EE6104/5104

Advanced/Adaptive Control Systems

Part I

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References

1. K.J. Astrom and B. Wittenmark, Adaptive Control, Addison Wesley, 1989 (1995).
2. K.S. Narendra and A.M. Annaswamy, Stable Adaptive Systems, Prentice-Hall, 1989.
3. G.C. Goodwin and K.S. Sin, Adaptive Filtering, Prediction and Control, Prentice, 1984.

Contents / Broad Outline

No. of Hours

1 **Introduction** 3 THL

Adaptive schemes. Adaptive control theory. Applications.

Situations when constant Gain feedback is insufficient.

Robust control. The adaptive control problem.

2 **Model Reference Adaptive Systems** 8 THL

The model following problem. MRAS based on stability theory. Model following when the full state is measurable. Direct MRAS for general linear systems. Prior knowledge in MRAS. MRAS for partially known systems. Use of robust estimation methods in MRAS.

3 **Self-Tuning Regulators** 8 THL

The basic idea. Indirect self-tuning regulators. Direct Self-tuning regulators. Linear Quadratic STR. Adaptive Predictive control. Prior knowledge in STR.

4 **Real-Time Parameter Estimation** 8 THL/WKH

Linear-in-the-parameters model. Least squares estimation. Experimental conditions. Recursive estimators. Extended least squares. Robust estimation methods (dead zone, projection). Implementation issues.

5 **Auto-Tuning** 3 WKH

PID Control. Transient response methods. Methods based on relay feedback. Relay oscillations.

6 **Gain Scheduling** 3 WKH

The Principle. Design of Gain-scheduling Regulators. Nonlinear Transformations. Applications of Gain-scheduling.

7 **Alternatives to Adaptive Control** 3 WKH

Situations where adaptive control should not be used. Robust high gain feedback control. Self-oscillating adaptive systems. Variable structure systems.

CA – 70% of module grade — 3 assignments (of which 1 is a hardware mini-project)

Exam – 30% of module grade

Pre-requisites for this course

- Linear Systems Theory
    1. Controllability, Observability
    2. General Observer Theory
- Digital Techniques
    1. Digital filter design and implementation considerations
    2. Sampling considerations
    3. Implementation of compensators, issues of finite word length coefficient representation, and rounding errors

The course will assume that you are already very familiar with these topics.

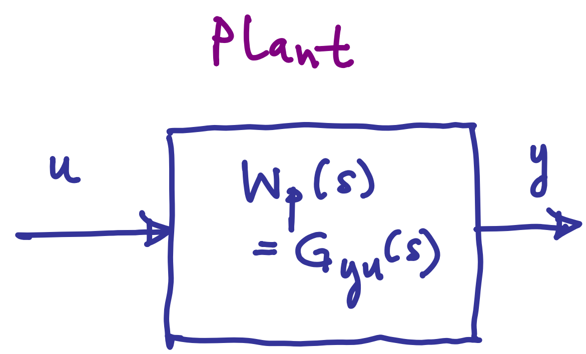
* Year 1 graduate students, &
  Exchange students — Highly recommended that you take EE5101 Linear Systems, or at least are sitting in!!

* Year 2 and above graduate students
  — Should be OK. But still good to check yourself, & review.

A simple adaptive control problem

(Narendra & Annaswamy, pp.111-114)
(Astrom & Wittenmark, pp.126-127)

Consider a system to be controlled



$$y = G_{yu} u$$

with transfer function

$$G_{yu}(s) = \frac{k_p}{s - a_p}$$

In time domain

$$(p - a_p) y(t) = k_p u(t)$$

This is the system we need to control!

where we use the notation

$$p \equiv \frac{d}{dt}$$

for the differential operator.

Thus

$$\dot{y}(t) = a_p y(t) + k_p u(t) \tag{1}$$

We usually refer to the system to be controlled as the "plant".

This is the "closed-loop" we want to achieve!

Control Objective:

- Drive y(t) to follow some reference command trajectory $y_m(t)$

For example, $y_m(t)$ may be generated by a reference system/model

$$\dot{y}_m(t) = a_m y_m(t) + k_m r(t)$$

$a_m < 0$, i.e. stable reference model.

Note: Review your Linear Systems & State-Variables. It is important that it is understood clearly that this condition $a_m < 0$ is a natural reasonable one!! Equivalent, more general conditions will appear in later, more difficult cases!!

Consider a control input of the form

$$u(t) = \theta^* y(t) + k^* r(t)$$

Then, the plant together with control (the closed-loop system) is

$$\dot{y}(t) = a_p y(t) + k_p \{\theta^* y(t) + k^* r(t)\}$$
$$= (a_p + k_p \theta^*) y(t) + k_p k^* r(t)$$

We want it to match the Ref Model

$$\dot{y}_m = a_m y_m + k_m r$$

If

$$\boxed{\theta^* = \frac{a_m - a_p}{k_p} \quad \text{and} \quad k^* = \frac{k_m}{k_p}}$$

"Matching Conditions" [Clearly valid here!]

then

$$\dot{y} = a_m y + k_m r$$

and the plant together with the control will match the reference model exactly because

$$e = y_{\cancel{m}} - y_{m}, \quad \dot{e} = a_m e$$

$$\boxed{e = y - y_m}$$

Tracking error

PROBLEM: We do not know a_p and k_p!

[Note: Also OK if we consider $\bar{e} = y_m - y$. Minor "details" change. Check for yourself!! :) :)]

Adaptive Control: Find a way to achieve this automatically.

- Estimate a_p and k_p on-line, and try to use these estimates; (Will work most of the time); and
- Use a rigorous procedure to "evolve" to the right values (Very powerful; but applies only to restricted classes).

Plant :

$$\dot{y}(t) = a_p\, y(t) + k_p\, u(t)$$

Reference Model :

$$\dot{y}_m(t) = a_m\, y_m(t) + k_m\, r(t)$$

Consider the

Tracking Error :

$$e(t) = y(t) - y_m(t)$$

In the course we will look at both types of approaches.

For now, let us consider the rigorous approach. :

Since θ^*, k^* not known, consider instead

$$\boxed{u(t) = \theta(t)y(t) + k(t)r(t)},$$ **Time-varying controller gains!**

"Control Law"

I.e. no longer linear-time-invariant (LTI) system!

How to choose $\{\theta(t), k(t), t \in [0, \infty)\}$?

"Adaptive law"

$$\dot{\theta}(t) = -\mathrm{sgn}(k_p)\gamma_1 e(t)y(t); \qquad \gamma_1 > 0$$

$$\dot{k}(t) = -\mathrm{sgn}(k_p)\gamma_2 e(t)r(t); \qquad \gamma_2 > 0$$

(Nonlinear updating mechanisms!)

Thus, adaptive controllers contain both time-varying and nonlinear dynamics!!

i.e. NLTV !!

Because of the NLTV characteristics, analysis and design usually difficult.

For this system, consider the quadratic form (recall your 3rd year mathematics!)

$$V(e,\phi,\psi)=\frac{1}{2}\left[e^2+|k_p|\left(\gamma_1^{-1}\phi^2+\gamma_2^{-1}\psi^2\right)\right]$$

$$= \frac{1}{2}[e \;\; \phi \;\; \psi]\begin{bmatrix} 1 & 0 & 0 \\ 0 & |k_p|\gamma_1^{-1} & 0 \\ 0 & 0 & |k_p|\gamma_2^{-1} \end{bmatrix}\begin{bmatrix} e \\ \phi \\ \psi \end{bmatrix}$$

with

$$\underbrace{e=y-y_m,}_{\text{tracking error}} \qquad \underbrace{\phi=\theta-\theta^*, \qquad \psi=k-k^*}_{\text{parameter errors}}$$

Re-membering that θ^* and k^* are constants

$$\dot{\phi}=\dot{\theta}-\dot{\theta}^*=-\text{sgn}(k_p)\gamma_1 ey$$

$$\dot{\psi}=\dot{k}-\dot{k}^*=-\text{sgn}(k_p)\gamma_2 er$$

Note that

"Reference Model:"

$$\boxed{\dot{y}_m = a_m y_m + k_m r}$$

"Plant:"

$$\boxed{\dot{y} = a_p y + k_p u} = a_p y + k_p \theta y + k_p k r$$
$$= a_p y + k_p(\phi + \theta^*)y + k_p(\psi + k^*)r$$
$$= (a_p + k_p\theta^*)y + k_p\phi y + k_p\psi r + k_p k^* r$$

we have

$$\boxed{\dot{e} = \dot{y} - \dot{y}_m = a_m e + k_p\phi y + k_p\psi r}$$

Error Signal Dynamics / Error Eqn

Consider derivative of quadratic form V

$$\dot{V} = e\dot{e} + |k_p|\left[\gamma_1^{-1}\phi\dot{\phi} + \gamma_2^{-1}\psi\dot{\psi}\right]$$
$$= a_m e^2 + k_p\phi e y + k_p\psi e r - |k_p|\left[\operatorname{sgn}(k_p)\phi e y + \operatorname{sgn}(k_p)\psi e r\right]$$
$$= a_m e^2 \le 0 \quad \because a_m < 0$$

Collecting together the intermediate results:

- Quadratic form $V(e,\phi,\psi) = \frac{1}{2}\left[e^2 + |k_p|\left(\gamma_1^{-1}\phi^2 + \gamma_2^{-1}\psi^2\right)\right]$

  results in $\dot{V} = a_m e^2 \le 0$

- This means that
  - $V(e,\phi,\psi)$ is bounded for all t
  - e^2 is bounded, ϕ^2 is bounded, and ψ^2 is bounded

- $\phi(t) \overset{\Delta}{=} \theta(t) - \theta^*$

  $\psi(t) \overset{\Delta}{=} k(t) - k^*$

  $\therefore$ **θ** (t) and k(t) are bounded

- $e(t) \overset{\Delta}{=} y(t) - y_m(t)$

  $y_m(t)$ is a reference signal, and obviously bounded

  $\therefore$ y(t) is bounded

- V(e, ϕ,ψ) positive definite

  $\dot{V} = a_m e^2 \le 0$

  $\therefore$ V is bounded over all time

$$\int_0^t \dot{V}(e(t), \phi(t), \psi(t)) d\tau = \int_0^t a_m e^2(\tau) d\tau$$

$$\Rightarrow -a_m \int_0^t e^2(\tau) d\tau = V(e(0), \phi(0), \psi(0)) - V(e(t), \phi(t), \psi(t))$$

i.e. there exists c_1 such that

$$\int_0^t e^2(\tau) d\tau \le c_1 \quad \text{for all } t$$

Since $V\left(e(t),\phi(t),\psi(t)\right)$ is always bounded,

$$\lim_{t\to\infty}\int_0^t e^2(\tau)d\tau \le c_1 \quad \text{a constant}$$

■ $\dot{e} = a_m e + k_p \phi y + k_p \psi r$

$\therefore$ $\dot{e}(t)$ is bounded ∀t

■ $$\left.\begin{array}{l} \dot{e} \quad \textit{bounded} \\ \int_0^\infty e^2(\tau)d\tau \le c_1 \end{array}\right\} \Rightarrow \lim_{t\to\infty} e(t) = 0$$

i.e. $\underline{\underline{\lim_{t\to\infty}\left\{y(t) - y_m(t)\right\} = 0}}$

The adaptive controller achieves the designed objective! !!

Rigorous analysis of adaptive control is typically as difficult.

Summary

Plant: $\dot{y}(t) = a_p y(t) + k_p u(t)$

As given.

Reference Model: $\dot{y}_m(t) = a_m y_m(t) + k_m r(t); \quad a_m < 0$

Control Law: $u(t) = \theta(t)y(t) + k(t)r(t)$, Time varying

Adaptive Law:

$$\dot{\theta}(t) = -\mathrm{sgn}(k_p)\gamma_1 e(t)y(t)$$

$$\dot{k}(t) = -\mathrm{sgn}(k_p)\gamma_2 e(t)r(t)$$

$$e(t) = y(t) - y_m(t)$$

$\theta(0) = k(0) = 0$, Arbitrary starting gains

User sets this up, with necessary choices.

Result:

If the adaptive controller is applied to the plant, then all signals $\{y, u, \theta, k\}$ are bounded and $\lim_{t\to\infty}[y(t) - y_m(t)] = 0$

Note: "Matching Condition" to be previously checked as valid by user.

A user-friendly picture

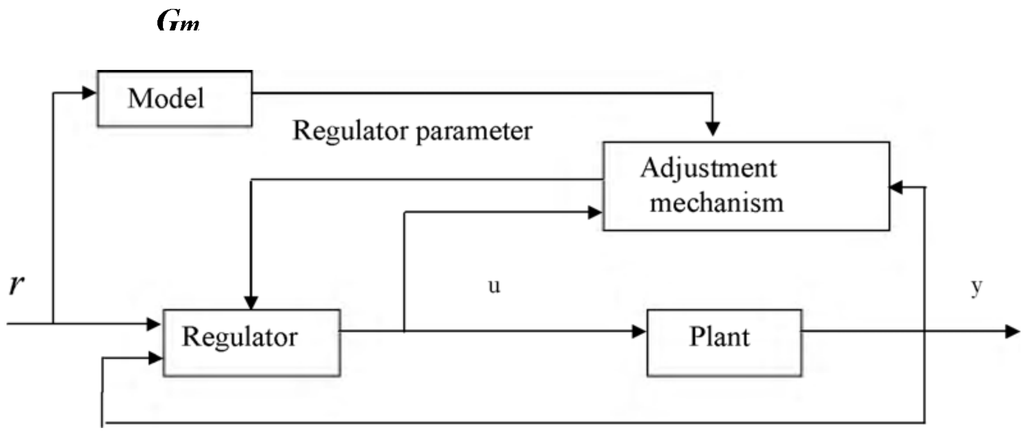


Figure 4.1 Block diagram of a model-reference adaptive system (MRAS)

Lyapunov's Direct Method

- Enables one to determine whether or not the equilibrium state of system

$$\dot{x} = f(x,t) \qquad x \in R^n \tag{2.1}$$

is stable without actually determining the solution $x(t;x_0,t_0)$.

- Involves finding a suitable scalar function V(x,t) and examining its time derivative $\dot{V}(x,t)$ along the trajectory of the system.

We will only need one result from this method, which we will state without proof.

<u>Theorem (Uniform Stability)</u>

Assume that a scalar function V(x,t), with continuous first partial derivatives w.r.t x and t, exists and that V(x,t) satisfies the following conditions:

(i) V(x,t) is positive-definite, i.e., $\exists$ a continuous non-decreasing scalar function α such that $\alpha(0)=0$ and $V(x,t) \geq \alpha(\|x\|) > 0 \quad \forall t$ and all $x \neq 0$;

(ii) V(x,t) is decrescent, i.e., $\exists$ a continuous non-decreasing scalar function such that $\beta(0)=0$ and $\beta(\|x\|)\geq V(x,t)$ for all t;

(iii) $\dot{V}(x,t)$ is negative semi-definite, i.e.,

$$\dot{V}(x,t)=\frac{\partial V}{\partial t}+(\nabla V)^T f(x,t)\leq 0$$

(iv) V(x,t) is radial unbounded, i.e.,
$\alpha(\|x\|)\to\infty$ as $\|x\|\to\infty$

then

(a) (i) and (iii) imply that the origin of differential equation (2-1) is stable;

(b) (i), (ii) and (iii) imply the origin of d.e.(2-1) is uniformly stable; (w.r.t. t)

(c) (i) - (iv) imply that the origin of d.e. (2-1) is uniformly stable in the large. i.e. all of $\mathbb{R}^n$

LTI System and Lyapunov Stability

Consider

$$\dot{x} = Ax; \qquad x(t_0) = x_0 \qquad x \in \mathbb{R}^n$$

Theorem 2-2 (Narendra ....)

All solutions of the above equation tend to zero as $t \to \infty$ if and only if all the eigenvalues of A have negative real parts,

Such a matrix A is referred to as an asympototically stable matrix.

Theorem 2-10 (Narendra, ...)

The equilibrium state x=0 of the LTI system

$$\dot{x} = Ax$$

is asymptotically stable if and only if, given any symmetric positive definite matrix Q, there exists a symmetric positive definite matrix P which is the unique solution of the $\frac{n(n+1)}{2}$ linear equations

$$A^T P + PA = -Q$$

Proof: *See Narendra & Annaswamy pp.60.*

Stability theory, in general, is a very wide field.

- See Narendra &Annaswamy, Chap.2 for a more comprehensive overview.

- We have only summarized the concepts and results that we need for adaptive control.

Adaptive control of plants with all state-variables measurable

(A similar treatment can be found in Narendra and Annaswamy pp.128 ff. However, note that some aspects are different.)

Assume that the plant is described by

g is a scalar

$$\dot{x}_p = A_p x_p + gbu$$

$$x_p \in \Re^n, \quad u \in \Re^1$$

Note: $x_p \in \mathbb{R}^n$ is measurable!

A_p is $(n \times n)$, b is $(n \times 1)$

b is known

This is possible in many situations.

Example 2.1: **b is known**

$$y = G_{yu} u$$

$$G_{yu}(s) = \frac{g}{s^2 + a_1 s + a_2}$$

Assume that position and velocity measurements are available.

Then, a suitable state-variable description is

$$x_1 = y$$
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \dot{y} = -a_1 x_2 - a_2 x_1 + gu$$

for y, $\dot{y}$ measurable

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{x}_p = A_p x_p + g b u$$

Thus $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is known exactly.

and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a measurable state-vector.

Consider plant:

$$\dot{x}_p = A_p x_p + gbu \quad (1)$$

Non-adaptive case:

$$u = \theta_x^{*T} x_p + \theta_r^* r \quad (2)$$

$$\theta_x^* \overset{\Delta}{=} \begin{bmatrix} \theta_1^* & \theta_2^* & \cdots & \theta_n^* \end{bmatrix}^T$$

Then

$$\dot{x}_p = [A_p + gb\theta_x^{*T}]x_p + b(g\theta_r^*)r$$

Assume that for these * gain (θ_x^* & θ_r^*) values, the feedback control (2) achieves model matching in the sense that

$$\boxed{\left.\begin{array}{ll} A_p + gb\theta_x^{*T} \equiv A_m & (n \times n) \\ g\theta_r^* \equiv g_m & scalar \end{array}\right\} \textbf{Matching conditions}}$$

Thus, control gains $\theta_x^* \in \Re^n$, and $\theta_r^* \in \Re$ exist to guarantee that the closed-loop system match the reference model

$$\dot{x}_m = A_m x_m + g_m br$$

[Note: In the system that you set-up & apply, be sure to check that the "Matching Conditions" are valid!!]

Recall the plant is:

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Example 2-1 (continued)

Choice of reference model

For example, the following might be desirable

$$y_m = G_{y_m r} r$$

$$G_{y_m r}(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$u = \theta_x^{*T} x_p + \theta_r^* r$$
$$= \begin{bmatrix} \theta_{1x}^* & \theta_{2x}^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \theta_r^* r$$

- unity steady-state gain
- speed of response defined by natural frequency ω_n
- damping specified by damping coefficient ξ

State-variable form

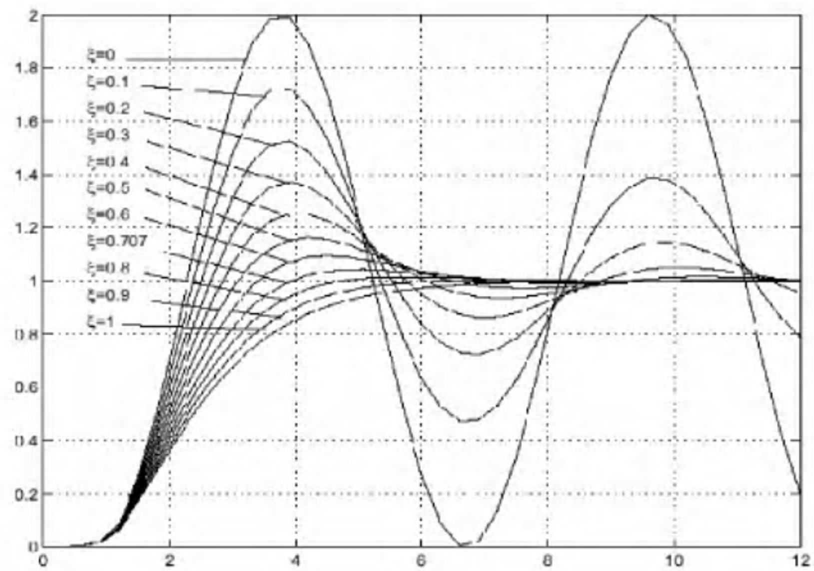
$$\begin{bmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix} + \omega_n^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$\dot{x}_m = A_m x_m + g_m b r$$

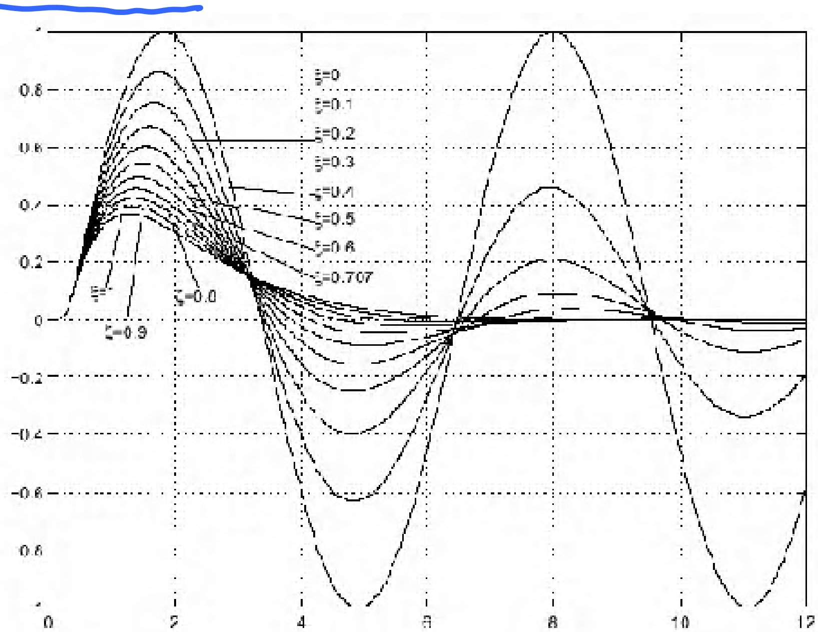
Check that for Example 2-1, the so-called "Matching Conditions" are valid!!

Useful time-response tables (helpful in choice of Ref Model)

Step Response



Impulse Response



Adaptive Case:

To adaptively match the reference model, consider the control law

$$u(t) = \theta_x^{\ T}(t)x_p(t) + \theta_r(t)r(t)$$

Define parameter errors

$$\phi_x(t) \overset{\Delta}{=} \theta_x(t) - \theta_x^*$$
$$\phi_r(t) \overset{\Delta}{=} \theta_r(t) - \theta_r^*$$

Then, the control law applied to the plant results in

$$\begin{aligned}
\dot{x}_p &= A_p x_p + gb\{\theta_x^T x_p + \theta_r r\} \\
&= \left[A_p + gb\theta_x^{*T}\right]x_p + gb\phi_x^T x_p + gb\theta_r r \\
&= A_m x_p + gb\phi_x^T x_p + gb\theta_r r
\end{aligned}$$

Compared with

$$\begin{aligned}
\dot{x}_m &= A_m x_m + g_m br \\
&= A_m x_m + gb\theta_r^* r
\end{aligned}$$

where $g_m = g\theta_r^*$.

Consider state error

$$e \stackrel{\Delta}{=} x_p - x_m$$

Then

$$\dot{e} = A_m e + gb\phi_x^T x_p + gb\phi_r r$$

$$\boxed{\dot{e} = A_m e + gb\phi^T x}$$

Error Signal Dynamics / Error Eqn

where

$$\phi \stackrel{\Delta}{=} \begin{bmatrix} \phi_x \\ \phi_r \end{bmatrix}; \qquad x \stackrel{\Delta}{=} \begin{bmatrix} x_p \\ r \end{bmatrix}$$

For a reference model, A_m must be chosen to be a stable matrix.

Thus, it satisfies the Lyapunov equation.

$$A_m^T P + P A_m = -Q$$

[Refer to earlier notes.]

i.e., for any symmetric positive definite matrix Q, there exists a symmetric positive definite P satisfying the above equation.

(From Linear Systems theory, or see Narendra & Annaswamy pp.60)

Consider a Lyapunov function candidate :

$$V(e(t),\phi(t)) = e(t)^T Pe(t) + |g|\,\phi(t)^T \Gamma^{-1}\phi(t)$$

From your undergrad "Linear Algebra", check that V is positive-def; decrescent; & radially unbounded!

where

Γ is a symmetric positive definite (s.p.d.) matrix

We will sometime write $V(e(t),\phi(t))$ as V(t) for short.

Evaluate $\dot{V}$ along the trajectory of the system

$$\begin{aligned}\dot{V} &= 2e^T P\dot{e} + 2|g|\phi^T\Gamma^{-1}\dot{\phi} \\ &= 2e^T P\{A_m e + gb\phi^T x\} + 2|g|\phi^T\Gamma^{-1}\dot{\phi} \\ &= 2e^T PA_m e + 2ge^T Pb\phi^T x + 2|g|\phi^T\Gamma^{-1}\dot{\phi} \\ &= e^T\left(A_m^T P^T + PA_m\right)e + 2ge^T Pb\phi^T x + 2|g|\phi^T\Gamma^{-1}\dot{\phi} \\ &= -e^T Qe + 2ge^T Pb\phi^T x + 2|g|\phi^T\Gamma^{-1}\dot{\phi} \qquad (3)\end{aligned}$$

Also, recall from your undergrad "Linear Algebra", that any square matrix M can be cast as

$$M = \underbrace{M_s}_{\text{symmetric part}} + \underbrace{M_{as}}_{\text{anti-symmetric part}} = \frac{1}{2}\{M+M^T\} + \frac{1}{2}\{M-M^T\}$$

- $-e^T Qe$ term always ≤ 0
- Use the design

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_r \end{bmatrix} = -\operatorname{sgn}(g)\Gamma x e^T Pb$$

and note that

$$\phi = \theta - \theta^*, \quad \Rightarrow \quad \dot{\phi} = \dot{\theta}$$

Equation (3) becomes

$$\begin{aligned} \dot{V} &= -e^T Qe + 2ge^T Pb\phi^T x - 2|g|\operatorname{sgn}(g)\phi^T x e^T Pb \\ &= -e^T Qe \leq 0 \end{aligned}$$

We can go through a similar analysis to show that

- $V(t)$ is positive definite and $\dot{V}(t) \le 0 \Rightarrow V(t)$ is bounded
- $\|e\|, \|\phi\|$ $\left(hence \quad \|\theta\|\right)$ are bounded
- $\dot{e}$ is bounded
- $\int_0^\infty e^T Q e d\tau \le c_1$
- $\lim_{t\to\infty} \|e\| = 0$ (or $\lim_{t\to\infty} \|x_p - x_m\| = 0$)

Summary

Plant

$$\dot{x}_p = A_p x_p + gbu$$

with $x_p \in \Re^n$ measurable, and b known

As given.

Matching Conditions

$$A_p + gb\theta_x^{*T} = A_m$$

$$g\theta_r^* = g_m$$

To be checked valid by user.

Reference Model

$$\dot{x}_m = A_m x_m + g_m br$$

Control Law

$$u(t) = \theta_x^T(t) x_p(t) + \theta_r(t) r(t)$$

Adaptive Law

$$e = x_p - x_m$$

$A_m^T P + PA_m = -Q$ *Choose* Q s.p.d., *Calculate* P s.p.d.

$$\begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_r \end{bmatrix} = -\operatorname{sgn}(g)\Gamma \begin{bmatrix} x_p \\ r \end{bmatrix} e^T Pb$$

User sets this up, with necessary choices.

Result: All signals $\{x_p, \theta_x, \theta_r\}$ are bounded, and $\lim_{t\to\infty} \|x_p - x_m\| = 0$

Exercise 1

Consider the plant $y = G_{yu}u$

where

$$G_{yu}(s) = \frac{g}{s^2 + a_1 s + a_2}$$

with $a_1 = 0,\quad a_2 = 1,\ y$ and $\dot{y}$ measurable.

Design an adaptive controller to match the reference model

$$y_m = G_{y_m r} r$$

$$G_{y_m r}(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

with $\omega_n = 2$ rad/s and $\xi = 0.9$.

Simulate and verify your design using MATLAB. (or any language)

Incorporation of integral control with all state variables measurable

Plant:

$$\dot{x}_p = A_p x_p + gbu$$
$$y = x_1 = [1 \quad 0 \quad \cdots \quad 0] x_p$$

$x_p \in \mathbb{R}^n$ is measurable

Pls review your Linear Dynamical Systems & State-Variables!

Matching condition:

$$A_p + gb\theta_x^{*T} = A_m$$

Integral control can be incorporated easily. Consider an additional state

$$\dot{x}_I = y - r$$

r(t) is the reference signal and $x_I = \int_0^t (y - r) d\tau$

Then, augmented plant

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A_p & \mathbf{0} \\ 1 \; 0 \ldots 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_I \end{bmatrix} + g \begin{bmatrix} b \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

Consider

$$u = \theta_x^{*T} x_p + \theta_I^* x_I$$

The closed-loop system is

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A_p + gb\theta_x^{*T} & gb\theta_I^* \\ 1 \quad 0 \ \dots \ 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_I \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

Thus, an additional matching condition is

$$\begin{bmatrix} A_p + gb\theta_x^{*T} & gb\theta_I^* \\ 1 \quad 0 \ \dots \ 0 & 0 \end{bmatrix} = \bar{A}_m$$

and

$$\dot{\bar{x}}_m = \bar{A}_m \bar{x}_m + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

For the corresponding adaptive controller, consider control law:

$$u(t) = \theta_x(t)^T x_p(t) + \theta_I(t) x_I(t)$$

and a reference model

$$\dot{\bar{x}}_m = \bar{A}_m \bar{x}_m + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r; \qquad \bar{x}_m \in \Re^{n+1}$$

Define

$$\bar{x}_p \stackrel{\Delta}{=} \begin{bmatrix} x_p \\ \cdots \\ x_I \end{bmatrix}$$

For $\bar{e} = \bar{x}_p - \bar{x}_m$, we have

$$\dot{\bar{e}} = \bar{A}_m \bar{e} + g \begin{bmatrix} b \\ \cdots \\ 0 \end{bmatrix} \left(\phi_x^T x_p + \phi_I x_I \right)$$

$$\boxed{\dot{\bar{e}} \quad = \bar{A}_m \bar{e} + g \bar{b}\, \phi^T \bar{x}_p}$$

Error Signal Dynamics / Error Eqn

Clearly, it is now a well-known error equation, and

$$\bar{b} = \begin{bmatrix} b \\ \cdots \\ 0 \end{bmatrix} \quad \textit{is known}$$

as b is known.

The adaptive law should be

$$\bar{A}_m^T \bar{P} + \bar{P} \bar{A}_m = -\bar{Q}$$

$$1\text{st } \textit{choose} \quad \bar{Q}, \ 2nd \ \textit{calculate} \quad \bar{P}$$

$$\begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_I \end{bmatrix} = -\operatorname{sgn}(g) \Gamma \begin{bmatrix} x_p \\ x_I \end{bmatrix} \bar{e}^T \bar{P} \bar{b}$$

Remarks: This follows as an extension of the previous case. Work this out & check!!

Adaptive control with all state-variables measurable: Incorporation of integral control

Example

Consider a plant

$$\dot{y}_p = a_p y_p + gu$$

To incorporate integral control, we know we must augment the state with

$$\dot{x}_I = y_p - r$$

The state of the new "plant" (including the augmented variable) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

with $x_1 \equiv y_p$ and $x_2 = x_I$

The feedback control is of the form

$$u(t) = \theta_1(t)x_1(t) + \theta_2(t)x_2(t)$$

(Refer to earlier notes.)

Consider constant gain feedback first. The plant with feedback is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_p + g\theta_1 & g\theta_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

This means that the reference model used must have the form

$$\begin{bmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \end{bmatrix} = \begin{bmatrix} a_{1m} & a_{2m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

- This is the only class of reference model that can be matched -------"matching conditions".
- The analysis then proceeds in the normal way. (Work this out too!)

What is the Adaptive Law?

Exercise 2

Consider the plant $y = G_{yu}u$

where

$$G_{yu}(s) = \frac{g}{s^2 + a_1 s + a_2}$$

with $a_1 = 0$, $a_2 = 1$, y and $\dot{y}$ measurable.

Design an adaptive controller incorporating integral action.

Explain carefully how you choose the reference model to track.

In the limit as $t \to \infty$, what is the equivalent relationship between y(t) and r(t)?

Hint: (Express this in transfer function form.)

(Optional. Problem is open-ended!!)

Exercise 3: Proportional plus integral adaptation

#1 Consider the plant

$$\dot{y} = a_p y + u$$

and the reference model

$$\dot{y}_m = a_m y_m + r$$

Then, the adaptive controller

$$e = y - y_m$$
$$u = \theta y + r$$
$$\dot{\theta} = -\gamma y e \qquad \gamma > 0$$

will result in asymptotic tracking.

#2 Astrom (pp.136,137) has suggested an adaptive law of the form

$$\theta(t) = -\gamma_1 y e - \gamma_2 \int_0^t y(\tau) e(\tau) d\tau$$

Using the methods we have presented, analyse this adaptive law.

Is it possible to make any clear conclusions?

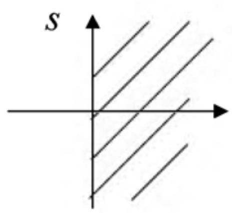
Positive Real Transfer Function

Definition: A rational transfer function H with real coefficients is positive real (PR) if *Re H(s)>0* for *Re s>0*.

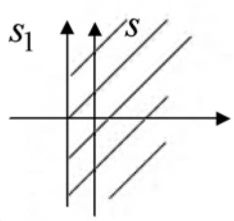
A transfer function H is strictly positive real (SPR) if H(s-ε) is positive real for some real ε>0.

Astrom…p129

Narendra…p65



PR



SPR

$s = s_1 + \varepsilon$

Kalman Yakuborich Lemma, Lemma 2.3 (Narendra…p66)

Given a scalar $\gamma \geq 0$, a vector h, an asymptotically stable matrix A, a vector b such that (A, b) is controllable, and a positive definite matrix L, there exist a scalar ε>0, a vector q and a symmetric positive-definite matrix P satisfying

$$A^T P + PA = -qq^T - \varepsilon L$$
$$Pb - h = \sqrt{\gamma} q$$

if and only if $H(s) \overset{\Delta}{=} \frac{1}{2}\gamma + h^T (sI - A)^{-1} b$ is SPR.

$$\dot{x} = Ax + bu$$
$$y = hx + \frac{1}{2}\gamma u$$

$$(A, b, c, d) = (A, b, h, \frac{1}{2}\gamma)$$

state-realisation

transfer function

Compare:

Given an asymp stable matrix A_m, and a p.d. matrix Q,

there exist a symm p.d. matrix P satisfying:

$$A_m^T P + P A_m = -Q$$

Lemma 4-2 (Astrom p129) is a special case of preceeding with

$$\gamma = 0$$

Positive real functions have been used extensively in network analysis.

The use of the concept in adaptive control is relatively recent.

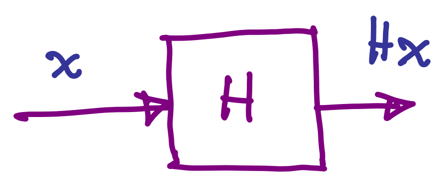
A passive network (consisting only of inductance, resistance and capacitance) is positive real. If in addition, the network is dissipative due to the presence of resistors, then the impedance function is strictly positive real.

Viewed from realization, any PR (SPR) impedance function can be realized by a passive (dissipative) network.

(Narendra pp.63)

"Impedance" operator

H : is said to be passive if $\exists\, \beta \in R$ such that



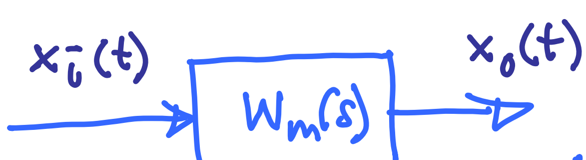
$$< Hx, x >_t = \int_0^t (Hx)^T x d\tau \geq \beta, \forall t \in R^+$$

$$(< y, x >_t = \int_0^t y^T x d\tau \geq \beta, \forall t \in R^+)$$

When H is a linear invariant operator, if H is passive, then H is PR.

(Narendra pp.71)

Example: Consider



with

$$W_m(s) = \frac{1}{s+1}$$

Any observations?

Continuous-time adaptive control using only input-output measurements

- Theory of SPR function will be used to prove stability.

References: Narendra… … Chapter 5
Astrom… … … Section 4.5

Consider a system described by

"plant"

$$R_p(p)y(t) = k_p Z_p(p)u(t) \qquad (1)$$

where

$$p \equiv \frac{d}{dt}$$

Note !!

$$R_p(p) = p^n + a_1 p^{n-1} + \cdots + a_n$$
$$Z_p(p) = p^m + b_1 p^{m-1} + \cdots + b_m$$

Z_p and R_p *is monic (i.e. leading coeff=1)*

$\deg(R_p) = n, \qquad \deg(Z_p) = m$

Define "relative degree" n^*

$$n^* \overset{\Delta}{=} n - m \qquad \textit{(i.e. excess poles over zeros.)}$$

"Big-sounding" name for polynomial division!! :) :)

Consider "Diophantine" identity

$$\boxed{T(p)R_m(p) = R_p(p)E(p) + F(p)}$$

$\left.\begin{aligned}\deg(T) &= n \\ \deg(R_m) &= n^*\end{aligned}\right\}$	*Design polynomials, chosen to be stable, and monic*

Info only at this stage. Not yet needed to verify "Diophantine" identity.

For given plant, i.e. R_p, and specified T and R_m

E and F are unique, and

$$\deg(E) = n^*, \quad E \quad monic, \quad \deg(F) = n-1$$

$$T(p) = p^n + t_1 p^{n-1} + \ldots + t_{n-1}p + t_n$$

Plant: $R_p(p)y = k_p Z_p(p)u$

Consider re-writing as: $ER_p y = k_p E Z_p u$

i.e.

$$TR_m y = Fy + k_p E Z_p u = k_p(\bar{F}y + \bar{G}u)$$

from Diophantine identity

$$\left(\bar{F} = \frac{F}{k_p}, \bar{G} = EZ_p \right)$$

$$\boxed{R_m y = k_p\left(\frac{\bar{F}}{T}y + \frac{\bar{G}}{T}u\right) \qquad (5)}$$

Note that (a) Z_p monic, and E monic, $\therefore$ $\bar{G}$ monic;

(b) $\deg(\bar{G}) = \underbrace{\deg(Z_p)}_{m} + \underbrace{\deg(E)}_{n^* = n-m} = n$

Accordingly, we have

$$\frac{\bar{G}}{T} = 1 + \frac{G_1}{T}$$

re-writing in this manner.

where

$$G_1(p) = g_1 p^{n-1} + g_2 p^{n-2} + \cdots + g_n$$

Similiarly, we can write

$$\bar{F}(p) = f_1 p^{n-1} + f_2 p^{n-2} + \cdots + f_n$$

$(\quad) = k^* r$

Equation (5) then becomes

$$\boxed{R_m y = k_p\left(\frac{\bar{F}}{T} y + \frac{G_1}{T} u + u\right) \qquad (7)}$$

Define $\quad y^{f_1} = \frac{1}{T} y \, , \qquad u^{f_1} = \frac{1}{T} u$

Consider what these signals would look like for $n=2$?

$$\bar{\theta}^* = \left[-f_1, -f_2, \cdots - f_n, -g_1, -g_2, \cdots - g_n, k^*\right]^T$$

$$\bar{\varpi} = \left[p^{n-1} y^{f_1}, p^{n-2} y^{f_1}, \cdots, y^{f_1}, p^{n-1} u^{f_1}, p^{n-2} u^{f_1}, \cdots, u^{f_1}, r\right]^T$$

and r(t) is the set-point signal.

Note that this is the "perfect control"

Consider

$$u(t) = -\frac{\bar{F}}{T} y - \frac{G_1}{T} u + k^* r = -\bar{F} y^{f_1} - G_1 u^{f_1} + k^* r \qquad (8)$$

$$\boxed{u(t) = \bar{\theta}^{*T} \bar{\varpi} = \theta_u^{*T} \omega_u + \theta_y^{*T} \omega_y + k^* r}$$

Equation (7) becomes

$$\boxed{\begin{aligned} R_m y &= k_p k^* r \\ &= k_m r \qquad \text{for} \quad k_p k^* = k_m \end{aligned}}$$

I.e.

The choice of the input (8) ensures that the plant with feedback becomes

$$\boxed{R_m(p)y(t) = k_m r(t)}$$

In other words, the plant output will track a reference output (reference model) given by

$$\boxed{R_m(p)y_m(t) = k_m r(t) \quad \Rightarrow \quad \frac{Y_m}{R} = \frac{k_m}{R_m(s)}}$$

I.e. there exists the input (8) which achieves this!

Remember,

$\deg(R_m) = n^*$

R_m is a design polynomial you choose.

Example:

n^*	Choice
$n^* = 1,$	$R_m(s) = (s + a_m)$ $k_m = a_m;$
$n^* = 2$	$R_m(s) = s^2 + 2\xi\omega_n s + \omega_n^2$ $k_m = \omega_n^2$

However,

while we can construct the signals in $\overline{\varpi}$,

we do not know the desired controller parameters $\overline{\theta}^*$. !!!

Try using

$$u(t) = \overline{\theta}(t)^T \overline{\varpi}(t)$$ (Control Law)

and adaptively determine $\overline{\theta}(t)$ *in some way.*

This is possible. Two separate cases:

(i) $n^* = 1$, *convergence proof easy (relatively)*

(ii) $n^* > 1$, *convergence proof possible but very difficult*

. . . .

Next change
Akan datang
(Malay dialect) :: ::

[Singapore dialect !! :: ::]

Recall equation (7)

$$R_m y = k_p\left(\frac{\overline{F}}{T}y + \frac{G_1}{T}u + u\right)$$

$$= k_p\left(\left[\frac{\overline{F}}{T}y + \frac{G_1}{T}u - k^* r\right] + k^* r + u\right)$$

$$= k_p\left(-\overline{\theta}^{*T}\overline{\varpi} + k^* r + u\right)$$

Control law:

$$u(t) = \overline{\theta}(t)^T \overline{\varpi}(t)$$

Define parameter error

$$\overline{\phi}(t) \overset{\Delta}{=} \overline{\theta}(t) - \overline{\theta}^*$$

Then above becomes

$$R_m y = k_p\left(\overline{\phi}(t)^T \overline{\varpi}(t) + k^* r(t)\right)$$

Comparing with reference model

$$R_m y_m = k_m r = k_p k^* r$$

gives error equation

$$\boxed{R_m e_1 = k_p \overline{\phi}(t)^T \overline{\varpi}(t)}$$

where $e_1 \overset{\Delta}{=} y - y_m$.

$\boxed{\text{Case (i) : } n^* = 1}$

Considering this case (i) for now...

At this point, remembering $n^* = 1$ and $\deg(R_m) = n^*$

we have $R_m(s) = s + a_m$ (stable reference model)

It is straightforward to check that

$$\frac{y_m}{r} = W_m = \frac{k_m}{R_m(s)} = \frac{k_m}{s + a_m}; \qquad k_m > 0$$

is SPR.

We will use this property together with the Kalman-Yakubovich Lemma to prove stability and convergence of $e_1(t)$ to zero.

However, we need first a state realization which includes

$$\omega(t) \stackrel{\Delta}{=} \left[y^{f_1}, \ldots, p^{n-2} y^{f_1}, p^{n-1} y^{f_1}, u^{f_1} \ldots, p^{n-2} u^{f_1}, p^{n-1} u^{f_1} \right]^T$$

as state~~s~~ (in order to prove boundedness of all state variables).

Proposition 1: The vector $\omega(t)$ is the state of a realization (nonminimal) of the plant

$$R_p y(t) = k_p Z_p u(t)$$

Proof:

Recall that, $T = p^n + t_1 p^{n-1} + \ldots + t_{n-1} p + t_n$

$$Tu^{f_1} = u \qquad (1)$$

and clearly

$$R_p y^{f_1} = k_p Z_p u^{f_1} \qquad (2)$$

Further, since $\deg(T) = \deg(R_p) = n$, clearly

$Ty^{f_1} = y$

$$\begin{aligned} y &= Ty^{f_1} = Ty^{f_1} + 0 \\ &= (T - R_p) y^{f_1} + k_p Z_p u^{f_1} \end{aligned} \qquad (3)$$

Equations (1), (2) and (3) form the basis for a 2n-dimensional state representation of the plant.

From (1), we have

$$\frac{d}{dt}\begin{bmatrix} u^{f_1} \\ pu^{f_1} \\ \vdots \\ p^{n-1}u^{f_1} \end{bmatrix} = A_T \begin{bmatrix} u^{f_1} \\ pu^{f_1} \\ \vdots \\ p^{n-1}u^{f_1} \end{bmatrix} + b_n u$$

where

$$A_T = \begin{bmatrix} \mathbf{0} & & \mathbf{I}_{(n-1)\times(n-1)} & \\ -t_n & -t_{n-1} & \cdots & -t_1 \end{bmatrix} \text{ and } b_n = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

Think about this for the n=2 case...

$$\dot{\omega} = \bar{\bar{F}} \omega + \bar{\bar{G}} u$$
$$y = \bar{\bar{H}} \omega$$

This can be combined with (2) and (3) to give

$$\frac{d}{dt}\omega(t) = \begin{bmatrix} \mathbf{A}_{R_p} & \mathbf{A}_{k_p Z_p} \\ \mathbf{0}_{n\times n} & \mathbf{A}_T \end{bmatrix} \omega(t) + \begin{bmatrix} \mathbf{0} \\ b_n \end{bmatrix} u(t)$$

$$y(t) = \lfloor t_n - a_n, t_{n-1} - a_{n-1}, \cdots, t_1 - a_1, 0, \cdots, 0, k_p b_1, \cdots, k_p b_m \rfloor \omega(t)$$

The forms of the $n \times n$ matrices $\mathbf{A}_{R_p}$ and $\mathbf{A}_{k_p Z_p}$ should be obvious. The above is a state realization of the plant with $\omega(t)$ as the state.

$$\omega(t) = \begin{bmatrix} y^{f_1} \\ p y^{f_1} \\ \vdots \\ p^{n-1} y^{f_1} \\ u^{f_1} \\ p u^{f_1} \\ \vdots \\ p^{n-1} u^{f_1} \end{bmatrix}$$

QED

Think about

$n=2$ case ...

Proposition 2: For the state realization given previously, there exist controller gains θ^* such that the closed-loop poles are at the roots of the 2n degree polynomial $TR_m Z_p$.

Proof:

Most of the materials have been introduced already. It only remains to put it into a formal basis.

Recall the Diophantine identity

$$TR_m = R_p E + F$$

and the plant equation

$$R_p y = k_p Z_p u$$

with exact parameters

From earlier, recall that our control law was

$$u = \theta^{*T} \omega + k^* r \qquad (1)$$

and it is equivalent to

$$\bar{F} y^{f_1} + \bar{G} u^{f_1} = k^* r \qquad (2)$$

with $k_p \bar{F} = F$.

[Why?

p.44, $u(t) = \theta^{*T} \omega + k^* r = -\dfrac{F}{T} y - \dfrac{G_1}{T} u + k^* r \quad \Rightarrow$

$\dfrac{F}{T} y + (1 + \dfrac{G_1}{T}) u = k^* r \quad \Rightarrow \quad \dfrac{F}{T} y + \dfrac{G}{T} u = k^* r$

]

Think of n=2 case to see the ideas clearly...

Observe that

$$
\begin{aligned}
TR_m Z_p &= R_p E Z_p + F Z_p \\
&= R_p \overline{G} + Z_p F \\
&= R_p \overline{G} + k_p Z_p \overline{F}
\end{aligned} \tag{3}
$$

If the control law (1) or equivalent (2) is applied to the plant, we have:

Plant: $R_p y = k_p Z_p u$

$$R_p y^{f_1} = k_p Z_p u^{f_1}$$

From (2)

$$
\begin{aligned}
R_p \overline{G} y^{f_1} &= k_p Z_p \overline{G} u^{f_1} \\
&= k_p Z_p \left\{ k^* r - \overline{F} y^{f_1} \right\}
\end{aligned}
$$

or

$$\left(R_p \overline{G} + k_p Z_p \overline{F} \right) y^{f_1} = k_p k^* Z_p r$$

From (3) $TR_m Z_p y^{f_1} = k_m Z_p r$

i.e. the closed loop poles are at $TR_m Z_p$.

In addition, if T and Z_p are stable, then the closed-loop transfer function is

$$R_m y = k_m r$$

QED

Corollary:

The reference model

$$R_m(p)y_m(t) = k_m r(t) \qquad i.e., \qquad \frac{Y_m}{R} = \frac{k_m}{R_m(s)}$$

admits a 2n dimensional state-representation

$$\dot{\omega}_m = A_m \omega_m + b_m r$$
$$y_m = c_m^T \omega_m$$

Proof:

Straightforward by combining previous two propositions. $\triangle\triangle\triangle$

Note that:

$R_m y_m = k_m r$	$\Rightarrow$	$W_m = \frac{y_m}{r} = \frac{k_m}{R_m}$
$\dot{\omega}_m = A_m \omega_m + b_m r$ $y_m = c_m^T \omega_m$	$\Rightarrow$	$W_m = \frac{y_m}{r} = k_m^T (sI - A_m)^{-1} b_m$

$\Rightarrow$

$$W_m = k_m^T (sI - A_m)^{-1} b_m = \frac{k_m}{R_m}$$

Stability Analysis

Use the 2n-dimensional state-representation

$$\dot{\omega} = A_p \omega + b_p u$$
$$y = c_p^T \omega$$

using result of Proposition 1

Consider

$$u = \bar{\theta}(t)^T \bar{\omega} = \theta(t)^T \omega + k(t) r$$
$$= \{\theta^* + \phi(t)\}^T \omega + \{k^* + \phi_k\} r$$

Then, C.L.

$$\dot{\omega} = \left(A_p + b_p \theta^{*T} \right) \omega + b_p \left(\phi^T \omega + \phi_k r \right) + k^* b_p r$$
$$y = c_p^T \omega$$

$$\dot{w}_m = A_m w_m + b_m r$$
$$y_m = c_m^T w_m$$

For the exact values θ^*, k^*, the plant with feedback matches the reference model,

$$A_p + b_p \theta^{*T} = A_m; \qquad k^* b_p = b_m$$

and $c_p = c_m$

from the analysis of Page 44, and Propositions 1 & 2

$$\bar{\phi} \triangleq \begin{bmatrix} \phi \\ \phi_k \end{bmatrix}$$

Consider now the state error

$$e \stackrel{\Delta}{=} \omega - \omega_m, \qquad e_1 = y - y_m$$

Then

$$\dot{e} = A_m e + \frac{1}{k^*} b_m \bar{\phi}^T \bar{\omega}$$

$$e_1 = c_m^T e$$

$$\boxed{e_1 = c_m^T [sI - A_m]^{-1} \frac{b_m}{k^*} \bar{\phi}^T \bar{\omega}}$$

!!

$e_1 = y - y_m$

Noting that (A_m, b_m, c_m) is a state representation of the reference model, we have

$$c_m^T (sI - A_m)^{-1} b_m = \frac{k_m}{R_m}$$

from the analysis of Page 44, and Propositions 1 & 2

This is strictly positive real, as is $c_m^T (sI - A_m)^{-1} \frac{1}{|k^*|} b_m = \frac{k_m}{|k^*| R_m}$.

Consider Lyapunov function condidate

$$V(e, \phi) = e^T P e + \bar{\phi}^T \Gamma^{-1} \bar{\phi}$$

where

$$A_m^T P + P A_m = -Q = -qq^T - \varepsilon L$$

$Q = qq^T + \varepsilon L$

Then

$$\dot{V} = -e^T Q e + 2 e^T P \frac{1}{k^*} b_m \bar{\phi}^T \bar{\omega} + 2 \bar{\phi}^T \Gamma^{-1} \dot{\bar{\phi}}$$

$\frac{1}{|k^*|} \operatorname{sgn}(k^*)$

Use the version with $(A, b, c, d) = (A, b, h, \frac{1}{2}\gamma)$ with $\gamma = 0$ ...

Since $\left(A_m, \frac{1}{|k^*|} b_m, c_m\right)$ is SPR, KY-Lemma ensures that

$$P \frac{1}{|k^*|} b_m = c_m$$

which means that

$$\begin{aligned}\dot{V} &= -e^T Q e + 2e^T c_m \operatorname{sgn}(k^*) \bar{\phi}^T \bar{\varpi} + 2\bar{\phi}^T \Gamma^{-1} \dot{\bar{\phi}} \\ &= -e^T Q e + 2e_1 \operatorname{sgn}(k^*) \bar{\phi}^T \bar{\varpi} + 2\bar{\phi}^T \Gamma^{-1} \dot{\bar{\phi}}\end{aligned}$$

If

$$\dot{\bar{\theta}} = \dot{\bar{\phi}} = -\operatorname{sgn}(k^*) \Gamma \bar{\varpi} e_1 = -\operatorname{sgn}(k_p) \Gamma \bar{\varpi} e_1 \quad \text{as} \quad k_m > 0$$

we have

$$\dot{V} = -e^T Q e \le 0$$

Thus

(a) $\|e\|$, $\|\bar{\phi}\|$ (hence $\|\omega\|$, $\|\bar{\theta}\|$) are bounded;

(b) $\dot{e}$ bounded, $\int_0^\infty e(\tau)^T Q e(\tau) d\tau \le c_1$

$$\therefore \lim_{t\to\infty} e(t) = 0$$

$$\Rightarrow \lim_{t\to\infty} e_1(t) = 0$$

Compare: Full state measurable

$\dot{\theta} = \dot{\phi} = -\operatorname{sgn}(q) \times \Gamma w \times e^T P b$

$e = x_p - x_m$

Case $n^* = 1$, Summary

Plant:

$$R_p y = k_p Z_p u$$

Control:

$$y^{f_1} = \frac{1}{T} y, \qquad u^{f_1} = \frac{1}{T} u$$

$$\varpi(t) \stackrel{\Delta}{=} \left[p^{n-1} y^{f_1}, p^{n-2} y^{f_1}, \cdots, y^{f_1}, p^{n-1} u^{f_1}, p^{n-2} u^{f_1}, \cdots, u^{f_1}, r\right]^T$$

$$u(t) = \bar{\theta}(t)^T \varpi(t)$$

Reference Model:

$$R_m(p)\, y_m = k_m r$$

Adaptive law

$$\dot{\bar{\theta}} = -\mathrm{sgn}(k_p) \Gamma \bar{\varpi} e_1$$

$$e_1 = y - y_m$$

Result: If

(a) order of R_p, n, is known;

(b) Z_p is stable polynomial;

(c) $\mathrm{sgn}(k_p)$ is known;

then the above system leads to y(t), u(t), $\bar{\theta}(t)$ bounded, and

$$\lim_{t \to \infty} \left(y(t) - y_m(t)\right) = 0$$

$\Delta\Delta\Delta$

EE5104 CA1 mini-project investigates this ...