EE5103/ME5403 Computer Control Systems: Homework #2 Solution

Semester 1 Y2021/2022

Q1 Solution

a)

Since the state vector is $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T = \begin{bmatrix} \dot{y}(t) & y(t) \end{bmatrix}^T$, it can be derived that

$$y(t) = x_2(t) \tag{1.1}$$

$$\dot{x}_2(t) = x_1(t) \tag{1.2}$$

From the DC motor model, the relationship between voltage and velocity is

$$X_1(s) = \frac{1}{Ts+1} U(s) \tag{1.3}$$

Then we have

$$TsX_1(s) + X_1(s) = U(s) \tag{1.4}$$

The corresponding ODE is,

$$T\dot{x}_1(t) + x_1(t) = u(t) \tag{1.5}$$

And further we have

$$\dot{x}_1(t) = -\frac{1}{T} x_1(t) + \frac{1}{T} u(t) \tag{1.6}$$

Thus, the state-space model is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -\frac{1}{T} & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{T} \\ 0 \end{bmatrix} u = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned} \tag{1.7}$$

Assume the discrete-time state-space model is

$$\begin{aligned} x(k+1) &= \Phi x(\mathrm{k}) + \Gamma u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned} \tag{1.8}$$

Then we have (*please check HW1 for details on how to solve matrix exponential.*)

$$\Phi = e^{Ah} = \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} \tag{1.9}$$

$$\Gamma = \int_0^h e^{At} dt B = \int_0^h \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} dt B = \begin{bmatrix} 1-e^{-h} \\ h+e^{-h}-1 \end{bmatrix} \tag{1.10}$$

Thus, the discrete state-space model is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1-e^{-h} \\ h+e^{-h}-1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned} \tag{1.11}$$

For deadbeat controller, the desired poles should be all zero. Thus, the closed-loop characteristic equation is $P(\mathrm{z}) = z^2$. The controllability matrix for system (1.11) is,

$$W_c = [\Gamma, \Phi\Gamma] = \begin{bmatrix} 1-e^{-h} & e^{-h}-e^{-2h} \\ h+e^{-h}-1 & h+e^{-2h}-e^{-h} \end{bmatrix} \tag{1.12}$$

It is apparent that W_c is nonsingular if $h > 0$. Thus, this sampled system is controllable. Now we can place the closed-loop poles arbitrarily. Assume the state-feedback gain is L, then the closed-loop characteristic polynomial is

$$\det(\mathrm{zI} - (\Phi - \Gamma L)) = z^2 \tag{1.13}$$

Let $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$, then according to Ackermann's formula, we have

$$L = \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} P(\Phi) = \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} \Phi^2 \tag{1.14}$$

Substituting (1.9) and (1.12), we can get

$$L = \frac{1}{h(1-e^{-h})^2} \begin{bmatrix} 1-he^{-2h}-e^{-h}, & 1-e^{-h} \end{bmatrix} \tag{1.15}$$

Thus, the deadbeat controller is

$$u(k) = -Lx(k). \tag{1.16}$$

b)

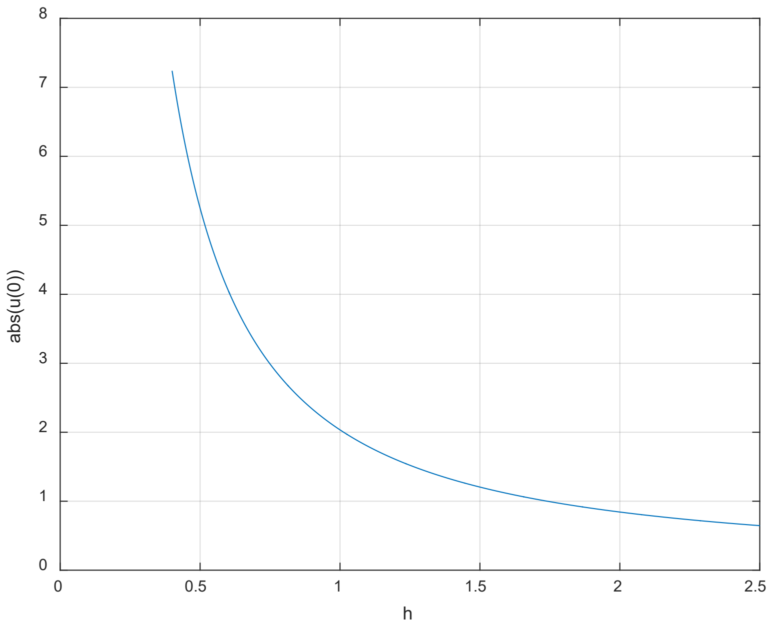
The maximum control signal is

$$u(0) = -Lx(0) = -L\begin{bmatrix}1 & 0.5\end{bmatrix}^T \tag{1.17}$$

To satisfy $|u(0)| < 1$, substitute equation (1.15) into (1.17) and we have

$$\left|\frac{1.5 - he^{-2h} - 1.5e^{-h}}{h(1-e^{-h})^2}\right| < 1 \tag{1.18}$$

If you plot the left-hand side part of (1.18), it is obviously monotonically decreasing. Focus on the range where $h \in [0.4, \quad 2.5]$ and the following figure is acquired:



Therefore, to make $|u(0)| < 1$, it is required that

$$h > 1.7404 \tag{1.19}$$

Since deadbeat control is implemented, the state will be brought to zero in 2 steps. Smaller sampling period implies faster speed, and hence higher energy cost. Therefore, smaller sampling period results in bigger control input.

Note: If you just give one specific value of h instead of the range of h, for example, $h = 2$ *, you will not get full marks.*

Q2 Solution

Since the given system is a linear one, according to the superposition principle, to eliminate the influence of the disturbance v on the system output y at steady state, we just need to make the *response* of y caused by v, say y_v, be zero at steady state. Assuming the transfer function from v to y is $H_v(z)$, the final value theorem tells that the value of y_v at steady state is

$$\lim_{k\to\infty} y_v[k] = \lim_{z\to 1}(z-1)\frac{vz}{z-1}H_v(z) = vH_v(1), \tag{2.1}$$

where v is the value of the constant disturbance $v(k)$. Thus, to make equation (2.1) equal to zero, we require $H_v(1)=0$. In other words, the steady state gain has to be zero. In the next three questions, we'll try to design some state feedback control law as well as a state observer, if needed, to achieve $H_v(1)=0$ in the closed-loop system.

a) The state and v can be measured.

Assume the state feedback gain for x and v is respectively L and L_v, that is

$$u(k) = -\begin{bmatrix} L & L_v \end{bmatrix}\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} = -Lx(k) - L_v v(k) \tag{2.2}$$

where $L \in \Re^2$ and $L_v \in \Re$. Then closed-loop state-space model is

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)x(k) + (\Phi_{xv} - \Gamma L_v)v(k) \\ y &= Cx(k) \end{aligned} \tag{2.3}$$

where $\Phi = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.8 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \Phi_{xv} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

After Z transform on equation (2.3), we can get

$$\begin{aligned} zX(z) &= (\Phi - \Gamma L)X(z) + (\Phi_{xv} - \Gamma L_v)V(z) \\ Y(z) &= CX(z) \end{aligned} \tag{2.4}$$

Then we can get the transfer function between the disturbance and output as

$$H_v(z) = \frac{Y(z)}{V(z)} = C(zI - (\Phi - \Gamma L))^{-1}(\Phi_{xv} - \Gamma L_v). \tag{2.5}$$

Now to achieve $H_v(1)=0$, there are actually two tasks involved in this question: (1) stabilize the closed-loop system with state feedback since the final value theorem only holds if the system is stable; (2) reject the disturbance on steady state. The first task should be done by choosing a proper state feedback gain L to place the closed-loop poles, while the second task can be attempted by finding a proper value for L_v.

For the first task, we can place the closed-loop poles both at 0 (a deadbeat controller). Then the closed-loop characteristic equation is

$$A_m(z)=z^2 \tag{2.6}$$

The controllability matrix is

$$W_c=\begin{bmatrix}\Gamma & \Phi\Gamma\end{bmatrix}=\begin{bmatrix}0.2 & 0.2\\ 0.1 & 0.18\end{bmatrix} \tag{2.7}$$

It is nonsingular. We can place the closed-loop poles arbitrarily with state feedback. From Ackermann's formula, we have

$$L=\begin{bmatrix}0 & 1\end{bmatrix}W_c^{-1}A_m(\Phi)=\begin{bmatrix}3.4375, & 6.1250\end{bmatrix} \tag{2.8}$$

Then substitute L into $H_v(z)$ in (2.5), which yields

$$\begin{bmatrix}0.8125 & \text{-}0.225\end{bmatrix}\left(\begin{bmatrix}1\\0\end{bmatrix}-L_v\begin{bmatrix}0.2\\0.1\end{bmatrix}\right)=0 \tag{2.9}$$

That is,

$$0.8125=0.14L_v \tag{2.10}$$

Thus, we can solve

$$L_v=\frac{0.8125}{0.14}=5.8036 \tag{2.11}$$

Thus, the state feedback controller is

$$u(k)=-Lx(k)-L_v v(k)=-\begin{bmatrix}3.4375, & 6.1250\end{bmatrix}x(k)-5.8036v(k) \tag{2.12}$$

b) The state can be measured

Since the disturbance cannot be measured, we need to design some kind of observer to estimate the disturbance for state feedback. Note that the state can be measured, and the disturbance is constant. From the state transition equation, it holds that

$$\Phi_{xv} v(k) = x(k+1) - \Phi x(k) - \Gamma u(k) \tag{2.13}$$

Since $v(k)$ is constant, i.e., $v(k) = v, \forall k \geq 0$. Inserting $k = 0$ into (2.13), we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} v = x(1) - \Phi x(0) - \Gamma u(0) = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \tag{2.14}$$

Since the state $x(1)$ and $x(0)$ can be measured and $u(0)$ is known, the right side of equation (2.14) can be acquired. Suppose it is $[\gamma \quad 0]^T$. Then the constant scalar disturbance v can be solved.

$$v(k) = v = \gamma, \quad \forall k \geq 0 \tag{2.15}$$

After the estimation of the disturbance is available, the controller is designed as (2.12), i.e.,

$$u(k) = -[3.4375, \quad 6.1250] x(k) - 5.8036 v \tag{2.16}$$

This example shows that a full order observer is not always necessary to estimate the disturbance if you have extra information.

c) Only the output can be measured.

Since only the output can be measured, we should implement an observer for both the state variables and the disturbance. Define the augmented state variables as

$$z(k) = \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \tag{2.17}$$

And the state-space model now turns into

$$\begin{aligned} z(k+1) &= \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & 1 \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) \\ y(k) &= [C \quad 0] z(k) \end{aligned} \tag{2.18}$$

For convenience, simplify the notations as follows

$$\Phi_z = \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & 1 \end{bmatrix}_{3\times 3}, \Gamma_z = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}_{3\times 1}, C_z = [C \quad 0]_{1\times 3} \tag{2.19}$$

The augmented state observer is described as

$$\hat{z}(k+1)=\Phi_z\hat{z}(k)+\Gamma_z u(k)+K(y-\hat{y})$$
$$\hat{y}=C_z\hat{z}(k) \tag{2.20}$$

The state estimation error is defined as $e(k)=z(k)-\hat{z}(k)$ and it follows that

$$e(k+1)=(\Phi_z-KC_z)e(k) \tag{2.21}$$

The objective is to make the error dynamics (2.21) stable and it should be faster than the controller dynamics. The observation matrix of system (Φ_z, C_z) is

$$W_o=\begin{bmatrix} C_z \\ C_z\Phi_z \\ C_z\Phi_z^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 1 \\ 0.75 & 1.3 & 1.5 \end{bmatrix} \tag{2.22}$$

which is nonsingular. Therefore, we can place the poles arbitrarily for (2.21) as we like.

Here, for convenience, we can design a deadbeat state observer, that is, let all the eigenvalues of (2.21) be zero. Then the characteristic polynomial is $f(z)=z^3$. The feedback gain K is

$$K=f(\Phi_z)W_o^{-1}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T=\begin{bmatrix} 2.3 \\ -2.06 \\ 5 \end{bmatrix} \tag{2.23}$$

From the above observer (2.20), we can estimate both the state and the disturbance to be $\hat{x}(k)$ and $\hat{v}(k)$. Then the controller is designed again as (2.12) but using the estimated variables, that is

$$u(k)=-L\hat{x}(k)-L_v\hat{v}(k)=-\begin{bmatrix} 3.4375, & 6.1250 \end{bmatrix}\hat{x}(k)-5.8036\hat{v}(k). \tag{2.24}$$

Q3 Solution

a)

Assume the continuous-time reference model is

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.25}$$

where ζ is the damping ratio and ω_n is the natural frequency.

According to the performance specifications, overshoot satisfies

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.1 \tag{2.26}$$

Then we have $0.591 \le \zeta < 1$, let $\zeta = 0.6$. (*You can choose any value inside the range.)*

The 1% settling time is required that

$$t_s \cong \frac{4.6}{\zeta\omega_n} < 10 \tag{2.27}$$

Thus, we have

$$\omega_n \ge \frac{4.6}{10\zeta} = 0.7667 \tag{2.28}$$

We choose $\omega_n = 0.7667$. The reference model in the continuous-time is

$$G(s) = \frac{0.5878}{s^2 + 0.92s + 0.5878} \tag{2.29}$$

Sampling period is $h = 0.1$. From the table 2.1, we can get the discrete-time reference model as

$$H_m(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{0.00285z + 0.002763}{z^2 - 1.906z + 0.9121} = \frac{B_m(z)}{A_m(z)} \tag{2.30}$$

b)

First, we find the continuous-time state-space model. Since $x(t) = [y(t), \dot{y}(t)]^T$, we have

$$\dot{x}_1(t) = x_2(t) \tag{2.31}$$

And the dynamics is $m\dot{x}_2(t) + bx_2(t) = u(t)$, it can be derived that

$$\dot{x}_2(t) = -\frac{b}{m}x_2(t) + \frac{1}{m}u(t) \tag{2.32}$$

Thus, the continuous-time state-space model is

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{aligned} \tag{2.33}$$

For the sampled system, the state matrix and input matrix are respectively

$$\Phi = e^{Ah} = \begin{bmatrix} 1 & 0.0995 \\ 0 & 0.99 \end{bmatrix} \tag{2.34}$$

$$\Gamma = \int_0^{0.1} e^{At} dt B = \begin{bmatrix} 0.0498 \\ 0.995 \end{bmatrix} \times 10^{-4} \tag{2.35}$$

Thus, the state-space model for the sampled system is

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned} \tag{2.36}$$

c)

From reference model (2.30) we can get the closed-loop characteristic polynomial as

$$A_m(z) = z^2 - 1.906z + 0.9121 \tag{2.37}$$

Now considering the state-space model (2.36), the controllability matrix is

$$W_c = [\Gamma, \Phi\Gamma] = \begin{bmatrix} 0.0498 & 0.1488 \\ 0.995 & 0.995 \end{bmatrix} \times 10^{-4} \tag{2.38}$$

And the rank of W_c is 2, which means the system is controllable. Therefore, we can place the poles arbitrarily through state feedback. And from Ackermann's formula, we have

$$L = [0 \quad 1] W_c^{-1} A_m(\Phi) \tag{2.39}$$

From equations (2.38) and (2.37),(2.34), we can get

$$L = [613.0, \ 821.6] \tag{2.40}$$

And the state feedback controller is

$$u_{fb}(k) = -Lx(k) = -[613.0, \ 821.6]\, x(k) \tag{2.41}$$

d)

We first derive the transfer function from $u_{ff}(k)$ to $y(k)$. It is obvious that the closed-loop system is

$$x(k+1) = \Phi x(k) + \Gamma\left(u_{ff}(k) - Lx(k)\right) = \left(\Phi - \Gamma L\right)x(k) + \Gamma u_{ff}(k)$$
$$y(k) = Cx(k) \tag{2.42}$$

where L is the state feedback law we found in c). The transfer function is

$$\frac{Y(z)}{U_{ff}(z)} = C[zI - (\Phi - \Gamma L)]^{-1}\Gamma = \frac{B(z)}{A_{cl}(z)} \tag{2.43}$$

Note that in the above state feedback c), we have designed the feedback gain L to match the poles of the reference model $H_m(z) = \frac{B_m(z)}{A_m(z)}$ in (2.30). Therefore, we already have $A_{cl}(z) = A_m(z)$.

Since $U_{ff}(z) = U_c(z)\mathrm{H}_{ff}(z)$, it is further derived from (2.43) that

$$\frac{Y(z)}{U_c(z)} = C[zI - (\Phi - \Gamma L)]^{-1}\Gamma H_{ff}(z) = \frac{B(z)H_{ff}(z)}{A_{cl}(z)} = \frac{B(z)H_{ff}(z)}{A_m(z)} \tag{2.44}$$

Now it can be seen that the feedforward controller $H_{ff}(z)$ is used to match the numerator of the reference model. It follows that

$$B(z)H_{ff}(z) = B_m(z) \quad \Rightarrow \quad H_{ff}(z) = \frac{B_m(z)}{B(z)} \tag{2.45}$$

From equations (2.30) and (2.43), we can get $H_{ff}(z)$. (You should work out the specific value of $H_{ff}(z)$ here instead of just listing the formula.)

$$H_{ff}(z) = \frac{B_m(z)}{B(z)} = \frac{2850z + 2763}{4.983z + 4.967} = \frac{571.9z + 556.3}{z + 0.9968} \tag{2.46}$$

(Summary: in this case, state feedback is to match poles and feed-forward controller is to match zeros; you need to conduct these two tasks one by one.)

e)

Since we have used state feedback, if only the output is measurable, then we have to estimate the state with an observer.

$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + K[y(k) - \hat{y}(k)]$$
$$\hat{y}(k) = C\hat{x}(k) \tag{2.47}$$

And define the estimation error as $e(k) = x(k) - \hat{x}(k)$. Equation (2.47) and (2.36) yields

$$e(k+1) = (\Phi - KC)e(k) \tag{2.48}$$

To make $e(k)$ converge to zero, it is required to place the eigenvalues of $\Phi - KC$ inside the unit circle on the z plane. And this requires the system (Φ, C) to be observable. The observability matrix is

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0.0995 \end{bmatrix} \tag{2.49}$$

W_o is nonsingular, so this system is observable. We can make a deadbeat observer, that is, to place all the eigenvalues at origin. In this case, the expected characteristic polynomial is

$$f(\mathrm{z}) = z^2 \tag{2.50}$$

And the observer gain K is

$$K = f(\Phi)W_o^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1.99 & 9.85 \end{bmatrix}^T \tag{2.51}$$

Then we can use the above K and the observer (2.47) to estimate the state variable vector though it is not directly measurable. Hence it is still possible to use this two-degree-of-freedom controller to meet the performance specification with estimated state feedback by a state observer.

In conclusion, the two-degree-of-freedom control method is designed as

$$u(k) = H_{ff}(q)u_c(k) - L\hat{x}(k) \tag{2.52}$$

where q is the time shift operator.

Note:

- If you cannot measure the state variables directly, then you have to design a state observer. After that, your control law should depend on the estimated state $\hat{x}$ instead of x because x is unavailable to you.