{d}{dt}\left(\mathbf{x}^T\mathbf{P}\mathbf{x}\right)dt &= \mathbf{x}^T(\infty)\mathbf{P}\mathbf{x}(\infty) - \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) \\
&= -\mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0),
\end{aligned}
$$

because $\mathbf{x}(\infty) \to \mathbf{0}$ from the assumed stability; The RHS is given by

$$\int_0^\infty \left(\mathbf{x}^T\mathbf{PB}+\mathbf{u}^T\mathbf{R}\right)\mathbf{R}^{-1}\left(\mathbf{B}^T\mathbf{Px}+\mathbf{Ru}\right)dt$$

$$-\int_0^\infty \left(\mathbf{x}^T\mathbf{Qx}+\mathbf{u}^T\mathbf{Ru}\right)dt$$

to get

$$J=\frac{1}{2}\mathbf{x}^T(0)\mathbf{Px}(0)+\frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{PB}+\mathbf{u}^T\mathbf{R}\right)\mathbf{R}^{-1}\left(\mathbf{B}^T\mathbf{Px}+\mathbf{Ru}\right)dt. \qquad (13)$$

But $\mathbf{x}(0)$ and $\mathbf{P}$ are independent of the control input $\mathbf{u}$ to be chosen. Thus J can be minimized by considering only the integrand in (13). This is nonnegative and therefore has a minimum of zero. This minimum occurs at

$$\mathbf{B}^T\mathbf{Px}+\mathbf{Ru}=\mathbf{0},$$

and this implies that

$$\mathbf{u}=-\mathbf{R}^{-1}\mathbf{B}^T\mathbf{Px}=-\mathbf{Kx},$$

that is, the optimal control law is a linear feedback of the state vector $\mathbf{x}$, indeed.

Theorem 1. *Consider the system:*

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \neq 0, \tag{14}$$

with the cost index:

$$J = \frac{1}{2}\int_0^\infty \left(x^T Qx + u^T Ru\right)dt, \tag{15}$$

where R is a positive definite and Q is semi-positive definite with $Q = H^T H$. If (A,B) is controllable and (A,H) observable, the optimal control minimizing (15) *is given by*

$$u(t) = -R^{-1}B^T Px(t), \tag{16}$$

where P is the symmetric positive definite solution of the algebraic Matrix Riccati equation:

$$A^T P + PA + Q - PBR^{-1}B^T P = 0. \tag{17}$$

Example 1. Consider the system:

$$\dot{x} = x + u,$$

with the cost function:

$$J = \frac{1}{2}\int_0^{\infty}(x^2 + u^2)dt.$$

The ARE is

$$2P - P^2 + 1 = 0,$$

and its solution is

$$P = 1 \pm \sqrt{2}.$$

We choose the positive definite solution, namely,

$$P = 1 + \sqrt{2}.$$

Since $R = 1$ and $B = 1$, the optimal control is

$$u = -R^{-1}BPx = -(1 + \sqrt{2})x.$$

Example 2. Find the optimal control with the cost function:

$$J = \frac{1}{2}\int_0^\infty \left[x^T \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} x + ru^2 \right] dt, \tag{18}$$

for the system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad x(0) = x_0. \tag{19}$$

Let

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}.$$

The ARE is

$$\begin{pmatrix} 0 & -\alpha_0 \\ 1 & -\alpha_1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$$

$$-r^{-1}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = 0. \tag{20}$$

$$-2\alpha_0 p_{12} + q_1 - p_{12}^2 r^{-1} = 0,$$

$$p_{11} - \alpha_1 p_{12} - \alpha_0 p_{22} - p_{12} p_{22} r^{-1} = 0,$$

$$2p_{12} - 2\alpha_1 p_{22} + q_2 - p_{22}^2 r^{-1} = 0,$$

The solutions are

$$p_{12} = -r\alpha_0 \pm r\sqrt{\alpha_0^2 + q_1/r},$$

$$p_{22} = -r\alpha_1 \pm r\left[\alpha_1^2 + q_2/r + 2p_{12}/r\right]^{\frac{1}{2}},$$

$$\begin{aligned} p_{11} &= \alpha_1 p_{12} + \alpha_0 p_{22} + p_{12} p_{22} r^{-1} \\ &= \left(\alpha_1 r + p_{22}\right)\left(\alpha_0 r + p_{12}\right) r^{-1} - \alpha_1 \alpha_0 r. \end{aligned}$$

The positive definite P is

$$p_{11} = r\left\{\alpha_1^2 + q_2/r + 2\left[-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\,\right]\right\}^{\frac{1}{2}} \sqrt{\alpha_0^2 + q_1/r} - \alpha_0\alpha_1 r,$$

$$p_{12} = -r\alpha_0 + r\sqrt{\alpha_0^2 + q_1/r},$$

$$p_{22} = -r\alpha_1 + r\left[\alpha_1^2 + q_2/r + 2\left(-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right)\right]^{\frac{1}{2}}.$$

The optimal control is given by

$$u = -r^{-1}\begin{pmatrix}0 & 1\end{pmatrix}Px$$

$$= -\left[-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r},\quad -\alpha_1 + \left[\alpha_1^2 + q_2/r + 2\left(-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right)\right]^{1/2}\right]x. \tag{21}$$

Selection of the Weighting Matrices. The selection of the weighting matrices Q and R is usually made on the basis of experience together with simulations of the results for different trial values. The following guidelines have emerged.

- Usually **Q** and **R** are selected to be diagonal so that specific state and control variables are penalized individually with higher weightings if their response is undesirable. To this end, the state variables and manipulated variables should represent sets of variables that are easily identifiable physically, rather than a set of transformed variables such as the model variables. This enables the designer to visualize the effects of the trial values for **Q** and **R**.

- The larger the elements of **Q** are, the larger are the elements of the gain matrix **K**, and the faster the state variables approach zero. On the other hand, the larger the elements of **R**, the smaller the elements of **K** and the slower the response.

- Often some state-variables are simply derivatives of others, such as the displacement and velocity of a mass. If only the displacement variable is penalized by **Q**, the response tends to be oscillatory. To reduce overshoots, therefore, include in **Q** a weighting term for the velocity variable.

- Often the linear state equations are the result of a linearization. In this case, it is sometimes possible to obtain a clue as to the proper weights by examining the second-order term in the Taylor expansion used to obtain the linearization. These terms are quadratic forms, and the relative size of their components can indicate which variables are most likely to jeopardize the accuracy of the linearization. These variables would then be weighted more heavily in **Q** or **R**.

§8.3 Stability of the Optimal Control System

In section 8.2, we assume that the optimal control assures the system is stable. Now let's prove this rigorously.

For the system

$$\dot{x} = Ax + Bu,$$

with the optimal control:

$$u = -Kx, \tag{22}$$

and

$$K = R^{-1}B^T P, \tag{23}$$

the closed-loop system is

$$\dot{x} = (A - BK)x, \quad x(0) = x_0. \tag{24}$$

(17) is repeated here

$$A^T P + PA + Q - PBR^{-1}B^T P = 0.$$

It follows from $K = R^{-1}B^TP$, $P = P^T$, $Q = Q^T$ and $R = R^T$, that

$$\begin{aligned} PBR^{-1}B^TP &= PB(R^{-1}R)R^{-1}B^TP \\ &= \left(R^{-1}B^TP\right)^T R(R^{-1}B^TP) \\ &= K^TRK. \end{aligned}$$

Thus,

$$A^TP + PA + Q - K^TRK = 0$$

$$A^TP + PA - 2K^TRK = -Q - K^TRK$$

$$(A^TP - PBR^{-1}B^TP) + (PA - PBR^{-1}B^TP) = -Q - K^TRK$$

$$(A^T - PBR^{-1}B^T)P + P(A - BR^{-1}B^TP) = -Q - K^TRK$$

$$(A - BR^{-1}B^TP)^TP + P(A - BR^{-1}B^TP) = -Q - K^TRK$$

$$(A - BK)^TP + P(A - BK) = -Q - K^TRK. \tag{25}$$

Theorem 2. *If the triple* (A,B,H) *is controllable and observable, then the closed-loop system using the optimal control* (16) *which minimizes the quadratic criterion function* (15) *is asymptotically stable.*

Proof: Define a functional:

$$V\left(x,t\right)=\frac{1}{2}x^{T}Px>0,\quad \text{if } x\neq 0, \tag{26}$$

since P is positive. One evaluates

$$\dot{V}(x,t) = \frac{1}{2}\left\{\left(\frac{d}{dt}x^T\right)Px + x^T P\frac{d}{dt}x\right\}$$

$$= \frac{1}{2}[x^T(A-BK)^T Px + x^T P(A-BK)x]$$

$$= -\frac{1}{2}[x^T Qx + x^T K^T RKx]$$

$$= -\frac{1}{2}[(Hx)^T I(Hx) + (Kx)^T R(Kx)] \le 0 . \qquad (27)$$

We need $\dot{V} < 0$ to ensure asymptotical stability. By (27), this is the case if $\dot{V} \neq 0$. Assume that $\dot{V}(x,t) = 0$ for some $x(t) \neq 0$, then $-Kx = 0$ and $Hx = 0$. $\dot{x} = Ax - BKx = Ax$ and $x = e^{At}x_0$. Then $Hx = He^{At}x_0 = 0$, which contradicts the fact that (A,H) is observable. Therefore, $\dot{V}\left(x,t\right) \neq 0,\ \forall x \neq 0$. Then, (24) is asymptotically stable. The proof is complete.

The poles of the closed-loop system (24) are the roots of

$$\det(sI - A + BK) = \det(sI - A + BR^{-1}B^T P) = 0. \tag{28}$$

One may also verify stability by finding these poles.

Example 3. Consider the LQR problem with

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \tag{29}$$

$$J = \int_0^\infty \left(x^T \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} x + ru^2 \right) dt. \tag{30}$$

How will the poles vary with r?

Substituting $\alpha_0 = -1$, $\alpha_1 = 0$, $q_1 = 9$ and $q_2 = 1$ into (21) gives

$$u = -\left\{1+\left(1+9/r\right)^{1/2}, \left[2+1/r+2\left(1+9/r\right)\right]^{1/2}\right\}x, \tag{31}$$

and the poles are

$$\frac{1}{2}\left\{-\left[1/r+2+2\sqrt{1+9/r}\right]^{1/2} \pm \left[1/r+2-2\sqrt{1+9/r}\right]^{1/2}\right\}, \tag{32}$$

which always have negative real parts for all r based on the root locus in Fig.1.

For $r = \infty$: very expensive control u (the control input should be as small as possible), we have

$$u = -\begin{bmatrix}2 & 2\end{bmatrix}x,$$

$$\{\text{closed-loop poles}\} = \frac{1}{2}\left\{-2 \pm 0\right\} = \left\{-1, -1\right\}.$$

Note that {open-loop poles}={ –1, 1}.

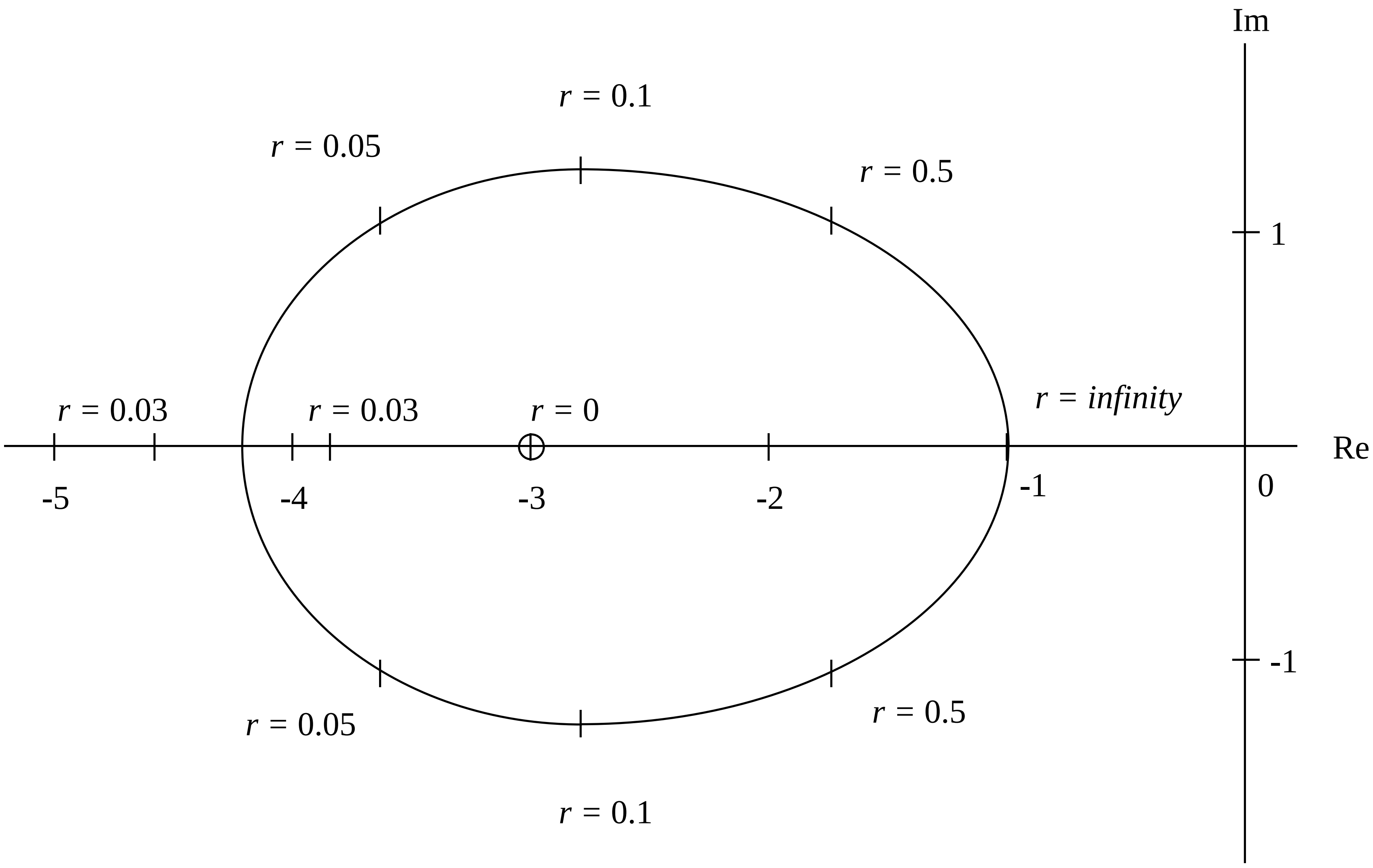
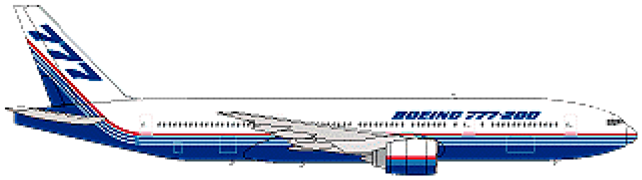


Figure 1 Optimal pole locations with variation of r.

An Industrial Application: A pitch controller design for an aircraft revisited



Model:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} \delta_e,$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}.$$

Design requirements

- Rise time: Less than 2 seconds;
- Overshoot: Less than 10%;
- 2% settling time: Less than 10 seconds.

Solution:

Let $R=1$, $Q=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 50 \end{bmatrix}$. The optimal control is

$$u(t)=-R^{-1}B^{T}Px(t),$$

where

$$k^{T}=R^{-1}B^{T}P=\begin{bmatrix} -0.6435 & 169.6950 & 7.0711 \end{bmatrix}.$$

The step response of the closed-loop system is shown in Figure 3. The rise time, overshoot, and settling time look satisfactory.

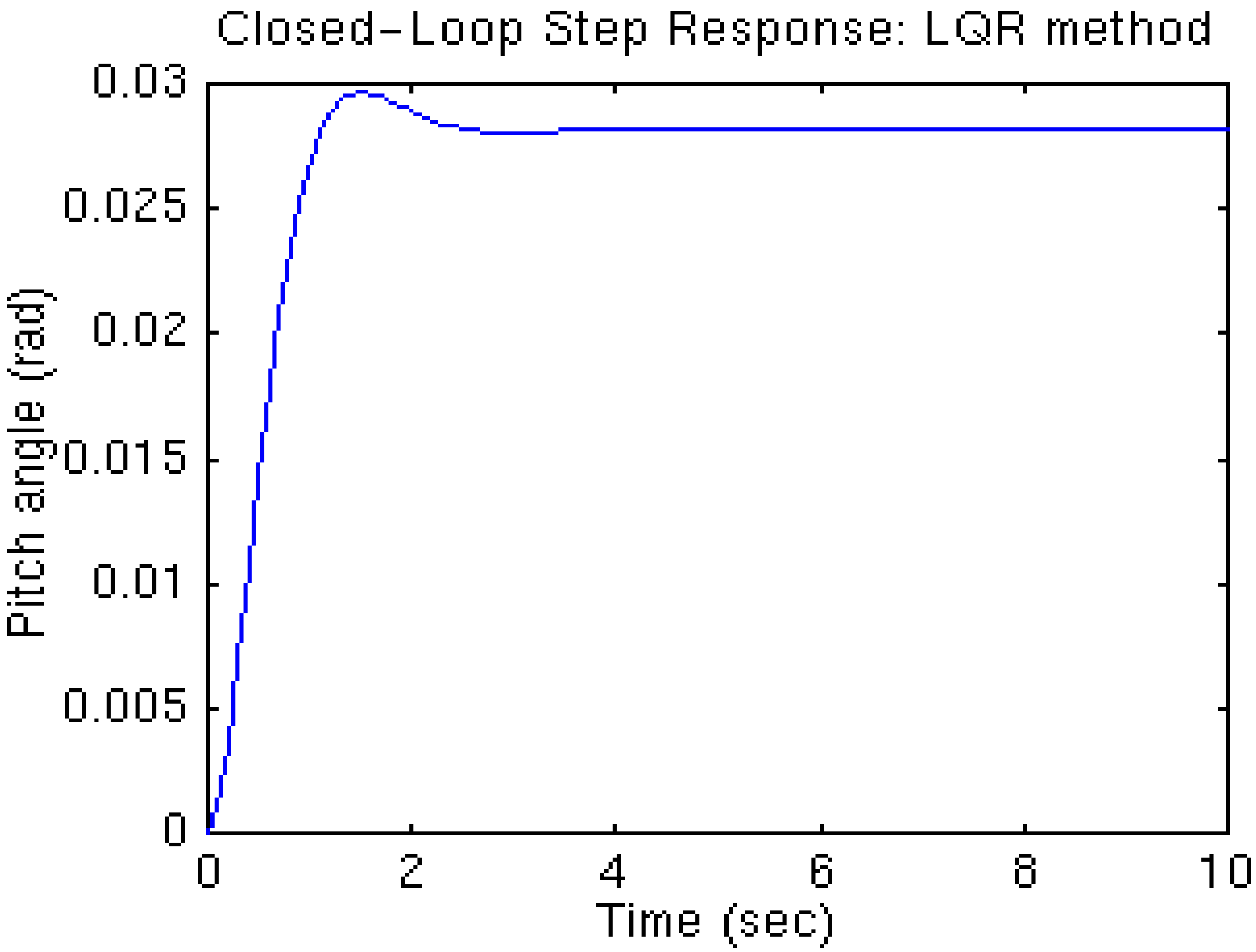


Figure 3 Optimal control for aircraft.

§8.4. How to Solve ARE?

To find the positive definite solution of the Riccati equation:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0,$$

one may use the following eigenvalue-eigenvector based algorithm.

Step 1: Form the $2n \times 2n$ matrix:

$$\Gamma = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}, \tag{33}$$

and find its n stable eigenvalues.

Lemma 3. *Let* $\phi_f(s) = \det(sI - A + BR^{-1}B^T P)$ *with its n roots being* $\{\lambda_i\}$*. Then,* $\{\lambda_i\}$ *and* $\{-\lambda_i\}$ *are the 2n eigenvalues of* Γ*.*

Proof: Let

$$T = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}, \quad \text{then,} \quad T^{-1} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}.$$

Perform the similarity transformation:

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} sI - A & BR^{-1}B^T \\ Q & sI + A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} sI - A + BR^{-1}B^T P & BR^{-1}B^T \\ Q + (sI + A^T)P & sI + A^T \end{bmatrix}$$

$$= \begin{bmatrix} sI - A + BR^{-1}B^T P & BR^{-1}B^T \\ Q + (sI + A^T)P - P(sI - A + BR^{-1}B^T P) & sI + A^T - PBR^{-1}B^T \end{bmatrix}$$

$$= \begin{bmatrix} sI - A + BR^{-1}B^T P & BR^{-1}B^T \\ 0 & sI + A^T - PBR^{-1}B^T \end{bmatrix},$$

where the last equality is due to

$$Q + A^T P + PA - PBR^{-1}B^T P = 0.$$

Let

$$\phi_f(s) = \det(sI - A + BR^{-1}B^T P).$$

One sees that

$$\det(sI + A^T - PBR^{-1}B^T)$$

$$= (-1)^n \det(-sI - A^T + PBR^{-1}B^T)$$ for $\det M = \det(-(-M)) = (-1)^n \det(-M)$

$$= (-1)^n \det(-sI - A + BR^{-1}B^T P)$$ for $\det M = \det M^T$

$$= (-1)^n \phi_f(-s)$$

Hence, we have

$$\det\left[sI-\Gamma\right]=\det\begin{pmatrix} sI-A & BR^{-1}B^T \\ Q & sI+A^T \end{pmatrix}$$

$$=\det\begin{pmatrix} sI-A+BR^{-1}B^TP & BR^{-1}B^T \\ 0 & sI+A^T-PBR^{-1}B^T \end{pmatrix}$$

$$=\det\left(sI-A+BR^{-1}B^TP\right)\det\left(sI+A^T-PBR^{-1}B^T\right)$$

$$=\phi_f(s)\left[(-1)^n\phi_f(-s)\right].$$

If λ_i are such that $\phi_f(\lambda_i)=0$, then one sees

$$\phi_f(-s)\Big|_{s=-\lambda_i}=\phi_f(\lambda_i)=0,$$

that is, $(-\lambda_i)$ are roots of $\phi_f(-s)$.

Step 2**:** Let the eigenvector or generalized eigenvector of Γ corresponding to **stable** λ_i , i = 1,..., n, be

$$\begin{pmatrix} \nu_i \\ \mu_i \end{pmatrix}, i = 1, 2, \cdots, n.$$

Then, P is given by

$$P = [\mu_1, \ldots, \mu_n][\nu_1, \ldots, \nu_n]^{-1}. \tag{34}$$

Proof: The ARE,

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 ,$$

is re-written as

$$\left(sI + A^T\right)P - P\left(sI - A + BR^{-1}B^T P\right) + Q = 0. \tag{35}$$

The eigenvalues λ_i of $\bar{A} = \left(A - BR^{-1}B^T P\right)$ and their corresponding eigenvectors v_i satisfy $\bar{A}v_i = \lambda_i v_i$:

$$Av_i - BR^{-1}B^T Pv_i = \lambda_i v_i, \tag{36}$$

Post-multiplying (35) by v_i, letting $s = \lambda_i$ and using the above relationship yields

$$\left(\lambda_i I + A^T\right)Pv_i + Qv_i = 0,$$

$$-Qv_i - A^T Pv_i = \lambda_i Pv_i, \tag{37}$$

Let $Pv_i = \mu_i$. Then, (36) and (37) combined yield:

$$\begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}\begin{pmatrix} v_i \\ \mu_i \end{pmatrix} = \lambda_i \begin{pmatrix} v_i \\ \mu_i \end{pmatrix},$$

showing $\begin{bmatrix} \nu_i \\ \mu_i \end{bmatrix}$ is an eigenvector of Γ corresponding to the eigenvalue λ_i. $P\nu_i = \mu_i$, $i = 1, 2, \cdots, n$, are arranged so that

$$P\begin{bmatrix} \nu_1 & \nu_2 & \cdots & \nu_n \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix},$$

which leads to solution (34).

Example 4. Consider

$$\dot{x} = -3x + u\,,$$

$$J = \frac{1}{2}\int_0^\infty \left(x^2 + u^2\right) dt.$$

Clearly, $A = -3$, $B = 1$, $Q = R = 1$.

Solution (i). The Riccati equation is

$$-6P - P^2 + 1 = 0,$$

$$P = -3 \pm \sqrt{10}\,.$$

The positive definite P of $\sqrt{10} - 3$ = 0.1622 gives

$$u = -Kx = -0.162x\,.$$

Solution (ii) (Eigenvectors). Form

$$\Gamma = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}.$$

Its eigenvalues are $\pm\sqrt{10}$ and the eigenvector corresponding to $-\sqrt{10}$ is

$$\begin{bmatrix} \nu \\ \mu \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{10} - 3 \end{bmatrix},$$

giving $P = \mu\nu^{-1} = 0.1622$, the same result.