Discrete-Time Adaptive Control

- Read Astrom ….. Sections 5.1 to 5.4

  Sections 6.1 and 6.2

  Structures for estimation and control used are very similar to continuous-time

- Proofs of boundedness, while still difficult, are slightly less complicated

Self-tuning Minimum Variance Control

(also called minimum prediction error control , d-step ahead control)

Plant

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + e(t) \tag{3.1}$$

where

$$A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_m q^{-m}$$

$\{e(t)\}$ is uncorrelated noise sequence with variance σ^2.

Consider first the case when the plant is known exactly.

Prediction identity (rather similar to continuous-time):

$$1 = A(q^{-1})E(q^{-1}) + q^{-d}F(q^{-1}) \tag{3.2}$$

$$\deg(E) = d-1$$

$$\deg(F) = n-1$$

(There exist polynomials E and F

$$E(q^{-1}) = e_0 + e_1 q^{-1} + \ldots + e_{d-1}q^{-(d-1)}$$

$$F(q^{-1}) = f_0 + f_1 q^{-1} + \ldots + f_{n-1}q^{-(n-1)}$$

such that the prediction identity holds)

Proof:

Define $$A^*(q) = q^n + a_1 q^{n-1} + \ldots + a_n$$

$$E^*(q) = e_0 q^{d-1} + e_1 q^{d-2} + \ldots + e_{d-1}$$

$$(e_0 = 1)$$

$$F^*(q) = f_0 q^{n-1} + f_1 q^{n-2} + \ldots + f_{n-1}$$

Then, from polynomial division, we have

$$q^{n+d-1} = A^*(q)E*(q) + F^*(q)$$

Noticing that

$$A^*(q) = q^n A(q^{-1})$$

$$E^*(q) = q^{d-1}E(q^{-1})$$

$$F^*(q) = q^{n-1}F(q^{-1})$$

We can see that the prediction identity (3.2) holds.

QED

Multiplying E to plant (3.1), $Ay = q^{-d}Bu + e$, gives

$$EAy = q^{-d}EBu + Ee \tag{3.3}$$

Using (3.2), equation (3.3) becomes

$$y(t) = q^{-d}F(q^{-1})y(t) + q^{-d}E(q^{-1})B(q^{-1})u(t) + E(q^{-1})e(t)$$

i.e.,

$$q^{d}y(t) = F(q^{-1})y(t) + E(q^{-1})B(q^{-1})u(t) + q^{d}E(q^{-1})e(t)$$

$$\Downarrow$$

$$\begin{aligned} y(t+d) = F(q^{-1})y(t) + E(q^{-1})B(q^{-1})u(t) \\ + E(q^{-1})e(t+d) \end{aligned} \tag{3.4}$$

Recall that *deg(E)=d-1,* the terms in $E(q^{-1})e(t+d)$ cannot be affected by present input *u(t).*

(Future noise cannot be affected by current input u(t))

The control that minimises the output variance

$$J = E\{y(t+d)^2\}$$

is clearly given by

$$F(q^{-1})y(t) + E(q^{-1})B(q^{-1})u(t) = 0 \qquad (3.5)$$

This gives

$$y(t+d) = E(q^{-1})e(t+d)$$

or

$$J_{\min} = \left(1 + e_1^2 + e_2^2 + \cdots + e_{d-1}^2\right)\sigma^2$$

$$(e_0^2 = 1^2)$$

Optimal predictor interpretation

Recall that

$$y(t+d) = F(q^{-1})y(t) + E(q^{-1})B(q^{-1})u(t) \\ + E(q^{-1})e(t+d)$$

Using only data up to time t, obviously the optimal predictor for y(t+d) is

$$\hat{y}(t+d/t) = F(q^{-1})y(t) + E(q^{-1})B(q^{-1})u(t)$$

as terms in $E(q^{-1})e(t+d)$ are future.

The minimum variance control (3.5) thus actually sets

$$\hat{y}(t+d/t) = 0$$

Summary

Plant: $A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + e(t)$

{e(t)} uncorrelated , variance σ^2

Prediction identity: $1 = AE + q^{-d}F$

The optimal predictor, using data up to time t only, is

$$\hat{y}(t+d/t) = F(q^{-1})y(t) + E(q^{-1})B(q^{-1})u(t)$$

The minimum variance control law for criterion $J = E\left\{y(j)^2\right\}$ is

$$\hat{y}(t+d/t) = 0$$

with

$$J_{\min} = \left(1 + e_1^2 + e_2^2 + \cdots + e_{d-1}^2\right)\sigma^2$$

The closed-loop is

$$y(t) = E(q^{-1})e(t) \tag{3.6}$$

and $E(q^{-1})B(q^{-1})u(t) = -F(q^{-1})y(t)$

or $B(q^{-1})u(t) = -F(q^{-1})e(t)$ (3.7)

From (3.7), it implies that minimum variance control is only applicable to plants where $B(q^{-1})$ is stable.

Incorporation of set-point tracking

Consider the plant

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + e(t)$$

Set-point tracking is achieved by modifying the control law to be

$$\hat{y}(t+d/t) = F(q^{-1})y(t) + E(q^{-1})B(q^{-1})u(t) = r(t) \quad (3.8)$$

Comparing with (3.4), we have

$$y(t+d) - r(t) = E(q^{-1})e(t+d)$$

or

$$E\left\{\left[y(t+d) - r(t)\right]^2\right\} = \left(1 + e_1^2 + e_2^2 + \cdots + e_{d-1}^2\right)\sigma^2$$

Thus the control law (3.8) minimises

$$J = E\left\{\left[y(t+d) - r(t)\right]^2\right\}$$

with

$$J_{\min} = \left(1 + e_1^2 + e_2^2 + \cdots + e_{d-1}^2\right)\sigma^2$$

The requirement for stable $B(q^{-1})$ obviously applies.

Self-tuning (or adaptive) minimum variance controller

Consider the plant

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) \tag{3.9}$$

If the parameters of $A(q^{-1})$ and $B(q^{-1})$ are not known, then an adaptive version has to be used.

For historical reasons, the adaptive minimum variance regulator is usually referred to as the self-tuning regulator.

Using the prediction identity, (3.9) becomes

$$y(t) = F(q^{-1})y(t-d) + G(q^{-1})u(t-d) \tag{3.10}$$

with $\quad G(q^{-1}) = E(q^{-1})B(q^{-1})$.

$$F(q^{-1}) = f_0 + f_1 q^{-1} \cdots + f_{n-1} q^{-(n-1)}$$

$$G(q^{-1}) = g_0 + g_1 q^{-1} + \cdots + g_{(m+d-1)} q^{-(m+d-1)}$$

Equation (3.10) can be written in the LIP form as

$$y(t) = \theta^{*T} \varphi(t-d) \tag{3.11}$$

where

$$\theta^* = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-1} & g_0 & g_1 & \cdots & g_{(m+d-1)} \end{bmatrix}^T$$

$$\varphi(t) = [\mathrm{y(t)}\ \ \mathrm{y(t-1)}\ \ \ldots\ \ \mathrm{y(t-(n-1))}\ \ \mathrm{u(t)}\ \ \mathrm{u(t-1)}\ \ \ldots\ \ \mathrm{u(t-(m+d-1))}]^T$$

which is in a suitable structure to construct an estimate $\hat{\theta}(t)$.

Proposition:

Consider the plant (3.11). Let the estimator be given by

$$\hat{\theta}(t) = \hat{\theta}(t-1) - \frac{\gamma\varphi(t-d)}{\alpha + \varphi^T(t-d)\varphi(t-d)} e_1(t) \qquad (3.12)$$

where

$$e_1(t) = \hat{y}(t) - y(t) = \varphi^T(t-d)\tilde{\theta}(t-1)$$

$\alpha \geq 0$ and $0 < \gamma < 2$.

$$\hat{y}(t) = \varphi^T(t-d)\hat{\theta}(t-1)$$

Then the estimator results in

(i) $\left\|\hat{\theta}(t) - \theta^*\right\| \leq \left\|\hat{\theta}(t-1) - \theta^*\right\| \leq \left\|\hat{\theta}(0) - \theta^*\right\|$

(ii) $\lim_{t\to\infty} \frac{e_1(t)}{\sqrt{\alpha + \varphi^T\varphi}} = 0$

(iii) $\lim_{t\to\infty} \left\|\hat{\theta}(t) - \hat{\theta}(t-k)\right\| = 0$ for any finite k.

Proof:

(Refer also to Astrom … Section 3.5)

Write

$$\begin{aligned} e_1(t) &= \hat{y}(t) - y(t) \\ &= \varphi^T(t-d)\hat{\theta}(t-1) - \varphi^T(t-d)\theta^* \\ &= \tilde{\theta}(t-1)^T \varphi(t-d) \end{aligned}$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$.

Where it is obvious, we will omit the time arguments.

Consider it is quadratic form

$$V(t) \equiv V(\tilde{\theta}(t)) = \tilde{\theta}(t)^T \tilde{\theta}(t)$$

The difference is given by

$$\Delta V = V(t) - V(t-1) = \tilde{\theta}(t)^T \tilde{\theta}(t) - \tilde{\theta}^T(t-1)\tilde{\theta}(t-1)$$

(--- $a^2 - b^2 = (a-b)(a+b)$----)

$$\begin{aligned} &= \left(\tilde{\theta}(t) - \tilde{\theta}(t-1)\right)^T \left(\tilde{\theta}(t) + \tilde{\theta}(t-1)\right) \\ &= \left(\tilde{\theta}(t) - \tilde{\theta}(t-1)\right)^T \left(\left(\tilde{\theta}(t) - \tilde{\theta}(t-1)\right) + 2\tilde{\theta}(t-1)\right) \\ &= \Delta\tilde{\theta}^T(t)\left(\Delta\tilde{\theta}(t) + 2\tilde{\theta}(t-1)\right) \qquad (3.13) \end{aligned}$$

From (3.12), by subtracting θ^* from both sides, we have

$$\Delta\tilde{\theta}(t) = \tilde{\theta}(t) - \tilde{\theta}(t-1) = -\frac{\gamma\varphi(t-d)}{\alpha + \varphi^T(t-d)\varphi(t-d)} e_1(t)$$

Analyzing (3.13) term by term, we have

$$\left(\tilde{\theta}(t)-\tilde{\theta}(t-1)\right)^T\left(2\tilde{\theta}(t-1)\right)=-\frac{2\gamma e_1(t)}{\alpha+\varphi^T\varphi}\varphi^T(t-d)\tilde{\theta}(t-1)$$

$$=-\frac{2\gamma e_1(t)^2}{\alpha+\varphi^T(t-d)\varphi(t-d)}$$

$$\left(\tilde{\theta}(t)-\tilde{\theta}(t-1)\right)^T\left(\tilde{\theta}(t)-\tilde{\theta}(t-1)\right)=\left\|\tilde{\theta}(t)-\tilde{\theta}(t-1)\right\|^2$$

$$=\frac{\gamma^2 e_1(t)^2}{\left(\alpha+\varphi^T\varphi\right)^2}\varphi^T\varphi=\frac{\gamma^2 e_1(t)^2}{\alpha+\varphi^T\varphi}*\frac{\varphi^T\varphi}{\alpha+\varphi^T\varphi}$$

$$\leq\frac{\gamma^2 e_1(t)^2}{\alpha+\varphi^T\varphi}$$

Equation (3.13) becomes

$$\begin{aligned}V(t)-V(t-1)&\leq-\frac{2\gamma e_1{}^2}{\alpha+\varphi^T\varphi}+\frac{\gamma^2 e_1{}^2}{\alpha+\varphi^T\varphi}\\&=-\gamma(2-\gamma)\frac{e_1{}^2}{\alpha+\varphi^T\varphi}\leq 0\end{aligned}\qquad(3.14)$$

as $0<\gamma<2$

Thus, $V(t)\leq V(t-1)$

or $\left\|\tilde{\theta}(t)\right\|^2\leq\left\|\tilde{\theta}(t-1)\right\|^2$

This proves (i).

Also, this implies V(t) bounded for all t.

From (3.14),

$$\gamma(2-\gamma)\frac{e_1^{\,2}}{\alpha+\varphi^T\varphi}\le V(t-1)-V(t)$$

$$\therefore\quad \sum_{j=0}^{t}\gamma(2-\gamma)\frac{e_1^{\,2}(j)}{\alpha+\varphi^T(j-d)\varphi(j-d)}\le V(0)-V(t)\le V(0)$$

Thus, $\displaystyle \lim_{t\to\infty}\sum_{j=0}^{t}\gamma(2-\gamma)\frac{e_1^{\,2}(j)}{\alpha+\varphi^T(j-d)\varphi(j-d)}\le V(0)$

$$\therefore\quad \lim_{j\to\infty}\frac{e_1(j)}{\sqrt{\alpha+\varphi^T(j-d)\varphi(j-d)}}=0$$

This proves (ii).

To prove (iii), note that

$$\hat{\theta}(t)-\hat{\theta}(t-1)=-\gamma\frac{\varphi(t-d)}{\alpha+\varphi(t-d)^T\varphi(t-d)}e_1(t)$$
$$=-\gamma\frac{\varphi(t-d)}{\sqrt{\alpha+\varphi^T\varphi}}\frac{e_1(t)}{\sqrt{\alpha+\varphi^T\varphi}}$$

Clearly, we have

$$\lim_{t\to\infty}\left\|\hat{\theta}(t)-\hat{\theta}(t-1)\right\|=0$$

$$\left\|\hat{\theta}(t)-\hat{\theta}(t-k)\right\|=\left\|\sum_{j=1}^{t}(\hat{\theta}(t-j+1)-\hat{\theta}(t-j))\right\|$$
$$\le\sum_{j=1}^{t}\left\|\hat{\theta}(t-j+1)-\hat{\theta}(t-j)\right\|$$

for finite k.

Using (ii), obviously

$$\lim_{t\to\infty}\left\|\hat{\theta}(t)-\hat{\theta}(t-k)\right\|=0$$

This proves (iii).

Thus, using the estimator, the adaptive minimum variance controller is achieved by using the control law

$$\hat{y}(t+d/t)=\hat{\theta}(t)^T\varphi(t)=r(t)$$

(Notice that $\varphi(t)$ is used in the control, while $\varphi(t-d)$ is used in estimation.)

In implementation, the above control is

$$\begin{bmatrix}\hat{f}_0 & \hat{f}_1 & \cdots & \hat{f}_{n-1} & \hat{g}_0 & \hat{g}_1 & \cdots & \hat{g}_{(m+d-1)}\end{bmatrix}\begin{bmatrix} y(t) \\ y(t-1) \\ \vdots \\ y(t-[n-1]) \\ u(t) \\ u(t-1) \\ \vdots \\ u(t-[m+d-1]) \end{bmatrix}=r(t)$$

or

$$u(t)=\frac{1}{\hat{g}_0}\left\{r(t)-\hat{f}_0y(t)-\hat{f}_1y(t-1)-\cdots-\hat{f}_{n-1}y(t-[n-1])\right.$$
$$\left.-\hat{g}_1u(t-1)-\cdots-\hat{g}_{(m+d-1)}u(t-[m+d-1])\right\}$$

To prove the stability of the control, we need the following Lamma.

Lemma 6.2 (Astrom) Key Technical Lemma

Let $\{s_t\}$ be a sequence of real numbers and $\{\sigma_t\}$ be a sequence of vectors such that

$$\|\sigma_t\| \le c_1 + c_2 \max_{0\le k\le t} |s_k|$$

Assume that

$$z_t = \frac{s_t^2}{\alpha_1 + \alpha_2 \sigma_t^T \sigma_t} \to 0$$

where $\alpha_1 > 0, \quad \alpha_2 > 0$. Then $\|\sigma_t\|$ is bounded.

Application:

$$\Rightarrow \quad \|\varphi(t-d)\| \le c_1 + c_2 \max_{0\le k\le t} |\varepsilon(k)|$$

Proof:

Case 1: $\{s_t\}$ is a bounded sequence.

Then the result follows trivially.

Case 2: $\{s_t\}$ is not bounded.

Then there must exist a subsequence $\{t_n\}$ such that

$$\left|s_{t_n}\right| \to \infty \text{ and } |s_t| \le \left|s_{t_n}\right| \text{ for } t \le t_n .$$

Along this subsequence, it follows that

$$\left|\frac{s_t^2}{\alpha_1 + \alpha_2 \sigma_t^T \sigma_t}\right| \ge \frac{s_t^2}{\alpha_1 + \alpha_2 \left(c_1 + c_2 |s_t|\right)^2} = \frac{s_t^2}{\alpha_1 + \alpha_2 c_1^2 + 2\alpha_2 c_1 c_2 |s_t| + \alpha_2 c_2^2 |s_t|^2}$$

$$\ge \frac{s_t^2}{\alpha_3 c_2^2 |s_t|^2} \ge \frac{1}{\alpha_3 c_2^2} \ge \varepsilon_1 > 0$$

where $0 < \alpha_2 < \alpha_3$. But this contradicts the assumption that

$$\frac{s_t^2}{\alpha_1 + \alpha_2 \sigma_t^T \sigma_t} \to 0 \text{ as } t \to \infty .$$

Thus, $\{s_t\}$ must be bounded. Hence also $\{\sigma_t\}$.

Note that, if $\alpha_3 > \alpha_2 > 0$, sometimes in the future,

$$(\alpha_3 - \alpha_2) c_2^2 |s_t|^2 \ge \alpha_1 + \alpha_2 c_1^2 + 2\alpha_2 c_1 c_2 |s_t|$$

i.e.,

$$\alpha_3 c_2^2 |s_t|^2 \ge \alpha_1 + \alpha_2 c_1^2 + 2\alpha_2 c_1 c_2 |s_t| + \alpha_2 c_2^2 |s_t|^2$$

Stability properties of the adaptive controller

We have already proved that $\hat{\theta}(t)$ is bounded.

Next, we need to prove boundedness of $y(t)$ and $u(t)$ $\forall t$, and say something about the convergence of $y(t+d)$ to $r(t)$.

For the plant, we have

$$y(t) = \theta^{*T} \varphi(t-d)$$

The control law is equivalent to

$$r(t) = \hat{\theta}(t)^T \varphi(t)$$

or

$$r(t-d) = \hat{\theta}(t-d)^T \varphi(t-d)$$

This gives

$$\begin{aligned} y(t) - r(t-d) &= -\tilde{\theta}(t-d)^T \varphi(t-d) \\ &= -\varepsilon(t) \end{aligned}$$

for $\varepsilon(t) \stackrel{\Delta}{=} \tilde{\theta}(t-d)^T \varphi(t-d)$.

Since the reference signal must clearly be bounded, we have

$$
\begin{aligned}
|y(t)| &\leq \alpha_1 + \beta_1 |\varepsilon(t)| \\
&\leq \alpha_1 + \beta_1 \max_{0\leq j\leq t} |\varepsilon(j)|
\end{aligned}
$$

Furthermore

$$B(q^{-1})u(t-d) = A(q^{-1})y(t)$$

and $B(q^{-1})$ is stable, we have

$$|u(t-d)| \leq \alpha_2 + \beta_2 \max_{0\leq j\leq t} |y(j)|$$

See also Astrom…. Section 6.2, especially pp.223-225.

This, therefore, means that

$$\|\varphi(t-d)\| \leq \alpha_3 + \beta_3 \max_{0\leq j\leq t} |\varepsilon(j)| \tag{3.15}$$

for

$$\varphi(t) = [y(t)\ \ y(t\text{-}1)\ \ \ldots\ \ y(t\text{-}(n\text{-}1))\ \ u(t)\ \ u(t\text{-}1)\ \ \ldots\ \ u(t\text{-}(m+d\text{-}1))]^T$$

Next, observe that

$$
\begin{aligned}
\varepsilon(t) &\stackrel{\Delta}{=} \tilde{\theta}(t-d)^T \varphi(t-d) \\
&= \tilde{\theta}(t-1)^T \varphi(t-d) + \left\{\tilde{\theta}(t-d) - \tilde{\theta}(t-1)\right\}^T \varphi(t-d) \\
&= e_1(t) - \left\{\tilde{\theta}(t-1) - \tilde{\theta}(t-d)\right\}^T \varphi(t-d)
\end{aligned}
$$

Using properties (ii) and (iii) of the estimator, we have

$$
\lim_{t\to\infty} \frac{\varepsilon(t)}{\sqrt{\alpha + \varphi(t-d)^T \varphi(t-d)}} = 0 \qquad (3.16)
$$

Then, using (3.15) and (3.16) and Lemma 6.2 (pp.223 in Astrom, notes in pp.15)

Identify ε(t) with s_t and $\varphi(t-d)$ with σ_t.

Then, we have $\|\varphi(t-d)\|$ bounded $\forall t$.

$$\Rightarrow \quad \{y(t)\}, \quad \{u(t)\} \text{ are bounded.}$$

Equation (3.16) further implies that $\lim_{t\to\infty} \varepsilon(t) = 0$.

Summary

Plant: $A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t)$

Estimator: $\hat{y}(t) = \hat{\theta}(t-1)^T \varphi(t-d)$

$$\varphi(t) = [\mathrm{y(t)\ \ y(t-1)\ \ ...\ \ y(t-(n-1))\ \ u(t)\ \ u(t-1)\ \ ...\ \ u(t-(m+d-1))}]^T$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) - \frac{\gamma\varphi(t-d)}{\alpha + \varphi(t-d)^T\varphi(t-d)}\left(\hat{y}(t) - y(t)\right)$$

$$\alpha > 0, \quad 0 < \gamma < 2$$

Control:

$$\hat{\theta}(t)^T\varphi(t) = r(t)$$

Result:

For the adaptive control applied to the plant above, if

(a1) the order of A and B are known,

(a2) the delay d is known, and

(a3) $B(q^{-1})$ is a stable polynomial,

then y(t) and u(t) are bounded $\forall t$, and

$$\lim_{t\to\infty}\{y(t+d) - r(t)\} = 0$$

General Minimum Variance Control

Clarke and Gawthrop, (1975), “A Self-tuning Controller”, IEEE Proceedings, Vol 122, pp929-934.

See also Astrom pp188 ff

Note that MV control was not applicable to non-minimum phase systems.

GMV controller modifies it to handle a restricted class of non-minimum phase systems.

Plant

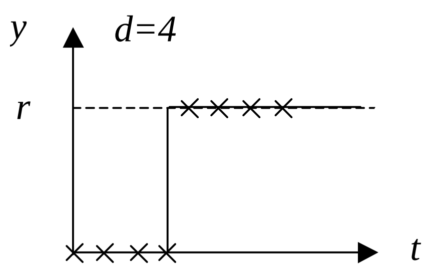
$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) + e(t)$$

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \cdots + b_m q^{-m}$$

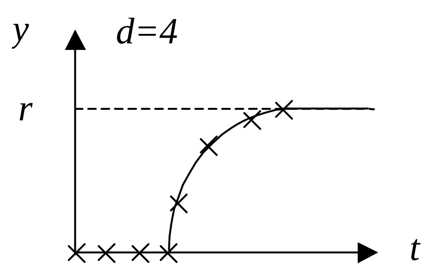
$$J = E\left\{\left(y(t+d) - r(t)\right)^2\right\}$$

$$\Rightarrow y(t) = q^{-d} r(t)$$



$$J = E\left\{\left(p(q^{-1})y(t+d) - r(t)\right)^2\right\}$$

$$\Rightarrow y(t) = \frac{q^{-d}}{p(q^{-1})} r(t)$$



GMV control minimises

$$J' = E\left\{\left(P(q^{-1})y(t+d) - k_m r(t)\right)^2 + \left(Q_0(q^{-1})u(t)\right)^2\right\}$$

Based on modified prediction identity:

$$P(q^{-1}) = A(q^{-1})E(q^{-1}) + q^{-d}F(q^{-1})$$
$$\deg(E) = d - 1$$
$$\deg(F) = n - 1$$

it can be shown that GMV equivalently minimises

$$J' = E\left\{\left(P(q^{-1})y(t+d) - k_m r(t) + Q(q^{-1})u(t)\right)^2\right\}$$

where Q_0 and Q differ only by a constant.

Thus, the GMV control law is

$$Q(q^{-1})u(t) = -P(q^{-1})y(t+d) + k_m r(t) \qquad (4.1)$$

Closed-loop properties

$$Q(q^{-1})q^{-d}B(q^{-1})u(t) = -P(q^{-1})B(q^{-1})y(t) + k_m q^{-d}B(q^{-1})r(t)$$

i.e.

$$Q\{Ay - e\} + PBy = k_m q^{-d}Br$$

or

$$(PB + QA)y = q^{-d}k_m Br + Qe$$

That is

- The closed-loop poles are $(PB + QA)$
- It does not matter if B is unstable
- Transfer function from r to y is

$$\frac{q^{-d}k_m B}{PB + QA}$$

Pole Placement:

Solve $PB + QA = A^*$ for P and Q where A^* is a stable reference polynomial.

The control law as given in (4.1) is not realisable

Instead, define

$$\psi(t) = P(q^{-1})y(t)$$

Using modified prediction identity:

$$\psi(t+d) = F(q^{-1})y(t) + G(q^{-1})u(t) + E(q^{-1})e(t+d)$$

where $G = EB$.

Since *deg(E)=d-1*, the optimal estimator for $\psi(t+d)$ using data up to t is

$$\hat{\psi}(t+d/t) = F(q^{-1})y(t) + G(q^{-1})u(t)$$

Therefore, we have a realisable version of (4.1) using the optimal predictor given by

$$Q(q^{-1})u(t) = -\{F(q^{-1})y(t) + G(q^{-1})u(t)\} + k_m r(t)$$

Any self-tuning version typically uses the above.

Self-Tuning GMV Controller

(1) Direct algorithm

- Estimation model

$$\psi(t) = P(q^{-1})y(t)$$
$$\hat{\psi}(t) = \hat{F}(q^{-1})y(t-d) + \hat{G}(q^{-1})u(t-d)$$

Use a suitable estimator, e.g. RLS,

to obtain $\hat{F}$, $\hat{G}$

- Control law

$$Q(q^{-1})u(t) = -\left\{\hat{F}(q^{-1})y(t) + \hat{G}(q^{-1})u(t)\right\} + k_m r(t)$$

<u>(Estimate the controller parameters directly)</u>

(2) Indirect algorithm

- Estimation model

  $\hat{A}(q^{-1})y(t) = \hat{B}(q^{-1})u(t-d)$

  Use RLS to obtain $\hat{A}, \quad \hat{B}$

- Control

  Solve prediction identity

  $$P(q^{-1}) = \hat{A}(q^{-1})\hat{E}(q^{-1}) + q^{-d}\hat{F}(q^{-1})$$

  to obtain $\hat{E}, \quad \hat{F}$.

  Calculate

  $$Q(q^{-1})u(t) = -\left\{\hat{F}(q^{-1})y(t) + \hat{E}(q^{-1})B(q^{-1})u(t)\right\} + k_m r(t)$$

(Estimate the plant parameters & solve for controller coeeficients)

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(3) Adaptive Pole Placement:

- Estimate: $\hat{\theta} \Rightarrow \hat{A}, \quad \hat{B}$
- Solve $\hat{P}\hat{B} + \hat{Q}\hat{A} = A^*$ for $\hat{P}$ and $\hat{Q}$ where A^* is a stable reference polynomial.
- Implement:

$$\hat{Q}(q^{-1})u(t) = -\hat{P}(q^{-1})y(t+d) + k_m r(t)$$

Inclusion of integral action in STR's

Simplest method:

In GMV controller, choose

$$Q(q^{-1}) = q_0\left(1 - q^{-1}\right)$$

then control law

$$q_0(1 - q^{-1})u(t) = -\left\{\hat{F}(q^{-1})y(t) + \hat{G}(q^{-1})u(t)\right\} + k_m r(t)$$

which contains integrator.

Closed loop:

$$\left(PB + q_0(1 - q^{-1})A\right)y(t) = q^{-d}k_m Br(t) + q_0(1 - q^{-1})e(t)$$

So that if $\lim\limits_{t\to\infty} r(t) = r_0$ a constant

$$\lim_{t\to\infty} y(t) = \frac{k_m}{P(z=1)} r_0$$

for $\{e(t)\} \equiv 0$.

Thus steady-state tracking is possible.

- $Z(r(t) = 1) = \dfrac{z}{z - 1} = \dfrac{1}{1 - z^{-1}}$
- $\lim\limits_{t\to\infty} y(t) = \lim\limits_{z\to 1}(1 - z^{-1})Y(z)$, Final Value Theorem
- $z = e^{Ts}$, $z^{-1} = e^{-Ts}$ represents a delay of one sample period

<u>Alternative method:</u> modify plant equation

$$A(q^{-1})y = q^{-d}B(q^{-1})u + e$$

Introduce $\Delta = 1 - q^{-1}$.

$$A\Delta y = q^{-d}B\Delta u + \Delta e$$

or

$$A_1 y = q^{-d}B\Delta u + \varepsilon$$

where $A_1 \equiv A\Delta, \quad \varepsilon \equiv \Delta e$.

Obviously $\deg(A_1) = n + 1$

Everything else follows as above except the prediction identity is

$$P(q^{-1}) = A_1(q^{-1})E_1(q^{-1}) + q^{-d}F_1(q^{-1})$$

$$\deg(E_1) = d\,, \quad \deg(F_1) = n$$

Control law is

$$Q\Delta u = -\left\{\hat{F}_1(q^{-1})y + \hat{G}_1(q^{-1})\Delta u\right\} + k_m r$$

The closed-loop for exact parameters is given by

$$Q(q^{-1})\Delta u(t) = -P(q^{-1})y(t+d) + k_m r(t)$$

$$q^{-d} BQ\Delta u = -BPy + q^{-d} k_m Br$$

or $$Q\{A\Delta y - \Delta e\} = -PBy + q^{-d} k_m Br$$

or $$\{PB + Q(1-q^{-1})A\}y = q^{-d} k_m Br + Q\Delta e$$

Thus, in fact, the closed-loop properties are the same, for exact parameters.

However, the performance of the self-tuning versions are not necessarily the same as they use different estimation models.

You will get to check this in a simulation exercise.

Modified GMV to include feedback control of disturbances that are measurable

Assume the plant is given by

$$Ay = q^{-d}Bu + q^{-d_2}Dv$$

The assumption has to be made that $d_2 \geq d$.

This disturbance v(t) is measurable

Thus consider

$$A\Delta y = q^{-d}B\Delta u + q^{-d_2}D\Delta v$$

or $$A_1 y = q^{-d}B\Delta u + q^{-d_2}D\Delta v$$

Use prediction identity:

$$P(q^{-1}) = A_1 E + q^{-d}F$$

Then,

$$Py = q^{-d}Fy + q^{-d}EB\Delta u + q^{-d_2}ED\Delta v \qquad (4.2)$$

Equation (4.2) can be used as the <u>estimation model</u>

If it is stepped d increments in time t, then we obtain the <u>prediction model</u>

$$\psi(t+d) = P(q^{-1})y(t+d)$$
$$= Fy(t) + G\Delta u(t) + H\Delta v(t-(d_2-d))$$

For the prediction model to be realisable at time t, it is clear that we must have

$$d_2 \geq d$$

The control law is thus

$$Q\Delta u = -\{Fy + G\Delta u + H\Delta v(t-[d_2-d])\} + k_m r$$

Comparing with (4.1), pp.22,

$$Q\Delta u = -q^d Py + k_m r$$

$$Qq^{-d}B\Delta u = -PBy + q^{-d}k_m Br$$

the closed-loop is

$$Q\{A\Delta y - q^{-d_2}D\Delta v\} = -PBy + q^{-d}k_m Br$$

$$(PB + Q\Delta A)y = q^{-d}k_m Br + q^{-d_2}QD\Delta v$$

Exercise 3.1

Consider the discrete-time system described by

$$A(q^{-1})y(t) = B(q^{-1})u(t)$$

$$A(q^{-1}) = 1 - 1.61q^{-1} + 0.61q^{-2}$$

$$B(q^{-1}) = 0.107\left(q^{-1} + 0.84q^{-2}\right)$$

Using the methods described, design

(a) an adaptive minimum prediction error controller ***
(p.1, p.8 of notes);

(b) an adaptive GMV controller ***;

(c) an adaptive pole-placement controller --- ;

(d) an adaptive GPC (General Predictive Control) controller (Nu=1, N=3) ---- p204, Astrom.

(See also Example 5.1 in Astrom…. Pp168, which is similar but not identical.)

Exercise 3.2

Refer to the plant described in Exercise 3.1.

Assume now that the plant is given by

$$A(q^{-1})y(t) = B(q^{-1})u(t) + D(q^{-1})v(t)$$

where $D(q^{-1}) = q^{-3}$

and v(t) is measurable.

Discuss how you would incorporate feedforward/feedback in your adaptive GMV controller.

Adaptive Control

Thus far, we have considered the following

(a) Continuous-time, all states measurable + matching conditions, rigorous solution;

(b) Continuous-time, only input-output measurable, minimum-phase plant, rigorous solution;

(c) Continuous-time, only input-output measurable, combine estimation with a suitable control, gradient estimator, non-rigorous;

(d) Discrete-time, only input-output measurable, minimum variance controller, no noise, rigorous solution; uses gradient estimator;

(e) Discrete-time, only input-output measurable, GMV controller, no noise,

direct approach + gradient estimator

----- can be shown to be rigorous;

indirect approach +gradient estimator

----- non-rigorous, combination of est and cont

We have mostly considered the gradient estimator.

Continuous-time

$$y(t) = \theta^{*T} \varphi(t) \qquad t \in \Re$$

Estimator is

$$\hat{y}(t) = \hat{\theta}(t)^T \varphi(t)$$

$$e_1(t) = \hat{y}(t) - y(t)$$

$$\dot{\hat{\theta}}(t) = -\gamma\varphi(t)e_1(t)$$

Discrete-time

$$y(t) = \theta^{*T} \varphi(t) \qquad t \in Z$$

Estimator is

$$\hat{y}(t) = \hat{\theta}(t-1)^T \varphi(t)$$

$$e_1(t) = \hat{y}(t) - y(t)$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) - \frac{\gamma\varphi(t)e_1(t)}{\alpha + \varphi(t)^T \varphi(t)}$$

$$\alpha > 0, \qquad 0 < \gamma < 2$$