EE5101/ME5401:
Linear Systems: Part II

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Teaching Arrangement

Part I: System Analysis: Prof Ong

Part II: System Design: Dr. Xiang Cheng

Overview of Part II

Chapter 7. Pole Placement

Chapter 8. Quadratic Optimal Control

Chapter 9. Decoupling Control

Chapter10. Servo Control

Chapter 11. State Estimation

Revisions & Tutorials & Project briefing

Total of six sessions

The Course Materials

The lecture notes, Tutorials and Powerpoint Slides are available at LumiNUS

References:

[1] T. Kailath, *Linear System*, PrenticeHall, 1980.

[2] Chi-Tsong Chen, *Linear System Theory and Design*, 3rd ed, New York : Oxford University Press, 1999.

[3] Umez Eronini, *System dynamics and control*, Pacific Grove, Calif. : International Thomson Publishing, 1999.

[4] Paul M. DeRusso, Rob J. Roy and Charles M. Close, *State variables for engineers*, New York : Wiley, c1998.

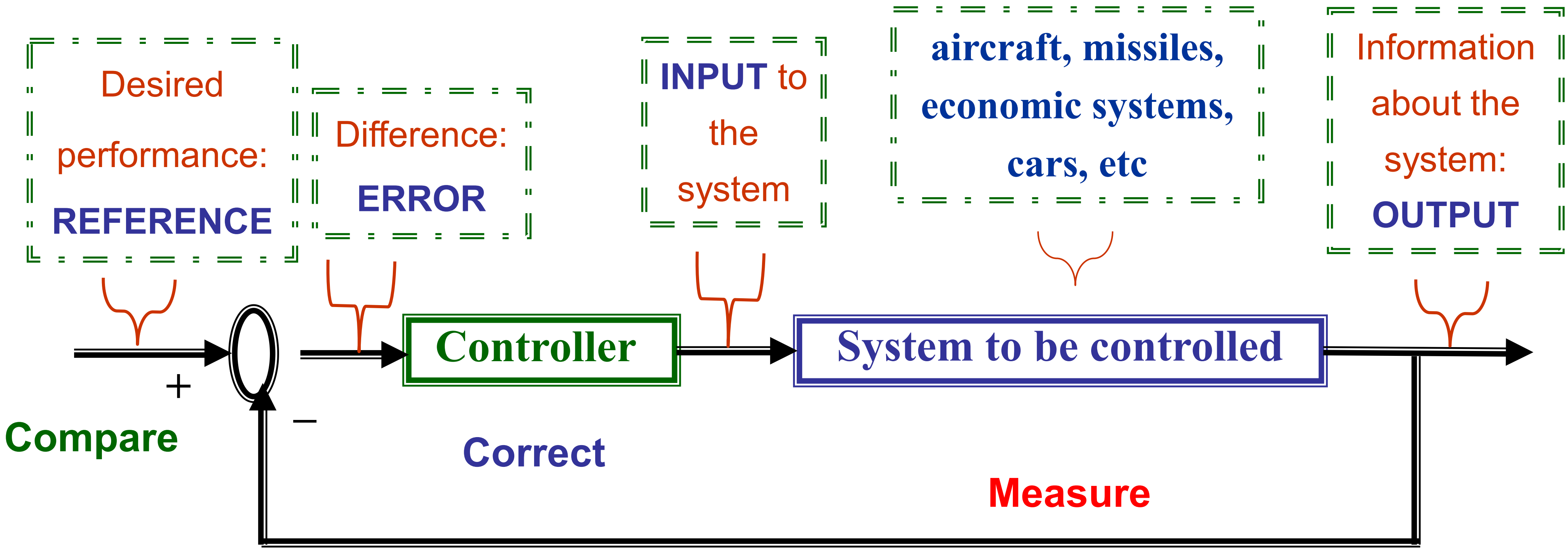
[5] P. N. Paraskevopoulos, *Modern Control Engineering*. Marcel Dekker, Inc, New York, 2002.

Assessment:

- Continuous Assessment (CA): 30%, one mini-project.
- Final Exam: 70%

Simulation Tools: MATLAB with SIMULINK toolbox

•You have learned many basic concepts and analysis in part I. Now it is time to apply these to controller design—the core of part II.



•**Feedback: Measure —Compare —-Correct**

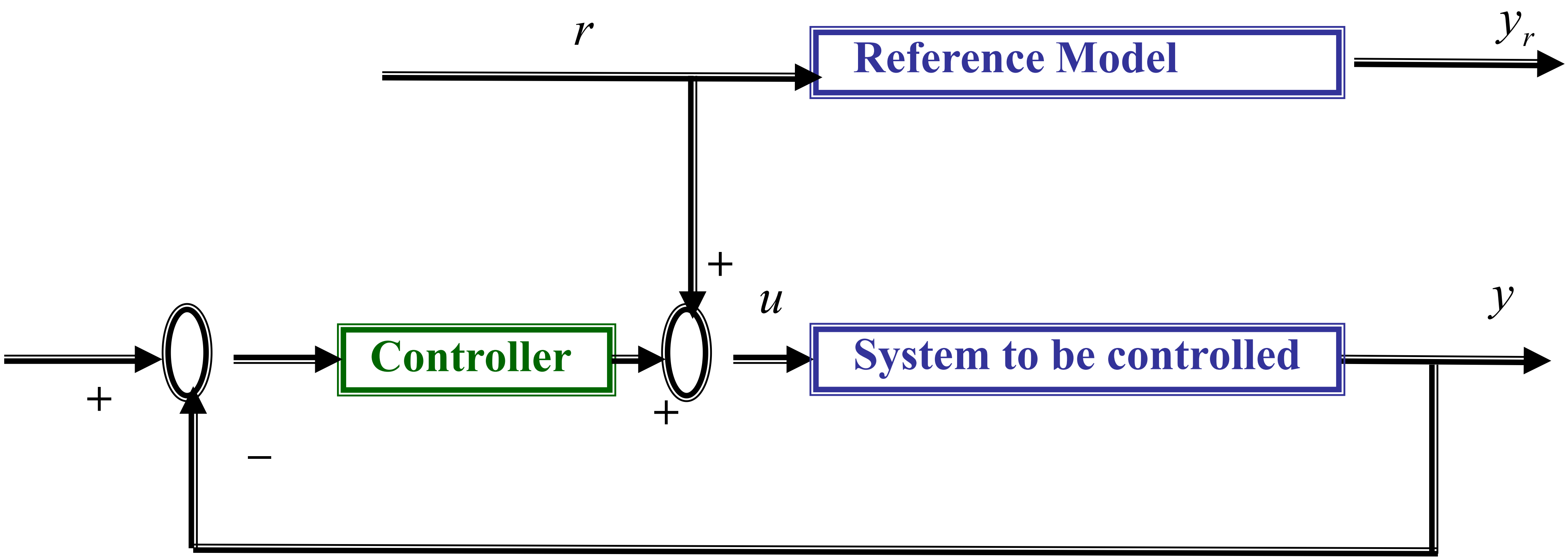
Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

How to specify the reference signal, or the desired output?

•Can we make the output follow arbitrary command signal? Impossible!

•The reference signal is usually generated by a well-behaved reference model.

- **First build a reference model which can meet all the performance requirements:**



Objective: To make the closed loop model follow the reference model as close as possible.

How to specify the appropriate reference model in practice?

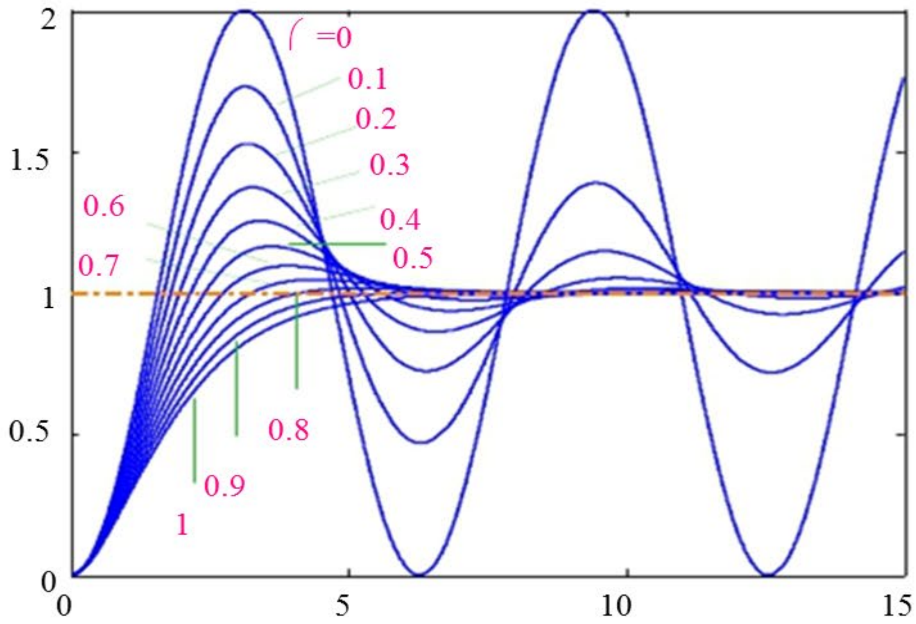
It depends upon the design specifications.

Transient behavior of 2nd order systems

Consider the following block diagram with a standard 2nd order system under unit step

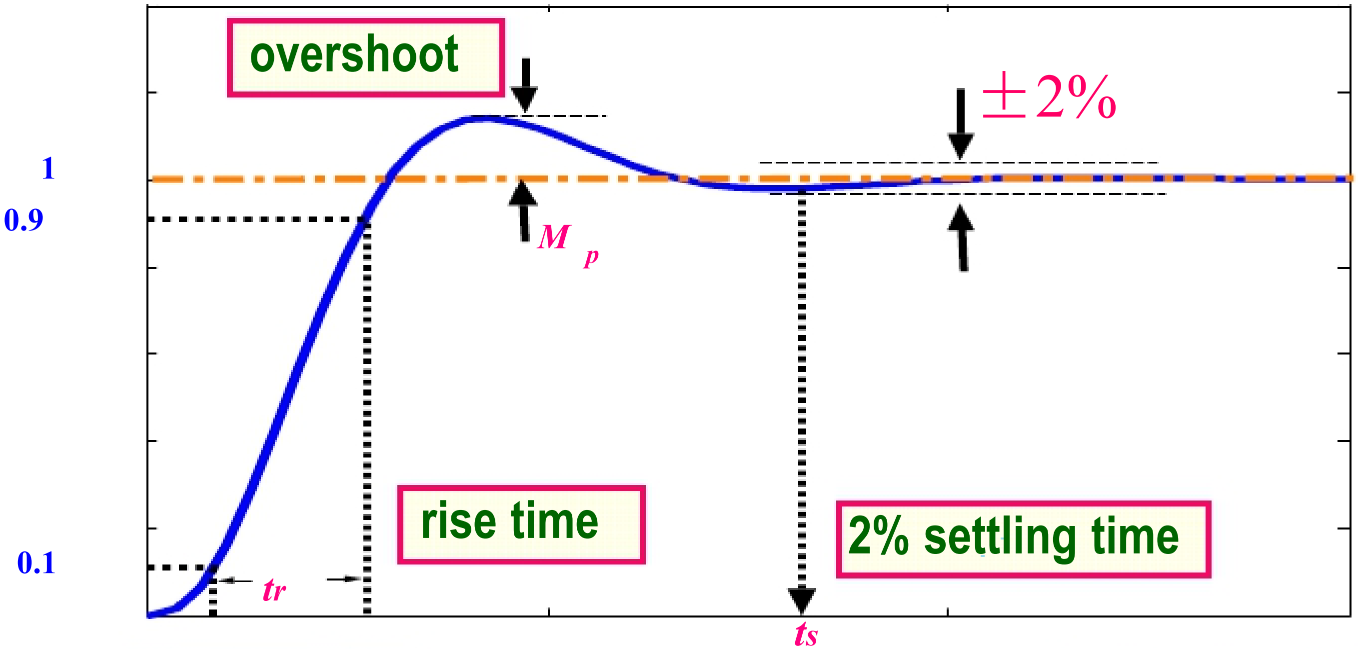
$$R(s)=\frac{1}{s} \qquad r(t)=1 \longrightarrow \boxed{H(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \longrightarrow Y(s)$$

The behavior of the system is as follows:



The system behavior is fully characterized by 2 factors: the damping ratio ζ, and the natural frequency ω_n .

Settling time, overshoot and rise time — time domain specifications



$$t_r \cong \frac{1.8}{\omega_n}$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

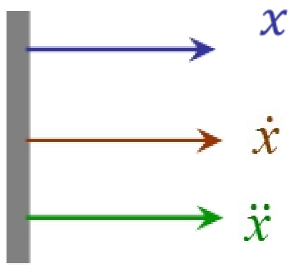
$$t_s \cong \frac{4.0}{\zeta\omega_n}$$

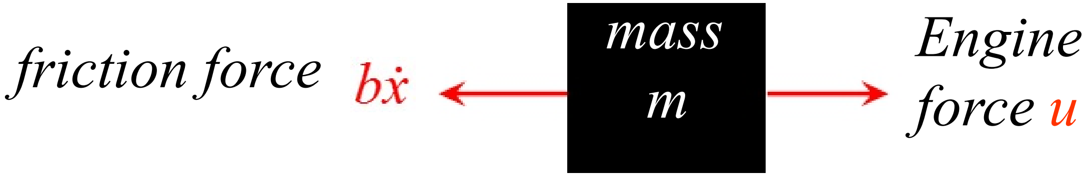
$(t_s, M_p, t_r) \Leftrightarrow (\zeta, \omega_n) \Leftrightarrow$ Reference Model

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

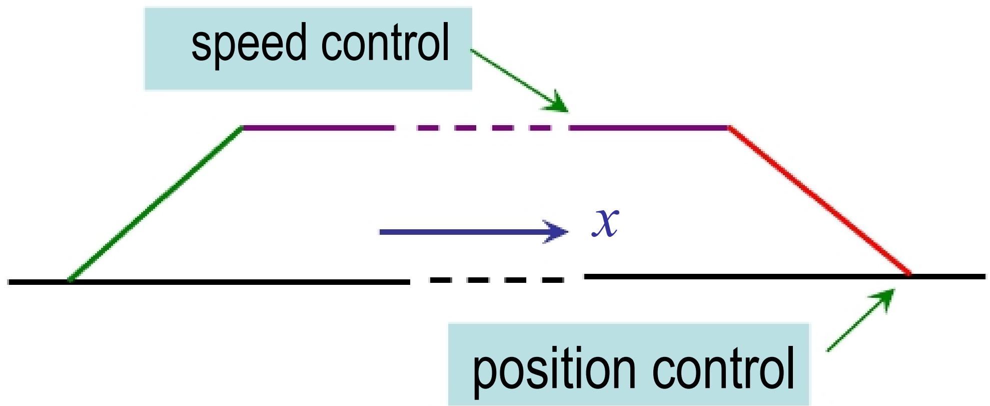
Design Example: speed control system

Consider the vehicle, which has a weight m = 1000 kg. Assuming the average friction coefficient b= 100, design a speed control system such that the vehicle can reach 100 km/h from 0 km/h in 8 s with an overshoot less than 5%.





x – *displacement*
$\dot{x}$ – *speed*
$\ddot{x}$ – *acceleration*
u – *engine force*



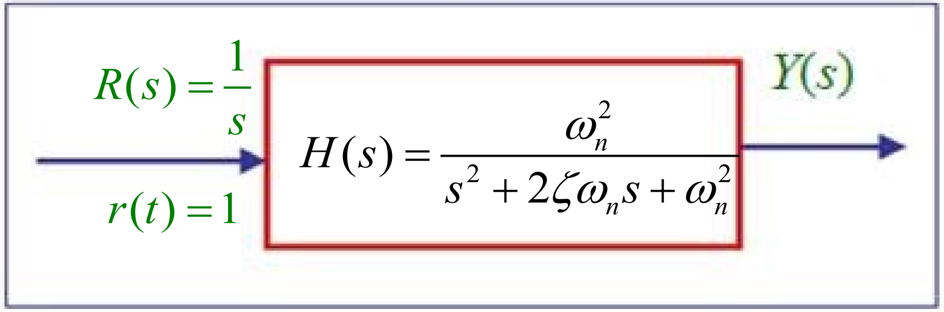
$$m\ddot{x} = u - b\dot{x} \quad \Rightarrow \quad m\ddot{x} + b\dot{x} = u$$

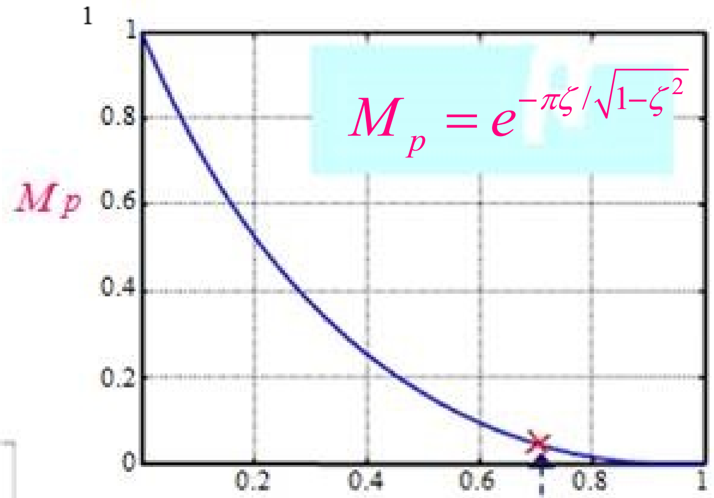
speed $\quad y = \dot{x} \quad \Rightarrow \quad m\dot{y} + by = u \quad \Rightarrow \quad \dfrac{Y(s)}{U(s)} = \dfrac{1}{ms+b}$

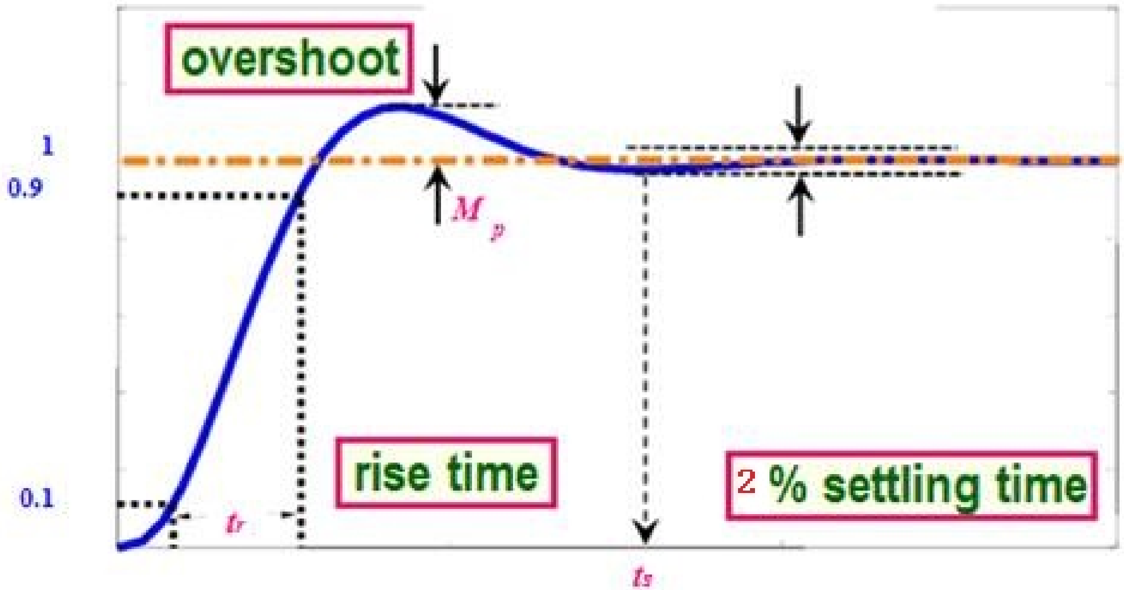
Build the reference model:

First derive ξ and ω_n from the design specifications:

Overshoot: 5%
Settling Time: 8s







$$\zeta \geq 0.6901 \quad \Rightarrow \quad \zeta = 0.7$$

$$t_s \cong \frac{4.0}{\zeta\omega_n} \Rightarrow \omega_n \cong \frac{4.0}{t_s \zeta}$$

$$\Rightarrow \omega_n = \frac{4.0}{8 \times 0.7} = 0.71$$

Reference Model $\Rightarrow \quad H_d(s) = \dfrac{0.5}{s^2 + s + 0.5}$

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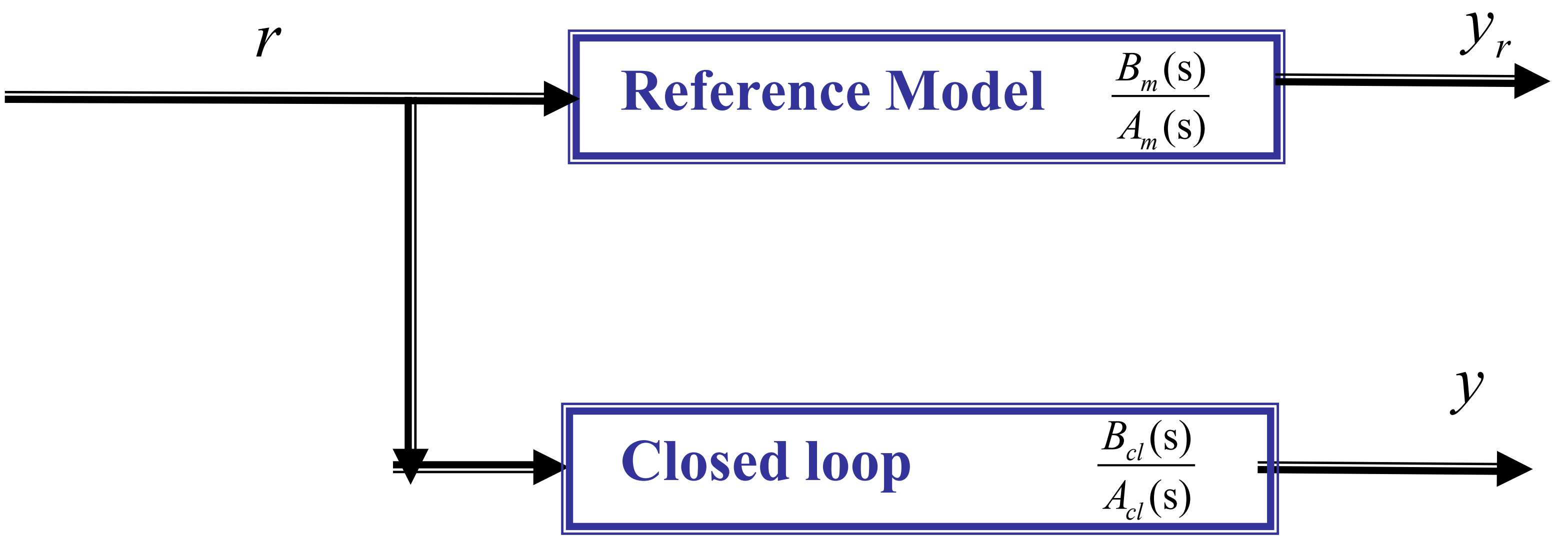


•Ideal case: perfect match

$$\frac{B_{cl}(s)}{A_{cl}(s)} = \frac{B_m(s)}{A_m(s)}$$

$$A_{cl}(s) \Rightarrow A_m(s)$$ Pole Placement

$$B_{cl}(s) \Rightarrow B_m(s)$$ Zero Placement

Which one is more important? And Why?

Pole placement since stability is the primary requirement!

Chapter 7 Pole Placement

§7.1 State Feedback

Consider a multiple-input and multiple-output (MIMO) n-dimensional linear plant with m-inputs and p-outputs described by

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx. \tag{1b}$$

with constant matrices A, B, and C of appropriate dimensions,

$$x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T,$$

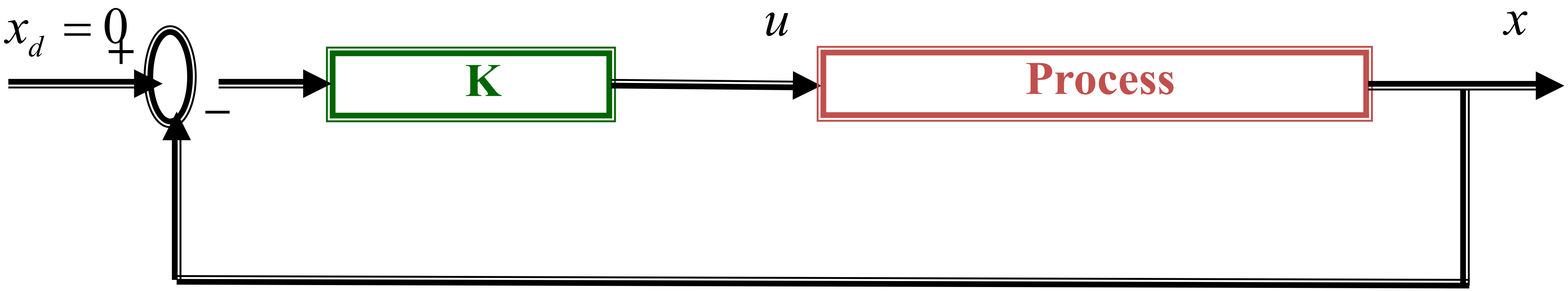
$$u = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}^T,$$

$$y = \begin{bmatrix} y_1 & \cdots & y_p \end{bmatrix}^T.$$

What's the simplest feedback-controller you can imagine?

Proportional Control

Let's try the simplest one first:



$$u(\mathrm{t}) = Ke(\mathrm{t}) = K(0 - x(\mathrm{t})) = -Kx(\mathrm{t})$$

We know that u is m-dimensional, x is n-dimensional. What is the dimension of K?

K is a mxn matrix.

In addition to the purely feedback term, we may also add feedforward signals such that

$$u = -Kx + Fr,$$

A control law:

$$u = -Kx + Fr, \tag{2}$$

is called the state feedback, where r is the reference input,
K *and* F are constant matrices to be designed.
For Pole Placement problem, we only need to design K!

If we plug it into the system equation (1), we have the closed-loop system

$$\dot{x} = (A - BK)x + BFr, \tag{3a}$$

$$y = Cx. \tag{3b}$$

It is important to stress that the system matrix has been changed by the feedback:

$$A \quad \Rightarrow (A - BK)$$

This is where all the magic power lies:
Feedback can change the dynamic property of the system!

The positions of the poles, which are important property of the system, can be changed by feedback.

§7.2 Why Pole Placement?

Time-domain Specifications

Specifications for a control system design often involve certain requirements associated with the time response of the system. The requirements for a step response of the stable system output are often expressed in terms of

The **rise time:** t_r is the time for the system output to reach 90% of its final value from its 10% value;

The **settling time:** t_s is the time for the system transients to decay to and stay in a small percentage around its final value, say 2%;

The **overshoot:** M_p is the maximum amount by which the system output overshoots its final value divided by its final value (and often expressed as a percentage),

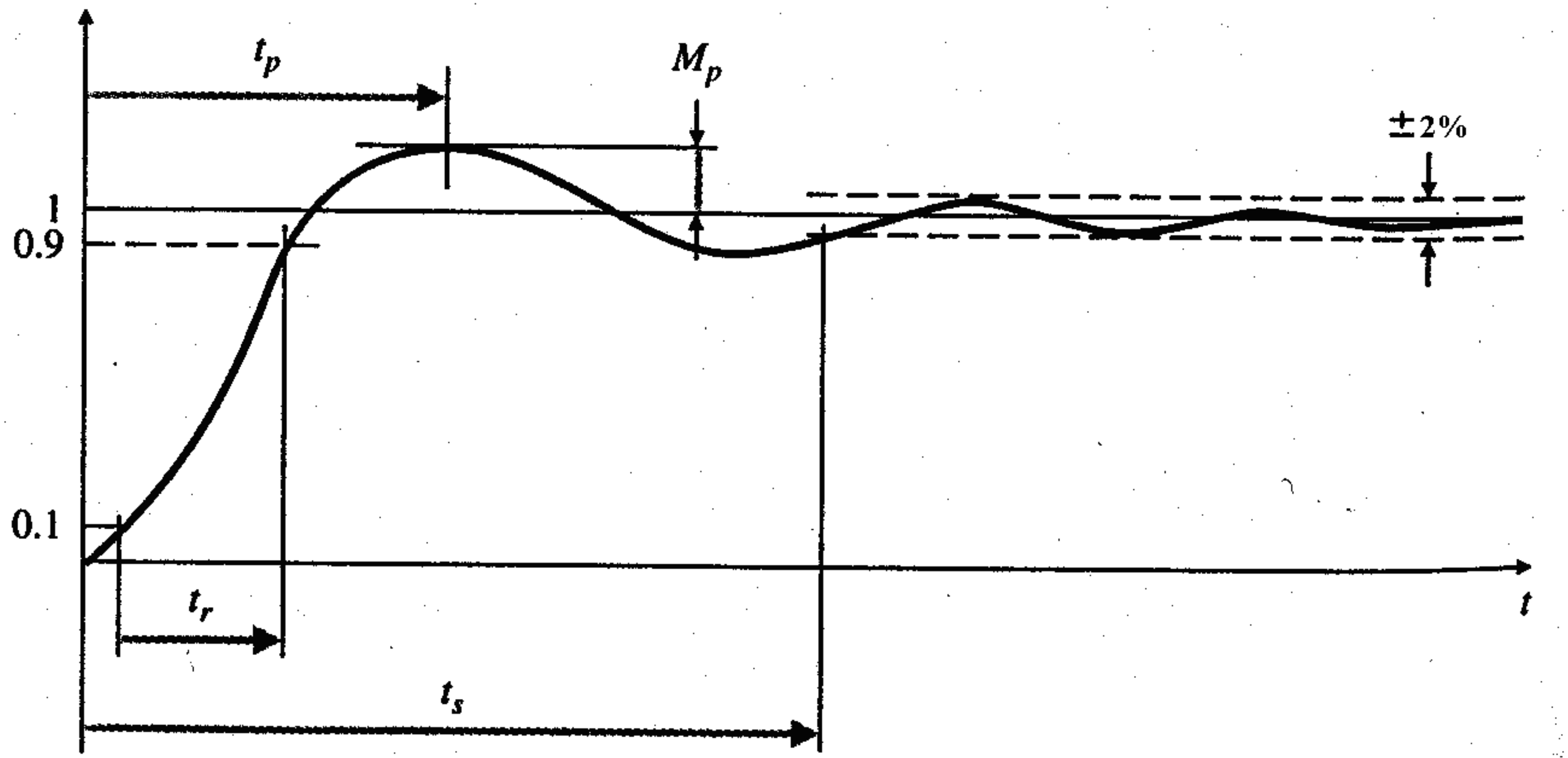


Figure 3 Control specifications.

Consider a standard second-order system:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Its poles are at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ (with $0 < \zeta < 1$). Two parameters, or the poles determine performance completely:

(a) Rise time, $t_r = \dfrac{1.8}{\omega_n}$;

(b) Peak overshoot, $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$;

(c) 2% settling time, $t_s = \dfrac{4.0}{\zeta\omega_n}$.

How to deal with higher order system?

The key is to use the dominant mode to approximate!

Example :

$$H(s) = \frac{2}{(s^2+s+1)(s+2)}$$

Step one: Compute the poles

$$s=-2 \quad s=-0.5\pm j\frac{\sqrt{3}}{2}$$

Step two: identify the fast and slow modes by checking the real parts of the poles:

$$s=-2 \Longrightarrow \frac{1}{s+2} \to e^{-2t}$$

$$s=-0.5\pm j\frac{\sqrt{3}}{2} \Longrightarrow \frac{1}{(s+0.5\pm j\frac{\sqrt{3}}{2})} \to e^{-0.5t}e^{j\frac{\sqrt{3}}{2}t}$$

Which one decreases faster to zero? e^{-2t}

The bigger the absolute value of the real part, the faster it goes to zero!

To approximate the system, shall we ignore the fast mode or the slow mode?

The dynamics of the system is dominated by the slow mode, and the fast mode can be ignored.

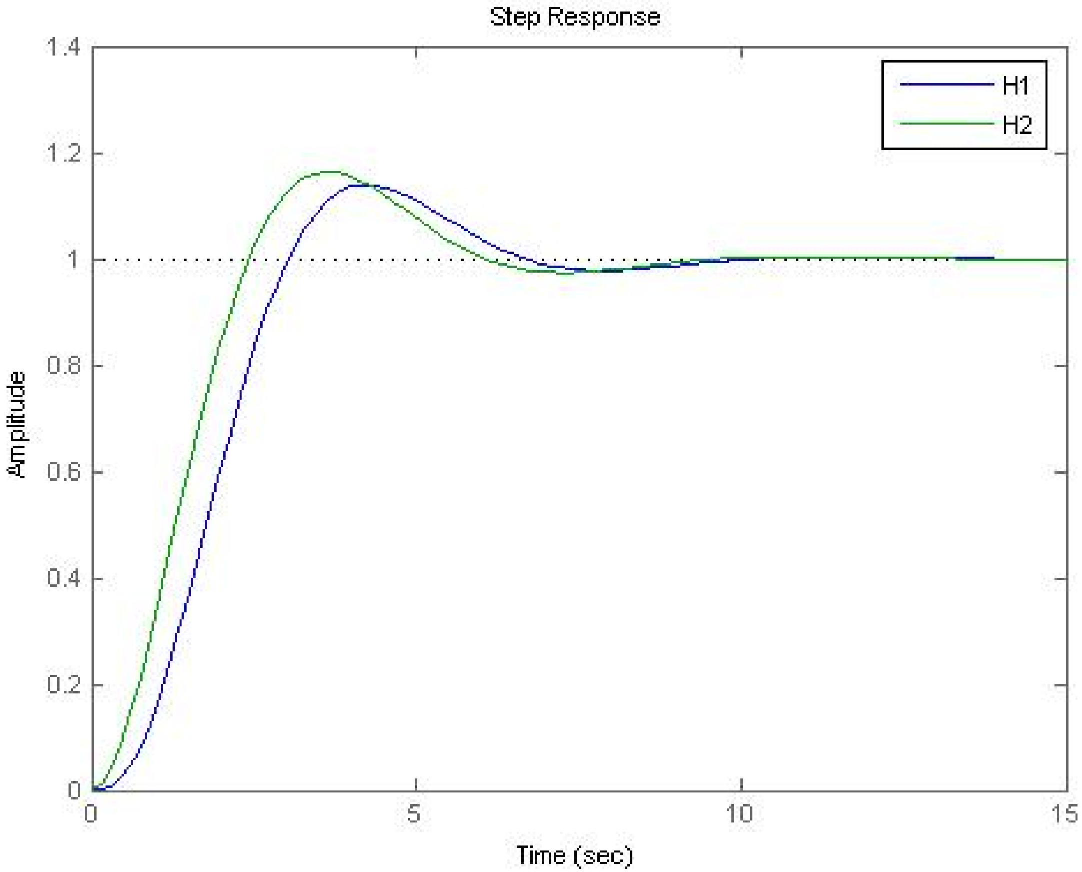
$$H(s) = \frac{2}{(s^2+s+1)(s+2)} \approx \frac{1}{(s^2+s+1)}$$

The higher order system can be approximated by second order or even first order system!

How good is the approximation?

The step responses of the two systems:

$$H_1(s) = \frac{2}{(s^2+s+1)(s+2)} \qquad H_2(s) = \frac{1}{(s^2+s+1)}$$



There are minor differences in the transient part.

The steady state responses are almost the same!

The higher order system can be approximated reasonably well by second order or even first order system!

An Industrial Motivation: DC Motor Position Control

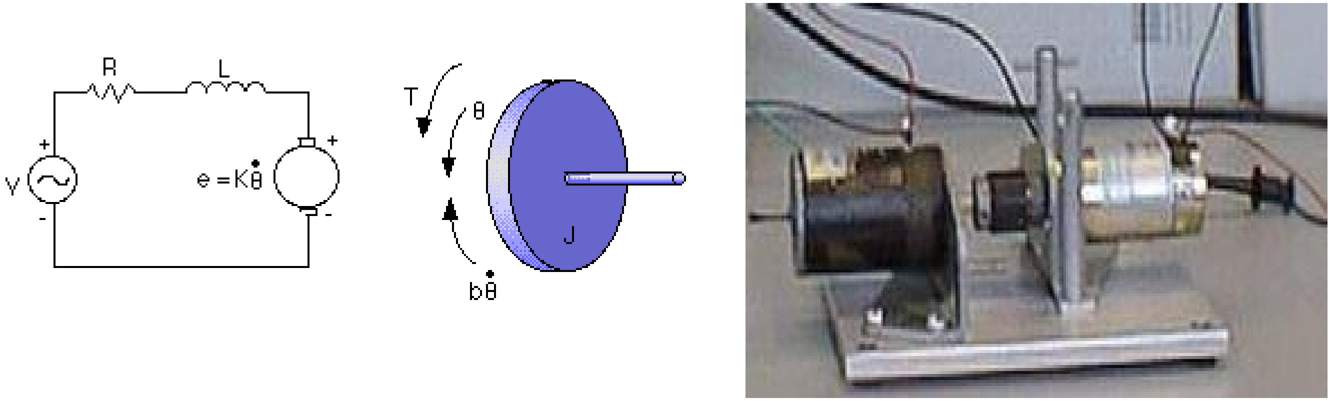


Figure 5 DC Motor.

1) Modeling. Suppose for this example:

- moment of inertia of the rotor, $J = 3.2284\times10^{-6}$ kg·m^2/s^2
- damping ratio of the mechanical system, $b = 3.5077\times10^{-6}$ N·m·s
- electromotive force constant, $K=K_e=K_t = 0.0274$ N·m/Amp
- electric resistance, $R = 4$ ohm
- electric inductance, $L = 2.75\times10^{-6}$ H

- input, V : Source Voltage
- output, θ: position of shaft

From the figure above we can write the following equations based on Newton's law combined with Kirchhoff's law:

$$J\ddot{\theta} + b\dot{\theta} = T,$$

$$L\frac{di}{dt} + Ri = V - e,$$

where the motor torque, T, is related to the armature current, i, by a constant factor K_t, and the back emf, e, is related to the rotational velocity by the following equations:

$$T = K_t i,$$

$$e = K_e \dot{\theta}.$$

In SI units (which we will use), K_t (armature constant) is equal to K_e (motor constant).

Is this the standard state space model? No.

What are the state variables? $i, \theta, \dot{\theta}$

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & -\frac{b}{J} & \frac{K_t}{J}\\ 0 & -\frac{K_e}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}V,$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix},$$

where the motor position, motor speed, and armature current are three state variables. V is the input voltage.

2) Performance requirement:.

Requirements on the closed-loop step response are that

- the settling time be less than 40 milliseconds, and
- the overshoot be less than 16%.

The plant has a pole at s=0, and is not asymptotically stable. Indeed, check the plant output step response in Figure 6, which goes with no bound and does not satisfy the design criteria at all.

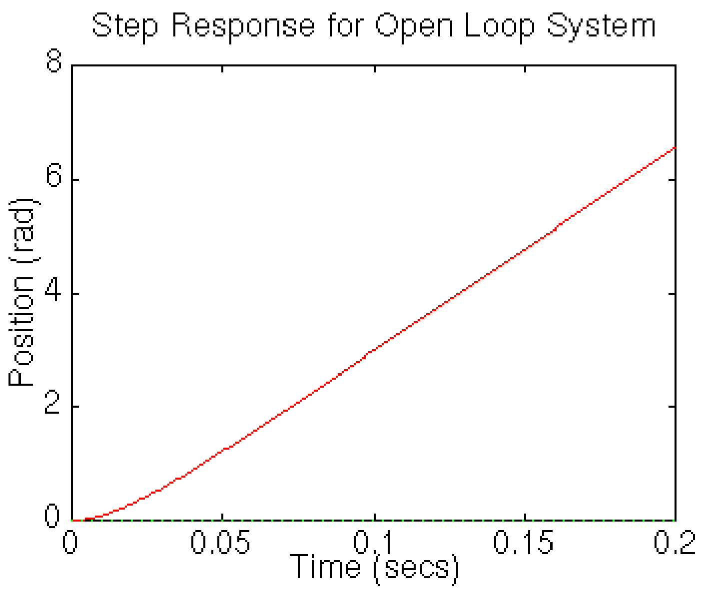


Figure 6 Step response of open loop.

3) Conclusion:

Feedback control is necessary to change the dynamic property of the system. In other words, good control (good closed-loop poles) is demanded.

Summing up:

Rationale of Pole Placement

Reasonable to represent system performance by the poles

Time domain ↘

Specifications $\Rightarrow$ Desired pole locations

Frequency domain ↗

The poles are critical indices for the property of the dynamic system.

If we want to follow the reference model, matching the poles is necessary.

But questions arise:

Q1: Is pole placement possible?

Q2: How can pole placement be done?

§7.3 Single-input Case

Consider a *single input* system:

$$\dot{x} = Ax + bu, \tag{4}$$

where *u is a scalar*. Use the state feedback:

$$u = r - k^T x. \tag{5}$$

Note $k = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$ is a column vector, $k^T = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$ is a row vector,

$$k^T x = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} k_i x_i,$$

Pole placement problem. For the plant (4) , let $\lambda_1, \lambda_2, \cdots, \lambda_n$, be the desired stable closed-loop poles, where for each complex pole, its conjugate must be also included in.

How to design the desired poles? From the performance requirements!

Define

$$\phi_d(s) = \prod_{i=1}^{n}(s-\lambda_i) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \gamma_0$$

as the desired closed-loop characteristic polynomial with real coefficients.

We want to determine the control law

$$u = -k^T x + r$$

such that the closed-loop system

$$\dot{x} = (A - bk^T)x + br$$

has the same poles as those desired ones:

$$\det(sI - (A - bk^T)) = \phi_d(s) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_0 \qquad (6)$$

Theorem 1. *The closed-loop poles of the system with* (4) *and* (5) *can be arbitrarily assigned if and only if* (*A*, *b*) *is controllable.*

Since (A,b) is controllable, we can transform it into controllable canonical form.
Use the state transformation,

$$\bar{x} = Tx$$

$$\dot{x} = Ax + bu, \quad \Longrightarrow$$

$$\dot{\bar{x}} = T\dot{x} = T(Ax + bu) =$$

$$= TAT^{-1}\bar{x} + Tbu = \bar{A}\bar{x} + \bar{b}u$$

$$\bar{A} = TAT^{-1} \qquad \bar{b} = Tb$$

What does the controllable canonical form look like? Do you still remember that from part I?

The controllable canonical form

$$\dot{\bar{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}}_{\bar{A}} \bar{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\bar{b}} u,$$

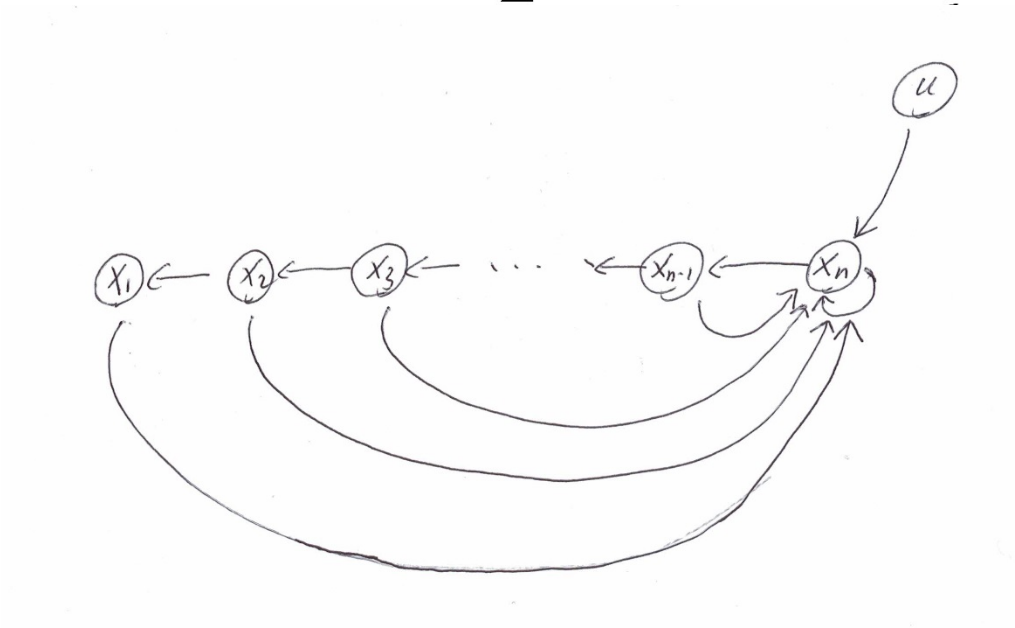
The characteristic polynomial of $\bar{A}$ (the denominator of the transfer function)is

$$\varphi_0(s) = \det\left(sI - \bar{A}\right) = \alpha_0 + \alpha_1 s + \cdots \alpha_{n-1} s^{n-1} + s^n = s^n + \sum_{i=0}^{n-1} \alpha_i s^i,$$

where α_i are from the last row of $\bar{A}$, the only non-trivial elements in $\bar{A}$. All the other elements are fixed, so called the trivial terms.

The graphical representation of the controllable canonical form

$$\dot{\bar{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}} \bar{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}} u,$$



1. The input affects only one state variable x_n.
2. The state variables are related to each other in a very special way such that if x_n are controlled properly, all the others will follow.

So we just need to focus upon how to control only one state variable, which makes the design easy!

Design the feedback controller for the controllable canonical form:

$$u = r - \bar{k}^T \bar{x} = r - \sum_{i=1}^{n} \bar{k}_i \bar{x}_i,$$

$$\overline{bk}^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \bar{k}^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{k}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_n \end{bmatrix}.$$

The input only affects the last state variable!

$$\bar{A} - \overline{bk}^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 - \bar{k}_1 & -\alpha_1 - \bar{k}_2 & -\alpha_2 - \bar{k}_3 & \cdots & \left(-\alpha_{n-1} - \bar{k}_n\right) \end{bmatrix}$$

Is it still in the controllable canonical form?

Yes! All the numbers in the non-trivial row can be tuned by the control parameters!

This is the best part about controllable canonical form.
The state feedback does not change the structure!

$$\dot{\bar{x}} = (\bar{A} - \bar{b}\bar{k}^T)\bar{x} + \bar{b}r$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 - \bar{k}_1 & -\alpha_1 - \bar{k}_2 & -\alpha_2 - \bar{k}_3 & \cdots & \left(-\alpha_{n-1} - \bar{k}_n\right) \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} r.$$

How to get the characteristic polynomial from this controllable canonical form?

Take out the last row!

$$\phi_f(s) = (\alpha_0 + \bar{k}_1) + (\alpha_1 + \bar{k}_2)\mathrm{s} + (\alpha_2 + \bar{k}_3)\mathrm{s}^2 + \cdots(\alpha_{n-1} + \bar{k}_n)\mathrm{s}^{n-1} + s^n = s^n + \sum_{i=0}^{n-1}(\alpha_i + \bar{k}_{i+1})s^i,$$

$$\phi_f(s) = (\alpha_0 + \bar{k}_1) + (\alpha_1 + \bar{k}_2)\mathrm{s} + (\alpha_2 + \bar{k}_3)\mathrm{s}^2 + \cdots (\alpha_{n-1} + \bar{k}_n)\mathrm{s}^{n-1} + s^n = s^n + \sum_{i=0}^{n-1} (\alpha_i + \bar{k}_{i+1}) s^i,$$

All the coefficients can be tuned by the control parameters!

If we compare it to any desired characteristic polynomial,

$$\phi_d(s) = \gamma_0 + \gamma_1 \mathrm{s} + \gamma_2 \mathrm{s}^2 + \cdots \gamma_{n-1} \mathrm{s}^{n-1} + s^n = s^n + \sum_{i=0}^{n-1} \gamma_i s^i,$$

Is it simple to choose $\bar{k}$ to match these two polynomials?

$$\phi_f(s) = \phi_d(s) \Leftrightarrow \alpha_i + \bar{k}_{i+1} = \gamma_i$$

$$\bar{k}_{i+1} = \gamma_i - \alpha_i$$

So the control gains are simply the difference between the coefficients of the desired characteristic polynomial and the actual one.

The controller for the original state space can be easily computed as

$$u = r - \bar{k}^T \bar{x} = r - \bar{k}^T T x = r - k^T x \rightarrow k^T = \bar{k}^T T$$

It can be shown by Cayley-Hamilton Theorem that

$$k^T = [0,\ 0,\ \cdots,\ 0,\ 1]\mathsf{C}^{-1}\phi_d(A),$$

$$\phi_d(A) = \phi_d(s)\Big|_{s=A} = A^n + \gamma_{n-1}A^{n-1} + \ldots + \gamma_0 I_n.$$

Ackermann's formula

"How to get K for pole placement?"

That is, ***given a controllable pair of (A, b) and a stable ϕ_d, find k to meet***

$$\det(sI - A + bk^T) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_0 \qquad (6)$$

You can use **Ackermann's formula.**

But what if you forget this formula?

The simplest method is to directly solve (6) by coefficient comparison.

Example 3. For
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

assign the desired closed-loop poles at $\lambda_1 = -1$ and $\lambda_2 = -2$.

Solution. Let $u = -k^T x = -[k_1\ k_2]x$. Then the closed-loop system becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$$
$$= \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x.$$

The closed-loop characteristic polynomial is

$$\det\left(sI - A + bk\right) = s^2 + k_2 s + k_1.$$

On the other hand, we have

$$\phi_d\left(s\right) = \left(s+1\right)\left(s+2\right) = s^2 + 3s + 2.$$

Comparing the corresponding coefficients of the above two gives

$$k_1 = 2,$$
$$k_2 = 3.$$

But the direct comparison might be difficult to solve if the system order is 3 or higher.

Ackermann formula is given by

$$k^T = [0,\ 0,\ \cdots,\ 0,\ 1]\mathrm{C}^{-1}\phi_d(A),$$

where

$$\phi_d(A) = \phi_d(s)\Big|_{s=A} = A^n + \gamma_{n-1}A^{n-1} + \ldots + \gamma_0 I_n.$$

Note $A^0 = I_n$.

Example 5. Let the desired poles be $\lambda_1 = -3$, $\lambda_2 = -4$, and the plant be

$$\dot{x} = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Then, it follows that

$$\phi_d(s) = (s+4)(s+3) = s^2 + 7s + 12,$$

$$\phi_d(A) = A^2 + 7A + 12I_2$$

$$= \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} + 7\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} + 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 3 \\ 12 & 4 \end{bmatrix} + \begin{bmatrix} 21 & 7 \\ 28 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 46 & 10 \\ 40 & 16 \end{bmatrix}.$$

$$\mathsf{C} = [b \quad Ab] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathsf{C}^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[0 \quad 1]\mathsf{C}^{-1} = [1 \quad 0].$$

Hence, using Ackermann's formula $k^T = [0,\ 0,\ \cdots,\ 0,\ 1]\mathsf{C}^{-1}\phi_d(A)$,

$$k^T = [1 \quad 0]\begin{bmatrix} 46 & 10 \\ 40 & 16 \end{bmatrix} = [46 \quad 10].$$

Summary on pole placement for the single-input case

What is the condition for solvability?

The system is controllable!

How many solutions can you find?

Unique

How many methods?

- Direct comparison using $\det\left[sI-(A-bk^T)\right]=\varphi_d(s)$
- Ackermann's formula

Which one to choose?

- Direct comparison recommended for systems of order 1 or 2
- Ackermann's formula recommended for systems of order 3 or higher.

Break

- ***Future Robots (1)***

How to design the desired poles?

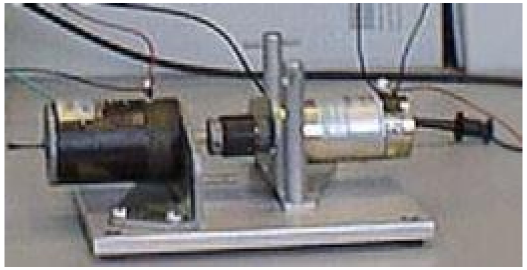
- Use performance specifications to determine a pair of dominant poles (i.e. slowest among all): λ_1, λ_2 with $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 < 0$. Say, $\lambda_{1,2} = -2 \pm j$, with the corresponding decay function of e^{-2t}.

- Locate extra poles other than the dominant ones to be 2-5 times faster than the dominant ones: $\lambda_i = (2 \sim 5)\operatorname{Re}\lambda_1, i = 3, 4, \ldots, n$. Say, e^{-10t}, which decays much faster than e^{-2t}.

- Fine-tune pole locations/performance by taking into account plant zeros and doing simulation.

An Industrial Application: DC Motor Position Control revisited

1) Model:



$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K_t}{J} \\ 0 & -\frac{K_e}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \frac{1}{L}\end{bmatrix} V,$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix}.$$

with

$J = 3.2284\times10^{-6}$, moment of inertia of the rotor

$b = 3.5077\times10^{-6}$, damping ratio of the mechanical system

$K = 0.0274$, electromotive force constant

$R = 4$, electric resistance

$L = 2.75\times10^{-6}$ electric inductance

2) Control Requirements

- Settling time less than 40 milliseconds
- Overshoot less than 16%

3) Design Solution:

How many poles to design?

Since the system is of order 3, there will be 3 poles to be placed for the closed-loop system.

We may decide two dominant poles from design requirements,

$$t_s = \frac{4}{\zeta\omega_n} \le 0.04 \quad \Rightarrow \quad \zeta\omega_n \ge 100,$$

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.16 \quad \Rightarrow \quad \zeta > 0.47,$$

choose $\zeta = 0.707$ and $\zeta\omega_n = 100$. Then two dominant poles are

$$\lambda_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -100 \pm j100.$$

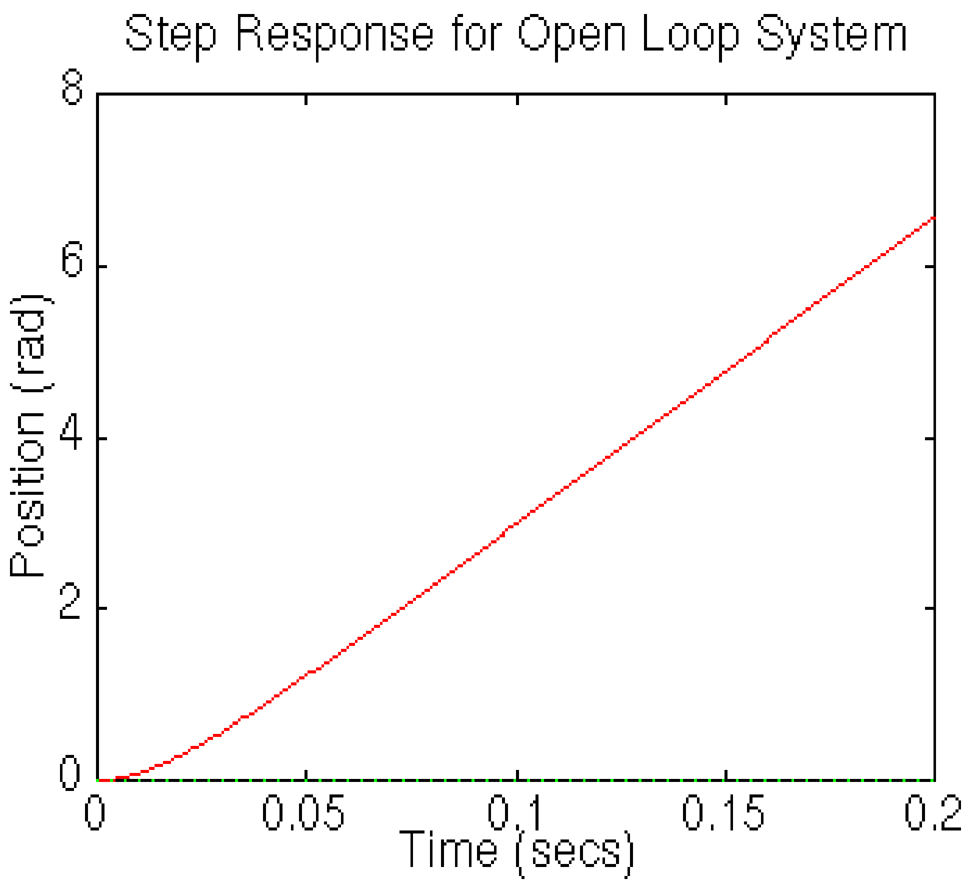
How to choose the third pole?

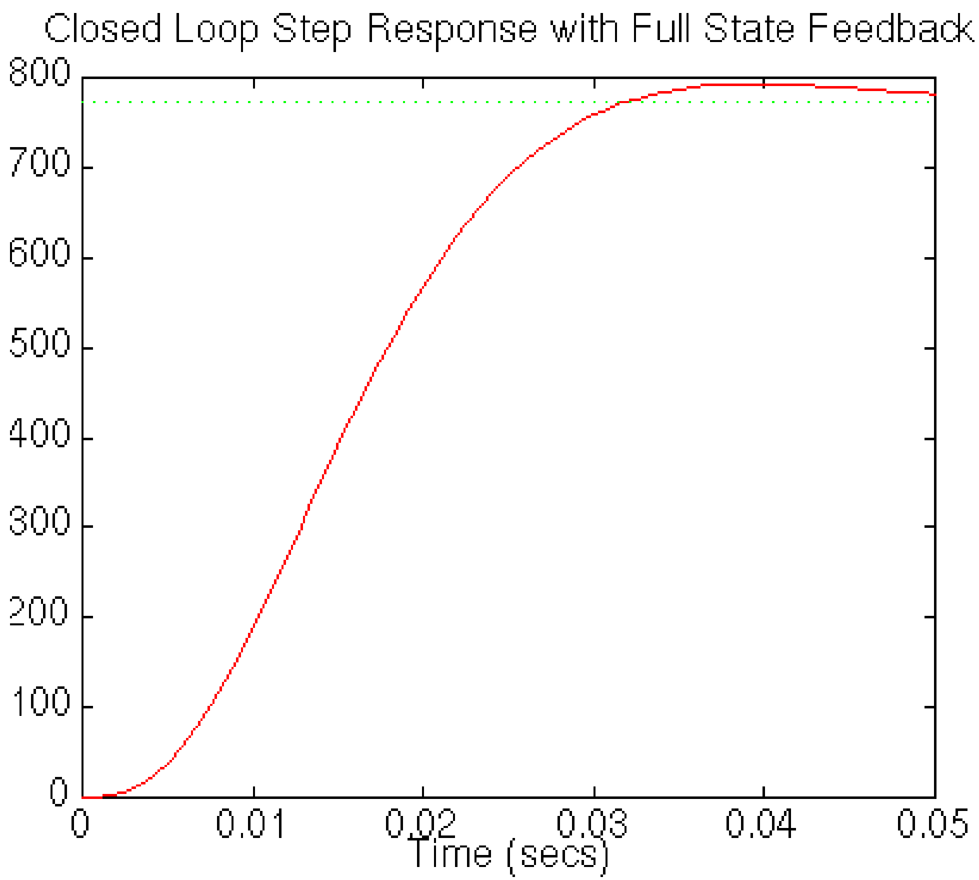
The fast pole has to be far away from the dominant one.
For example, 2 to 5 times left to these two dominant poles

The third pole is chosen as $\lambda_3 = -200$.
By Ackermann's formula, we obtain

$$K = \begin{bmatrix} 0.0013 & -0.0274 & -3.9989 \end{bmatrix}.$$

The step response of the closed-loop system.





Does it meet the performance requirements?

- Settling time less than 40 milliseconds Yes.
- Overshoot less than 16%

§7.4 Multi-input Case

Consider a multi-input plant:

$$\dot{x} = Ax + Bu, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

$$y = Cx.$$

where u has 2 or more elements, or B has 2 or more columns.

The control law is $\quad u = -Kx + Fr.$

The closed-loop system becomes

$$\dot{x} = (A - BK)x + BFr,$$
$$y = Cx.$$

Pole placement problem. For the plant (4) , let $\lambda_1, \lambda_2, \cdots, \lambda_n$, be the desired stable closed-loop poles, where for each complex pole, its conjugate must be also included in. Define

$$\phi_d(s) = \prod_{i=1}^{n}(s-\lambda_i) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \gamma_0$$

as the desired closed-loop characteristic polynomial with real coefficients. We want to determine the control law (5) such that the closed-loop system:

$$\dot{x} = (A-BK)x + BFr,$$

meets

$$\det(sI-(A-BK)) = s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_0 \tag{6}$$

Direct comparison

In the following let's first try to use direct comparison method.

Example 7. Consider the system:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u$$

The closed-loop poles are to be at $\lambda_{1,2} = -1 \pm j$.

$$\phi_d(s) = (s+1+j)(s+1-j) = s^2 + 2s + 2.$$

Check:

$$[B \quad AB] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \text{controllable } (A, B).$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u \qquad u = -Kx + Fr, \qquad K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$\det(sI - (A - BK))$$

$$= \det(sI - (\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix})) = \det(sI - \begin{bmatrix} -k_{11} & 1-k_{12} \\ -k_{21} & -k_{22} \end{bmatrix})$$

$$= \det(\begin{bmatrix} s+k_{11} & -1+k_{12} \\ k_{21} & s+k_{22} \end{bmatrix}) = (s+k_{11})(s+k_{22}) - (k_{12}-1)k_{21}$$

$$= s^2 + (k_{11}+k_{22})s + k_{11}k_{22} - k_{12}k_{21} + k_{21}$$

Compare with the desired C.P. $\phi_d(s) = (s+1+j)(s+1-j) = s^2 + 2s + 2.$

We have
$$\begin{cases} k_{11} + k_{22} = 2 \\ k_{11}k_{22} - k_{12}k_{21} + k_{21} = 2 \end{cases}$$

How many equations? 2!

How many design parameters (unknowns)? 4

How many solutions can you find?

Infinity!

Engineers do not like too many choices!

How many solutions can you find for SISO case?

Only one!

Since we already know how to do this for single-input system. Is it possible to convert the multi-input into single input system?

Multi-input means that you can choose multiple input signals at the same time.

But can we limit our choice to only one input signal instead?

The answer is simply YES because you can choose whatever input you like!

What is the simplest way to force the multi-input to single –input?

Just use one input and keep others as zero.

To convert multi input to single input, we can also design all the input signals as the same:

$$u = \begin{bmatrix} v \\ v \\ \vdots \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} v.$$

In general, we can have different weights for different inputs such that

$$u = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} v = qv,$$

where v is a scalar.

One sees that the system:

$$\begin{aligned}\dot{x} &= Ax - Bu \\ &= Ax - Bqv \\ &= Ax + \underbrace{(Bq)}_{b} v,\end{aligned} \tag{10}$$

Is it a single input system now?

Yes. So if the system pair (A, Bq) is controllable, we can apply the state feedback control law

$$v = -k^T x,$$

Overall, we have

$$u = qv = -qk^T x = -Kx.$$

$$K = qk^T = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}, \tag{11}$$

What is the rank of this gain matrix K?

One. That's why the method is called unity rank method.

The Unity Rank Method

Given a controllable pair $\{A, B\}$. Proceed with

Step 1. Choose the weight vector q such that the pair $\{A, Bq\}$ is controllable.

Step 2. Use any single-input pole placement algorithmfor the *pair* $\{A, Bq\}$ to determine k^T such that

$$A - Bqk^T$$

has the desired eigenvalues (closed-loop poles).

Overall, the required state feedback matrix is

$$K = qk^T. \tag{11}$$

Example 10. Consider the system:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u$$

The closed-loop poles are to be at $\lambda_{1,2} = -1 \pm j$.

$$\phi_d(s) = (s+1+j)(s+1-j) = s^2 + 2s + 2.$$

Check:

$$[B \quad AB] = \begin{bmatrix} 1 & 0 & \times & \times \\ 0 & 1 & \times & \times \end{bmatrix} \Rightarrow \text{controllable } (A, B).$$

Step 1. Set $q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $Bq = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} Bq & ABq \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has full rank. (A, Bq) is thus controllable.

Step 2. Let $k^T = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, thus

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{Bq} k^T = \underbrace{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}}_{A_1},$$

$$\det\left(sI - A_1\right) = s^2 + k_2 s + k_1,$$

$$\phi_d\left(s\right) = s^2 + 2s + 2,$$

and directly comparing coefficients yields

$$k^T = [2 \quad 2].$$

$$K = qk^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [2 \quad 2] = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$$

Is the solution unique?

If we choose another weight vector, $q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

then $Bq = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

and $\begin{bmatrix} Bq & ABq \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has full rank. $(A,\ Bq)$ is thus controllable.

Let $k^T = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. One has

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} - \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{Bq} k^T = \underbrace{\begin{bmatrix} -k_1 & 1-k_2 \\ -k_1 & -k_2 \end{bmatrix}}_{A_1},$$

$$\det\left(sI - A_1\right) = s^2 + \left(k_1 + k_2\right)s + k_1,$$
$$\phi_d\left(s\right) = s^2 + 2s + 2,$$

and directly comparing coefficients yields

$$k^T = [2 \quad 0].$$

This leads to the state feedback gain as

$$K = qk^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [2 \quad 0] = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$$

This case shows that K is not unique. Why?

There are infinite number of ways to choose the weight vector q such that {A, Bq} is controllable!

But we do not like too many choices. Therefore, is there any other systematic way?

For the single-input case, we showed that the controllable canonical form is the key for the solution.

Can we also rely on the controllable canonical form for multi-input system?

But how do we obtain the controllable canonical form for multi-input system?

Example 11.

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix} x$$

Solution: Obtain the controllable canonical form first.

How to obtain the controllable canonical form for MIMO system?

First compute the controllability matrix

$$W_c = \{B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\}$$

$$= \{b_1 \quad \cdots \quad b_m \quad Ab_1 \quad \cdots \quad Ab_m \quad \cdots \quad A^{n-1}b_1 \quad \cdots \quad A^{n-1}b_m\}$$

Check whether it is full rank or not. If it is controllable, then move on to the next step.

For single input system, the controllability matrix is a square matrix, all of the vectors are independent.

For MIMO system, we need to select the n independent vectors out of nm vectors from the controllability matrix in the strict order from left to right.

And group them in a square matrix C in the following form

$$C = \{b_1 \quad Ab_1 \quad \cdots \quad A^{d_1-1}b_1 \quad b_2 \quad Ab_2 \quad \cdots \quad A^{d_2-1}b_2 \cdots \quad b_m \quad Ab_m \quad \cdots \quad A^{d_m-1}b_m \quad \}$$

Where the indices di imply the number of vectors in C related to the i-th input, ui

Please note that the sequence of the vectors in C is different from Wc

The vectors associated with the same input are grouped together now!

Constructing this matrix C is the critical step.
The controllable canonical form can then be computed from this matrix C!

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$W_c = \{B \quad AB \quad A^2B\} = \{b_1 \quad b_2 \quad Ab_1 \quad Ab_2 \quad A^2b_1 \quad A^2b_2\}$$

$$= \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 & 4 \end{bmatrix}$$

Select 3 independent vectors from the controllability matrix in the order from left to right

And group them in a matrix C in the following form

$$C = \{b_1 \quad b_2 \quad Ab_2\} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Once we have the matrix C, we need to compute the inverse of C,

$$C^{-1} = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \quad \Longrightarrow \quad \text{transformation matrix T}$$

How to construct the transformation matrix T for the single input case?

$$\mathrm{C} = \{b \quad Ab \quad \cdots \quad A^{n-1}b\}$$

$$C^{-1}C = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} q_1^T b & q_1^T Ab & \cdots & q_1^T A^{n-1}b \\ q_2^T b & q_2^T Ab & \cdots & q_2^T A^{n-1}b \\ \vdots & \vdots & \vdots & \vdots \\ q_n^T b & q_n^T Ab & \cdots & q_n^T A^{n-1}b \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Take out the last row of this matrix, we have

$$\begin{bmatrix} q_n^T b & q_n^T Ab & \cdots & q_n^T A^{n-1}b \end{bmatrix} = [0 \quad 0 \quad \cdots \quad 1]$$

For single input case, just take the last row of C^{-1} , q_n^T , and construct T as

$$T = \begin{bmatrix} q_n^T \\ q_n^T A \\ \vdots \\ q_n^T A^{n-1} \end{bmatrix} \Rightarrow Tb = \begin{bmatrix} q_n^T b \\ q_n^T Ab \\ \vdots \\ q_n^T A^{n-1} b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \bar{b}$$

We take out only one row from C^{-1} because there is only one input.

Now, there are multiple inputs. If you want that all the inputs are applied to the system efficiently, then more rows will be taken out in addition to the last row!

For each input, there will be one row to be taken out to construct T!

For multi-input case, we need to take out m rows from C^{-1} corresponding to the m inputs, and form T as

$$
T = \begin{bmatrix} q_{d_1}^T \\ q_{d_1}^T A \\ \vdots \\ q_{d_1}^T A^{d_1-1} \\ q_{d_1+d_2}^T \\ q_{d_1+d_2}^T A \\ \vdots \\ q_{d_1+d_2}^T A^{d_2-1} \\ \vdots \end{bmatrix} \Rightarrow \overline{B} = TB = \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & \times & \cdots & \times \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \times \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{array}\right] \begin{array}{l} \\ \\ \\ d_1^{th} row \\ \\ \\ \\ (d_1+d_2)^{th} row \\ \\ \\ \\ n^{th} row \end{array}
$$

Where the indices di imply the number of vectors in C related the i-th input, ui

$$
C = \{ b_1 \quad Ab_1 \quad \cdots \quad A^{d_1-1}b_1 \quad b_2 \quad Ab_2 \quad \cdots \quad A^{d_2-1}b_2 \cdots \quad b_m \quad Ab_m \quad \cdots \quad A^{d_m-1}b_m \quad \}
$$

For this case,

$$C = \{b_1 \quad b_2 \quad Ab_2\} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

How many vectors in C are associated with u_1 ? $d_1 = 1$

How many vectors in C are associated with u_2 ? $d_2 = 2$

Which rows should we take out to form T?

The first (d_1)and third ($d_1 + d_2$)rows!

$$T = \begin{bmatrix} q_1^T \\ q_3^T \\ q_3^T A \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The key is to form the transformation matrix T. The rest is simple.

$$\bar{A} = TAT^{-1} = \left[\begin{array}{c|cc} -1 & 7 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -2 & 3 \end{array}\right], \qquad \bar{B} = TB = \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right],$$

How many non-trivial rows are there in the canonical form?

There are two inputs, so there are two nontrivial rows since the inputs only affect the nontrivial rows!

Design the feedback gain matrix for the controllable canonical form

$$\bar{K} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} \end{bmatrix}$$

Form the closed loop matrix:

$$\bar{A} - \bar{B}\bar{K} = \left[\begin{array}{c|cc} -\bar{k}_{11} - 1 & -\bar{k}_{12} + 7 & -\bar{k}_{13} \\ \hline 0 & 0 & 1 \\ -\bar{k}_{21} & -\bar{k}_{22} - 2 & -\bar{k}_{23} + 3 \end{array}\right].$$

Is it still in the canonical form?

Once again, we notice that the feedback control does not change the canonical form!
It only changes the non-trivial terms!

Let the desired eigenvalues be -1, -2, -3.

$$\det\left(sI - A_d\right) = \left(s+1\right)\left(s+2\right)\left(s+3\right)$$
$$= \left(s+1\right)\left(s^2+5s+6\right)$$

Then a possible desired closed-loop matrix is

$$A_d = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -6 & -5 \end{array}\right]$$

It is a combination of one first order and one second order system!

Compare $\bar{A} - \bar{B}\bar{K}$ and A_d ,

$$\bar{A} - \bar{B}\bar{K} = \left[\begin{array}{c|cc} -\bar{k}_{11} - 1 & -\bar{k}_{12} + 7 & -\bar{k}_{13} \\ \hline 0 & 0 & 1 \\ -\bar{k}_{21} & -\bar{k}_{22} - 2 & -\bar{k}_{23} + 3 \end{array}\right].$$

$$\bar{k}_{11} = \bar{k}_{13} = \bar{k}_{21} = 0,\ \bar{k}_{12} = 7,\ \bar{k}_{22} = 4,\ \bar{k}_{23} = 8.$$

$$K = \bar{K}T = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 12 & 8 \end{bmatrix}.$$

What is the rank of this matrix? Two. So It is full rank now!

Algorithm For Full rank pole placement

Given controllable $\{A, B\}$ and the desired $\phi_d(s)$.

(i) Obtain the controllable canonical form in the state x via

$$\bar{x} = Tx.$$

The most time-consuming part is computing T.

Note that the controllable canonical form depends upon the square matrix formed by the n independent vectors from the controllability matrix.

$$C = \{b_1 \quad Ab_1 \quad \cdots \quad A^{d_1-1}b_1 \quad b_2 \quad Ab_2 \quad \cdots \quad A^{d_2-1}b_2 \cdots \quad b_m \quad Ab_m \quad \cdots \quad A^{d_m-1}b_m \quad \}$$

Through the state transformation T,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$

is in the controllable canonical form.

(ii) Specify a desired closed-loop matrix A_d

such that $\det(sI - A_d) = \phi_d(s)$ and A_d is the same as $\bar{A}$ except non-trivial rows.

(iii) Compare the non-trivial rows of the canonical form with state feedback to the desired one, and compute $\bar{K}$.

(iv) Compute the original feedback gain K

$$K = \bar{K}T.$$

Is the solution unique? Why?

There are many ways to select the desired canonical form which share the same poles!

Let the desired eigenvalues be -1, -2, -3. Then a possible desired closed-loop matrix is

$$A_d = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -6 & -5 \end{array}\right]$$

since

$$\det(sI - A_d) = (s+1)(s+2)(s+3)$$
$$= (s+1)(s^2+5s+6)$$

Let's rewrite above characteristic polynomial

$$\det(sI - A_d) = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6.$$

We can also construct another controllable canonical form matrix

$$\tilde{A}_d = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ -6 & -11 & -6 \end{array}\right]$$

The trivial rows must be the same.
The nontrivial rows are different.

which will lead to a different state feedback controller.

***Pole placement*:**

- Reasons
- Solvability
- Methods

In particular, we focused on the single-input case and showed that

- the conditions for arbitrary assignment of closed-loop poles (*Theorem 1*);
- two methods for the determination of feedback gain: Ackermann's *Formula* and direct comparison.

We also extended the theorems to the multi-input case and discussed:

Three methods for the determination of feedback gain:
Direct Comparison,
Unity Rank (force MI to SI)
and Full-Rank (use controllable canonical form) Methods.

For single input case, is the solution unique?

YES!

For multi-input case, is the solution unique?

NO!

Q & A…

THANK YOU !