EE5137 Stochastic Processes: Problem Set 8
Assigned: 11/03/22, Due: 18/03/22

There are six (6) non-optional problems in this problem.

1. Exercise 4.10 (Gallager's book)

2. Consider a Markov chain with given transition probabilities and with a single recurrent class that is aperiodic. Assume that for $n \geq 500$, the n-step transition probabilities are very close to the steady-state probabilities.

   (a) Find an approximate formula for $\Pr(X_{1000} = j, X_{1001} = k, X_{2000} = k | X_0 = i)$.

   (b) Find an approximate formula for $\Pr(X_{1000} = i | X_{1001} = j)$

3. A coin having probability $p, 0 < p < 1$, of landing heads is tossed continually. We are interested in the length of consecutive heads in the tosses. Define $X_n = k$ if the coin comes up heads in all of the most recent k tosses (from the $(n-k+1)$-st up to the nth), but tails in the $(n-k)$-th toss. On the contrary, if the coin comes up tails in the n-th toss, then let $X_n = 0$. For example, for the outcome of 15 tosses HHHHTTHHTHTTHHH, the value of X_n's are

$$(X_1, \ldots, X_{15}) = (1, 2, 3, 4, 0, 0, 1, 2, 0, 1, 0, 0, 1, 2, 3)$$

   Observe that $\{X_n : n \geq 0\}$ is a Markov chain with infinite state-space $\mathcal{S} = \{0, 1, 2, \ldots\}$ and

$$X_n = \begin{cases} X_{n-1} + 1 & \text{with probability } p \text{ (if the coin comes up heads in the } n\text{th toss)} \\ 0 & \text{with probability } 1-p \text{ (if the coin comes up tails in the } n\text{th toss)} \end{cases}$$

   (a) Find the limiting distribution of this chain, i.e., find

$$\pi_i = \lim_{n \to \infty} \Pr(X_n = i), \quad i = 0, 1, 2, \ldots$$

   (b) Let T_k be the first time that k consecutive heads have appeared. In other words, $T_k = m$ if and only if at m-th toss the Markov chain $\{X_n : n \geq 0\}$ reaches state k for the first time. Explain why that $\mathbb{E}[T_k]$'s satisfy the recursive equation

$$p\mathbb{E}[T_k] = \mathbb{E}[T_{k-1}] + 1, \quad k = 2, 3, 4, \ldots$$

   and also show that $\mathbb{E}[T_1] = 1/p$.

   (c) Solve the recursive equation in part (b) to find $\mathbb{E}[T_k]$.

4. We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ϵ where $0 < \epsilon < 1$, or we select a ball at random so that each ball has probability $(1-\epsilon)/n$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the state-state distribution of the number of white balls?

5. Each of two urns contains m balls. Out of the total of $2m$ balls, m are white and m are black. A ball is simultaneously selected from each urn and moved to the other urn, and the process is indefinitely repeated. What is the steady-state distribution of the number of white balls in each urn?

6. An auto insurance company classifies its customers in three categories: bad, satisfactory and preferred. No one moves from bad to preferred or from preferred to bad in one year. 40% of the customers in the bad category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become bad; 20% of those in the preferred category are downgraded to satisfactory.

   (a) Write the state transition matrix for the model.

   (b) What is the limiting fraction of customers in each of these categories, i.e., the fraction of bad, satisfactory, and preferred customers after many years?

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7. (Optional) Exercise 4.5 (Gallager's book)

8. (Optional) Exercise 4.9 (Gallager's book)