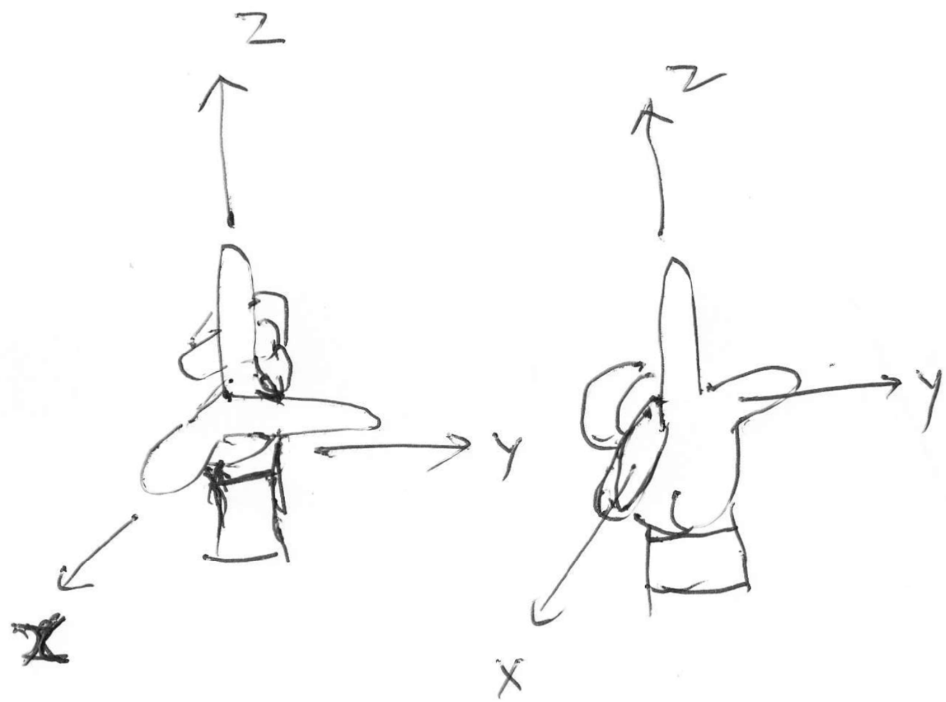
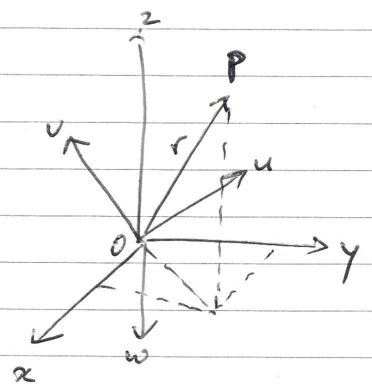
Right hand coordinate system


$x \cdot y = z$



# Pre and Post multiplication in Rotations.

Date No.




Ouvw : robot coordinate system (object)

Oxyz : reference coordinate system

$r = [r_x \; r_y \; r_z] = [r_u \; r_v \; r_w]$

Object may rotate about the principal axis of either Oxyz frame or Ouvw frame, order of rotation is always important.

composite rotation $R_1 \neq R_2$

$$R(y,\alpha)\,R(z,\theta)\,R(x,\beta) \neq R(x,\beta)\,R(z,\theta)\,R(y,\alpha)$$

Oxyz and Ouvw must coincide initially for multiplication rule to apply $\Rightarrow R = I$

Apply appropriate transformation to make them coincide if they are not

If frame Ouvw rotates about Oxyz, then "pre-multiply" the I matrix by appropriate rotation matrix

If frame Ouvw rotates about itself, then "post-multiply" the I matrix by appropriate rotation matrix.

Angle-set conventions — 24 conventions that perform 3 rotations about principal axes in a certain order.
12 – Euler-angle sets, 12 – fixed-angles sets.
Duality of fixed-angle sets w Euler-angle sets ⇒ 12 unique parameterizations.

APPENDIX B

# The 24 angle-set conventions

The 12 Euler angle sets are given by

' : Euler angle

$$R_{X'Y'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{X'Z'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Y'X'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{Y'Z'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Z'X'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix},$$

$$R_{Z'Y'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix},$$

$$R_{X'Y'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$R_{X'Z'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Y'X'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$



Source: Introduction to Robotics: Mechanics and Control, by John J. Craig, 2005.

$$R_{Y'Z'Y'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Z'X'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix},$$

$$R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.$$

The 12 fixed angle sets are given by

$$R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix},$$

$$R_{XZY}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{YXZ}(\gamma,\beta,\alpha) = \begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix},$$

$$R_{YZX}(\gamma,\beta,\alpha) = \begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{ZXY}(\gamma,\beta,\alpha) = \begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{ZYX}(\gamma,\beta,\alpha) = \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{XYX}(\gamma,\beta,\alpha) = \begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$R_{XZX}(\gamma,\beta,\alpha) = \begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{YXY}(\gamma,\beta,\alpha) = \begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

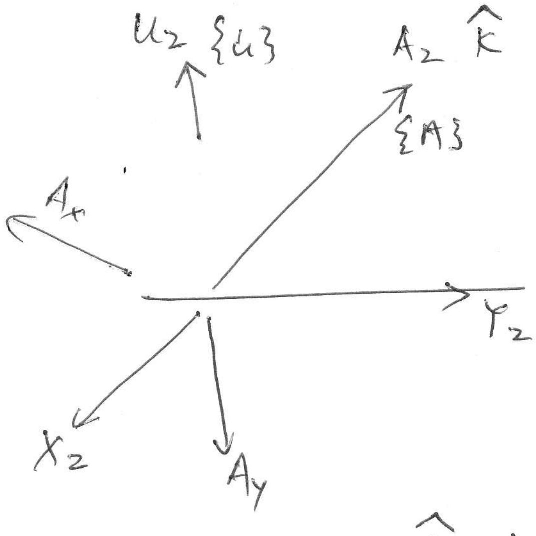
$$R_{YZY}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{ZXZ}(\gamma,\beta,\alpha)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix},$$

$$R_{ZYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.$$

## Equivalent angle-axis representation

Image a frame {A} whose $\hat{Z}$ axis is aligned with the direction $\hat{K}$:




Rotation which rotates vectors about $\hat{K}$ by $\theta$ degrees:

$$R = {}^{U}_{A}R \; rot({}^{A}\hat{Z}, \theta)\; {}^{A}_{U}R \qquad \text{—— (1)}$$

Description of {A} in {U}:

$$ {}^{U}_{A}R = \begin{bmatrix} A & D & K_x \\ B & E & K_y \\ C & F & K_z \end{bmatrix} $$

If we multiply out Eq (1), and simplify using

$$A^2 + B^2 + C^2 = 1, \quad D^2 + E^2 + F^2 = 1, \quad [A \; B \; C] \cdot [D \; E \; F] = 0$$

$$[A \; B \; C] \otimes [D \; E \; F] = [K_x \; K_y \; K_z]$$

$$R_K(\theta) = \begin{bmatrix} k_x k_y v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_y v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, ${}^{A}\hat{K} = [k_x \; k_y \; k_z]^T$

Source: R P Paul, Robot Manipulators, MIT Press, Cambridge, MA, 1981 (page 25).