Q.1 Consider the nonlinear system described by the second order equation

$$m\ddot{y} + f(\dot{y}) + k(y - y_0) = u\,,$$

where m is the unknown mass, $f(\dot{y})$ denotes the friction, k is the spring constant, y_0 is the equilibrium position, y is the position of the mass, and u is the control input.

The equilibrium position, y_0, is known, and y and $\dot{y}$ are measurable. Using Lyapunov's direct method or any other methods you are comfortable with, answer the following questions.

(a) Assume that the system is well lubricated and that only viscous friction exists in the system, i.e., $f(\dot{y}) = b\dot{y}$ with unknown coefficient, b. Design an adaptive control scheme such that the position, $y(t)$, converges to the output of the reference model of the form

$$\frac{Y_m(s)}{R(s)} = \frac{1}{s^2 + 10s + 25},$$

and ensure that all other signals in the closed-loop system are bounded. $Y_m(s)$ and $R(s)$ are Laplace transforms of the output signal, $y_m(t)$, and the reference signal, $r(t)$, respectively.

(10 marks)

(b) Assume that the frictional force is more complicated and is modelled as $f(\dot{y}) = b_1\dot{y} + b_2\dot{y}^2$ with unknown coefficients. Design an adaptive control scheme such that the output of the system, $y(t)$, tracks the same reference model as in Q1(a).

(10 marks)

In the development of your answers, show clearly your control laws, the closed-loop error equations, the adaptation mechanisms, and the stability properties of the closed-loop system. State clearly the assumptions you make.

Q.2 Consider the sampled-data system described by

$$(1+a_1 q^{-1} + a_2 q^{-2}) y(t) = (b_0 q^{-2} + 0.8 q^{-3}) u(t)$$

where a_1 and a_2 are unknown constants, b_0 is a constant to be specified, $y(t)$ is the measurable output, and $u(t)$ is the input.

Using any estimator of your choice, design adaptive controllers for the following cases:

(i) $b_0 > 1$ but unknown;

(10 marks)

(ii) $b_0 = 1$; and

(10 marks)

(iii) $0.2 < b_0 < 0.5$ but unknown.

(5 marks)

In your answers, appropriate reasons should be given with relevant estimation and/or control laws.

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The system is given by

$$m\ddot{y} + f(\dot{y}) + k(y - y_0) = u$$

Define $x = y - y_0$. The system is then written as

$$\sum_1: \qquad m\ddot{x} + f(\dot{x}) + kx = u$$

Define $x_1 = x, x_2 = \dot{x}$, the system can be described in state space forms as

$$\sum_2: \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ m\dot{x}_2 &= -f(x_2) - kx_1 + u \end{aligned}$$

$$\sum_3: \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{f(x_2)}{m} - \frac{k}{m}x_1 + \frac{1}{m}u \end{aligned}$$

(i) Linear Case:
In this case, the three models can be written as

$$\sum_1: \qquad m\ddot{x} + b\dot{x} + kx = u$$

$$\sum_2: \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ m\dot{x}_2 &= -bx_2 - kx_1 + u \end{aligned}$$

$$\sum_3: \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u \end{aligned}$$

Stable control system can then be designed by following the lecture notes for $\sum_3$ by re-writing into the standard form:

$$\Sigma_{3,1}: \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_2 x_2 - a_1 x_1 + gu \end{aligned}$$

where $a_1 = \dfrac{k}{m}, a_2 = \dfrac{b}{m}, g = \dfrac{1}{m}$.

Or, in the general form:

$$\Sigma_{3,2}: \quad \dot{x} = Ax + gbu:$$

where $A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The model that it can be matched:

$$\dot{x}_m = A_m x_m + g_m br, \qquad x_m = \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix}, A_m = \begin{bmatrix} 0 & 1 \\ -a_{m,2} & -a_{m,1} \end{bmatrix}, g_m = a_{m,2}$$

$$\frac{Y_m(s)}{R(s)} = \frac{a_{m,2}}{s^2 + a_{m,1}s + a_{m,2}} = \frac{25}{s^2 + 10s + 25}$$

which specifies the desired performance.

For the unknown plant, $\dot{x}_p = A_p x_p + gbu$

consider the non-adaptive case: $u = \theta_x^{*T} x_p + \theta_r^* r$, $\qquad \theta_x^* \stackrel{\Delta}{=} [\theta_1^* \quad \theta_2^*]^T$

Then the closed-loop system is $\dot{x}_p = [A_p + gb\theta_x^{*T}]x_p + b(g\theta_r^*)r$

Apparently, the following matching condition holds $A_p + gb\theta_x^{*T} \equiv A_m \in R^{2\times 2}, g\theta_r^* \equiv g_m \in R$

Thus, control gains $\theta_x^* \in \Re^n$, and $\theta_r^* \in \Re$ exist to guarantee that the closed-loop system match the reference model_ $\dot{x}_m = A_m x_m + g_m br$

Adaptive Case:

To adaptively match the reference model, consider the control law

$$u(t) = \theta_x^T(t)x_p(t) + \theta_r(t)r(t)$$

Define parameter errors, $\phi_x(t) \stackrel{\Delta}{=} \theta_x(t) - \theta_x^*$, $\qquad \phi_r(t) \stackrel{\Delta}{=} \theta_r(t) - \theta_r^*$

Then, the control law applied to the plant results in

$$\begin{aligned} \dot{x}_p &= A_p x_p + gb\{\theta_x^T x_p + \theta_r r\} = \left[A_p + gb\theta_x^{*T}\right]x_p + gb\phi_x^T x_p + gb\theta_r r \\ &= A_m x_p + gb\phi_x^T x_p + gb\theta_r r \end{aligned}$$

Compared with $\dot{x}_m = A_m x_m + g_m br = A_m x_m + gb\theta_r^* r$, we have the closed-loop error equation

$$\dot{e} = A_m e + gb\phi_x^T x_p + gb\phi_r r \qquad = A_m e + gb\phi^T x$$

where $g_m = g\theta_r^*$, $e = x_p - x_m$, $\phi \stackrel{\Delta}{=} \begin{bmatrix} \phi_x \\ \phi_r \end{bmatrix}$; $\qquad x \stackrel{\Delta}{=} \begin{bmatrix} x_p \\ r \end{bmatrix}$.

For a reference model, A_m must be chosen to be a stable matrix. Thus, it satisfies the Lyapunov equation $A_m^T P + PA_m = -Q$, i.e., for any symmetric positive definite matrix Q, there exists a symmetric positive definite P satisfying the above equation.

Consider a Lyapunov function candidate $V(e(t),\phi(t)) = e(t)^T Pe(t) + |g|\phi(t)^T \Gamma^{-1}\phi(t)$, where Γ is a symmetric positive definite (s.p.d.) matrix.

Evaluate $\dot{V}$ along the trajectory of the system

$$\dot{V} = 2e^T P\dot{e} + 2|g|\phi^T\Gamma^{-1}\dot{\phi} \qquad = 2e^T P\{A_m e + gb\phi^T x\} + 2|g|\phi^T\Gamma^{-1}\dot{\phi}$$
$$= e^T\left(A_m^T P^T + PA_m\right)e + 2ge^T Pb\phi^T x + 2|g|\phi^T\Gamma^{-1}\dot{\phi}$$
$$= -e^T Qe + 2ge^T Pb\phi^T x + 2|g|\phi^T\Gamma^{-1}\dot{\phi}$$

Letting $\dot{\phi} = \dot{\theta} = \begin{bmatrix}\dot{\theta}_x \\ \dot{\theta}_r\end{bmatrix} = -\operatorname{sgn}(g)\Gamma x e^T Pb$, for $\phi = \theta - \theta^*$, we have

$$\dot{V} = -e^T Qe + 2ge^T Pb\phi^T x - 2|g|\operatorname{sgn}(g)\phi^T xe^T Pb$$
$$= -e^T Qe \le 0$$

Note that (i) $V(t)$ is positive definite, (ii) $V(t)$ is decrescent, and (iii) $V(t)$ is radically unbounded. Accordingly, we have the following conclusion

- $V(t)$ is positive definite and $\dot{V}(t) \le 0 \Rightarrow V(t)$ is bounded
- $\|e\|, \|\phi\| \quad (hence \quad \|\theta\|)$ are bounded
- $\dot{e}$ is bounded, $\int_0^\infty e^T Qe d\tau \le c_1$, $\lim_{t\to\infty}\|e\| = 0$

All signals $\{x_p, \theta_x, \theta_r\}$ are bounded, and $\lim_{t\to\infty}\|x_p - x_m\| = 0$

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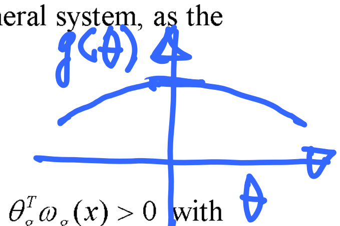
(ii) Nonlinear Case:

This could be handled easily one the student realize that the system can be transformed into the standard format. In what follows, a general description is given for a general system, as the system in this question is a special case. Consider



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u$$

where $x = [x_1, x_2]^T$ are the measurable states, $f(x) = \theta_f^T \omega_f(x)$ and $g(x) = \theta_g^T \omega_g(x) > 0$ with $\omega_f(x) \in R^n$ and $\omega_g(x) \in R^k$ being known functions, while $\theta_f^* \in R^n$ and $\theta_g^* \in R^k$ are unknown constant vectors.

angular position

Consider stable reference model, $\dot{x}_m = A_m x_m + g_m br$, and the control law

g_m

$$u = \frac{1}{\hat{g}(x)}[-\hat{f}(x) - a_1 x_1 - a_2 x_2 + r],\ i.e.,\ \hat{g}(x)u = -\hat{f}(x) - a_1 x_1 - a_2 x_2 + g_m r$$

with $\hat{f}(x) = \hat{\theta}_f^T \omega_f(x)$, and $\hat{g}(x) = \hat{\theta}_g^T \omega_g(x) > 0$.

The closed-loop error equation is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u - \hat{g}u + \hat{g}u = (f - \hat{f}) + (g - \hat{g})u - a_1 x_1 - a_2 x_2 + g_m r$$
$$= -a_1 x_1 - a_2 x_2 - \tilde{\theta}_f^T \omega_f - \tilde{\theta}_u^T \omega_u + g_m r \quad = -a_1 x_1 - a_2 x_2 - \tilde{\theta}^T \omega + g_m r$$

where $\tilde{\theta}_f = [\hat{\theta}_f - \theta_f^*], \tilde{\theta}_u = [\hat{\theta}_u - \theta_u^*], \tilde{\theta} = [\tilde{\theta}_f^T, \tilde{\theta}_u^T]^T, \omega = [\omega_f^T, \omega_u^T]^T$

Comparing with the reference model

$$\dot{x}_m = A_m x_m + g_m br, \qquad x_m = \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix}, A_m = \begin{bmatrix} 0 & 1 \\ -a_{m,2} & -a_{m,1} \end{bmatrix}, g_m = a_{m,2}$$

$$\dot{\tilde{\theta}} = -\Gamma e^T P b \omega$$

we have the error equation as

$$\boxed{\dot{e} = A_m e + b\tilde{\theta}^T \omega}$$

$$\Gamma > 0$$

Using the results in part (a), the following $\dot{\hat{\theta}} = M^{-1} e^T P b \omega$ ensures stability.

Measure has to be taken to avoid controller singularity $\hat{g} = 0$ in this approach.

Solution To Question No. : 2

The system is given by

$$(1 + a_1 q^{-1} + a_2 q^{-2}) y(t) = (b_0 q^{-2} + 0.8 q^{-3}) u(t)$$

Converting into the standard form $A(q^{-1}) y(t) = q^{-d} B(q^{-1}) u(t)$,

we have

$$A = 1 + a_1 q^{-1} + a_2 q^{-2},$$

$$B = b_0 + 0.8 q^{-1}$$

$$d = \cancel{1}\,2, n = 2, m = 1$$

10 marks

(i) $b_0 > 1$ but unknown: $B(q^{-1})$ is stable.

Applying the prediction identity:

$$1 = A(q^{-1}) E(q^{-1}) + q^{-d} F(q^{-1}), \text{ deg(E)=d-1=1, deg(F)=n-1=1}$$

leads to

$$1 = (1 + a_1 q^{-1} + a_2 q^{-2})(1 + e_1 q^{-1}) + q^{-2}(f_0 + f_1 q^{-1})$$

$$G = EB = b_0 + (0.8 + b_0 e_1) q^{-1} + 0.8 e_1 q^{-2}$$

Then, the prediction form of the process is

$$\begin{aligned} y(t) &= q^{-d} F y(t) + q^{-d} E B u(t) \\ &= (f_0 + f_1 q^{-1}) y(t-d) + G u(t-d), \deg(G) = 2 \\ &= f_0 y(t-2) + f_1 y(t-3) + g_0 u(t-2) + g_1 u(t-3) + g_2 u(t-3) \\ &= [f_0, f_1, g_0, g_1, g_2] \begin{bmatrix} y(t-2) \\ y(t-3) \\ u(t-2) \\ u(t-3) \\ u(t-3) \end{bmatrix} \end{aligned}$$

This is the basis for estimation, i.e.,

$$y(t) = \theta^{*T} \varphi(t-d) = \theta^{*T} \varphi(t-2)$$

Then the gradient estimator

$$\hat{\theta}(t) = \hat{\theta}(t-1) - \frac{\gamma\varphi(t-d)}{\alpha + \varphi^T(t-d)\varphi(t-d)} e_1(t)$$

where

$$e_1(t) = \hat{y}(t) - y(t) = \varphi^T(t-d)\tilde{\theta}(t-1)$$

$\alpha \geq 0$ and $0 < \gamma < 2$.

$$\hat{y}(t) = \varphi^T(t-d)\hat{\theta}(t-1)$$

Then the estimator results in

(i) $\left\|\hat{\theta}(t) - \theta^*\right\| \leq \left\|\hat{\theta}(t-1) - \theta^*\right\| \leq \left\|\hat{\theta}(0) - \theta^*\right\|$

(ii) $\lim\limits_{t\to\infty} \dfrac{e_1(t)}{\sqrt{\alpha + \varphi^T\varphi}} = 0$ $\quad \sum_{j=0}^{t} \dfrac{e_1^2(j)}{\{\alpha + \|\varphi(j)\|^2\}} \leq c_1$

for all $t > 0$

(iii) $\lim\limits_{t\to\infty}\left\|\hat{\theta}(t) - \hat{\theta}(t-k)\right\| = 0$ for any finite k.

Combining with the above estimator, the adaptive minimum variance controller is achieved by using the control law

$$\hat{y}(t+d/t) = \hat{\theta}(t)^T\varphi(t) = r(t)$$

(Notice that $\varphi(t)$ is used in the control, while $\varphi(t-d)$ is used in estimation.)

In implementation, the above control is

$$[f_0, f_1, g_0, g_1, g_2]\begin{bmatrix} y(t) \\ y(t-1) \\ u(t) \\ u(t-1) \\ u(t-2) \end{bmatrix} = r(t) \text{ or}$$

$$u(t) = \frac{1}{\hat{g}_0}\left\{r(t) - \hat{f}_0 y(t-1) - \hat{f}_1 y(t-1) - \hat{g}_1 u(t-1) - \hat{g}_2 u(t-2)\right\}, \hat{g}_0 \neq 0$$

The closed-loop system can be proven rigorously stable using technical lemma 6.2 in the lecture notes.

(ii) $b_0 = 1$: the system becomes

10 marks

$$B = 1 + 0.8q^{-1}, \text{ and } B(q^{-1}) \text{ is stable.}$$

Similar analysis can be conducted with some better insight.

Applying the prediction identity:

$$1 = A(q^{-1})E(q^{-1}) + q^{-d}F(q^{-1}), \ \deg(E)=d-1=1, \ \deg(F)=n-1=1$$

leads to

$$1 = (1 + a_1 q^{-1} + a_2 q^{-2})(1 + e_1 q^{-1}) + q^{-2}(f_0 + f_1 q^{-1})$$

$$G = EB = 1 + (0.8 + e_1)q^{-1} + 0.8 e_1 q^{-2}$$

As such, we have $g_0 = 1$, and there is no need to estimate it, and better controller is expected.

Then, the prediction form of the process is

$$\begin{aligned}
y(t) &= q^{-d}Fy(t) + q^{-d}EBu(t) \\
&= (f_0 + f_1 q^{-1})y(t-d) + Gu(t-d), \deg(G) = 2 \\
&= f_0 y(t-2) + f_1 y(t-3) + u(t-2) + g_1 u(t-3) + g_2 u(t-3) \\
&= [f_0, f_1, g_1, g_2]\begin{bmatrix} y(t-2) \\ y(t-3) \\ u(t-3) \\ u(t-3) \end{bmatrix} + u(t-2)
\end{aligned}$$

This is the basis for estimation, i.e.,

$$y'(t) = y(t) - u(t-2) = \theta^{*T}\varphi(t-d) = \theta^{*T}\varphi(t-2)$$

By taking $y'(t)$ as $y(t)$, then we can construct estimator as in (i) and the controller is changed to

$$u(t) = \left\{ r(t) - \hat{f}_0 y(t-1) - \hat{f}_1 y(t-1) - \hat{g}_1 u(t-1) - \hat{g}_2 u(t-2) \right\}$$

for which controller singularity will never occur.

5 marks

(iii) $0.2 < b_0 < 0.5$ but unknown:

Since $B(q^{-1})$ is unstable, and we should use general minimum variance (GMV) adaptive control. There are several methods possible.

MV : $1 = AE + q^{-d} F$

GMV : $P(q^{-1}) = AE + q^{-d} F$

One method is:

Estimation algorithm:

$$y(t) = \varphi(t-d)^T \theta_0$$

$$e(t) = y(t) - \varphi(t-d)^T \hat{\theta}(t-1)$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \frac{\varphi(t-d)}{1 + \varphi(t-d)^T \varphi(t-d)} e(t)$$

$\varphi(j) = (1 + p_1 q^{-1}) y(j)$

$\varphi(j+d) = (1 + p_1 q^{-1}) y(j+d)$

- Estimate: $\hat{\theta} \Rightarrow \hat{A}, \quad \hat{B}$
- Solve $\hat{P}\hat{B} + \hat{Q}\hat{A} = A^*$ for $\hat{P}$ and $\hat{Q}$ where A^* is a stable reference polynomial.
- Implement:

$$\boxed{\hat{Q}(q^{-1})u(t) = -\hat{P}(q^{-1})y(t+d) + k_m r(t)}$$

where $r(t)$ is the reference signal and k_m is the corresponding control gain.

MV : minimises

$$J_{MV} = E\left\{ [y(t+d)]^2 \right\}$$

GMV : minimise

$$J_{GMV} = E\left\{ [P(q^{-1}) y(t+d)]^2 + [Q(q^{-1}) u(t)]^2 \right\}$$

$$\approx E\left\{ [P(q^{-1}) y(t+d) + Q(q^{-1}) u(t)]^2 \right\}$$

for GMV control, closed-loop characteristic polynomial is $\left\{ P(q^{-1}) B(q^{-1}) + Q(q^{-1}) A(q^{-1}) \right\}$