EE5101/ME5401:
Linear Systems: Part II
State Estimation

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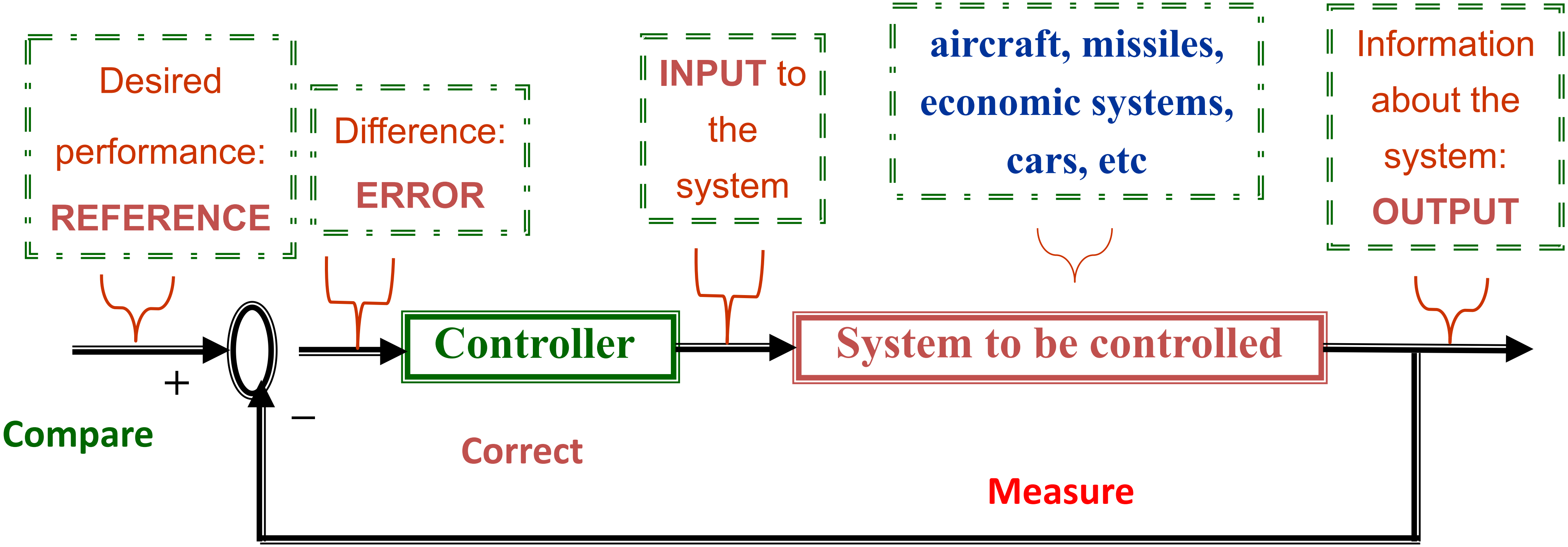
Where are we now?



•**Feedback: Measure —Compare —-Correct**

Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

How to specify the reference signal, or the desired output?

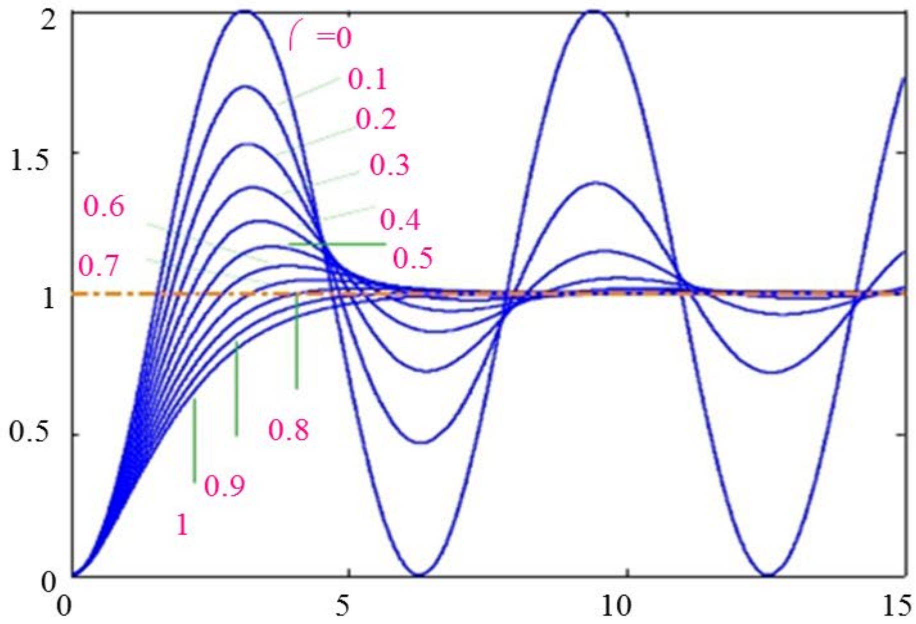
In practical control problems, there are certain performance specifications for transient response.

Transient behavior of 2nd order systems

Consider the following block diagram with a standard 2nd order system under unit step

$$R(s) = \frac{1}{s}, \quad r(t) = 1 \longrightarrow \boxed{H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \longrightarrow Y(s)$$

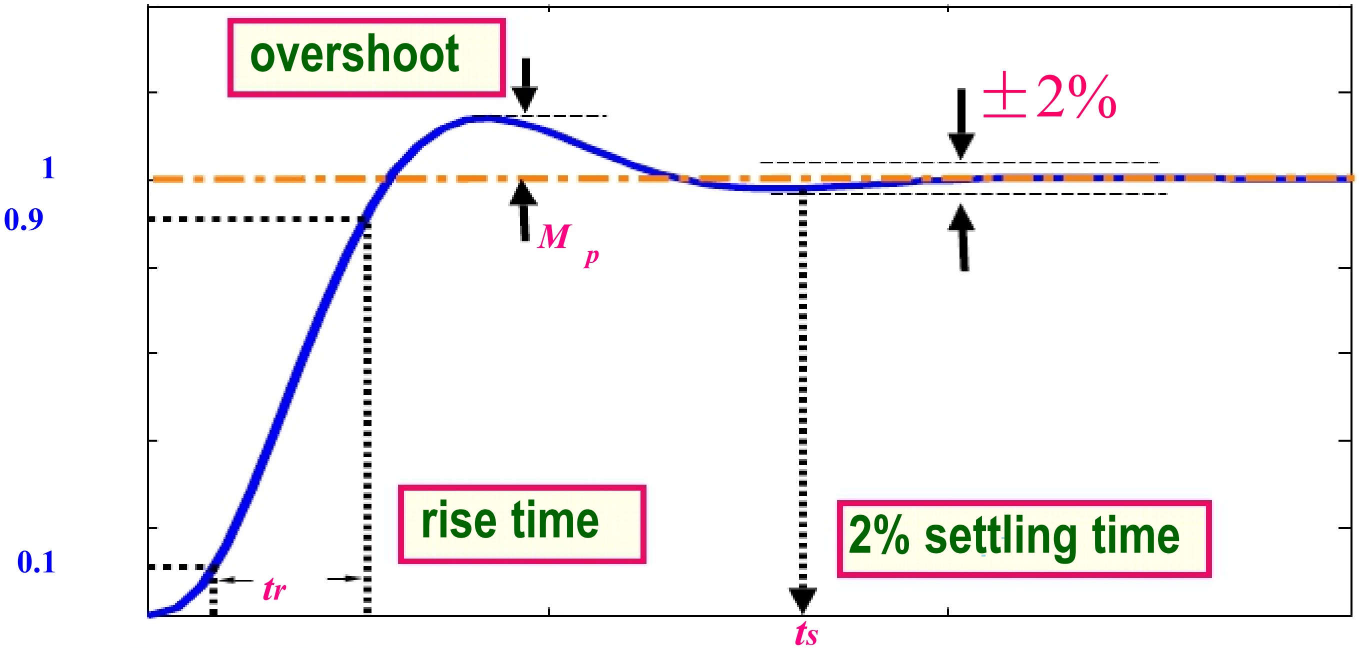
The behavior of the system is as follows:



The system behavior is fully characterized by 2 factors: the damping ratio ζ, and the natural frequency ω_n.

Settling time, overshoot and rise time — time domain specifications

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$t_r \cong \frac{1.8}{\omega_n}$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_s \cong \frac{4.0}{\zeta\omega_n}$$

$(t_s, M_p, t_r) \Leftrightarrow (\zeta, \omega_n) \Leftrightarrow$ Reference Model

For higher order system, choose the second order system as the dominant mode, and place all the other poles far away from the dominant one.

Where are we now?

From the transient performance specifications, you can design the positions of the desired poles. Then there are a number of controllers for you to choose.

What was the first type of controller we introduced?

The pole placement controller.

What kind of form does the controller take?

The state feedback controller which looks like a proportional controller

$$u = -Kx + Fr$$

Since there are many possible poles to meet the transient performance requirements, sometimes we want to choose the “best” one to strike the balance between speed and cost.

That is the motivation behind the optimal control -- LQR

Does LQR take the same form as that for pole placement?

$$u = -Kx + Fr$$

But in order to compute K, you need to solve the ARE first.

Where are we now?

After pole placement and optimal controller, what type of controller did we discuss?

Decoupling.

How many ways for decoupling?

There are two ways.

1) State Feedback
2) Output Feedback

For state feedback, it still takes the same form:

$$u = -Kx + Fr$$

What is the last type of controller we discussed?

Servo control.

What is the objective of the servo control?

It is for reference tracking and disturbance rejection at steady state.

How many ways to design servo control?

There are two ways.

1) State Feedback
2) Output Feedback

Where are we now?

For servo control, does the state feedback controller take the same form as

$$u = -Kx + Fr\,?$$

Not exactly. It is the state feedback. But the state refers to the augmented state:

$$u = -K\bar{x} = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

servo control= stabilization + servo mechanism!

We can use the same idea of augmented state space to deal with more general type of reference output. But we do not have time to cover it. Please read the reference book about this if you are interested.

In summary, for linear system, we can meet almost any performance requirement if there is no limit on the magnitude of the control input we can apply.

In reality, this is not true as there is always a limit on the control input!

And the most commonly used controller is the state feedback controller:

$$u = -Kx + Fr$$

What are the problems with state feedback?

When we try to implement the state feedback controller

$$u = -Kx + Fr$$

we need to know the values of all the state variables.

Therefore, we need to have sensors to measure all the state variables.

That is the biggest problem with the state feedback controller.

(i) Not all state variables are measurable; and

(ii) Even if all the state variables can be measured, some sensors might be too expensive to buy!

If you want to know how to save money, then you cannot skip this chapter!
Because there is no need to put sensors to measure all the state variables!
For those expensive ones or un-accessible ones, we can just estimate them!
The idea of state estimation is not only useful for control system design, it is also useful for other applications like signal processing and sensor fusion.

Why state estimation?

- The state space approach uses state feedback:

$$u = -Kx + Fr.$$

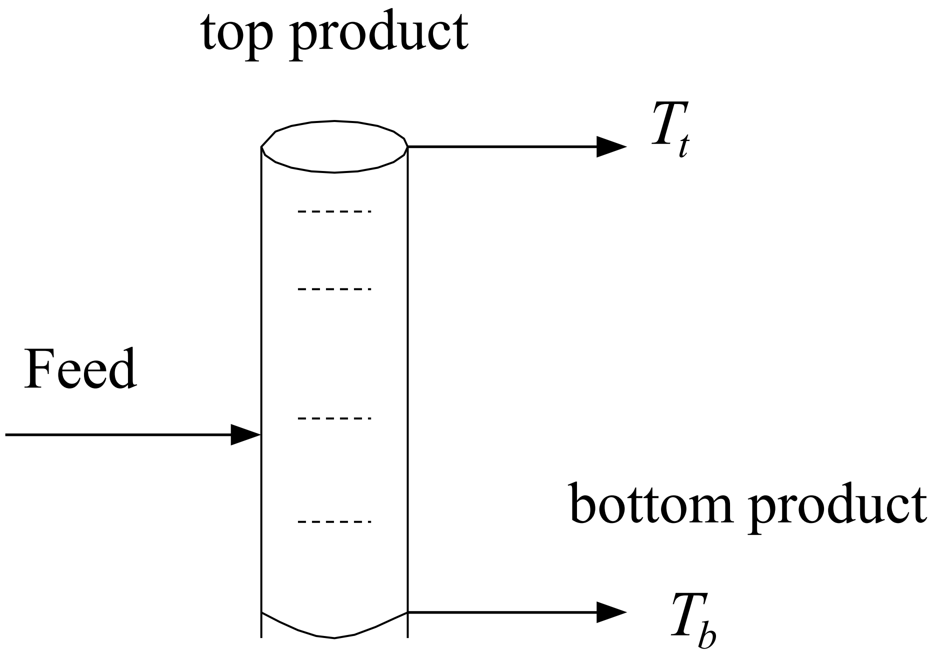
- But usually, x is not available as all state variables are not measurable.
- Only the output, y, is measurable.
- To implement state feedback without $x(t)$ necessitates use of state estimation.

There is no need to measure all the state variables, as they are related to each other.

If you want to know something, you can either observe it directly, or estimate it indirectly!

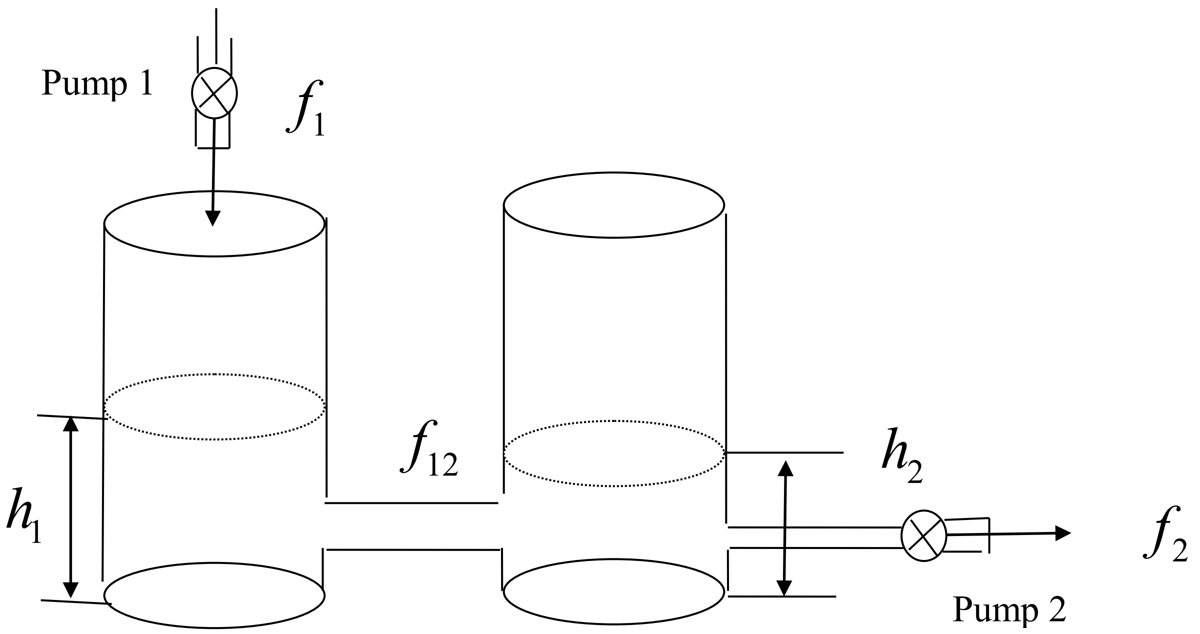
For instance, a full state space model for a Distillation Column will have each plate temperature T_i as a state variable. Not all T_i are measurable.

Only T_b and T_t are usually measured.



An Industrial Motivation: a coupled-tank level estimation

Process: Consider a coupled-tank (Goodwin G. C. etc., Control system design, Prentice Hall, USA, 2002) as follows



Water flows into the first tank through pump 1 at a rate of f_1 that affects the height of the water in tank 1. Water flows out of tank 1 into tank 2 at a rate of f_{12}, affecting both h_1 and h_2. Water flows out of tank 2 at a rate of f_2 controlled by pump 2.

The challenge is to build an observer to estimate the height of liquid in tank 1 from measurement of the height of liquid in tank 2 and the flows f_1 and f_2.

Model: The height of liquid in tank 1 can be described by equation

$$\frac{dh_1(t)}{dt} = \frac{1}{A}\left(f_1(t) - f_{12}(t)\right)$$

Similarly, $h_2(t)$ is described by

$$\frac{dh_2(t)}{dt} = \frac{1}{A}\left(f_{12}(t) - f_2(t)\right)$$

The flow between the two tanks can be approximated by the free-fall velocity for the difference in height between tanks:

$$f_{12}(t) = \sqrt{2g\left(h_1(t) - h_2(t)\right)}$$

Now, if we measure the liquid height in the tanks in % (where 0% is empty and 100% is full), we can convert the flow rates into equivalent in % per second (where $f_1(t)$ is the equivalent flow into tank 1 and $f_2(t)$ is the equivalent flow out of tank 2). The model for the system is then

$$\begin{bmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \end{bmatrix} = \begin{bmatrix} -K\sqrt{h_1(t) - h_2(t)} + f_1(t) \\ K\sqrt{h_1(t) - h_2(t)} - f_2(t) \end{bmatrix}$$

where $K = \dfrac{\sqrt{2g}}{A} = 0.26$. This nonlinear model can be linearized around a nominal steady-state, (H_1, H_2),

$$h_1(t) = H_1 + \Delta h_1(t),$$
$$h_2(t) = H_2 + \Delta h_2(t).$$

Now, we have

$$\begin{bmatrix}\Delta\dot{h}_1\\ \Delta\dot{h}_2\end{bmatrix}=\begin{bmatrix}\left.\dfrac{-0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}}\Delta h_1+\left.\dfrac{0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}}\Delta h_2+\Delta f_1\\ \left.\dfrac{0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}}\Delta h_1+\left.\dfrac{-0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}}\Delta h_2-\Delta f_2\end{bmatrix}$$

This yields the following linear model:

$$\begin{bmatrix}\Delta\dot{h}_1\\ \Delta\dot{h}_2\end{bmatrix}=\begin{bmatrix}-k & k\\ k & -k\end{bmatrix}\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\Delta f_1\\ \Delta f_2\end{bmatrix}$$

where $k=\dfrac{0.13}{\sqrt{H_1-H_2}}$. If we let

$$x=\begin{bmatrix}\Delta h_1\\ \Delta h_2\end{bmatrix},$$

$$u=\begin{bmatrix}\Delta f_1\\ \Delta f_2\end{bmatrix}=\begin{bmatrix}u_1\\ u_2\end{bmatrix},$$

the process model can be expressed as

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

where $A = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. If we assume that the operating point is at $H_1 = 50\%$ and $H_2 = 34\%$, then $k = 0.325$.

Task: With $h_2(t)$ measured, estimate $h_1(t)$.

State estimation problem: Let the system be

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1}$$

With knowledge of $y(t)$ and $u(t)$, one wants to estimate $x(t)$.

The questions arising are

Q1: Is it possible to estimate *x(t)?*

Q2: How to estimate *x(t)?*

Possible Estimation Schemes

One simple way of finding $x(t_1)$ is

$$
\begin{aligned}
y(t) &= Cx(t), \\
\dot{y}(t) &= C\dot{x}(t) \\
&= CAx(t) + CBu(t), \\
&\vdots \\
\frac{d^{n-1}y(t)}{dt^{n-1}} &= CA^{n-1}x(t) + \cdots
\end{aligned}
$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t_1) = \begin{bmatrix} y(t_1) \\ \dot{y}(t_1) - CBu(t_1) \\ \\ \vdots \end{bmatrix}$$

If the system is observable, then it is possible to find $x(t_1)$ by solving these linear equations.

Why do not we use this simple way in practice? Which signal may give us trouble in practice?

The measurement of output y inevitably has noise and differentiation of a noisy signal will cause large errors.

For instance, let the high frequency component of the noise be $\mathrm{d(t)} = \sin(\omega \mathrm{t})$,

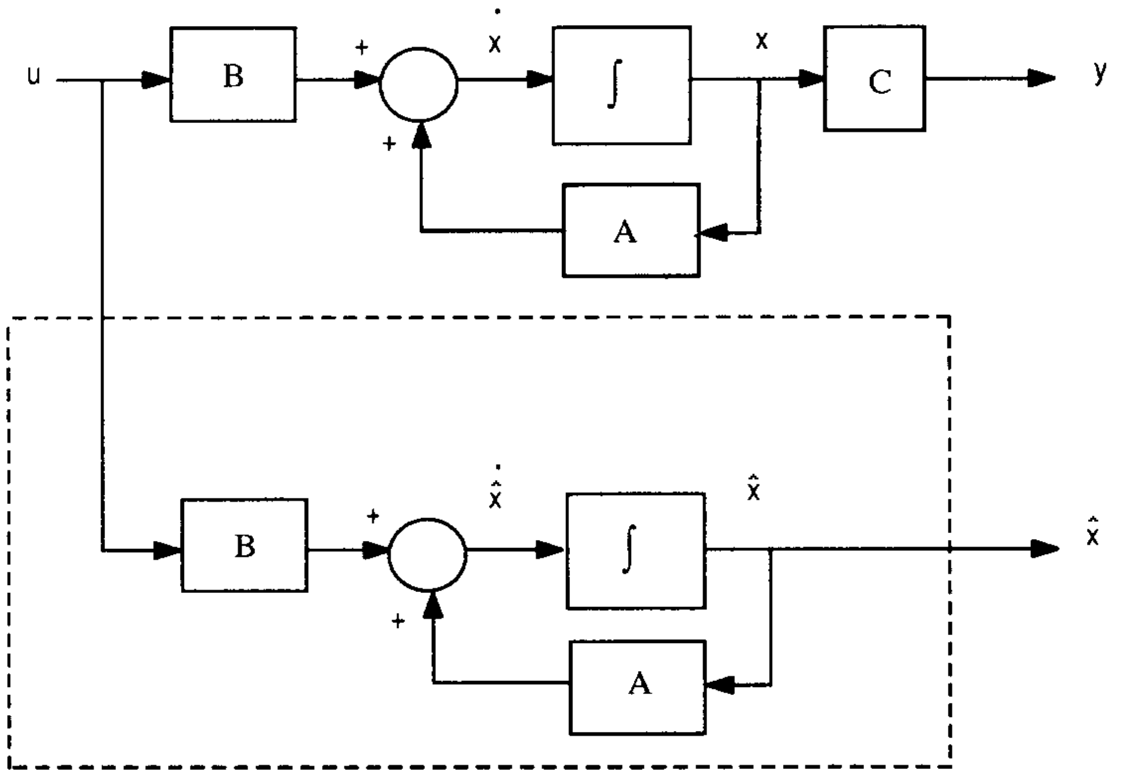
What is its derivative? $\dot{\mathrm{d}}(\mathrm{t}) = \omega\cos(\omega \mathrm{t})$,

Differentiation will amplify the noise! We should avoid this in practice whenever possible!

Another possible way to estimate the state is to simulate a model with the complete knowledge of A, B, and C, and construct a so-called open-loop estimator.

$$\dot{x} = Ax + Bu,$$
$$y = Cx.$$

$$\dot{\hat{x}} = A\hat{x} + Bu,$$
$$\hat{y} = C\hat{x}.$$



An open-loop estimator.

If one chooses $\hat{x}(0) = x(0)$, then, $\hat{x}(t) = x(t)$ for all t>0.

But do we know the initial state x(0)?

No.

Let the state estimate error be $\tilde{x}(t) = x(t) - \hat{x}(t).$ (2)

It follows that the dynamic behavior of $\tilde{x}(t)$ is governed by

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = Ax + Bu - (A\hat{x} + Bu) = A\tilde{x}(t) \quad (3)$$

and its solution is given by $\tilde{x}(t) = e^{At}\tilde{x}(0), t \geq 0.$ (4)

But, we do not have the initial state $x(0)$. Instant determination of the state: $\hat{x}(t) = x(t)$ is impossible. It only enables asymptotic estimation of the state: $\hat{x}(\infty) = x(\infty)$, if A is stable.

What would happen if A is unstable?

The error would increase exponentially for unstable matrix A and $\tilde{x}(0) \neq 0$

§11.2 Full-order Observers

Where does it come and how does it work?

Observations:

- $x(t)$ unknown $\rightarrow$ $x(0)$ unknown $\rightarrow$ state estimation error is expected!
- The only thing to do is to have some correction function for the state estimation error, and the correction carries on till, hopefully, the error settles to zero in the end.
- The state estimation error is NOT available, too, while the error between the measured output $y(t)$ and predicted output $\hat{y}(t) = C\hat{x}(t)$ from state estimate can be obtained.
- Thus, one may correct and reduce the error in the estimation $\hat{x}$ by some feedback based on the observed error between the measured output $y(t)$ and predicted output $\hat{y}(t) = C\hat{x}(t)$.

What is the simplest way to use the output estimation error:

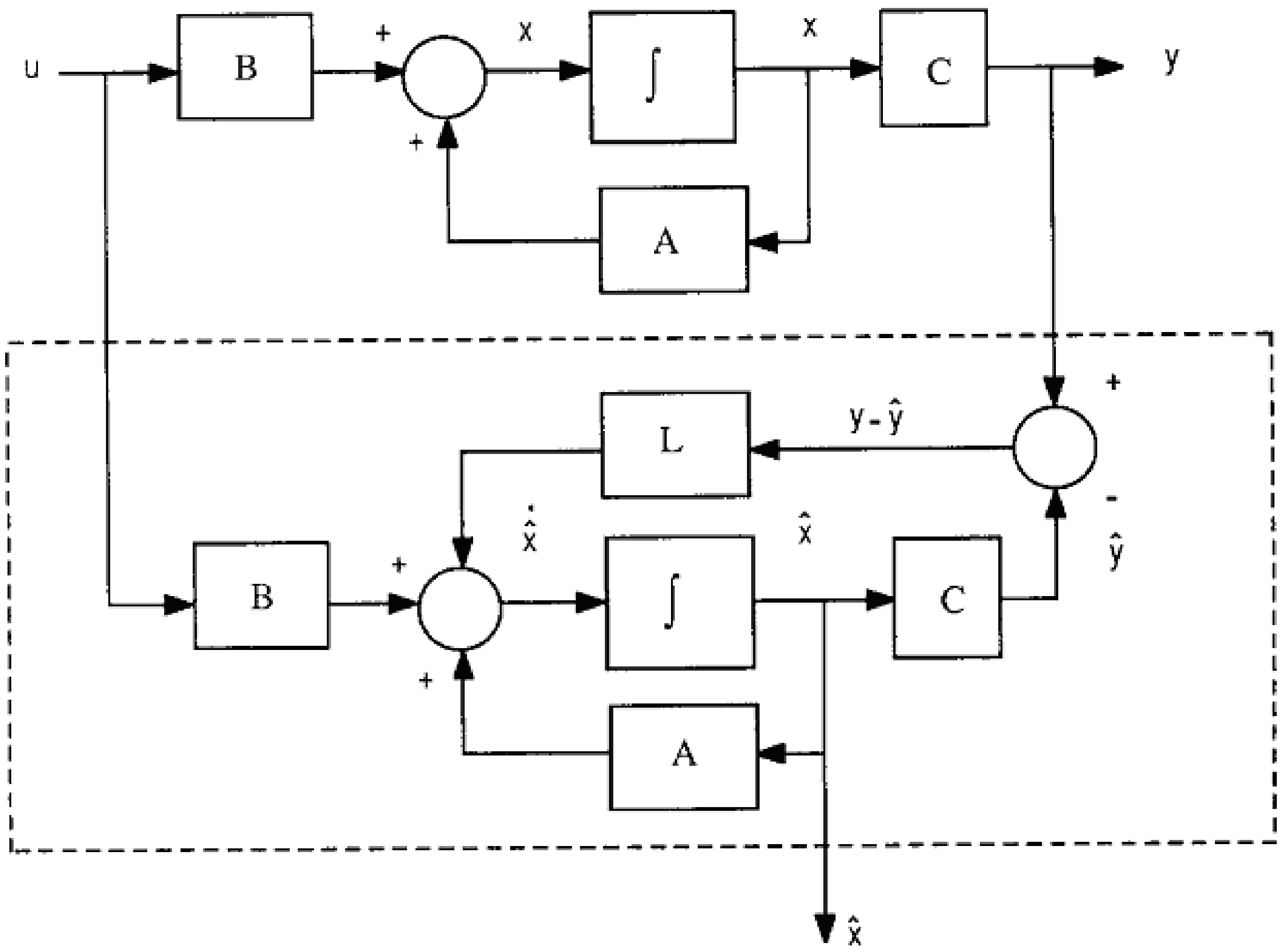
$$y(t) - \hat{y}(t)$$

to make the correction?

Make it proportional to the error! $L(y(t) - \hat{y}(t))$

§11.2 Full-order Observers

The observer, is a closed-loop estimator which was first introduced by Luenberger (1964).



Observer=Model + Feedback Correction Mechanism

Objectives:

- $\hat{x}(t) \rightarrow x(t)$ or $x(t) - \hat{x}(t) \xrightarrow[t\to\infty]{} 0$, or the estimation error is stable.
  - The rate of error convergence is adjustable.

Analysis

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

Consider an estimator:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L\left[y - \hat{y}\right] \\ \hat{y} &= C\hat{x} \end{aligned} \tag{5}$$

Observer=Model + Feedback Correction Mechanism

Let the estimation error in the state be

$$\tilde{x} = x - \hat{x}$$

Then, it follows that

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - \{A\hat{x} + Bu + L(y - C\hat{x})\}$$
$$= A(x - \hat{x}) - L(Cx - C\hat{x})$$
$$= A(x - \hat{x}) - LC(x - \hat{x})$$

$$\dot{\tilde{x}} = (A - LC)\tilde{x}, \qquad \tilde{x}(0) = x(0) - \hat{x}(0) \tag{6}$$

If we choose L such that $(A - LC) = A_1$ is stable, we have

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = A_1\tilde{x},$$

$$\tilde{x}(t) = e^{A_1 t}\tilde{x}(0).$$

Clearly $\tilde{x}(t) \to 0$ as $t \to \infty$. Thus the estimated state $\hat{x}(t)$ will 'track' $x(t)$ asymptotically.

Note that

- no derivatives of y are used;
- the error dynamics is adjustable by L;

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = A_1\tilde{x}$$

- the error could decay to zero even for unstable matrix A and $\tilde{x}(0) \neq 0$, as long as $(A - LC) = A_1$ is stable.

Then, when can $(A - LC) = A_1$ be made stable and have arbitrary stable eigenvalues?

For the observer problem, write its error characteristic polynomial:

$$\begin{aligned}
\det\left[sI - (A - LC)\right] &= \det\left[sI - (A - LC)\right]^T \\
&= \det\left[sI - (A - LC)^T\right] \\
&= \det[sI - \left(A^T - C^T L^T\right)] \\
&= \det\left[sI - (\tilde{A} - \tilde{B}\tilde{K})\right]
\end{aligned}$$

where $\tilde{A} = A^T$, $\tilde{B} = C^T$, $\tilde{K} = L^T$.

Therefore, the question becomes, given any pair $\{\tilde{A}, \tilde{B}\}$, can we find $\tilde{K}$ such that $(\tilde{A} - \tilde{B}\tilde{K})$ is stable?

Have we solved this problem before?

This is the pole placement problem in Chapter Seven!

- By duality, $(\tilde{A}, \tilde{B}) = (A^T, C^T)$ is controllable if the pair *(A, C)* is observable.

- By the pole placement theorem, L can be found to place the roots of $|sI - (A - LC)| = 0$ arbitrarily, or the observer can have any desired eigenvalues if the pair *(A, C)* is observable.

But where should we place the poles of the observer?

The desired poles of the controller can be designed from the transient response specifications.

Shall we use the transient response requirement to design the poles for observer?

Not really. For observer, what we care is how fast the estimation error converges to zero.

Let's take a closer look at the error dynamics.

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$

The rate of convergence to zero of the estimation error will be controlled by how negative the real parts of the pole of the observer are.

In order for this convergence to be fast, L must be large in magnitude.

It seems that the higher gain, the better! But there is a catch!

If the observation y has been corrupted by noise, say y=Cx+d, then the error equation becomes

$$\dot{\tilde{x}} = (A - LC)\tilde{x} - Ld$$

Therefore, a large L will lead to large estimation error if d is persistent.

There must be a trade-off in achieving small error: large L may be needed for fast convergence if there is no noise, but small L amplifies less the effect of noise.

This trade-off can be formulated as an optimization problem. The Kalman filter is an observer that has been optimized with respect to disturbances. The key difference is that the observer gain L is designed to be time-varying, L(t), instead of a constant gain.

But Kalman filter is beyond the scope of this module due to time constraint. It requires some knowledge on stochastic process to understand.

Instead, you can use the following simple empirical rule.

Design

The observer design is to find a suitable observer gain matrix L. One may use the pole placement to adjust the 'rate of convergence' of $\hat{x}(t)$ to $x(t)$. The choice of observer poles is a trade-off between speed and constraints imposed by noise, saturation and nonlinearity.

A simple guideline is to place observer poles 3-5 times faster than the closed loop poles designed by the controller.

However, for real world applications, it is better to implement Kalman filter for state estimation to improve the performance of the system.

A design procedure is summarized as follows.

(i) Choose the observer poles 3-5 times faster than controller poles;

(ii) Use a pole placement algorithm to obtain *L;*

(iii) Implement the observer,

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y - \hat{y}\right]$$

$$\hat{y} = C\hat{x}$$

Example 1 Design an observer for the plant,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Test $\{A, C\}$ for observability:

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is nonsingular, and the system is observable.

$$A_1 = A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}.$$

The observer poles are the roots of

$$\left|\lambda I - A_1\right| = \begin{vmatrix} \lambda + l_1 & -1 \\ l_2 & \lambda \end{vmatrix} = \lambda^2 + \lambda l_1 + l_2 = 0$$

Choose the observer poles at $\lambda_1 = \lambda_2 = -5$ and this choice gives $l_1 = 10$ and $l_2 = 25$. The designed observer is given by

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 10 \\ 25 \end{bmatrix} (y - \hat{y})$$

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -25 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 10 \\ 25 \end{bmatrix} y$$

The system and observer are shown in Figure 3.

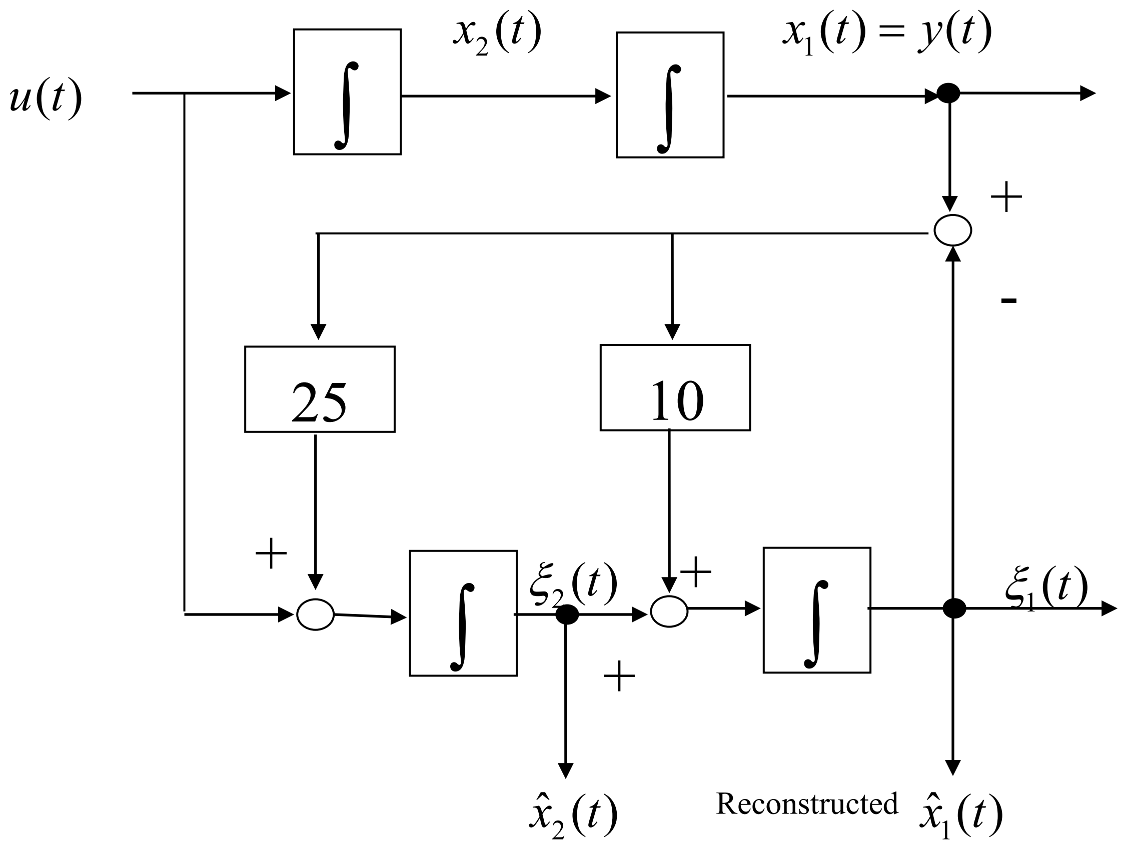


Figure 3 Observer design for Example 1.

The observer estimates both x_1 and x_2. But do we need to estimate x_1?

Note that in this case the observer is redundant in the sense that $x_1(t)$ is already available as $y(t)$ and so there is no need to 'reconstruct' it. In fact, only $x_2(t)$ needs to be reconstructed. This suggests that this observer only needs to be a first-order system, instead of the second-order one that was just designed. It is this observation that leads to the investigation of lower-order observers.

We will resolve this issue later.

An Industrial Application: the coupled-tank level estimation revisited

Model: The linearized model of the coupled-tank system at the operating point of $H_1 = 50\%, H_2 = 34\%$ is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.325 & 0.325 \\ 0.325 & -0.325 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$y = x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Observer: we need to estimate x_1 .

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left[y - C\hat{x}\right]$$

Observer design: The system is both controllable and observable. Let

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}, \qquad A_1 = A - LC = \begin{bmatrix} -0.325 & 0.325 - l_1 \\ 0.325 & -0.325 - l_2 \end{bmatrix}$$

The characteristic polynomial of the observer is readily seen to be

$$\det(\lambda I - (A - LC)) = \lambda^2 + (0.65 + l_2)\lambda + 0.325\left(l_2 + l_1\right)$$

We can choose the observer poles; that choice gives us values for l_1 and l_2.
If we want two poles at $\lambda = -1$, then $l_2 = 1.35$ and $l_1 = 1.727$
The designed observer is given by

$$\dot{\hat{x}}(t) = \begin{bmatrix} -0.325 & -1.402 \\ 0.325 & -1.675 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u(t) + \begin{bmatrix} 1.727 \\ 1.35 \end{bmatrix} y(t)$$

The estimator performance is shown in Figure 4 for the initial state of the process at $x(0)=\begin{bmatrix}0.5 & 0.34\end{bmatrix}^T$ and that of the observer set $\hat{x}(0)=\begin{bmatrix}0 & 0\end{bmatrix}^T$.

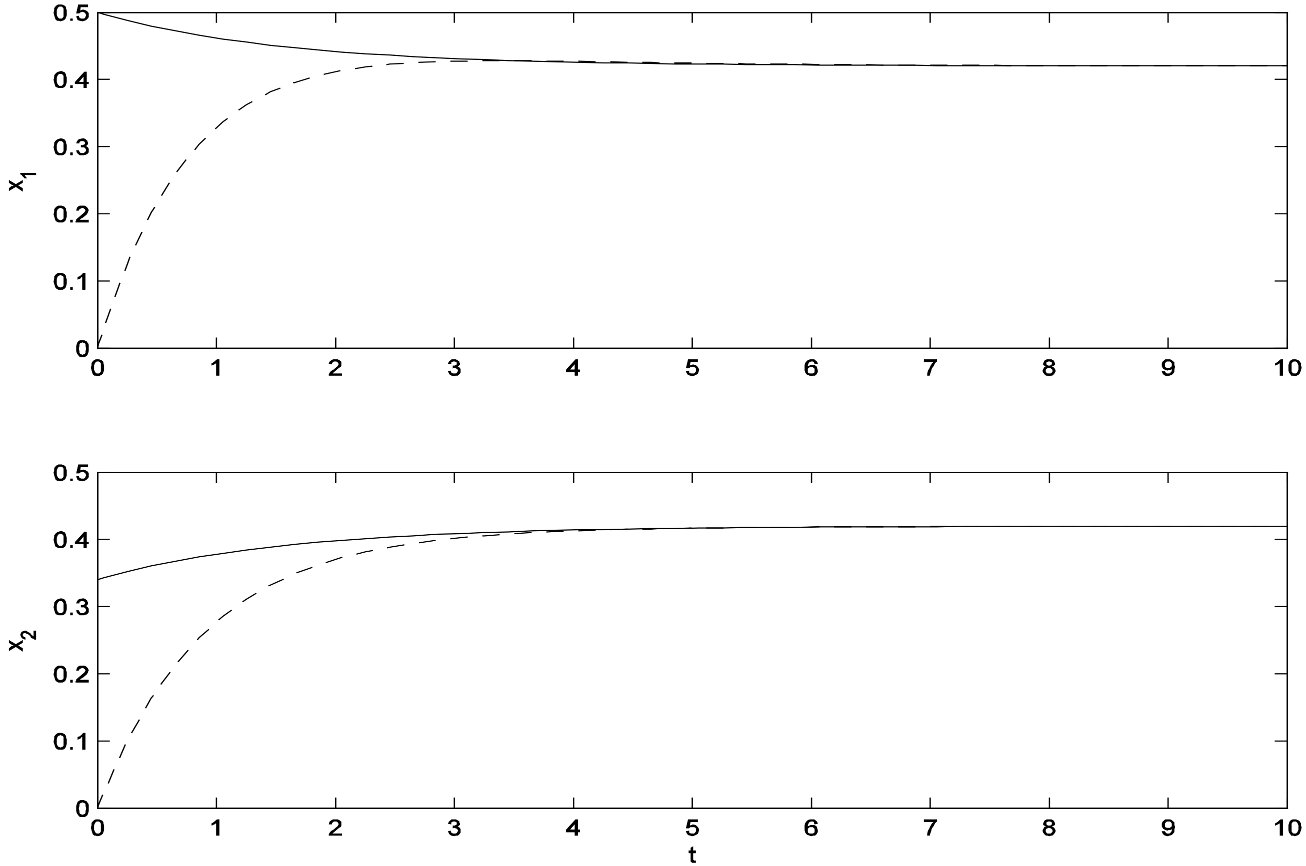


Figure 4 Observer performace for Tanks.
(solid-actual state variables; dash-estimated state variables)

Break

State-of-the-art control systems

Future Robots (40:50)

§11.3 Full Order Observer/Controller Combination

What happens when the state feedback control is implemented with $\hat{x}$ instead of x ?

It should be noted that the controller design assumes all the state variables are available, but the actual implementation uses only the estimated $\hat{x}(t)$. What is the overall performance of the resulting system with the observer compared with that with the true state feedback?

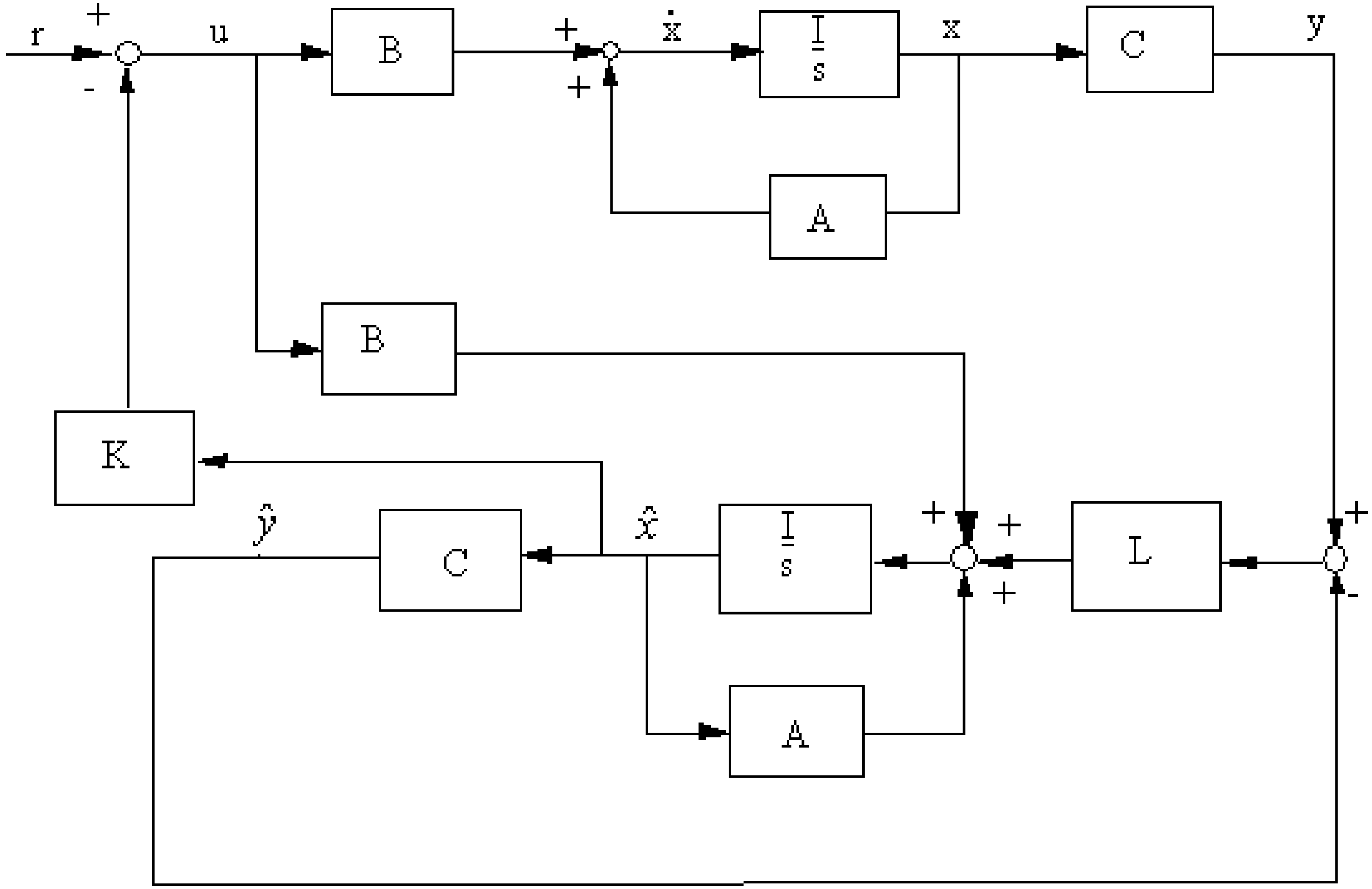


Figure 5 System of plant/observer/controller.

The observer control system has the plant:

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

the observer: $$\dot{\hat{x}} = A\hat{x} + Bu + L[y - C\hat{x}]$$

and the control law:

$$u = -K\hat{x} + r$$

It follows that

$$\dot{x} = Ax + B[r - K\hat{x}]$$
$$\dot{x} = Ax + B[r - K(x - \tilde{x})]$$
$$= (A - BK)x + BK\tilde{x} + Br \tag{7}$$

Also, we have the estimation error dynamics:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \tag{8}$$

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r, \qquad (9)$$

$$y = Cx.$$

The Laplace transform gives

$$sX(s) - x(0) = (A - BK)\,\mathrm{X(s)} + \mathrm{BK}\,\tilde{X}(s) + BR(s)$$

$$s\tilde{X}(s) - \tilde{x}(0) = (A - LC)\tilde{X}(s),$$

$$Y(s) = CX(s).$$

$$(sI - (A - BK))X(s) = x(0) + \mathrm{BK}\,\tilde{X}(s) + BR(s)$$

$$(sI - (A - LC))\tilde{X}(s) = \tilde{x}(0)$$

$$\begin{aligned} X(s) = {} & (sI - A + BK)^{-1}x(0) \\ & + (sI - A + BK)^{-1}BK(sI - A + LC)^{-1}\tilde{x}(0) \\ & + (sI - A + BK)^{-1}BR(s), \end{aligned}$$

$$\tilde{X}(s) = (sI - A + LC)^{-1}\tilde{x}(0),$$

and $$Y(s) = CX(s)$$

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r,$$
$$y = Cx.$$

$$X(s) = (sI - A + BK)^{-1} x(0) + (sI - A + BK)^{-1} BR(s)$$
$$+ (sI - A + BK)^{-1} BK(sI - A + LC)^{-1} \tilde{x}(0)$$

a) The closed-loop poles are the eigenvalues of (A-BK) and (A-LC). K and C are designed to make both (A-BK) and (A-LC) stable.

b) The transfer function matrix relating $X(s)$ (or $Y(s)$) to $R(s)$ is the same as obtained equivalently by using x as feedback.

c) There would have been no difference on performance at all if $\tilde{x}(0)$ were zero! Otherwise, the error in the initial state estimation, $\tilde{x}(0)$, propagates through the feedback loop until its effect goes to zero asymptotically.

The error in x with and without an observer is

$$X_E(s) = (sI - A + BK)^{-1} BK(sI - A + LC)^{-1} \tilde{x}_0$$

Example 2. To see the difference between the systems with x and $\hat{x}$,

Let the plant be

$$\dot{x} = 2x + 3u,$$

$$y = x,$$

with $r = 0$ and $x(0) = 1$.

The case of true state feedback: Suppose

$$u = -Kx\ ,$$

where K=2 will move the closed loop pole to – 4: $A - bK = 2 - 3\times 2 = -4$.

Then the closed-loop is described by

$$\dot{x} = 2x + 3(-2x) = -4x,$$

$$\dot{y} = -4y,\ y(0) = 1.$$

It yields the transient:

$$y(t) = e^{-4t}y(0) = e^{-4t} \tag{10}$$

Case of estimated state feedback: Use the observer:

$$\dot{\hat{x}} = 2\hat{x} + 3u + L(y - \hat{y})$$

where $L = 10$ will place the observer pole at -8. The control law becomes

$$u = -2\hat{x}.$$

Let $\hat{x}(0) = 0$. Then, $\tilde{x} = x(0) - \hat{x}(0) = 1$. The difference between the above two cases is

$$\begin{aligned} Y_E(s) &= C(sI - A + BK)^{-1} BK(sI - A + LC)^{-1} \tilde{x}(0) \\ &= 1 \cdot (s - 2 + 3 \times 2)^{-1} \cdot 3 \times 2 \cdot (s - 2 + 10 \times 1)^{-1} \cdot 1 \\ &= \frac{6}{(s+4)(s+8)} = \frac{1.5}{s+4} - \frac{1.5}{s+8}. \end{aligned}$$

We have

$$y_E(t) = 1.5e^{-4t} - 1.5e^{-8t} \tag{11}$$

So, the output with the observer is

$$y(t) = (10) + (11) = 2.5e^{-4t} - 1.5e^{-8t} .$$

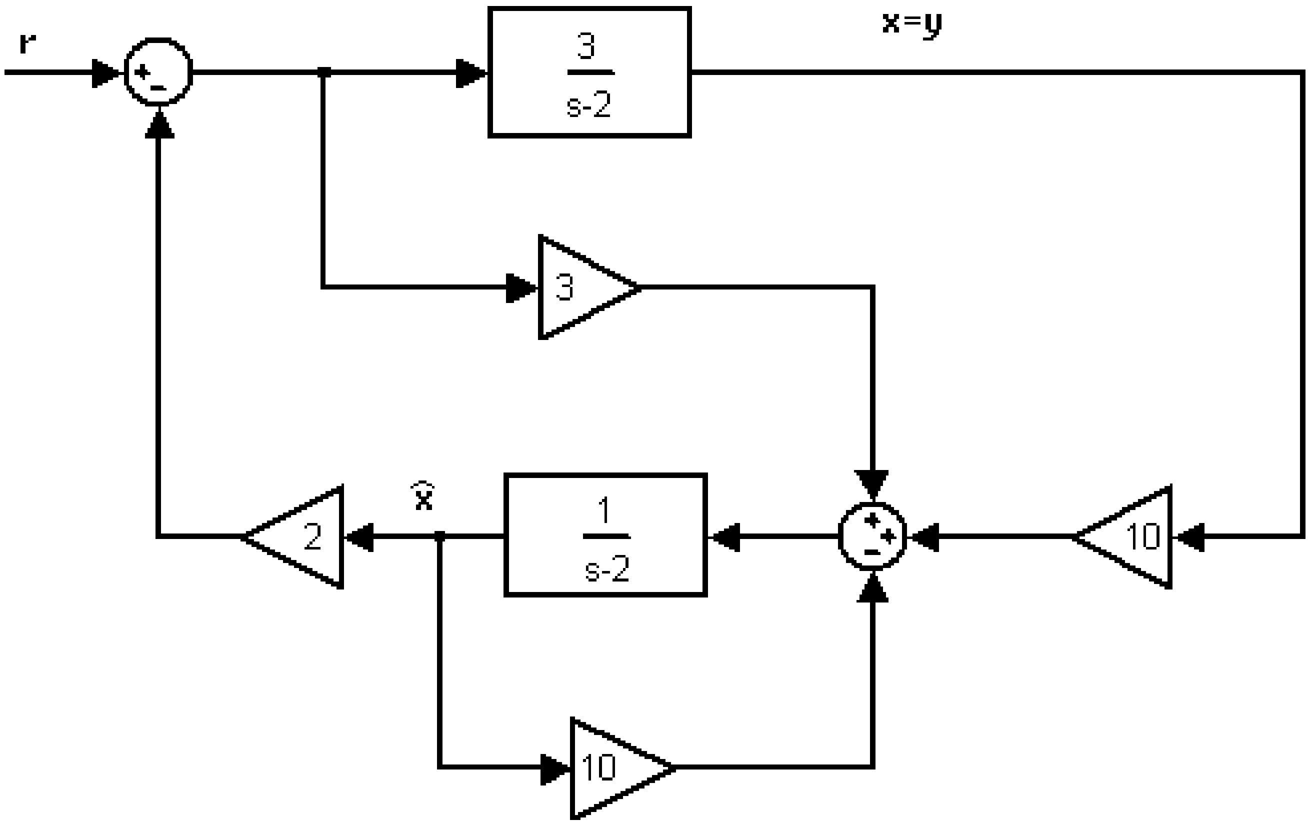


Figure 6 Oserver control system for Example 2.

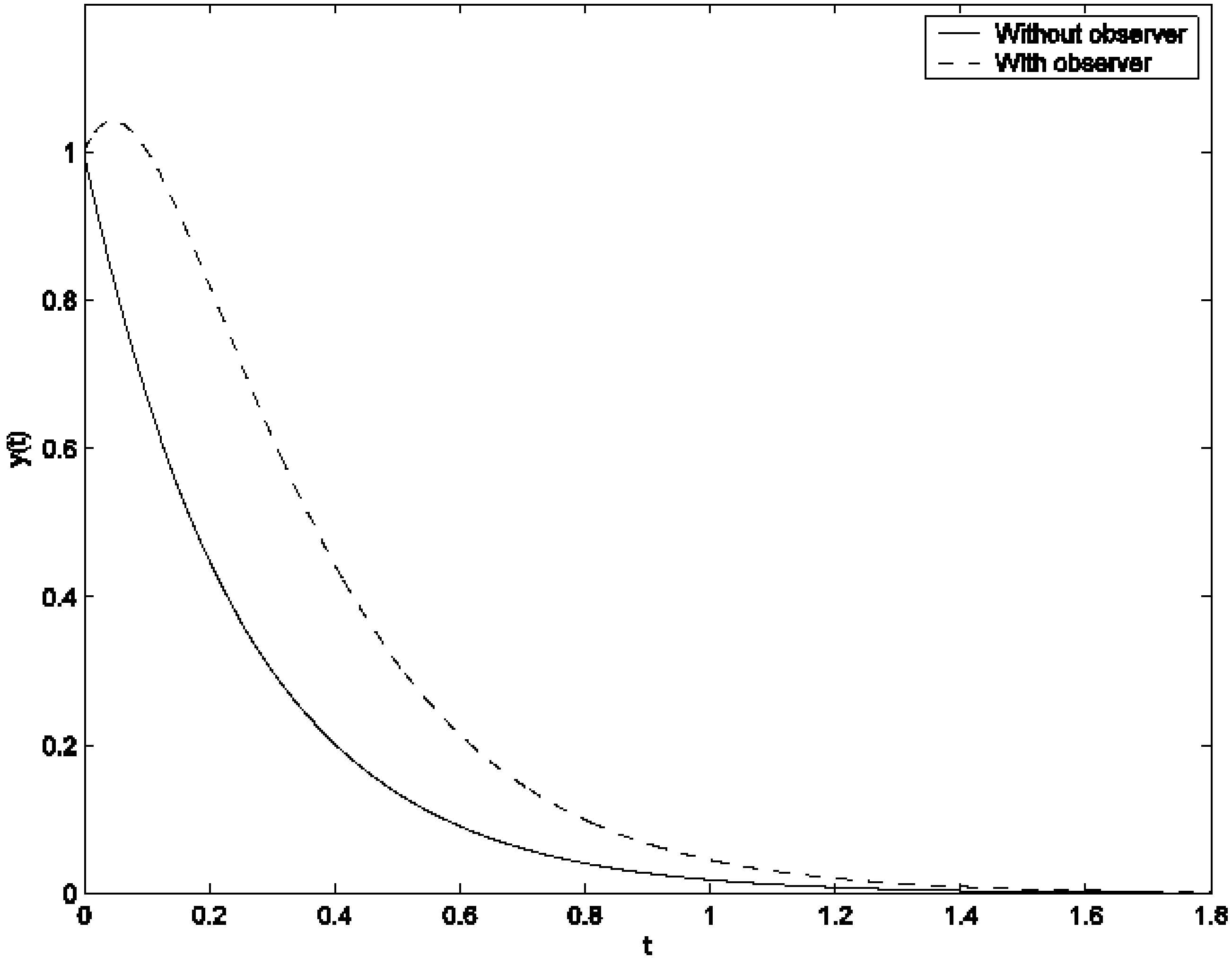


Figure 7 Control performance with and without observer.

In previous observer design examples, we notice that some state variables are already available from the output signal. Can we reduce the redundancy?

§11.4 Reduced-Order Observers

Idea: Look at $y = Cx$, where y is $m\times 1$, x $n\times 1$ and C $m\times n$. Suppose that C has rank of m. We want to take advantage of the m state variables that are available through y and construct an observer of order $n-m$, lower than n, to estimate the remaining (n-m) state variables.

Let $\xi = Tx$, where T is $(n-m)\times n$ with $(n-m)$ rows that are linearly independent of C. This is one of the keys to the design of a reduced-order observer.

Combining these two equations results in

$$\begin{bmatrix} y \\ \xi \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} x \qquad \begin{bmatrix} C \\ T \end{bmatrix} \text{ is of } n\times n \text{ and non-singular.} \qquad (12)$$

The problem becomes estimating the transformed state variables rather than the original ones!

There are infinite number of ways to transform the states, which can be used to our advantage.

The transformation T becomes the design variable in the observer!

The state can be obtained by inverting (12) as

$$x = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi \end{bmatrix} \tag{13}$$

This equation says that x is linear combination of plant's output and the observer's output.

In general, for a nth-order system with m outputs, an observer with n-m outputs is needed.

Observer Construction:

For the system:

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

Can we directly use the original model {A,B,C} as the model for the observer?

No. The model of the observer has to be constructed as the estimated states are the transformed states, not the original one.

We need to design the observer from scratch!

Construct an observer as

$$\dot{\xi} = D\xi + Eu + Gy \qquad (14)$$

such that

$$\xi \to Tx \qquad \text{as } t \text{ goes to infinity.} \qquad (15)$$

Let the estimation error

$$e = \xi - Tx$$

Objective: We need to construct the observer such that

$$[e = \xi - Tx] \to 0 \quad \text{As } \ t \to \infty$$

Let's use this goal to guide all the designs!

Let's find out the error dynamics,

$$\dot{e} = \dot{\xi} - T\dot{x}$$

since

$$\dot{\xi} = D\xi + Eu + Gy$$

$$T\dot{x} = TAx + TBu$$

We have

$$\dot{e} = D\xi + Eu + Gy - TAx - TBu \tag{16}$$

Since we are interested in $[e = \xi - Tx]$
we need to make e appear on the right side instead of ξ.

$$\dot{e} = D\xi + Eu + Gy - TAx - TBu$$

$$e = \xi - Tx \quad \Rightarrow \quad \xi = e + Tx$$

$$\dot{e} = D(e + Tx) + GCx + Eu - TAx - TBu$$

$$\dot{e} = De + (DT - TA + GC)x + (E - TB)u$$

How to make e → 0?

First of all, how to design D?

We can choose a stable D, but what else?

$$DT - TA + GC = 0,$$

$$E - TB = 0.$$

Such that $\dot{e} = De$

Therefore we need to design T,D,G and E together.

The natural way seems to choose the transformation T first, and then design D,G and E.

But it turns out that it is much easier to choose a stable D first and then design T and G from the equation

$$DT - TA + GC = 0$$

Design: Determine T, D, G and E as follows.

(i) Find the constraints on T such that $\begin{bmatrix} C \\ T \end{bmatrix}$ is non-singular; but do not choose T directly at this step.
Using this freedom is the key to success!

(ii) Choose D such that its eigenvalues have negative real parts, or desired decay rates;

(iii) Solve $DT - TA + GC = 0$ for T and G; and

(iv) Calculate E=TB.

Step 2 and 3 are key steps, as step 4 is trivial.

Finally, one needs to reconstruct the state variables from the plant's output and the observer's output based on (13):

$$\hat{x} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi \end{bmatrix} \tag{18}$$

Example 3 Design a first-order observer for the plant described in Example 1:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Only $x_2(t)$ needs to be reconstructed since $x_1(t) = y(t)$. The observer is of first-order and given by

$$\dot{\xi} = d\xi + eu + gy$$

where d, e and g are scalars.

(i) Find the constraints on T such that $\begin{bmatrix} C \\ T \end{bmatrix}$ is non-singular;

$T = \begin{bmatrix} t_1 & t_2 \end{bmatrix}$ is such that the matrix $\begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$ must be nonsingular.

This is the case iff $t_2 \neq 0$.

(ii) Choose $d = -3$ (or other desired decay rate).

(iii) Design T and g from $dT - TA + gC = 0$, or

$$
\begin{array}{ccccc}
TA & - & dT & = & gC \\
\begin{bmatrix} t_1 & t_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & + & 3\begin{bmatrix} t_1 & t_2 \end{bmatrix} & = & g\begin{bmatrix} 1 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & t_1 \end{bmatrix} & + & \begin{bmatrix} 3t_1 & 3t_2 \end{bmatrix} & = & \begin{bmatrix} g & 0 \end{bmatrix}
\end{array}
$$

$$
g = 3t_1 \qquad\qquad t_1 + 3t_2 = 0
$$

How many solutions can you find? Infinity!

This is not a surprise as there are infinite number of ways to transform the states!

What would happen if we design T first?

(i) $T=\begin{bmatrix} t_1 & t_2 \end{bmatrix}$ is such that the matrix $\begin{bmatrix} C \\ T \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$ must be nonsingular.

It seems that the most natural way to choose T is [0 1] such that the estimate ξ corresponds to state variable x_2 directly.

(iii) Then let's design d and g from $dT-TA+gC=0$, or

$$\begin{array}{ccccc} TA & - & dT & = & gC \\ [0 \quad 1]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & - & d[0 \quad 1] & = & g[1 \quad 0] \\ [0 \quad 0] & - & [0 \quad d] & = & [g \quad 0] \end{array}$$

How many stable solutions can you find?

d=g=0. No stable solution exists!

We cannot choose T in the most natural way!

We'd better keep the transformation T as a design variable to suit our needs rather than fix it.

We need to find out g and T from

$$g = 3t_1 \qquad\qquad t_1 + 3t_2 = 0$$

Try $g = 1$, and solve for t_1 and t_2 and check the rank of $\begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$.

This yields $t_1 = \frac{1}{3}$, $t_2 = -\frac{1}{9}$ and the transformation matrix will have full rank.

One then calculates

$$e = TB = \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = t_2 = -\frac{1}{9}$$

The observer is thus determined as

$$\dot{\xi} = -3\xi - \frac{1}{9}u + y$$

One then reconstructs the state variables from the plant's output and the observer's output from (18) as

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & -\frac{1}{9} \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix}$$

Or

$$\begin{aligned} \hat{x}_1(t) &= y(t) = x_1, \\ \hat{x}_2(t) &= 3y(t) - 9\xi(t). \end{aligned}$$

The plant and the first-order observer are shown in Figure 8.

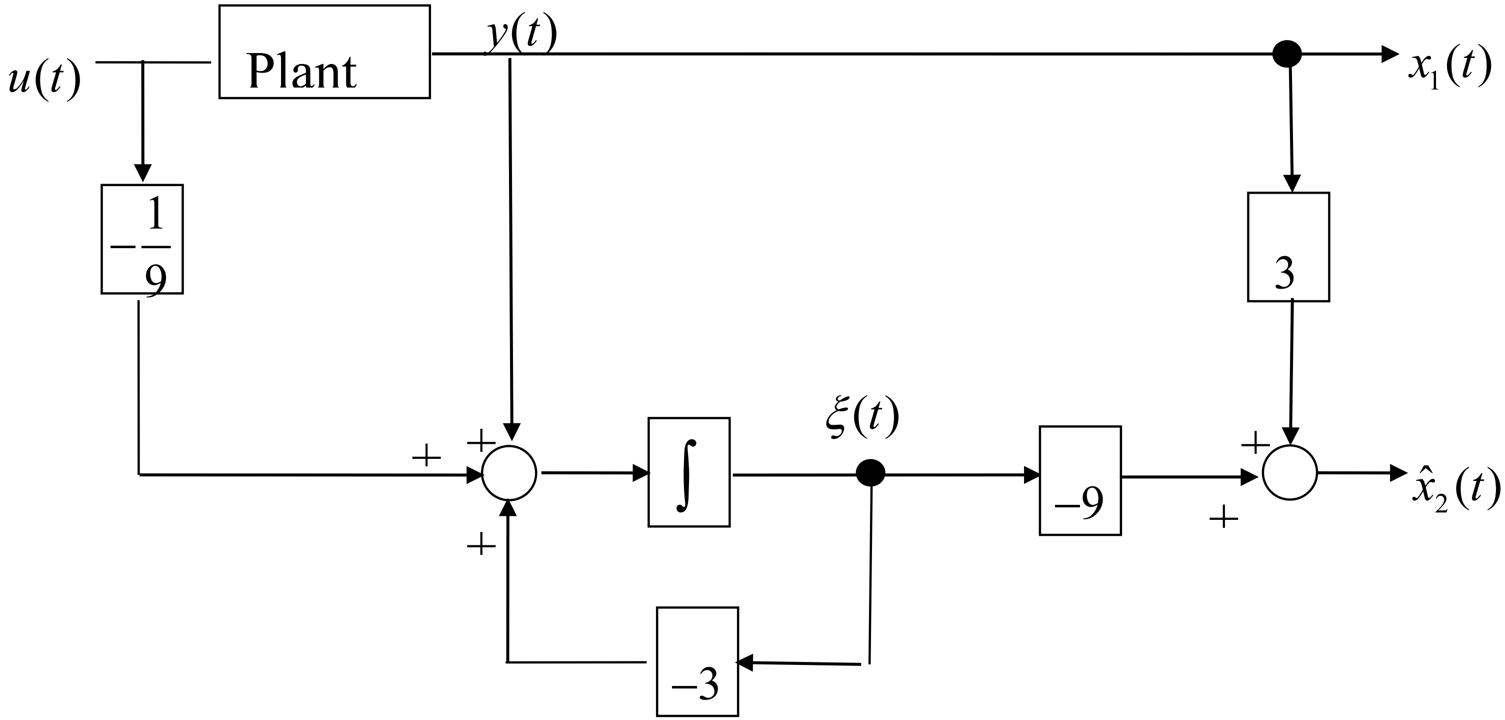


Figure 8 Reduced-order observer for Example 3.

Example 4 Design an observer of minimal order for the system with

$$A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\ \mathrm{B}=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \mathrm{C}=\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}$$

A check of $\begin{bmatrix}C^T & A^TC^T & (A^T)^2C^T\end{bmatrix}$ reveals that the system is observable.

How many state variables do we need to estimate?

Only one. Since the dimenstion of y is 2, we can reconstruct all three state variables of this system by a first-order observer,

$$\dot{\xi}=d\xi+eu+Gy$$

where d and e are scalars and $G=\begin{bmatrix}g_1 & g_2\end{bmatrix}$.

Let $T=\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}$ We have

$$\begin{bmatrix}y\\\xi\end{bmatrix}=\begin{bmatrix}C\\T\end{bmatrix}x=\begin{bmatrix}1&0&0\\1&1&0\\t_1&t_2&t_3\end{bmatrix}x \tag{19}$$

The rank of $\begin{bmatrix} C \\ T \end{bmatrix}$ must be 3, or

$$\det\begin{bmatrix} C \\ T \end{bmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ t_1 & t_2 & t_3 \end{vmatrix} = t_3 \neq 0$$

Choose the eigenvalue of the observer as $d = -2$.

The main design equation is

$$TA - dT = GC \tag{20}$$

$$\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - d\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = \begin{bmatrix} g_1 & g_2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} t_3 & t_1 & t_2 \end{bmatrix} + 2\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = \begin{bmatrix} g_1 + g_2 & g_2 & 0 \end{bmatrix}$$

This results in

$$
\begin{aligned}
&t_3 + 2t_1 = g_1 + g_2,\\
&t_1 + 2t_2 = g_2,\\
&t_2 + 2t_3 = 0,\ t_3 \neq 0.
\end{aligned}
\tag{21}
$$

How many solutions? Infinity!

Choosing $g_1 = 10$ and $g_2 = 1$, the solution is $t_1 = 5$, $t_2 = -2$, and $t_3 = 1$, which satisfies $t_3 \neq 0$.

One then gets

$$
e = TB = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -2
$$

The observer becomes

$$
\dot{\xi} = -2\xi - 2u + \begin{bmatrix} 10 & 1 \end{bmatrix} y
$$

The reconstructed state variables are now formed from combination of the system output and the observer output as

$$\hat{x} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -7 & 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ \xi(t) \end{bmatrix}$$

or

$$\begin{aligned} \hat{x}_1(t) &= y_1(t) = x_1(t) \\ \hat{x}_2(t) &= -y_1(t) + y_2(t) = x_2(t) \\ \hat{x}_3(t) &= -7y_1(t) + 2y_2(t) + \xi(t) \end{aligned}$$

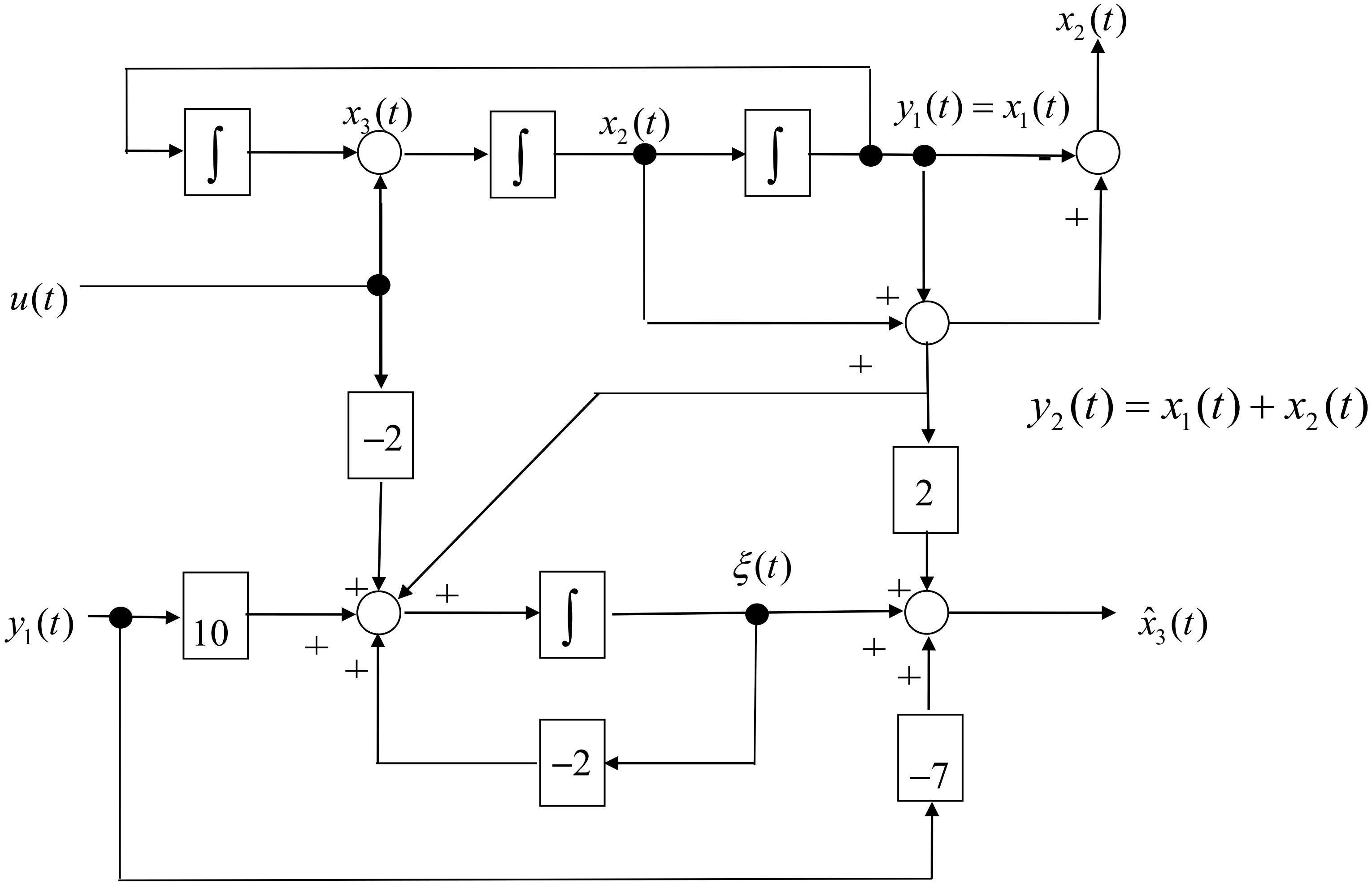


Figure 9 Reduced-order observer for Example 4.

Relationship to full order observer: For now, assume that we make no use of outputs in (12). Then, T will be of $n \times n$ and can be taken as identity matrix *I*. This will lead to the full-order observer discussed before:

- $\xi = Tx = x$ produces:

$$x = T^{-1}\xi = \xi \tag{22}$$

- The design equations:

$$DT - TA + GC = 0$$
$$E - TB = 0$$

become

$$\begin{aligned} D &= A - GC \\ E &= B \end{aligned} \tag{23}$$

The eigenvalues of D should have negative real parts. Since A and C are known and observable, the matrix G can be used to specify the eigenvalues of D.

- The observer:

$$\dot{\xi} = D\xi + Eu + Gy$$

becomes

$$\dot{\xi} = (A - GC)\xi + Bu + Gy,$$

So the reduced order observer becomes the standard full order observer if we force T to be identity matrix and do not use any outputs as the state estimates directly.

State Estimation

- Reasons
- Solvability
- Methods

In particular, we showed that

- The condition for building the observer
- Two methods for constructing the observer
- The stability proof of the overall system

When you design your control system with state space model, you can always follow the following design procedure:

Step One: Assume all the state variables are available, build the state feedback controller. No need to worry whether you can buy all the sensors!

Step Two: Decide the type of sensors to monitor the system, and build observer to estimate those state variables without any sensors. If cost is really critical, then just use the cheapest sensor!

Step Three: Replace those state variables with their estimates and implement the controller.

Now you are ready to complete the whole mini-project!

Q & A…

THANK YOU !