Chapter 2 – Differential Kinematics and Statics

CHUI Chee Kong, PhD
Control & Mechatronics Group
Mechanical Engineering, NUS

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2. Kinematic Modeling of Instantaneous Motions
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1. Translational and Rotational Velocities

Translational Velocities

- Differentiation of *position vectors* ^{B}Q (with reference to frame B)

$$^{B}V_{Q} \equiv \frac{d}{dt}\,{}^{B}Q = \lim_{\Delta t \to 0} \frac{{}^{B}Q(t+\Delta t) - {}^{B}Q(t)}{\Delta t}$$

Note:

- Important: frame in which the vector is differentiated.
- Velocity vector may also be described in terms of any frame
- Notation:

$$^{A}\left({}^{B}V_{Q}\right) = \frac{{}^{A}d}{dt}\,{}^{B}Q \qquad {}^{B}\left({}^{B}V_{Q}\right) = {}^{B}V_{Q}$$

Translational Velocities

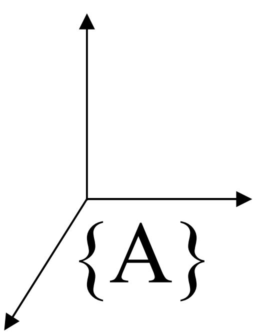
Change in representation frame:

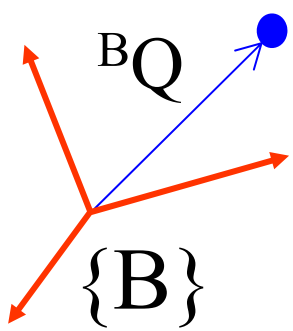
$$^{A}\left({}^{B}V_{Q}\right) = {}^{A}_{B}R\,{}^{B}V_{Q}$$

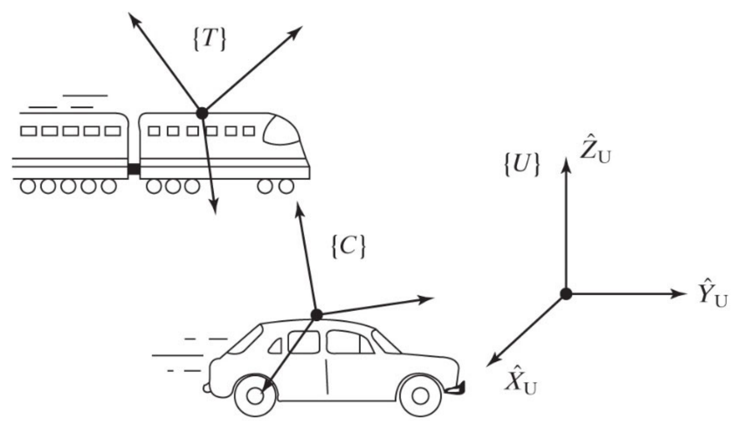
Velocity of point moving relative to a *translating* and *rotating* frame

- Consider two frames {A} and {B}
  - Orientation of frame {B} changing in time with respect to frame {A}.
    - ${}^{A}\omega_{B}$ is the rotational velocity of {B} relative to {A}.
  - Given motion of Q defined with respect to {B}
  - Given motion of origin of {B} wrt {A}

Find: Velocity of Q wrt {A}







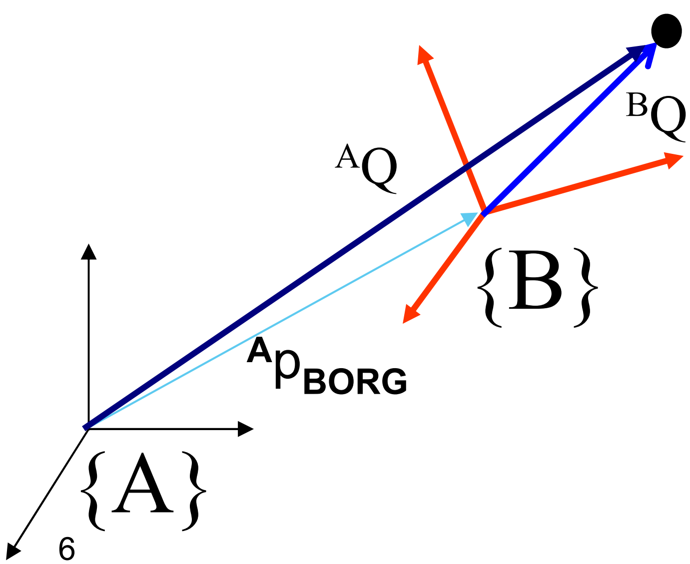
Velocity of point moving relative to a translating and rotating frame

- Find: Velocity of Q wrt {A}

  Starting from: ${}^{A}Q = {}^{A}p_{BORG} + {}^{A}_{B}R\,{}^{B}Q$

  Differentiating wrt time:

$$\boxed{{}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + \underbrace{{}^{A}_{B}\dot{R}\,{}^{B}Q}}$$



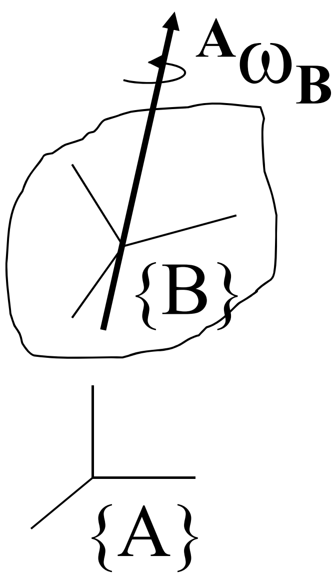
Contribution from rotational motion of frame B. Will show that it is equal to: ${}^{A}\omega_{B} \times ({}^{A}_{B}R\,{}^{B}Q)$

Rotational Velocities

- Linear velocity describes an attribute of a point.
- Angular velocity describes an attribute of a body.

Represented by the rotational (angular) velocity vector:

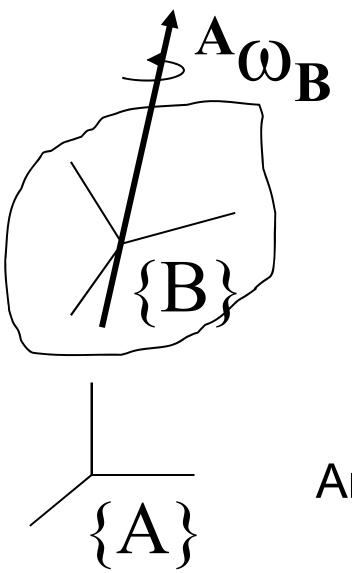
$$^{A}\omega_{B} \in \Re^{3x1}$$



magnitude of $^{A}\omega_{B}$
= speed of rotation

unit vector along $^{A}\omega_{B}$
= instantaneous axis of rotation = $^{A}\mathbf{k}$

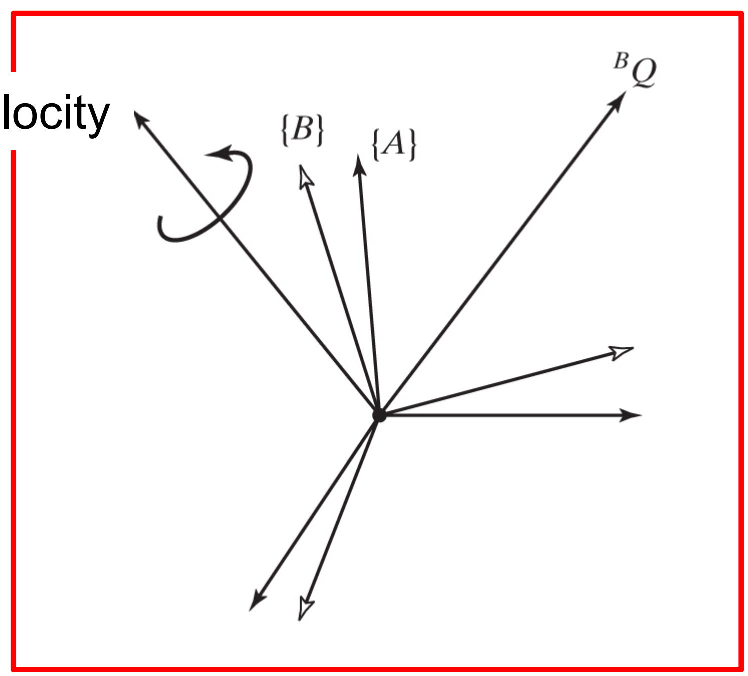
Rotational Velocities



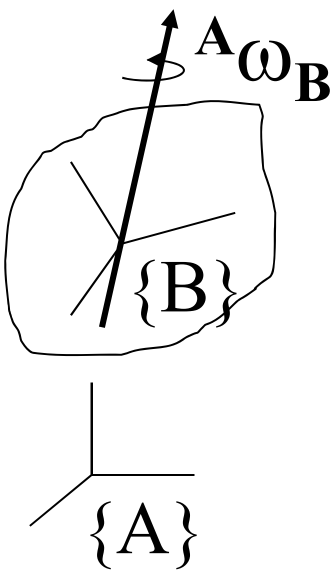
$^A\omega_B$ is related to $\dfrac{d}{dt}\,{}^A_B R$:

$$\frac{d}{dt}\,{}^A_B R = {}^A_B \dot{R} = \lim_{\Delta t \to 0} \frac{{}^A_B R(t + \Delta t) - {}^A_B R(t)}{\Delta t}$$

Angular velocity



Rotational Velocities

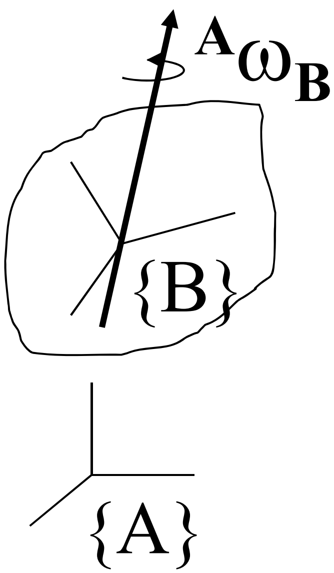


${}^{A}\omega_{B}$ is related to $\frac{d}{dt}\,{}^{A}_{B}R$:

$$\frac{d}{dt}\,{}^{A}_{B}R = {}^{A}_{B}\dot{R} = \lim_{\Delta t \to 0} \frac{\boxed{{}^{A}_{B}R(t+\Delta t)} - {}^{A}_{B}R(t)}{\Delta t}$$

$$ {}^{A}_{B}R(t+\Delta t) = \underbrace{{}^{A}R_{\mathbf{k}}(\Delta\theta)}_{\text{Small rotation occurs during } \Delta t}\,{}^{A}_{B}R(t) \qquad (2.1)$$

Rotational Velocities



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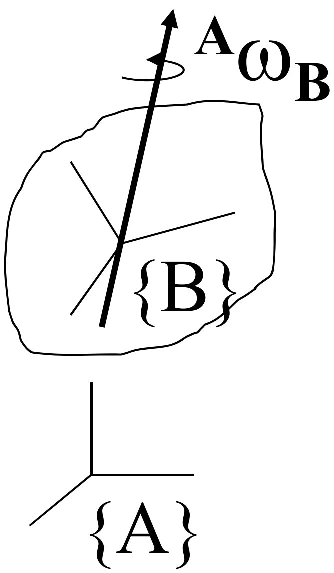
$${}^{A}_{B}R(t+\Delta t) = \underbrace{{}^{A}R_{\mathbf{k}}(\Delta\theta)}_{\text{Small rotation occurs during } \Delta t}\,{}^{A}_{B}R(t) \qquad (2.1)$$

From previous chapter:

$$^{A}R_{\mathbf{k}}(\theta) = \begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z s\theta & k_z k_x v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix}$$

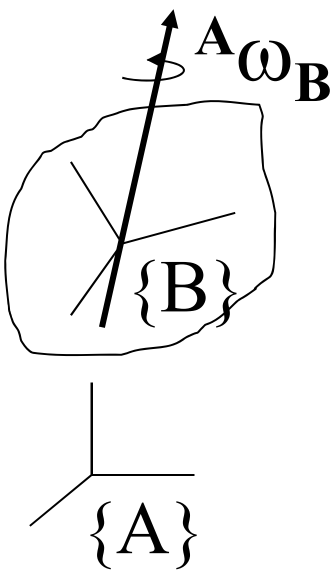
Rotational Velocities



For small $\Delta\theta$,

$$^{A}R_{\mathbf{k}}(\Delta\theta) = \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 1 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 1 \end{bmatrix}$$

Rotational Velocities



For small $\Delta\theta$,

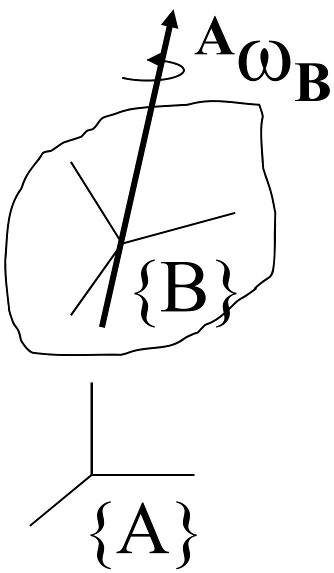
$$ {}^{A}R_{\mathbf{k}}(\Delta\theta) = \begin{bmatrix} 1 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 1 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 1 \end{bmatrix} $$

$$ \frac{d}{dt}{}^{A}_{B}R = {}^{A}_{B}\dot{R} = \lim_{\Delta t \to 0} \frac{\boxed{{}^{A}_{B}R(t+\Delta t)} - {}^{A}_{B}R(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\boxed{{}^{A}R_{\mathbf{k}}(\Delta\theta){}^{A}_{B}R(t)} - {}^{A}_{B}R(t)}{\Delta t} $$

$$ = \lim_{\Delta t \to 0} \frac{\left[{}^{A}R_{\mathbf{k}}(\Delta\theta) - I\right]{}^{A}_{B}R(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\begin{bmatrix} 0 & -k_z\Delta\theta & k_y\Delta\theta \\ k_z\Delta\theta & 0 & -k_x\Delta\theta \\ -k_y\Delta\theta & k_x\Delta\theta & 0 \end{bmatrix}{}^{A}_{B}R(t)}{\Delta t} $$

$$ = \begin{bmatrix} 0 & -k_z\dot{\theta} & k_y\dot{\theta} \\ k_z\dot{\theta} & 0 & -k_x\dot{\theta} \\ -k_y\dot{\theta} & k_x\dot{\theta} & 0 \end{bmatrix}{}^{A}_{B}R(t) $$

Rotational Velocities



That is,

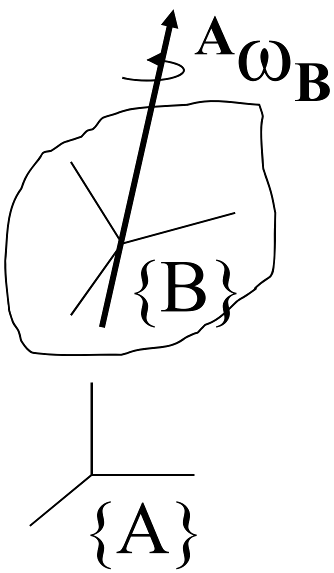
$$ {}^A_B\dot{R} = \begin{bmatrix} 0 & -{}^A\omega_{Bz} & {}^A\omega_{By} \\ {}^A\omega_{Bz} & 0 & -{}^A\omega_{Bx} \\ -{}^A\omega_{By} & {}^A\omega_{Bx} & 0 \end{bmatrix} {}^A_B R \qquad (2\text{-}2) $$

where $$ {}^A\omega_B = \begin{bmatrix} {}^A\omega_{Bx} \\ {}^A\omega_{By} \\ {}^A\omega_{Bz} \end{bmatrix} = \begin{bmatrix} k_x\dot{\theta} \\ k_y\dot{\theta} \\ k_z\dot{\theta} \end{bmatrix} = \dot{\theta}\mathbf{k} $$

Let $$ \lfloor {}^A\omega_{\mathbf{B}} \times \rfloor = \begin{bmatrix} 0 & -{}^A\omega_{Bz} & {}^A\omega_{By} \\ {}^A\omega_{Bz} & 0 & -{}^A\omega_{Bx} \\ -{}^A\omega_{By} & {}^A\omega_{Bx} & 0 \end{bmatrix} $$

- angular velocity tensor of $^A\omega_B$

Rotational Velocities



Now, Eq (2-2) can be written as

$$\boxed{{}^{A}_{B}\dot{R} = \lfloor {}^{A}\omega_B \times \rfloor {}^{A}_{B}R}$$

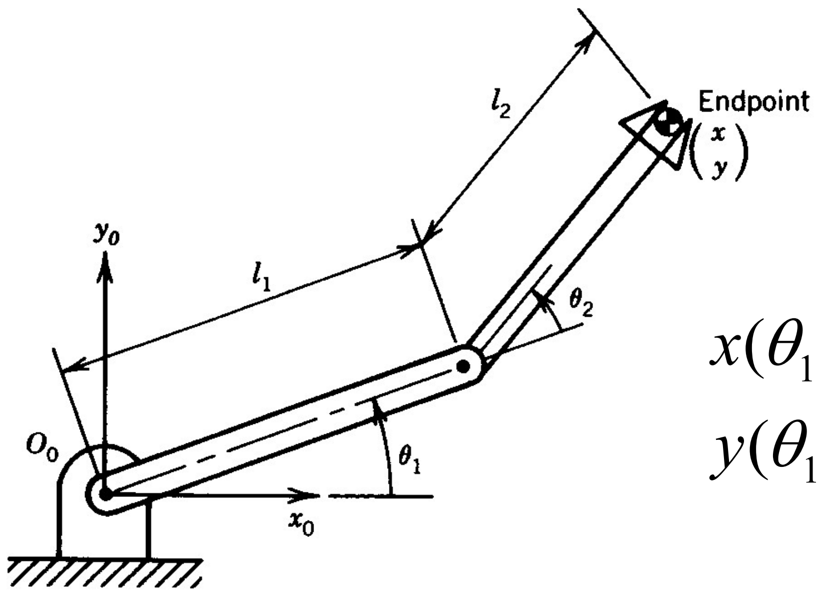
Note: Can verify that, $\lfloor {}^{A}\omega_B \times \rfloor \mathbf{p} = {}^{A}\omega_B \times \mathbf{p}$

where **p** is any (3x1) vector

2. Kinematic Modeling of Instantaneous Motions

- Differential Relationships

Consider the following 2 dof planar manipulator:



$$x(\theta_1, \theta_2) = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y(\theta_1, \theta_2) = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Differential Relationships

Differential form:

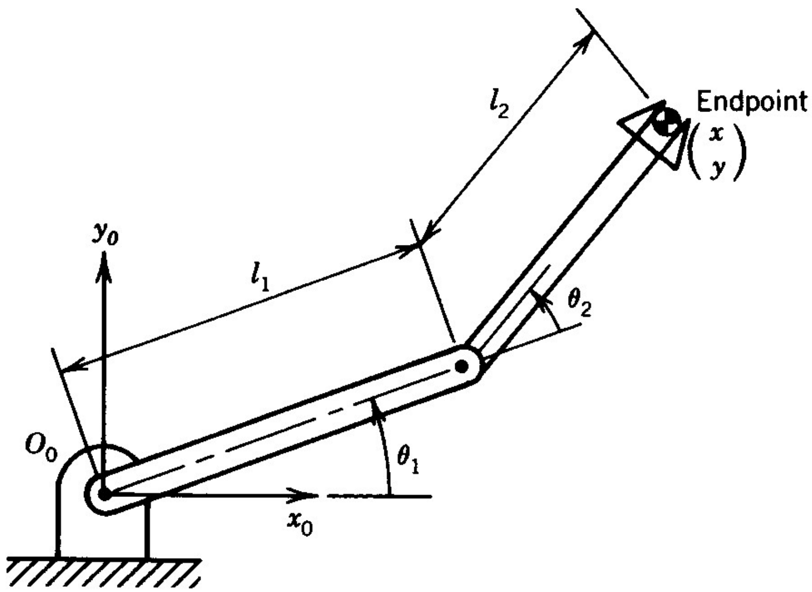
$$dx = \frac{\partial x(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial x(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

$$dy = \frac{\partial y(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial y(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

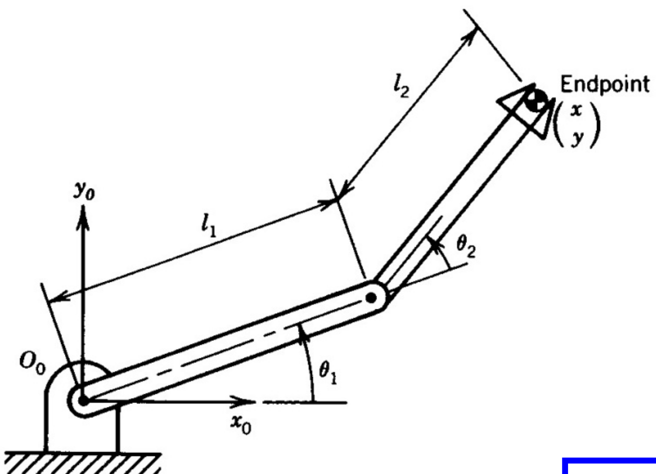
$$d\mathbf{x} = J d\theta$$

where

$$d\mathbf{x} = \begin{bmatrix} dx \\ dy \end{bmatrix}, d\theta = \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$



Differential Relationships



$$d\mathbf{x} = J d\theta \quad \text{or} \quad \mathbf{v} = \mathbf{J}\dot{\theta}$$

Infinitesimal joint displacements

$$d\theta$$

$$\left(\dot{\theta}\right)$$

Manipulator Jacobian

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{bmatrix}$$

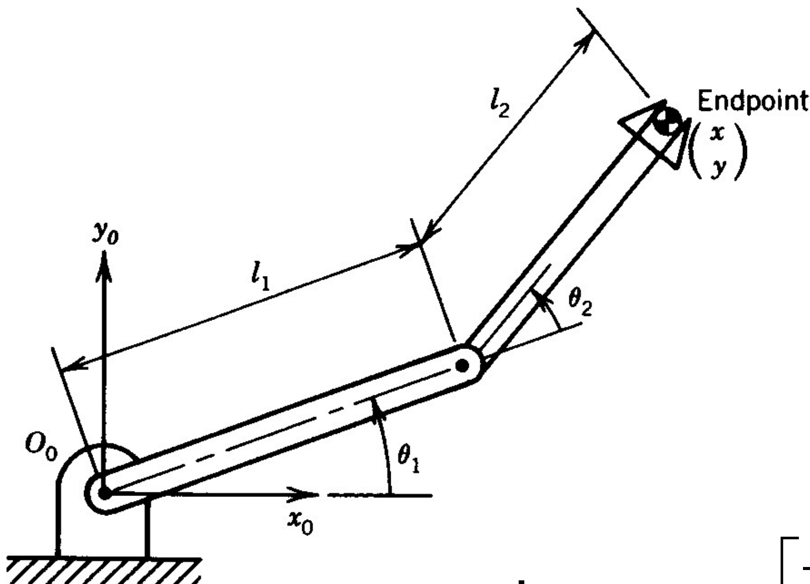
Infinitesimal end-effector displacement

$$d\mathbf{x}$$

$$\left(\mathbf{v}\right)$$

Differential Relationships

- For the 2 dof planar manipulator,



$$\mathbf{J} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\mathbf{v} = \mathbf{J}\dot{\theta} \qquad (3.1)$$

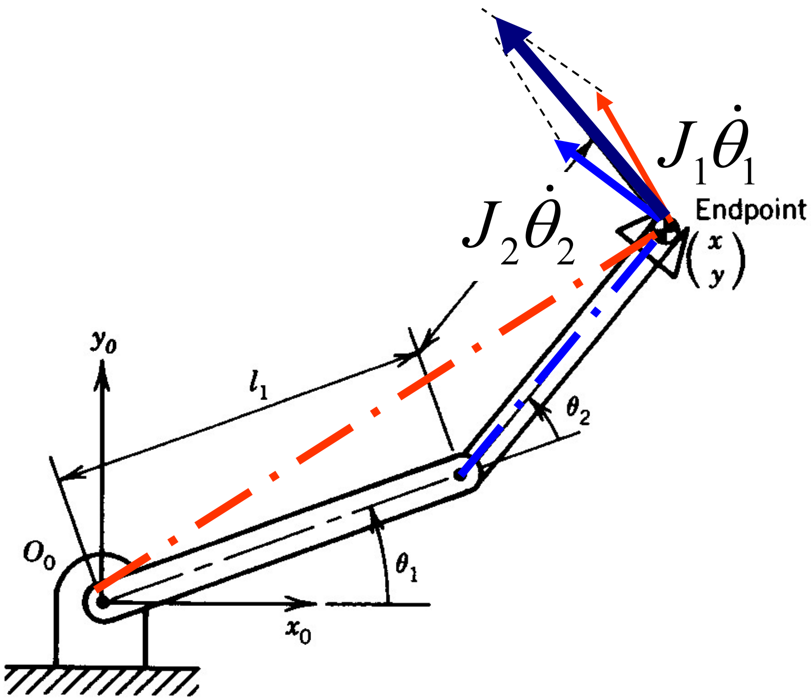
$$\mathbf{v} = J_1\dot{\theta}_1 + J_2\dot{\theta}_2$$

where $J_1 = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}, J_2 = \begin{bmatrix} -l_2 \sin(\theta_1 + \theta_2) \\ l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$

Differential Relationships

- For the 2 dof planar manipulator,

$$\mathbf{v} = J_1\dot{\theta}_1 + J_2\dot{\theta}_2$$



Infinitesimal Rotations

- Difference between finite and infinitesimal rotations

3x3 rotation matrix representing infinitesimal rotation $d\phi_x$ about the x axis:

$$R_x(d\phi_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(d\phi_x) & -\sin(d\phi_x) \\ 0 & \sin(d\phi_x) & \cos(d\phi_x) \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{bmatrix}$$

Similarly, for infinitesimal rotations about y & z axes:

$$d\phi_y \rightarrow R_y(d\phi_y)$$
$$d\phi_z \rightarrow R_z(d\phi_z)$$

Infinitesimal Rotations

- For consecutive rotations about x & y axes,

$$R_x(d\phi_x)R_y(d\phi_y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_x \\ 0 & d\phi_x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d\phi_y \\ 0 & 1 & 0 \\ -d\phi_y & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & d\phi_y \\ 0 & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix}$$

Note: Higher order term $d\phi_x \, d\phi_y$ is neglected.

Infinitesimal Rotations

- Can easily check that:

$$R_x(d\phi_x)R_y(d\phi_y) = R_y(d\phi_y)R_x(d\phi_x)$$

=> Infinitesimal rotations do not depend on the order of rotations (they commute)

In general,

$$\boxed{R(d\phi_x, d\phi_y, d\phi_z) = \begin{bmatrix} 1 & -d\phi_z & d\phi_y \\ d\phi_z & 1 & -d\phi_x \\ -d\phi_y & d\phi_x & 1 \end{bmatrix}}$$

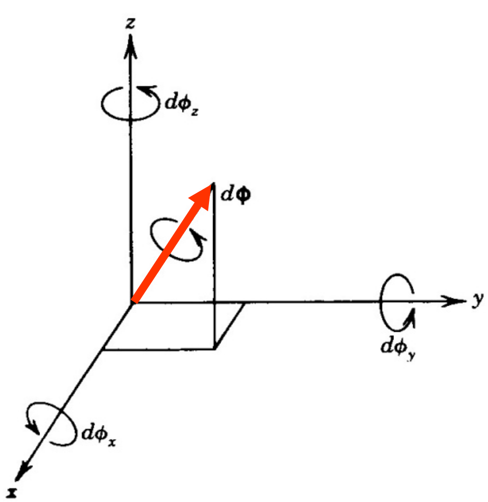
Infinitesimal Rotations

- Note:
  - Order of rotations is not important
  - Infinitesimal rotations are also additive:

$$R(d\phi_x, d\phi_y, d\phi_z)R(d\phi'_x, d\phi'_y, d\phi'_z)$$
$$= R(d\phi_x + d\phi'_x, d\phi_y + d\phi'_y, d\phi_z + d\phi'_z)$$

Infinitesimal Rotations

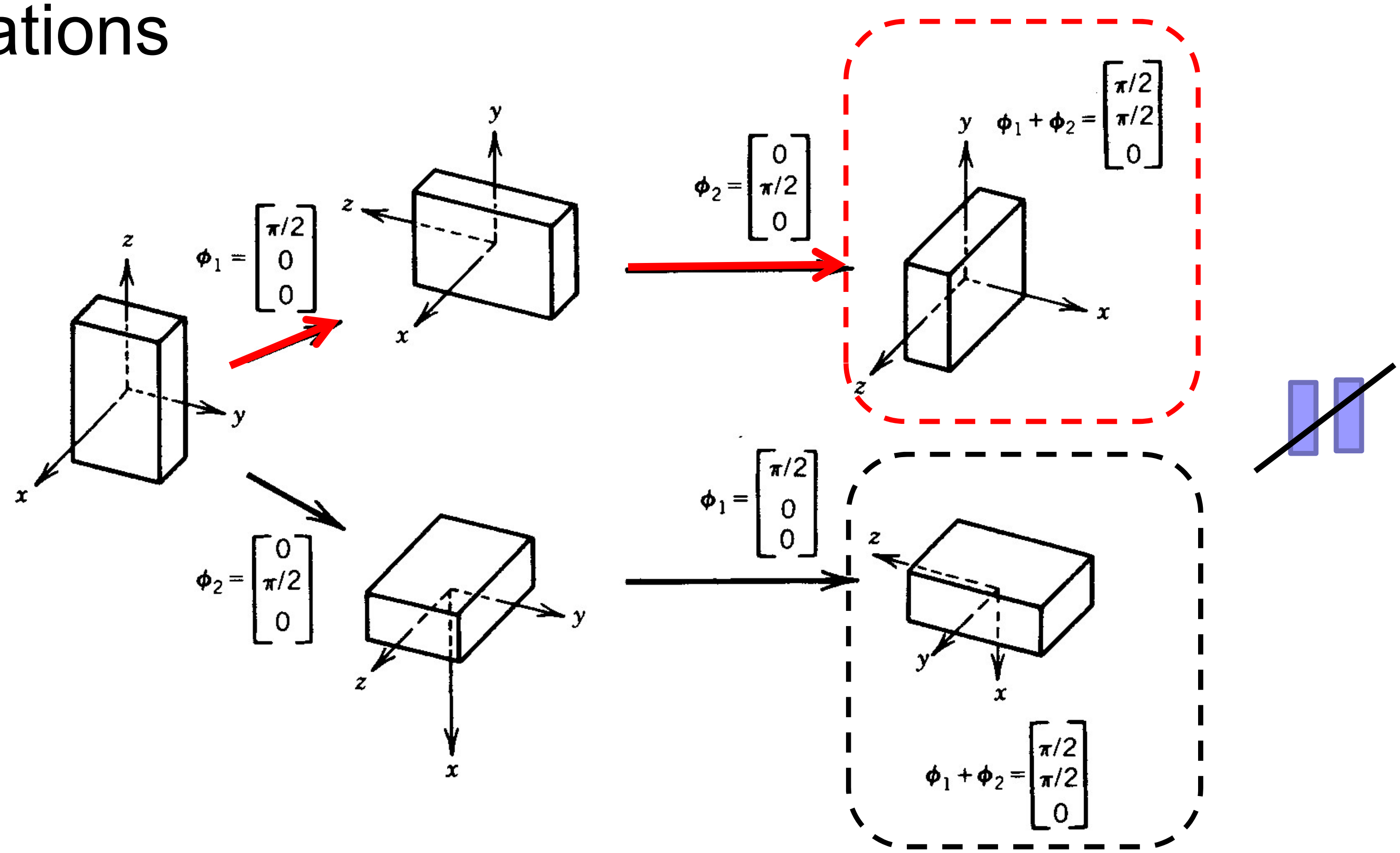
- Note: (cont)
  - $d\phi = \begin{bmatrix} d\phi_x \\ d\phi_y \\ d\phi_z \end{bmatrix}$ can be treated as vector in a vector field (because it possesses properties of a vector, eg. Associativity of addition: **u** + (**v** + **w**) = (**u** + **v**) + **w**, Commutativity of addition: **u** + **v** = **v** + **u**, etc)



- Direction of arrow represents axis of rotation
- Length represents magnitude of the rotation

Infinitesimal Rotations

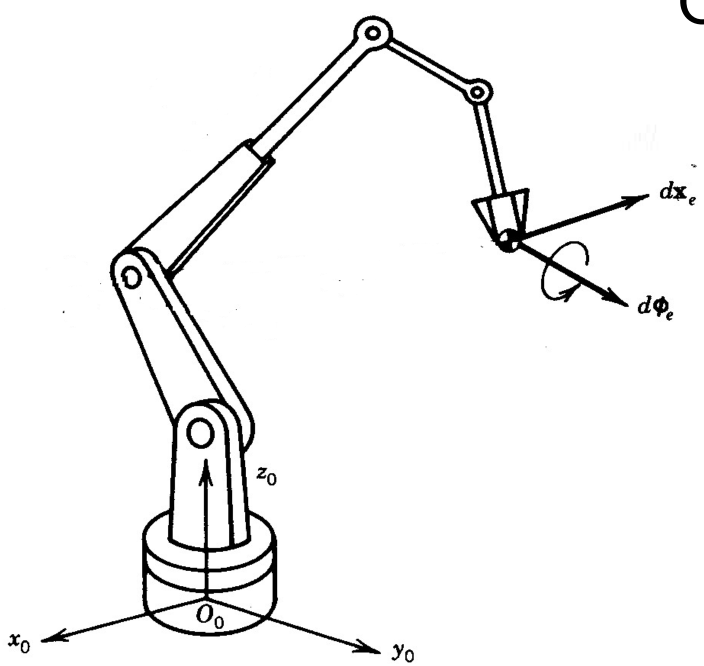
- Note: (cont)
  - Vector representation not allowed for finite rotations



Computation of Manipulator Jacobian

$$d\mathbf{p} = \begin{bmatrix} d\mathbf{x}_e \\ d\phi_e \end{bmatrix} \begin{matrix} \} \text{ Infinitesimal end-effector translation vector} \\ \} \text{ Infinitesimal end-effector rotation vector} \end{matrix}$$

Represented wrt O_o-$x_o y_o z_o$



Or $\quad \dot{\mathbf{p}} = \begin{bmatrix} \mathbf{v}_e \\ \omega_e \end{bmatrix}$

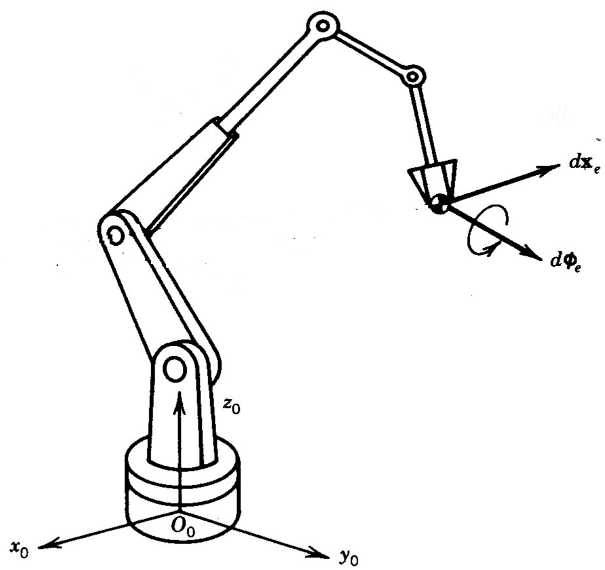
$$\dot{\mathbf{p}} = \mathbf{J}\dot{\mathbf{q}}$$

where $\dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$ is nx1 joint velocity vector

Computation of Manipulator Jacobian

J is 6xn

- first 3 row vectors => $\mathbf{v}_e$
- last 3 row vectors => ω_e
- each column vector => velocity & angular velocity generated by corresponding individual joint



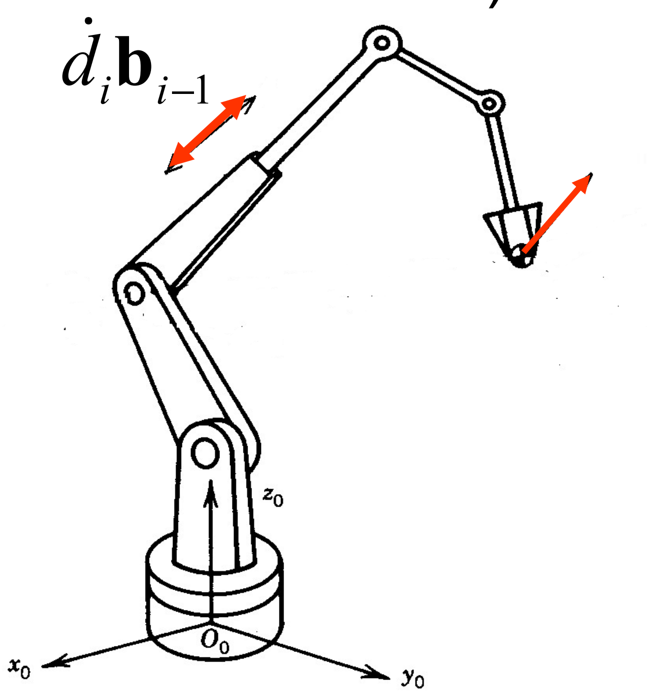
$$\begin{array}{cl} \mathbf{J} = \left[\begin{array}{c:c:c:c} J_{L1} & J_{L2} & \cdots & J_{Ln} \\ J_{A1} & J_{A2} & & J_{An} \end{array}\right] & \begin{array}{l} \text{Linear, } \mathbf{v}_e \\ \text{Angular, } \boldsymbol{\omega}_e \end{array} \\ \begin{array}{cccc} \quad\quad q_1 & q_2 & & q_n \end{array} & \end{array} \quad (3.2)$$

Computation of Manipulator Jacobian

- Linear velocity of end-effector:

$$\mathbf{v}_e = \mathbf{J}_{L1}\dot{q}_1 + \cdots + \mathbf{J}_{Ln}\dot{q}_n$$

a) For a prismatic joint:



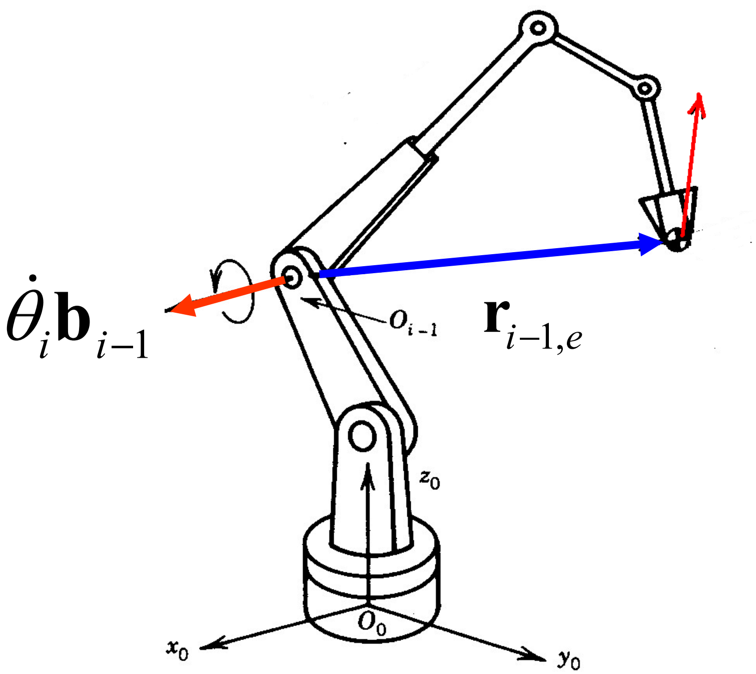
$$\boxed{\mathbf{J}_{Li}\dot{q}_i = \dot{d}_i\mathbf{b}_{i-1}}$$

where $\mathbf{b}_{i-1}$ is the unit vector along z-axis of frame {i-1} (along joint axis *i*) expressed in $O_ox_oy_oz_o$ and $\dot{d}_i$ is the scalar joint velocity

Computation of Manipulator Jacobian

- Linear velocity of end-effector (cont):

  b) For a revolute joint (rotates the composite of distal links from links *i* to *n* at angular velocity $\omega_i = \dot{\theta}_i \mathbf{b}_{i-1}$):



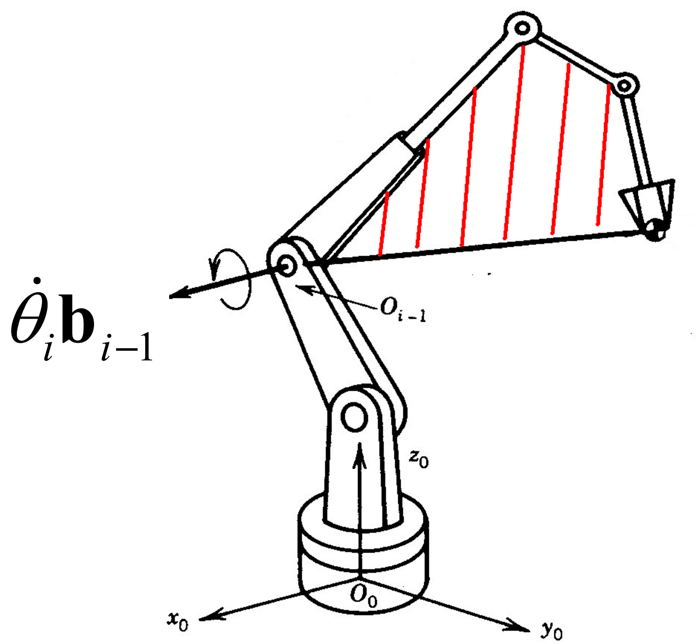
$$\boxed{\mathbf{J}_{Li}\dot{q}_i = \omega_i \times \mathbf{r}_{i-1,e} = \left(\mathbf{b}_{i-1} \times \mathbf{r}_{i-1,e}\right)\dot{\theta}_i}$$

where $\mathbf{r}_{i-1,e}$ is the position vector from O_{i-1} to end-effector (expressed in $O_o x_o y_o z_o$)

Computation of Manipulator Jacobian

- Angular velocity of end-effector:

$$\omega_e = \mathbf{J}_{A1}\dot{q}_1 + \cdots + \mathbf{J}_{An}\dot{q}_n$$



a) Prismatic joint:

$$\mathbf{J}_{Ai}\dot{q}_i = \mathbf{0}$$

b) Revolute joint:

$$\mathbf{J}_{Ai}\dot{q}_i = \omega_i = \mathbf{b}_{i-1}\dot{\theta}_i$$

Computation of Manipulator Jacobian

■<u>Summary:</u>

a) Prismatic joint: $$\begin{bmatrix} \mathbf{J}_{Li} \\ \mathbf{J}_{Ai} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{i-1} \\ \mathbf{0} \end{bmatrix} \qquad (3\text{-}3)$$

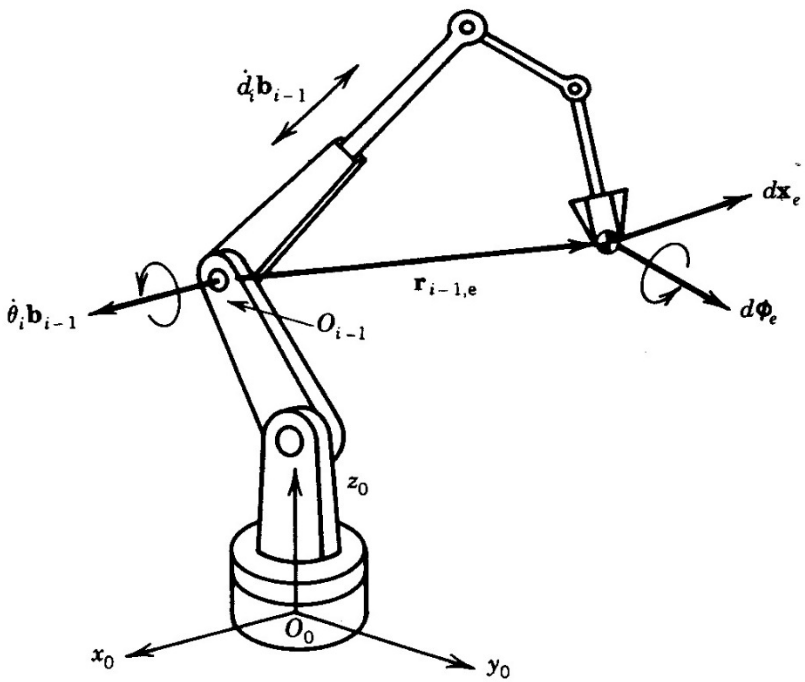
b) Revolute joint: $$\begin{bmatrix} \mathbf{J}_{Li} \\ \mathbf{J}_{Ai} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{i-1} \times \mathbf{r}_{i-1,e} \\ \mathbf{b}_{i-1} \end{bmatrix} \qquad (3\text{-}4)$$

Note: Elements of Jacobian are in general functions of joint displacements (i.e., Jacobian is configuration-dependent).

Computation of Manipulator Jacobian

- General approach to obtain $\mathbf{b}_{i-1}$ and $\mathbf{r}_{i-1,e}$:

$$\mathbf{b}_{i-1} = {}^{0}_{1}\mathbf{R}(q_1)\cdots{}^{i-2}_{i-1}\mathbf{R}(q_{i-1})\bar{\mathbf{b}} = {}^{0}_{i-1}\mathbf{R}\bar{\mathbf{b}}$$



where $\bar{\mathbf{b}} = [0 \quad 0 \quad 1]^T$

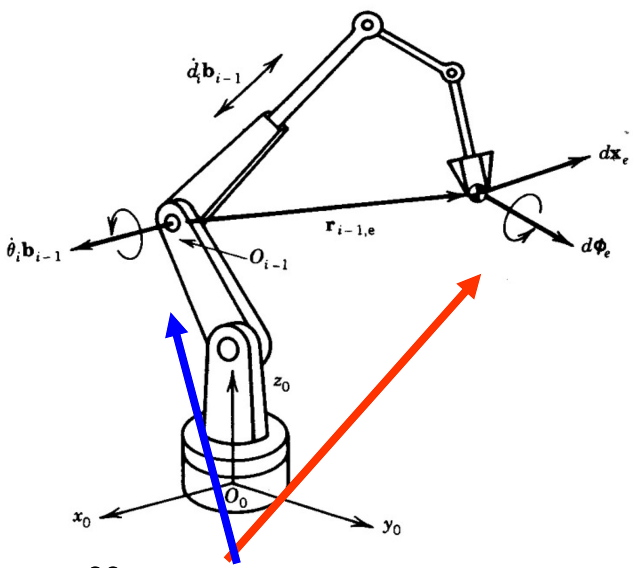
$\mathbf{b}_{i-1}$ is the third column of ${}^{0}_{i-1}\mathbf{R}$

Computation of Manipulator Jacobian

$\mathbf{r}_{i-1,e}$ can be computed using 4x4 homogeneous matrices $\,{}^{j-1}_{\ \ j}\mathbf{A}(q_j)$

Let $\mathbf{X}_{i-1,e}$ be 4x1 augmented vector of $\mathbf{r}_{i-1,e}$ and $\overline{\mathbf{X}} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$

$$\mathbf{X}_{i-1,e} = \underbrace{{}^{0}_{1}\mathbf{A}(q_1)\cdots{}^{n-1}_{\ \ n}\mathbf{A}(q_n)\overline{\mathbf{X}}}_{\text{Position vector from origin } O_o \text{ to end-effector}} - \underbrace{{}^{0}_{1}\mathbf{A}(q_1)\cdots{}^{i-2}_{i-1}\mathbf{A}(q_{i-1})\overline{\mathbf{X}}}_{\text{Position vector from origin } O_o \text{ to } O_{i-1}}$$

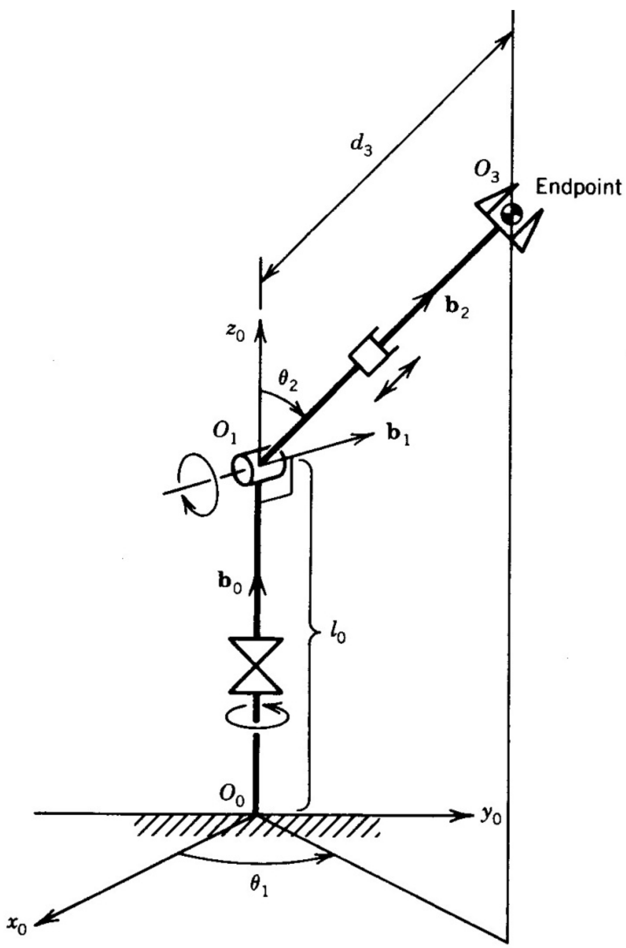


Position vector from origin O_o to end-effector

Position vector from origin O_o to O_{i-1}

Computation of Manipulator Jacobian

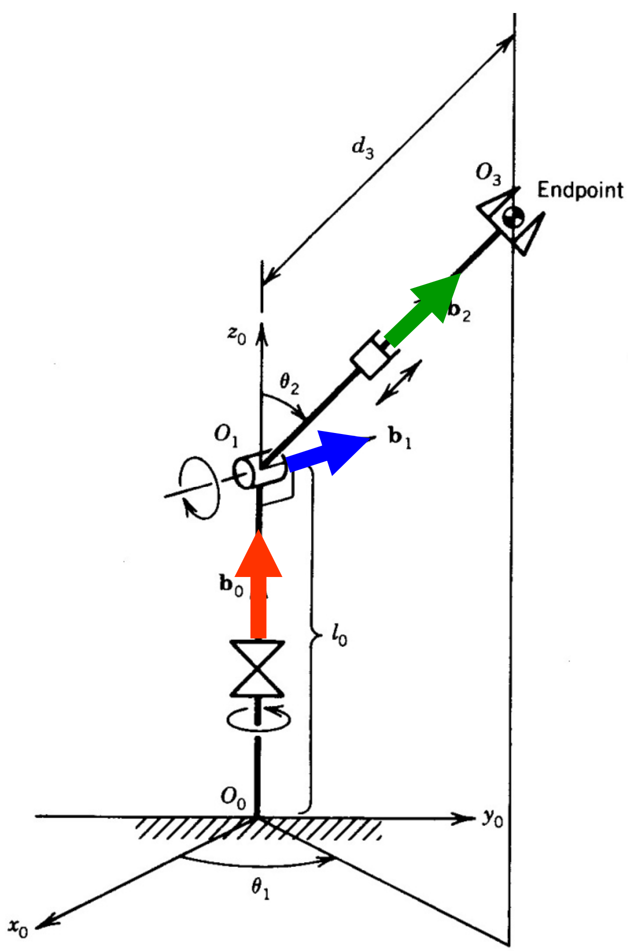
Example 3-1: To find the Jacobian matrix of the following polar coordinate manipulator:



Joint displacements are θ_1, θ_2 and d_3

Computation of Manipulator Jacobian

Solution:

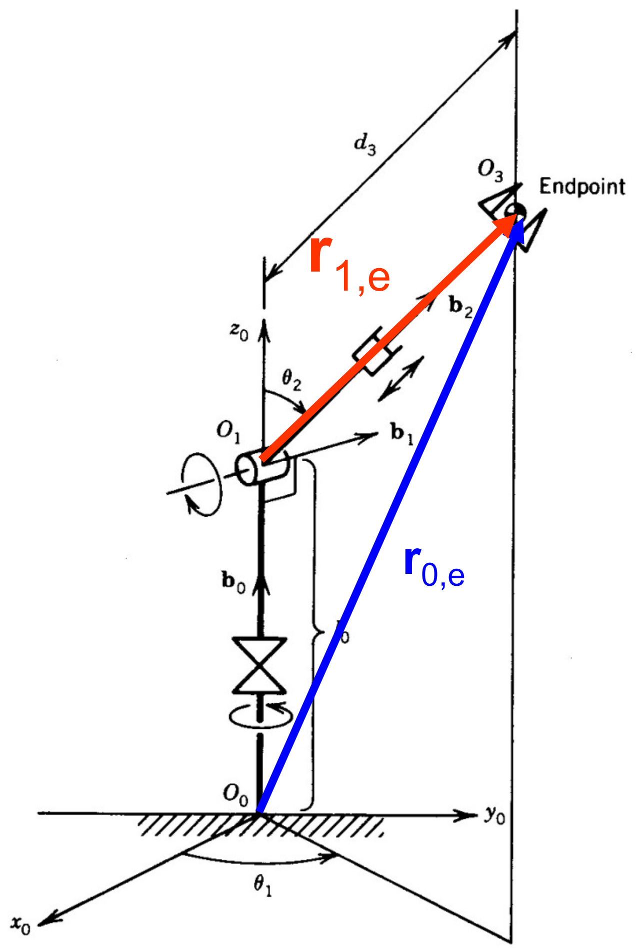


First determine the joint axes directions (expressed in $O_o x_o y_o z_o$):

$$\mathbf{b}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

Computation of Manipulator Jacobian

Solution: (cont)



For revolute joints, need to find $\mathbf{r}_{i\text{-}1,e}$:

For θ_1, $\mathbf{r}_{0,e} = l_0\mathbf{b}_0 + d_3\mathbf{b}_2$

For θ_2, $\mathbf{r}_{1,e} = d_3\mathbf{b}_2$

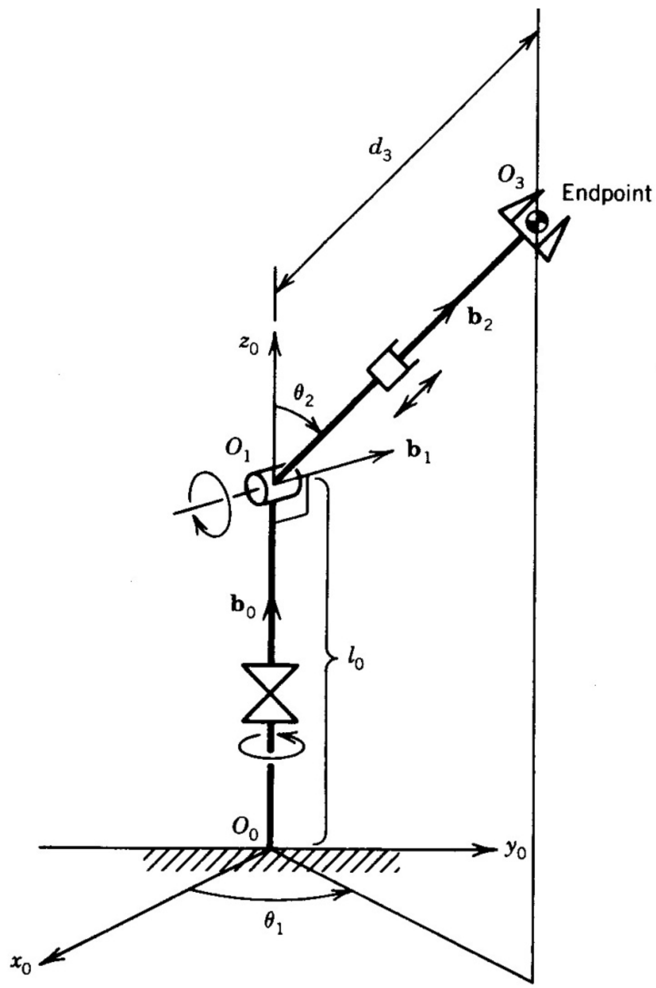
Substitute above into Eqs (3-3) and (3-4)

For θ_1 ,

$$\begin{bmatrix} \mathbf{J}_{L1} \\ \mathbf{J}_{A1} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \times \mathbf{r}_{0,e} \\ \mathbf{b}_0 \end{bmatrix} = \begin{bmatrix} -d_3 s_1 s_2 \\ d_3 c_1 s_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Computation of Manipulator Jacobian

Solution: (cont)

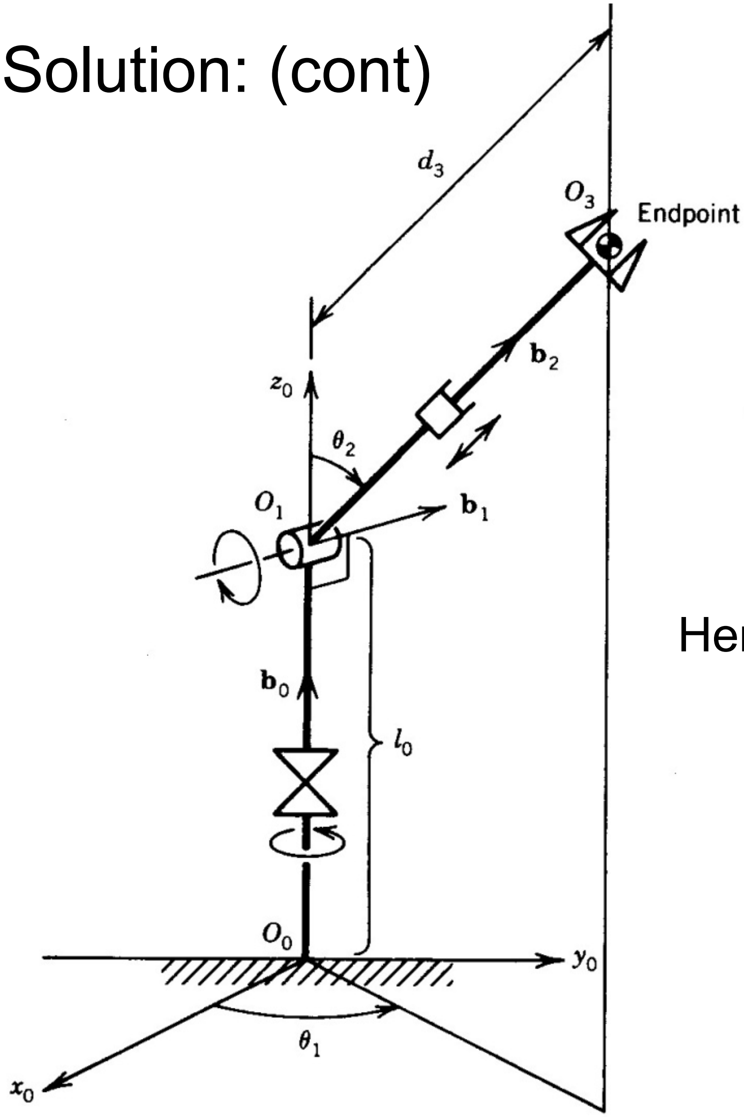


For θ_2 ,
$$\begin{bmatrix} \mathbf{J}_{L2} \\ \mathbf{J}_{A2} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \times \mathbf{r}_{1,e} \\ \mathbf{b}_1 \end{bmatrix} = \begin{bmatrix} d_3 c_1 c_2 \\ d_3 s_1 c_2 \\ -d_3 s_2 \\ -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

For d_3,
$$\begin{bmatrix} \mathbf{J}_{L3} \\ \mathbf{J}_{A3} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_2 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Computation of Manipulator Jacobian

Solution: (cont)



Hence,

$$\mathbf{J} = \begin{bmatrix} -d_3 s_1 s_2 & d_3 c_1 c_2 & c_1 s_2 \\ d_3 c_1 s_2 & d_3 s_1 c_2 & s_1 s_2 \\ 0 & -d_3 s_2 & c_2 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Inverse Instantaneous Kinematics

■Resolved Motion Rate

Consider a 6 degree-of-freedom manipulator,
From earlier, $\dot{\mathbf{p}} = \mathbf{J}\dot{\mathbf{q}}$, where **J** is 6x6 square matrix

If **J** is non-singular, $\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{p}}$

This gives the required individual joint velocities
to obtain a given end-effector velocity $\dot{\mathbf{p}}$

The control scheme to generate the end-effector velocity based on this approach is called "Resolved motion rate control" (Whitney, 1969)

Resolved Motion Rate

At singular configuration, **J** not full rank

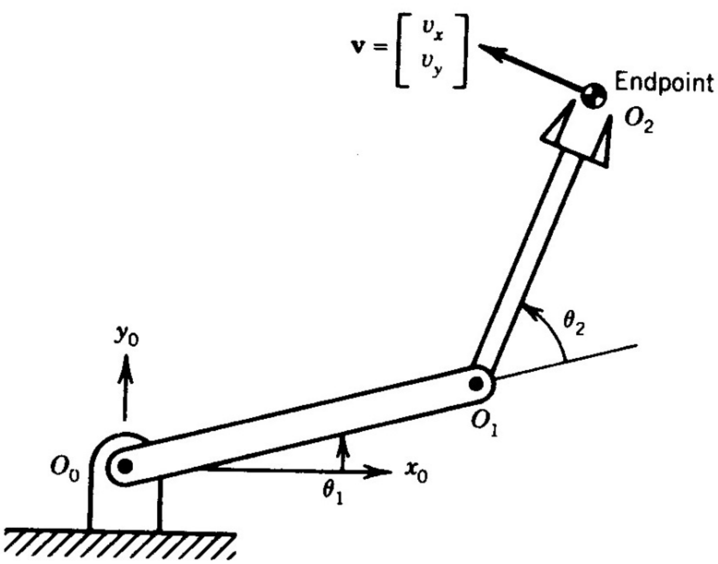
=> column vectors are linearly dependent (do not span the whole 6-dimensional vector space of $\dot{\mathbf{p}}$)

=> **J** cannot be inversed

Resolved Motion Rate

- **Example 4-1:** Consider the 2 dof planar manipulator as shown below with length of each link equal 1 and endpoint velocity denoted by v =[v_x, v_y]^T :

a) Find the joint velocities that produce the desired endpoint velocity;



From Eq (3-1),

$$\mathbf{J} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

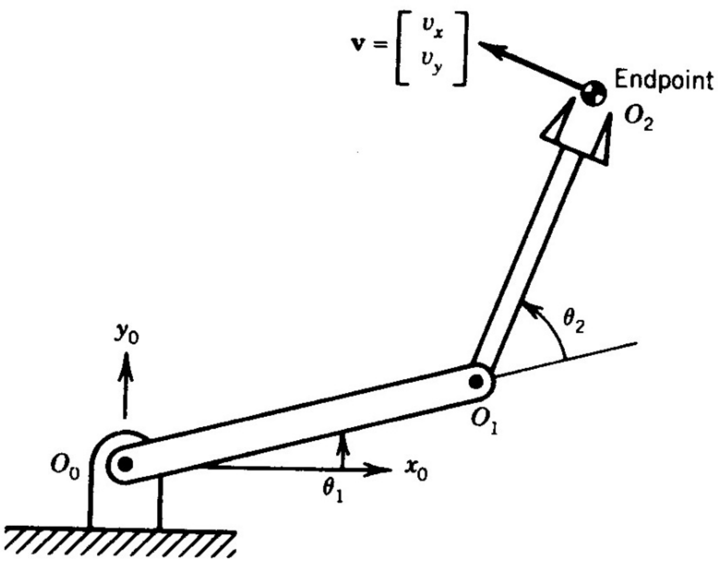
$$= \begin{bmatrix} -\sin\theta_1 - \sin(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \cos\theta_1 + \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\mathbf{J}^{-1} = \frac{1}{\det(\mathbf{J})} \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\cos\theta_1 - \cos(\theta_1+\theta_2) & -\sin\theta_1 - \sin(\theta_1+\theta_2) \end{bmatrix}$$

$$= \frac{1}{\sin\theta_2} \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\cos\theta_1 - \cos(\theta_1+\theta_2) & -\sin\theta_1 - \sin(\theta_1+\theta_2) \end{bmatrix}$$

Resolved Motion Rate

- Example 4-1(a) (cont)



$$\dot{\theta} = \mathbf{J}^{-1}\mathbf{v}$$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \frac{1}{\sin\theta_2} \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) \\ -\cos\theta_1 - \cos(\theta_1+\theta_2) & -\sin\theta_1 - \sin(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{v_x\cos(\theta_1+\theta_2) + v_y\sin(\theta_1+\theta_2)}{\sin\theta_2} \\ -\dfrac{v_x[\cos\theta_1 + \cos(\theta_1+\theta_2)] + v_y[\sin\theta_1 + \sin(\theta_1+\theta_2)]}{\sin\theta_2} \end{bmatrix}$$

Resolved Motion Rate

- Example 4-1: b) Find all the singular configurations, and determine the direction along which the endpoint cannot move for each of these configurations.

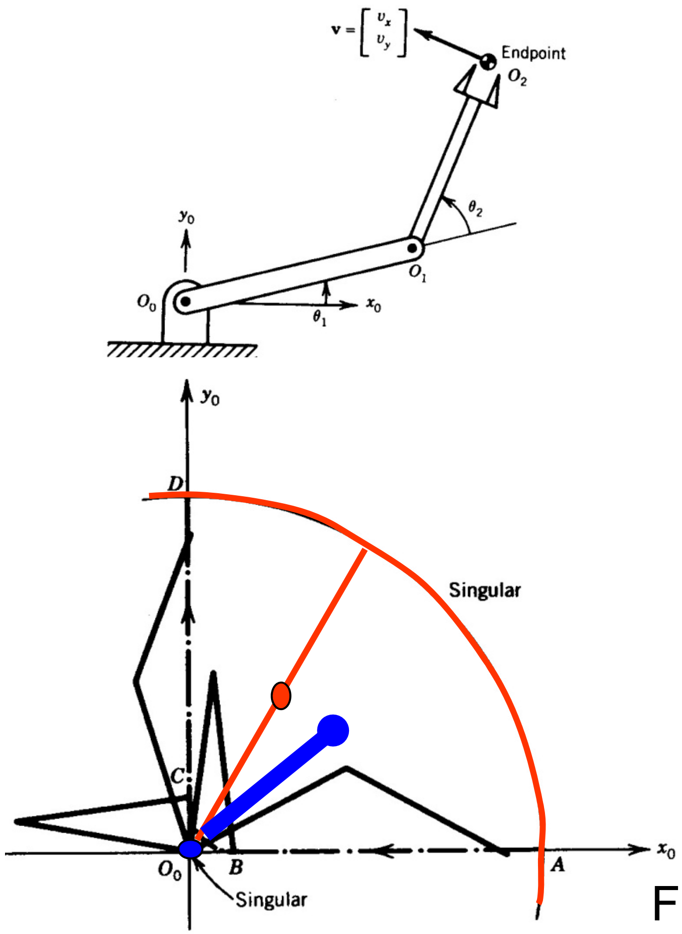


Figure 4-1

Singularity → when det(**J**) = 0

det(**J**) = sin θ_2 = 0

=> θ_2 = 0 or π

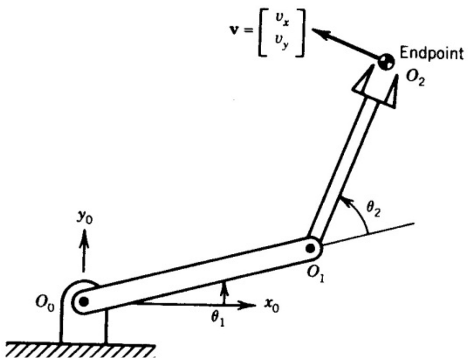
=> arm fully extended or fully contracted (see Figure 4-1)

(singular configurations when endpoint is at origin O_O and boundary of reachable space)

Resolved Motion Rate

- Example 4-1: b) (cont)

At singular configuration which corresponds to $\theta_2 = 0$,

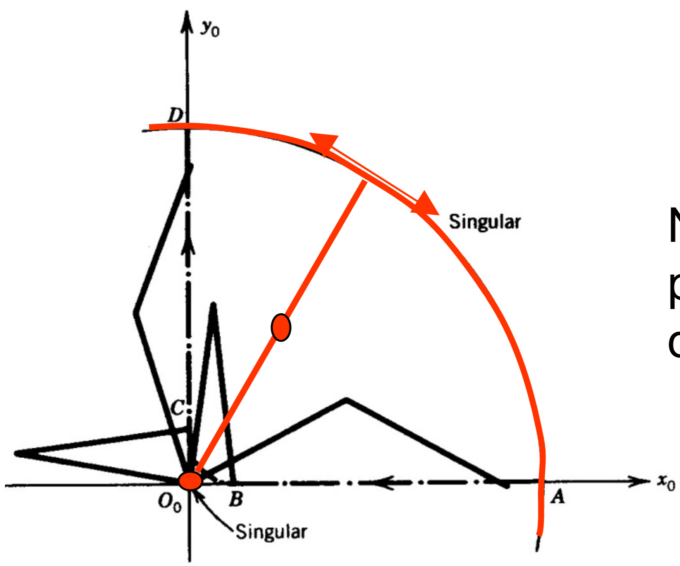


$$\mathbf{J} = \begin{bmatrix} -\sin\theta_1 - \sin(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \cos\theta_1 + \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$= \begin{bmatrix} -2\sin\theta_1 & -\sin(\theta_1) \\ 2\cos\theta_1 & \cos(\theta_1) \end{bmatrix}$$

That is,

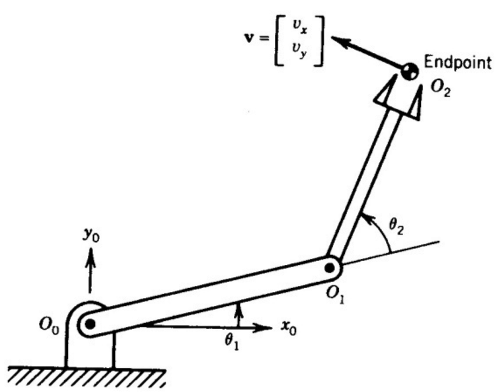
$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -2\sin\theta_1 \\ 2\cos\theta_1 \end{bmatrix}\dot{\theta}_1 + \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}\dot{\theta}_2 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix}\left(2\dot{\theta}_1 + \dot{\theta}_2\right)$$



Note: The two columns of Jacobian matrix become parallel and the endpoint can only move in the direction perpendicular to the arm links

Resolved Motion Rate

- Example 4-1: c) Find the profiles of joint velocities when the endpoint is required to track the trajectory ABCD (as shown in Figure 4-1) at a constant tangential speed.



First obtain the joint angles that correspond to each endpoint position on the trajectory.
Then,

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{v_x \cos(\theta_1 + \theta_2) + v_y \sin(\theta_1 + \theta_2)}{\sin\theta_2} \\ -\dfrac{v_x[\cos\theta_1 + \cos(\theta_1 + \theta_2)] + v_y[\sin\theta_1 + \sin(\theta_1 + \theta_2)]}{\sin\theta_2} \end{bmatrix}$$

Results:

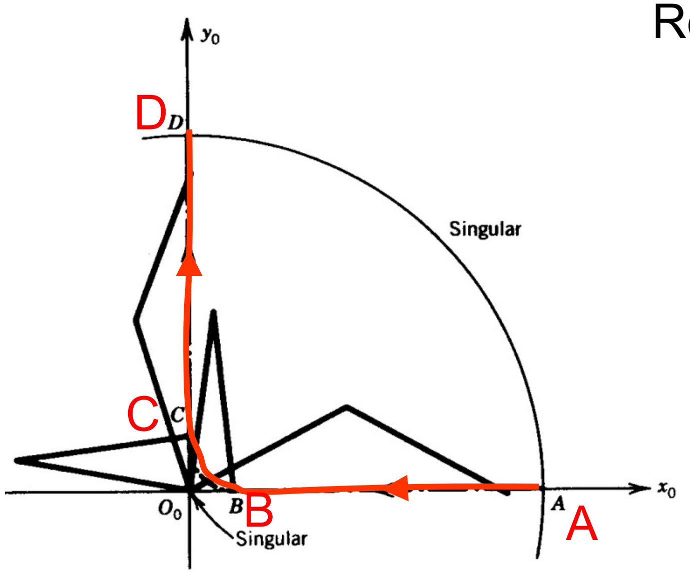
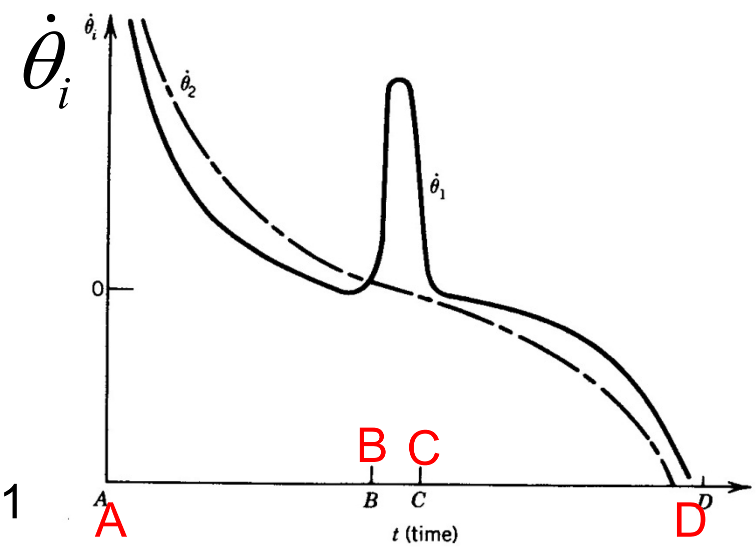
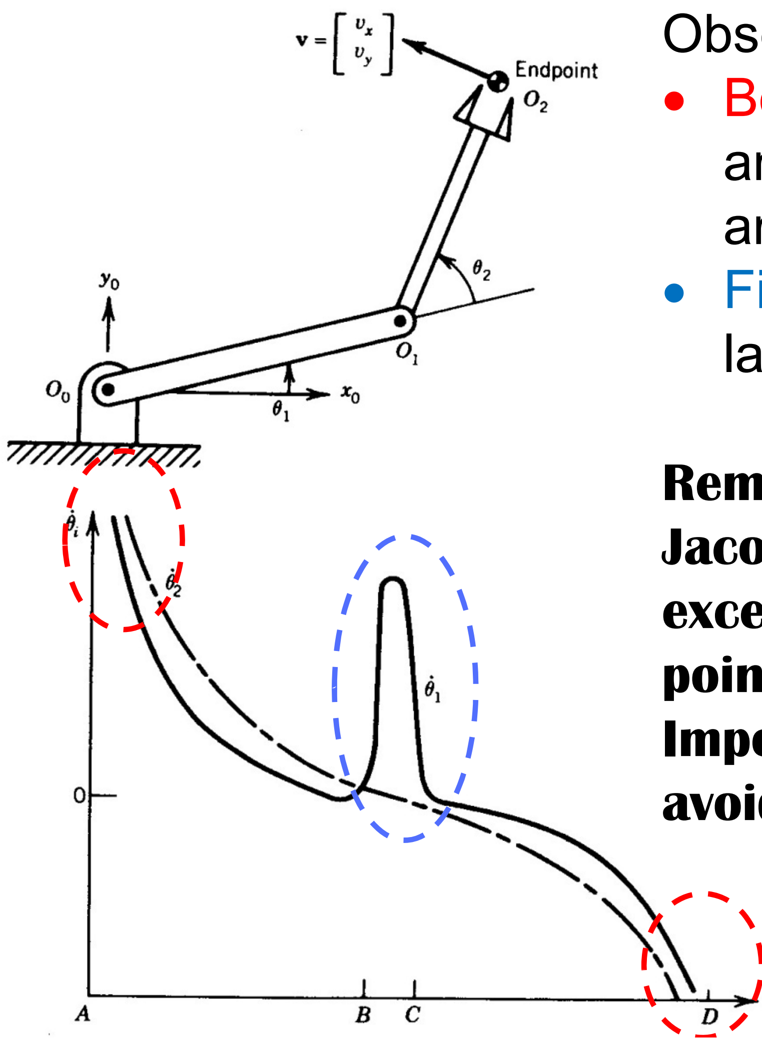


Figure 4-1



Resolved Motion Rate

- Example 4-1: c) (cont)



Observations:

- Both joint velocities near singular points A and D are excessively large (denominators are almost zero)
- First joint velocity becomes excessively large between B and C

Remark: Even if inverse of manipulator Jacobian exist, the joint velocities may become excessively large in the vicinity of singular points =>
Important to address singularities and how to avoid them

Redundancy

kinematic equation

instantaneous kinematic equation

manipulator arm having exactly 6 dof → Finite number of solutions | Unique solutions

manipulator arm having > 6 dof → Infinite no. of solutions | Infinite no. of solutions

e.g. Human arm – 7 dof
(excluding fingers' joints)

Redundancy

- Consider a general n dof manipulator,
- Given a task, it is usually required to specify m independent variables, d**p** =[dp_1 … dp_m]^T for the end-effector motion, where $m \leq 6$.

E.g. in arc welding, only 5 independent variables of torch motion need be specified (due to symmetry about its centre line)

Instantaneous kinematic equation for the n dof manipulator arm:

$$\underbrace{d\mathbf{p}}_{mx1} = \underbrace{\mathbf{J}}_{mxn} \underbrace{d\mathbf{q}}_{nx1} \qquad (4\text{-}2)$$

Redundancy

When $n>m$ **and J** is full rank

→

How many redundant dof for the given task?

Redundancy

When n>m **and J** is full rank

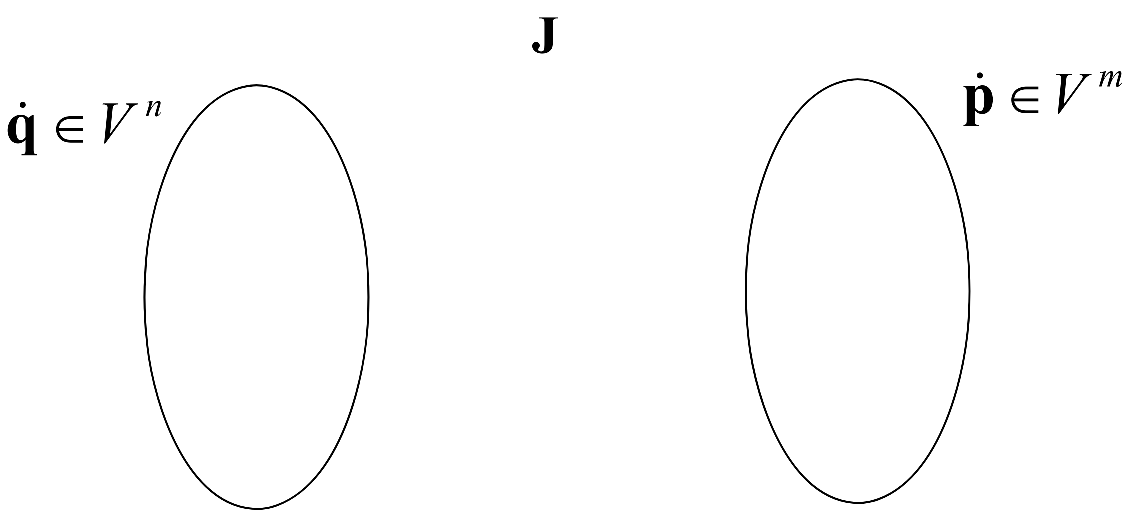
$\Rightarrow$

(n-m) redundant dof for the given task

Redundancy

$$\dot{\mathbf{p}} = \mathbf{J}\dot{\mathbf{q}} \qquad (4\text{-}3)$$

linear mapping from n-dimensional vector space V^n to m-dimensional vector space V^m

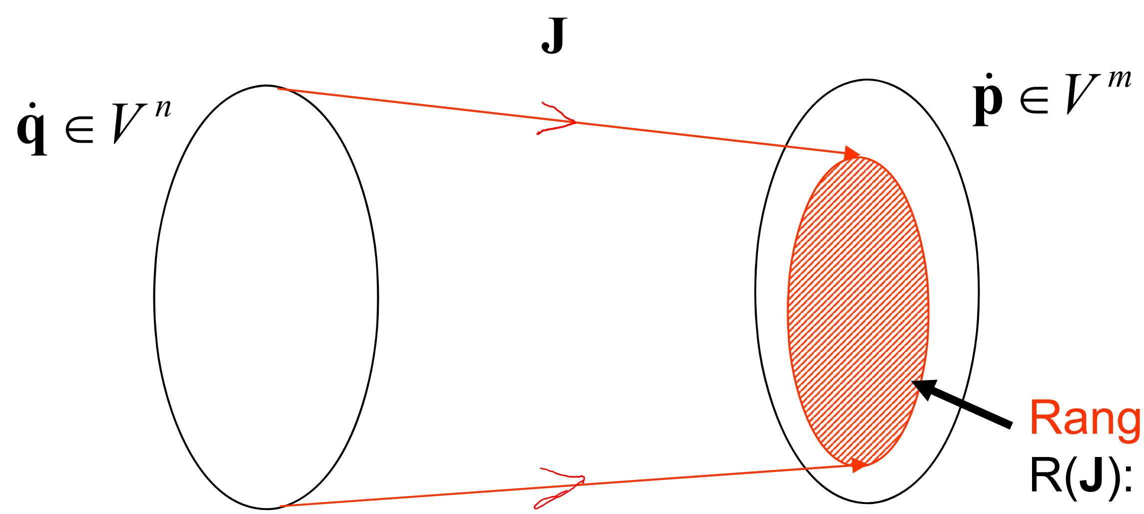


Dimension of a vector space = No. of vectors in every basis
(A basis for a vector space is a set of vectors that has 2 properties:
a) The vectors are linearly independent; b) The vectors span the vector space.)

Redundancy

$$\dot{\mathbf{p}} = \mathbf{J}\dot{\mathbf{q}} \qquad (4\text{-}3)$$

linear mapping from n-dimensional vector space V^n to m-dimensional vector space V^m

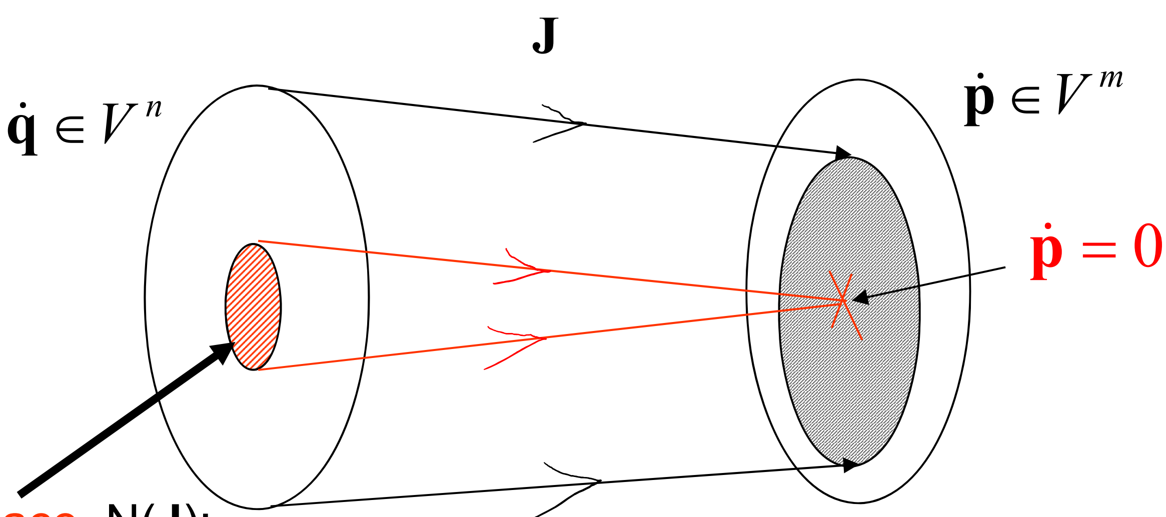


Range space, R(**J**): all possible end-effector velocities at the given manipulator posture

Redundancy

$$\dot{\mathbf{p}} = \mathbf{J}\dot{\mathbf{q}} \qquad (4\text{-}3)$$

linear mapping from n-dimensional vector space V^n to m-dimensional vector space V^m

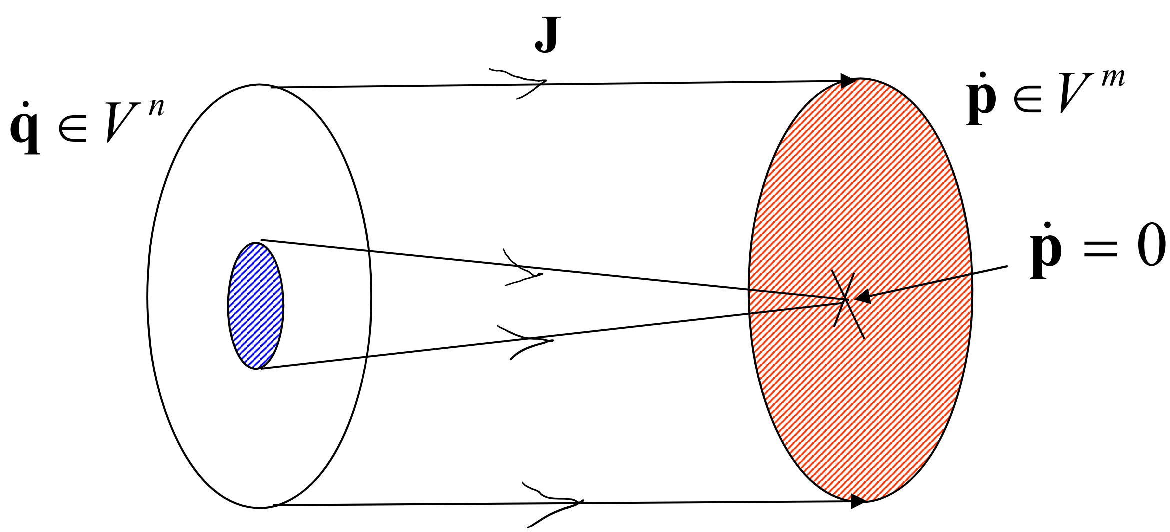


Null space, N(**J**): elements in this space mapped to zero vector in V^m, ie. $\mathbf{J}\dot{\mathbf{q}} = \mathbf{0}$

Redundancy

Assume n≥m,

- If **J** is of full rank (or full row rank), range space covers entire V^m
  - dim[1] R(**J**)=m
  - dim N(**J**) = n-m (= no. of redundant dof)

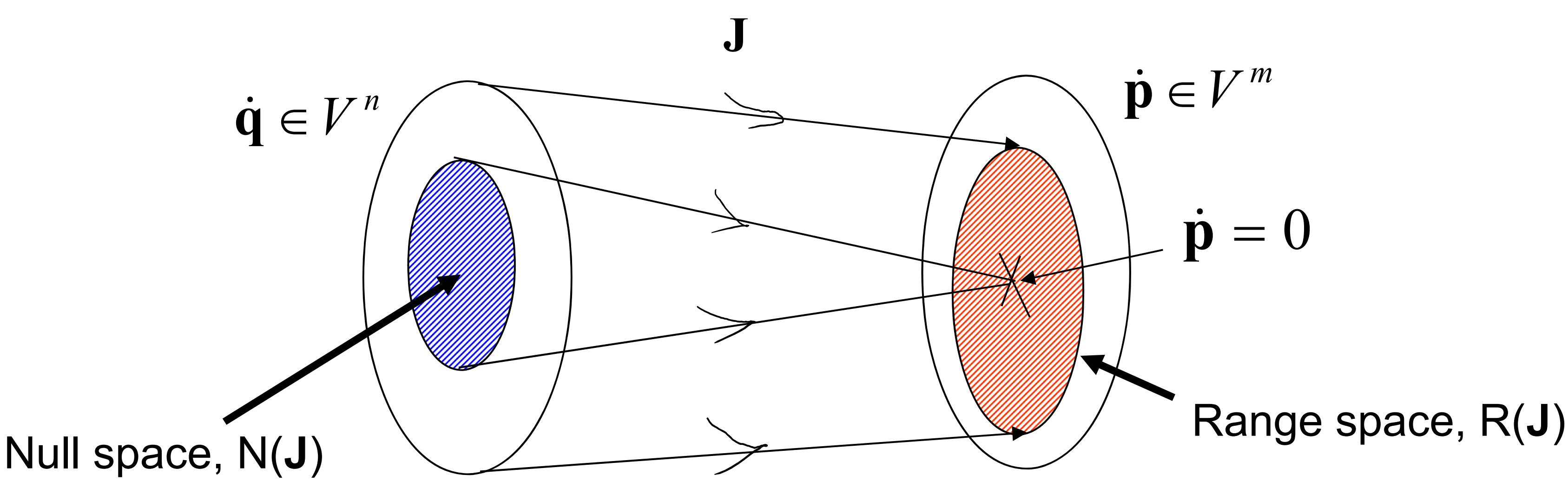


[1] "dim (vector space)" denotes "Dimension of the vector space"

Redundancy

Assume n≥m,

- If **J** is degenerate (not of full rank, singularity),
  - dim R(**J**) decreases and dim N(**J**) increases by same amount
  - dim R(**J**)+dim N(**J**) = n (always holds independent of the rank of **J**)

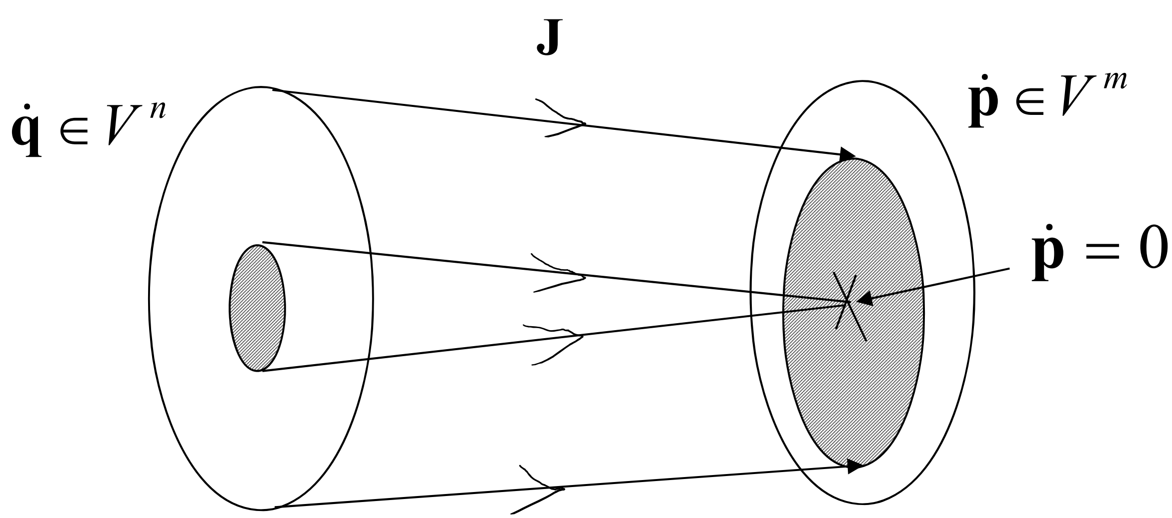


Redundancy

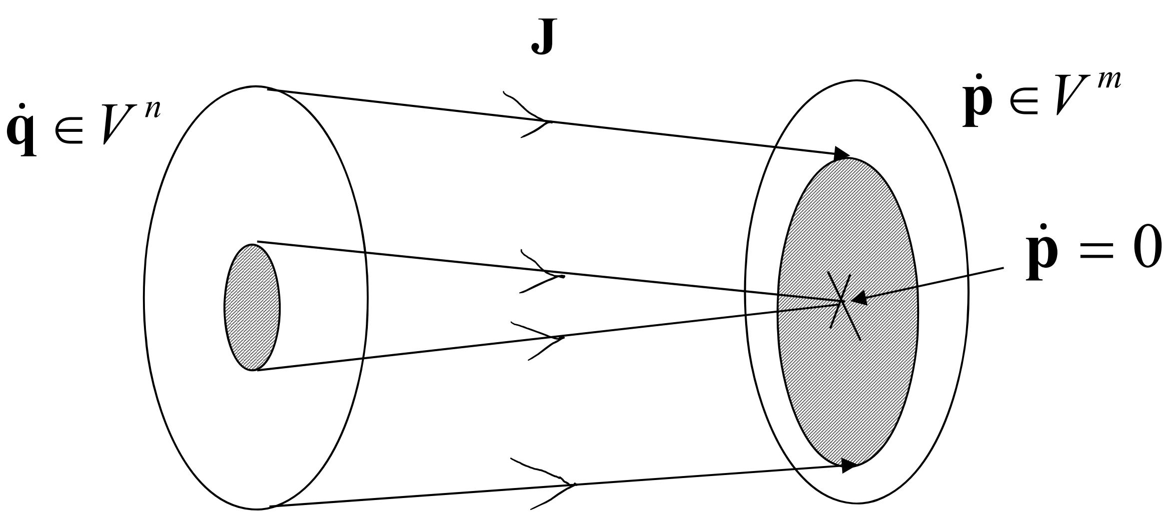
- If N(**J**) is not an empty set:

Let $\dot{\mathbf{q}}^*$ be a solution of Eq (4-3) and $\dot{\mathbf{q}}_0 \in$ N(**J**).
Then, $\dot{\mathbf{q}} = \dot{\mathbf{q}}^* + k\dot{\mathbf{q}}_0$ is also a solution of Eq (4-3)
(k - arbitrary scalar constant)

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{q}}^* + k\mathbf{J}\dot{\mathbf{q}}_0 = \mathbf{J}\dot{\mathbf{q}}^* = \dot{\mathbf{p}}$$



- **Remark:**
  - Redundancy can be utilized to avoid singular point and for optimal motion control.



Optimal Solutions

Assume n > m and **J** is of full row rank

Problem: Find $\dot{\mathbf{q}}$ that satisfies Eq (4-3) for a given $\dot{\mathbf{p}}$ and $\mathbf{J}$ while minimizing quadratic cost function of joint velocity vector:

$$E(\dot{\mathbf{q}}) = \dot{\mathbf{q}}^T \mathbf{W} \dot{\mathbf{q}}$$

where **W** - nxn symmetric positive definite weighting matrix

Method to solve this problem: Lagrange multipliers

Optimal Solutions

- Modified cost function:

$$E(\dot{\mathbf{q}},\lambda)=\dot{\mathbf{q}}^T\mathbf{W}\dot{\mathbf{q}}-\lambda^T\left(\mathbf{J}\dot{\mathbf{q}}-\dot{\mathbf{p}}\right)$$

where λ is a mx1 unknown vector (Lagrange multipliers)

Necessary conditions for optimal solution:

$$\frac{\partial E}{\partial \dot{\mathbf{q}}}=\mathbf{0}\Rightarrow 2\mathbf{W}\dot{\mathbf{q}}-\mathbf{J}^T\lambda=\mathbf{0} \quad \Rightarrow \quad \dot{\mathbf{q}}=\frac{1}{2}\mathbf{W}^{-1}\mathbf{J}^T\lambda$$

$$\frac{\partial E}{\partial \lambda}=\mathbf{0}\Rightarrow \mathbf{J}\dot{\mathbf{q}}-\dot{\mathbf{p}}=\mathbf{0} \quad \Rightarrow \frac{1}{2}(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)\lambda-\dot{\mathbf{p}}=\mathbf{0}$$

$$\lambda=2(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T)^{-1}\dot{\mathbf{p}}$$

$\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^T$ a full-rank square matrix (invertible) because **J** is assumed to be of full row-rank

Optimal Solutions

$$\dot{\mathbf{q}} = \mathbf{W}^{-1}\mathbf{J}^{T}(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^{T})^{-1}\dot{\mathbf{p}}$$

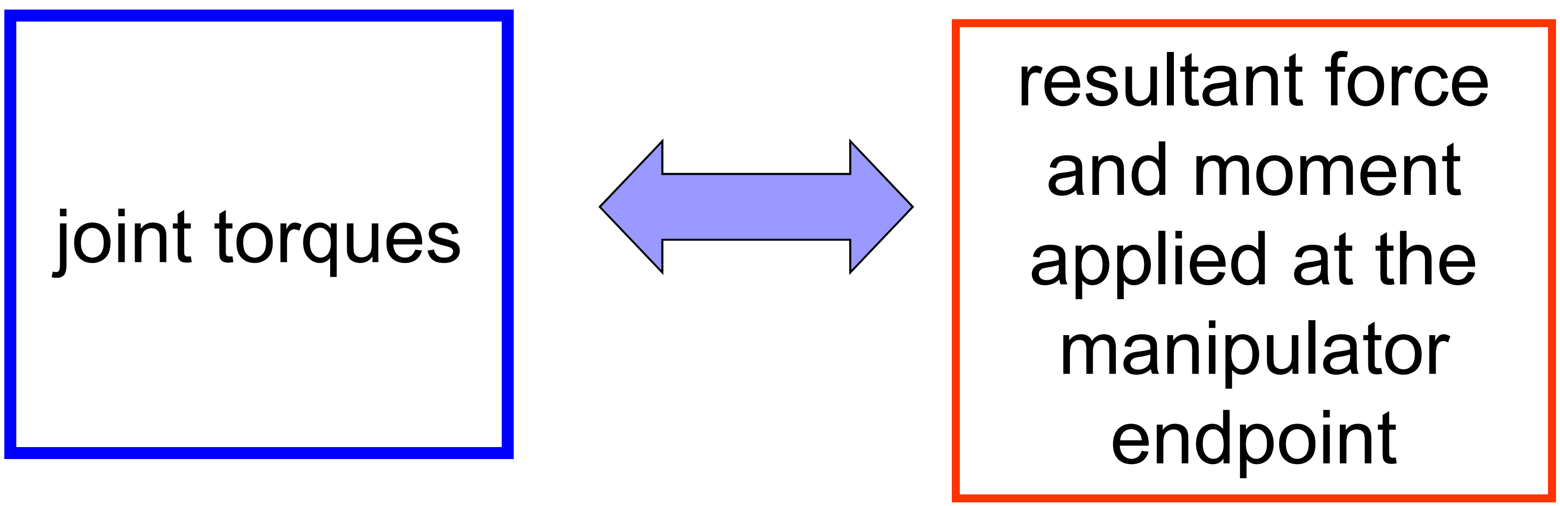
If **W** is mxm identity matrix,

$$\dot{\mathbf{q}} = \underbrace{\mathbf{J}^{T}(\mathbf{J}\mathbf{J}^{T})^{-1}}\dot{\mathbf{p}}$$

Pseudo-inverse of **J**

3. Statics

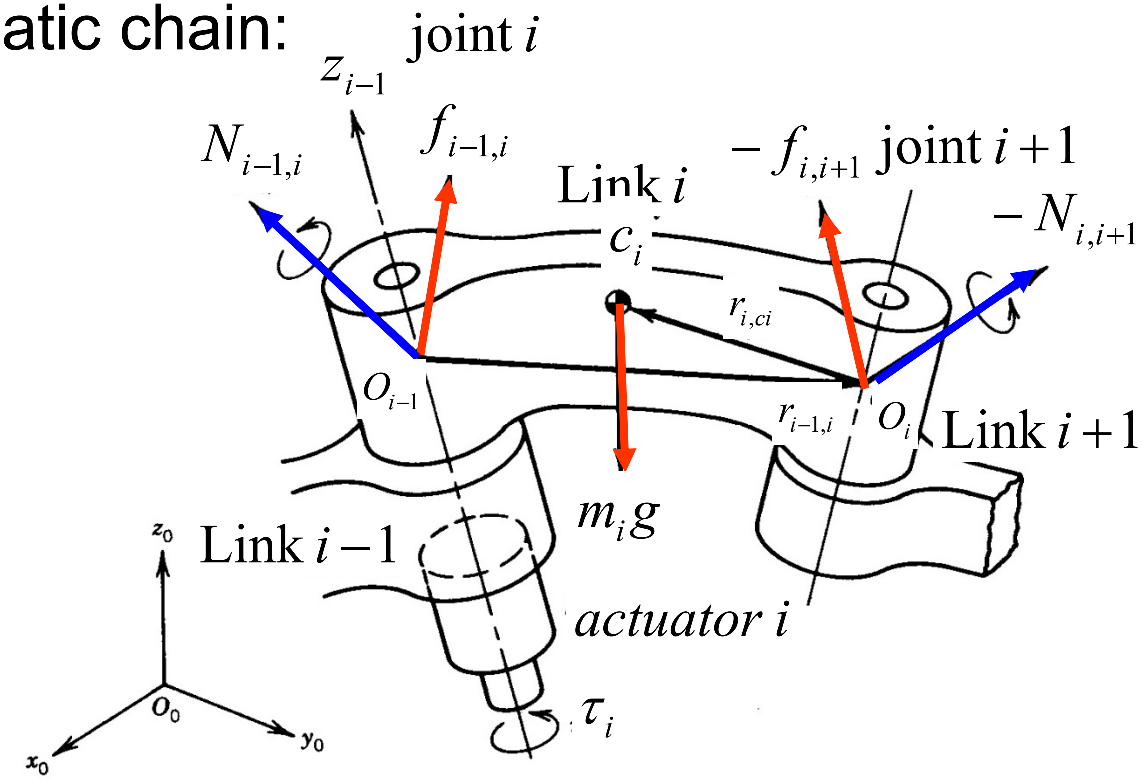
- Concern with Forces and moments which act on a manipulator arm when it is at rest
- Also:



Balance of Forces and Moments

- To derive basic equations that govern the static behavior of a manipulator arm

Consider free body diagram of an individual link of an open kinematic chain:

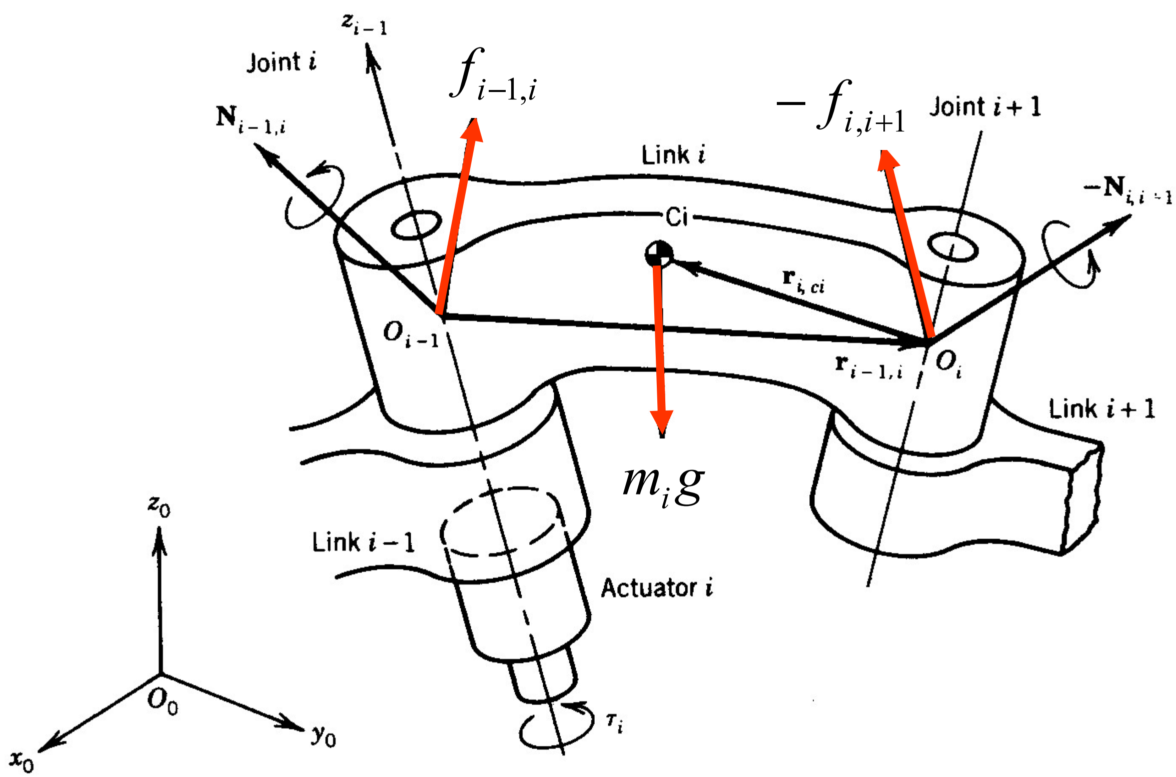


Balance of Forces and Moments

- Balance of linear forces:

$$\mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} + m_i\mathbf{g} = \mathbf{0} \qquad i = 1,\ldots,n \qquad (5\text{-}1)$$

(all vectors expressed in O_o-$x_oy_oz_o$)

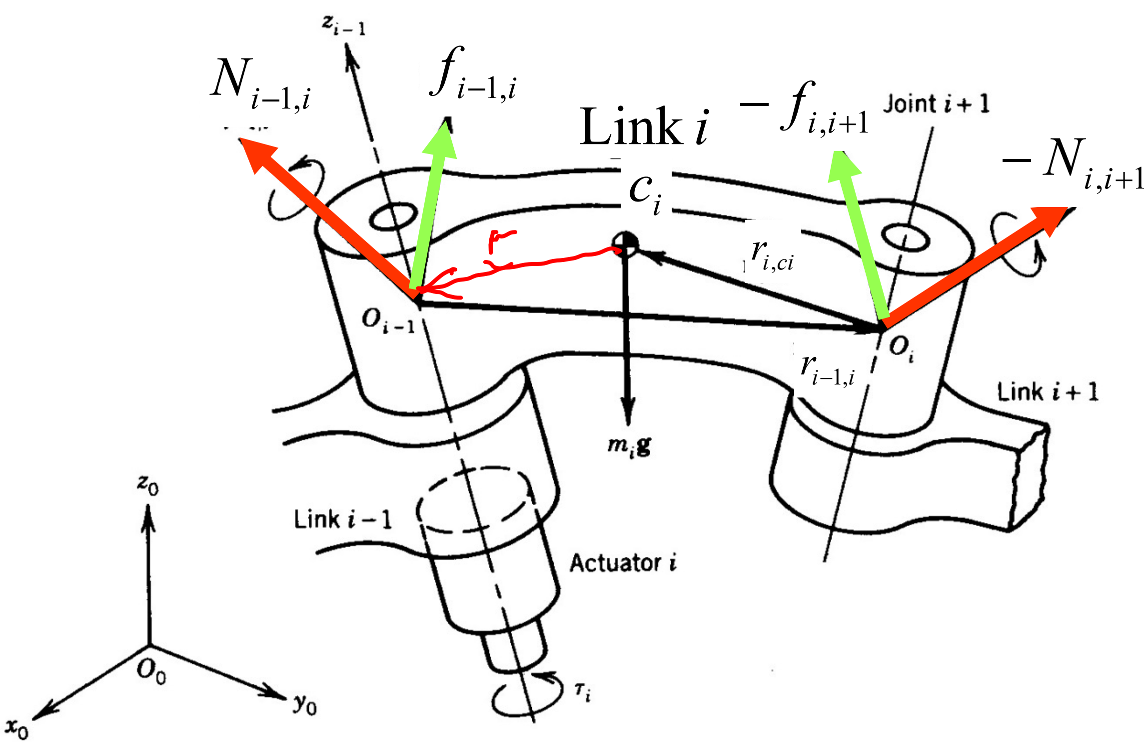


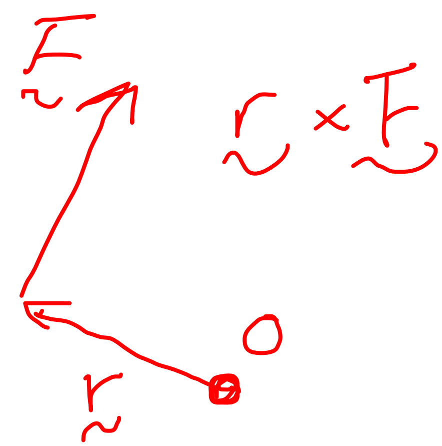
Balance of Forces and Moments

Balance of moments about centroid C_i:

$$\mathbf{N}_{i-1,i} - \mathbf{N}_{i,i+1} - \underbrace{(\mathbf{r}_{i-1,i} + \mathbf{r}_{i,ci})}_{\underset{\sim}{r}} \text{x } \mathbf{f}_{i-1,i} + (-\mathbf{r}_{i,ci}) \text{x} (-\mathbf{f}_{i,i+1}) = 0,\ i = 1,\ldots,n \qquad (5\text{-}2)$$

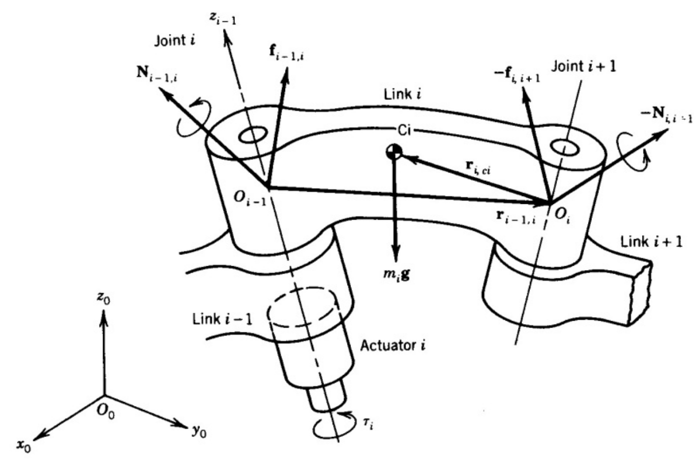
(all vectors expressed in O_o-$x_oy_oz_o$)





Balance of Forces and Moments

- $\mathbf{f}_{i-1,i}$ and $\mathbf{N}_{i-1,i}$ are called the coupling force and moment, respectively, between the adjacent links i and i-1
- Above two equations can be derived for all the link members except base link, $i = 1,\ldots,n$. => 2n vector equations
- Number of coupling forces and moments involved is 2(n+1) => two of the coupling forces and moments must be specified for the equations to be solved.

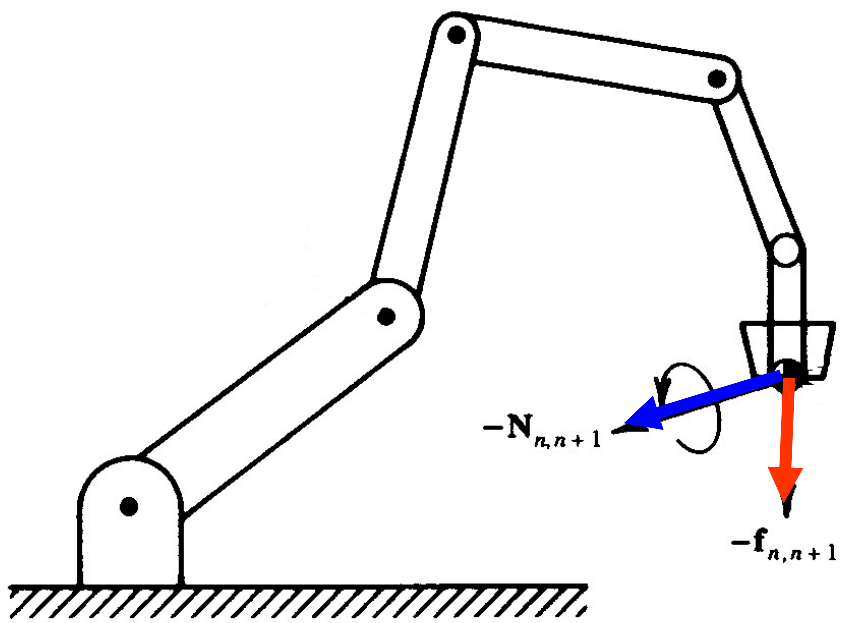


Balance of Forces and Moments

- Commonly, specify the force $f_{n,n+1}$ and moment $N_{n,n+1}$ that the manipulator arm applies to the environment:

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_{n,n+1} \\ \mathbf{N}_{n,n+1} \end{bmatrix}$$

6x1 endpoint force vector



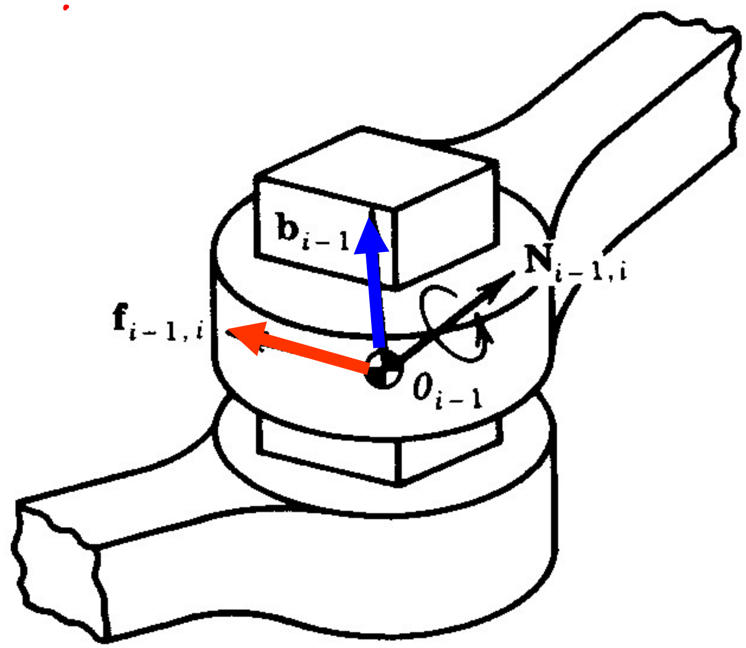
Static Force/Torque Relationship

- To derive relationship between input torques exerted by actuators and resultant endpoint force.
  - Assume that each joint *i* is driven by an individual actuator that exerts a drive torque or force τ_i between adjacent links.
  - Assume joint mechanism is frictionless.

Static Force/Torque Relationship

- For prismatic joint, drive force τ_i is a linear force exerted along *i*th joint axis:

$$\tau_i = \mathbf{b}_{i-1}^T . \mathbf{f}_{i-1,i}$$



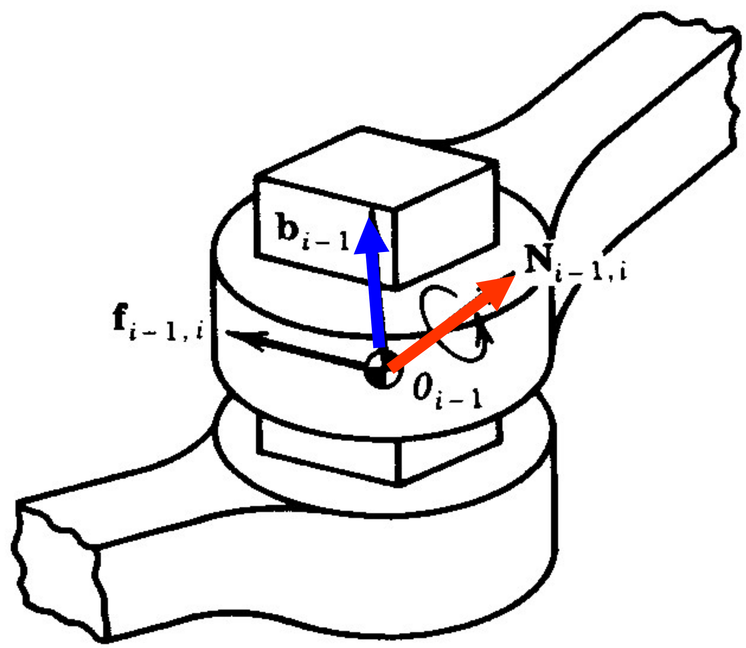
where $\mathbf{b}_{i-1}$ is unit vector pointing along *i*th joint axis

Static Force/Torque Relationship

- For a revolute joint, we have drive torque τ_i which is used to balance coupling moment component $N_{i-1,i}$ along its joint axis :

$$\tau_i = \mathbf{b}_{i-1}^T . \mathbf{N}_{i-1,i}$$

where $\mathbf{b}_{i-1}$ is unit vector pointing along *i*th joint axis

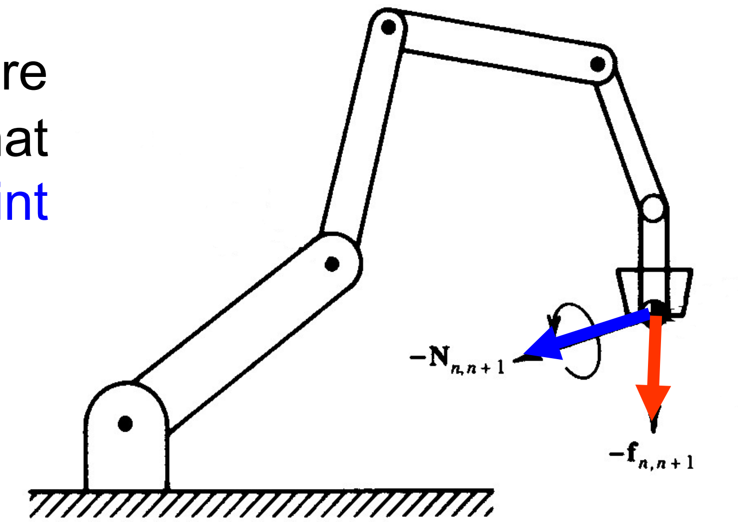


Static Force/Torque Relationship

- **Theorem 5-1:**

  Assume that the joint mechanisms are frictionless, then the joint torques[1] τ that are required to bear an arbitrary endpoint **force –F** ($= -[f_{n,n+1}, N_{n,n+1}]^T$) are given by

$$\tau = \mathbf{J}^T \mathbf{F} \qquad (5\text{-}3)$$



where **J** is the 6xn manipulator Jacobian relating infinitesimal joint displacement d**q** to infinitesimal end-effector displacement d**p**:

$$\mathbf{dp} = \mathbf{J}\ \mathbf{dq} \qquad (5\text{-}4)$$

[1]Also called the equivalent joint torques corresponding to the endpoint force F

Static Force/Torque Relationship

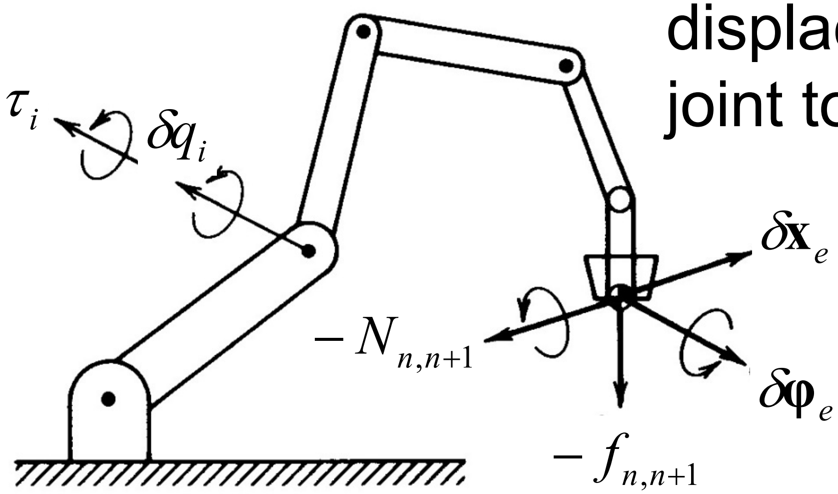
- Proof:

  Based on the principle of virtual work. Let's consider virtual displacements[1] at individual joints, δq_i and the corresponding virtual displacements at the end-effector, $\delta \mathbf{x}_e$ and $\delta \phi_e$

[1] Virtual displacements are arbitrary displacements of a mechanical system that conform to the geometric constraints of the system.

Static Force/Torque Relationship

- Proof: (cont)



Let's assume that joint torques τ_i (i = 1,…,n) and endpoint force and moment, $-\mathbf{f}_{n,n+1}$ and $-\mathbf{N}_{n,n+1}$, act on the manipulator while the joints and the end-effector are displaced. Then the virtual work done by joint torques, forces and moments is :

$$\delta W = \tau_1 \delta q_1 + \ldots + \tau_n \delta q_n - \mathbf{f}_{n,n+1}{}^T \delta \mathbf{x}_e - \mathbf{N}_{n,n+1}{}^T \delta \boldsymbol{\phi}_e$$

$\delta W = \boldsymbol{\tau}^T \delta \mathbf{q} - \mathbf{F}^T \delta \mathbf{p}$, where

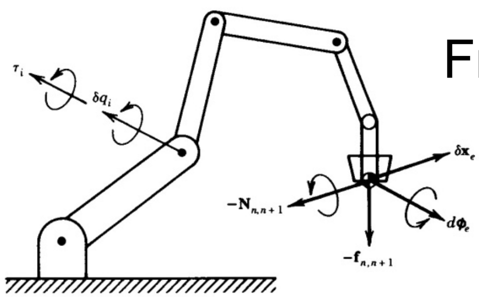
$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{bmatrix} \qquad \delta \mathbf{q} = \begin{bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \mathbf{f}_{n,n+1} \\ \mathbf{N}_{n,n+1} \end{bmatrix} \qquad \delta \mathbf{p} = \begin{bmatrix} \delta \mathbf{x}_e \\ \delta \phi_e \end{bmatrix}$$

Static Force/Torque Relationship

- <u>Proof: (cont)</u>

According to the principle of virtual work, the arm linkage is **in equilibrium** if, and only if, the virtual work **δW vanishes** for arbitrary virtual displacements which conform to geometric constraints.

$$\delta \mathbf{W} = \boldsymbol{\tau}^T \delta \mathbf{q} - \mathbf{F}^T \delta \mathbf{p} \equiv \mathbf{0}$$



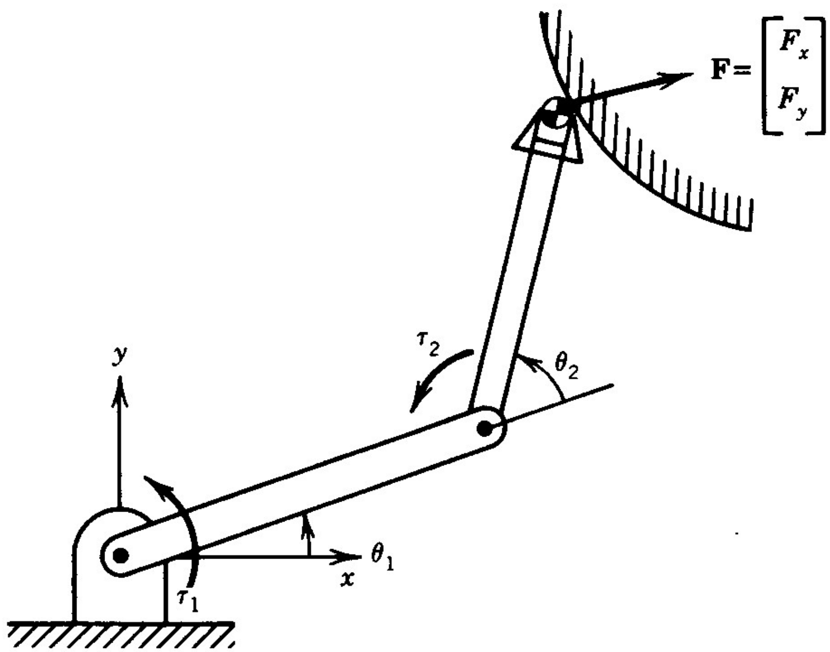
From Eq **(5-4)**, $\delta \mathbf{W} = \boldsymbol{\tau}^T \delta \mathbf{q} - \mathbf{F}^T \mathbf{J}\, \delta \mathbf{q} = (\boldsymbol{\tau} - \mathbf{J}^T \mathbf{F})^T\, \delta \mathbf{q} \equiv \mathbf{0}$

Since δ**q** is a vector with independent elements, we must have

$$(\boldsymbol{\tau} - \mathbf{J}^T \mathbf{F}) = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}$$

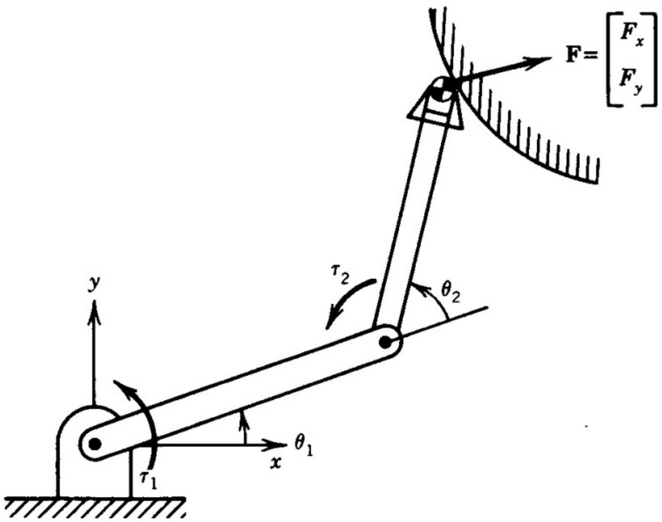
Static Force/Torque Relationship

- **Example 5-1**: The figure shows a 2 degree-of-freedom planar manipulator. At the endpoint, the arm is in contact with the external surface and applies the force **F**$=[F_x, F_y]^T$. Find the equivalent joint torques $\tau = [\tau_1, \tau_2]^T$ corresponding to the endpoint force **F**, assuming that the joint mechanisms are frictionless.



Static Force/Torque Relationship

- Solution:



First obtain Jacobian matrix that maps $\delta\theta = [\delta\theta_1 , \delta\theta_2]^T$ to $\delta\mathbf{p} = [\delta x , \delta y]^T$:

$$\mathbf{J} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

Equivalent joint torques:

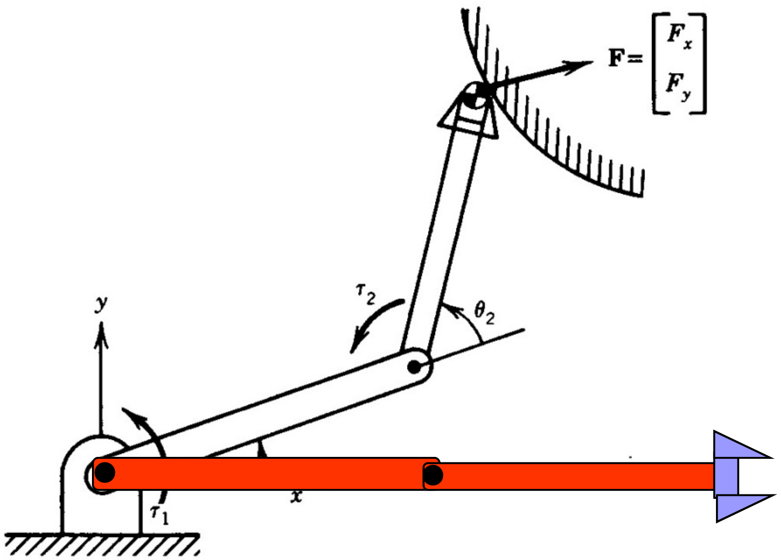
$$\boldsymbol{\tau} = \mathbf{J}^T \mathbf{F}$$

=>

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

Static Force/Torque Relationship

- Solution:



Note: If $\theta_1 = 0$ and $\theta_2 = 0$,

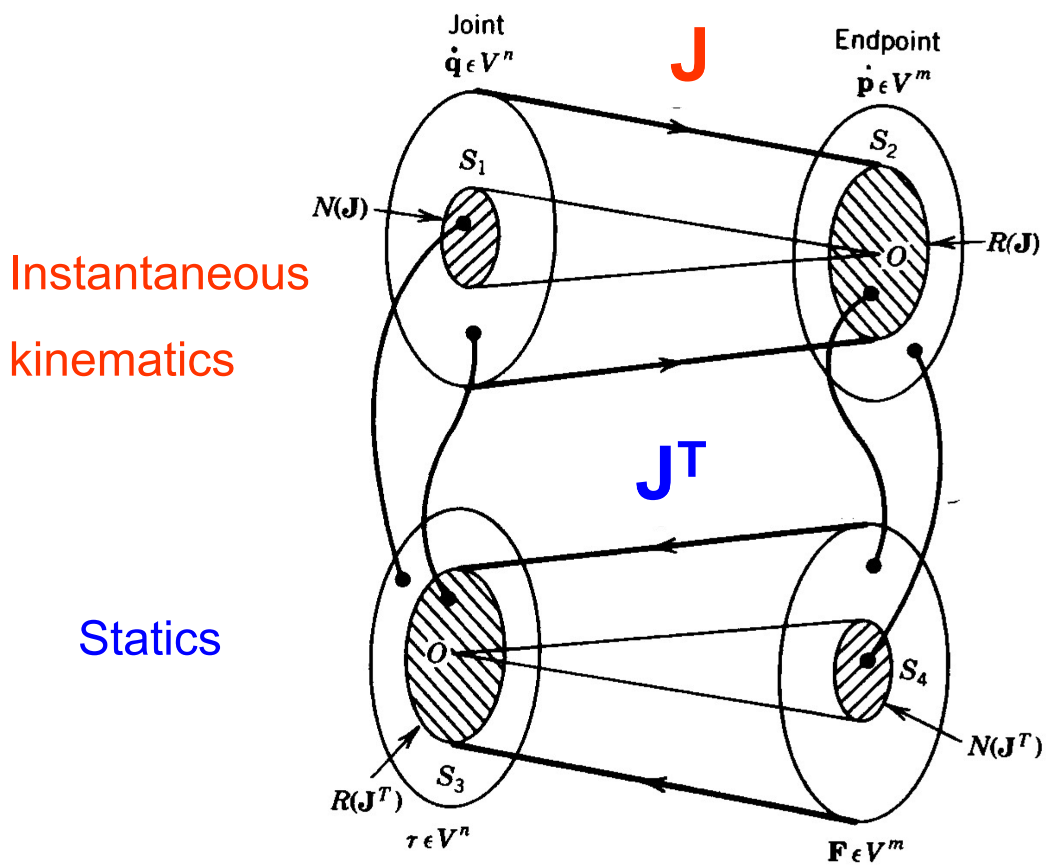
$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} 0 & l_1 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} (l_1 + l_2) F_y \\ l_2 F_y \end{bmatrix}$$

That is, F_x does not require any torque

F_x

Duality of instantaneous kinematics and static force relationships

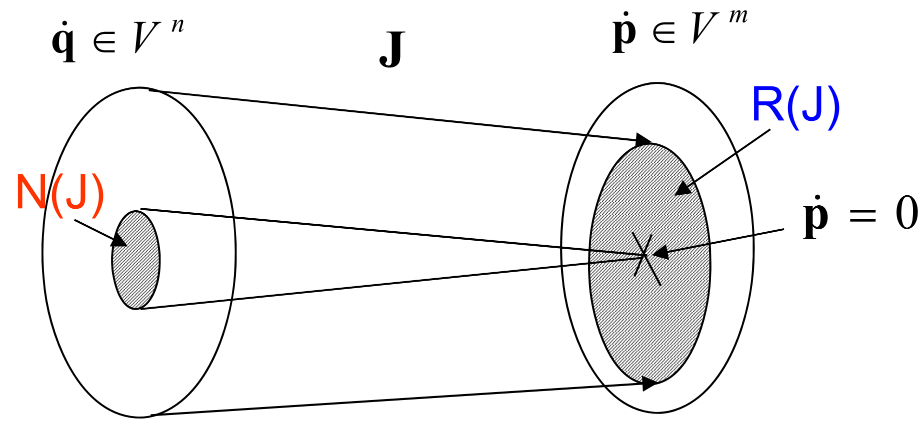
- Static force relationship (equivalent torques/endpoint force) is closely related to the instantaneous kinematics.



Duality of instantaneous kinematics and static force relationships

From previous, for linear mapping of instantaneous kinematics:

1. Range space R(J): set of all possible end-effector velocities generated by joint motions.
2. When Jacobian matrix degenerates (arm configuration is singular), the range space does not span the whole vector space V^m. → There exists a direction in which the end-effector cannot move.
3. Null space N(J): If it is not an empty set → there exists a set of joint velocities that do not produce a velocity at the end-effect (infinite number of solutions that cause the same end-effector velocity).



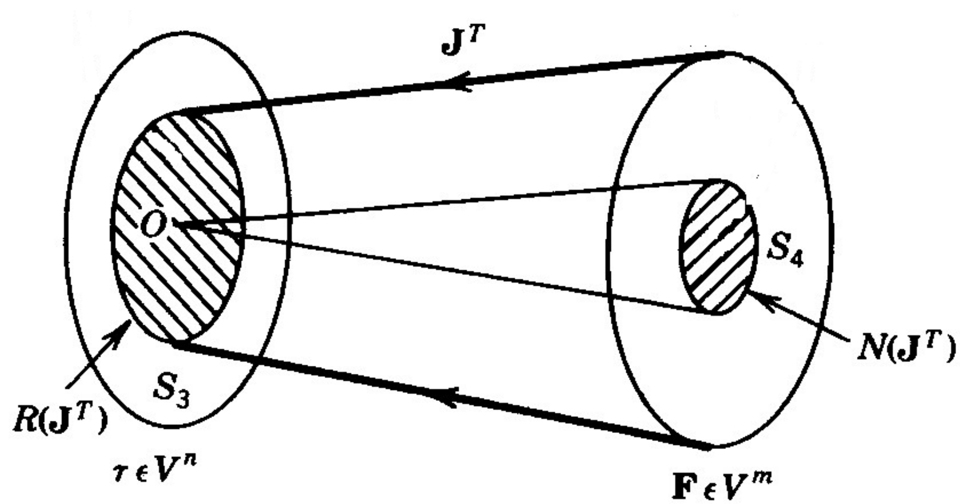
Instantaneous kinematics

Duality of instantaneous kinematics and static force relationships

For linear mapping of statics:

- Mapping is from m-dimensional vector space V^m associated with the end-effector coordinates, to n-dimensional vector space V^n associated with joint coordinates.
- Joint torques τ always determined uniquely for any arbitrary endpoint force F
- For given joint torques, endpoint force does not always exist.

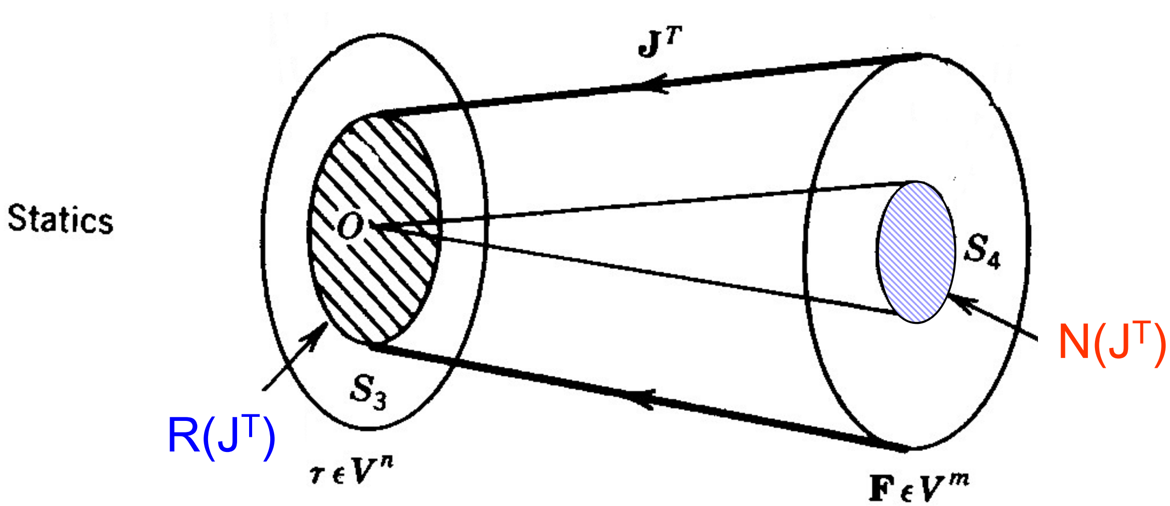
Statics



Duality of instantaneous kinematics and static force relationships

For linear mapping of statics (cont):

- Null space $N(J^T)$: represents the set of all endpoint forces that do not require any torques at the joints to bear the corresponding load. (The endpoint force is borne entirely by the structure of the arm linkage).
- Range space $R(J^T)$: represents the set of all possible joint torques that can balance the endpoint forces.

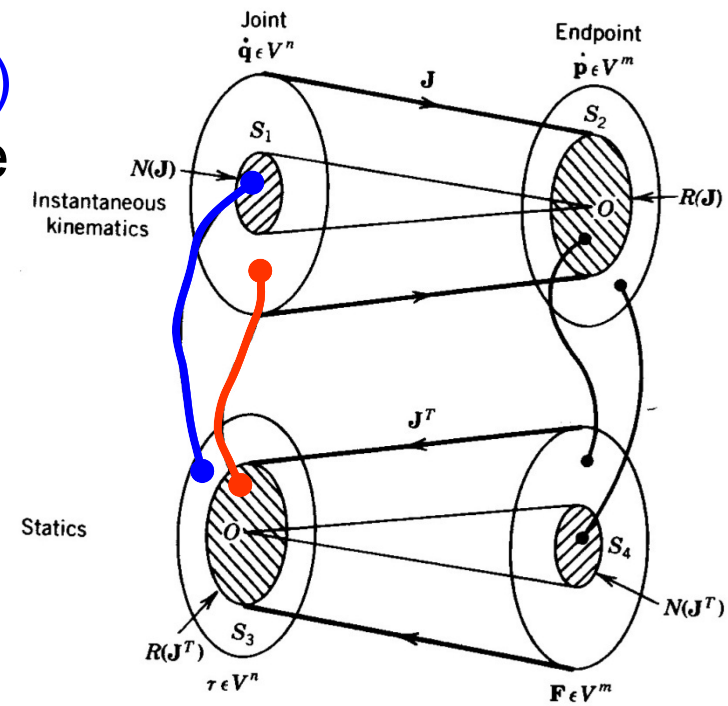


Duality of instantaneous kinematics and static force relationships

Note:

- From linear algebra: Null space N(J) is **orthogonal complement** of range space $R(J^T)$. (i.e., if a non-zero n-vector x is in N(J), it cannot also belong to $R(J^T)$, and vice-versa)

  => In the direction in which joint velocities do not cause any end-effector velocity, the joint torques cannot be balanced with any endpoint force.

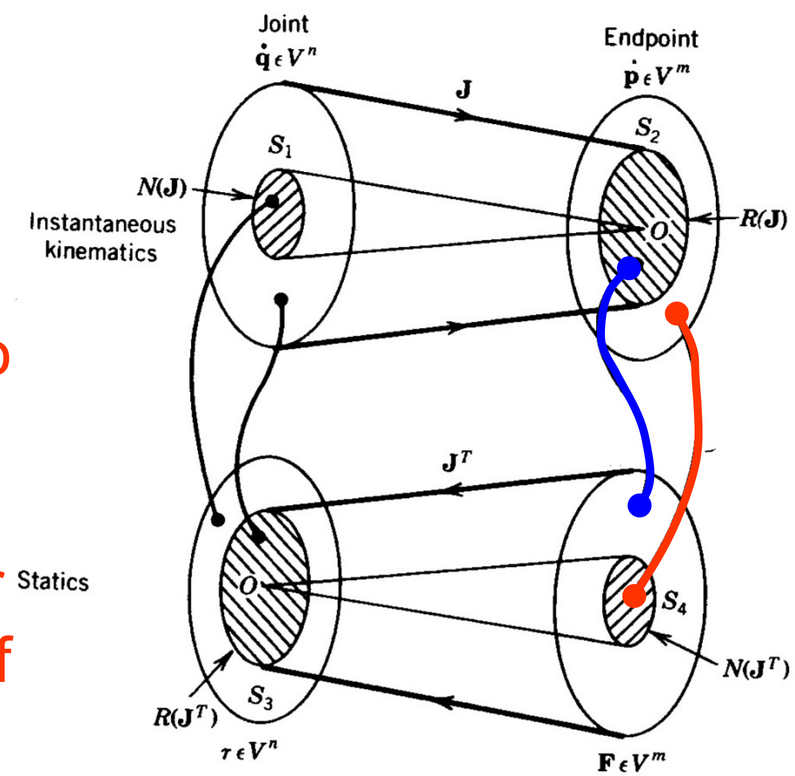


Duality of instantaneous kinematics and static force relationships

Note (cont):

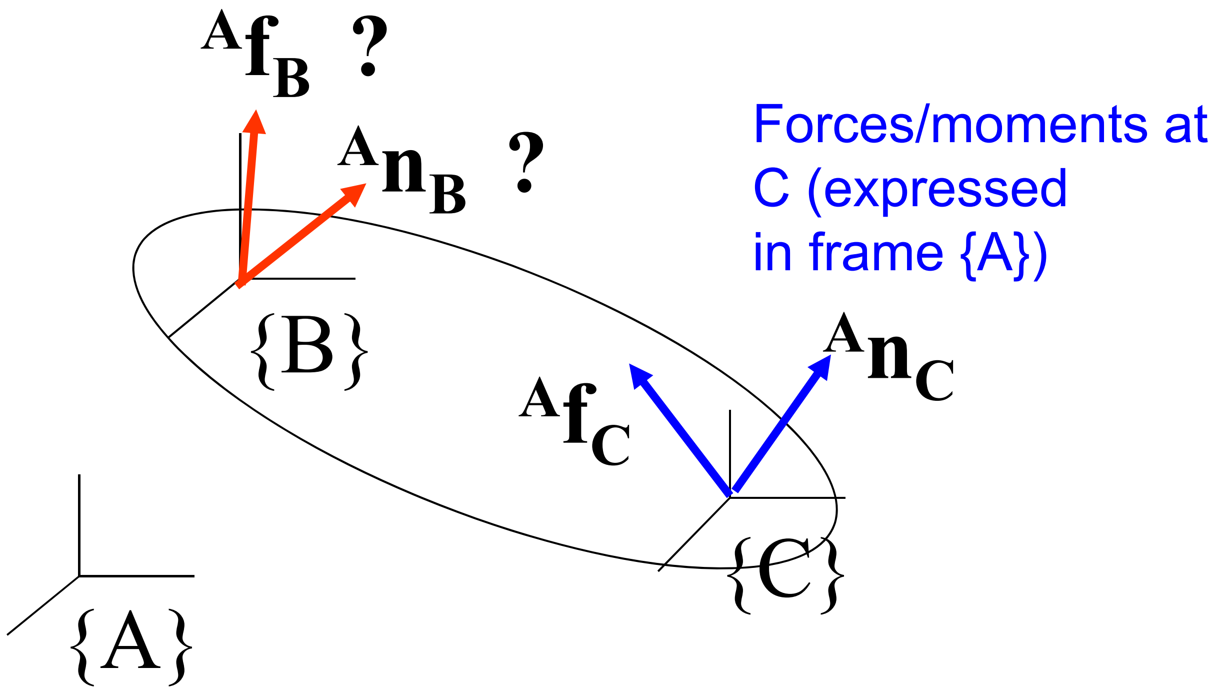
- Similarly, Range space R(J) is **orthogonal complement** to the null space N(J^T)

  => no joint torques are required to balance the end point force when the external force acts in the direction in which the end-effector cannot be moved by the motion of the arm joints



Transformation of Forces and Moments

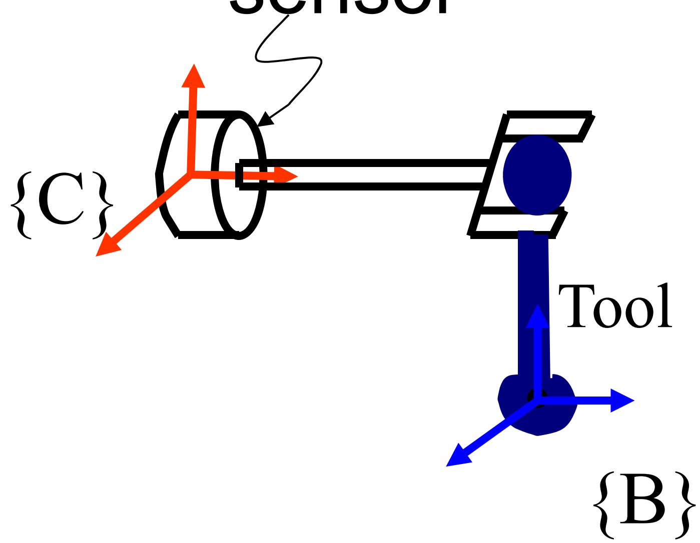
- Consider two frames C and B attached to a rigid body. Given: ${}^{A}f_{C}$, ${}^{A}n_{C}$. Find: ${}^{A}f_{B}$, ${}^{A}n_{B}$ (That is, find the equivalent force/moment at B if force/moment is known at C)



Transformation of Forces and Moments

- Why is this important?

Force/torque sensor

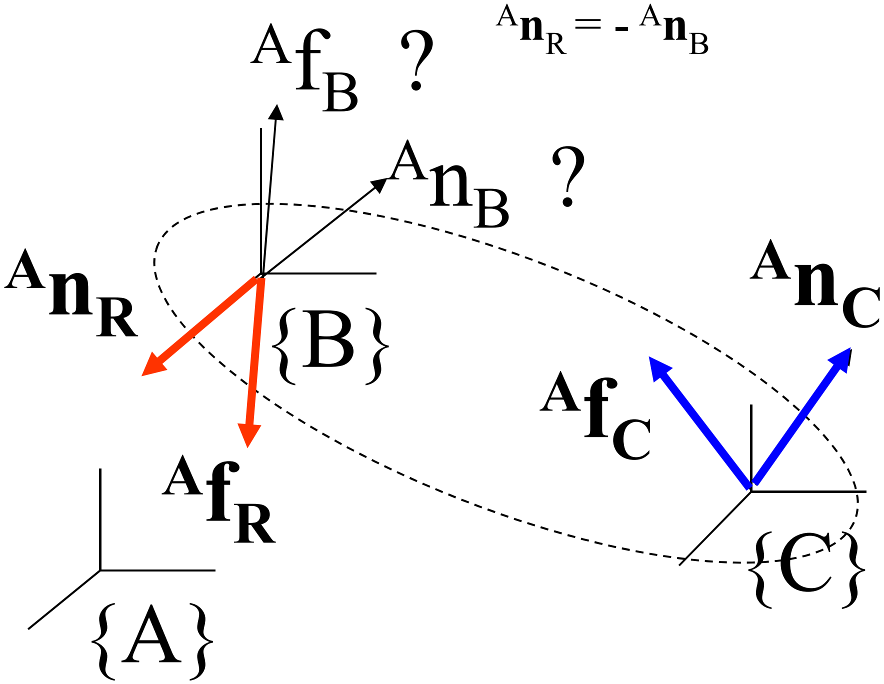


$^C\mathbf{f}_C$ & $^C\mathbf{n}_C$ can be force sensor readings. But our primary interest is $^B\mathbf{f}_B$ & $^B\mathbf{n}_B$ (force/moments at tool tip)

Transformation of Forces and Moments

Reaction forces/moments applied at B under *equilibrium*

$${}^A\mathbf{f}_R = -\,{}^A\mathbf{f}_B$$
$${}^A\mathbf{n}_R = -\,{}^A\mathbf{n}_B$$



For equilibrium:

$\sum \mathbf{F} = \mathbf{0}$

$${}^A\mathbf{f}_C + {}^A\mathbf{f}_R = 0$$
$${}^A\mathbf{f}_R = -\,{}^A\mathbf{f}_C$$

But ${}^A\mathbf{f}_B = -\,{}^A\mathbf{f}_R \Rightarrow \boxed{{}^A\mathbf{f}_B = {}^A\mathbf{f}_C}$

$\sum \mathbf{N} = \mathbf{0}$ (about origin of {C})

$${}^A\mathbf{n}_C + ({}^A\mathbf{p}_B - {}^A\mathbf{p}_C) \times {}^A\mathbf{f}_R + {}^A\mathbf{n}_R = 0$$
$$\Rightarrow {}^A\mathbf{n}_R = -\,{}^A\mathbf{n}_C - ({}^A\mathbf{p}_B - {}^A\mathbf{p}_C) \times {}^A\mathbf{f}_R$$

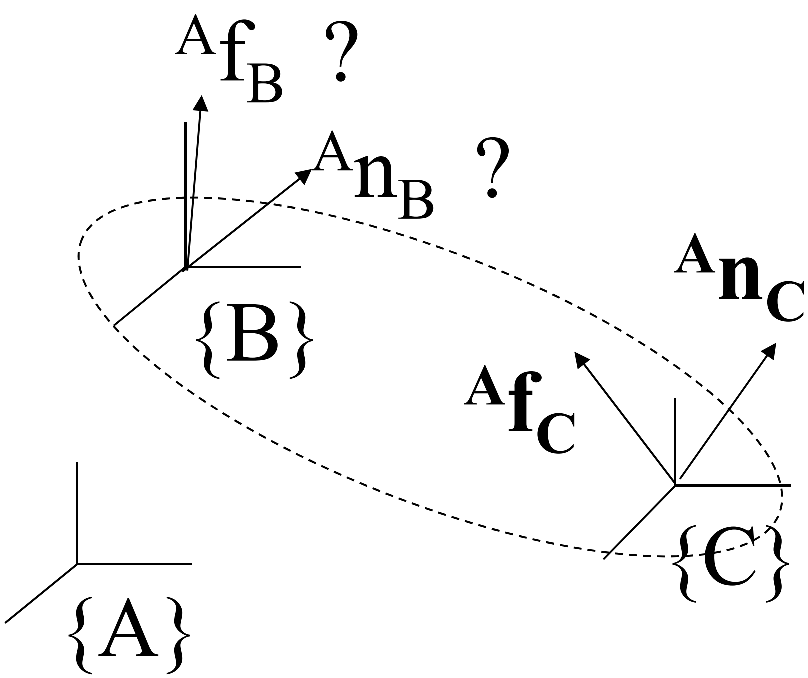
$${}^A\mathbf{n}_B = -{}^A\mathbf{n}_R = {}^A\mathbf{n}_C + ({}^A\mathbf{p}_B - {}^A\mathbf{p}_C) \times {}^A\mathbf{f}_R$$
$$= {}^A\mathbf{n}_C + ({}^A\mathbf{p}_B - {}^A\mathbf{p}_C) \times (-\,{}^A\mathbf{f}_C)$$

$${}^A\mathbf{n}_B = {}^A\mathbf{n}_C + ({}^A\mathbf{p}_C - {}^A\mathbf{p}_B) \times {}^A\mathbf{f}_C$$

OR

$$\boxed{{}^A\mathbf{n}_B = {}^A\mathbf{n}_C + \lfloor {}^A({}^B\mathbf{p}_C) \times \rfloor\ {}^A\mathbf{f}_C}$$

Transformation of Forces and Moments



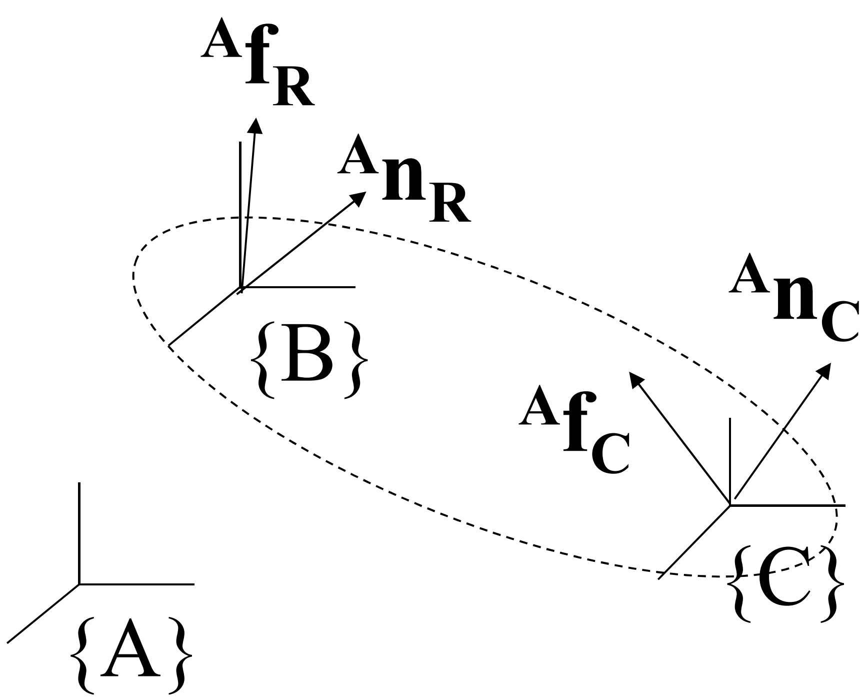
In Matrix Form

$$\begin{bmatrix} {}^A f_B \\ {}^A n_B \end{bmatrix} = \begin{bmatrix} I & 0 \\ \underbrace{\lfloor {}^A R_B \, {}^B p_C \times \rfloor}_{\lfloor {}^A({}^B P_C)\times \rfloor} & I \end{bmatrix} \begin{bmatrix} {}^A f_C \\ {}^A n_C \end{bmatrix}$$

But in typical applications, we would like to relate

$$\begin{bmatrix} {}^C f_C \\ {}^C n_C \end{bmatrix} \text{ with } \begin{bmatrix} {}^B f_B \\ {}^B n_B \end{bmatrix}$$

Transformation of Forces and Moments



We can transform vectors **f** & **n** like any other vector via Rotation Matrices

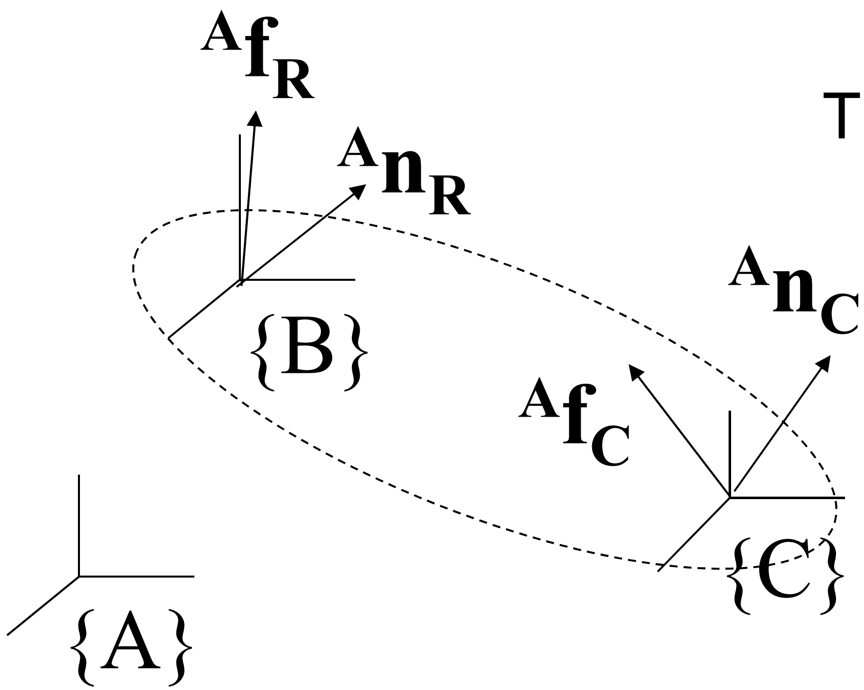
$$\begin{bmatrix} {}^{A}f_{C} \\ {}^{A}n_{C} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}_{C}R & 0 \\ \hdashline 0 & {}^{A}_{C}R \end{array}\right] \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} {}^{A}f_{B} \\ {}^{A}n_{B} \end{bmatrix} = \left[\begin{array}{c:c} I & 0 \\ \hdashline \lfloor {}^{A}_{B}R\,{}^{B}p_{C}\times \rfloor & I \end{array}\right] \left[\begin{array}{c:c} {}^{A}_{C}R & 0 \\ \hdashline 0 & {}^{A}_{C}R \end{array}\right] \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$

$$\begin{bmatrix} {}^{A}f_{B} \\ {}^{A}n_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{A}_{C}R & 0 \\ \hdashline \lfloor {}^{A}_{B}R\,{}^{B}p_{C}\times \rfloor\,{}^{A}_{C}R & {}^{A}_{C}R \end{array}\right] \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$

Transformation of Forces and Moments



Also,

$$\begin{bmatrix} {}^{B}f_{B} \\ {}^{B}n_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{B}_{A}R & 0 \\ \hdashline 0 & {}^{B}_{A}R \end{array}\right] \begin{bmatrix} {}^{A}f_{B} \\ {}^{A}n_{B} \end{bmatrix}$$

Therefore,

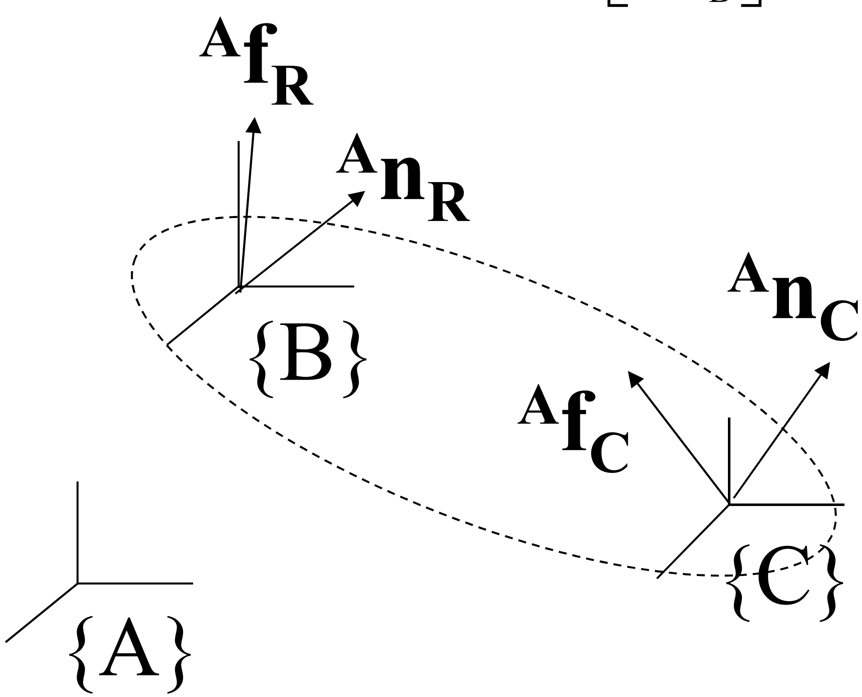
$$\begin{bmatrix} {}^{B}f_{B} \\ {}^{B}n_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & 0 \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}_{C}R & 0 \\ {}^{A}_{B}R \lfloor {}^{B}p_{C} \times \rfloor {}^{A}_{C}R & {}^{A}_{C}R \end{bmatrix} \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$

$$= \begin{bmatrix} {}^{B}_{A}R\, {}^{A}_{C}R & 0 \\ {}^{B}_{A}R \lfloor {}^{A}_{B}R\, {}^{B}p_{C} \times \rfloor {}^{A}_{C}R & {}^{B}_{A}R\, {}^{A}_{C}R \end{bmatrix} \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$

Transformation of Forces and Moments

Therefore,

$$\begin{bmatrix} {}^{B}f_{B} \\ {}^{B}n_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R\,{}^{A}_{C}R & 0 \\ {}^{B}_{A}R\,\lfloor {}^{A}_{B}R\,{}^{B}p_{C}\times \rfloor\,{}^{A}_{C}R & {}^{B}_{A}R\,{}^{A}_{C}R \end{bmatrix} \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$



$$\left({}^{B}_{A}\mathbf{R}\lfloor {}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{p}_{C}\times\rfloor {}^{A}_{C}\mathbf{R}\right){}^{C}\mathbf{f}_{C}$$

$$= {}^{B}_{A}\mathbf{R}\left[\left({}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{p}_{C}\right)\times\left({}^{A}_{C}\mathbf{R}\,{}^{C}\mathbf{f}_{C}\right)\right]$$

$$= \left({}^{B}_{A}\mathbf{R}\,{}^{A}_{B}\mathbf{R}\,{}^{B}\mathbf{p}_{C}\right)\times\left({}^{B}_{A}\mathbf{R}\,{}^{A}_{C}\mathbf{R}\,{}^{C}\mathbf{f}_{C}\right)$$

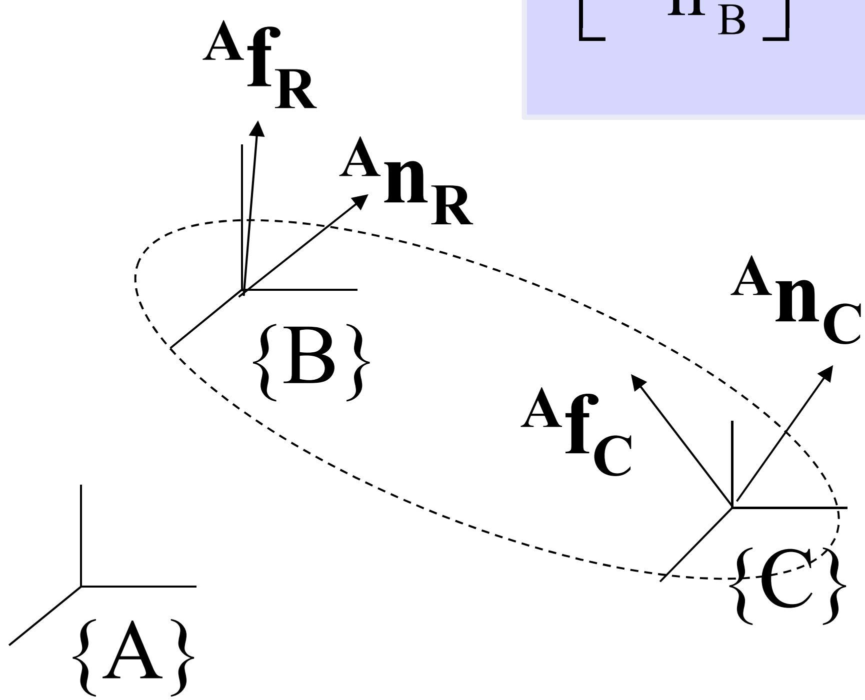
$$= \left({}^{B}\mathbf{p}_{C}\right)\times\left({}^{B}_{C}\mathbf{R}\,{}^{C}\mathbf{f}_{C}\right)$$

$$= \underline{\lfloor {}^{B}\mathbf{p}_{C}\times\rfloor {}^{B}_{C}\mathbf{R}\,{}^{C}\mathbf{f}_{C}}$$

Transformation of Forces and Moments

Therefore,

$$\begin{bmatrix} {}^{B}f_{B} \\ {}^{B}n_{B} \end{bmatrix} = \left[\begin{array}{c:c} {}^{B}_{C}R & 0 \\ \hdashline \lfloor {}^{B}p_{C}\times \rfloor {}^{B}_{C}R & {}^{B}_{C}R \end{array}\right] \begin{bmatrix} {}^{C}f_{C} \\ {}^{C}n_{C} \end{bmatrix}$$



Summary

- Expressions for translational and angular velocities
- Transform velocities in different spaces
  - Relate joint velocities with end-effector velocities
  - Concept of Jacobians
- Solve the forward and inverse instantaneous (or differential) kinematics
- Understand robot singularities
- Static force/torque transformations between task space and joint space
- Static force/torque transformations between frames