EE5137 2020/21 (Sem 2): Quiz 2 (Total 25 points)

Name: ______________________________

Matriculation Number: ____________________

Score: Q1: Q2: Q3: Total:

You have 1.0 hour for this quiz. There are FIVE (5) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. (a) (4 pts) Taxis arrive at a taxi stand according to a Poisson process with rate of 10 an hour. Every taxi is empty with probability 1/5 and empty taxis can pick up passengers if empty. You are standing at a taxi stand with 1 person ahead of you. What's the mean and variance of your waiting time to get onto a taxi?

(b) (6 pts) For a Poisson process with rate λ, compute

$$\Pr\big(1 \text{ arrival in } [1,4] \text{ and } 3 \text{ arrivals in } [3,5]\big).$$

Express your answer as

$$e^{-b\lambda} \sum_{i=0}^{4} a_i \lambda^i$$

by finding the constants b and $\{a_i\}_{i=0}^{4}$. In your calculation, carefully state which properties of the Poisson process you are using.

2. Insurance claims arrive according to a Poisson process $\{N(t) : t > 0\}$ with rate λ. Let

   - S_n be the time of the n^{th} claim;
   - C_n amount of the n^{th} claim and $\{C_n\}_{n=1}^{\infty}$ are i.i.d. random variables with mean μ independent of $\{N(t) : t > 0\}$.

   Let $\alpha > 0$ be a fixed constant. The total discounted cost is defined as

$$D(t) := \sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}.$$

   (a) (2 pts) For a uniform random variable U on $[0, t]$ find $\mathbb{E}[e^{-\alpha U}]$.

   (b) (6 pts) By utilizing the theory of conditional arrivals find $\mathbb{E}[D(t) \mid N(t) = n]$.

(c) (2 pts) Find $\mathbb{E}[D(t)]$ using part (b) and the law of iterated expectations.

3. Let $\{N(t) : t > 0\}$ be the Poisson counting process with rate λ. The *compensated Poisson process* is defined as $M(t) = N(t) - \lambda t$. Let $\mathcal{F}_t := \{M(\tau) : 0 < \tau \leq t\}$ be the process up to and including time t. A *continuous-time martingale* $\{X(t) : t > 0\}$ is a stochastic process satisfying

$$\mathbb{E}[|X(t)|] < \infty \quad \text{and} \quad \mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s) \quad \text{a.s.} \quad \forall\, t > s > 0.$$

Define the process

$$X(t) = M(t)^2 - \lambda t = (N(t) - \lambda t)^2 - \lambda t.$$

(5 points) Show that $\{X(t) : t > 0\}$ is a continuous-time martingale.

Hint: You might find it useful to figure out $\mathbb{E}[N(t) \mid \mathcal{F}_s]$ and $\mathbb{E}[N(t)^2 \mid \mathcal{F}_s]$ first. Calculating these two quantities correctly will get you some partial credit.