EE5110: Special Topics in Automation and Control

Segment C: Control Optimization

Lecture One

Xiang Cheng

Associate Professor

Department of Electrical & Computer Engineering
The National University of Singapore

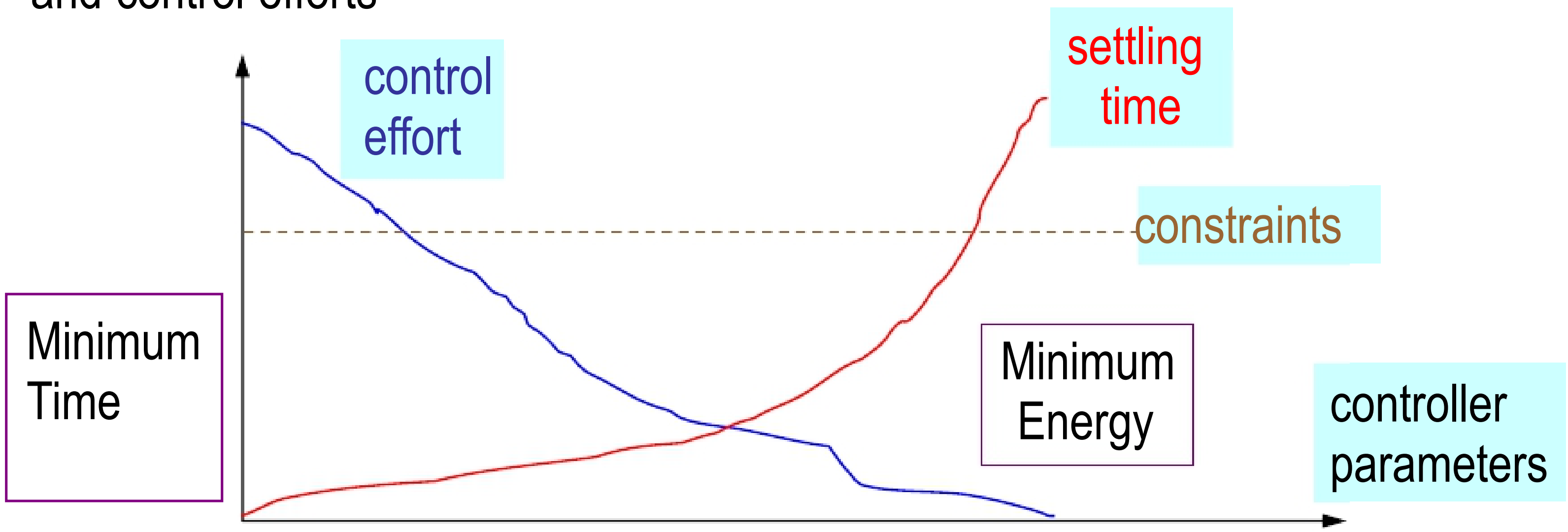
Phone: 65166210 Office: Block E4-08-07
Email: elexc@nus.edu.sg

Why do we need optimization for control problem?

Control specifications often have multiple objectives which conflict, e.g.

- reduce the settling time $\Rightarrow$ reduce the damping $\Rightarrow$ increase the overshoot
- reduce the steady state error $\Rightarrow$ increase gain $\Rightarrow$ higher energy cost

Optimization is to achieve the best trade-off among all the desired objectives. It is to make balance between the convergence speed (settling time) and control efforts



Brief Overview: Two approaches will be introduced to deal with the optimization problem. The first one is the calculus of variations, in which the minimization function is regarded as a point in a function space. The second one is the dynamic programming approach, in which the optimal policy is computed at every state. Due to time constraint, only scalar case will be discussed in the class.

CA: A set of four problems related to calculus of variations and dynamic programming will be given.

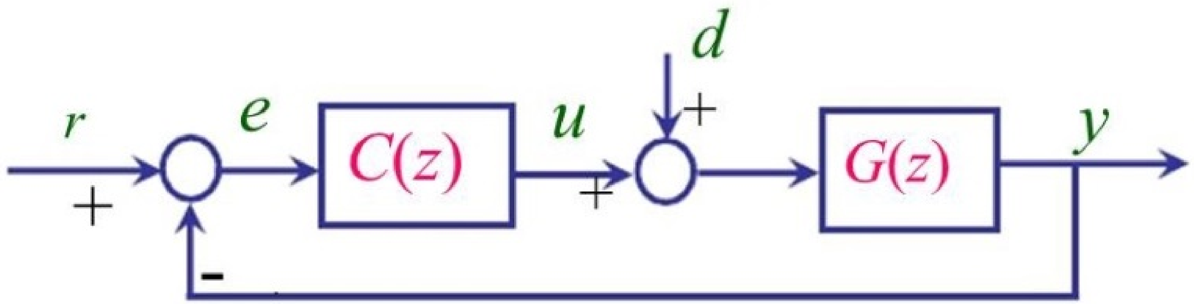
Research Project: “How effective is approximate dynamic programming?"

Students will carry out a critical evaluation of the latest technique of data-driven control: approximate dynamic programming. The technical report should consist of at least three parts:

1. A comprehensive literature study of approximate dynamic programming (ADP)
2. A critical evaluation of the strength and weakness of the ADP.
3. Simulation studies to validate the analysis in part 2.

Why optimal control?

- Let’s recall what we have learnt about designing a controller for a system.



<u>If you are given a system without a mathematical model, and you want to design a controller to improve its performance, what would you do first?</u>

Try the model free method! The PID controller should be attempted first.

<u>If the performance of the PID controller is not good enough, what would you do?</u>

 Try to use physical laws and identification techniques to build a model.

<u>Once a model is available, what is the simplest model based controller you have learned in the past?</u>

You may try pole-placement controller to make transfer function of the closed-loop match some desired reference model.

When you design pole-placement controller, do you ever care about the size of control signals?

The object of pole placement is just try to place the poles of the closed loop to some desired ones. We never asked the question if the control input can be supplied by the actuator or what the cost of the control is.

In real control system design, should we care about the size of the control signals?

However, in reality, we must weigh the cost of undesirable performance of the system against the cost of control. This is one of the basic facts of control theory: nothing for nothing, or No Free Lunch.

- Let's take a look at one example: the single-link robotic arm, or the inverted pendulum.

- Consider the pendulum example, after linearization around the equilibrium point (upright position), we have the model

$$\ddot{y}(t) - y(t) = u(t)$$

Objective: To bring the system OUTPUT, y(t), to zero for any small nonzero initial conditions.

If feedback control is used, then

$$u = g(y, \dot{y})$$

and we would like to choose the function $g(y, \dot{y})$

in such a way that y and $\dot{y}$ approach zero reasonably rapidly without violating the condition that $|y|$ and $|\dot{y}|$ must not be too large.

This is clearly not a precise analytic formulation. Let's try to put it into a more precise way.

How to translate the requirement of real world application into mathematical language is the most important skills for engineer!

Let us then ask for a control law that makes y and $\dot{y}$ small on average.

<u>How to compute average for numbers $\{x_1, x_2, x_3, \cdots, x_n\}$?</u>

$$\frac{\sum_{1}^{n} x_i}{n}$$

<u>How to take average for continuous variable, f(t) over the interval [0,T]?</u>

$$\frac{\sum_{1}^{n} \mathrm{f}(x_i)}{n} = \frac{\sum_{1}^{n} \mathrm{f}(x_i)\Delta}{n\Delta} = \frac{\int f(x)dx}{T}$$

So integration implies averaging!

Let us try to determine the feedback control function u=g(y, $\dot{y}$), so that the quantity

$$\int_0^T (y^2 + \dot{y}^2)dt$$

is minimal.

Now we have translated the requirement that the signals are small on average into some mathematical form!

What about input?

We have, however, omitted to impose any cost of control. Different types of control laws require different amounts of effort to implement. How should we estimate this type of cost? One way is to use some average cost, such as

$$\int_0^T u^2 \, dt$$

If we add the two costs together, we have a total cost of

$$\int_0^T \left(y^2 + \dot{y}^2 + u^2\right)dt = \int_0^T \left(y^2 + \dot{y}^2 + g^2(y, \dot{y})\right)dt$$

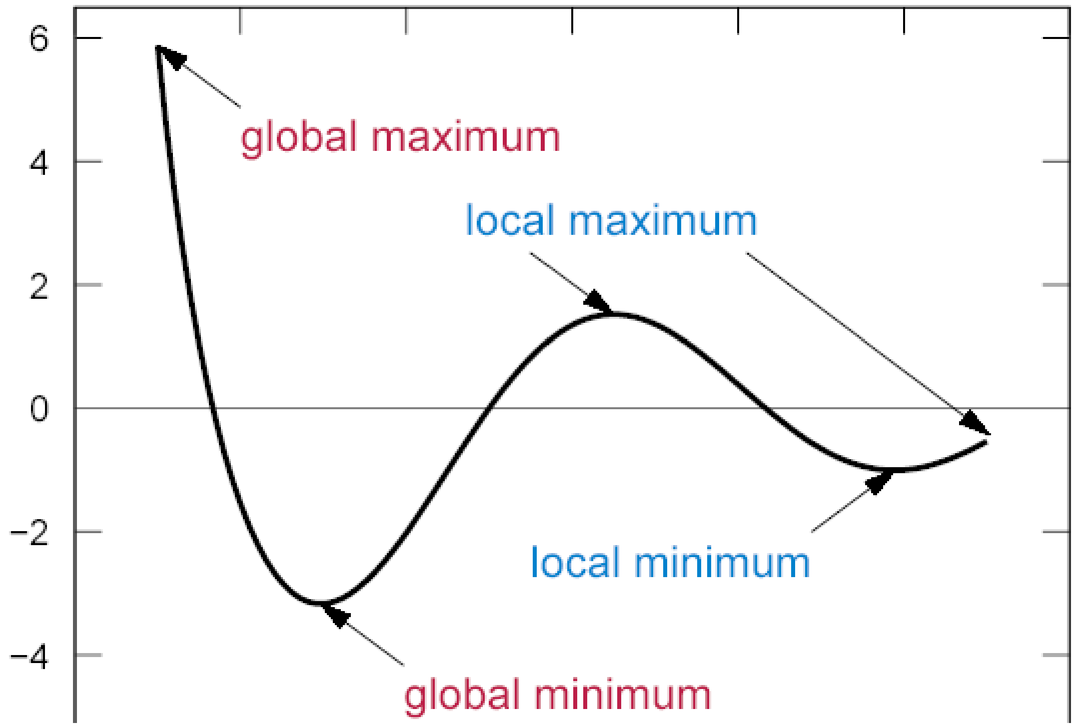
How do we choose the feedback control function g so as to minimize this total cost?

Is this an optimization problem?

- **How to find out the maxima or minima of a function f(x) in [a,b]? Where do we search?**

If you are secondary school student, you may answer to search the whole interval [a,b]!

Since you are a graduate student now, you should know a better way.



The optimal solution should satisfy the condition:

$$f'(x) = 0$$

But how did we derive this condition?

The key idea is to evaluate the change of the function f(x) in the neighbourhood of the optimal solution.

So we take the problem as solved and assume that x is the optimal point, we need to compute the variation of the function values in the neighbourhood of x, $f(x + \Delta x)$.

<u>How to evaluate $\boldsymbol{f(x + \Delta x)}$ around x?</u>

Use Taylor series

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2}\Delta x^2 + \cdots$$

It is clear that if x is the optimal point, then $f'(x)$ must be 0!

The problem is converted into solving algebraic equations!

$$f'(x) = 0$$

- To find out the optimal solution for

$$\int_0^T (y^2 + \dot{y}^2 + u^2)dt = \int_0^T (y^2 + \dot{y}^2 + g^2(y, \dot{y})dt$$

This is a map from the space of functions to real number.
So it is not a simple function f(x). We call it functional.

<u>Where do we search?</u>

Search the best candidate in the space of functions instead of a variable in a real line!

Can we directly use the standard way $f'(x) = 0$ to solve this problem?

No.

This is the subject of calculus of variations, which you are going to learn today!

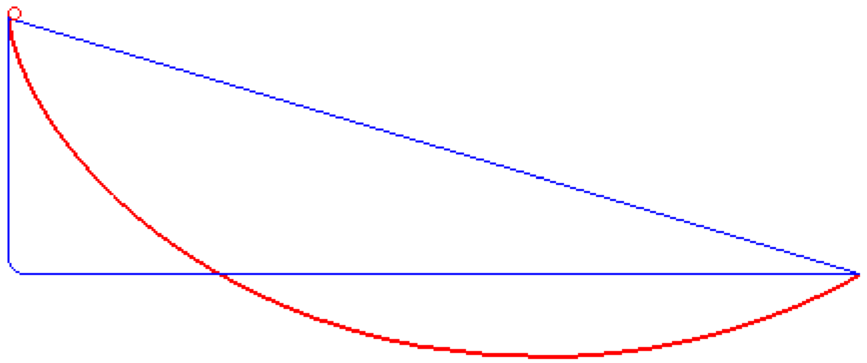
Who’s Who in Mathematics!

The list of mathematicians who made contributions to this problem.

- Fermat, Bernoulli brothers, Leonhard Euler, Lagrange,, Legendre,
- Isaac Newton and Gottfried Leibniz,
- Carl Friedrich Gauss (1829), Siméon Poisson
- Carl Jacobi, Cauchy,
- Weierstrass.
- David Hilbert, Emmy Noether, Leonida Tonelli, Henri Lebesgue and Jacques Hadamard
- Richard Bellman

The most famous problem in the history of Calculus of Variations

brachistochrone curve (from Ancient Greek βράχιστος χρόνος *(brákhistos khrónos)*, meaning "shortest time"), or curve of fastest descent



You need to overshoot to be fastest!

Let's go back to our optimal control problem!

$$\int_0^T (y^2 + \dot{y}^2 + u^2)dt = \int_0^T (y^2 + \dot{y}^2 + g^2(y, \dot{y}))dt$$

- It is always a good strategy by considering the simple system first.

$$\frac{dy}{dt} = ay + u, y(0) = c,$$

and a quadratic approximation of the cost function

$$J(y,u) = \int_0^T [y^2 + u^2]dt$$

<u>The simplest case</u>

Let a = 0, then $\dot{y} = u$. We thus have the specific problem of minimizing the functional

$$J(y) = \int_0^T (y^2 + \dot{y}^2)dt \qquad \text{where } y(0) = c.$$

In the following, we will try to find the optimal solution y(t) to minimize the functional

$$J(y) = \int_0^T (y^2 + \dot{y}^2) dt \qquad \text{where } y(0) = c.$$

How to deal with minimization of a functional?

When we try to find out the minimal of a function, we use Taylor series:

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2}\Delta x^2 + \cdots$$

to compute the variations of a function.

Similarly, we need to know how to compute the variations of functional in the neighborhood of the optimal solution.

Once again, let's take the problem as solved, and let y_o denote a hypothetical solution to the minimization of the functional

$$J(y) = \int_0^T \left(y^2 + \dot{y}^2\right) dt \tag{3.1}$$

<u>How do we compute the change of a function f(x) around a point xo?</u>

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2}\Delta x^2 + \cdots$$

We need to know how the function changes in the neighborhood of x: $f(x + \Delta x)$

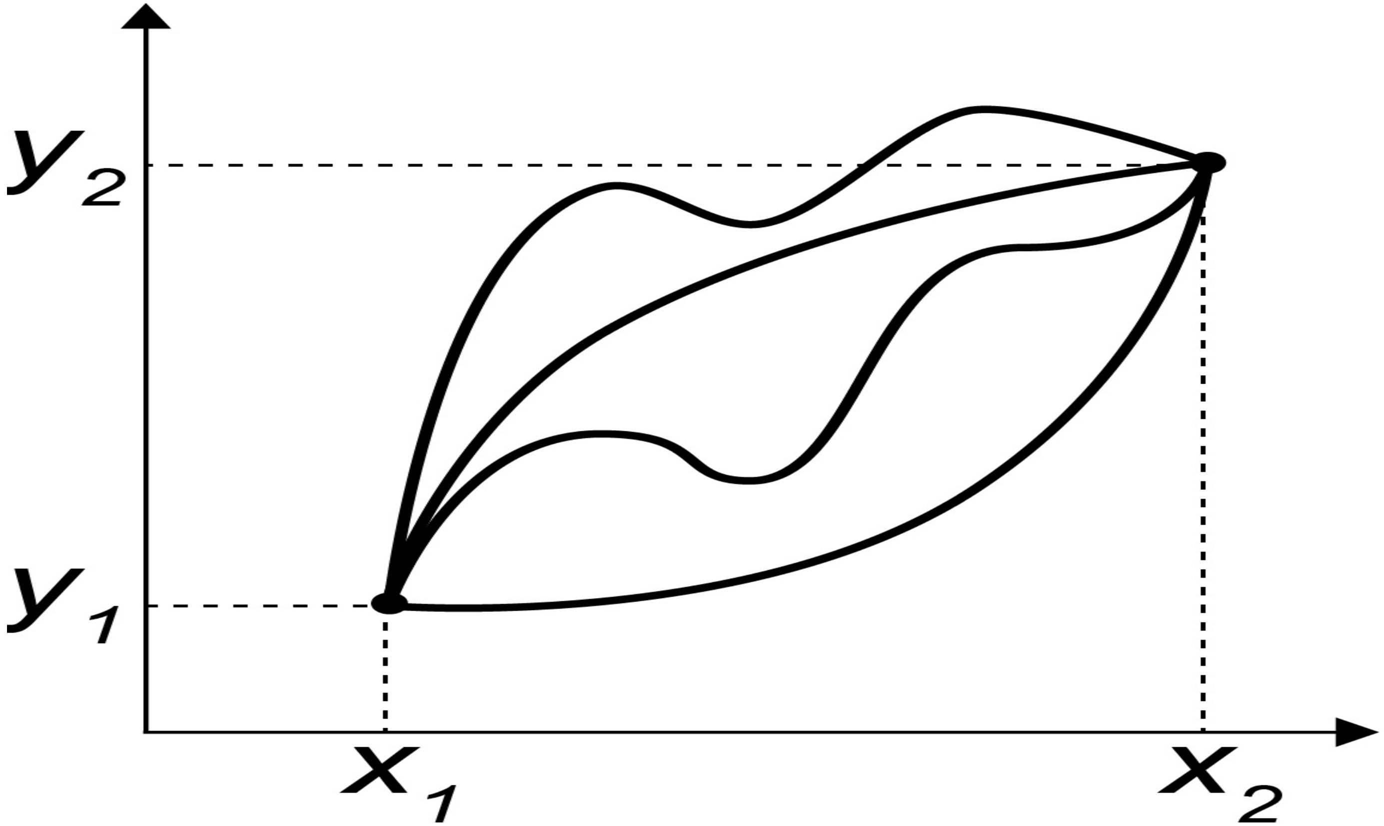
Similarly, we need to find out a way to compute the change of the functional around the optimal solution

$$J(y + \delta y)$$

<u>But how do we express the change of a function</u> δy?

$J(y+\delta y)$

<u>Key question: how do we express the change of a function</u> δy?



How many ways can you change the function?

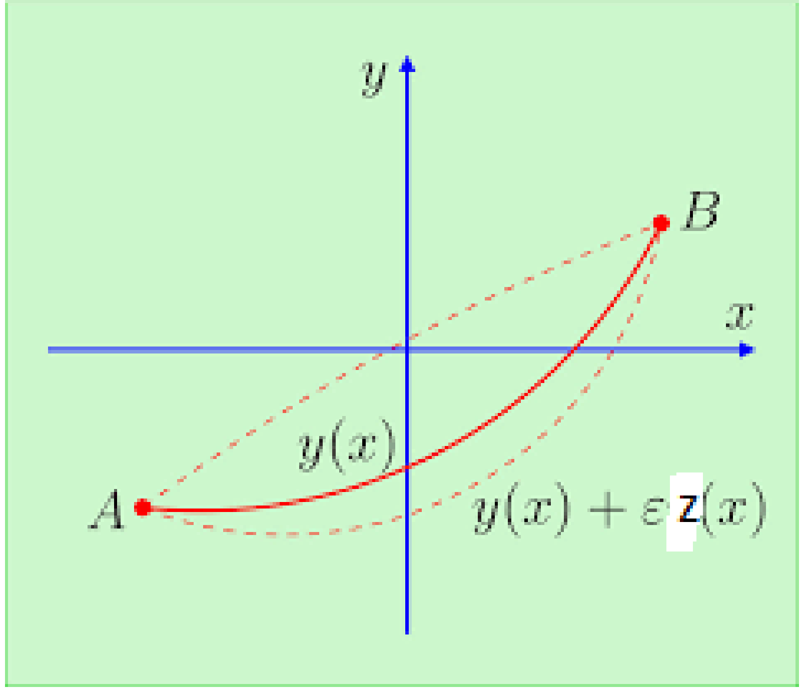
There are infinite number of ways to vary the function.

There are infinite number of directions to consider in the functional space. Which direction to choose?

Key question: how do we express the change of a function δy?

There are infinite number of directions to consider in the functional space. So let's just take any direction, i.e. any function z, and compute the change of the function as

$$\delta y = \varepsilon z$$



We introduce ε as a scalar parameter to control the size of the change.

This is the most important step in understanding the rest of the theory!

Now we compute the variation of the functional in the neighbourhood of the optimal solution, y_o, in the direction of z(t).

$$J(y_o + \varepsilon z) = \int_0^T [(y_o + \varepsilon z)^2 + (\dot{y}_o + \varepsilon \dot{z})^2]\, dt \tag{3.2}$$

When y_o and z are fixed, i.e. fixed functions, is this a function or functional?

It is just a function of the scalar variable ε !

Does this function have a minimal value? where is it?

If y_o is indeed the function that yields the absolute minimum of J(y), then $J(y_o + \varepsilon z)$ as a function of the scalar parameter ε must have an absolute minimum at ε=0.

Then we can just use the standard result from calculus.

- Using $\delta y = \varepsilon z$ changes the problem into the minimization of a scalar function !

What is the necessary condition on a scalar function $J(y_o + \varepsilon z)$ to get a minimum?

$$\left.\frac{d}{d\varepsilon} J(y_0 + \varepsilon z)\right|_{\varepsilon=0} = 0 \qquad (3.3)$$

$$J(y_o + \varepsilon z) = \int_0^T [(y_o + \varepsilon z)^2 + (\dot{y}_o + \varepsilon \dot{z})^2]\, dt$$

$$= \int_0^T ({\dot{y}_o}^2 + {y_o}^2) dt + 2\varepsilon \int_0^T (\dot{y}_o \dot{z} + y_o z)\, dt + \varepsilon^2 \int_0^T (\dot{z}^2 + z^2) dt$$

From the optimality condition (3.3), we can easily obtain

$$\int_0^T (\dot{y}_o \dot{z} + y_o z)\, dt = 0 \qquad (3.4)$$

$$\int_0^T (\dot{y}_o \dot{z} + y_o z)\, dt = 0 \qquad (3.4)$$

If both integrands contain z, then it is easy to proceed.

However, one integrand contains $\dot{z}$, and we need to deal with it.

Can we change the $\int_0^T \dot{y}_o \dot{z} dt$ into a form like $\int_0^T (**) z dt$?

Did you ever encounter such problem in calculus? What is the standard technique?

integration by parts

$$(f(x)g(x))' = f'g + fg'$$

$$\int_0^T (f(x)g(x))' dx = \int_0^T f' g dx + \int_0^T f g' dx$$

$$\int_0^T f g' dx = \int_0^T (f(x)g(x))' dx - \int_0^T f' g dx = f(x)g(x)\big|_0^T - \int_0^T f' g dx$$

So $$\int_0^T \dot{y}_o \dot{z} dt = \dot{y}_o z\big|_0^T - \int_0^T \ddot{y}_o z dt$$

$$\int_0^T (\dot{y}_o \dot{z} + y_o z)\, dt = 0 \tag{3.4}$$

Integrating by parts, we obtain from (3.4) the relation

$$\dot{y}_o(T)z(T) - \dot{y}_o(0)z(0) + \int_0^T z(-\ddot{y}_o + y_o)\,dt = 0 \tag{3.5}$$

Let's first look at the boundary conditions.

What is z(0)?

Since $y_0 + \varepsilon z$ is an admissible function, it satisfies the boundary condition

$$y_0(0) + \varepsilon z(0) = c$$

We see z(0)=0 since y(0)=c.

Can we also show z(T)=0 in a similar fashion?

No, we only know y(0)=c, and nothing about y(T).

$$\int_0^T (\dot{y}_o \dot{z} + y_o z)\, dt = 0 \tag{3.4}$$

Integrating by parts, we obtain from (3.4) the relation

$$\dot{y}_o(T)z(T) - \dot{y}_o(0)z(0) + \int_0^T z(-\ddot{y}_o + y_o)dt = 0 \tag{3.5}$$

Since z(0)=0, we have

$$\dot{y}_o(T)z(T) + \int_0^T z(-\ddot{y}_o + y_o)dt = 0$$

In order to derive the equation, we need to impose additional condition.

Let's assume $\dot{y}_o(T) = 0.$ Then we have

$$\int_0^T z(-\ddot{y}_o + y_o)dt = 0$$

Since this equation holds for any admissible function z(t), we can show that

$$-\ddot{y}_o + y_o = 0 \tag{3.6}$$

And this is the famous Euler Equation!

- If f(x) is a simple function, the minimization problem can be turned into solving an algebraic equation as the candidates are points.

To minimize a functional, the problem becomes solving an ODE as the candidates are functions instead of points!

<u>***Break***</u>

State-of-the-art control systems

<u>*Future Technology*</u>

Euler equation

$$\ddot{y}_o - y_o = 0 \tag{3.6}$$

<u>How to solve this equation? What is the powerful tool to solve linear ODE?</u>

Let's try the famous Laplace transform, and we will have

$$s^2 Y_o(s) - s y_o(0) - \dot{y}_o(0) - Y_o(s) = 0$$

And we get

$$Y_o(s) = \frac{s y_o(0) - \dot{y}_o(0)}{(s^2 - 1)}$$

But do we know the initial conditions $y_o(0)$ and $\dot{y}_o(0)$?

No. We know $y_o(0)$=c and $\dot{y}_o(T)$=0 !

The desired solution satisfies a two-point boundary condition instead of initial conditions. Therefore, we cannot use Laplace Transform to get the solution.

$$\ddot{y}_o - y_o = 0 \qquad (3.6)$$

$$y_o(0) = c \text{ and } \dot{y}_o(T) = 0$$

From the Linear ODE theory, we know that we should first get the roots of the characteristic equation associated with the ODE (3.6):

$$\lambda^2 - 1 = 0$$

There are two roots: $\lambda = 1$ and $\lambda = -1$

There are two fundamental solutions in the form of $e^{\lambda t}$:

e^t and e^{-t}.

The general solution of the Euler equation is

$$y = c_1 e^t + c_2 e^{-t}$$

$$\ddot{y}_o - y_o = 0 \tag{3.6}$$

$y_o(0)$=c and $\dot{y}_o(T)$=0

The general solution of the Euler equation is $y = c_1 e^{t} + c_2 e^{-t}$

How do we figure out the coefficients c_1 and c_2 ?

Using the boundary conditions, we have the two equations

$$c = c_1 + c_2$$

$$0 = c_1 e^{T} - c_2 e^{-T}$$

to determine the coefficients c_1 and c_2

Finally, we obtain the solution:

$$y = c\left(\frac{e^{t-T} + e^{-(t-T)}}{e^{-T} + e^{T}}\right) = c\left(\frac{\cosh(t-T)}{\cosh(T)}\right)$$

where the hyperbolic cosine function cosh(t) is defined as $\cosh(t) = \dfrac{e^{t} + e^{-t}}{2}$

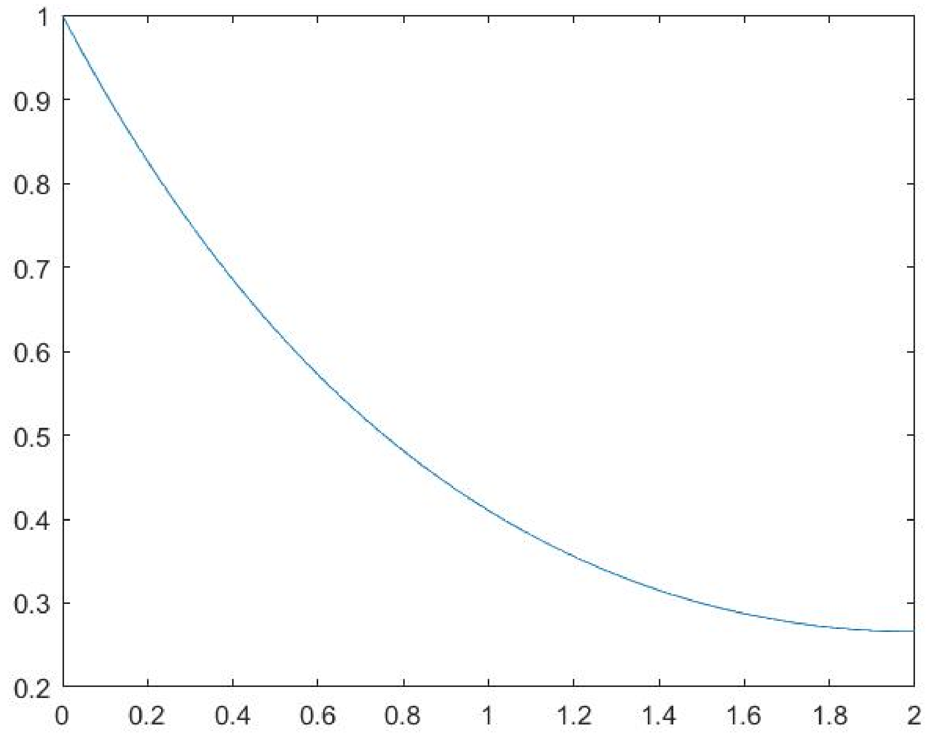
The optimal solution y(t) to minimize the cost function

$$J(y) = \int_0^T \left(y^2 + \dot{y}^2\right) dt$$

with initial condition y(0)=c, turns out to be

$$y = c\left(\frac{e^{t-T} + e^{-(t-T)}}{e^{-T} + e^{T}}\right) = c\left(\frac{\cosh(t-T)}{\cosh(T)}\right)$$

Below is an example when c=1 and T=2.



The optimal solution:

$$y = c(\frac{e^{t-T} + e^{-(t-T)}}{e^{-T} + e^{T}}) = c(\frac{\cosh(t-T)}{\cosh(T)})$$

The control input: $u(t) = \dot{y}(t) = c(\frac{e^{t-T} - e^{-(t-T)}}{e^{-T} + e^{T}})$

- **For control problem, we are interested in the case** $T \to \infty$

Let $T \to \infty$ We have

$$y(t) = c\left(\frac{e^{t-T} + e^{-(t-T)}}{e^{-T} + e^{T}}\right) = c\left(\frac{e^{t-2T} + e^{-t}}{e^{-2T} + 1}\right) \to ce^{-t}$$

$$u(t) = \dot{y}(t) = c\left(\frac{e^{t-T} - e^{-(t-T)}}{e^{-T} + e^{T}}\right) = c\left(\frac{e^{t-2T} - e^{-t}}{e^{-2T} + 1}\right) \to -ce^{-t}$$

So we have $u(t) = -y(t)$

Is this a feedback controller?

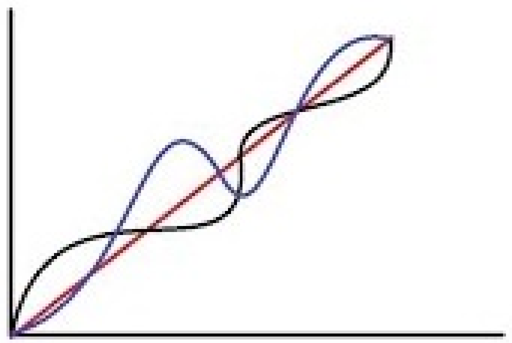
Yes. The optimal controller is in the form of a feedback controller. If you take the course of linear systems, EE5101, you can obtain the same solution by solving ARE for LQR problem!

In many optimization problem, if the search space is a functional space, then you can use calculus of variations to find the optimal solution.

The minimal curve problem is to find the shortest path between two specified locations.

We are given two distinct points, A and B,

$A = (x_a, y_a)$ and $B = (x_b, y_b)$



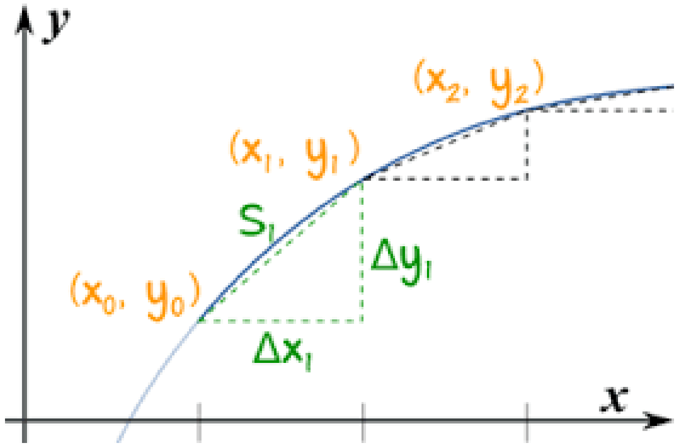
From common sense, we know that the line segment connecting A and B is the curve with minimal distance.

$$y = cx + d = \frac{(y_b - y_a)}{(x_b - x_a)}(x - x_a) + y_a \tag{3.6.2}$$

But how to prove this mathematically?

Let us see how we might formulate the minimal curve problem in a mathematically precise way.

For simplicity, we assume that the minimal curve is given as the graph of a smooth function y(x). Then, the length of the curve is given by the standard arc length integral



$$
\begin{aligned}
J(y) &= \int ds = \int \sqrt{dx^2 + dy^2} \\
&= \int dx \sqrt{1 + (\frac{dy}{dx})^2} \\
&= \int_{x_a}^{x_b} \sqrt{1 + y'(x)^2}\, dx
\end{aligned}
\tag{3.6.3}
$$

where $y'(x) = \frac{dy}{dx}$. The function y(x) is required to satisfy the boundary conditions

$$
y(x_a) = y_a, y(x_b) = y_b \tag{3.6.4}
$$

- Let us now discuss the most basic analytical techniques for solving such minimization problem.

Find a suitable function y(x) that minimizes the objective functional

$$J(y) = \int_a^b L(x, y, y')dx \qquad (3.6.5)$$

The integrand is known as the ***<u>Lagrangian</u>*** for the variational problem, in honor of Lagrange. We usually assume that the Lagrangian L(x,y,p) is a reasonably smooth function of all three of its (scalar) arguments x, y and p, which represents the derivative dy/dx.

For example, the arc length functional (3.6.3) has Lagrangian function

$$L(x, y, p) = \sqrt{1 + p^2} \qquad (3.6.6)$$

The boundary conditions are

$$y(a) = y_a, y(b) = y_b \qquad (3.6.7)$$

Following the same way for solving the optimal control problem, we need to compute the variations of the functional around the optimal solution.

Given a function y(x), we consider a variation of this function in any direction of v(x) by $y+\varepsilon v$

We have
$$J(y+\varepsilon v)=J(x,y+\varepsilon v,y'+\varepsilon v') \qquad (3.6.9)$$

Take derivative with respect to ε, we have

$$\frac{dJ(y+\varepsilon v)}{d\varepsilon}=\int_a^b \frac{dL(x,y+\varepsilon v,y'+\varepsilon v')}{d\varepsilon}dx \qquad (3.6.10)$$

But how to compute the derivative $\frac{dL}{d\varepsilon}$? Which rule should we use?

chain rule $F(x)=f(g(x))\Rightarrow F'(\mathrm{x})=\mathrm{f'(g)\,g'(x)}$

Therefore

$$\frac{dL(x,y+\varepsilon v,y'+\varepsilon v')}{d\varepsilon}=\frac{\partial L}{\partial y}(x,y+\varepsilon v,y'+\varepsilon v')v+\frac{\partial L}{\partial p}(x,y+\varepsilon v,y'+\varepsilon v')v' \qquad (3.6.11)$$

So we get

$$\frac{dJ(y+\varepsilon v)}{d\varepsilon} = \int_a^b [\frac{\partial L}{\partial y}(x, y+\varepsilon v, y'+\varepsilon v')v + \frac{\partial L}{\partial p}(x, y+\varepsilon v, y'+\varepsilon v')v']dx$$

Once again, we want to change the form of the second term into $\int_{\mathrm{a}}^{b}(**)v dx$

What should we do?

Integrate by parts we have

$$\frac{dJ(y+\varepsilon v)}{d\varepsilon} = \int_a^b [\frac{\partial L}{\partial y}(x, y+\varepsilon v, y'+\varepsilon v')v + \frac{\partial L}{\partial p}(x, y+\varepsilon v, y'+\varepsilon v')v']dx$$

$$= v\frac{\partial L}{\partial p}(x, y+\varepsilon v, y'+\varepsilon v')\Big|_a^b + \int_a^b v[\frac{\partial L}{\partial y}(x, y+\varepsilon v, y'+\varepsilon v') - \frac{d}{dx}(\frac{\partial L}{\partial p}(x, y+\varepsilon v, y'+\varepsilon v'))]dx$$

Since y+εv satisfies the boundary conditions, we have

$$v(a)=0, \text{ and } v(b)=0 \qquad (3.6.14)$$

So we obtain

$$\frac{dJ(y+\varepsilon v)}{d\varepsilon} = \int_a^b v[\frac{\partial L}{\partial y}(x, y+\varepsilon v, y'+\varepsilon v') - \frac{d}{dx}(\frac{\partial L}{\partial p}(x, y+\varepsilon v, y'+\varepsilon v'))]dx$$

Set ε=0, we have

$$\int_a^b v[\frac{\partial L}{\partial y}(x, y, y') - \frac{d}{dx}(\frac{\partial L}{\partial p}(x, y, y'))]dx = 0 \qquad (3.6.15)$$

Since v(x) is arbitrary function satisfying v(a)=0 and v(b)=0, we have

$$\frac{\partial L}{\partial y}(x, y, y') - \frac{d}{dx}(\frac{\partial L}{\partial p}(x, y, y')) = 0 \qquad (3.6.16)$$

Euler–Lagrange equation

The optimal solution , y(x), to minimize the functional

$$J(y) = \int_a^b L(x, y, y')dx \tag{3.6.5}$$

satisfies the ODE

$$\frac{\partial L}{\partial y}(x, y, y') - \frac{d}{dx}(\frac{\partial L}{\partial p}(x, y, y')) = 0 \tag{3.6.16}$$

Let us return to the most elementary problem in the calculus of variations: finding the curve of shortest length connecting two points in the plane.

As we derived earlier, such planar geodesics minimize the arc length integral

$$J(y) = \int_{x_a}^{x_b} \sqrt{1 + y'(x)^2}\, dx$$

subject to the boundary conditions

$$y(a) = y_a, y(b) = y_b$$

So the corresponding Lagrangian is

$$L(x, y, p) = \sqrt{1 + p^2}$$

Let's find out the corresponding Euler–Lagrange equation

$$\frac{\partial L}{\partial y}(x, y, y') - \frac{d}{dx}\left(\frac{\partial L}{\partial p}(x, y, y')\right) = 0$$

In this case, $L(x, y, p) = \sqrt{1 + p^2}$

So we have

$$\frac{\partial L}{\partial y}(x, y, y') = 0$$

and $$\frac{\partial L}{\partial p}(x, y, y')) = \frac{p}{\sqrt{1 + p^2}}$$

The **Euler–Lagrange equation** in this case takes the form

$$-\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = -\frac{y''}{(1+(y')^2)^{\frac{3}{2}}} = 0$$

$$\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = \frac{d}{dx}[((1+(y')^2)^{-\frac{1}{2}})(y')]$$

$$= -\frac{1}{2}((1+(y')^2)^{-\frac{3}{2}}(2y')y'')(y') + ((1+(y')^2)^{-\frac{1}{2}})y''$$

$$= \frac{y''}{(1+(y')^2)^{\frac{3}{2}}}(-(y')^2+1+(y')^2)$$

$$= \frac{y''}{(1+(y')^2)^{\frac{3}{2}}}$$

So finally we have $$y''=0$$

We deduce that the solutions to the Euler–Lagrange equation are all affine functions, y=cx+d, whose graphs are straight lines.

$$y = \frac{(y_b - y_a)}{(x_b - x_a)}(x - x_a) + y_a$$

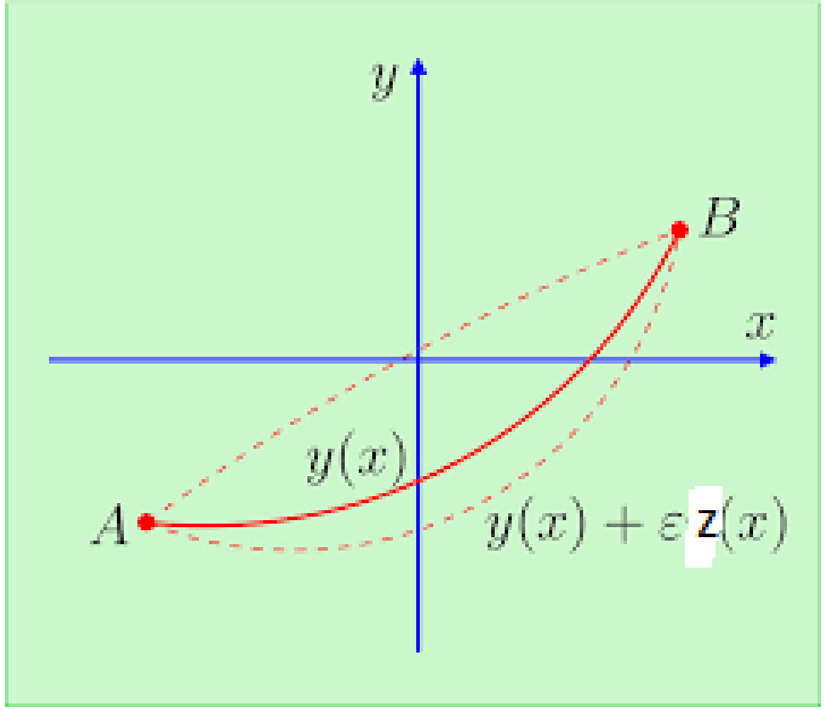
Summary: the key to calculus of variations

- **I hope you have learned how to compute the variations of the functional around the critical point!**

$$J(y+\delta y)$$

How to specify the change of the function, δy?

$$\delta y = \varepsilon z$$



where z can be any admissible function and ε is introduced to control the size of the change, which is the key!

Then we can use the standard techniques from classical calculus to solve the problem.

Q & A…

THANK YOU !