EE5137 Stochastic Processes: Problem Set 5
Assigned: 11/02/22, Due: 18/02/22

There are six (6) non-optional problems in this problem set.

1. Exercise 2.1(b) (Gallager's book) Find the moment generating function of X (or find the Laplace transform of $f_X(x)$), and use this to find the moment generating function (or Laplace transform) of $S_n = X_1 + X_2 + \ldots + X_n$.

   Solution: Since X has exponential distribution, the formula for the MGF is almost trivial here,

$$g_X(r) = \int_0^\infty \lambda e^{-\lambda x} e^{rx} dx = \frac{\lambda}{\lambda - r}, \quad \text{for} \quad r < \lambda. \tag{1}$$

   Since S_n is the sum of n IID rv's,

$$g_{S_n}(r) = [g_X(r)]^n = \left(\frac{\lambda}{\lambda - r} \right)^n. \tag{2}$$

2. Exercise 2.2(a) and 2.2(b) (Gallager's book)

   (a) Find the mean, variance, and moment generating function of $N(t)$, as given by (2.17).

   (b) Show by discrete convolution that the sum of two independent Poisson rv's is again Poisson.

   Solution:

   (a) We have

$$\mathbb{E}[N(t)] = \sum_{n=0}^\infty \frac{n(\lambda t)^n e^{-\lambda t}}{n!} = (\lambda t) \sum_{n=1}^\infty \frac{(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!} = (\lambda t) \sum_{n=0}^\infty \frac{(\lambda t)^n e^{-\lambda t}}{n!} = \lambda t, \tag{3}$$

   where in the last step, we recognized the terms in the sum as the Poisson PMF of parameter λt; since the PMF must sum to 1, the mean is λt is shown. Using the same approach, the second moment is $(\lambda t)^2 + \lambda t$, so the variance is λt. For the MGF,

$$\mathbb{E}[e^{rN(t)}] = \sum_{n=0}^\infty \frac{(\lambda t e^r)^n e^{-\lambda t}}{n!} = e^{-\lambda t} \sum_{n=0}^\infty \left[\frac{(\lambda t e^r)^n e^{-\lambda t e^r}}{n!} \right] e^{\lambda t e^r}. \tag{4}$$

   Recognizing the term in brackets as the PMF of a Poisson rv of parameter λe^r in this expression, we get

$$\mathbb{E}[e^{rN(t)}] = e^{-\lambda t} e^{\lambda t e^r} = \exp[\lambda t (e^r - 1)]. \tag{5}$$

(b) Let X and Y be independent Poisson with parameter λ and μ respectively. Then,

$$p_{X+Y}(m) = \sum_{n=0}^{m} p_X(n)p_Y(m-n) = e^{-\lambda-\mu}\sum_{n=0}^{m}\frac{\lambda^n\mu^{m-n}}{n!(m-n)!} \tag{6}$$

$$= \frac{e^{-\lambda-\mu}}{m!}\sum_{n=0}^{m}\binom{m}{n}\lambda^n\mu^{m-n} = \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^m}{m!}, \tag{7}$$

where we recognized the final sum as a binomial sum.

3. Exercise 2.4 (Gallager's book) Assuming that a counting process $\{N(t) : t > 0\}$ has the independent and stationary increment properties and satisfies (2.17) (for all $t > 0$). Let X_1 be the epoch of the first arrival and X_n be the interarrival time between the $(n-1)$st and the nth arrival. Use only these assumptions in doing the following parts of this exercise.

(a) Show that $\Pr\{X_1 > x\} = e^{-\lambda x}$.

Solution: The event $\{X_1 > x\}$ is the same as the event $\{N(x) = 0\}$. Thus, from (2.17), $\Pr\{X_1 > x\} = \Pr\{N(x) = 0\} = e^{-\lambda x}$.

(b) Let S_{n-1} be the epoch of the $(n-1)$st arrival. Show that $\Pr\{X_n > x|S_{n-1} = \tau\} = e^{-\lambda x}$.

Solution: The condition event $\{S_{n-1} = \tau\}$ is somewhat messy to deal with in terms of the parameters of the counting process, so we start by solving an approximation of the desired result and then go to a limit as the approximation becomes increasing close. Let $\delta > 0$ be a small positive number which we later allow to approach 0. We replace the event $\{S_{n-1} = \tau\}$ with $\{\tau - \delta < S_{n-1} \leq \tau\}$. Since the occurrence of two arrivals in small interval of size δ is very unlikely (of order $o(\delta)$), we also include the condition that $S_{n-2} \leq \tau - \delta$ and $S_n \geq \tau$. With this, the approximation conditioning event becomes

$$\{S_{n-2} \leq \tau - \delta < S_{n-1} \leq \tau < S_n\} = \{N(\tau - \delta) = n-2, \tilde{N}(\tau - \delta, \tau) = 1\}. \tag{8}$$

Since we are irretrievably deep in approximations, we also replace the event $\{X_n > x\}$ (conditional on this approximating condition) with $\{\tilde{N}(\tau, \tau + x) = 0\}$. Note that this approximation is exact for $\delta = 0$, since in that case $S_{n-1} = \tau$, so $X_n > x$ means that no arrivals occur in $(\tau, \tau + x)$.

We can now solve this approximate problem precisely,

$$\Pr\{\tilde{N}(x, \tau + x) = 0|N(\tau - \delta) = n-2, \tilde{N}(\tau - \delta, \tau) = 1\} = \Pr\{\tilde{N}(\tau, \tau + x) = 0\} \tag{9}$$

$$= e^{-\lambda x}. \tag{10}$$

In the first step, we used the independent increment property and in the second, the stationary increment property along with (a).

In the limit $\delta \to 0$, the conditioning event becomes $S_{n-1} = \tau$ and the conditioned event becomes $X_n > x$. The argument is very convincing, and becomes more convincing the more one thinks about it. At the same time, it is somewhat unsatisfactory since both the conditioned and conditioning event are being approximated. One can easily upper and lower bound the probability that $X_n > x$ for each δ but the "proof" then requires many uninsightful and tedious details.

(c) For each $n > 1$, show that $\Pr\{X_n > x\} = e^{-\lambda x}$ and that X_n is independent of S_{n-1}.

Solution: We have seen that $\Pr\{X_n > x|S_{n-1} = \tau\} = e^{-\lambda x}$. Since the value of this probability conditioned on $\{S_{n-1} = \tau\}$ does not depend on τ, X_n must be independent of S_{n-1}.

(d) Argue that X_n is independent of $X_1, X_2, \cdots, X_{n-1}$.

Solution: Equivalently, we show that X_n is independent of $\{S_1 = s_1, S_2 = s_2, \cdots, S_{n-1} = s_{n-1}\}$ for all choices of $0 < s_1 < s_2 < \cdots s_{n-1}$. Using the same artifice as in (b), this latter event is the same as the limit as $\delta \to 0$ of the event

$$\{N(s_1-\delta)=0, \tilde{N}(s_1-\delta, s_1)=1, \tilde{N}(s_1, s_2-\delta)=0,$$
$$\tilde{N}(s_2-\delta, s_2)=1, \cdots, \tilde{N}(s_{n-1}-\delta, s_{n-1})=1\}. \tag{11}$$

From the independent increments property, the above event is then independent of the random variable $\tilde{N}(s_{n-1}, s_{n-1}+x)$ for each $x>0$. As in (b), this shows that X_n is independent of $S_1, S_2, \cdots, S_{n-1}$ and thus of $X_1, X_2, \cdots, X_{n-1}$.

The most interesting part of this entire exercise is that the Poisson CDF was used only to derive the fact that X_1 has an exponential CDF. In other words, we have shown quite a bit more than Definition 2 of a Poisson process. We have shown that if X_1 is exponential and the stationary and independent increment properties hold, then the process is Poisson. On the other hand, we have shown that a careful derivation of the properties of the Poisson process from this definition requires a great deal of intricate, uninsightful, and tedious analysis.

4. Exercise 2.7 (Gallager's book) Assume that a counting process $\{N(t); t>0\}$ has the independent and stationary increment properties and, for all $t>0$, satisfies

$$\Pr\{\tilde{N}(t, t+\delta)=0\} = 1-\lambda\delta+o(\delta), \tag{12}$$
$$\Pr\{\tilde{N}(t, t+\delta)=1\} = \lambda\delta+o(\delta), \tag{13}$$
$$\Pr\{\tilde{N}(t, t+\delta)>1\} = o(\delta). \tag{14}$$

(a) Let $F_1^c(\tau)=\Pr\{N(\tau)=0\}$ and show that $dF_1^c(\tau)/d\tau = -\lambda F_1^c(\tau)$.

(b) Show that X_1, the time of the first arrival, is exponential with parameter λ.

(c) Let $F_n^c(\tau)=\Pr\{\tilde{N}(t, t+\tau)=0 | S_{n-1}=t\}$ and show that $dF_n^c(\tau)/d\tau = -\lambda F_n^c(\tau)$.

(d) Argue that X_n is exponential with parameter and independent of earlier arrival times.

Hint for Part(a): Recall that the derivative of a function $f(t)$ at the point τ is

$$\left.\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right|_{t=\tau} = \lim_{\delta\downarrow 0}\frac{f(\tau+\delta)-f(\tau)}{\delta}.$$

Solution:

(a) Note that F_1^c is the complementary CDF of X_1. Using the fundamental definition of derivative,

$$\frac{dF_1^c(\tau)}{d\tau} = \lim_{\delta\to 0}\frac{F_1^c(\tau+\delta)-F_1^c(\tau)}{\delta} = \lim_{\delta\to 0}\frac{\Pr\{N(\tau+\delta)=0\}-\Pr\{N(\tau)=0\}}{\delta} \tag{15}$$
$$= \lim_{\delta\to 0}\frac{\Pr\{N(\tau)=0\}\Big(\Pr\{\tilde{N}(\tau, \tau+\delta)=0\}-1\Big)}{\delta} \tag{16}$$
$$= \lim_{\delta\to 0}\frac{\Pr\{N(\tau)=0\}(1-\lambda\delta+o(\delta)-1)}{\delta} \tag{17}$$
$$= \Pr\{N(\tau)=0\}(-\lambda) = -\lambda F_1^c(\tau), \tag{18}$$

where (16) resulted from the independent increment property and (17) resulted from (13).

(b) The complementary CDF of X_1 is $F_1^c(\tau)$, which satisfies the first order linear differential equation in (a). The solution to that equation, with the boundary point $F_1^c(0)=1$ is $e^{-\lambda\tau}$ for $\tau\ge 0$.

(c) By the independent increment property, $\{\tilde{N}(t, t+\tau)=0\}$ is independent of $\{S_{n-1}=t\}$, and by the stationary increment property, it has the same probability as $N(\tau)=0$. Thus, $dF_n^c(\tau)/d\tau = -\lambda F_n^c(\tau)$ follows by the argument in (a).

(d) Note that $F_n^c(\tau)$ is the complementary CDF of X_n conditional on $\{S_{n-1} = t\}$, and as shown in (c), it's distribution does not depended on S_{n-1}, and as shown in (c), it's distribution does not depend on S_{n-1}. In other words, $F_n^c(\tau)$ as found in (c) is the complementary CDF of X_n. It is exponential by the argument given in (b) for X_1, and it is independent of earlier since $\{\tilde{N}(t, t+\tau) = 0\}$ is independent of arrivals before t.

This was also shown in the solution to Exercise 2.4 (c) and (d). In other words, definitions 2 and 3 of a Poisson process both follow from the assumption that X_1 is exponential and that the stationary and independent increment properties hold.

5. Transmitters A and B independently send messages to a single receiver in a Poisson manner with rates λ_{A} and λ_{B} respectively. All the messages are so brief that we may assume that they occupy single points in time. The number of words in a message, regardless of the source that is transmitting it, is a random variable with PMF

$$p_W(w) = \begin{cases} 2/6 & w = 1 \\ 3/6 & w = 2 \\ 1/6 & w = 3 \\ 0 & \text{otherwise} \end{cases}$$

and is independent of everything else.

(a) What is the probability that during an interval of duration t, a total of exactly 9 messages will be received?

Solution: Let R be the total number of messages received during an interval of duration t. Note that R is a Poisson random variable with arrival rate $\lambda_{\mathrm{A}} + \lambda_{\mathrm{B}}$. Therefore, the probability that exactly nine messages are received is

$$\Pr(R = 9) = \frac{((\lambda_{\mathrm{A}} + \lambda_{\mathrm{B}})t)^9 e^{-(\lambda_{\mathrm{A}} + \lambda_{\mathrm{B}})t}}{9!}. \tag{19}$$

(b) Let N be the total number of words received during an interval of duration t. Determine the expected value of N.

Solution: Let R be defined as in part (a), and let W_i be the number of words in the ith message. Then,

$$N = N_1 + N_2 + \ldots + N_R, \tag{20}$$

which is a sum of a random number of random variables. Thus,

$$\begin{aligned} \mathbb{E}[N] &= \mathbb{E}[W]\mathbb{E}[R] && (21) \\ &= \left(1 \times \frac{2}{6} + 2 \times \frac{3}{6} + 3 \times \frac{1}{6}\right)(\lambda_{\mathrm{A}} + \lambda_{\mathrm{B}})\, t && (22) \\ &= \frac{11}{6}(\lambda_{\mathrm{A}} + \lambda_{\mathrm{B}})\, t. && (23) \end{aligned}$$

(c) Determine the PDF of the time from $t = 0$ until the receiver has received exactly eight three-word messages from transmitter A.

Solution: Three-word messages arrive from transmitter A in a Poisson manner, with average rate $\lambda_{\mathrm{A}} p_W(3) = \lambda_{\mathrm{A}}/6$. Therefore, the random variable of interest is Erlang of order 8, and its PDF is given by

$$f(x) = \frac{(\lambda_{\mathrm{A}}/6)^8 x^7 e^{-\lambda_{\mathrm{A}} x/6}}{7!}. \tag{24}$$

(d) What is the probability that exactly 8 out of the next 12 messages received will be from transmitter A?

Solution: Every message originates from either transmitter A or B, and can be viewed as an independent Bernoulli trial. Each message has probability $\lambda_{\mathrm{A}}/(\lambda_{\mathrm{A}}+\lambda_{\mathrm{B}})$ of originating from transmitter A (view this as a "success"). Thus, the number of messages from transmitter A (out of the next twelve) is a binomial random variable, and the desired probability is equal to

$$\binom{12}{8}\left(\frac{\lambda_{\mathrm{A}}}{\lambda_{\mathrm{A}}+\lambda_{\mathrm{B}}}\right)^{8}\left(\frac{\lambda_{\mathrm{B}}}{\lambda_{\mathrm{A}}+\lambda_{\mathrm{B}}}\right)^{4}. \tag{25}$$

6. Consider a Poisson process of rate $\lambda > 0$. Let t^* be a fixed time instant and consider the length of the interarrival interval $[U, V]$ that contains t^*. In this question, we would like to determine the distribution of

$$L = (t^* - U) + (V - t^*).$$

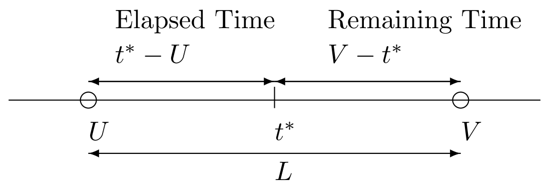


Figure 1: Here U and V are successive arrival epochs and t^* is a fixed time instance between U and V.

(i) Give a one sentence answer as to why $V - t^*$ is independent of $t^* - U$.

(ii) In class, we determined the distribution of $V - t^*$. What is this distribution?

(iii) Consider the event

$$\{t^* - U > x\}.$$

This event is the same as

$$\{\text{there are } k \text{ arrivals in the interval } [t^* - x, t^*]\}.$$

Find the integer k. No explanation is needed.

(iv) Hence, find the distribution of $t^* - U$.

(v) By using the preceding parts, find the distribution of L.

(vi) What is the distribution of an interarrival time of a Poisson process? Why is this the same or different from that of L in part (v)?

Correct Solutions:

To keep notation simple, write $t = t^*$. We rewrite $V_t = S_{N(t)}$ and $U_t = S_{N(t)+1}$ where $S_0 = 0$. Define the *age* (elapsed time in Fig. 1) and *exceedance* (remaining time in Fig. 1) by $A_t = t - S_{N(t)}$ and $Z_t = S_{N(t)+1} - t$. We want to make the dependence of A_t and Z_t on t explicit as it turns out the distribution of A_t depends on t.

We saw in lecture (Theorem 2.2.5 in Gallager's book) that $Z_t \sim \mathrm{Exp}(\lambda)$ and it is independent of all the preceding arrivals. Hence, it is also independent of V_t and A_t. There is nothing wrong with this.

Now, finding the distribution A_t is not as simple as the incorrect solution below. The problem is that with $S_0 = 0$, x cannot go beyond t^*. So we need to split the calculation into different cases.

For $0 \le s < t$,

$$\begin{aligned}
\Pr(A_t < s) &= \Pr(N(t) - N(t-s) \ge 1) \\
&= \Pr(N(s) \ge 1) \\
&= 1 - \Pr(N(s) = 0) \\
&= 1 - \exp(-\lambda s).
\end{aligned}$$

However, note that $A_t < t$ almost surely (because $S_{N(t)} > 0$ almost surely and $A_t = t - S_{N(t)}$). Hence, for $s \ge t$, $\Pr(A_t < s) = 1$. Thus, we have the cumulative distribution function of the age A_t as

$$F_{A_t}(s) = \begin{cases} 0 & s < 0 \\ 1 - \exp(-\lambda s) & 0 \le s < t \\ 1 & s \ge t. \end{cases}.$$

In other words, A_t has the same distribution as $\min\{\tilde{A}, t\}$ where $\tilde{A} \sim \mathrm{Exp}(\lambda)$. Note that A_t converges in distribution to $\tilde{A}$ as $t \to \infty$. Furthermore, the expectation of A_t is

$$\begin{aligned}
\mathbb{E}[A_t] &= \int_0^\infty (1 - F_{A_t}(s)) \, \mathrm{d}s \\
&= \int_0^t \exp(-\lambda s) \, \mathrm{d}s \\
&= \frac{1}{\lambda}(1 - \exp(-\lambda t)).
\end{aligned}$$

Now, we find the distribution of $A_t + Z_t$. Consider $0 \le s < t$ so we take the second clause in $F_{A_t}(s)$. We have

$$\begin{aligned}
\Pr(A_t + Z_t \le s) &= \int_0^\infty f_{Z_t}(z) \Pr(A_t + Z_t \le s \mid Z_t = z) \, \mathrm{d}z \\
&= \int_0^\infty \lambda \exp(-\lambda z) \Pr(A_t + z \le s \mid Z_t = z) \, \mathrm{d}z \\
&= \int_0^\infty \lambda \exp(-\lambda z) \Pr(A_t \le s - z \mid Z_t = z) \, \mathrm{d}z \\
&= \int_0^\infty \lambda \exp(-\lambda z) \Pr(A_t \le s - z) \, \mathrm{d}z \\
&= \int_0^s \lambda \exp(-\lambda z) \big(1 - \exp(-\lambda(s-z))\big) \, \mathrm{d}z \\
&= 1 - \exp(-\lambda s) - \lambda s \exp(-\lambda s).
\end{aligned}$$

On the other hand, if $s \ge t$, we have to separately consider the second and third clauses in $F_{A_t}(s)$ as follows:

$$\begin{aligned}
\Pr(A_t + Z_t \le s) &= \int_0^{s-t} \lambda \exp(-\lambda y) \, \mathrm{d}y + \int_{s-t}^s \lambda \big(1 - \exp(-\lambda(s-y))\big) \exp(-\lambda y) \, \mathrm{d}y \\
&= 1 - \exp(-\lambda s) - \lambda t \exp(-\lambda s)
\end{aligned}$$

Thus, in sum,

$$F_{A_t + Z_t}(s) = \begin{cases} 0 & s < 0 \\ 1 - \exp(-\lambda s) - \lambda s \exp(-\lambda s) & 0 \le s < t \\ 1 - \exp(-\lambda s) - \lambda t \exp(-\lambda s) & s \ge t \end{cases}.$$

Differentiating this with respect to s yields the PDF

$$f_{A_t+Z_t}(s) = \begin{cases} 0 & s < 0 \\ \lambda^2 s \exp(-\lambda s) & 0 \le s < t \\ (\lambda^2 t + \lambda)\exp(-\lambda s) & s \ge t \end{cases}.$$

Notice that this looks like the Erlang of order 2 in the interval $[0, t)$ only.

The expectation of $A_t + Z_t$ is

$$\mathbb{E}[A_t + Z_t] = \frac{1}{\lambda}(1 - \exp(-\lambda t)) + \frac{1}{\lambda}.$$

This is longer than that of a single interarrival time which is $1/\lambda$. It converges to $2/\lambda$ as $t \to \infty$ as we discussed in office hours.

Not Completely Correct Solutions:

(i) Independent increments property of the Poisson process.

(ii) This distribution is exponential with rate λ.

(iii) $k = 0$.

(iv) We need to evaluate

$$\Pr(\text{no arrivals in interval } [t^* - x, t^*]) = \Pr(N(x) = 0) = e^{-\lambda x}.$$

We have shown that

$$\Pr(t^* - U > x) = e^{-\lambda x}, \qquad \Pr(t^* - U \le x) = 1 - e^{-\lambda x}$$

so $t^* - U$ is also exponential with rate λ.

(v) Since $t^* - U$ and $V - t^*$ are independent exponentials with rate λ, their sum L is Erlang of order 2 with rate λ.

(vi) The interarrival time of a Poisson process is an exponential with rate λ. It is more likely that t^*, being observed, is in a longer interarrival interval.

---

7. (Optional) Exercise 2.8 (Gallager's book) For a Poisson process, let $t > 0$ be arbitrary and let Z_1 be the duration of the interval from t until the next arival after t. Let Z_m, for each $m > 1$, be the interarrival time from the epoch of the $(m-1)$st arrival after t until the mth arrival after t.

(a) Given that $N(t) = n$, explain why $Z_1 = X_{n+1} - t + S_n$ and, for each $m > 1$, $Z_m = X_{m+n}$.
Solution: Given $N(t) = n$, the mth arrival after t for $m \ge 1$ must be the $(n+m)$th arrival overall. Its arrival epoch is then

$$S_{n+m} = S_{n+m-1} + X_{n+m}. \tag{26}$$

By definition for $m > 1$, Z_m is the interval from S_{n+m-1} (the time of the $(m-1)$st arrival after t) to S_{n+m}. Thus, from (26), $Z_m = X_{n+m}$ for $m > 1$. For $m = 1$, Z_1 is the interval from t until the next arrival, i.e. ,$Z_1 = S_{n+1} - t$. Using $m = 1$ in (26), $Z_1 = S_n + X_{n+1} - t$.

(b) Conditional on $N(t) = n$ and $S_n = \tau$, show that $Z_1, Z_2, \cdots$ are IID.
Solution: The condition $N(t) = n$ and $S_n = \tau$ implies that $X_{n+1} > t - \tau$. Given this condition, $X_{n+1} - (t - \tau)$ is exponential, so $Z_1 = X_{n+1} - (t - \tau)$ is exponential. For $m > 1$, and for the given condition, $Z_m = X_{n+m}$. Since X_{n+m} is exponential and independent of $X_1, X_2, \cdots, X_{n+m-1}$, we see that Z_m is also exponential and independent of $Z_1, \cdots, Z_{m-1}$. Since these exponential distributions are the same, $Z_1, Z_2, \cdots$, are IID conditional on $N(t) = n$ and $S_n = \tau$.

(c) Shows that $Z_1, Z_2, \cdots$ are IID.

Solution: We have shown that $\{Z_m : m \geq 1\}$ are IID and exponential condition on $N(t) = n$ and $S_n = \tau$. The joint distribution of $Z_1, Z_2, \cdots, Z_m$ is thus specified as a function of $N(t) = n$ and $S_n = \tau$. Since this function is constant in n and τ, the joint conditional distribution must be the same as the joint unconditional distribution, and therefore $Z_1, Z_2, \cdots, Z_m$ are IID for all $m > 0$.

8. Exercise 2.9 (Gallager's book) Consider a "shrinking Bernoulli" approximation $N_\delta(m\delta) = Y_1 + Y_2 + \ldots + Y_m$ to a Poisson process as described in Subsection 2.2.5.

(a) Show that

$$\Pr\{N_\delta(m\delta) = n\} = \binom{m}{n}(\lambda\delta)^n(1-\lambda\delta)^{m-n}. \tag{27}$$

Solution: This is just the binomial PMF in (1.23).

(b) Let $t = m\delta$, and the t be fixed for the remainder of this exercise. Explain why

$$\lim_{\delta\to 0} \Pr\{N_\delta(t) = n\} = \lim_{m\to\infty} \binom{m}{n}\left(\frac{\lambda t}{m}\right)^n\left(1-\frac{\lambda t}{m}\right)^{m-n}, \tag{28}$$

where the limit on the left is taken over values of δ that divide t.

Solution: This is just the binomial PMF in (a) with $\delta = t/m$.

(c) Derive the following two inequalities:

$$\lim_{m\to\infty} \binom{m}{n}\frac{1}{m^n} = \frac{1}{n!}; \quad \text{and} \quad \lim_{m\to\infty}\left(1-\frac{\lambda t}{m}\right)^{m-n} = e^{-\lambda t}. \tag{29}$$

Solution: Note that

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{1}{n!}\prod_{i=0}^{n-1}(m-i). \tag{30}$$

When this is divided by m^n, each term in the product above is divided by m, so

$$\binom{m}{n}\frac{1}{m^n} = \frac{1}{n!}\prod_{i=0}^{n-1}\frac{(m-i)}{m} = \frac{1}{n!}\prod_{i=0}^{n-1}\left(1-\frac{i}{m}\right). \tag{31}$$

Taking the limit as $m \to \infty$, each of the n terms in the product approaches 1, so the limit is $1/n!$, verifying the first equality in (c). For the second,

$$\left(1-\frac{\lambda t}{m}\right)^{m-n} = \exp\left[(m-n)\ln\left(1-\frac{\lambda t}{m}\right)\right] = \exp\left[(m-n)\left(\frac{-\lambda t}{m}+o(1/m)\right)\right] \tag{32}$$

$$= \exp\left[-\lambda t + \frac{n\lambda t}{m} + (m-n)o(1/m)\right]. \tag{33}$$

In the second equality, we expanded $\ln(1-x) = -x + x^2/2\ldots$. In the limit $m \to \infty$, the final expression is $\exp(-\lambda t)$, as was to be shown.

If one wishes to see how the limit in (31) is approached, we have

$$\frac{1}{n!}\prod_{i=0}^{n-1}\left(1-\frac{i}{m}\right) = \frac{1}{n!}\exp\left(\sum_{i=0}^{n-1}\ln\left(1-\frac{i}{m}\right)\right) = \frac{1}{n!}\exp\left(\frac{-n(n-1)}{2m}+o(1/m)\right). \tag{34}$$

(d) Conclude from this fact that for every t and every n, $\lim_{\delta\to 0}\Pr\{N_\delta(t) = n\} = \Pr\{N(t) = n\}$ where $\{N(t) : t > 0\}$ is a Poisson process of rate λ.

Solution: We simply substitute the results of (c) into the expression in (b), getting

$$\lim_{\delta\to 0}\Pr\{N_\delta(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}. \tag{35}$$

This shows that the Poisson PMF is the limit of shrinking Bernoulli PMF's, but recall from Exercise 2.5 that this is not quite enough to show that a Poisson process is the limit of shrinking Bernoulli processes. It is also necessary to show that the stationary and independent increment properties hold in the limit $\delta \to 0$. It can be seen that the Bernoulli process has these properties at each increment δ, and it is intuitively clear that these properties should hold in the limit, but it seems that carrying out all the analytical details to show this precisely is neither warranted or interesting.

9. (Optional) Exercise 2.12 (Gallager's book) Starting from time 0, northbound buses arrive at 77 Mass. Avenue according to a Poisson process of rate λ. Customers arrive according to an independent Poisson process of rate μ. When a bus arrives, all waiting customers instantly enter the bus and subsequent customers wait for the the next bus.

(a) Find the PMF for the number of customers entering a bus (more specifically, for any given m, find the PMF for the number of customers entering the mth bus).

(b) Find the PMF for the number of customers entering the mth bus given that the interarrival interval between bus $m - 1$ and bus m is x.

(c) Given that a bus arrives at time $10 : 30$ PM, find the PMF for the number of customers entering the next bus.

(d) Given that a bus arrives at $10 : 30$ PM and no bus arrives between $10 : 30$ and 11, find the PMF for the number of customers on the next bus.

(e) Find the PMF for the number of customers waiting at some given time, say $2 : 30$ PM (assume that the processes started infinitely far in the past). Hint: think of what happens moving backward in time from $2 : 30$ PM.

(f) Find the PMF for the number of customers getting on the next bus to arrive after $2 : 30$. Hint: this is different from (a); look carefully at (e).

(g) Given that I arrive to wait for a bus at $2 : 30$ PM, find the PMF for the number of customers getting on the next bus.

Solution:

(a) Since the customer arrival process and the bus arrival process are independent Poisson processes, the sum of the two counting processes is a Poisson counting process of rate $\lambda + \mu$. Each arrival for the combined process is a bus with probability $\lambda/(\lambda + \mu)$ and a customer with probability $\mu/(\lambda + \mu)$. The sequence of choices between bus or customer arrivals is an IID sequence. Thus, starting immediately after bus $m - 1$ (or at time 0 for $m = 1$), the probability of n customers in a row followed by a bus, for any $n \geq 0$, is $[\mu/(\lambda + \mu)]^n\lambda/(\lambda + \mu)]$. This is the probability that n customers enter the mth bus, i.e., defining N_m as the number of customers entering the mth bus, the PMF of N_m is

$$p_{N_m}(n) = \left(\frac{\mu}{\lambda + \mu}\right)^n \frac{\lambda}{\lambda + \mu}. \tag{36}$$

(b) For any given interval of size x (i.e., for the interval $(s, s + x]$ for any given s), the number of customer arrivals in that interval has a Poisson distribution of rate μ. Since the customer arrival process is independent of the bus arrivals, this is also the distribution of customer arrivals between the arrival of bus $m-1$ and that of bus m given that the interval X_m between these bus arrivals is x. Thus letting X_m be the interval between the arrivals of bus $m-1$ and m,

$$p_{N_m|X_m}(n|x) = (\mu x)^n e^{-\mu x}/n!. \tag{37}$$

(c) First assume that for some given m, bus $m-1$ arrives at $10:30$. The number of customers entering bus m is still determined by the argument in (a) and has the PMF in (36). In other words, N_m is independent of the arrival time of bus $m-1$. From the formula in (36), the PMF of the number entering a bus is also independent of m. Thus the desired PMF is that on the right side of (36).

(d) Using the same reasoning as in (b), the number of customer arrivals from $10:30$ to 11 is a Poisson rv, say N' with PMF $p_{N'}(n) = (\mu/2)^n e^{-\mu/2}/n!$ (we are measuring time in hours so that μ is the customer arrival rate in arrivals per hour.) Since this is independent of bus arrivals, it is also the PMF of customer arrivals in $(10:30$ to $11]$ given no bus arrival in that interval.

The number of customers to enter the next bus is N' plus the number of customers N" arriving between 11 and the next bus arrival. By the argument in (a), N'' has the PMF in (36). Since N' and N'' are independent, the PMF of $N'+N''$ (the number entering the next bus given this conditioning) is the convolution of the PMF's of N' and N'', i.e.,

$$p_{N'+N''}(n) = \sum_{k=0}^{n} \left(\frac{\mu}{\lambda+\mu}\right)^k \frac{\lambda}{\lambda+\mu} \frac{(\mu/2)^{n-k} e^{-\mu/2}}{(n-k)!}. \tag{38}$$

This does not simplify in any nice way.

(e) Let $\{Z_i; -\infty < i < \infty\}$ be the (double infinite) IID sequence of bus/customer choices where $Z_i = 0$ if the ith combined arrival is a bus and $Z_i = 1$ if it is a customer. Indexing this sequence so that -1 is the index of the most recent combined arrival before $2:30$, we see that if $Z_{-1} = 0$, then no customers are waiting at $2:30$. If $Z_{-1} = 1$ and $Z_{-2} = 0$, then one customer is waiting. In general, if $Z_{-n} = 0$ and $Z_{-m} = 1$ for $1 \le m < n$, then n customers are waiting. Since the Z_i are IID, the PMF of the number N_{past} waiting at $2:30$ is

$$p_{N_{past}}(n) = \left(\frac{\mu}{\lambda+\mu}\right)^n \frac{\lambda}{\lambda+\mu}. \tag{39}$$

This is intuitive in one way, i.e., the number of customers looking back toward the previous bus should be the same as the number of customers looking forward to the next bus since the bus/customer choices are IID. It is paradoxical in another way since if we visualize a sample path of the process, we see waiting customers gradually increasing until a bus arrival, then going to 0 and gradually increasing again, etc. It is then surprising that the number of customers at an arbitrary time is statistically the same as the number immediately before a bus arrival. This paradox is partly explained at the end of (f) and fully explained in Chapter 5.

Mathematically inclined readers may also be concerned about the notion of 'starting infinitely far in the past.' A more precise way of looking at this is to start the Poisson process at time 0 (in accordance with the definition of a Poisson process). We can then find the PMF of the number waiting at time t and take the limit of this PMF as $t \to \infty$. For very large t, the number M of combined arrivals before t is large with high probability. Given $M = m$, the geometric distribution above is truncated at m, which is a neglibible correction for t large. This type of issue is handled more cleanly in Chapter 5

(f) The number getting on the next bus after $2:30$ is the sum of the number N_p waiting at $2:30$ and the number of future customer arrivals N_f (found in (c)) until the next bus after $2:30$. Note that N_p and N_f are IID. Convolving these PMF's, we get

$$p_{N_p+N_f}(n) = \sum_{m=0}^{n} \left(\frac{\mu}{\lambda+\mu}\right)^m \frac{\lambda}{\lambda+\mu} \left(\frac{\mu}{\lambda+\mu}\right)^{n-m} \frac{\lambda}{\lambda+\mu} \tag{40}$$

$$= (n+1) \left(\frac{\mu}{\lambda+\mu}\right)^n \left(\frac{\lambda}{\lambda+\mu}\right)^2 . \tag{41}$$

This is very surprising. It says that the number of people getting on the first bus after 2 : 30 is the sum of two IID rv's, each with the same distribution as the number to get on the mth bus. This is an example of the 'paradox of residual life,' which we discuss very informally here and then discuss carefully in Chapter 5.

Consider a very large interval of time $(0, t_0]$ over which a large number of bus arrivals occur. Then choose a random time instant T, uniformly distributed in $(0, t_0]$. Note that T is more likely to occur within one of the larger bus interarrival intervals than within one of the smaller intervals, and thus, given the randomly chosen time instant T, the bus interarrival interval around that instant will tend to be larger than that from a given bus arrival, $m-1$ say, to the next bus arrival m. Since 2 : 30 is arbitrary, it is plausible that the interval around 2 : 30 behaves like that around T, making the result here also plausible.

(g) My arrival at 2 : 30 is in addition to the Poisson process of customers, and thus the number entering the next bus is $1 + N_p + N_f$. This has the sample value n if $N_p + N_f$ has the sample value $n-1$, so from (f),

$$p_{1+N_p+N_f}(n) = n \left(\frac{\mu}{\lambda+\mu}\right)^{n-1} \left(\frac{\lambda}{\lambda+\mu}\right)^2 . \tag{42}$$

10. Customers depart from a bookstore according to a Poisson process with rate λ per hour. Each customer buys a book with probability p, independent of everything else.

(a) Find the distribution of the time until the first sale of a book.

Solution: We note that the process of departures of customers who have bought a book is obtained by splitting the Poisson process of customer departures, and is itself a Poisson process, with rate $p\lambda$. Hence, the time until the first sale of a book is the time until the first customer departure in the split Poisson process. It is therefore exponentially distributed with parameter $p\lambda$.

(b) Find the probability that there are no books sold during a particular hour.

Solution: This is the probability of no customers in the split Poisson process during an hour, and using the result of part (a), equals $e^{-p\lambda}$.

(c) Find the expected number of customers who buy a book during a particular hour.

Solution: This is the expected number of customers in the split Poisson process during an hour, and is equal to $p\lambda$.

11. (Optional) Let S_1 and S_2 be independent and exponentially distributed with parameters λ_1 and λ_2, respectively. Show that the expected value of $\max\{S_1, S_2\}$ is

$$\mathbb{E}[\max\{S_1, S_2\}] = \frac{1}{\lambda_1 + \lambda_2} \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right)$$

using Poisson Processes.

Hint: Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. We interpret S_1 as the first arrival time in the first process, and S_2 the first arrival time in the second process. Let $V = \min\{S_1, S_2\}$ be the first time when one of the processes registers an arrival. Let $W = \max\{X_1, X_2\} - V$

be the additional time until both have registered an arrival. Now calculate the expectations of V and W to find the expectation of the desired $\max\{S_1, S_2\}$.

Solution: Since the merged process is Poisson with rate $\lambda_1 + \lambda_2$, we have

$$\mathbb{E}V = \frac{1}{\lambda_1 + \lambda_2} \tag{43}$$

Concerning W, there are two cases to consider

(a) The first arrival comes from the first process; this happens with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$. We then have to wait for an arrival from the second process which takes $\frac{1}{\lambda_2}$ time on average.

(b) The first arrival comes from the second process; this happens with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$. We then have to wait for an arrival from the first process which takes $\frac{1}{\lambda_1}$ time on average.

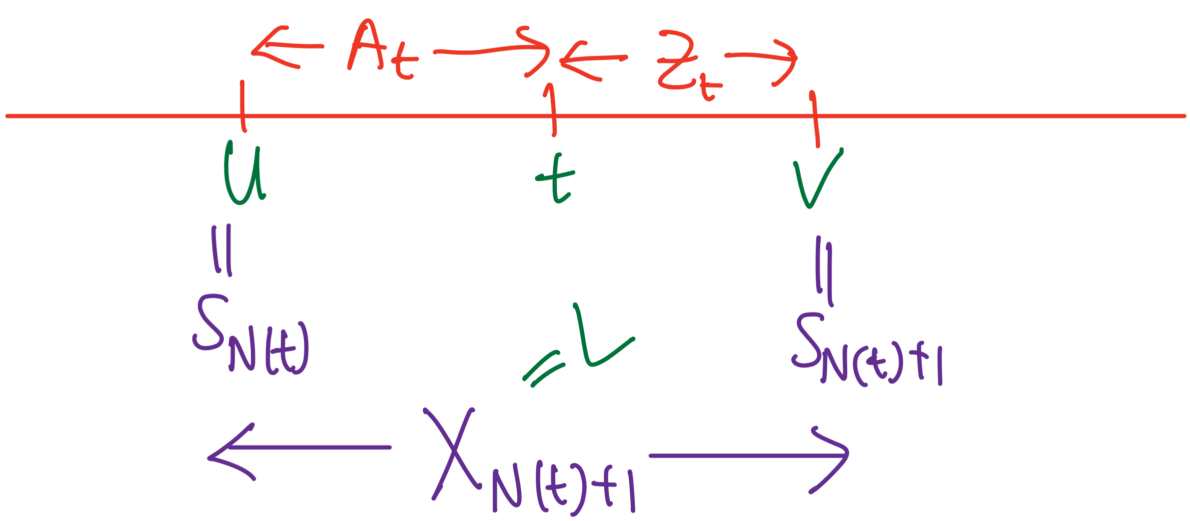
Putting everything together, we have

$$\mathbb{E}\max\{X_1, X_2\} = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{1}{\lambda_1} \tag{44}$$

$$= \frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right). \tag{45}$$

Alternative Method for Q5.

We want $\mathbb{E}[A_t + Z_t]$. This is the interarrival time that contains time t.



It suffices for us to find $\mathbb{E}[X_{N(t)+1}] = \mathbb{E}[A_t + Z_t]$.

$= \mathbb{E}[(t-U)+(V-t)]$.

We now find $\mathbb{E}[X_{N(t)+1}]$ by the law of iterated expectations.

$$\mathbb{E}[X_{N(t)+1} \mid N(t)=n]$$

$$= \mathbb{E}[A_t + Z_t \mid N(t)=n]$$

$$= \underbrace{\mathbb{E}[A_t \mid N(t)=n]}_{\frac{t}{n+1}} + \underbrace{\mathbb{E}[Z_t \mid N(t)=n]}_{1/\lambda}.$$

$\because Z_t$ is $\mathrm{Exp}(\lambda)$ & independent of $N(t)$

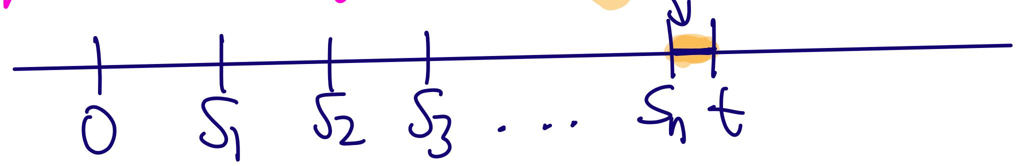
$\Updownarrow$

$$\mathbb{E}[A_t \mid N(t)=n] = \frac{t}{n+1}$$

by conditional arrival theory.
(Eqn (2.47) in Gallager's book)

A_t

At t there are n arrivals



$$\Rightarrow \mathbb{E}[X_{N(t)+1} \mid N(t)=n] = \frac{t}{n+1} + \frac{1}{\lambda}.$$

$$\mathbb{E}[X_{N(t)+1}] = \mathbb{E}[\mathbb{E}[X_{N(t)+1} \mid N(t)]]$$

$$= \sum_{n=0}^{\infty} \Pr(N(t)=n)\left(\frac{t}{n+1} + \frac{1}{\lambda}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!}\left(\frac{t}{n+1} + \frac{1}{\lambda}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n t^{n+1} e^{-\lambda t}}{(n+1)!} + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1} e^{-\lambda t}}{(n+1)!} + \frac{1}{\lambda}$$

$m = n+1$

$$= \frac{1}{\lambda} \sum_{m=1}^{\infty} \frac{(\lambda t)^m e^{-\lambda t}}{m!} + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right) + \frac{1}{\lambda},$$

which is the same answer as the method I used in the solutions (sol_5.pdf).