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Q1

a)

The state space is as follow:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

After sampling, the state space changes into discrete-time form:

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k)$$
$$\Phi = e^{Ah}, \quad \Gamma = \int_0^h e^{Av} dvB$$

We can caculate Φ and Γ via Caley-Hamilton Theorem. The eigenvalues of A are $\lambda_0 = -1,\ \lambda_1 = 0$

$$\begin{cases} h(\lambda) = \beta_0 + \beta_1 \lambda \\ e^{\lambda_0 h} = \beta_0 + \beta_1 \lambda_0 \\ e^{\lambda_1 h} = \beta_0 + \beta_1 \lambda_1 \end{cases} \Rightarrow \begin{cases} e^{-h} = \beta_0 - \beta_1 \\ 1 = \beta_0 \end{cases} \Rightarrow \begin{cases} \beta_0 = 1 \\ \beta_1 = 1 - e^{-h} \end{cases}$$

So:

$$\Phi = e^{Ah} = \beta_0 I + \beta_1 A = \begin{bmatrix} e^{-h} & 0 \\ 1 - e^{-h} & 1 \end{bmatrix}$$

$$\begin{aligned} \Gamma &= \int_0^h e^{Av} dvB \\ &= \int_0^h \begin{bmatrix} e^{-v} & 0 \\ 1 - e^{-v} & 1 \end{bmatrix} dv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 - e^{-h} \\ e^{-h} + h - 1 \end{bmatrix} \end{aligned}$$

Assuming that $u(k) = -Lx(k)$, using the deadbeat controller:

$$A_c(\Phi) = \Phi^2 = \begin{bmatrix} e^{-2h} & 0 \\ 1-e^{-2h} & 1 \end{bmatrix}$$

$$W_c = [\Gamma \quad \Phi\Gamma] = \begin{bmatrix} 1-e^{-h} & e^{-h}-e^{-2h} \\ e^{-h}+h-1 & e^{-2h}-e^{-h}+h \end{bmatrix}$$

$$\begin{aligned} L &= [0 \quad 1] W_c^{-1} A_c(\Phi) \\ &= [0 \quad 1] \begin{bmatrix} e^{-2h}-e^{-h}+h & e^{-2h}-e^{-h} \\ -e^{-h}-h+1 & 1-e^{-h} \end{bmatrix} \begin{bmatrix} e^{-2h} & 0 \\ 1-e^{-2h} & 1 \end{bmatrix} / det(W_c) \\ &= [-he^{-2h}-e^{-h}+1 \quad 1-e^{-h}]/det(W_c) \\ &= \begin{bmatrix} \frac{e^{2h}-e^{h}-h}{h(e^h-1)^2} & \frac{e^h}{h(e^h-1)} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} u(k) &= -Lx(k) \\ &= -\begin{bmatrix} \frac{e^{2h}-e^{h}-h}{h(e^h-1)^2} & \frac{e^h}{h(e^h-1)} \end{bmatrix} x(k) \end{aligned}$$

b)

The control signal at k=0 can be expressed as

$$\begin{aligned} u(0) &= -Lx(0) \\ &= -\begin{bmatrix} \frac{e^{2h}-e^{h}-h}{h(e^h-1)^2} & \frac{e^h}{h(e^h-1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \\ &= -\frac{3e^{2h}-3e^{h}-2h}{2h(e^h-1)^2} \end{aligned}$$

We want the control signal less than one at k=0, so:

$$\begin{gathered} |u(0)| < 1 \\ |\frac{3e^{2h}-3e^{h}-2h}{2h(e^h-1)^2}| < 1 \\ h > 0 \\ \Rightarrow h > 1.74 \end{gathered}$$

Q2

Augmented state vector:

$$z(k) = \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

$$z(k+1) = \begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & \Phi_v \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [C \quad 0] z(k)$$

$$\Phi = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.8 \end{bmatrix},\ \Phi_{xv} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \Phi_v = 1,\ \Gamma = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},\ C = [1 \quad 0]$$

a)

Beacuse the state and v can be measured, assuming that $u(k) = -Lx(k) - L_v v(k)$,

$$\begin{aligned} x(k+1) &= \Phi x(k) - \Gamma(Lx(k) + L_v v(k)) + \Phi_v v(k) \\ &= (\Phi - \Gamma L)x(k) + (\Phi_v - \Gamma L_v) v(k) \end{aligned}$$

Deadbeat controller:

$$A_c(\Phi) = \Phi^2 = \begin{bmatrix} 0.75 & 1.3 \\ 0.65 & 1.14 \end{bmatrix}$$

$$W_c = [\Gamma \quad \Phi\Gamma] = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.18 \end{bmatrix}$$

$$\begin{aligned} L &= [0 \quad 1] W_c^{-1} A_c(\Phi) \\ &= [0 \quad 1] \begin{bmatrix} 0.18 & -0.2 \\ -0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 0.75 & 1.3 \\ 0.65 & 1.14 \end{bmatrix} / det(W_c) \\ &= [3.4375 \quad 6.1250] \end{aligned}$$

Then, z-transfor:

$$zX(z) = (\Phi - \Gamma L)X(z) + (\Phi_v - \Gamma L_v)V(z)$$

$$X(z) = (zI - \Phi + \Gamma L)^{-1}(\Phi_v - \Gamma L_v)V(z)$$

$$Y(z) = CX(z) = C(zI - \Phi + \Gamma L)^{-1}(\Phi_v - \Gamma L_v)V(z) = H_v(z)V(z)$$

If we want to eliminate the influence of v, $H_v(1)$ should be 0

$$\begin{aligned} H_v(z) &= C(zI - \Phi + \Gamma L)^{-1}(\Phi_v - \Gamma L_v) \\ &= [1 \quad 0]\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.8 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} [3.4375 \quad 6.1250]\right)^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} L_v\right) \\ &= \frac{-80zL_v + 24L_v + 400z - 75}{400z^2} \end{aligned}$$

$$H_v(1) = \frac{-80L_v + 24L_v + 400 - 75}{400} = 0$$

In concludes, $L = [3.4375 \quad 6.1250]$, $L_v = 5.803$, The state space can be expressed as follow:

$$u(k) = -Lx(k) - L_v v(k) = -\begin{bmatrix}3.4375 & 6.1250\end{bmatrix} x(k) - 5.803 v(k)$$

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & \Phi_v \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} - \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \begin{bmatrix} L & L_v \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$
$$= \begin{bmatrix} \Phi - \Gamma L & \Phi_{xv} - \Gamma L_v \\ 0 & \Phi_v \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$
$$= \begin{bmatrix} -0.1875 & -0.2250 & -0.1606 \\ 0.1562 & 0.1875 & -0.5803 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix}$$

b)

Build an observe to estimate the disturbance:

$$\hat{z}(k+1) = \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & \Phi_v \end{bmatrix} \hat{z}(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) + K(y(k) - \hat{y}(k))$$
$$\hat{y}(k) = C\hat{z}(k)$$

Put all pols to 0, design a Dead-beat Observer, let $\Phi' = \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & \Phi_v \end{bmatrix}$, $C' = \begin{bmatrix} C & 0 \end{bmatrix}$

$$A_o(\Phi') = \Phi'^3 = \begin{bmatrix} 1.025 & 1.79 & 2.25 \\ 0.895 & 1.562 & 1.15 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W_o = \begin{bmatrix} C' \\ C'\Phi' \\ C'\Phi'^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 1 \\ 0.75 & 1.3 & 1.5 \end{bmatrix}$$

$$K = A_o(\Phi') W_o^{-1} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2.3 \\ -2.06 \\ 5 \end{bmatrix}$$

Because the disturbance v can not be measured directly, so

$$u(k) = -Lx(k) - L_v \hat{v}(k)$$

c)

The observe is same as **b)**, because only output signal can be measured, the input signal will change to:

$$u(k) = -L\hat{x}(k) - L_v \hat{v}(k)$$

Q3

a)

We can use overshoot and stabilization time:

$$\begin{cases} \sigma = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\% \leq 10\% \\ t_s \approx \frac{4.6}{\zeta\omega_n} \leq 10.0s \end{cases}$$

$$\Rightarrow \begin{cases} \zeta \geq 0.591 \\ \zeta\omega_n \geq 0.46 \end{cases} \Rightarrow \begin{cases} \zeta = 0.6 \\ \omega_n = 0.77 \end{cases}$$

The reference model in the continuous-time is:

$$H_m(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{0.593}{s^2 + 0.924s + 0.593}$$

Then we can get discrete-time transfer function:

$$\begin{aligned} H_m(z) &= (1 - z^{-1})Z[\mathscr{L}^{-1}(\frac{G(s)}{s})] \\ &= \frac{z-1}{z}Z[\mathscr{L}^{-1}(\frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2})] \\ &= \frac{z-1}{z}Z[\mathscr{L}^{-1}(\frac{1}{s} + \frac{p_2}{p_1 - p_2} \cdot \frac{1}{s - p_1} + \frac{p_1}{p_2 - p_1} \cdot \frac{1}{s - p_2})] \\ &= \frac{z-1}{z}(\frac{z}{z-1} + \frac{p_2}{p_1 - p_2} \cdot \frac{z}{z - e^{p_1 T}} + \frac{p_1}{p_2 - p_1} \cdot \frac{z}{z - e^{p_2 T}}) \\ &= 1 + \frac{p_2}{p_1 - p_2} \cdot \frac{z-1}{z - e^{p_1 T}} + \frac{p_1}{p_2 - p_1} \cdot \frac{z-1}{z - e^{p_2 T}} \end{aligned}$$

Because $T = 0.1s,\ p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},\ p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$, then

$$H_m(z) = \frac{0.002874z + 0.002787}{z^2 - 1.906z + 0.9117} = \frac{B_m(z)}{A_m(z)}$$

b)

Because we need to use position control system, the output signal is the position $y(t)$.

$$\ddot{y}(t) = -\frac{b}{m}\dot{y}(t) + \frac{1}{m}u(t) = -0.1\dot{y} + 0.001u(t)$$

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

The continuous-time state-space is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, the discrete-time model is

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k)$$
$$\Phi = e^{AT} = \begin{bmatrix} 1 & 10 - 10e^{-0.01} \\ 0 & e^{-0.01} \end{bmatrix} = \begin{bmatrix} 1 & 0.0995 \\ 0 & 0.990 \end{bmatrix}$$
$$\Gamma = \int_0^T e^{Av} dv B = \int_0^{0.1} \begin{bmatrix} 1 & 10 - 10e^{-0.1v} \\ 0 & e^{-0.1v} \end{bmatrix} dv \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} = \begin{bmatrix} 4.983 \times 10^{-6} \\ 9.95 \times 10^{-5} \end{bmatrix}$$
$$C = [1 \quad 0]$$

c)

We need to put poles to $A_m(z)$:

$$A_m(z) = z^2 - 1.906z + 0.9117$$
$$A_m(\Phi) = \Phi^2 - 1.906\Phi + 0.9117I = \begin{bmatrix} 0.0057 & 0.008363 \\ 0 & 0.004864 \end{bmatrix}$$
$$W_c = [\Gamma \quad \Phi\Gamma] = \begin{bmatrix} 4.983 \times 10^{-6} & 1.488 \times 10^{-5} \\ 9.95 \times 10^{-5} & 9.851 \times 10^{-5} \end{bmatrix}$$
$$L = [0 \quad 1] W_c^{-1} A_c(\Phi) = [572.85 \quad 816.02]$$

d)

The continuous-time transfer function is

$$H(s) = \frac{1}{1000s^2 + 100s}$$

We can get discrete-time T.F from continuous-time T.F

$$H(z) = \frac{4.983 \times 10^{-6} z + 4.967 \times 10^{-6}}{z^2 - 1.99z + 0.99} = \frac{B(z)}{A(z)}$$

The output signal $Y(z)$ can be expressed:

$$Y(z) = CX(z) = C(zI - \Phi + \Gamma L)^{-1} H_{ff}(z) U_c(z) = H_{ff}(z) \frac{B(z)}{A_m(z)} U_c(z)$$
$$H_{ff}(z) = \frac{B_m(z)}{B(z)} = \frac{0.002874z + 0.002787}{4.983 \times 10^{-6} z + 4.967 \times 10^{-6}} = \frac{576.76z + 559.3}{z + 0.997}$$

e)

The observability matrix is:

$$W_o = \begin{bmatrix} 1 & 0 \\ 1 & 0.0995 \end{bmatrix}$$

$rank(W_o) = 2$, so ervey state can be esstimate via output signal. We need to design an Dead-bead Observer.

$$\hat{x}(k+1) = \Phi x(k) + \Gamma u(k) + K(y(k) - \hat{y}(k))$$
$$\hat{y}(k) = C\hat{x}(k)$$
$$A_o(\Phi) = \Phi^2 = \begin{bmatrix} 1 & 0.1980 \\ 0 & 0.9802 \end{bmatrix}$$
$$K = A_o(\Phi) W_o^{-1} [0 \quad 1]^T = \begin{bmatrix} 1.990 \\ 9.851 \end{bmatrix}$$

The input signal is:

$$U(z) = -L\hat{X}(z) + H_{ff}(z)U_c(z)$$