Chapter 1 - Kinematics

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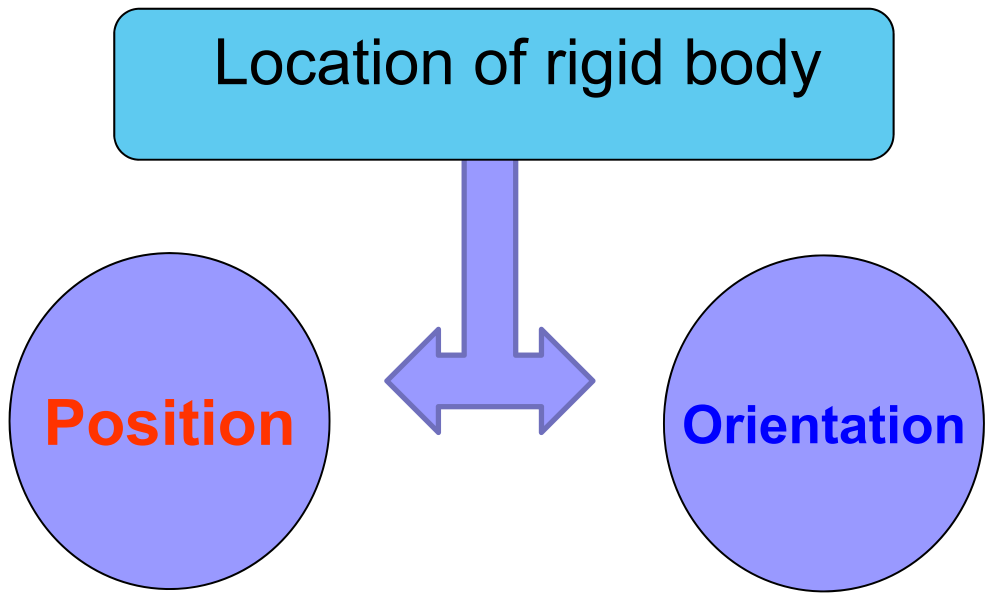
1. Positions and Orientations

Spatial descriptions and Transformations

- Need to specify spatial attributes of various objects with which a manipulation system deals
- Cartesian coordinate frames are used

Spatial descriptions and Transformations

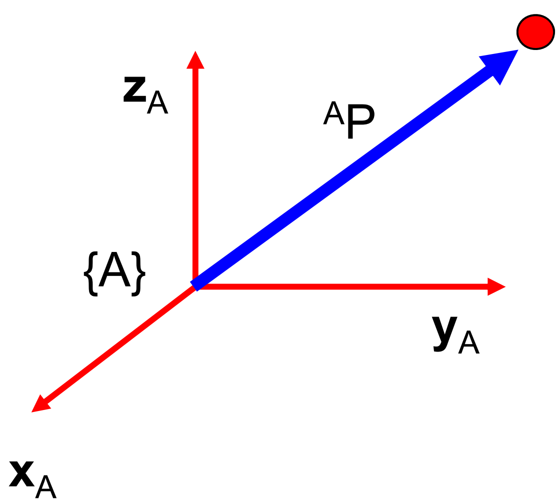
- Position & orientation of a Rigid Body



Position & orientation of a Rigid Body

- Position: Attribute of a point

With reference to a coordinate frame



Represented by a position vector

$$^AP = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \in \Re^3$$

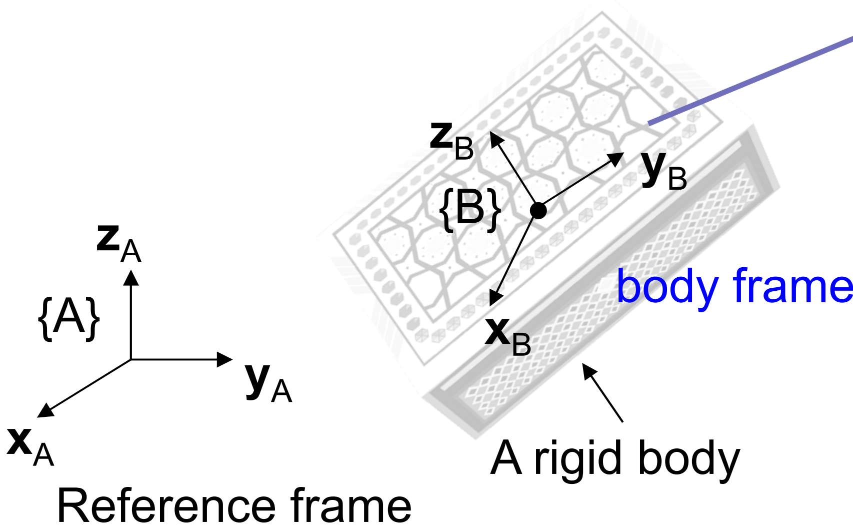
Position & orientation of a Rigid Body

- Orientation: Attribute of a body

Represented by Matrices
(Rotation Matrices)

$$ {}^{A}_{B}R = [{}^{A}\mathbf{x}_B \ {}^{A}\mathbf{y}_B \ {}^{A}\mathbf{z}_B] \in \Re^{3x3} $$

By attaching a coordinate frame to the body and then give a description of this coordinate frame relative to the reference coordinate frame



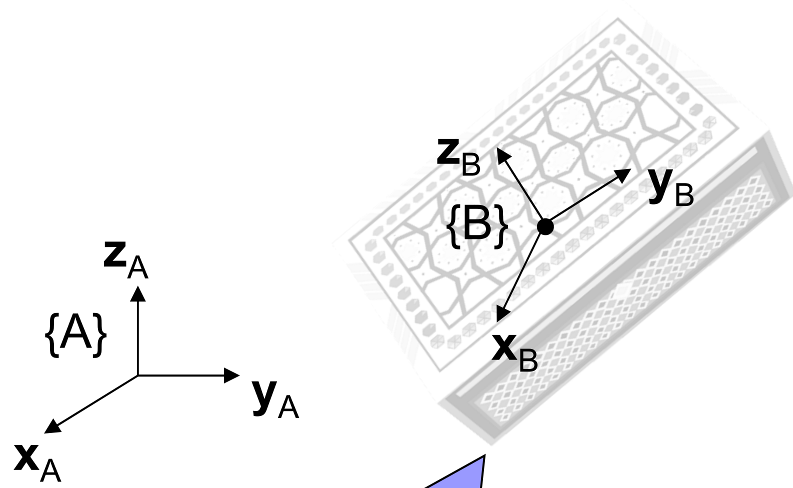
Position & orientation of a Rigid Body

- Orientation representation

$$ {}^{A}_{B}R = [\underbrace{{}^{A}\mathbf{x}_{B} \; {}^{A}\mathbf{y}_{B} \; {}^{A}\mathbf{z}_{B}}] \in \Re^{3x3} $$

Orthogonal unit vectors
(orthonormal matrix)

$$ \det({}^{A}_{B}R) = 1 $$



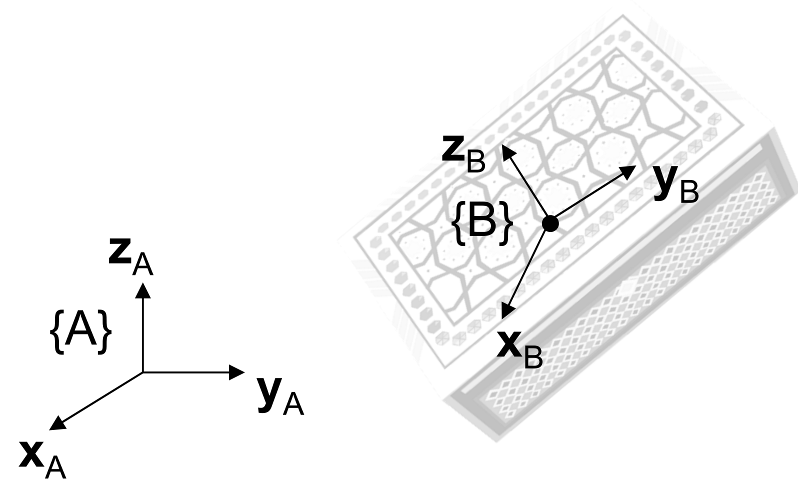
Multiplication of rotation matrices generally not commutative
i.e. $R_1R_2 \neq R_2R_1$

Note: ${}^{A}\mathbf{x}_B$ means $\mathbf{x}_B$ expressed in {A}

Position & orientation of a Rigid Body

■ Orientation representation

$$
{}^{A}_{B}R = [{}^{A}\mathbf{x}_{B}\ {}^{A}\mathbf{y}_{B}\ \ {}^{A}\mathbf{z}_{B}\ ]
= \begin{bmatrix} \mathbf{x}_B \cdot \mathbf{x}_A & \mathbf{y}_B \cdot \mathbf{x}_A & \mathbf{z}_B \cdot \mathbf{x}_A \\ \mathbf{x}_B \cdot \mathbf{y}_A & \mathbf{y}_B \cdot \mathbf{y}_A & \mathbf{z}_B \cdot \mathbf{y}_A \\ \mathbf{x}_B \cdot \mathbf{z}_A & \mathbf{y}_B \cdot \mathbf{z}_A & \mathbf{z}_B \cdot \mathbf{z}_A \end{bmatrix}
$$



Remark:

- Components are direction cosines
  (dot product of two unit vectors yields the cosine of the angle between them)
- Choice of frame is arbitrary for the vectors, but must be the same

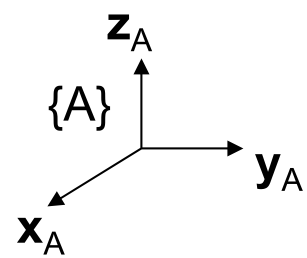
Position & orientation of a Rigid Body

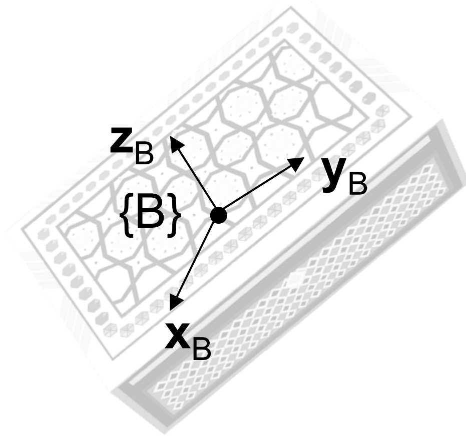
- Orientation

$$
{}^{A}_{B}R = [{}^{A}\mathbf{x}_{B}\ {}^{A}\mathbf{y}_{B}\ \ {}^{A}\mathbf{z}_{B}\,]
$$

$$
= \begin{bmatrix} \mathbf{x}_B \cdot \mathbf{x}_A & \mathbf{y}_B \cdot \mathbf{x}_A & \mathbf{z}_B \cdot \mathbf{x}_A \\ \mathbf{x}_B \cdot \mathbf{y}_A & \mathbf{y}_B \cdot \mathbf{y}_A & \mathbf{z}_B \cdot \mathbf{y}_A \\ \mathbf{x}_B \cdot \mathbf{z}_A & \mathbf{y}_B \cdot \mathbf{z}_A & \mathbf{z}_B \cdot \mathbf{z}_A \end{bmatrix}
$$

transpose of "z_A expressed in {B}"





Note: Rows of the matrix are the unit vectors of {A} expressed in {B}

$$
= \begin{bmatrix} {}^{B}\mathbf{x}_A^T \\ {}^{B}\mathbf{y}_A^T \\ {}^{B}\mathbf{z}_A^T \end{bmatrix} = \begin{bmatrix} {}^{B}\mathbf{x}_A & {}^{B}\mathbf{y}_A & {}^{B}\mathbf{z}_A \end{bmatrix}^T = {}^{B}_{A}R^T
$$

Position & orientation of a Rigid Body

- Orientation

Hence, description of frame {A} relative to {B}, ${}^{B}_{A}R = {}^{A}_{B}R^{T}$

This suggests: $\boxed{{}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}}$

Verification:

$$
{}^{A}_{B}R^{T}\,{}^{A}_{B}R = \begin{bmatrix} {}^{A}\mathbf{x}_B^T \\ {}^{A}\mathbf{y}_B^T \\ {}^{A}\mathbf{z}_B^T \end{bmatrix} \begin{bmatrix} {}^{A}\mathbf{x}_B & {}^{A}\mathbf{y}_B & {}^{A}\mathbf{z}_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3
$$

I_3 denotes the 3x3 identity matrix

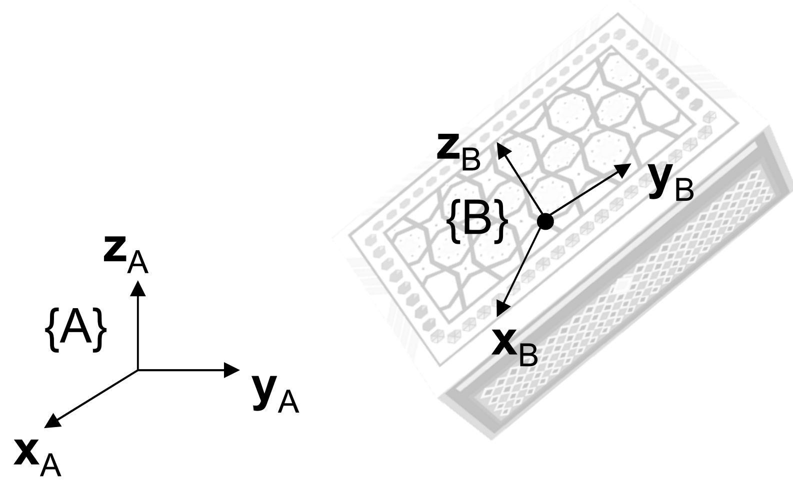
Linear algrebra: inverse of a matrix with orthonormal columns is equal to its transpose.

Position & orientation of a Rigid Body

- Orientation representation

Uniqueness

3 independent parameters



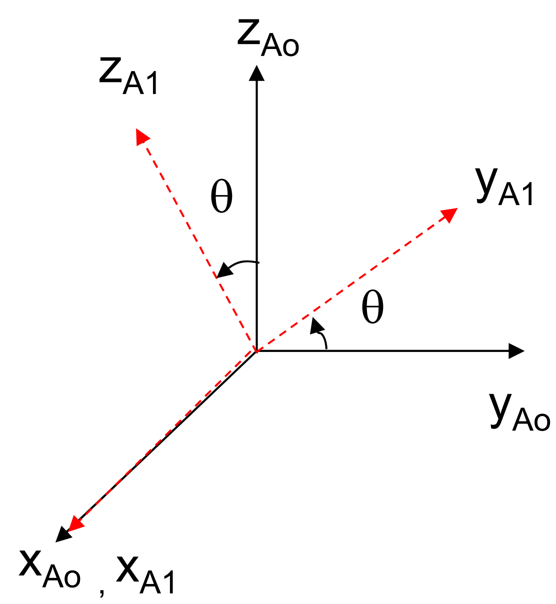
$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Position & orientation of a Rigid Body

- Elementary (Basic, Fundamental) Rotation matrices

Rotation about x_{Ao} by angle θ

$$ {}^{A_0}_{A_1}R = R_X(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} $$

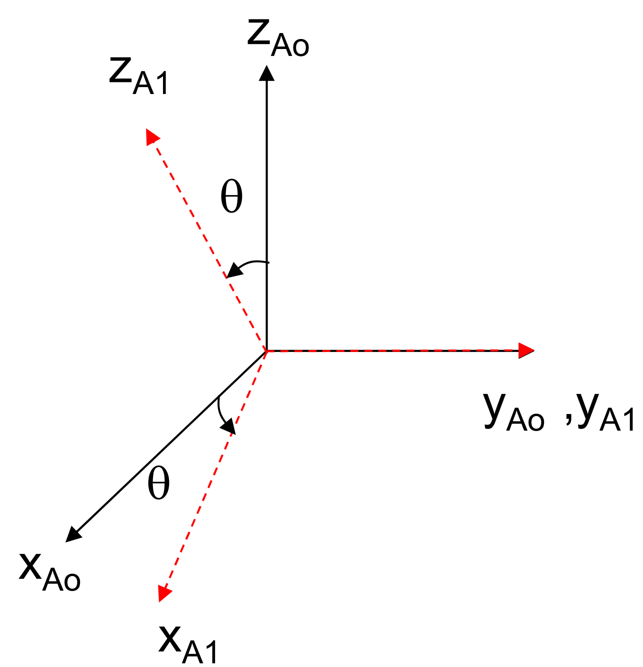


Position & orientation of a Rigid Body

- Elementary (Basic, Fundamental) Rotation matrices

Rotation about y_{Ao} by angle θ

$$ {}^{A_0}_{A_1}R = R_Y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} $$

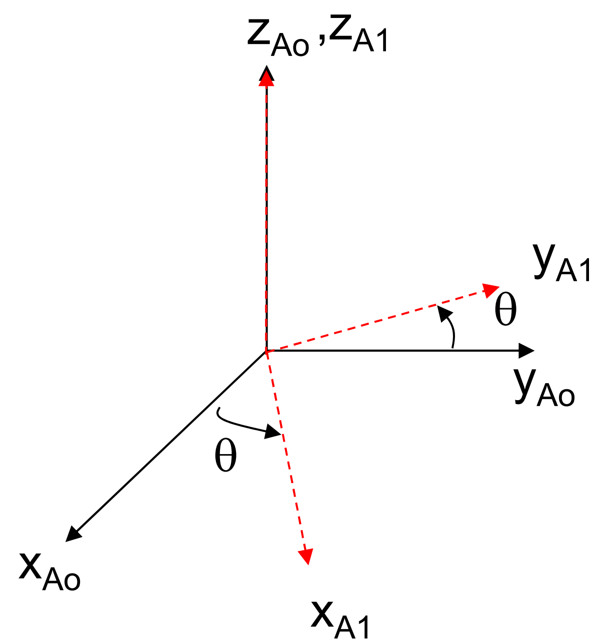


Position & orientation of a Rigid Body

- Elementary (Basic, Fundamental) Rotation matrices

Rotation about z_{Ao} by angle θ

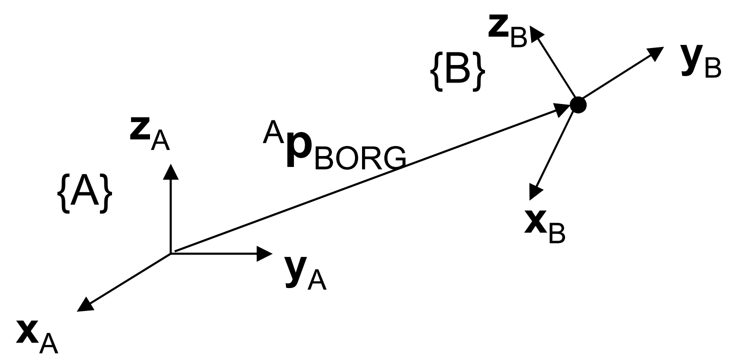
$$ {}^{A_0}_{A_1}R = R_Z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} $$



Position & orientation of a Rigid Body

- Description of a frame

Once we have attached a frame to a rigid body, how to represent the frame location and orientation?



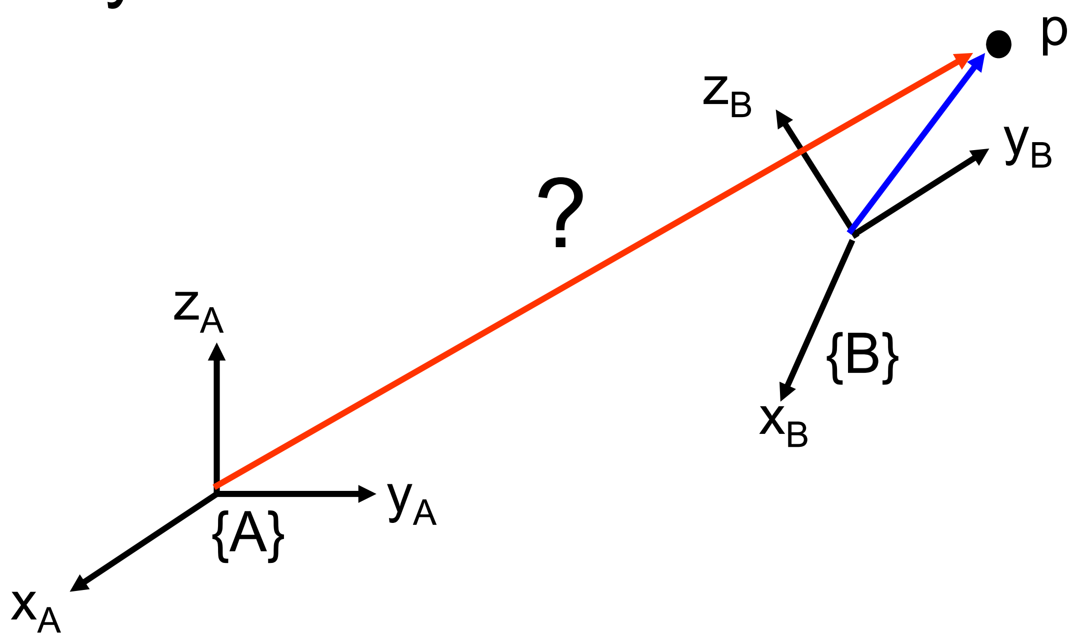
→ A set of four vectors giving position (typically of the origin) and orientation information of the frame relative to the reference frame

E.g., Frame {B} relative to frame {A} is described by

$$\left\{ {}^A_B R, {}^A\mathbf{p}_{BORG} \right\}$$

Coordinate Transformations (Mappings)

- Mapping: Changing descriptions from frame to frame
- Expressing the **position vector** of a point in space in terms of various reference coordinate systems

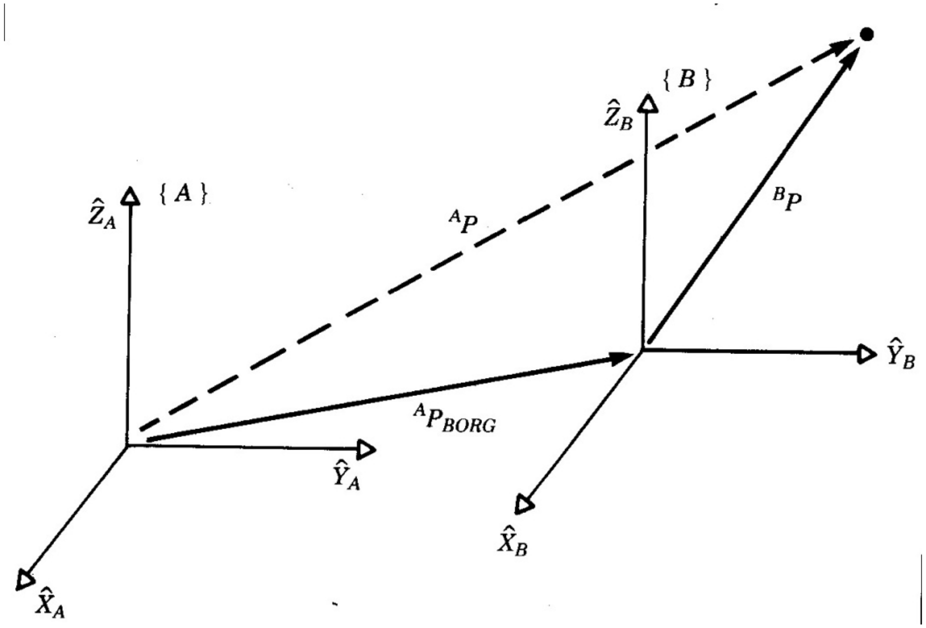


Given ^{B}P

Find ^{A}P

Coordinate Transformations (Mappings)

- Mappings involving translated frames (i.e., frames having same orientations)

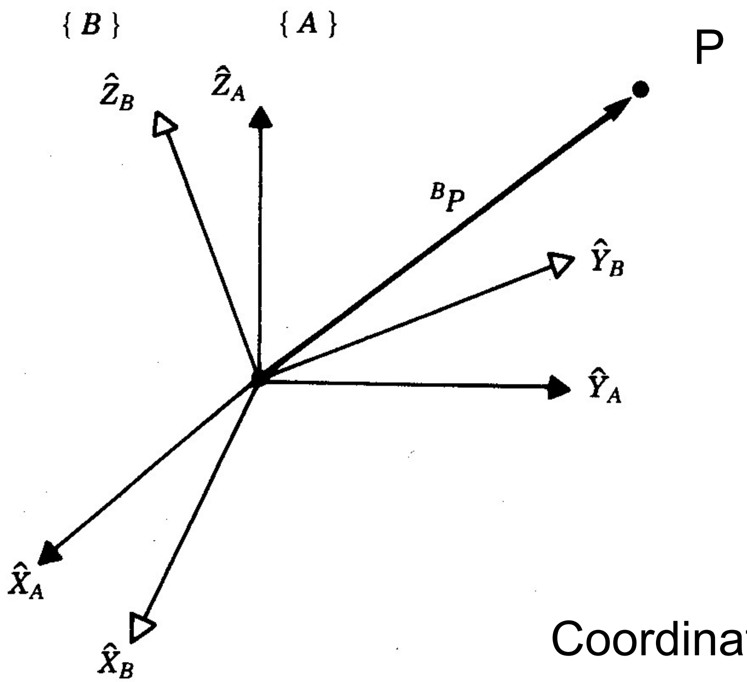


$$ {}^{A}P = {}^{B}P + {}^{A}P_{BORG} $$

Only applicable when orientations of frames A and B are the same

Coordinate Transformations (Mappings)

- Mappings involving rotated frames



Coordinates of P in B

$$^{A}P = {}^{A}_{B}R \, {}^{B}P$$

Coordinates of P in A

Above equation assumes origins of Frames A & B are coincident.
What if the two origins are not coincident?

Coordinate Transformations (Mappings)

- Example 1-1: Figure 1-1 shows a frame {B} which is rotated relative to frame {A} about **z** by 30 degrees. Here, **z** is pointing out of the page. There is a point P whose position vector expressed in {B} is ${}^{B}\mathbf{p} = [0\ 2\ 0]^T$. What is the position vector of the point P expressed in {A}?

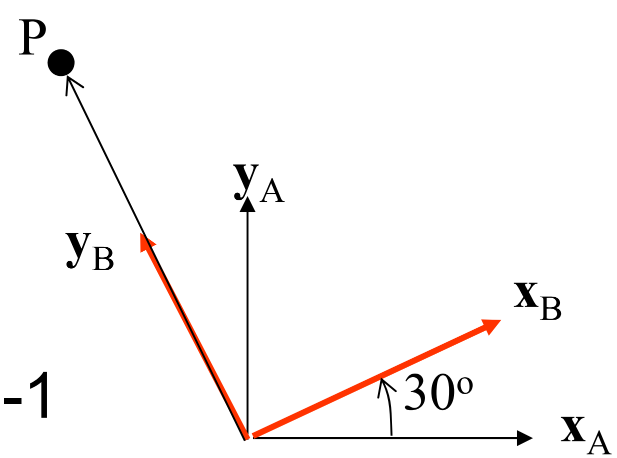


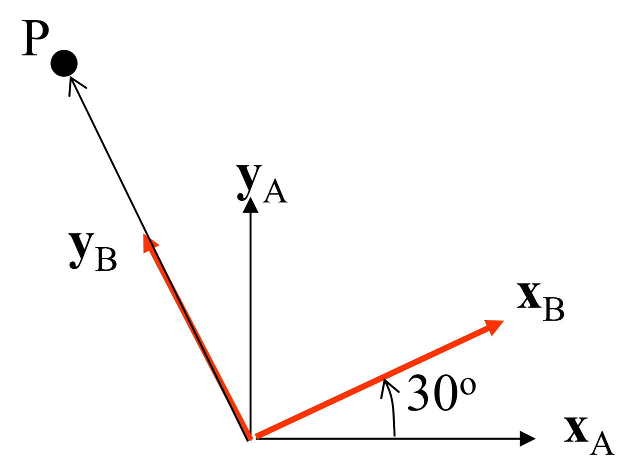
Figure 1-1

Coordinate Transformations (Mappings)

- Solution:

Writing the unit vectors of {B} in terms of {A} and stacking them as columns of the rotation matrix we obtain:

$$ {}^A_B R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



Given ${}^B\mathbf{p} = [0\ 2\ 0]^T$,

$$ {}^A\mathbf{p} = {}^A_B R\, {}^B\mathbf{p} = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix} $$

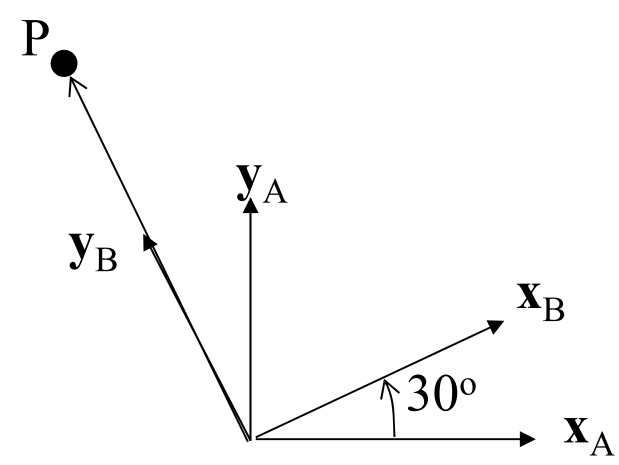
Coordinate Transformations (Mappings)

- Solution:

$$
{}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

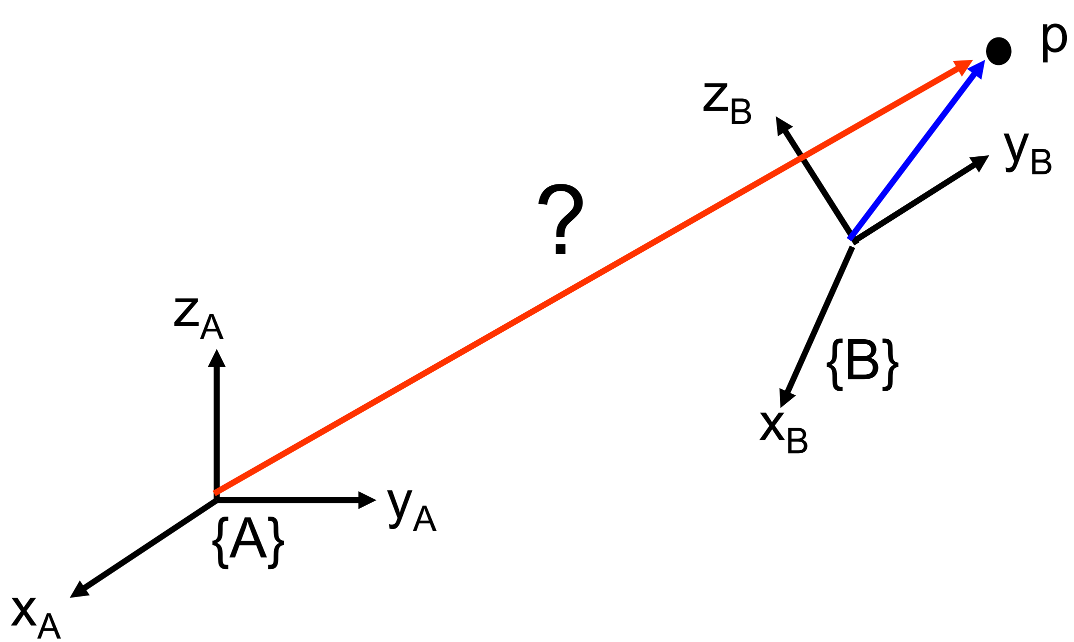
Remark:

- ${}^{A}_{B}R$ acts as a mapping which is used to describe **p** relative to frame {A} given ${}^{B}\mathbf{p}$.
- The original vector **p** is not changed in space. We simply compute the new description of the vector relative to a new frame.



Coordinate Transformations (Mappings)

- Mappings involving general frames

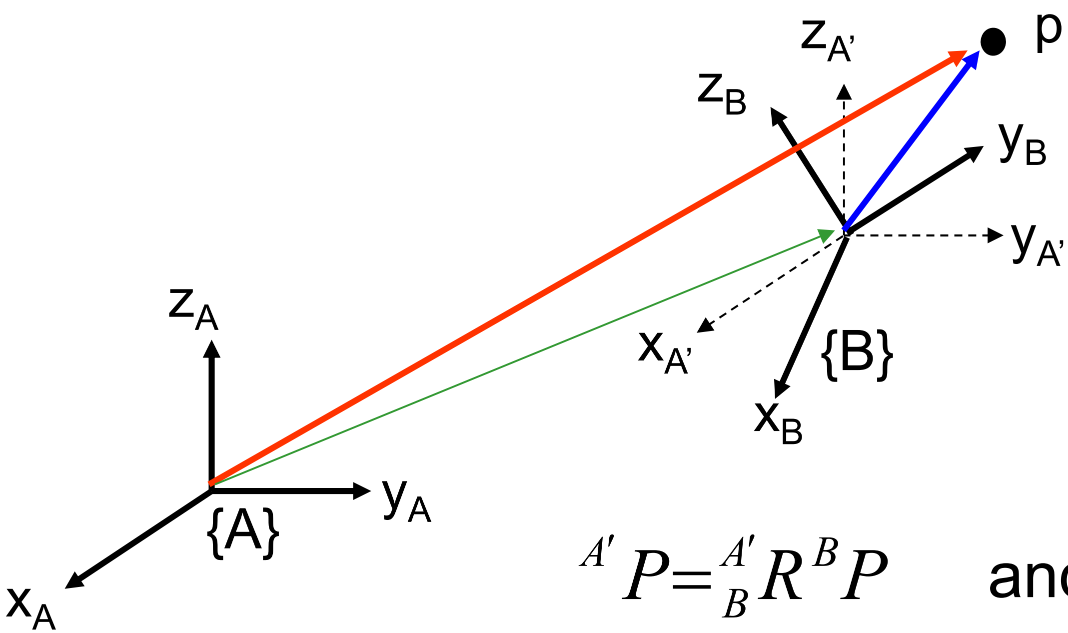


Given $\;{}^{B}P$

Find $\;{}^{A}P$

Coordinate Transformations (Mappings)

- Mappings involving general frames



{A} and {A'} same orientation:

$$ {}^{A}_{A'}R = I_3 $$

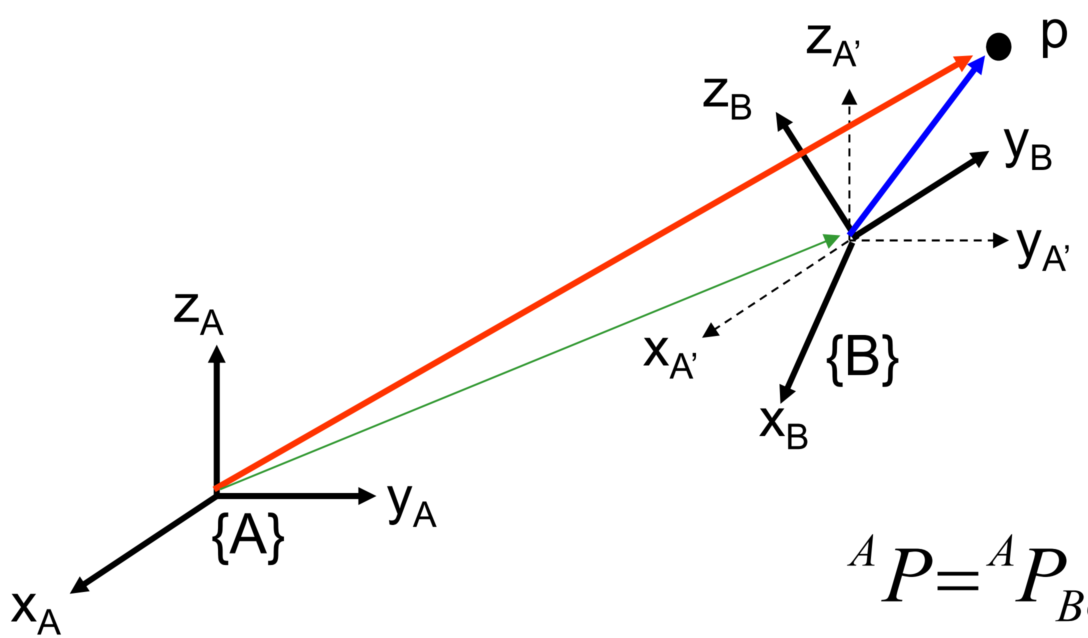
$$ {}^{A'}_{B}R = {}^{A}_{B}R $$

$$ \underbrace{{}^{A'}P = {}^{A'}_{B}R\,{}^{B}P \quad \text{and} \quad {}^{A}P = {}^{A}P_{BORG} + {}^{A'}P} $$

$$ {}^{A}P = {}^{A}P_{BORG} + {}^{A'}_{B}R\,{}^{B}P = {}^{A}P_{BORG} + {}^{A}_{B}R\,{}^{B}P $$

Coordinate Transformations (Mappings)

- Mappings involving general frames

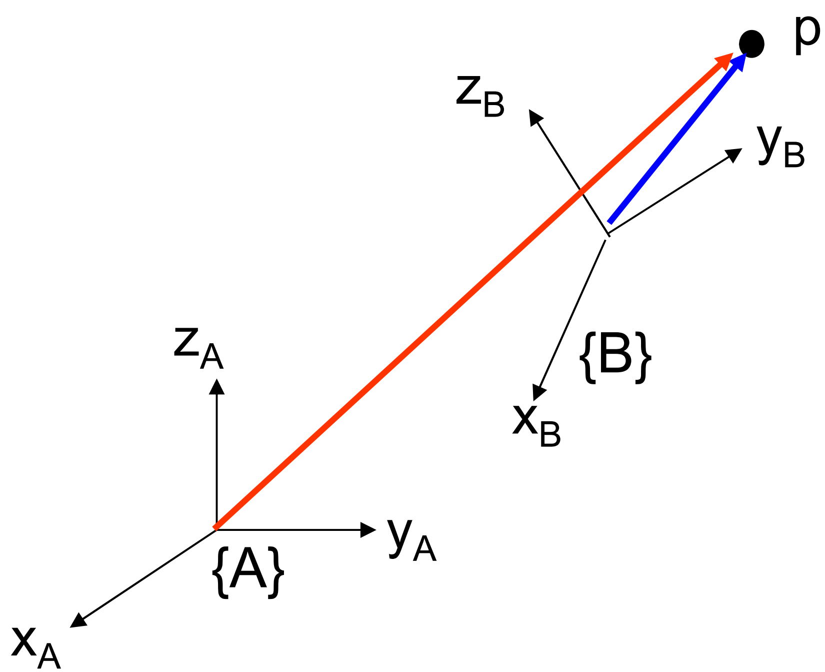


$$ {}^{A}P = {}^{A}P_{BORG} + {}^{A}_{B}R\,{}^{B}P $$

Translational transformation

Rotational transformation

2. Homogeneous Transformation



$$^{A}P = {}^{A}P_{BORG} + {}^{A}_{B}R\,{}^{B}P$$

$$^{A}P = {}^{A}_{B}T\,{}^{B}P$$

Homogeneous transformation

Homogeneous Transformation

$$ {}^{B}P = \begin{bmatrix} {}^{B}p_x \\ {}^{B}p_y \\ {}^{B}p_z \\ 1 \end{bmatrix} \Rightarrow {}^{A}_{B}T = \left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hdashline 0 & 1 \end{array}\right] \Rightarrow {}^{A}P = \begin{bmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \\ 1 \end{bmatrix} $$

Homogeneous transformation matrix

4x4 matrix

Augmented vector

Homogeneous Transformation

- Example 1-2

Figure 1-2 shows a frame {B} which is rotated relative to frame {A} about **z** by 30 degrees, and translated 10 units in $\mathbf{x}_\mathbf{A}$, and 5 units in $\mathbf{y}_A$. Find ${}^A\mathbf{p}$ where ${}^B\mathbf{p} = [3\ 7\ 0]^T$.

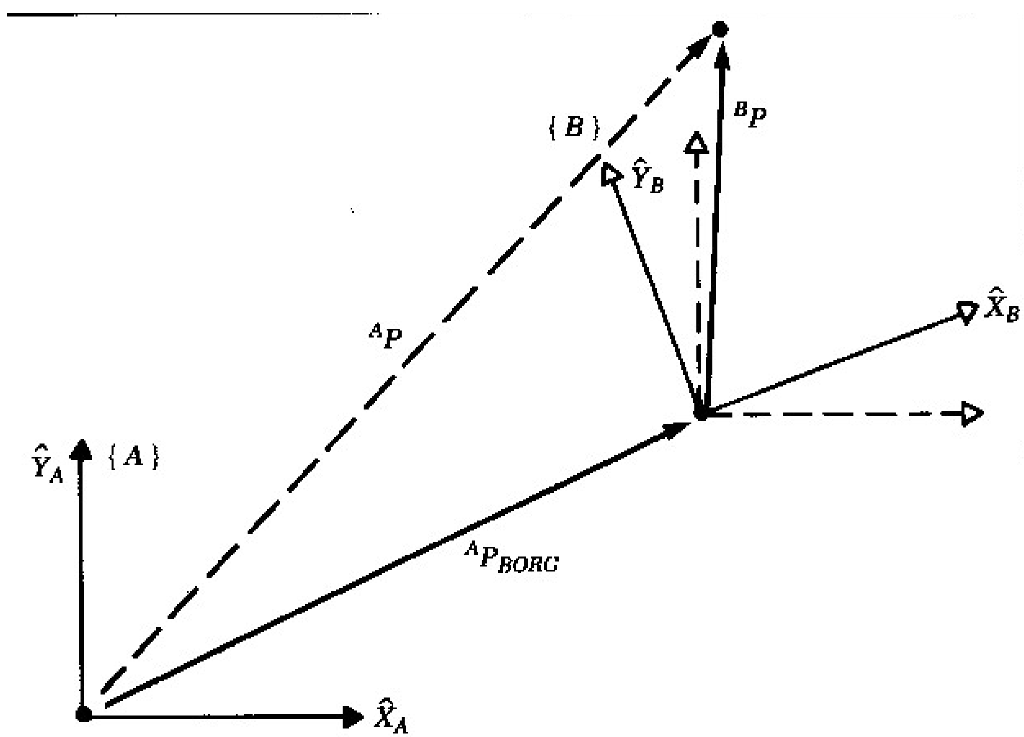


Figure 1-2

Homogeneous Transformation

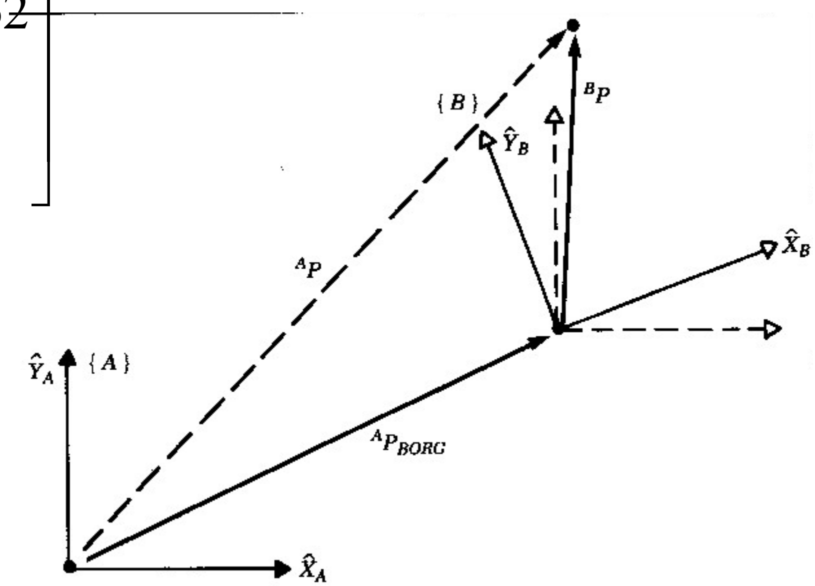
- Solution:

$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}\mathbf{p} = {}^{A}_{B}T\,{}^{B}\mathbf{p} = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix} $$

That is,

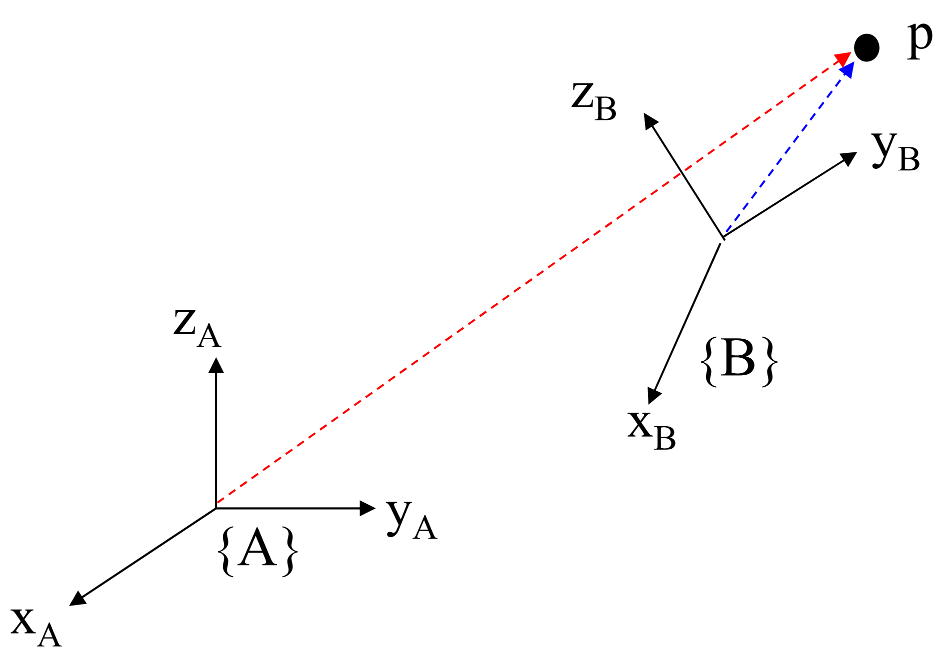
$$ {}^{A}\mathbf{p} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \end{bmatrix} $$



Homogeneous Transformation

- Interpretation 1:

  T represents coordinate transformations in compact form

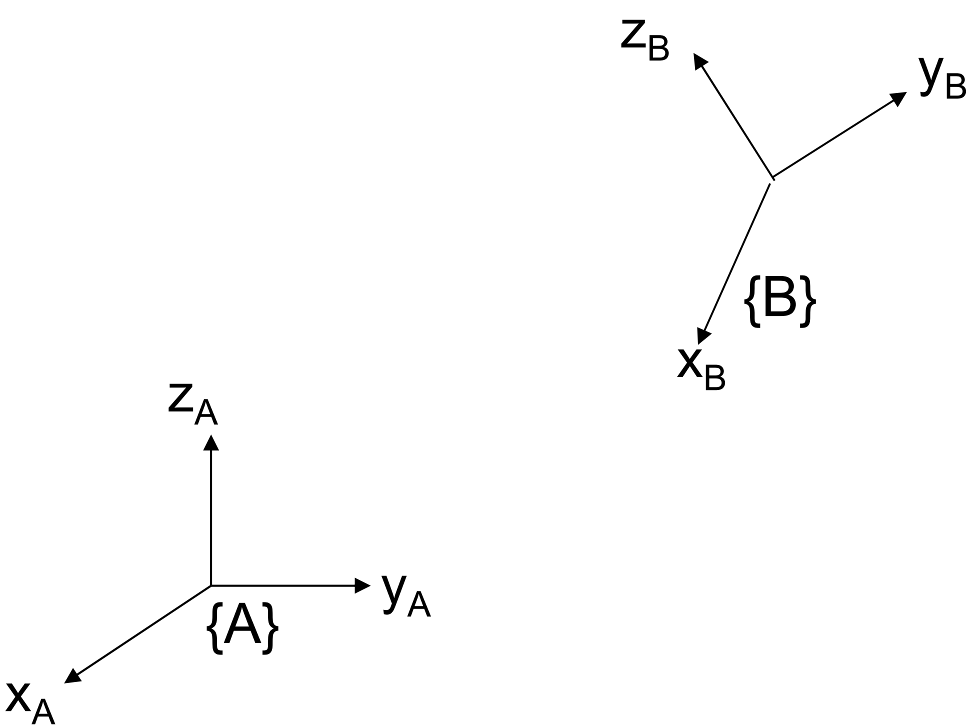


$$ {}^{A}_{B}T = \left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hdashline 0 & 1 \end{array}\right] $$

Homogeneous Transformation

- Interpretation 2:

  T represents position & orientation of the coordinate frame {B} relative to {A}

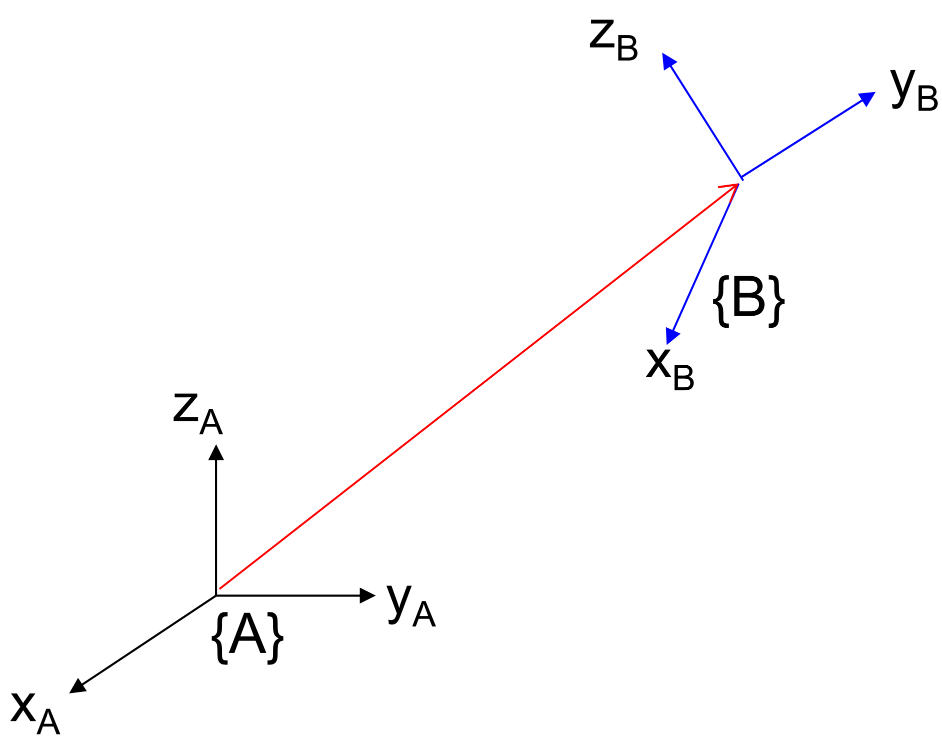


$$
{}^{A}_{B}T = \left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hdashline 0 & 1 \end{array}\right]
$$

Homogeneous Transformation

- Interpretation 3:

  T represents rotation and translation of the coordinate frame {A} to {B}



$$
{}^{A}_{B}T = \left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hdashline 0 & 1 \end{array} \right]
$$

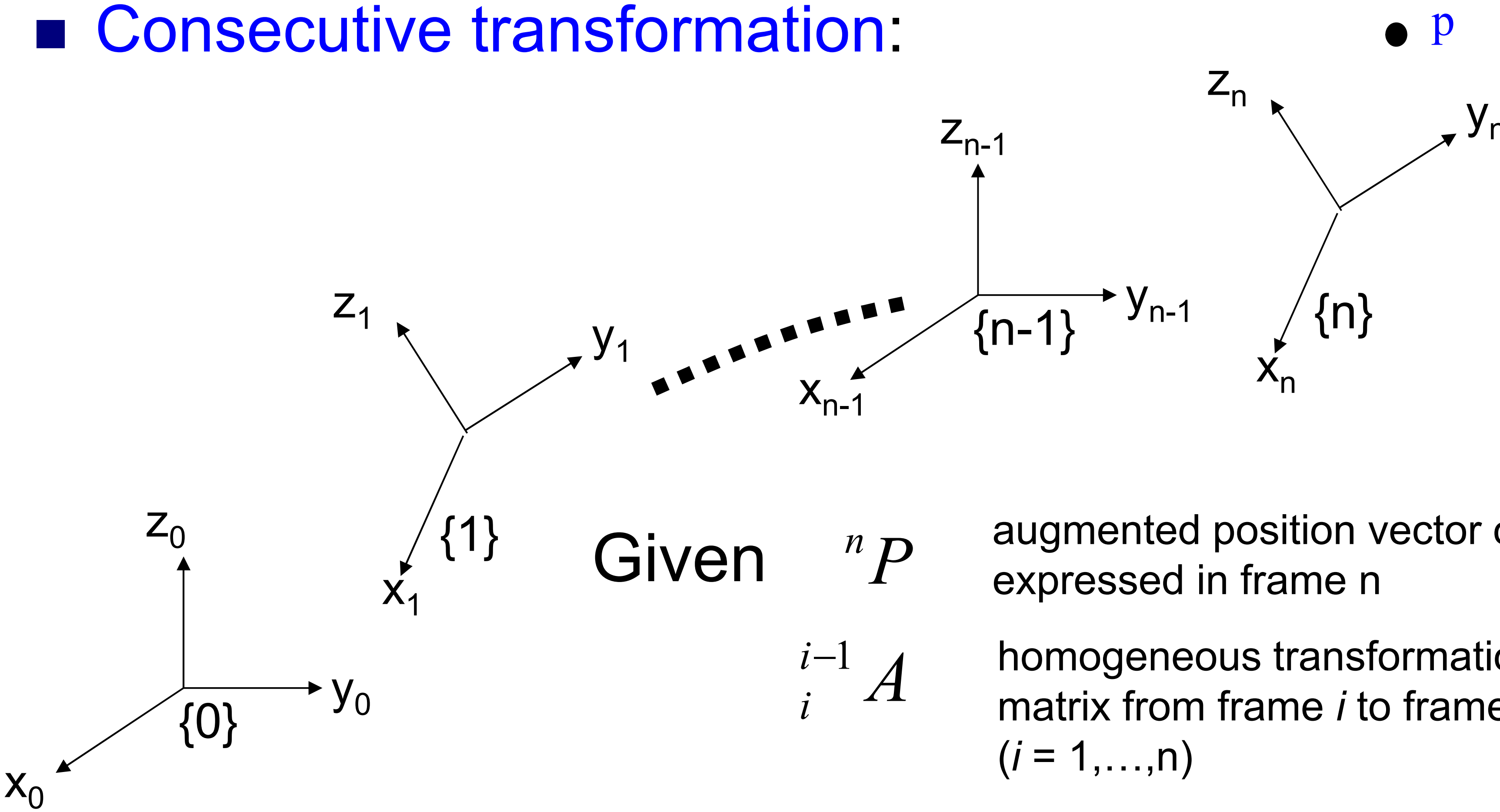
Homogeneous Transformation

- For free vectors (e.g. force, velocity, etc), augment the vectors with 0 rather than 1:

$$
{}^{A}P = \begin{bmatrix} {}^{A}p_x \\ {}^{A}p_y \\ {}^{A}p_z \\ 0 \end{bmatrix}
$$

Homogeneous Transformation

- Consecutive transformation:



Given ^{n}P augmented position vector of P expressed in frame n

$^{i-1}_{i}A$ homogeneous transformation matrix from frame *i* to frame *i-1* (*i* = 1,…,n)

Find ^{0}P augmented position vector of P expressed in frame 0

Homogeneous Transformation

- Consecutive transformation:

$$ {}^{0}P = {}^{0}_{1}A\,{}^{1}_{2}A \cdots {}^{n-1}_{n}A\,{}^{n}P = {}^{0}_{n}A\,{}^{n}P $$

where

${}^{i-1}_{i}A$ homogeneous transformation matrix from frame *i* to frame *i-1* (*i* = 1,…,n)

${}^{n}P$ augmented position vector of P expressed in frame n

${}^{o}P$ augmented position vector of P expressed in frame 0

$$ {}^{0}_{1}A\,{}^{1}_{2}A \cdots {}^{n-1}_{n}A = {}^{0}_{n}A $$ homogeneous transformation from frame n to frame 0

Homogeneous Transformation

- Inverse of a Homogeneous Transformation Matrix

Given ${}^A_BT$, find ${}^B_AT$ or ${}^A_BT^{-1}$

By definition, ${}^A_BT\,{}^A_BT^{-1} = I$

i.e. $$\left[\begin{array}{c|c} {}^A_BR & {}^AP_{BORG} \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} X & Y \\ \hline 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} I_{3\times 3} & 0 \\ \hline 0 & 1 \end{array}\right]$$

$$\begin{cases} {}^A_BRX = I & \Rightarrow \quad X = {}^A_BR^{-1} = {}^A_BR^T = {}^B_AR \\ {}^A_BRY + {}^AP_{BORG} = 0 & \Rightarrow \quad Y = -{}^A_BR^{-1}\,{}^AP_{BORG} = -{}^A_BR^T\,{}^AP_{BORG} \end{cases}$$

Homogeneous Transformation

- Inverse of a Homogeneous Transformation Matrix

$$
{}^{B}_{A}T = {}^{A}_{B}T^{-1} = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 1 \end{array}\right]
$$

Homogeneous Transformation

- Example 1-4

Figure 1-5 shows a frame {B} which is rotated relative to frame {A} about **z** by 30 degrees, and translated four units in $\mathbf{x}_A$, and three units in $\mathbf{y}_A$. Find ${}^B_A T$.

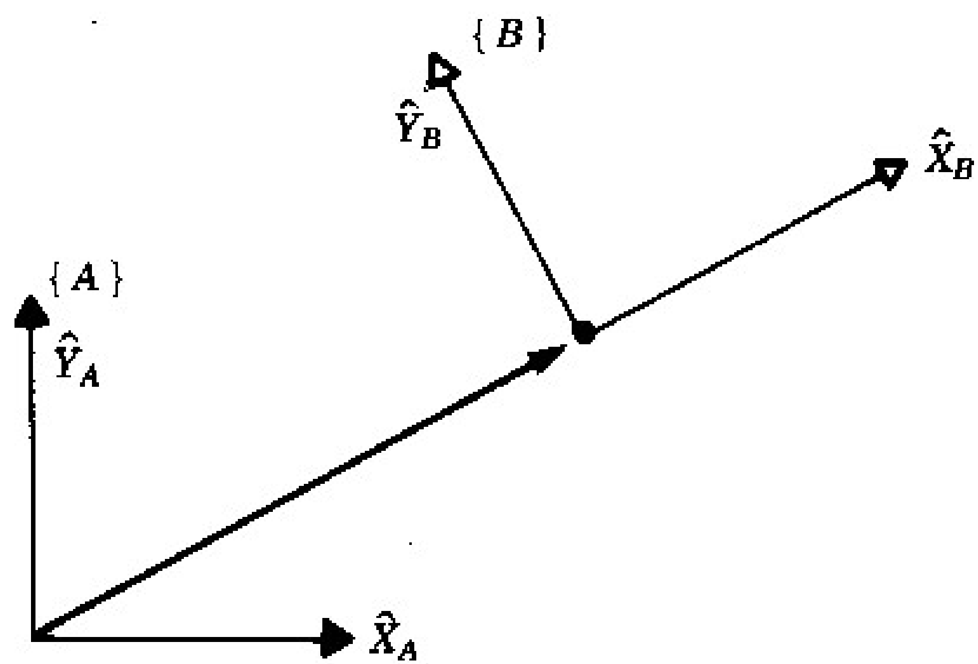
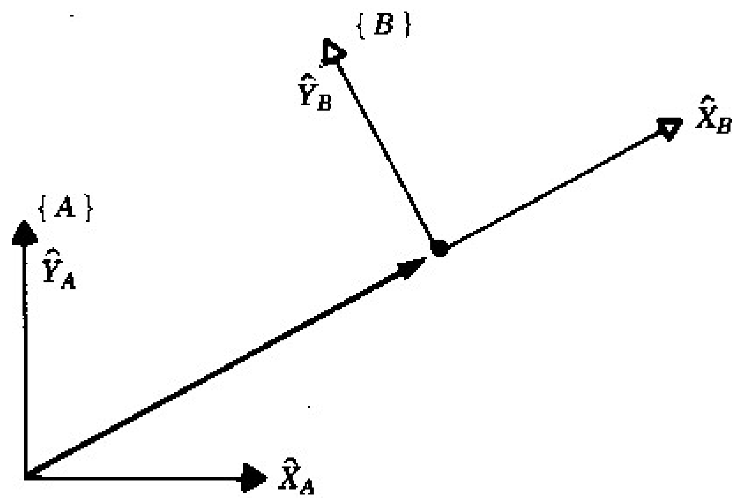


Figure 1-5

Homogeneous Transformation

- Solution:

$$
{}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$
{}^{B}_{A}T = {}^{A}_{B}T^{-1} = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}\mathbf{p}_{BORG} \\ \hline \mathbf{0} & 1 \end{array}\right] = \begin{bmatrix} 0.866 & 0.5 & 0 & -4.964 \\ -0.5 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Example 1-5

Assume we know:

${}^{B}_{T}T$ (which describes the frame at the manipulator's fingertips {T} relative to the base of the manipulator, {B}).

${}^{B}_{S}T$ (which describes the frame {S}, which is attached to the table, relative to the base of the manipulator, {B}.

${}^{S}_{G}T$ (which describes the frame attached to the bolt lying on the table relative to the table frame).

Calculate ${}^{T}_{G}T$ (position and orientation of the bolt relative to the manipulator's hand)

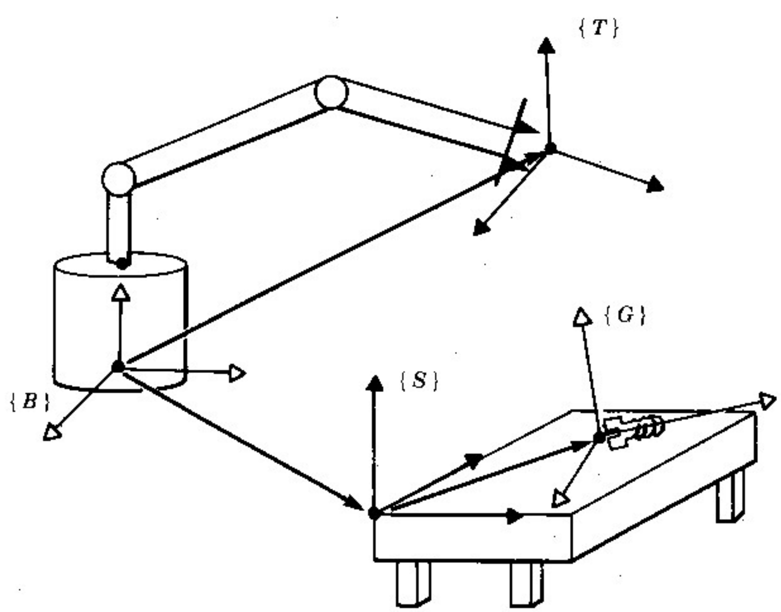
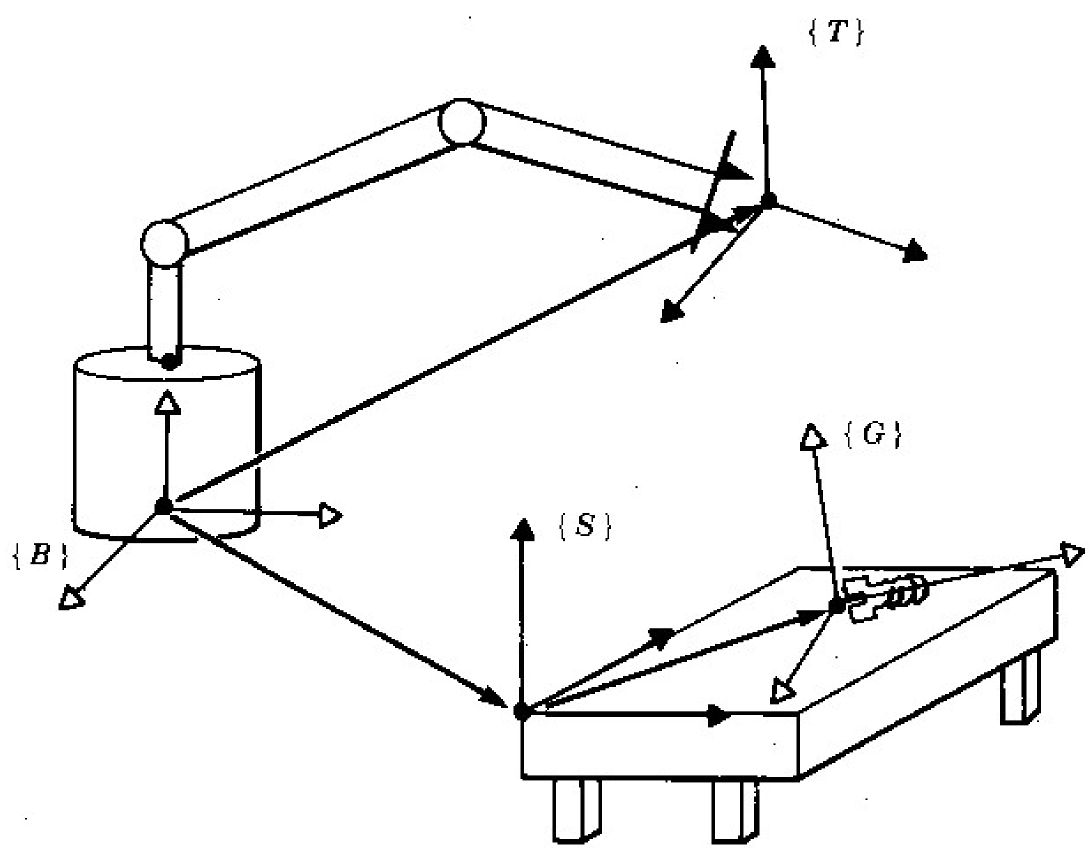


Figure 1-6

- Solution:

$$ {}^{T}_{G}T = {}^{T}_{B}T \, {}^{B}_{S}T \, {}^{S}_{G}T = {}^{B}_{T}T^{-1} \, {}^{B}_{S}T \, {}^{S}_{G}T $$



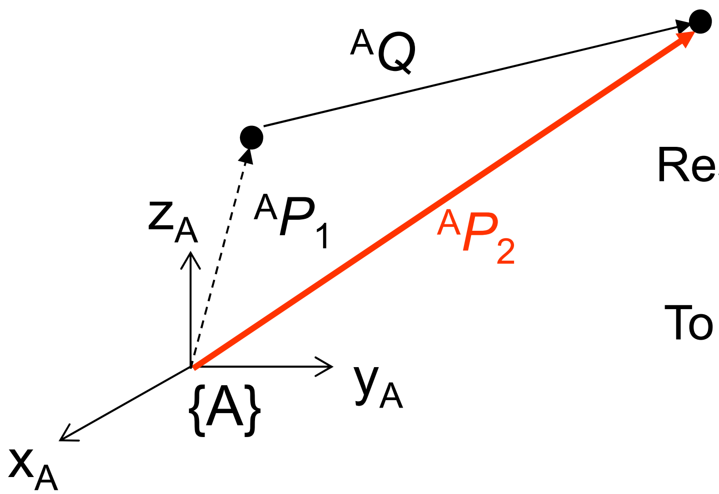
Operators: Translations, Rotations, and Transformation

- Operators :
  - Another interpretation of the mathematical forms used for coordinate transformations
  - Only one coordinate frame is involved

Operators: Translations, Rotations, and Transformation

- Translational operators: Moves a point in space a finite distance along a given vector direction

How a vector ${}^{A}P_1$ is translated by a vector ${}^{A}Q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$:



Result of the operation is a new vector ${}^{A}P_2 = {}^{A}P_1 + {}^{A}Q$

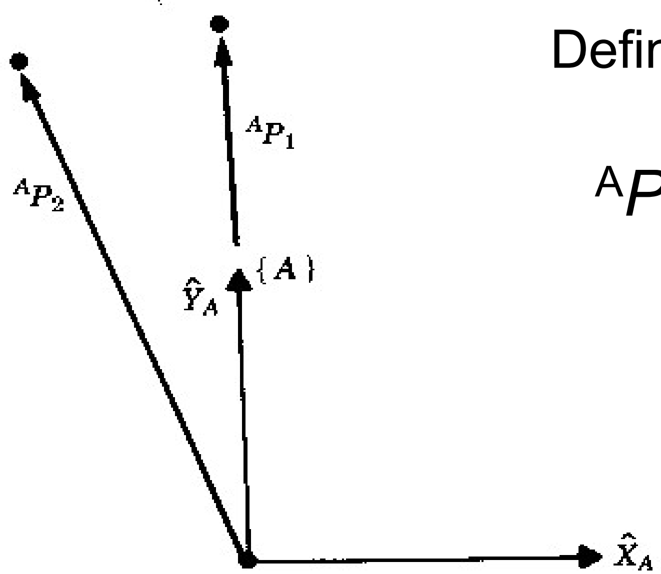
To write the translation operation as a matrix operator,

$$
{}^{A}P_2 = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{A}P_1
$$

where q_x, q_y and q_z are components of the translation vector Q

Operators: Translations, Rotations, and Transformation

- Rotational operators:
  - Operates on a vector $^{A}P_1$ and changes that vector to a new vector $^{A}P_2$, by means of a rotation, R (about origin)
  - Same as the rotation matrix that describes a frame rotated by R relative to the reference frame



Define a rotational operator $\mathbf{R_K}(\theta)$:

$$^{A}P_2 = \mathbf{R_K}(\theta)\,^{A}P_1$$

- K is the axis of rotation

- θ (in degrees) is the amount of rotation about K

Operators: Translations, Rotations, and Transformation

- Rotational operators:

E.g. operator that rotates about the *Z* axis by θ can be written as:

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3x3 rotation matrix

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

homogeneous transformation matrix

Operators: Translations, Rotations, and Transformation

- Example 1-6

Figure 1-3 shows a vector ${}^{A}\mathbf{p}_1 = [0\ 2\ 0]^T$. We wish to compute the vector obtained by rotating this vector about $\mathbf{z}$ by 30 degrees. Call the new vector ${}^{A}\mathbf{p}_2$.

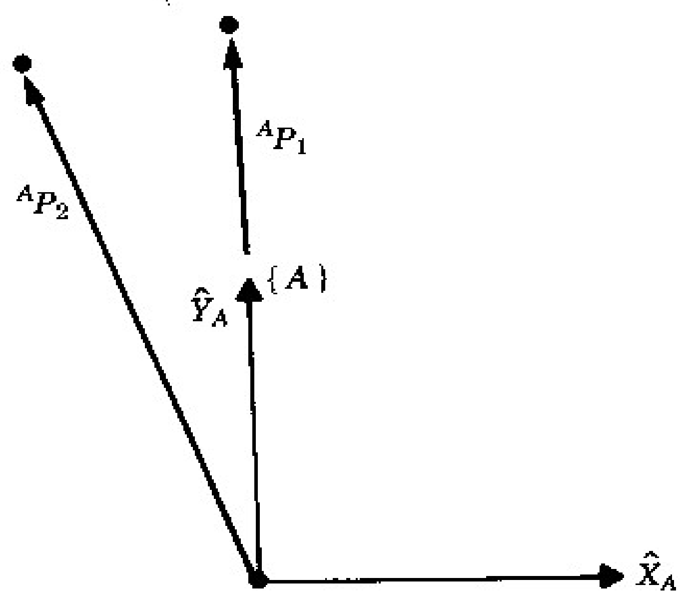
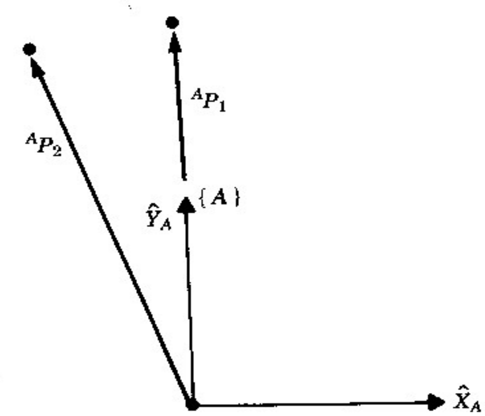


Figure 1-3

Operators: Translations, Rotations, and Transformation

- Solution:

Note: Rotation matrix which rotates vectors by 30 degrees about **z** = Rotation matrix which describes a frame rotated 30 degrees about **z** relative to the reference frame.



$$R_{\mathbf{z}}(30.0) = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^A\mathbf{p}_2 = R_{\mathbf{z}}(30.0)\,^A\mathbf{p}_1 = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix}$$

Operators: Translations, Rotations, and Transformation

- Transformation operators
  - Rotate and translate

Define operator **T** to be one which rotates and translates a vector ${}^{A}P_{1}$ to a new vector ${}^{A}P_{2}$:

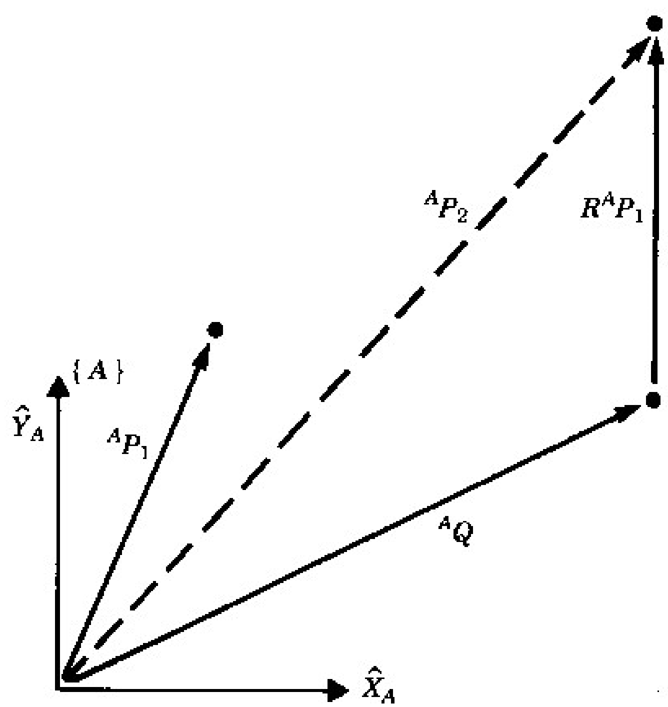
$$ {}^{A}P_{2} = \mathbf{T}\,{}^{A}P_{1} \qquad (1\text{-}3) $$

The transform that rotates by R and translates by Q is the same as the transform that describes a frame rotated by R and translated by Q relative to the reference frame.

Operators: Translations, Rotations, and Transformation

- Example 1-7

Figure 1-4 shows a vector ${}^{A}\mathbf{p}_1$. We wish to rotate it about $\mathbf{z}$ by 30 degrees, and translate it 10 units in $\mathbf{x}_A$, and 5 units in $\mathbf{y}_A$. Find ${}^{A}\mathbf{p}_2$ where ${}^{A}\mathbf{p}_1 = [3\ 7\ 0]^T$.



Operators: Translations, Rotations, and Transformation

- Solution:

The operator T, which performs the rotation and translation, is

$$T = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{A}\mathbf{p}_2 = T\,^{A}\mathbf{p}_1 = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}$$

Note that this example is numerically similar to Example 1-2, but the interpretation is different.

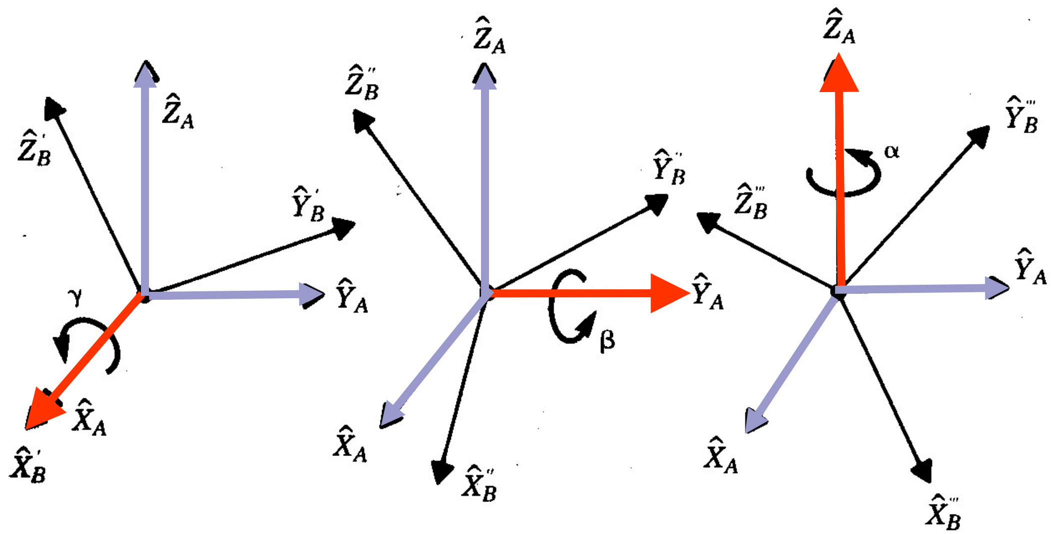
3. Other Orientation Representation

- Orientation of rigid body represented by 3x3 rotation matrix **R** → 9 elements
- Subject to:
  - *orthogonality* conditions (3 equations) and
  - *unit length* conditions (3 equations)

  => only 3 of 9 elements are independent (ie. there is redundancy in **R**)
- Different representations of orientation which requires only three or four parameters

Other Orientation Representation

- X-Y-Z fixed angles



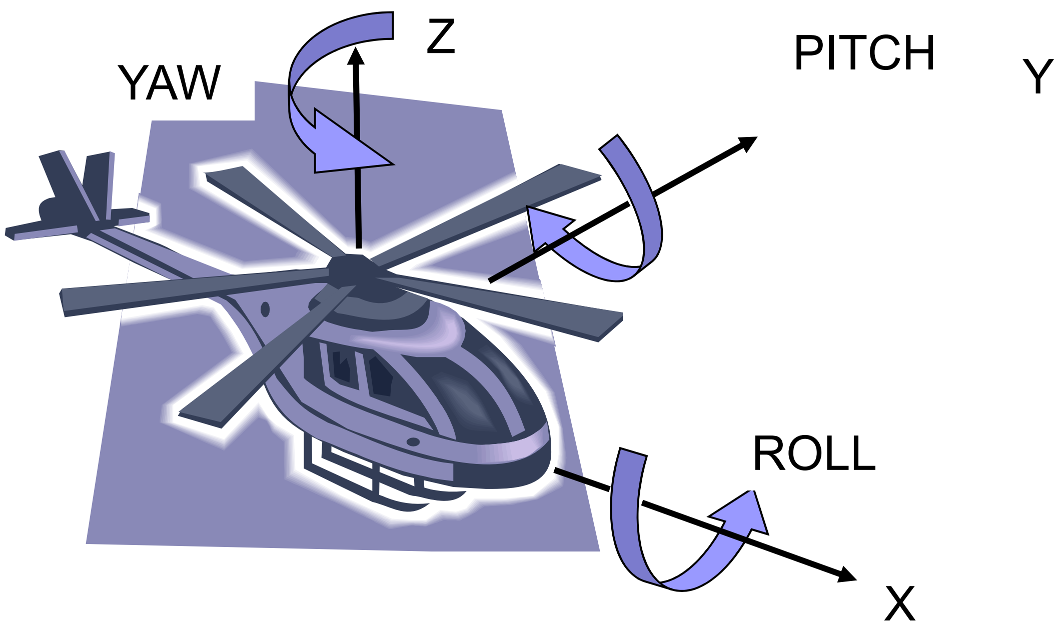
X $\longrightarrow$ Y $\longrightarrow$ Z

(roll, γ) (pitch, β) (yaw, α)

Each of the three rotations takes place about an axis in the **fixed** reference frame, {A}

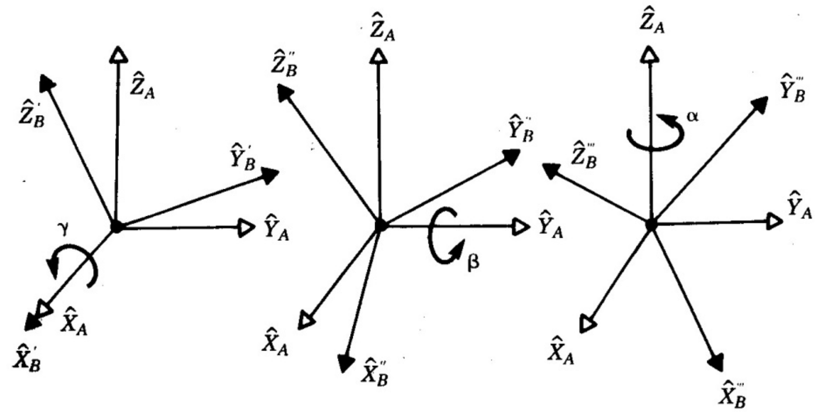
Other Orientation Representation

- X-Y-Z fixed angles



Other Orientation Representation

- X-Y-Z fixed angles



$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)
$$

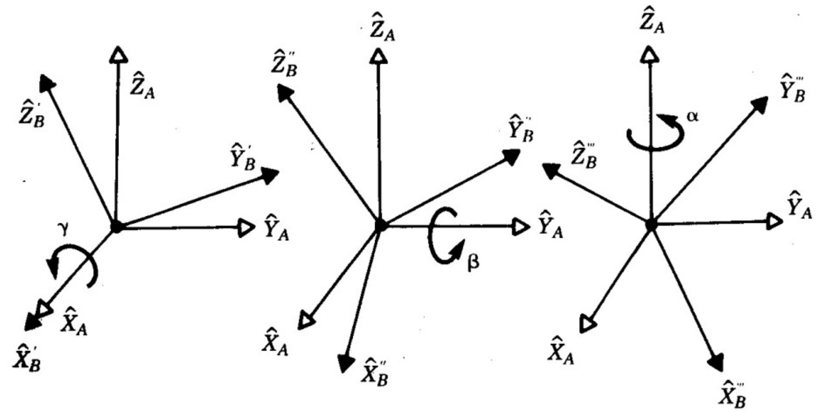
$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

where *c* and *s* denote *cosine* and *sine* functions, respectively

Other Orientation Representation

- X-Y-Z fixed angles
  - Inverse transformation



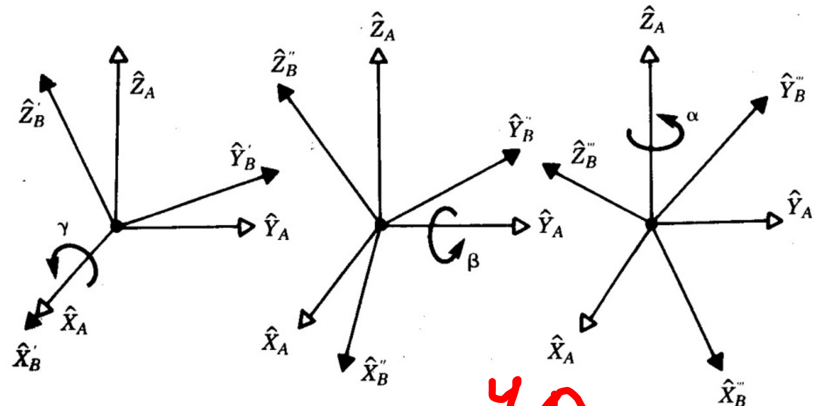
Given $${}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$, find α, β, γ

Nine equations (6 dependencies) and three unknowns

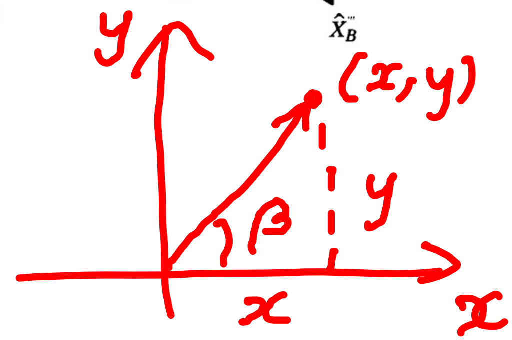
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

Other Orientation Representation

- X-Y-Z fixed angles
  - Inverse transformation



$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ \boxed{-s\beta} & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$



$$\beta = A\tan 2(s\beta, c\beta)$$

$$= A\tan 2(-r_{31}, \pm\sqrt{1-r_{31}^2})$$

$\tan^{-1} 1 = 45°, 225°$

$A\tan 2(1,1) = 45°$

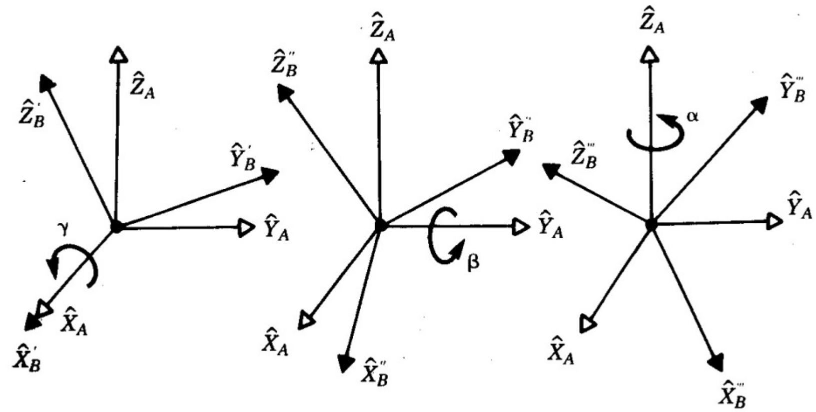
$A\tan 2(-1,-1) = 225°$

Remark:

•Atan2(y,x) computes $\tan^{-1}(y/x)$ and uses signs of both x and y to determine the quadrant in which the angle lies

Other Orientation Representation

- X-Y-Z fixed angles
  - Inverse transformation



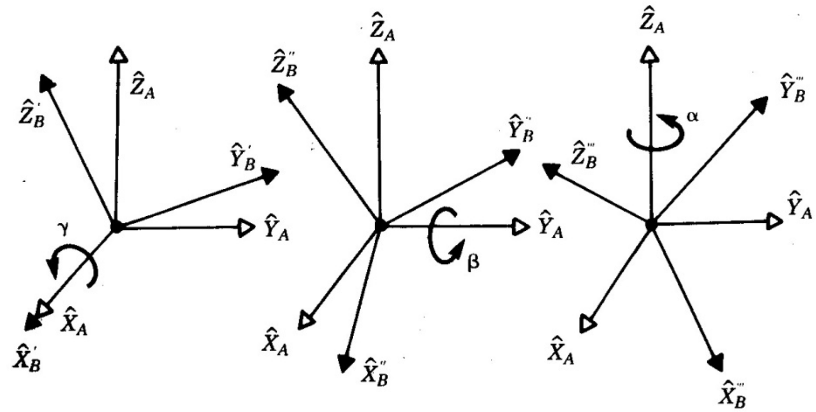
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$$\alpha = A\tan 2(s\alpha, c\alpha)$$

$$= A\tan 2\left(\frac{r_{21}}{c\beta}, \frac{r_{11}}{c\beta}\right)$$

Other Orientation Representation

- X-Y-Z fixed angles
  - Inverse transformation



$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$$\gamma = A\tan 2(s\gamma, c\gamma)$$

$$= A\tan 2\left(\frac{r_{32}}{c\beta}, \frac{r_{33}}{c\beta}\right)$$

Other Orientation Representation

- X-Y-Z fixed angles
  - Inverse transformation

$$\beta = A\tan 2(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}) \qquad \alpha = A\tan 2(\frac{r_{21}}{c\beta}, \frac{r_{11}}{c\beta}) \qquad \gamma = A\tan 2(\frac{r_{32}}{c\beta}, \frac{r_{33}}{c\beta})$$

Remark:

- 2 solutions in general
- If $|r_{31}|=1$ (implies $r_{11} = r_{21} = r_{32} = r_{33} = 0$), the solution degenerates (mathematical singularity) such that only the difference of α and γ may be computed:

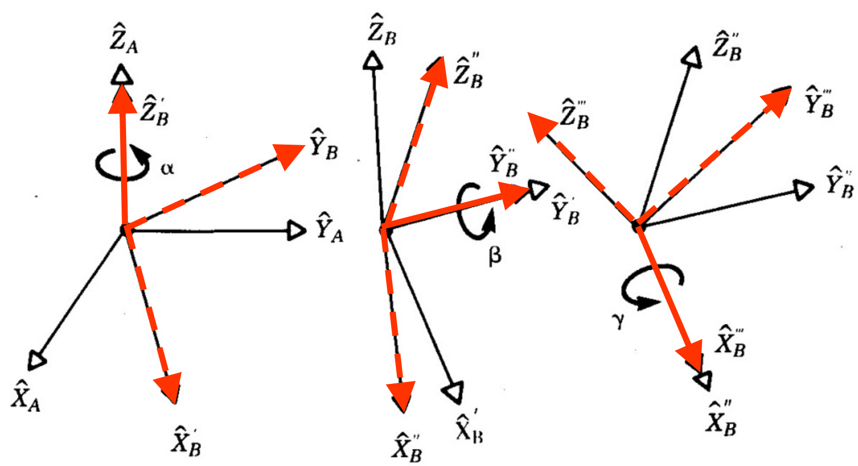
$$\begin{bmatrix} 0 & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ r_{31} & 0 & 0 \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

If $r_{31} = +1$: $\beta = A\tan 2(-1,0) = -90^0$ => $\alpha+\gamma$ = Atan2 ($-r_{23}$, r_{22})

If $r_{31} = -1$: $\beta = A\tan 2(1,0) = 90^0$ => $\alpha-\gamma$ = Atan2 (r_{23} , r_{22})

Other Orientation Representation

- Z-Y-X Euler angles



Z $\longrightarrow$ Y $\longrightarrow$ X

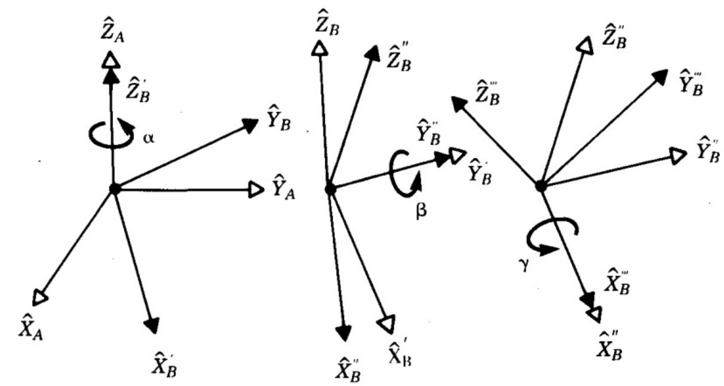
Each rotation is performed about an axis of the *moving* frame {B}, rather than the fixed reference frame, {A}

Other Orientation Representation

- Z-Y-X Euler angles

$$ {}^{A}_{B}R = {}^{A}_{B'}R \, {}^{B'}_{B''}R \, {}^{B''}_{B}R $$

Recall:
Consecutive
transformation



$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

Note:

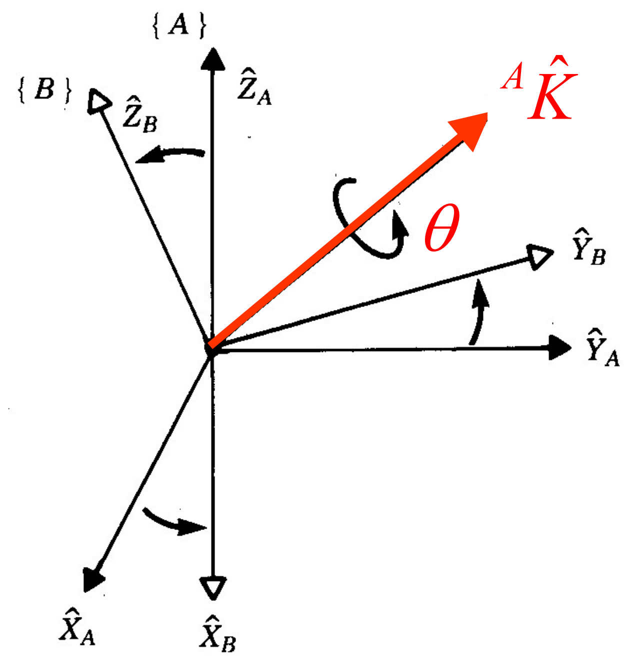
- Same as the result obtained for the same three rotations taken in the opposite order about fixed axes.

Other Orientation Representation

- Four-parameter representations for orientation
  - Equivalent angle-axis
  - Euler parameters or Unit Quaternion
- Nonminimal representations of orientation

Other Orientation Representation

- Equivalent angle-axis
  - *Any* rotation matrix can be represented by choosing a proper axis and angle (*Euler's theorem on rotation*)
  - quadruple of ordered real parameters consisting of one scalar θ (angle of rotation) and one *unit* vector $\hat{K}$ (axis of rotation) (according to *right-hand* rule)

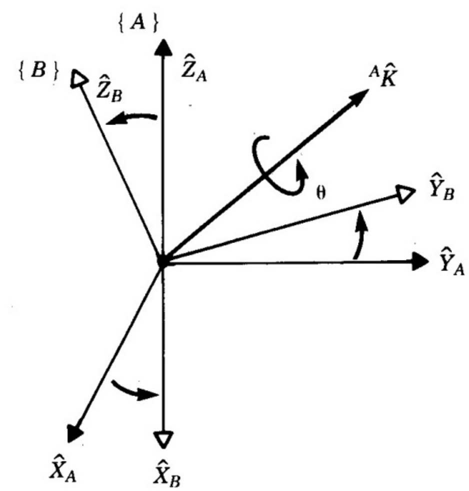


Other Orientation Representation

- Equivalent angle-axis

$$R_{\hat{K}}(\theta) =$$

$$\begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z s\theta & k_z k_x v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix} \quad (1\text{-}4)$$



where $v\theta = (1 - \cos\theta)$, ${}^A\hat{K} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}^T$, $\left| {}^A\hat{K} \right| = 1$ (unit vector)

, sign of θ is determined by right-hand rule

Remark:

•For *small angular rotations*, axis of rotation becomes *ill-defined*.

•*Non-unique* representation: $R_{-\hat{K}}(-\theta) = R_{\hat{K}}(\theta)$

Other Orientation Representation

- Example 1-8

A frame {B} is described as follow: initially coincident with {A} we rotate {B} about the vector ${}^{A}\mathbf{k} = [0.707\ 0.707\ 0]^T$ (passing through the origin) by an amount $\theta = 30$ degrees. Give the frame description of {B} with reference to {A} after the rotation.

Other Orientation Representation

- Solution:

$$
{}^{A}R_{\mathbf{k}}\left(30^{o}\right)=\begin{pmatrix}
k_x k_x v30^{o} + c30^{o} & k_y k_x v30^{o} - k_z\ s30^{o} & k_z k_x\ v30^{o} + k_y\ s30^{o} \\
k_x k_y v30^{o} + k_z\ s30^{o} & k_y k_y v30^{o} + c30^{o} & k_z k_y v30^{o} - k_x s30^{o} \\
k_x k_z v30^{o} - k_y\ s30^{o} & k_y k_z v30^{o} + k_x s30^{o} & k_z k_z v30^{o} + c30^{o}
\end{pmatrix}
$$

where $v30^{o} = 1\text{-}\cos 30^{0}$; $\mathbf{k} = [k_x, k_y, k_z]^T = [0.707\ 0.707\ 0]^T$

Since there is no translation of the origin,

$$
{}_{B}^{A}T=\left[\begin{array}{ccc|c}
 & & & 0 \\
 & {}^{A}R_{\mathbf{k}}\left(30^{o}\right) & & 0 \\
 & & & 0 \\
\hline
0 & 0 & 0 & 1
\end{array}\right]=\begin{bmatrix}
0.933 & 0.067 & 0.354 & 0 \\
0.067 & 0.933 & -0.354 & 0 \\
-0.354 & 0.354 & 0.866 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Other Orientation Representation

- Example 1-9

A frame {B} is described as follows: initially coincident with {A}, we rotate {B} about the vector ${}^{A}\mathbf{k} = [0.707 \quad 0.707 \quad 0.0]^T$, passing through the point ${}^{A}P = [1 \quad 2 \quad 3]^T$, by $\theta = 30^o$. Give the frame description of {B} with reference to {A}.

Solution:

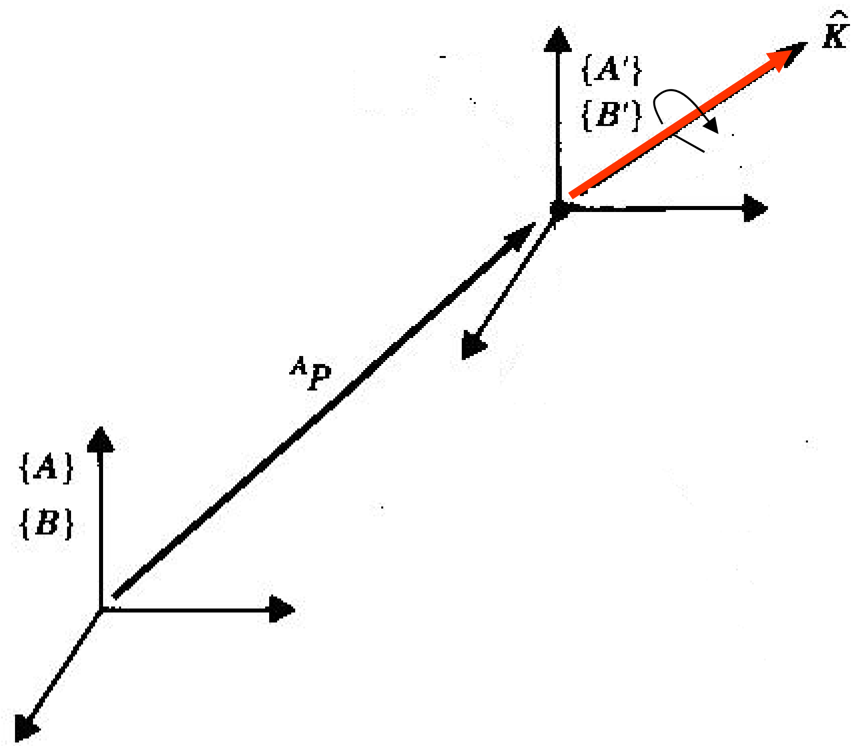
Before performing the rotation, {A} and {B} are coincident. We define two new frames, {A'} and {B'} which are obtained by translating {A} and {B}, respectively, to point P. {B} and {B'} can be treated as if they are mounted on the same rigid body, whereas {A} and {A'} are fixed in the space.

Other Orientation Representation

- Solution:

$$
{}^{A}_{A'}T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{B'}_{B}T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Other Orientation Representation

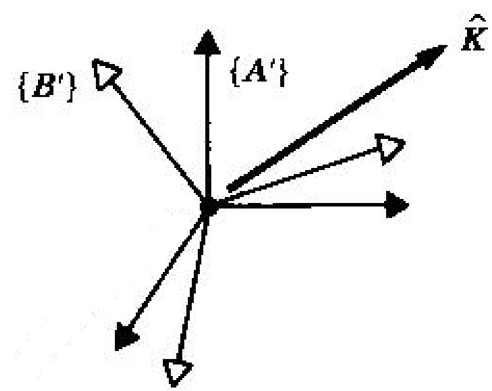
- Solution:

Note that the point P is on the axis of rotation. Now, let's rotate {B'} relative to {A'}. This is a rotation about an axis which passes through the origin, so we may use

$$^{A'}R_{\mathbf{k}}(\theta)=\begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z\, s\theta & k_z k_x\, v\theta + k_y\, s\theta \\ k_x k_y v\theta + k_z\, s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y\, s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix}$$

to compute {B'} relative to {A'} after the rotation.

$$^{A'}_{B'}T=\left[\begin{array}{ccc|c} & ^{A'}R_{\mathbf{k}}(30^o) & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]=\begin{bmatrix} 0.933 & 0.067 & 0.354 & 0 \\ 0.067 & 0.933 & -0.354 & 0 \\ -0.354 & 0.354 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

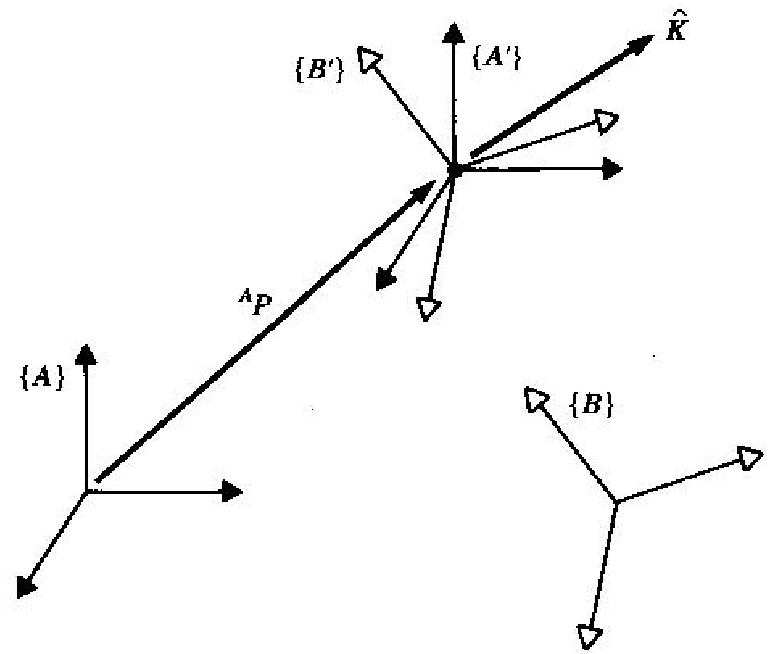


Other Orientation Representation

- Solution:

Finally,

$$
{}^{A}_{B}T = {}^{A}_{A'}T\,{}^{A'}_{B'}T\,{}^{B'}_{B}T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & -1.13 \\ 0.067 & 0.933 & -0.354 & 1.13 \\ -0.354 & 0.354 & 0.866 & 0.05 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

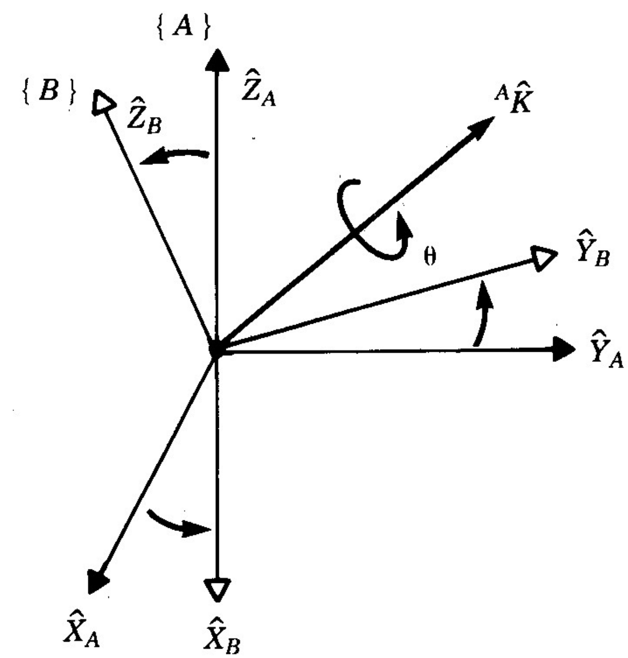


Note: After rotating about the axis **k** , {B} becomes separated from {A} as shown in Fig. 1-9.

Figure 1-9

Other Orientation Representation

- Equivalent angle-axis
  - *Inverse Problem*



Given ${}^A_B R$ → Find ${}^A\hat{K}$, θ

$$\text{Let } {}^A_B R = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} \longleftarrow \text{given}$$

$$= \begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z\, s\theta & k_z k_x\, v\theta + k_y\, s\theta \\ k_x k_y v\theta + k_z\, s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y\, s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix}$$

where $v\theta$ = (1 - cosθ)

Other Orientation Representation

- Equivalent angle-axis
  - *Inverse Problem*

$$
{}^{A}R_{\hat{K}}(\theta) = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z s\theta & k_z k_x v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix}
$$

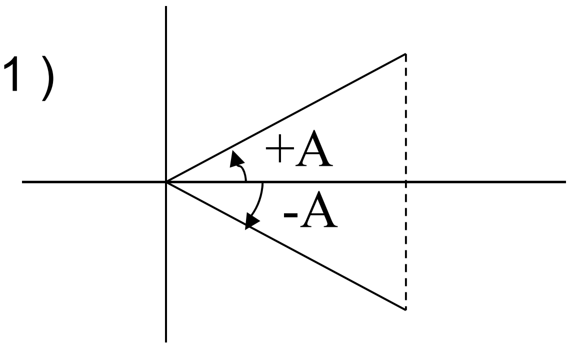
where $v\theta = (1 - \cos\theta)$

From the diagonal elements,

Trace $({}^{A}_{B}R) = 1 + 2\cos\theta$

=> $\cos\theta = \frac{1}{2}(\text{Trace}({}^{A}_{B}R) - 1)$

(Two solutions for $\theta = \pm A$)



Other Orientation Representation

- Equivalent angle-axis
  - *Inverse Problem*

$$
{}^{A}R_{\hat{K}}(\theta) = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z s\theta & k_z k_x v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix}
$$

If $\sin\theta \neq 0$:

From : n_z & a_x elements

$$k_y = \frac{a_x - n_z}{2\sin\theta}$$

From : o_z & a_y elements

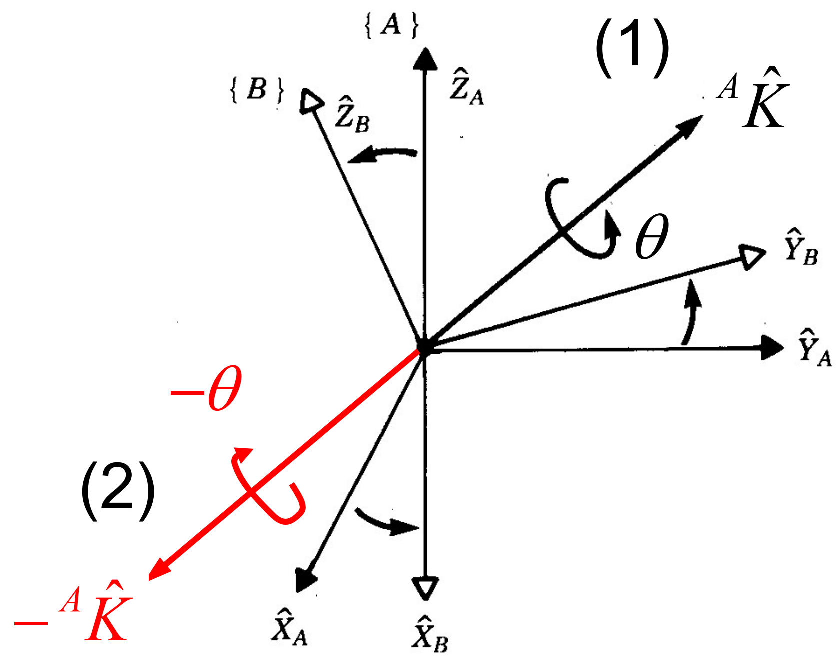
$$k_x = \frac{o_z - a_y}{2\sin\theta}$$

From : n_y & o_x elements

$$k_z = \frac{n_y - o_x}{2\sin\theta}$$

Other Orientation Representation

- Equivalent angle-axis
  - *Inverse Problem*



In general:
2 solutions

Other Orientation Representation

- Equivalent angle-axis
  - Inverse Problem

•If $\theta = 0°$

$$ {}^{A}_{B}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

Identity (null rotation)

$\Rightarrow$ $\hat{K}$ = any axis (infinite solutions)

•If $\theta = 180°$

$$ {}^{A}_{B}R = \begin{pmatrix} -1+2k_x^{\ 2} & 2k_xk_y & 2k_xk_z \\ 2k_xk_y & -1+2k_y^{\ 2} & 2k_yk_z \\ 2k_xk_z & 2k_yk_z & -1+2k_z^{\ 2} \end{pmatrix} $$

(symmetric)

Other Orientation Representation

- Equivalent angle-axis
  - Inverse Problem

•If θ = 180° (cont)

$$ {}^{A}_{B}R = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} -1 + 2k_x^{2} & 2k_xk_y & 2k_xk_z \\ 2k_xk_y & -1 + 2k_y^{2} & 2k_yk_z \\ 2k_xk_z & 2k_yk_z & -1 + 2k_z^{2} \end{pmatrix} $$

$$ \left.\begin{aligned} k_x &= \pm\sqrt{\frac{n_z n_y}{2o_z}} \\ k_y &= \frac{n_y}{2k_x} \\ k_z &= \frac{a_z}{2k_x} \end{aligned}\right\} \Rightarrow $$

2 solutions for $\hat{K}$
(If one solution is $\hat{K}_1$,
the other is $-\hat{K}_1$)

Other Orientation Representation

- Equivalent angle-axis
  - Inverse Problem

•If θ = 180º (cont)

If all off-diagonal terms are 0, one of k_x, k_y or k_z = 1 & the rest are 0s. The component of $\hat{K}$ that is 1 has a value of 1 in the diagonal of ${}^A_B R$.

E.g. $k_x = 1$, $k_y = k_z = 0$

$$ {}^A_B\mathrm{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} $$

Other Orientation Representation

- Euler parameters (Unit Quaternion)
  - Able to solve the non-uniqueness problem encountered by angle/axis representation

Euler parameters:

$$\varepsilon_1 = k_x \sin\frac{\theta}{2}$$
$$\varepsilon_2 = k_y \sin\frac{\theta}{2}$$
$$\varepsilon_3 = k_z \sin\frac{\theta}{2}$$
$$\varepsilon_4 = \cos\frac{\theta}{2}$$

4 parameters not independent because:

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

where k_x, k_y and k_z are the coordinates of the equivalent axis $\hat{K}$

and θ is the equivalent angle

Other Orientation Representation

- Euler parameters (Unit Quaternion)

Based on Eq (1-4) (equivalent angle-axis) and the definition of unit quaternion:

$$
{}^{A}_{B}R = \begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z s\theta & k_z k_x v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix}
$$

$$
\frac{\varepsilon_1}{\sin\frac{\theta}{2}} = k_x \qquad \frac{\varepsilon_2}{\sin\frac{\theta}{2}} = k_y \qquad \frac{\varepsilon_3}{\sin\frac{\theta}{2}} = k_z
$$

$$
\cos\theta = 2\cos^2\left(\theta/2\right) - 1 = 2\varepsilon_4^2 - 1
$$

$$
2\sin^2\left(\theta/2\right) = 1 - \cos\theta
$$

, etc

$$
\boxed{R_\varepsilon = \begin{bmatrix} 2(\varepsilon_4^2 + \varepsilon_1^2) - 1 & 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) & 2(\varepsilon_4^2 + \varepsilon_2^2) - 1 & 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) & 1 - 2(\varepsilon_1^2 + 2\varepsilon_2^2) \end{bmatrix}} \qquad (1\text{-}5)
$$

Other Orientation Representation

- Euler parameters (Unit Quaternion)
  - Inverse Problem

Given $${}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$, find ε_i

$$R_\varepsilon = \begin{bmatrix} 2(\varepsilon_4^2+\varepsilon_1^2)-1 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 2(\varepsilon_4^2+\varepsilon_2^2)-1 & \boxed{2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4)} \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & \boxed{2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)} & 1-2(\varepsilon_1^2+\varepsilon_2^2) \end{bmatrix}$$

$$\begin{cases} \varepsilon_1 = \frac{1}{2}\boxed{\operatorname{sgn}(r_{32}-r_{23})}\sqrt{r_{11}-r_{22}-r_{33}+1} \\ \varepsilon_2 = \frac{1}{2}\operatorname{sgn}(r_{13}-r_{31})\sqrt{r_{22}-r_{33}-r_{11}+1} \\ \varepsilon_3 = \frac{1}{2}\operatorname{sgn}(r_{21}-r_{12})\sqrt{r_{33}-r_{11}-r_{22}+1} \\ \varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}} \end{cases}$$

Note: sgn(x) = 1 for $x \geq 0$ and sgn(x) = -1 for $x < 0$

Other Orientation Representation

- Euler parameters (Unit Quaternion)
  - Inverse Problem

$$\varepsilon_1 = \frac{1}{2}\operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$\varepsilon_2 = \frac{1}{2}\operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1}$$

$$\varepsilon_3 = \frac{1}{2}\operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1}$$

$$\varepsilon_4 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

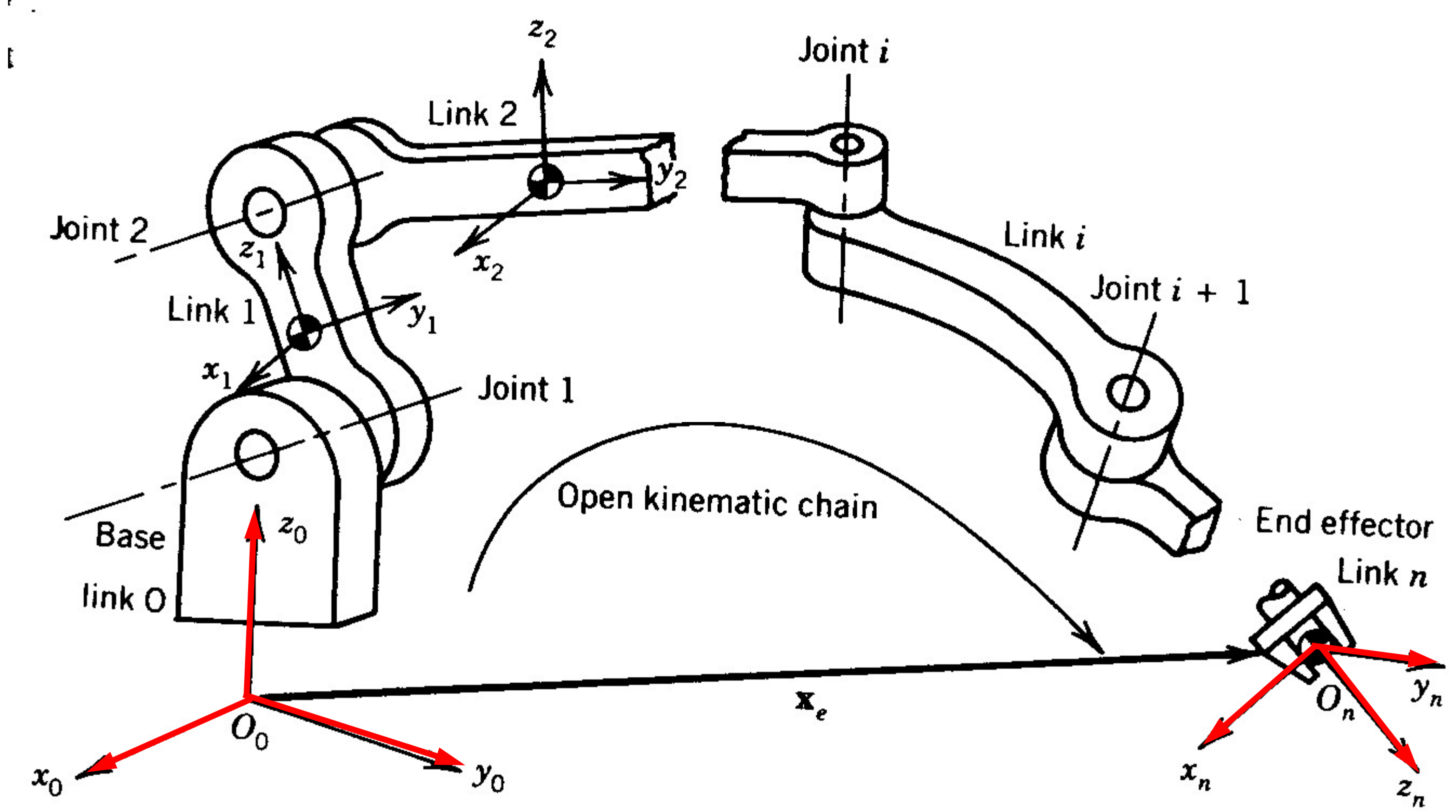
Note:

•Implicitly assumed $\varepsilon_4 \geq 0$ (by considering $\theta \in [-\pi, \pi]$)

•No singularity occurs (compared to angle/axis representation)

4. Kinematic Modeling of Manipulator Arms

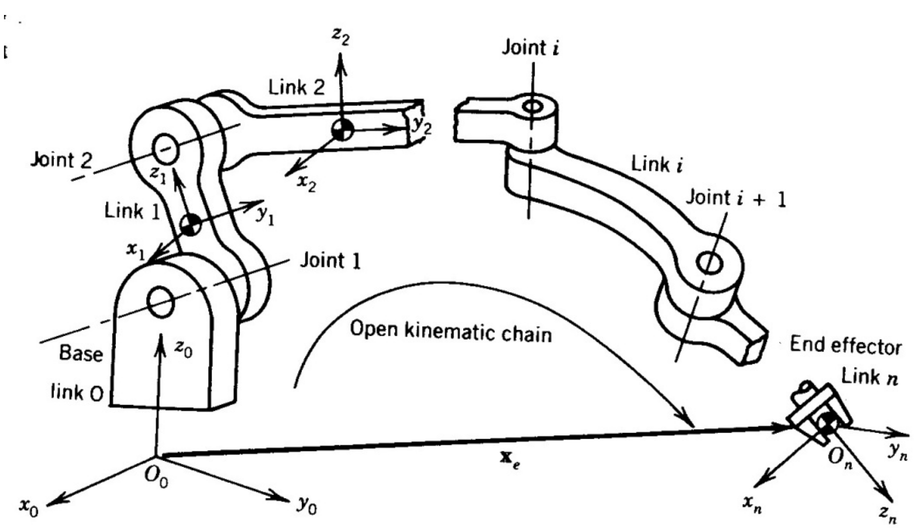
- Open Kinematic Chains



How to express the end-effector's position and orientation with reference to base frame?

Kinematic Modeling of Manipulator Arms

■ Overview

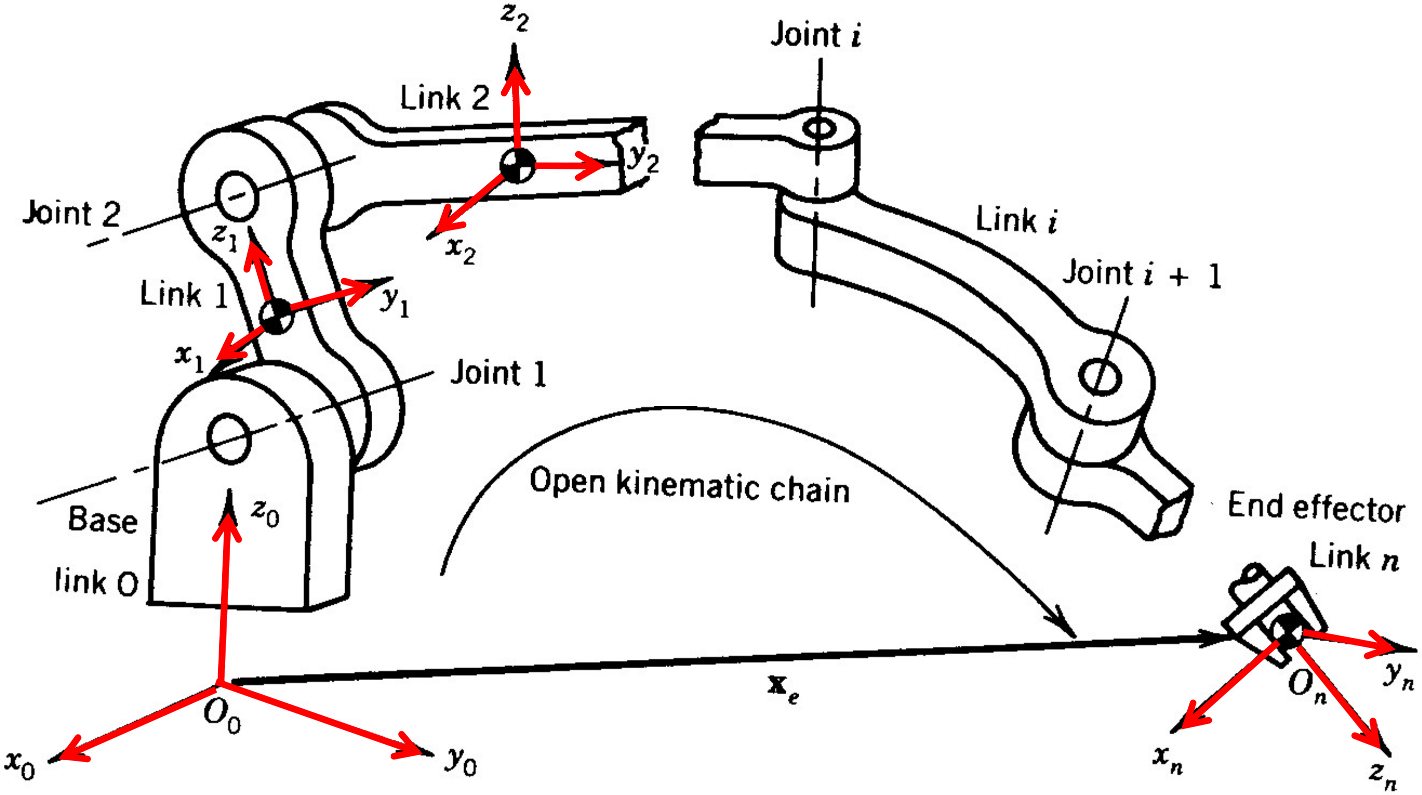


- Each link numbered in series from **0** (base) to **n** (end-effector)
- Joint between link *i-1* & link *i* is labeled as joint *i*
- Attach a coordinate frame 0_i-$x_i y_i z_i$ to each link *i*
- Using 4x4 Homogeneous transformation matrix to describe the position & orientation of frame 0_i-$x_i y_i z_i$ relative to the previous frame 0_{i-1}-$x_{i-1} y_{i-1} z_{i-1}$

End-effector position & orientation obtained by *consecutive* homogeneous transformations from last frame back to the base frame

Kinematic Modeling of Manipulator Arms

- How should the frames be attached?

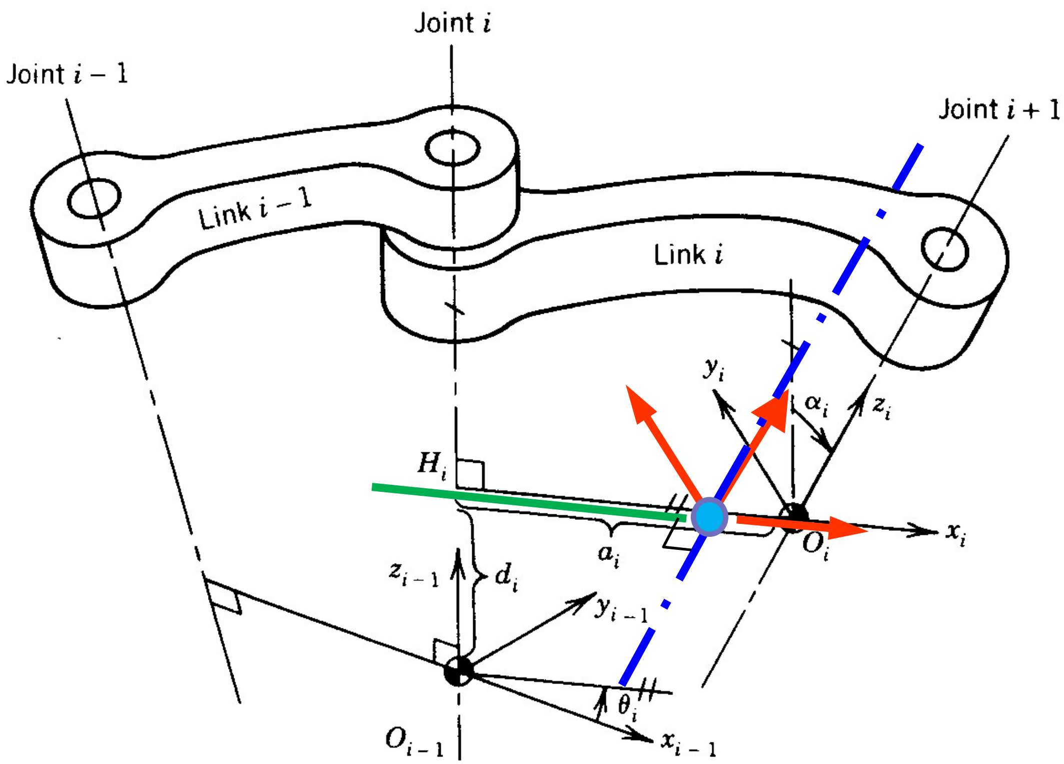


Denavit-Hartenberg representation

- *Systematic method* of describing kinematic relationship between a pair of adjacent links in an *open* kinematic chain
- Based on 4x4 homogeneous matrix representation of rigid body position and orientation
- Use *minimum number* of parameters to describe the kinematic relationship

Denavit-Hartenberg representation

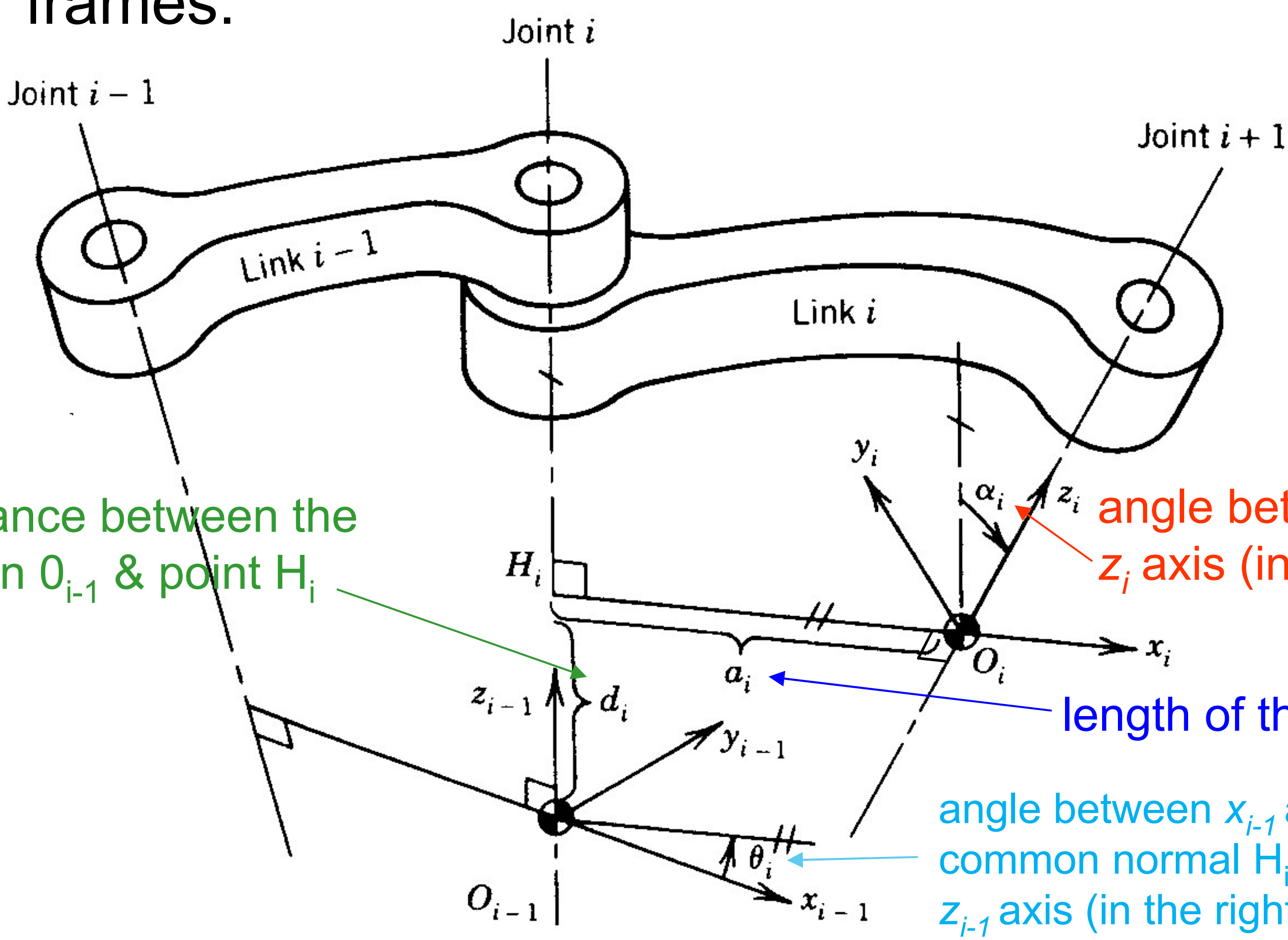
Procedure to form frame 0_i-$x_iy_iz_i$ (attached to link *i*):



1. Origin of the *i*th coordinate frame 0_i is located at the intersection of joint axis *i+1* and the common normal between joint axes *i* & *i+1*
2. x_i axis is directed along the extension line of the common normal
3. z_i axis is along the joint axis *i*+1
4. y_i axis is chosen s.t. the resultant frame 0_i-$x_iy_iz_i$ forms a right-hand coordinate system

Denavit-Hartenberg representation

4 parameters to determine the relative location of the 2 frames:



distance between the origin 0_{i-1} & point H_i

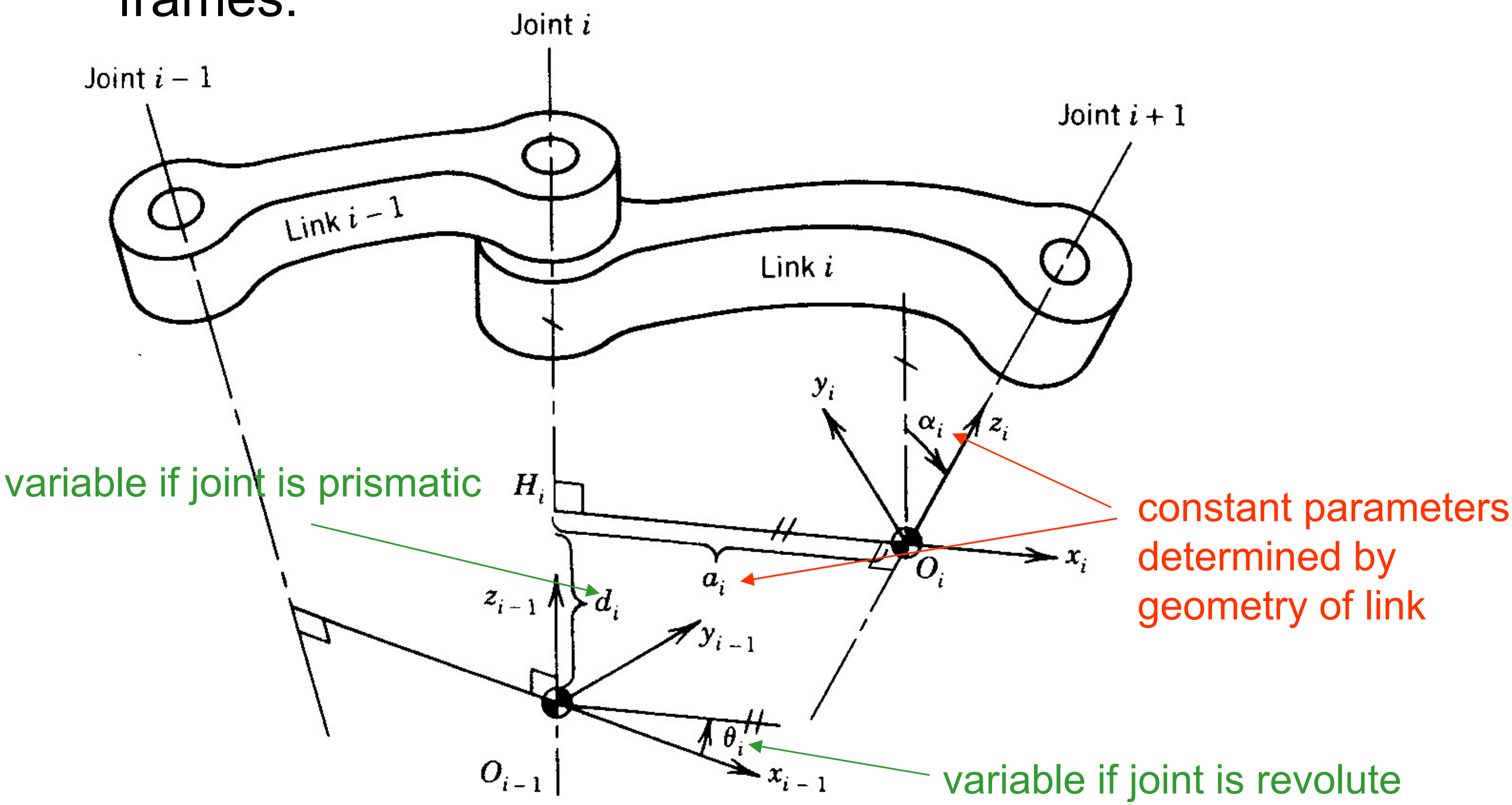
angle between the joint axis i & z_i axis (in the right hand sense)

length of the common normal

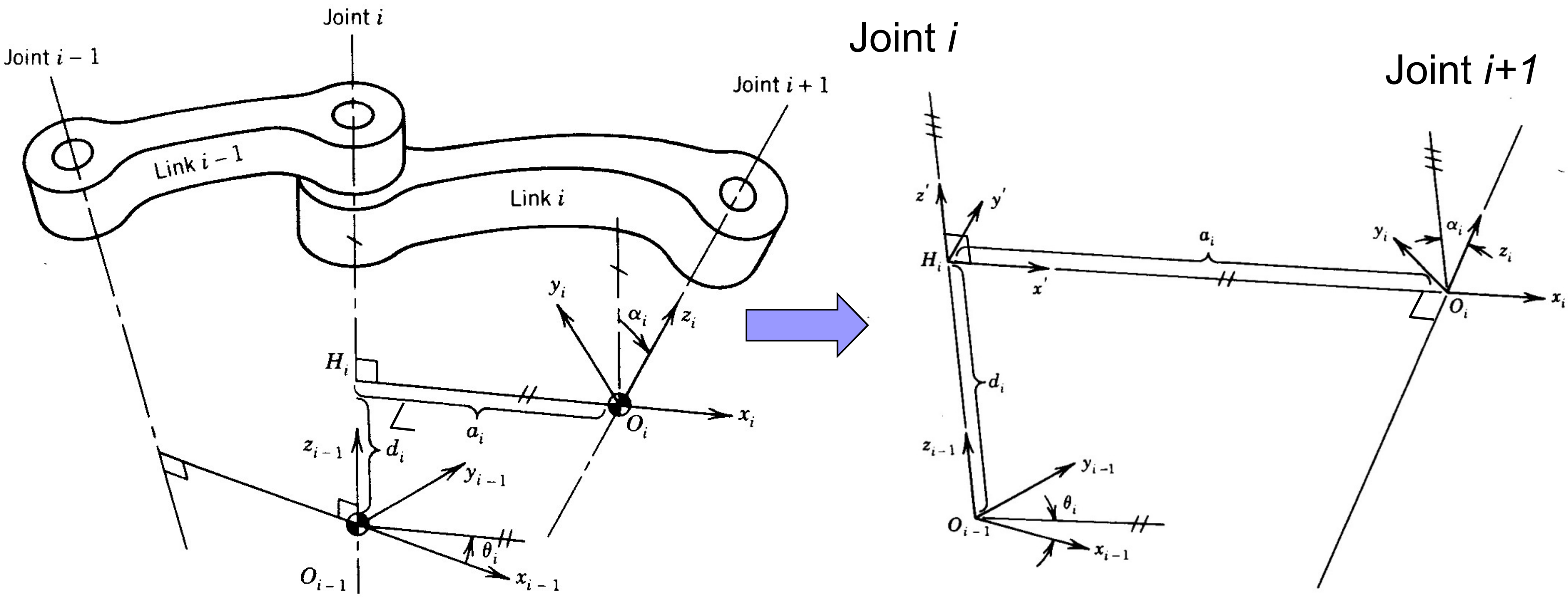
angle between x_{i-1} axis and the common normal H_iO_i measured abt z_{i-1} axis (in the right hand sense)

Denavit-Hartenberg representation

4 parameters to determine the relative location of the 2 frames:

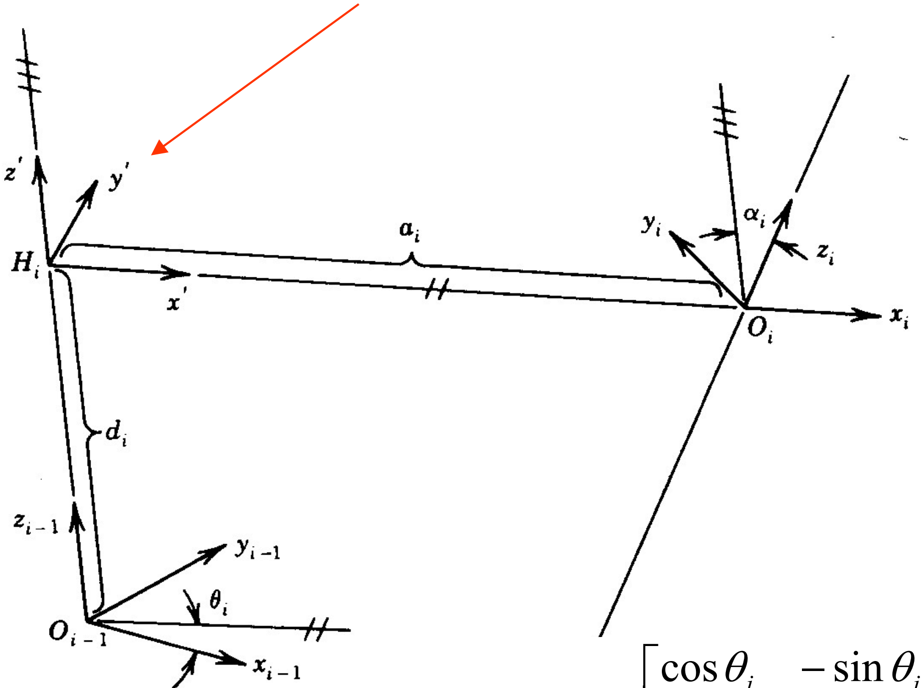


Denavit-Hartenberg representation



Denavit-Hartenberg representation

intermediate coordinate frame



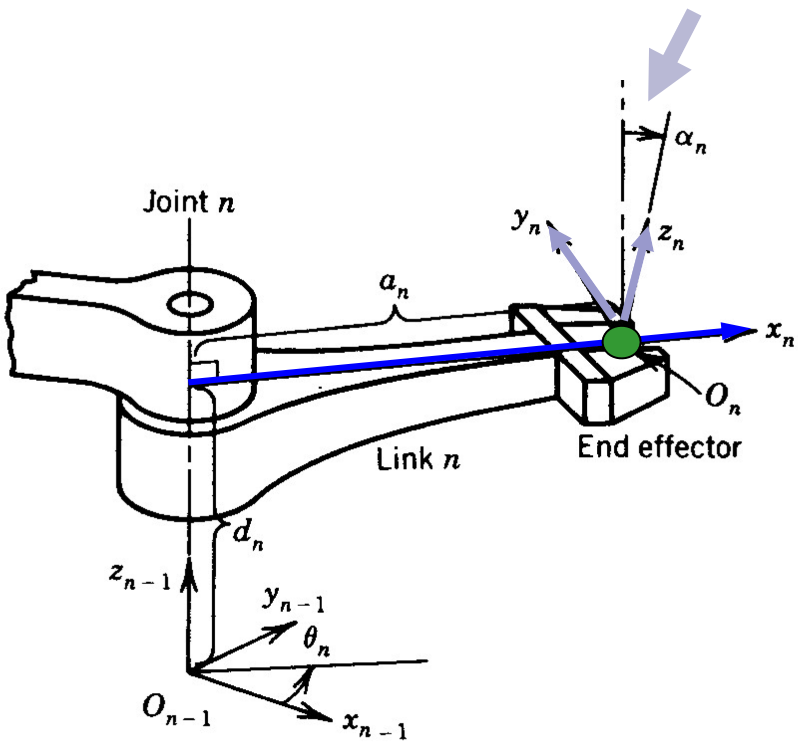
$$ {}^{\text{int}}_{i}A = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{\text{int}}A = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{i}A = {}^{i-1}_{\text{int}}A \; {}^{\text{int}}_{i}A = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2\text{-}1) $$

Denavit-Hartenberg representation

- Several exceptions to D-H notation rule:



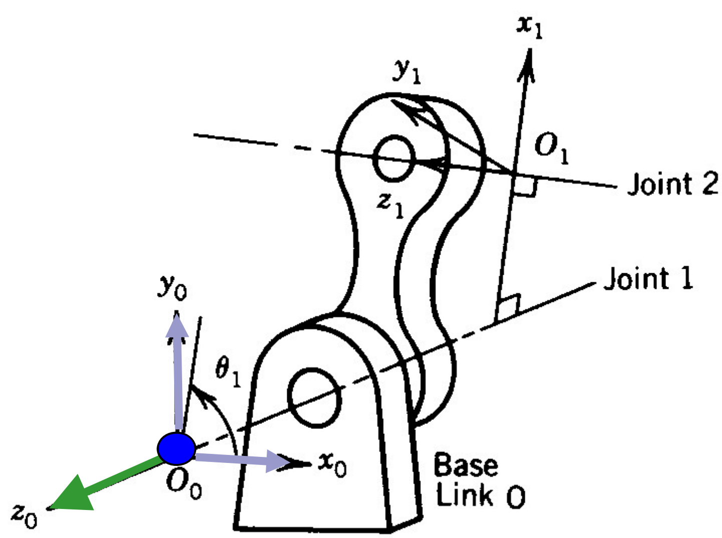
Location of the end coordinate frame.

Last link (frame {n}):

- origin of coordinate frame chosen at any convenient point of the end-effector;
- x_n axis intersects last joint axis at right angles;
- α_n is arbitrary

Denavit-Hartenberg representation

- Several exceptions to D-H notation rule:



Location of the base coordinate frame.

Base link (frame {0}):

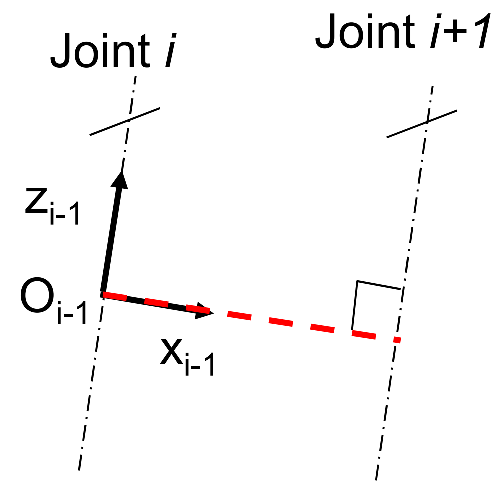
- z_0 axis along joint axis 1
- origin chosen at an arbitrary point on the joint axis 1
- x_0 & y_0 axes is arbitrary (as long as the frame is right-handed)

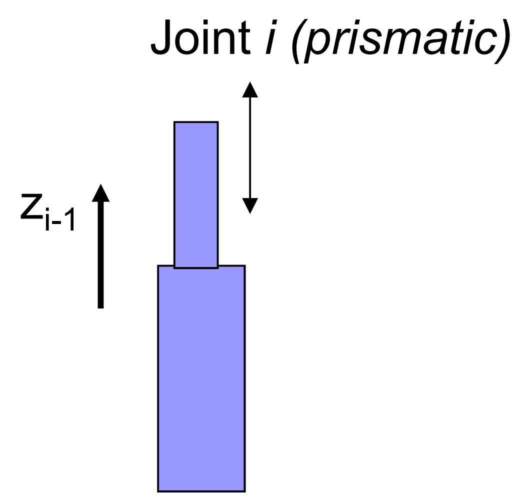
Denavit-Hartenberg representation

- Several exceptions to D-H notation rule:

▪ For intermediate links

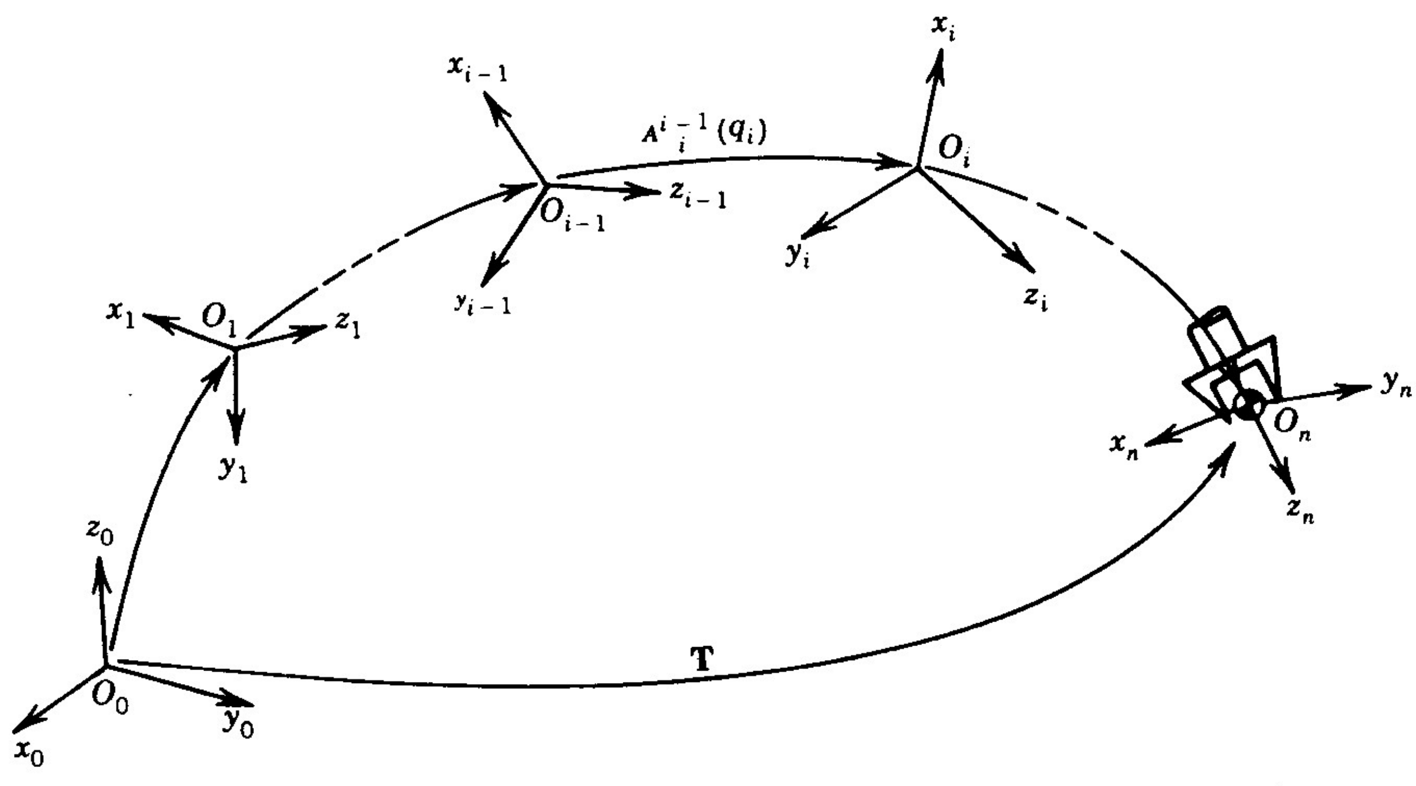
- When 2 joints axes of an intermediate link are parallel => common normal not unique.
  - Choice of common normal is arbitrary, typically make it passes through 0_{i-1} , s.t. $d_i = 0$
- For prismatic joints, only direction of the joint axis is meaningful => position of joint axis can be chosen arbitrarily.





5. Forward (Direct) Kinematic Equations

- Express position and orientation of end-effector as a function of joint displacement (using D-H convention)



Forward (Direct) Kinematic Equations

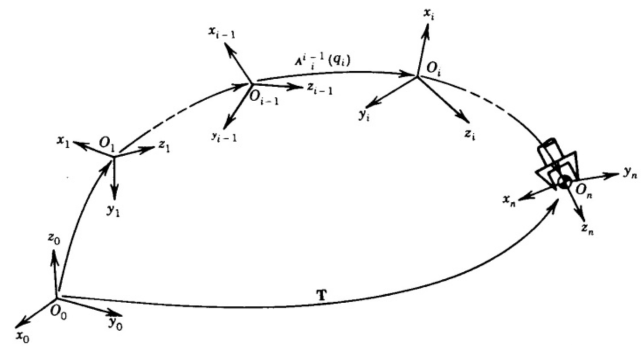
- Procedure:

1. Identify the joint variables and link kinematic parameters:

   Denote joint displacement by q_i:

   $q_i = \theta_i$ (revolute joint)

   $q_i = d_i$ (prismatic joint)



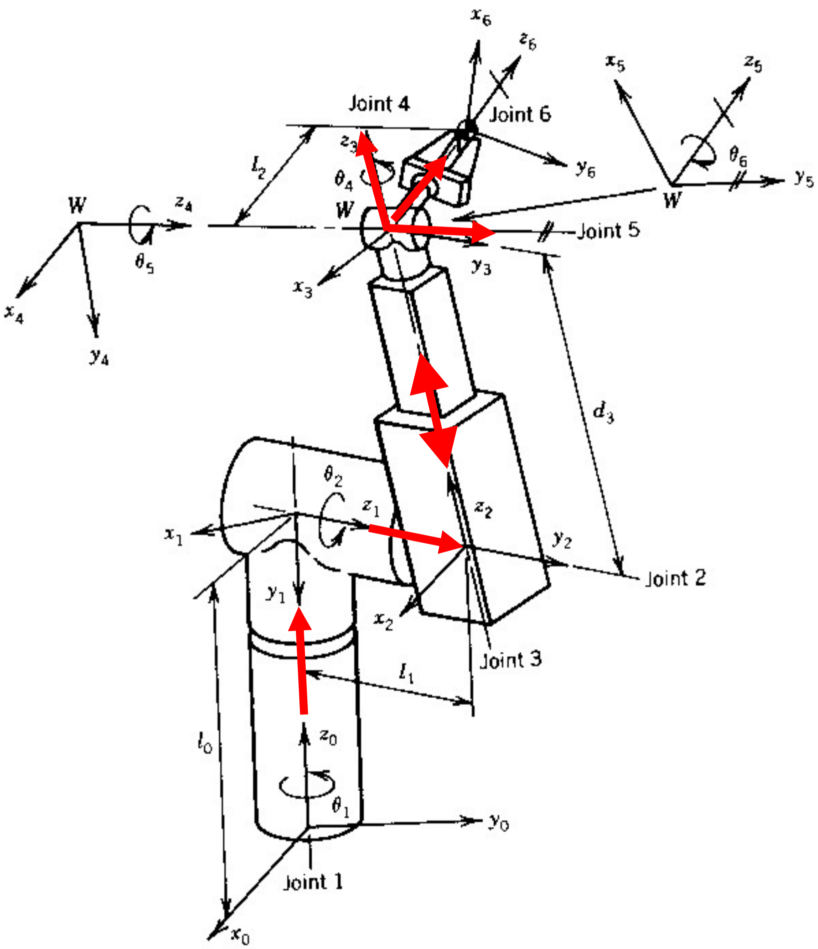
2. Assign Cartesian Coordinate frames to each link (including the base 0 & end-effector n)
3. Define the link transformation matrices.
4. Compute the forward transformation

$$ {}^{0}_{n}T = {}^{0}_{1}A(q_1)\,{}^{1}_{2}A(q_2)\cdots{}^{n-1}_{n}A(q_n) $$

- Kinematic equation of the manipulator arm

Forward (Direct) Kinematic Equations

Example 2-1: Find the Kinematic Model of the following 5-R-1P Manipulator Arm.

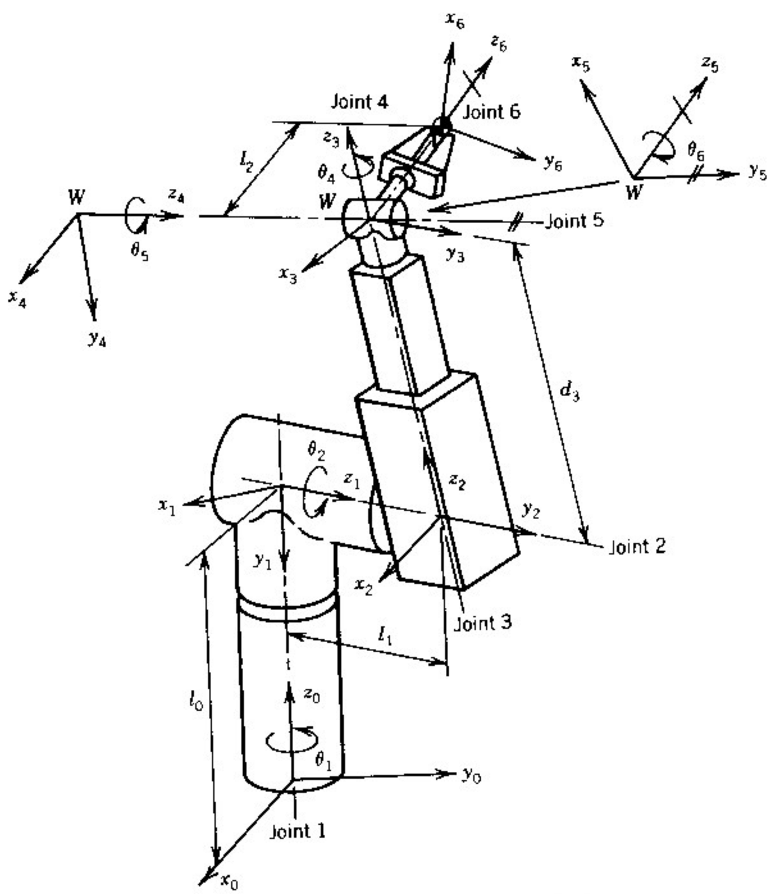


1. Identify all the joints:

   Joint 1: Revolute joint
   Joint 2: Revolute joint
   Joint 3: Prismatic joint (joint axis chosen to be coincides with joint 4)
   Joint 4, 5, 6: Revolute joints whose axes intersect at the single point W.

Forward (Direct) Kinematic Equations

Example 2-1: (cont)



2. Attach frames to all the links:

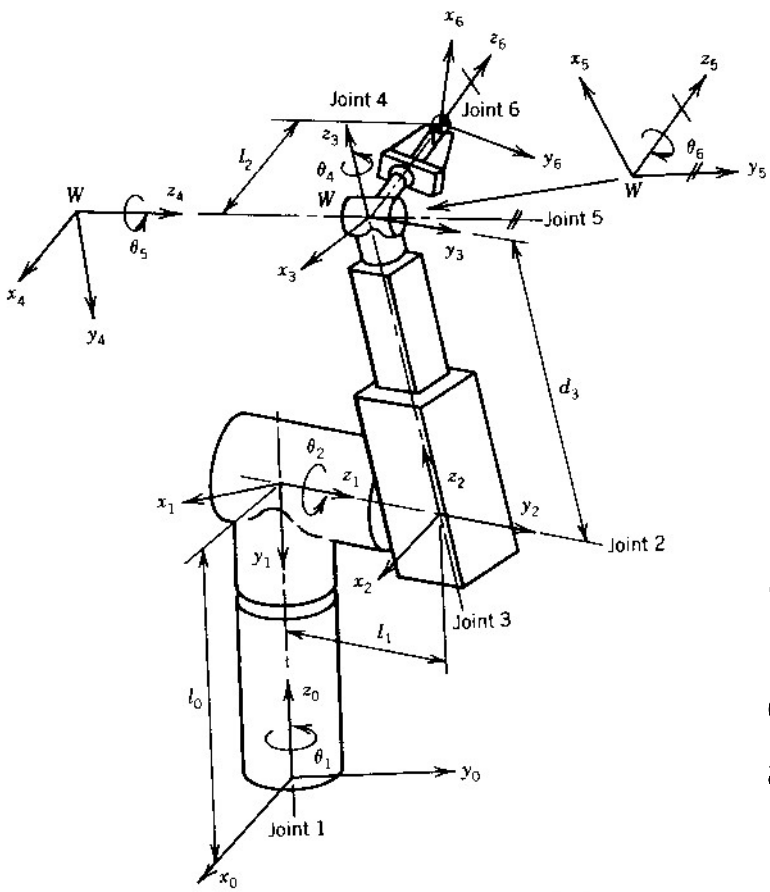
   a. Base frame chosen on the table surface with z_o axis along the joint axis 1
   b. The origin of the final frame can be selected arbitrarily (we choose an appropriate point on the last joint axis at which a workpiece will be grasped).
   c. Other frames are assigned according to the Denavit-Hartenberg rule

Remark:

- Try to define coordinate frames so that minimal number of non-zero parameters is resulted

Forward (Direct) Kinematic Equations

Example 2-1: (cont)



3. The Denavit-Hartenberg parameters for these frames are listed below:

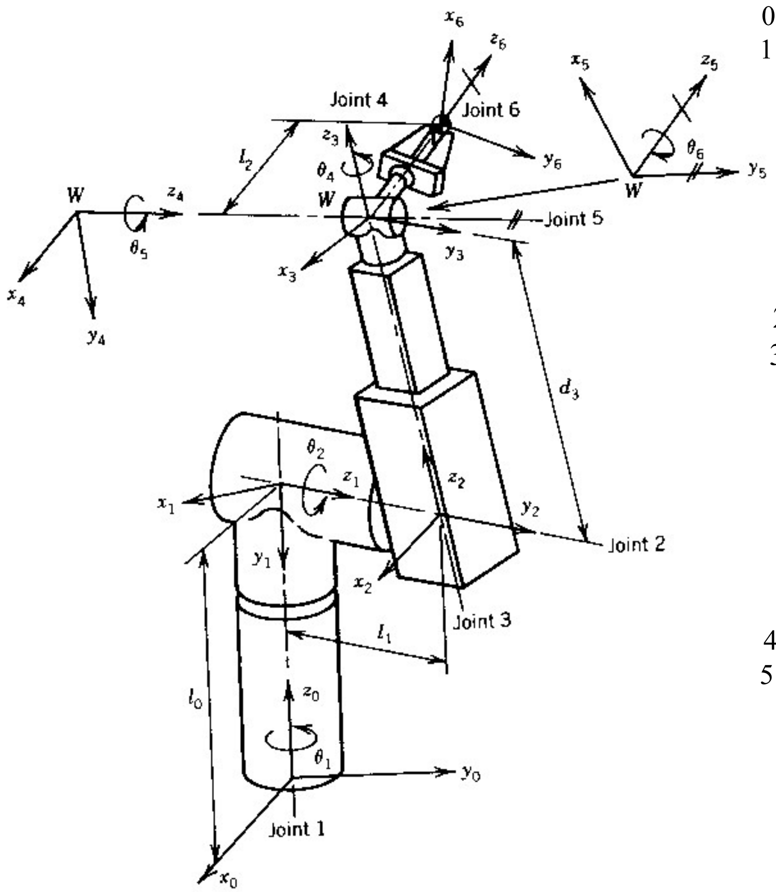
link number	α_i	a_i	d_i	θ_i
1	-90°	0	l_0	θ_1
2	+90°	0	l_1	θ_2
3	0	0	d_3	0
4	-90°	0	0	θ_4
5	+90°	0	0	θ_5
6	0	0	l_2	θ_6

The 4x4 matrix $\,{}^{i-1}_{i}A(q_i)$

can be obtained by substituting the above parameters into Eq. (2-1):

Forward (Direct) Kinematic Equations

Example 2-1: (cont)



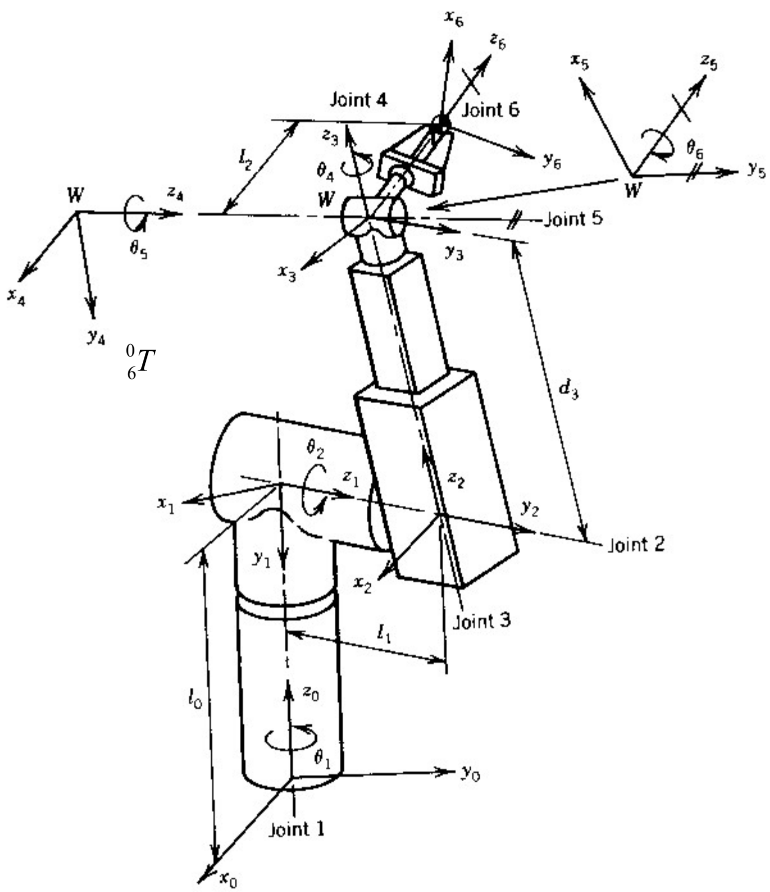
$$ {}^{0}_{1}A(\theta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}_{2}A(\theta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}A(d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{4}A(\theta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}A(\theta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{5}_{6}A(\theta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward (Direct) Kinematic Equations

Example 2-1: (cont)



4. The kinematic equation of this manipulator arm is:

$$^0_6T = {}^0_1A(\theta_1)\,{}^1_2A(\theta_2)\,{}^2_3A(d_3)\,{}^3_4A(\theta_4)\,{}^4_5A(\theta_5)\,{}^5_6A(\theta_6)$$

represents the end-effector position and orientation as a function of joint displacements, θ_1, θ_2, d_3, θ_4, θ_5, θ_6.

6. Inverse Kinematics

Given end-effector position & orientation, ${}^{o}_{n}T$

Joint displacements: $q_1, q_2, q_3, \ldots, q_n$

$$
{}^{0}_{n}T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{n}T = {}^{0}_{1}A(q_1)\,{}^{1}_{2}A(q_2)\cdots{}^{n-1}_{n}A(q_n)
$$

$$
{}^{i-1}_{i}A = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse Kinematics

$$ {}^{0}_{n}T = {}^{0}_{1}A(q_1)\,{}^{1}_{2}A(q_2)\cdots{}^{n-1}_{\;\;n}A(q_n) $$

LHS(i,j) = RHS(i,j), where i and j are row and column indices

12 Equations

- 9 eqns (rotation) – only 3 independent eqns
- 3 eqns (position)

Solve for n unknowns: $\mathbf{q} = [q_1, q_2, q_3, \ldots , q_n]^T$

Inverse Kinematics

- General Analytical Inverse Kinematic Formula

  General Approach: Isolate one joint variable at a time

$$\underbrace{{}^{0}_{1}A^{-1}}_{\text{function of } q_1}\,{}^{0}_{n}T = {}^{1}_{2}A \cdots {}^{n-1}_{n}A = \underbrace{{}^{1}_{n}T}_{\text{function of } q_2,\ldots,q_n}$$

Steps:

- Look for constant elements in ${}^{1}_{n}T$
- Equate LHS(i,j) = RHS(i,j)
- Solve for q_1

Inverse Kinematics

- General Analytical Inverse Kinematic Formula (cont)

  Next

$$ {}^{1}_{2}A^{-1}\, {}^{0}_{1}A^{-1}\, {}^{0}_{n}T = {}^{2}_{3}A \cdots {}^{n-1}_{n}A = {}^{2}_{n}T $$

function of q_1 and q_2 function of $q_3,\ldots,q_n$

- Look for constant elements in ${}^{2}_{n}T$
- Equate LHS(i,j) = RHS(i,j)
- Solve for q_2

May find equation involving q_1 only

Inverse Kinematics

- General Analytical Inverse Kinematic Formula (cont)

- No algorithmic approach that is 100% effective
- Geometric intuition may help to simplify the process

Inverse Kinematics

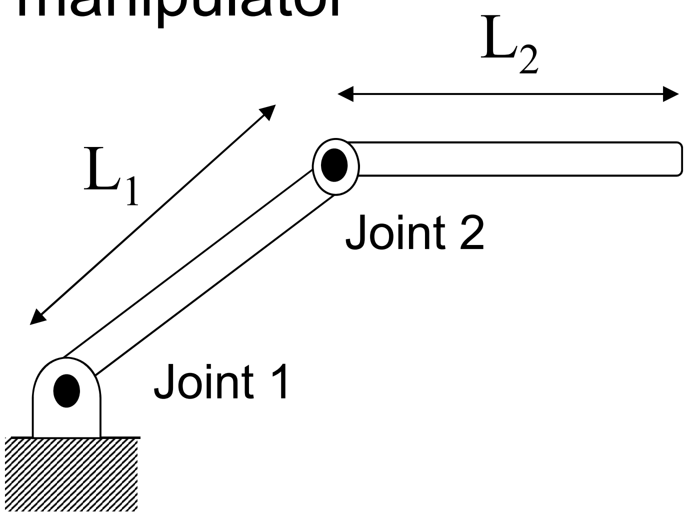
Issues:

- *Existence* of solutions: Specified goal point must lie within workspace
  - Workspace definition:
    - Dextrous Workspace (all orientations)
    - Reachable Workspace (at least one orientation)
    - Dextrous workspace $\subset$ reachable workspace

Inverse Kinematics

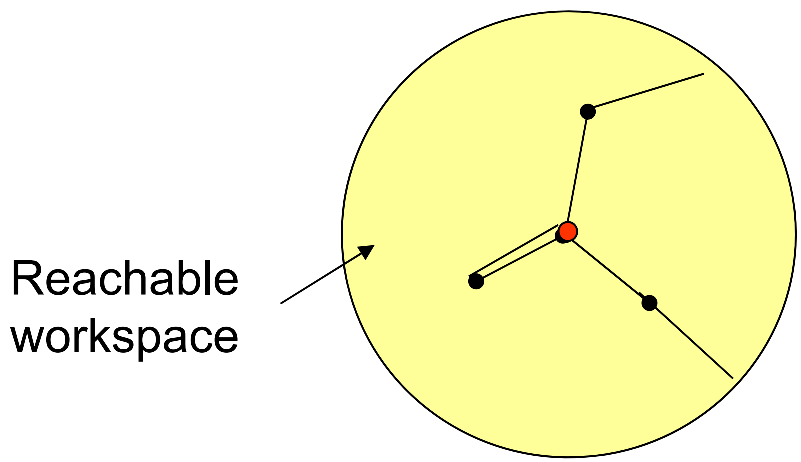
- Existence of solutions (cont):

  Example 2-2: Consider the workspace of a 2-link planar manipulator



If $L_1 = L_2$

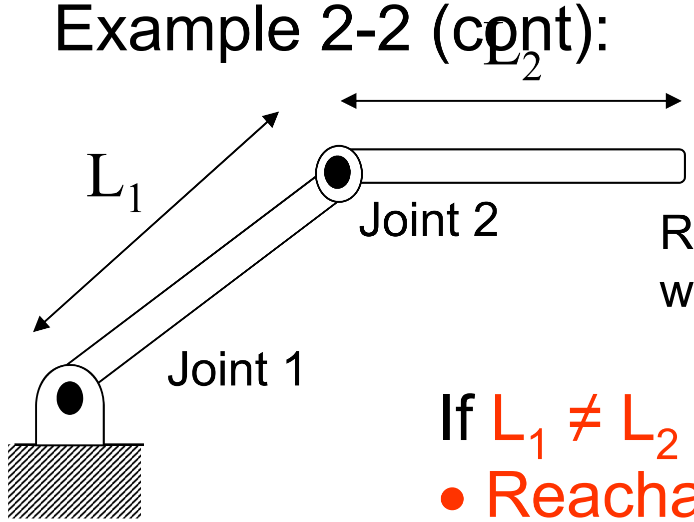
- Reachable workspace: A disc of radius $2L_1$.
- Dextrous workspace: Only a single point (at joint 1)



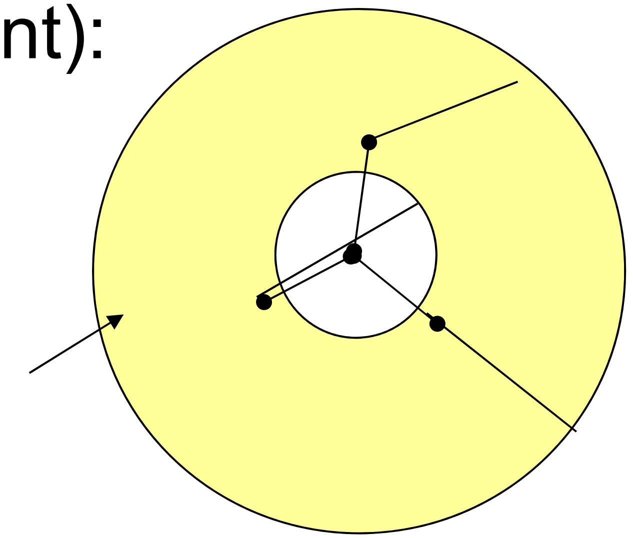
Inverse Kinematics

- Existence of solutions (cont):

Example 2-2 (cont):



Reachable workspace



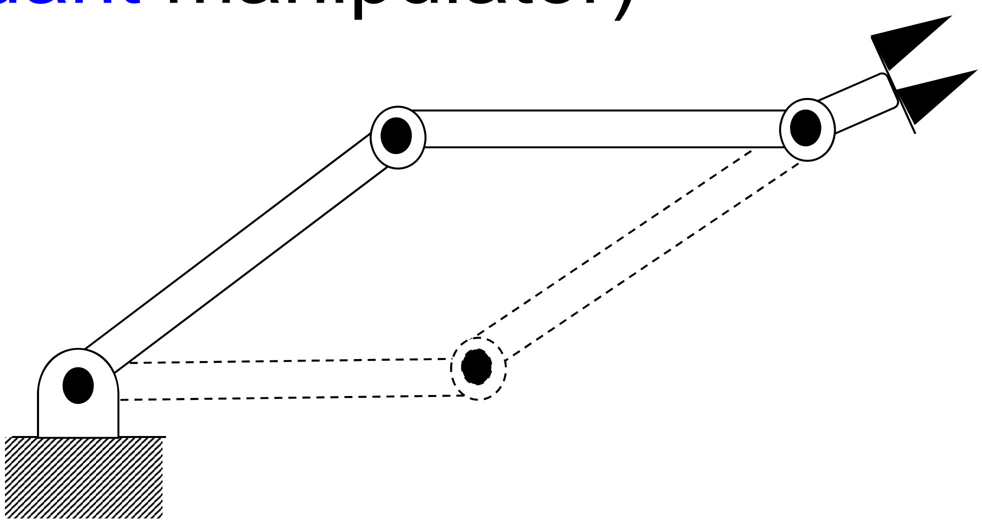
If $L_1 \neq L_2$

- Reachable workspace: Ring of outer radius $L_1 + L_2$ and inner radius $| L_1 - L_2 |$
- Dextrous workspace: Nil

Inverse Kinematics

Issues:

- *Multiple solutions* may exist (infinite solution may exist, e.g. in kinematically redundant manipulator)



Inverse Kinematics

- Multiple solutions (cont)

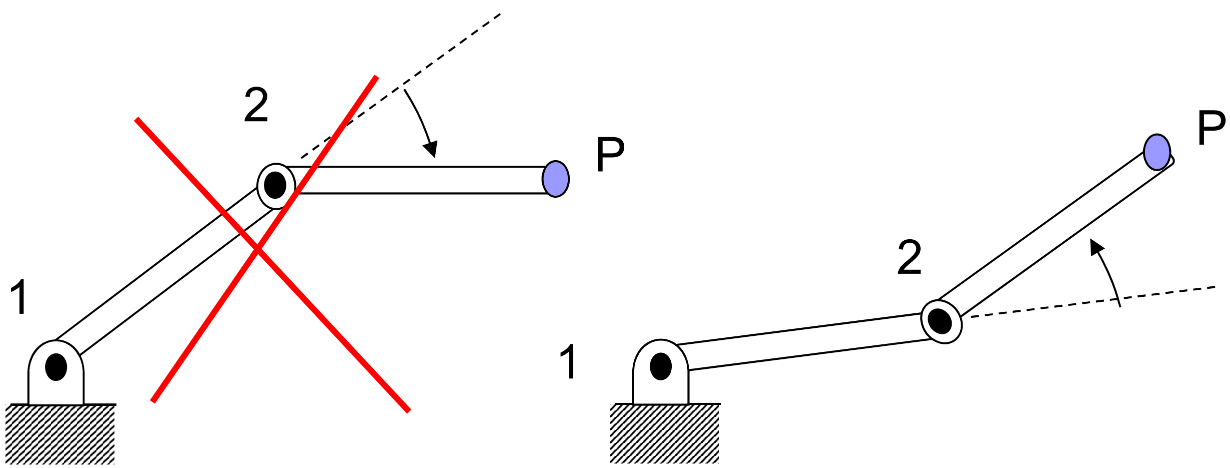
  General notes:

  - A manipulator arm must have at least 6 dof to locate its end-effector at an arbitrary point & with an arbitrary orientation in space
  - If number of dof > 6, infinite no. of solutions may exist to the kinematic eqn, e.g. human arm -> redundant manipulator
  - Physical constraints (e.g. Joint limits) may reduce the number of solutions

Inverse Kinematics

- Multiple solutions (cont)

If *joint limits* occur, workspace and/or no of solutions may be reduced, or number of possible orientations may be reduced.
E.g. If joint 1 is [0^o 360^o] and joint 2 is [0^o 180^o], reachable workspace has the same extent, but only one configuration is attainable at each point.



Inverse Kinematics

Issues:

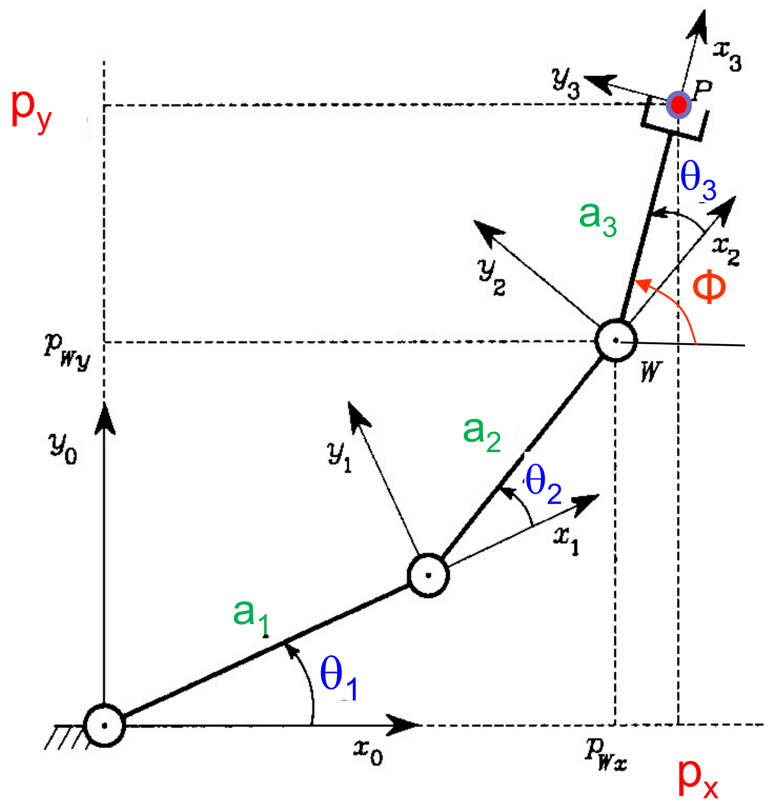
- Solvability: Not always possible to find closed form solution due to nonlinear (transcendental) equations solving
- Alternative: Numerical methods (iterative, slow)

Inverse Kinematics

Example 2-3: Inverse Kinematics of a planar arm

Consider the three-link planar arm shown below whose direct kinematics is given by Eq.(2 2).

$$ {}^0_3T(\theta_1,\theta_2,\theta_3) = {}^0_1A\,{}^1_2A\,{}^2_3A = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{(2-2)} $$



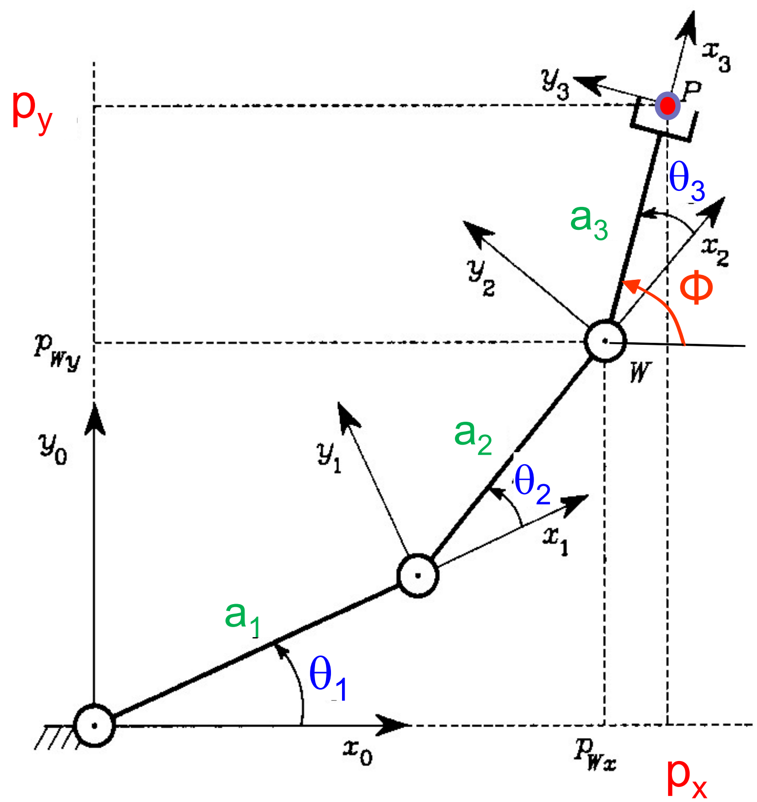
Find the joint variables: θ_1 , θ_2 , θ_3 corresponding to a given end-effector position (p_x , p_y) and orientation ϕ (with reference to axis x_o).

Inverse Kinematics

…Example 2-3:

- Solution:

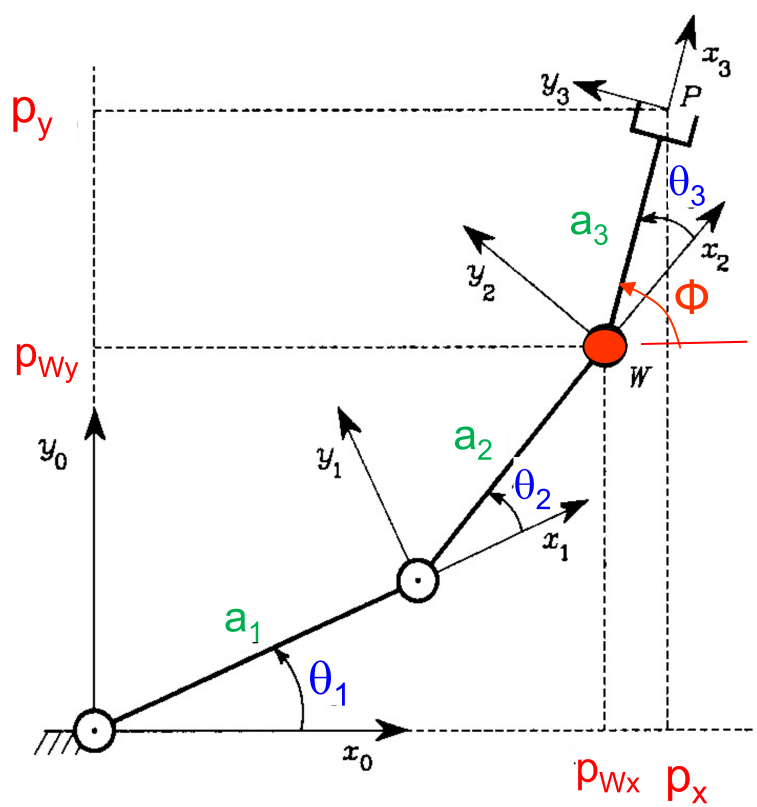
The direct kinematics equation can be written in the following form:



$$\mathbf{x} = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix} \quad \textbf{(2-3)}$$

Inverse Kinematics

…Example 2-3:



To find θ_2:

From (2-3), position of point W (origin of Frame 2):

$$\begin{aligned} p_{Wx} &= p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12} \\ p_{Wy} &= p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12} \end{aligned} \qquad (2\text{-}4)$$

=>

$$p_{Wx}{}^2 + p_{Wy}{}^2 = a_1{}^2 + a_2{}^2 + 2a_1 a_2 c_2$$

=>

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

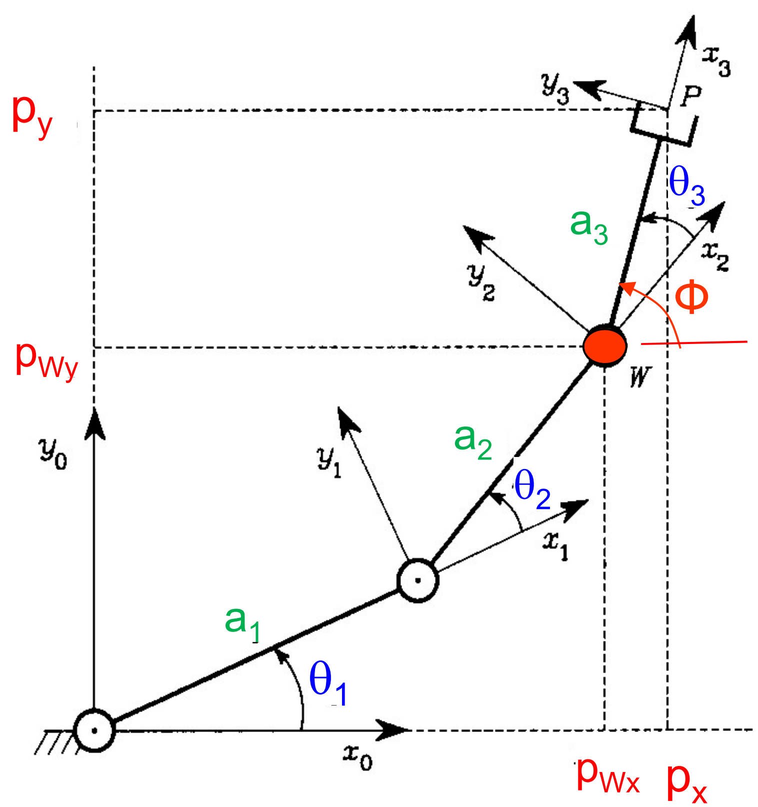
Existence of a solution $\Leftrightarrow$ $-1 \le \dfrac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2} \le 1$

Set $\quad s_2 = \pm\sqrt{1 - c_2^2}$

And $\quad \theta_2 = A\tan 2(s_2, c_2)$

Inverse Kinematics

…Example 2-3:



To find θ_1:

Sub θ_2 into (2-4) yields an algebraic system of two equations in 2 unknowns s_1 and c_1, whose solution is:

$$s_1 = \frac{(a_1 + a_2 c_2) p_{Wy} - a_2 s_2 p_{Wx}}{p_{Wx}^2 + p_{Wy}^2}$$

$$c_1 = \frac{(a_1 + a_2 c_2) p_{Wx} - a_2 s_2 p_{Wy}}{p_{Wx}^2 + p_{Wy}^2}$$

Thus $$\theta_1 = A\tan 2(s_1, c_1)$$

To find θ_3 :

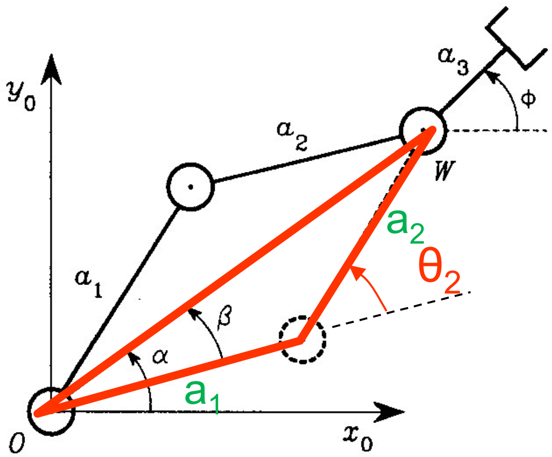
From (2-3), $\phi = \theta_1 + \theta_2 + \theta_3$

$\Rightarrow \theta_3 = \phi - \theta_1 - \theta_2$

Inverse Kinematics

…Example 2-3:

<u>Alternative approach:</u>



By cosine theorem:

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos(\pi - \theta_2) = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\theta_2)$$

$$\Rightarrow c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

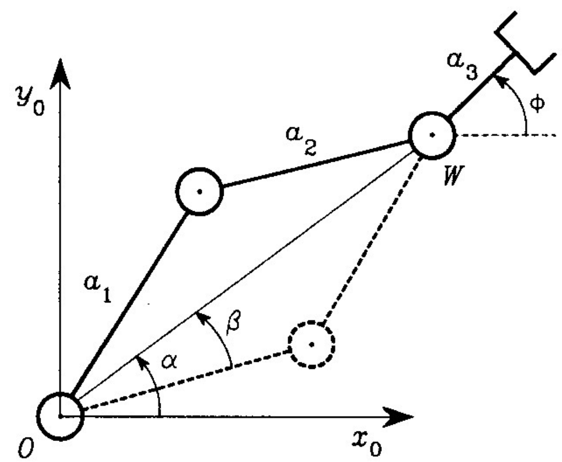
$$\Rightarrow \theta_2 = \cos^{-1}(c_2)$$

Note:

- Existence of the triangle: $\sqrt{p_{Wx}^2 + p_{Wy}^2} \le a_1 + a_2$
- Two admissible configurations of the triangle.
  - elbow-up posture is obtained for $\theta_2 \in (-\pi, 0)$
  - elbow-down posture is obtained for $\theta_2 \in (0, \pi)$

Inverse Kinematics

…Example 2-3:



Now to find θ_1,

$$\alpha = A\tan 2(p_{Wy}, p_{Wx})$$

From cosine theorem,

$$\beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

$$\text{where} \quad \beta \in (0, \pi)$$

$$\theta_1 = \begin{cases} \alpha + \beta & \text{for } \theta_2 < 0 \\ \alpha - \beta & \text{for } \theta_2 > 0 \end{cases}$$

Finally, θ_3 is computed from (2-3).

Inverse Kinematics

- Solvability

(Roth, 1975): A manipulator will be considered **solvable** if all the sets of joint variables can be determined by an algorithm which are associated with a given position & orientation.

(Asada and Slotine, 1986): The kinematic structure for which a closed-form solution exists is referred to as a **solvable structure**.

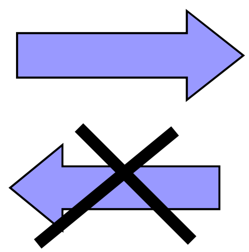
Note: Most industrial robots have solvable structures.

Inverse Kinematics

- Solvability (cont)

(Pieper, 1968): For a 6 dof manipulator arm

Joint axes of 3 consecutive revolute joints intersect at a single point



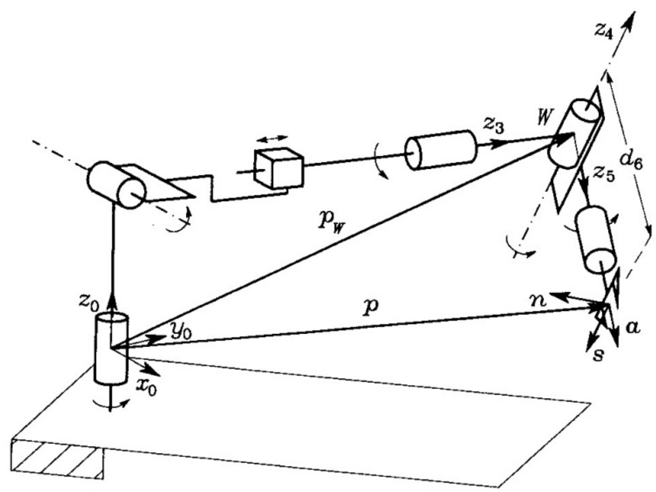
kinematic structure of the manipulator arm is solvable

In general, a 6 dof kinematic structure has closed-form inverse kinematics solutions if:

- 3 consecutive revolute joint axes intersect at a common point, e.g. those with spherical wrist
- 3 consecutive revolute joint axes are parallel

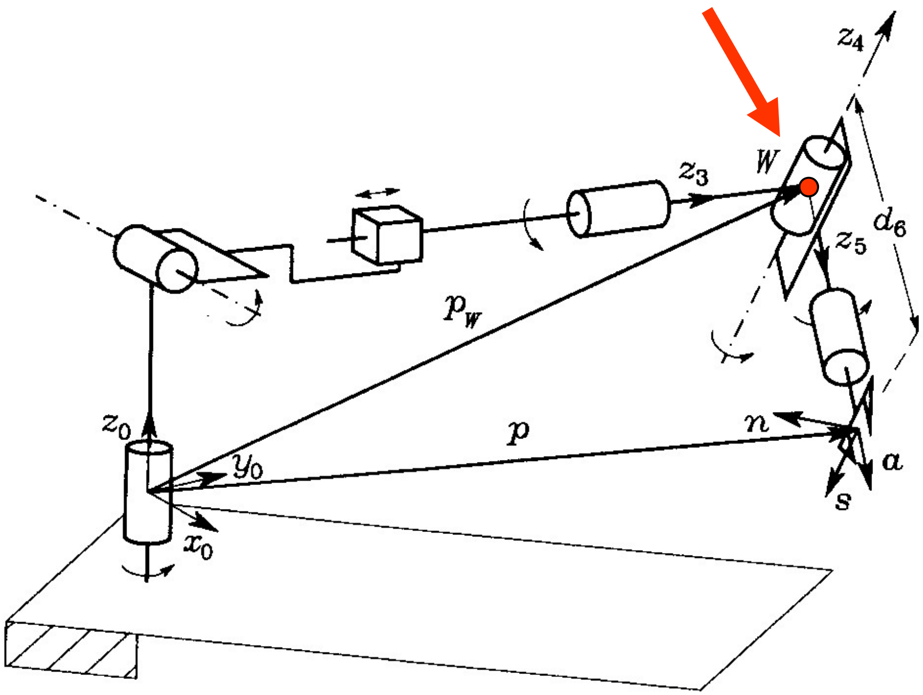
Inverse Kinematics

- Solution of 6 DOF Manipulators with Spherical Wrist
  - 3 consecutive revolute joint axes intersect at a common point => solvable
  - Inverse kinematics problem can be broken down into two subproblems (kinematic decoupling)



Inverse Kinematics

- Solution of 6 DOF Manipulators with Spherical Wrist (cont)

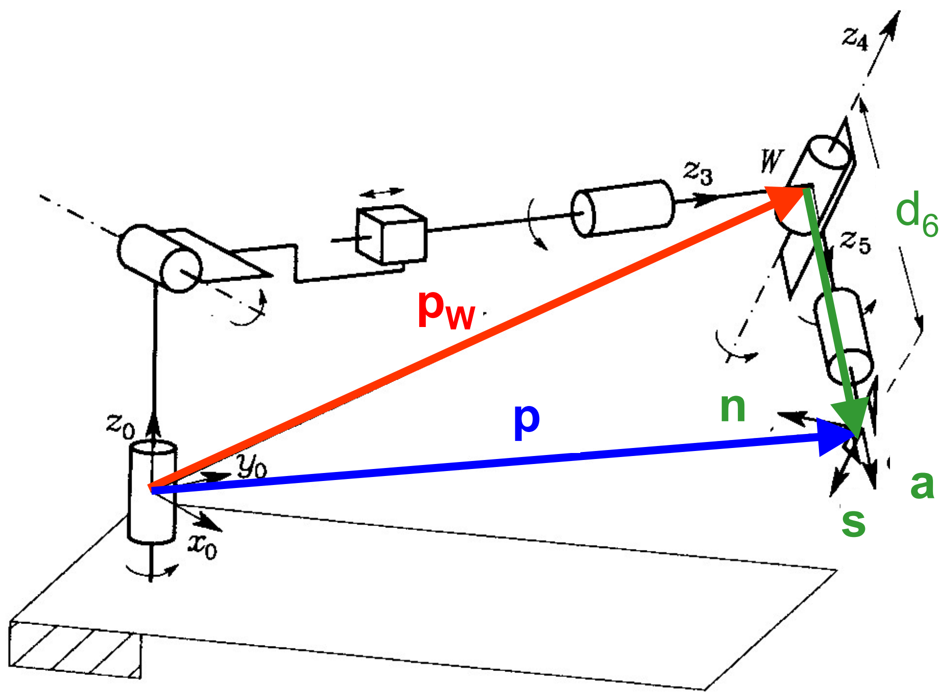


Simple to work with point W which:

- is at the intersection of the three terminal revolute axes
- can be treated as the position of the wrist

Inverse Kinematics

- Solution of 6 DOF Manipulators with Spherical Wrist (cont)



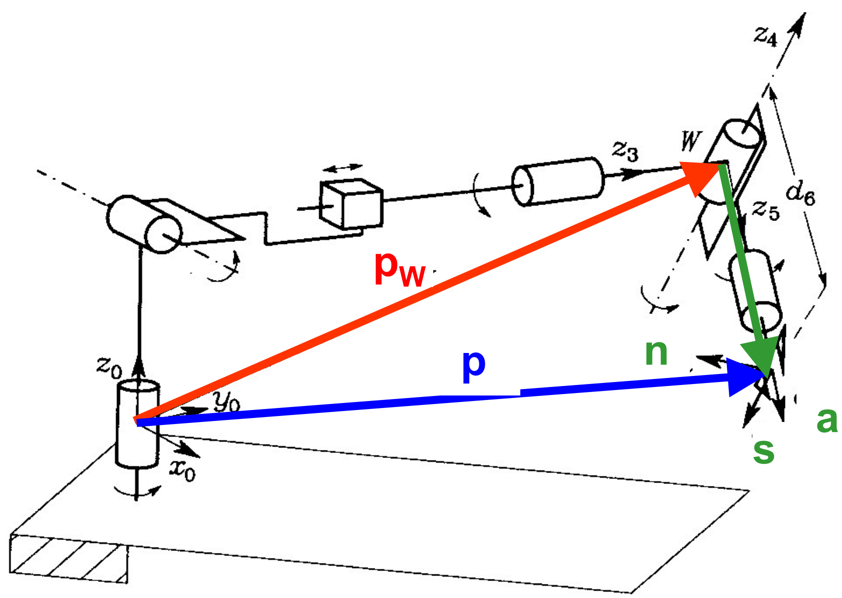
End-effector position and orientation are specified in terms of **p** and ${}^{0}_{6}R$ = **[n s a]**

$$\mathbf{p_W} = \mathbf{p} - d_6\mathbf{a} \qquad (2\text{-}5)$$

- function of the 3 joint variables (q_1, q_2, q_3)

Inverse Kinematics

- Solution of 6 DOF Manipulators with Spherical Wrist



Procedure to solve the inverse kinematics:

•Compute the wrist position $\mathbf{p}_W$ as in (2-5)

•Solve inverse kinematics for (q_1, q_2, q_3) (from ${}^0_3T$, assume that we have a nonredundant 3-dof arm)

•Compute ${}^0_3R(q_1, q_2, q_3)$ (from ${}^0_3T$)

•Compute ${}^3_6R(q_4, q_5, q_6) = {}^3_0R \; {}^0_6R$

$$= {}^0_3R^{-1} \; {}^0_6R$$

$$= {}^0_3R^T \; {}^0_6R$$

•Solve inverse kinematics for orientation (q_4, q_5, q_6)

Summary

- Definitions of positions and orientations of rigid bodies
- Analysis of different representations for orientation
- Transformation of coordinates
- Determination of new positions and orientations after a sequence of rigid body motions
- Kinematics modeling of robotic manipulators
- Forward and inverse kinematics of position