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1.(a) If a row vector π is a left-eigenvector of $[P]$, when can get

$$\lambda_k \cdot \pi^{(k)} = \pi_k[P]. \iff \lambda_k \cdot \pi_j^{(k)} = \sum_i \pi_i^{(k)} \cdot P_{ij}$$

So: $\sum_i \pi_i^{(k)} P_{ij}^2 = \sum_i \pi_i P_{ij} \cdot P_{ij}$

$= \lambda_k \cdot \pi_j^{(k)} \cdot \sum_i P_{ij}$

$= \lambda_k \cdot \sum_i \pi_{ij}^{(k)} \cdot P_{ij}$

$= \lambda_k^2 \cdot \pi_j^{(k)}$

By using induction, we can get

$$\sum_i \pi_i^{(k)} P_{ij}^n = \lambda_k^n \cdot \pi_j^{(k)}.$$

(b) From Q1.(a), we know:

$$\lambda_k^n \cdot \pi_j^{(k)} = \sum_i \pi_i^{(k)} P_{ij}^n.$$

$$\lambda_k^n = \frac{\sum_i \pi_i^{(k)} P_{ij}^n}{\pi_j^{(k)}} \qquad P_{ij}^n \to \pi_j$$

~~Because $\sum_i \pi_i$~~ We can choose j to maximize $|\pi_j^{(k)}|$ for the given k.

$$\Rightarrow |\lambda_k|^n \le \frac{\sum_i \pi_i^{(k)} P_{ij}^n}{\max|\pi_j^{(k)}|} = M. = \pi_j^{n-1} \cdot \sum_i \pi_i = \pi_j^{n-1}$$

(c) Because $P_{ij}^n \longrightarrow \pi_j^{(k)}$.

So: $|\lambda_k|^n \le \sum_i \pi_i^{(k)} \pi_j^{n-1} = \pi_j^{n-1} \cdot \sum_i \pi_i^{(k)}$.

$\sum_i \pi_i^{(k)} = 1$,

$LHS = \pi_j^{n-1}$. Because $\pi_j \le 1$

So. $|\lambda_k|^n \le 1$

$|\lambda_k| \le 1$

2. (a) $\cancel{[[A]-\lambda v\pi]^2 = [A^2] - 2\lambda v\pi[A]}$

$$[[A]-\lambda v\pi]^2 = [A^2] - \lambda v\pi[A] - \lambda[A]v\pi + \lambda^2 v\pi v\pi$$

$$= [A^2] - \lambda^2 v\pi - \lambda^2 v\pi + \lambda^2 v\pi = [A^2] - \lambda^2 v\pi$$

(Because $\pi[A] = \lambda\pi$, $[A]v = \lambda v$)

(b) $$[[A^n] - \lambda^n v\pi][[A] - \lambda v\pi] = [A^{n+1}] - \lambda^n v\pi[A] - \lambda[A^n]v\pi + \lambda^{n+1} v\pi v\pi$$

$$= [A^{n+1}] - \lambda^{n+1} v\pi.$$

(c): (a) gives the base of the induction and (b) gives the inductive step.

3. (a) $[P] = \begin{bmatrix} [P_T] & [P_{TR}] \\ 0 & [P_R] \end{bmatrix}$ $\qquad [P^2] = \begin{bmatrix} [P_T^2] \cdot & [P_x^2] \\ 0 & [P_R^2] \end{bmatrix}$

We can use induction

$$[P^{n+1}] = [P^n]\cdot[P] = \begin{bmatrix} [P_T^n] & [P_x^n] \\ 0 & [P_R^n] \end{bmatrix} \begin{bmatrix} [P_T] & [P_{TR}] \\ 0 & [P_R] \end{bmatrix} = \begin{bmatrix} [P_{TR}^{n+1}] & [P_x^{n+1}] \\ 0 & [P_R^{n+1}] \end{bmatrix}.$$

(b) Let $T = \{1, \dots, t\}$ be the transient states
$R = \{t+1, \dots, t+r\}$ be the recurrent class.

for all $i \in T$, there exists a walk of length $\le t$ to a recurrent state.

$\forall i \in T$, $\sum_{j \in R} P_{ij}^t > 0$. and $\sum_{j \in T} P_{ij}^t < 1$

$\Downarrow$

$\sum_{t \le j \le t+r} P_{ij}^t > 0$ for any $i \in T$.

for $\forall i \in R$, it also hold. $\sum_{t+1 \le j \le t+r} P_{ij}^t > 0$.

(c) $q = \min_{i\in T} \sum_{j\in R} P_{ij}^t > 0$. ~~因~~ $\sum_{j\in R} P_{ij}^t + \sum_{j\in T} P_{ij}^t = 1 \Rightarrow \min_{i\in T} \sum_{j\in R} P_{ij}^t + \max_{i\in T} \sum_{j\in T} P_{ij}^t = 1$

~~$1-q\le 1$~~ $\Rightarrow \max_{i\in T} \sum_{j\in T} P_{ij}^t = 1-q$.

$$\max_{i\in T} \sum_{j\in T} P_{ij}^{(n+1)t} = \sum_{j\in T}\left(\sum_{k\in T} P_{ik}^t P_{kj}^{nt}\right)$$

$$= \sum_{k\in T} P_{ik}^t \left(\sum_{j\in T} P_{kj}^{nt}\right)$$

$$\le \sum_{k\in T} P_{ik}^t \max_{l\in T} \sum_{j\in T} P_{lj}^{nt}$$

$$\le \left(\max_{i\in T} \sum_{k\in T} P_{ik}^t\right)\left(\max_{l\in T} \sum_{j\in T} P_{lj}^{nt}\right)$$

$$= (1-q)\left(\max_{l\in T} \sum_{j\in T} P_{lj}^{nt}\right)$$

We can use induction, and get

$$\max_{i\in T} \sum_{j\in T} P_{ij}^{nt} \le (1-q)^n \cdot \sum_{j\in T} P_{ij}$$

So: $P_{ij}^{nt} \le (1-q)^n$.

Because $q>0$, $(1-q)<0$, ~~因~~ so $\lim_{n\to\infty} (1-q)^n \to 0$

$[P_T^n]$ approaches to 0 .

(d) ~~因~~ $\lambda = 1$, $\pi = \pi[P]$, $\pi = \pi[P^n]$

$$(\pi_T \quad \pi_R) = (\pi_T \quad \pi_R)\begin{pmatrix} P_T^n & P_{TR}^n \\ 0 & P_R^n \end{pmatrix}$$

$$= (\pi_T P_T^n , \quad \pi_T P_x^n + \pi_R P_R^n)$$

~~From~~ $\begin{cases} \pi_T = \pi_T P_T^n \\ \pi_R = \pi_T P_x^n + \pi_R P_R^n \end{cases}$. in Q.3 (c), we know that $[P_T^n] = 0$.

So. $\pi_T = 0$. , $\pi_R = \pi_R P_R^n$ · (π_R must be positive and a left eigenvector of P_R)

(e) let $e = \begin{pmatrix} e_T \\ e_R \end{pmatrix}$

$e = P e$, $e = P^n e$

$\begin{pmatrix} e_T \\ e_R \end{pmatrix} = \begin{bmatrix} P_T^n & P_X^n \\ 0 & P_R^n \end{bmatrix} \begin{pmatrix} e_T \\ e_R \end{pmatrix}$

$\Rightarrow \begin{cases} e_T = P_T^n e_T + P_X^n e_R = P_X^n e_R. \\ e_R = P_R^n e_R \end{cases}$

So, e_R is the right eigenvector of P_R.

$e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. suppose u be any other steady state vector,

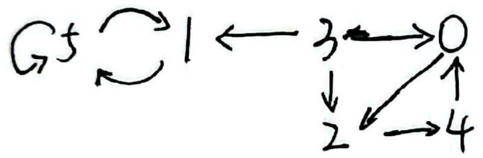
$u[P] = u.$, $u = u[P^n]$.

$u = \lim_{n\to\infty} u[P^n] = u \lim_{n\to\infty} [P^n] = u e \pi$

since $\sum_i u_i = 1 \Rightarrow [u_1, u_2 \dots] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1$

$u e \pi = \pi$. $\therefore$ steady-state vector is unique

4. (a) Based on $[P]$, we can get



$\{0, 2, 4\}$, $\{3\}$, $\{1, 5\}$

(b) recurrent states: $\{0, 1, 2, 4, 5\}$

transient states: $\{3\}$

(c) $\{0, 2, 4\}$: gcd: $\{3, 6, 9, \dots\} = 3$

$\{1, 5\}$: gcd: $\{1, 2, 3, 4 \dots\} = 1$.

(d) 1 and 5.

(e) Because state can't go to 0, 2, 3, 4. so: $\pi_0 = \pi_2 = \pi_3 = \pi_4 = 0$.

$\pi_1 + \pi_5 = 1$, $\pi = \pi[P] \Rightarrow \pi_1 = \frac{1}{2}\pi_5$

So, $\pi_1 = \frac{1}{3}$, $\pi_5 = \frac{2}{3}$

4.(f) Rate of convergence of $[P^n]$ to $e\pi$ is governed by the absolute value of the second largest eigen-value of $[P]$.

$$\log |[P^n]_{11} - \pi_1| \cong \log(|\lambda_2|^n) + \log C.$$

$$= n \log |\lambda_2| + \log C$$

$$\frac{1}{n} \log |[P^n]_{11} - \pi_1| = \log |\lambda_2| + \underline{\frac{1}{n} \log C} \to 0.$$

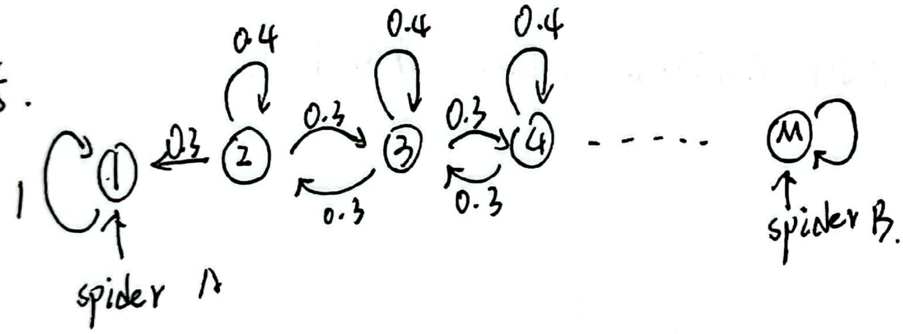
$$\therefore \lim_{n\to\infty} -\frac{1}{n} \log |[P^n]_{11} - \pi_1| = -\log |\lambda_2|$$

Because $|\lambda_2| = 1$.

$$\therefore -\log |\lambda_2| = 0$$

$$\therefore \lim_{n\to\infty} -\frac{1}{n} \log |[P^n]_{11} - \pi_1| = 0.$$

5.



For $M=4$. $[P] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

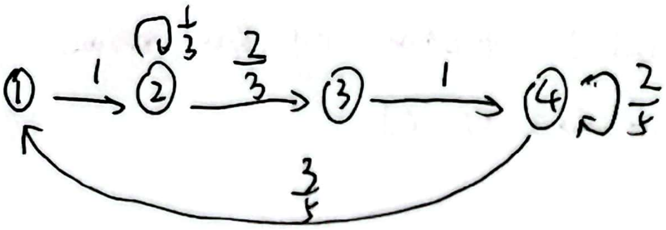
$$V = r + [P]V.$$

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} V_1 \\ 1 + 0.4V_2 + 0.3V_3 \\ 1 + 0.3V_2 + 0.4V_3 \\ V_4 \end{pmatrix}$$

$$\Rightarrow \begin{cases} V_2 = 1 + 0.4V_2 + 0.3V_3 \\ V_3 = 1 + 0.3V_2 + 0.4V_3 = V_3 \end{cases}$$

$$\Rightarrow V_2 = V_3 = \frac{10}{3} = 3.33$$

6.



(a) The initial state $\pi_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Pi_6 = \Pi_0 [P^6] = \begin{bmatrix} \frac{182}{1125} \\ \frac{529}{1228} \\ \frac{334}{1215} \\ \frac{121}{928} \end{bmatrix} \qquad r_{11}(6) = \frac{182}{1125}$$

(b) Let T denotes the time takes from state 1 back to state 1.

$T = T_{12} + T_{23} + T_{34} + T_{41}$

$T_{12} = T_{34} = 1 \Rightarrow$ like geometrically distributed variable with $p=1$.

$T_{23}: \ P_1 = \frac{2}{3}$

$T_{41}: \ P_2 = \frac{3}{5}$

$\therefore E[T] = 2 + E[T_{23}] + E[T_{41}] = 2 + \frac{3}{2} + \frac{5}{3} = \frac{31}{6}$

Since $T_{12}, T_{23}, T_{34}, T_{41}$ are independent

$Var(T) = (1-\frac{2}{3}) \cdot \frac{3^2}{2^2} + (1-\frac{3}{5}) \frac{5^2}{3^2} = \frac{67}{36}$

(c) Let A be the event that X_{999}, X_{1000} and X_{1001} are all different.

$$P(A | X_{999} = i) = \begin{cases} \frac{2}{3}, & \text{for } i = 1,2 \\ \frac{3}{5}, & \text{for } i = 3,4. \end{cases}$$

Thus, using the total probability theorem assume that the process is in steady state at 999, we obtain

$$P(A) = \frac{2}{3}(\pi_1 + \pi_2) + \frac{3}{5}(\pi_3 + \pi_4) = \frac{98}{105}$$