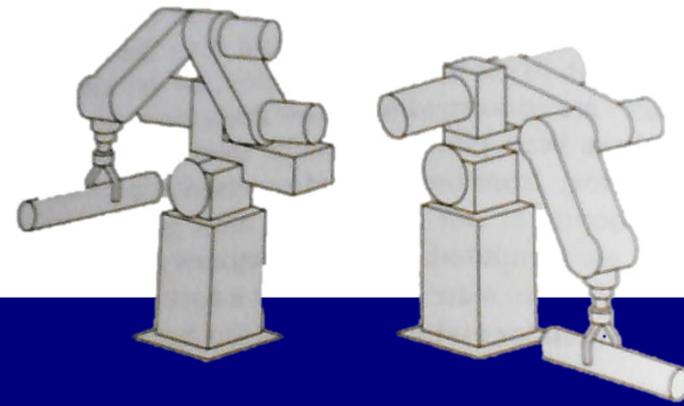

Chapter 3 – Robot Trajectory Planning

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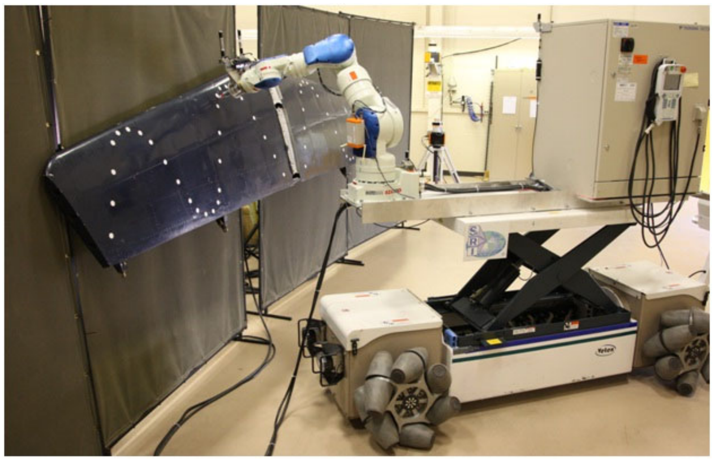
- Trajectory: Time history of *position*, *velocity*, and *acceleration* for each dof
- User simply specifies desired goal position and orientation of the end-effector; system decides the trajectory
- Desired to have smooth path: one that is continuous and has a continuous first derivation, or even a continuous second derivative.

  Note: Rough or jerky motions increase wear on mechanism

Joint space schemes

- Path shapes described in terms of "functions of joint angles"



Source:http://www.swri.org/3pubs/ird2010/synopses/108019.htm

Identify path points[1]

Desired position and orientation of tool frame {T} relative to the station frame {S}

Using inverse kinematics

A set of desired joint angles

A smooth function is used for each joint

All joints reach the via points at the same time

[1] Path points includes all the via points plus the initial and final points; via point is a point located midway between the starting and stopping positions of a robot tool tip, through which the tool tip passes without stopping.

Joint space schemes

Remark:

- Joint space schemes
  - achieve desired position and orientation only at via points.
  - Discrete correspondence between joint space and Cartesian space => No problem with singularities

Cubic polynomials

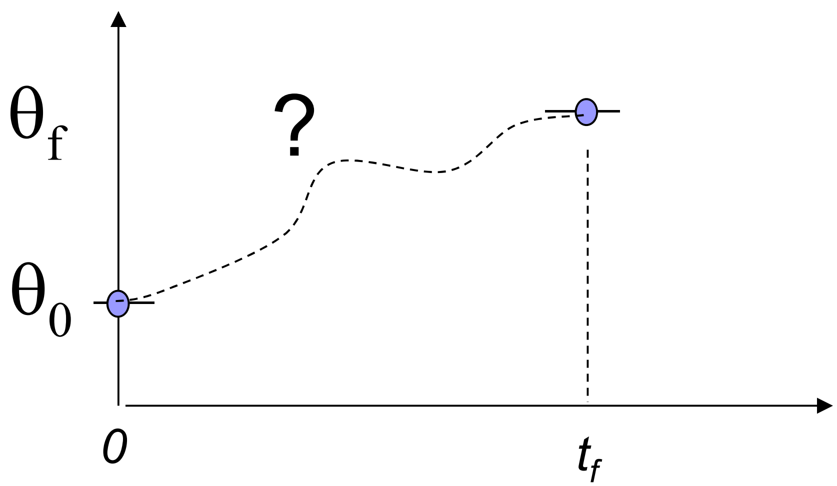
- Say we wish to move the tool from its initial position to a goal position in a certain amount of time:
  - Inverse kinematics to calculate the set of joint angles corresponding to the initial and goal positions and orientations.
  - Adopt a smooth function to interpolate each joint value from initial value to goal value.

Cubic polynomials

- 4 constraints for each of the joints' motion:

Initial & Final Positions $\theta(0) = \theta_0$ $\theta(t_f) = \theta_f$

Initial & Final Velocities $\dot{\theta}(0) = 0$ $\dot{\theta}(t_f) = 0$



Cubic polynomials

- The constraints can be satisfied by a third degree polynomial:

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \qquad \textbf{(2-1)}$$

Joint velocity and acceleration along this path:

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 \qquad \textbf{(2-2)}$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t \qquad \textbf{(2-3)}$$

Cubic polynomials

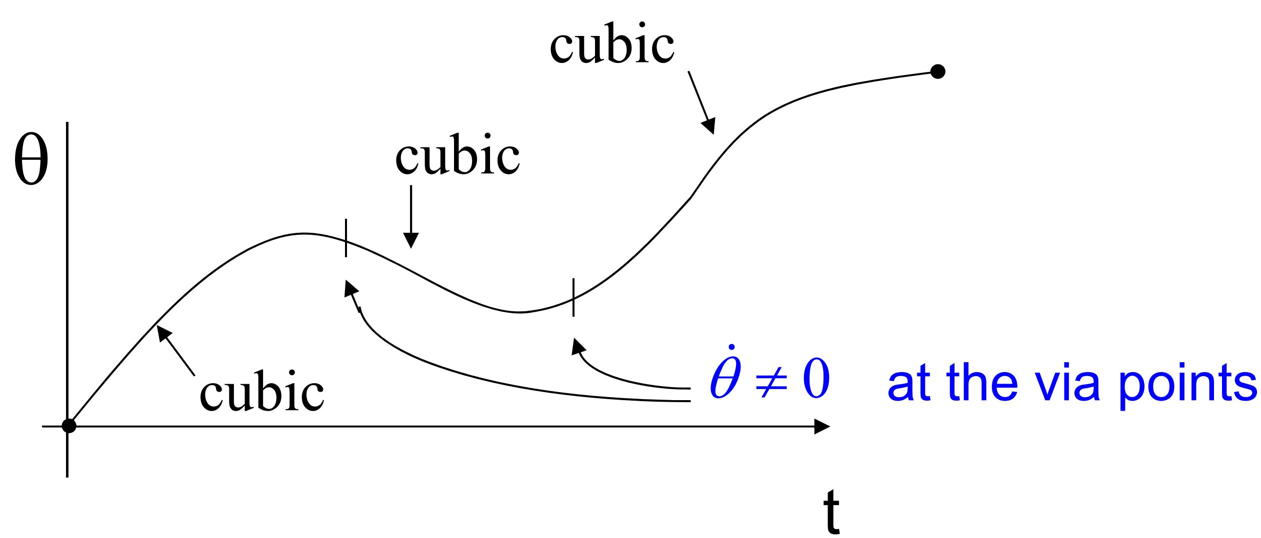
- Four equations (constraints) in four unknowns $(a_0, \ldots, a_3)$ =>

$$a_0 = \theta_0$$
$$a_1 = 0$$
$$a_2 = \frac{3}{t_f^{\,2}}(\theta_f - \theta_0)$$
$$a_3 = \frac{-2}{t_f^{\,3}}(\theta_f - \theta_0)$$

Remark:
This solution is for the case when the joint starts and finishes at zero velocity

Cubic Polynomials with via Points

- In general, path specification may include intermediate via points.
- Desired velocities of the joints at the via points may not be zero.



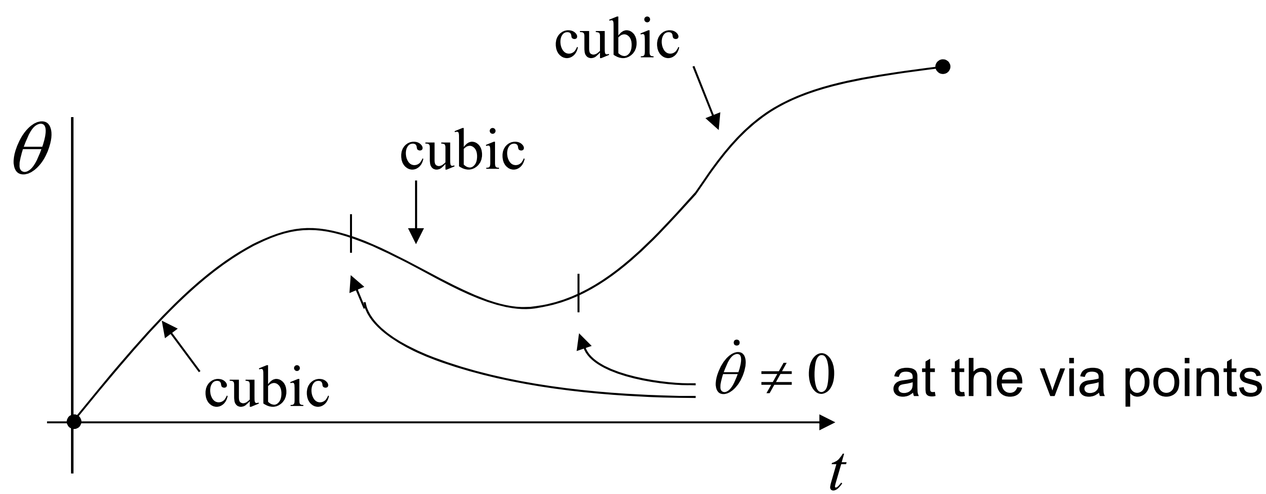
Cubic Polynomials with via Points

- Using a cubic polynomial for each segment
  Velocity constraints at each end:

$$\dot{\theta}(0) = \dot{\theta}_0$$

$$\dot{\theta}(t_f) = \dot{\theta}_f$$

(here, velocity constraints need not be zero)



Cubic Polynomials with via Points

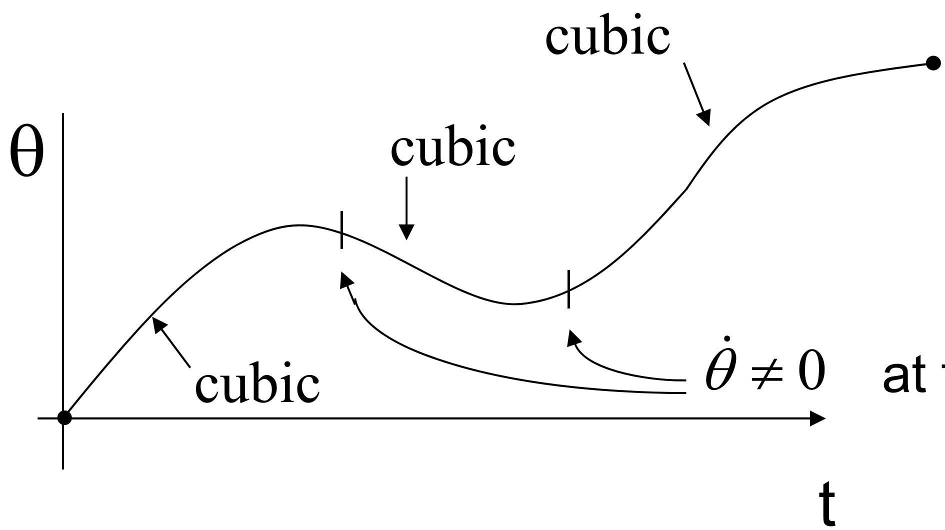
- Applying the position and velocity constraints at each end to Eqs (2-1) and (2-2), we obtain:

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{3}{t_f^2}\left(\theta_f - \theta_0\right) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

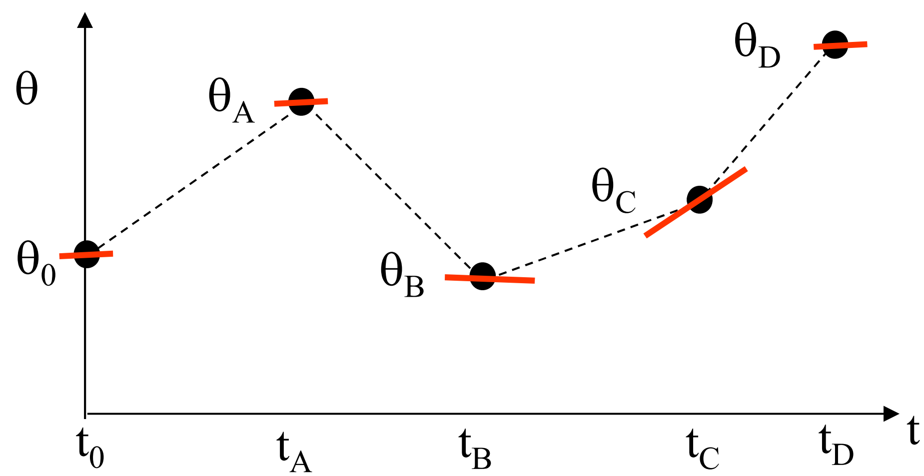
$$a_3 = -\frac{2}{t_f^3}\left(\theta_f - \theta_0\right) + \frac{1}{t_f^2}\left(\dot{\theta}_f + \dot{\theta}_0\right)$$



at the via points

Cubic Polynomials with via Points

- How to specify the joint velocities at via points?
  - User specifies desired velocity at each via point in terms of a Cartesian linear and angular velocity of the tool frame at that instant. Then, apply inverse Jacobian (singularity issue).
  - System automatically chooses the velocities at the via points by suitable heuristic:



Higher order polynomials

- To specify the position, velocity, and acceleration at the beginning and end of a path segment, a quintic polynomial is required:

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\left.\begin{array}{l}\text{Initial \& final positions} \\ \text{''} \quad \text{''} \quad \text{''} \quad \text{velocities} \\ \text{''} \quad \text{''} \quad \text{''} \quad \text{accelerations}\end{array}\right\} \text{6 constraints}$$

Remark:

- Cubic trajectory has discontinuities in acceleration => Leads to impulsive **jerk (derivative of acceleration)** => May excite vibrational modes => Reduce tracking accuracy
- With fifth order polynomial, can specify constraints (say zero) for acceleration at the start and end

Higher order polynomials

- 6 Equations in 6 unknowns:

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{\ddot{\theta}_0}{2}$$

$$a_3 = \frac{20\,\theta_f - 20\,\theta_0 - (8\,\dot{\theta}_f + 12\,\dot{\theta}_0)t_f - (3\,\ddot{\theta}_0 - \ddot{\theta}_f)t_f^{\,2}}{2t_f^{\,3}}$$

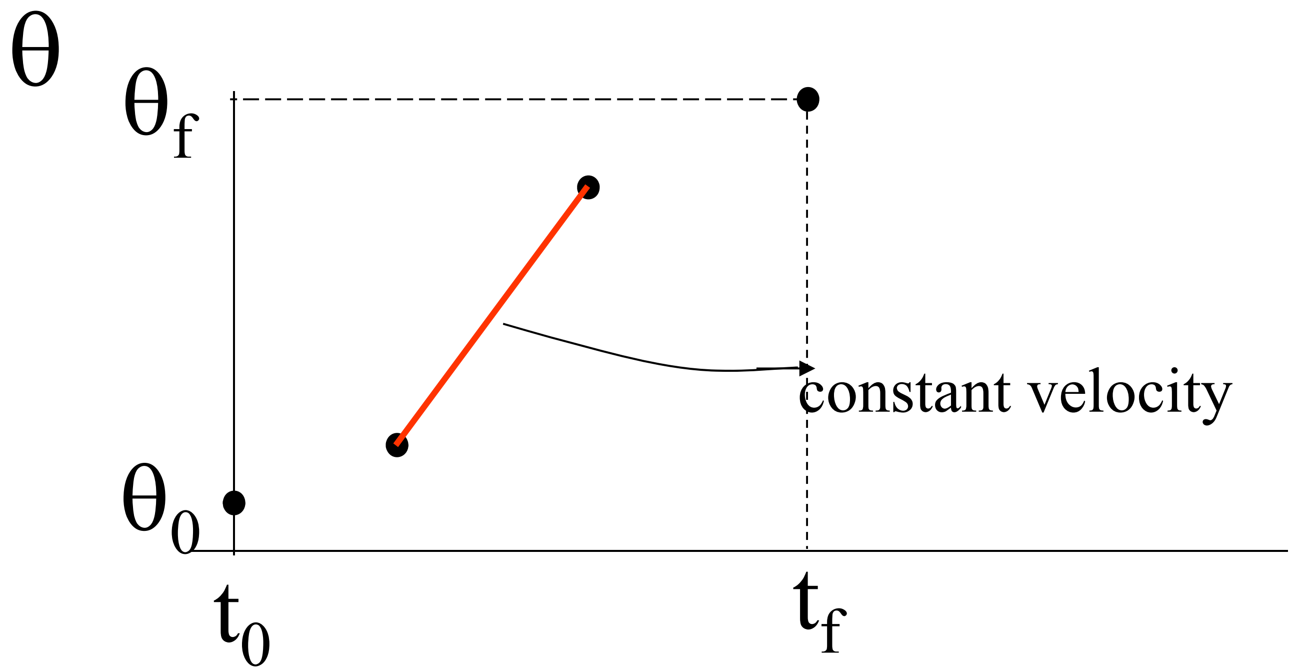
$$a_4 = \frac{30\,\theta_0 - 30\,\theta_f + (14\,\dot{\theta}_f + 16\,\dot{\theta}_0)t_f + (3\,\ddot{\theta}_0 - 2\ddot{\theta}_f)t_f^{\,2}}{2t_f^{\,4}}$$

$$a_5 = \frac{12\,\theta_f - 12\,\theta_0 - (6\,\dot{\theta}_f + 6\dot{\theta}_0)t_f - (\ddot{\theta}_0 - \ddot{\theta}_f)t_f^{\,2}}{2t_f^{\,5}}$$

Linear Segment with Parabolic Blend

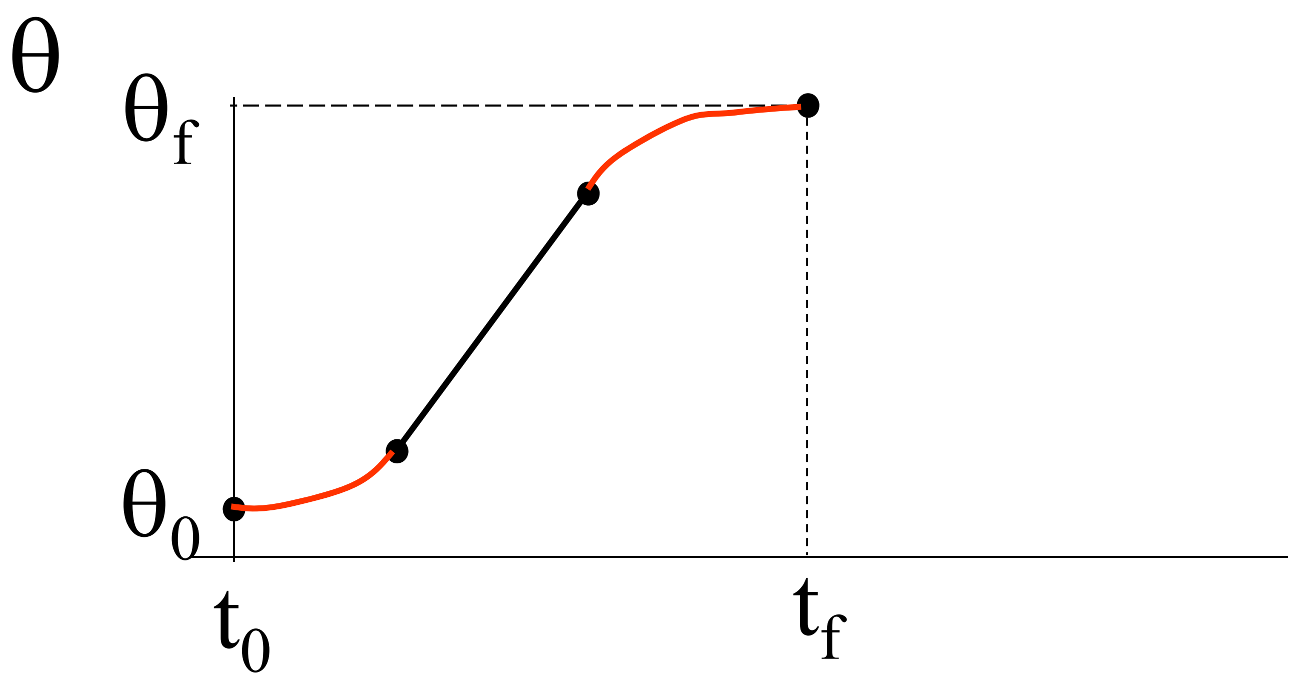
- Another choice of path shape is linear

Let's assume $\dot{\theta}(t_o) = \dot{\theta}(t_f) = 0$



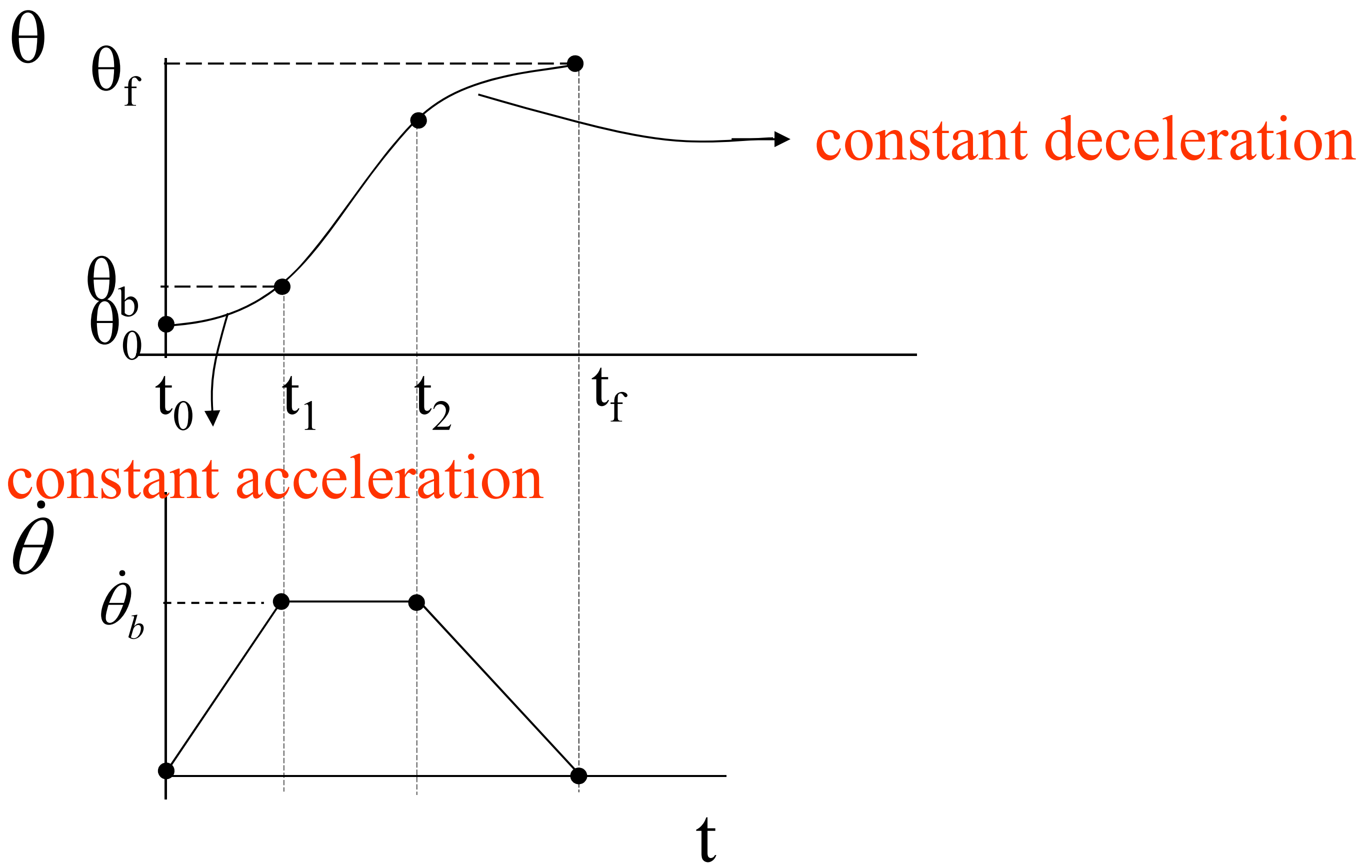
Linear Segment with Parabolic Blend

- Add parabolic blend region at each path point to ensure smooth path with continuous position and velocity



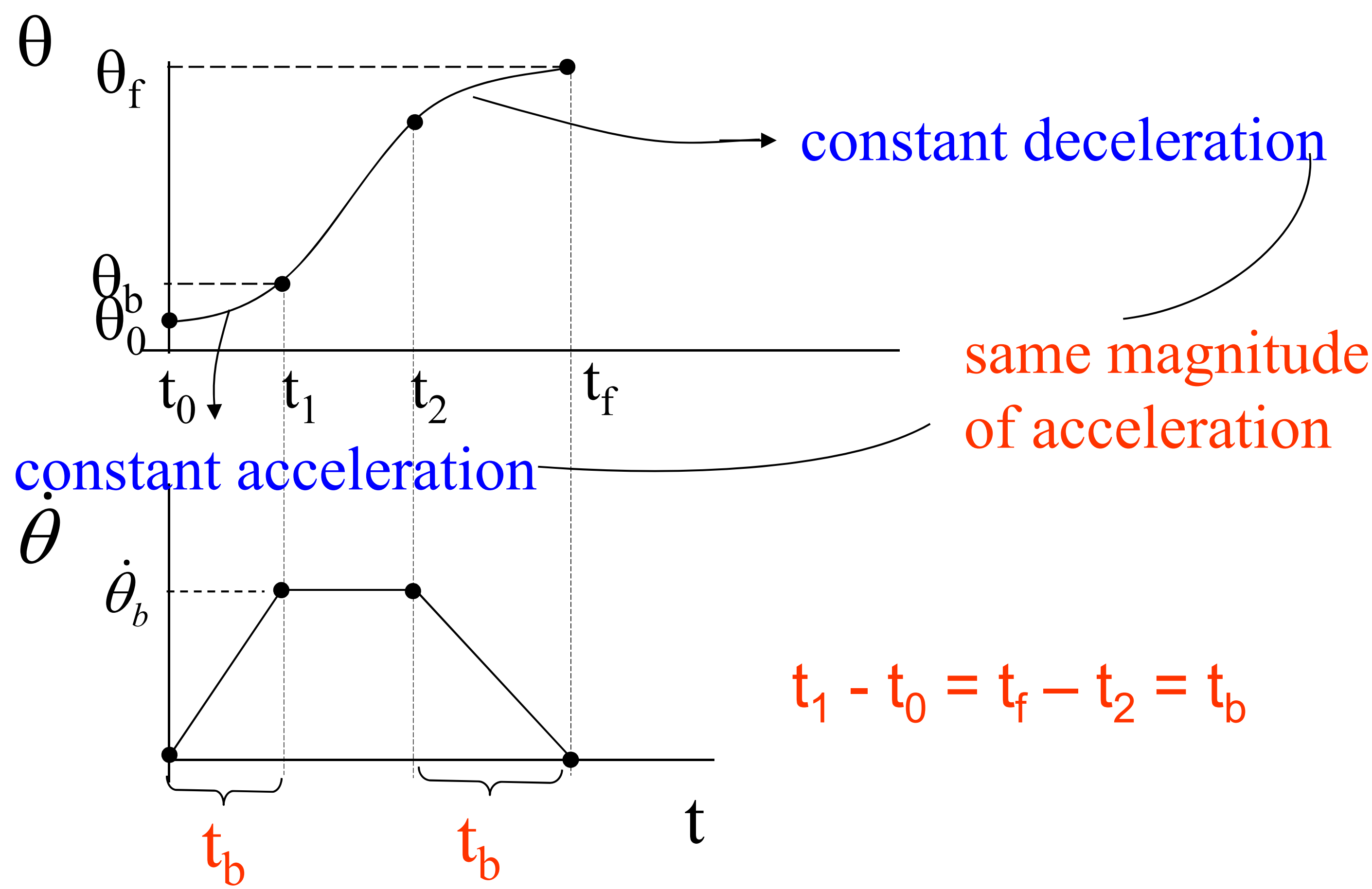
Linear Segment with Parabolic Blend

- During blend portion, constant acceleration is used



Linear Segment with Parabolic Blend

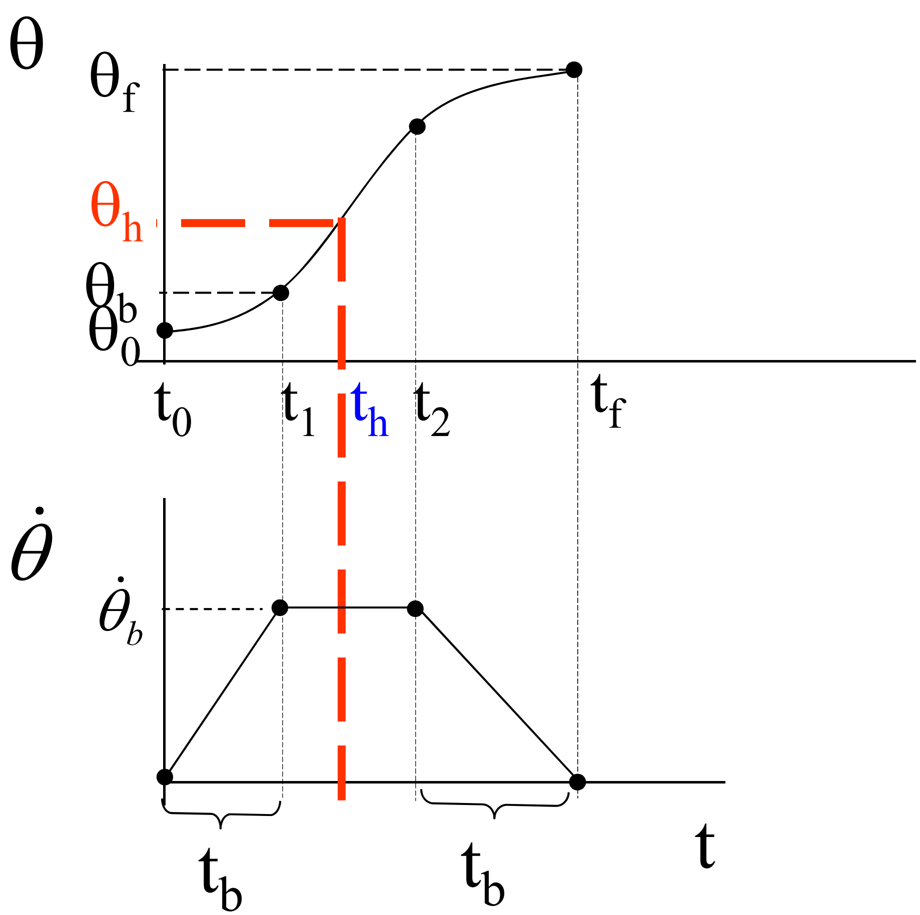
- Assume both parabolic blends have same duration (=> same magnitude for the acceleration)



$t_1 - t_0 = t_f - t_2 = t_b$

Linear Segment with Parabolic Blend

- Halfway point position θ_h occurs at halfway point in time t_h (some form of symmetry)



$\theta_h = \theta(t_h)$

$\theta_h = ½(\theta_f + \theta_0)$

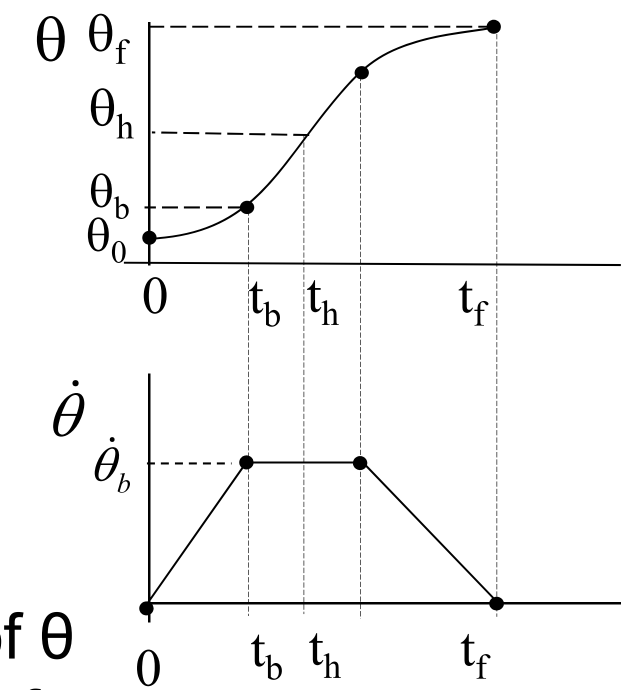
$t_h = ½(t_f + t_0)$

Linear Segment with Parabolic Blend

- There are many possible solutions, given

$$\theta(0) = \theta_0 \qquad \dot{\theta}(0) = 0 \quad (\text{assume } t_o = 0)$$

$$\theta(t_f) = \theta_f \qquad \dot{\theta}(t_f) = 0$$



Assume blend acceleration $\ddot{\theta}$ is given, find t_b

Velocity at end of first blend region = velocity of linear section

$$\dot{\theta}_b = \cancelto{0}{\dot{\theta}_0} + \ddot{\theta} t_b = \frac{\theta_h - \theta_b}{t_h - t_b} \qquad \textbf{(2-4)}$$

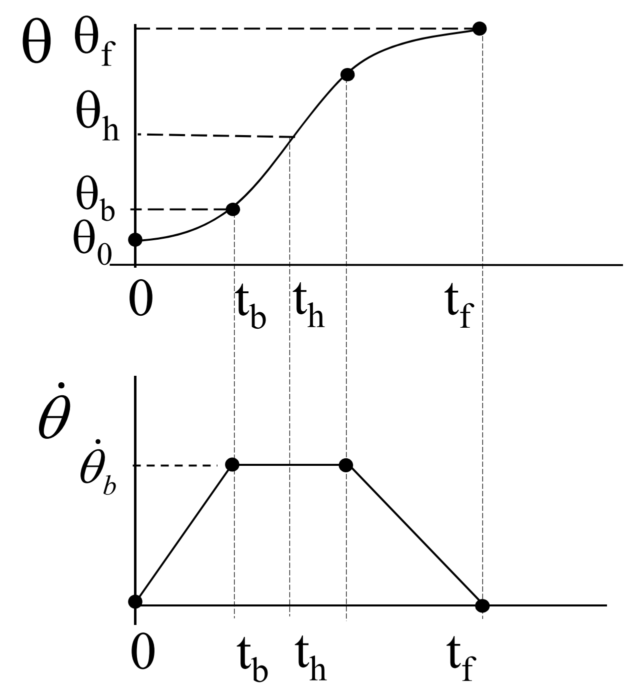
where $\theta_h = \left(\frac{\theta_f + \theta_0}{2}\right)$, $t_h = t_f/2$, θ_b is value of θ at the end of blend region, t_b is the duration of the parabolic blends

Linear Segment with Parabolic Blend

- θ_b is given by:

$$\theta_b = \theta_0 + \frac{1}{2}\ddot{\theta}t_b^2 \qquad \textbf{(2-5)}$$

where $\ddot{\theta}$ is the constant acceleration at the blends



Given θ_0, θ_f and t_f (total move time), and substitute Eq (2-5) into (2-4), we get:

$$\ddot{\theta}t_b^2 - \ddot{\theta}t_f t_b + \left(\theta_f - \theta_0\right) = 0$$

Linear Segment with Parabolic Blend

- Hence,

$$t_b = \frac{t_f}{2} - \frac{\sqrt{\ddot{\theta}^2 t_f^{\,2} - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

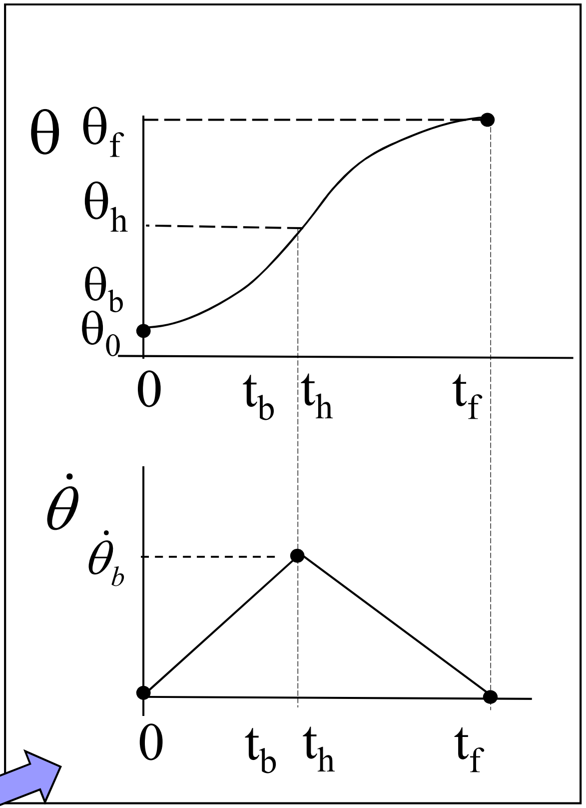
For t_b to exists, $\ddot{\theta} \geq \frac{4(\theta_f - \theta_0)}{t_f^{\,2}}$

(acceleration must be sufficiently high)

When equality holds:

$$t_b = \frac{t_f}{2} = t_h$$

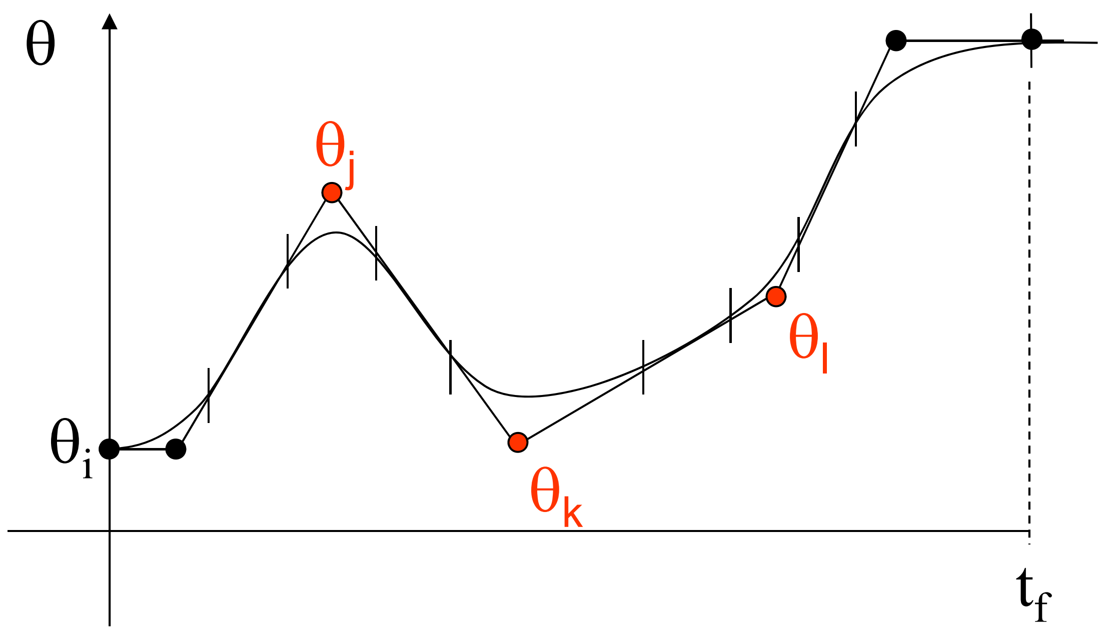
no constant velocity or linear segment



Linear Segments with Parabolic Blends & with Via Points

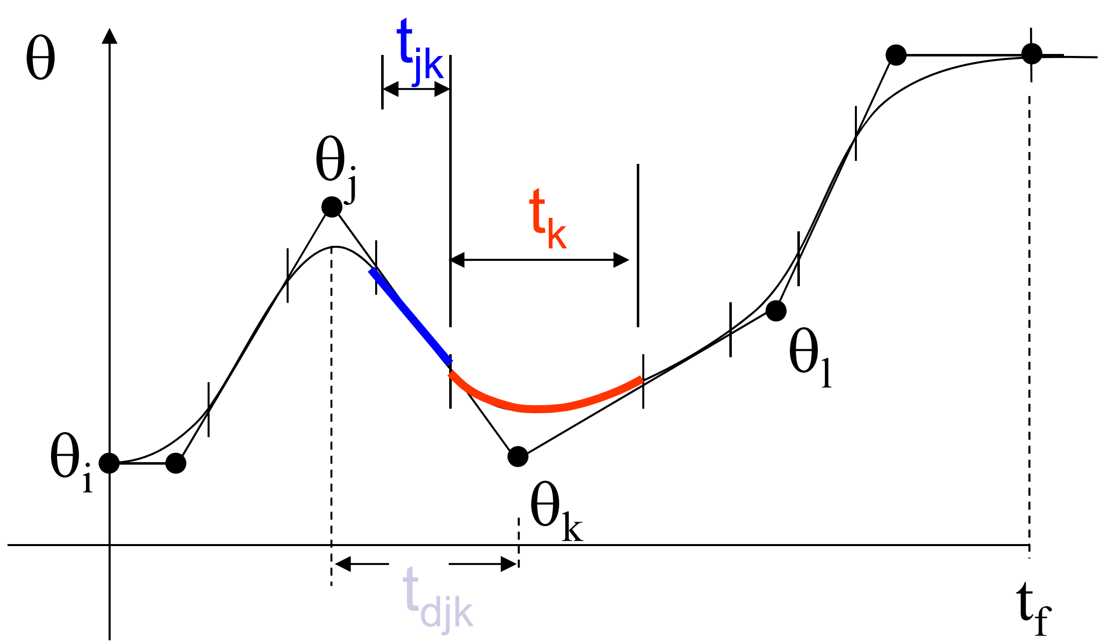
- Linear functions connect via points and parabolic blend regions added around each via point (or pseudo via point)

  Considering three neighbouring path points which we call points j, k and l



Linear Segments with Parabolic Blends & with Via Points

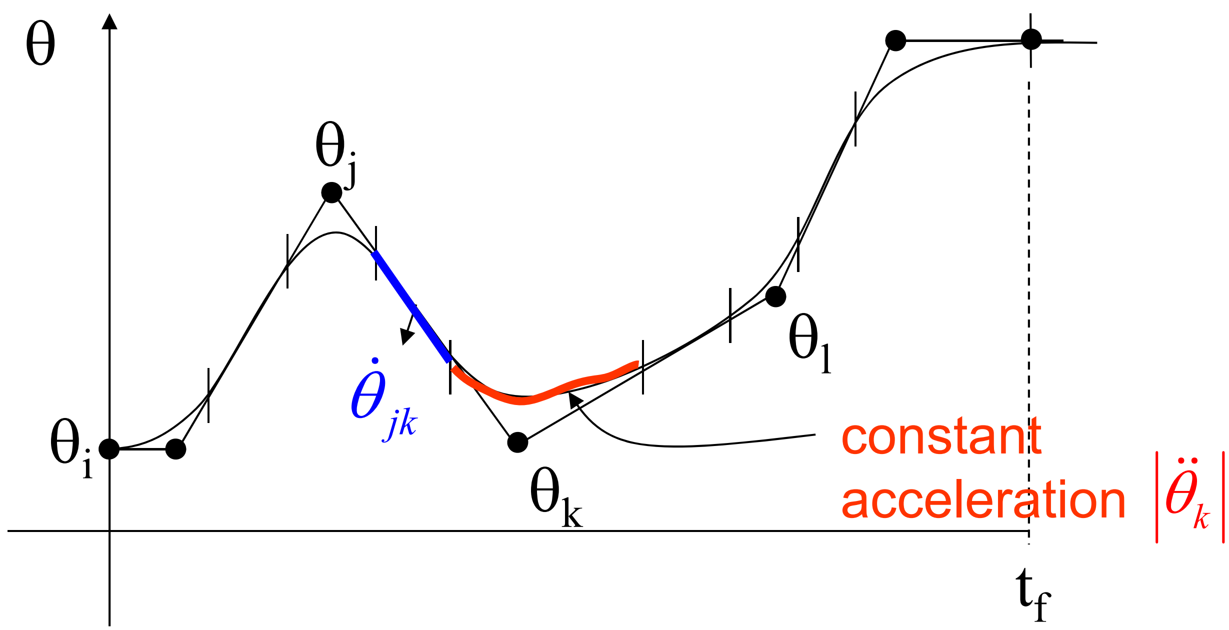
Given: t_{djk} – overall duration of the segment connecting points j and k
Find: t_k – duration of blend region at path point k
and t_{jk} – duration of the linear portion between points j and k



Linear Segments with Parabolic Blends & with Via Points

$\dot{\theta}_{jk}$ - velocity during the linear portion

$\ddot{\theta}_k$ - acceleration during the blend at point k



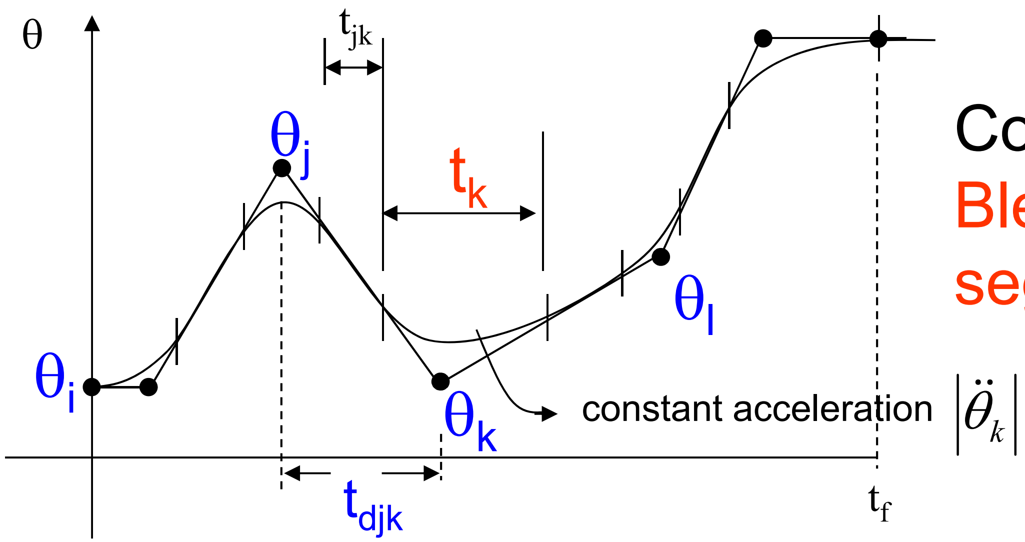
Linear Segments with Parabolic Blends & with Via Points

- Given:

- all path points θ_k

- all durations t_{djk}

- magnitude of acceleration at each path point $|\ddot{\theta}_k|$



Compute:
Blend times t_k and linear segment times t_{jk}

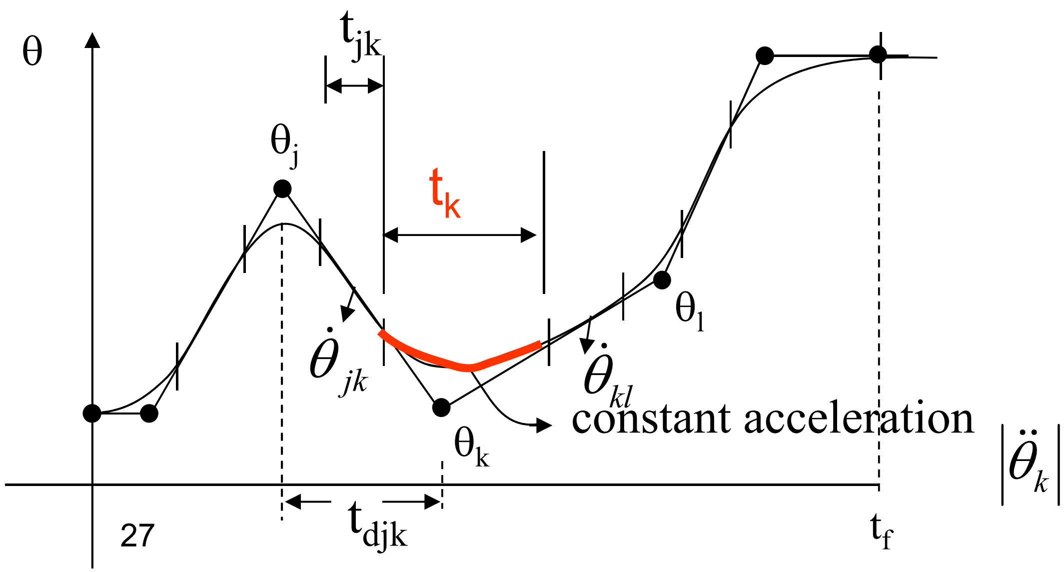
Linear Segments with Parabolic Blends & with Via Points

- For interior path points (say between j & k):

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk}} \qquad \dot{\theta}_{kl} = \frac{\theta_l - \theta_k}{t_{dkl}}$$

$$\ddot{\theta}_k = \mathrm{sgn}(\dot{\theta}_{kl} - \dot{\theta}_{jk})\left|\ddot{\theta}_k\right|$$

$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k}$$



Similarly, t_j can be obtained.

Then, duration of linear segment,

$$t_{jk} = t_{djk} - \tfrac{1}{2}\, t_j - \tfrac{1}{2}\, t_k$$

Assuming parabolic blend is symmetric around the path points

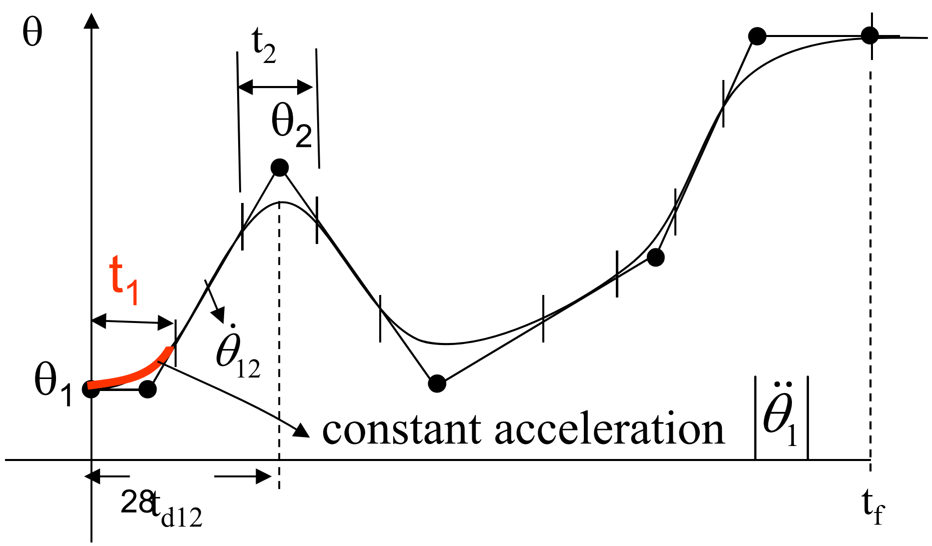
Linear Segments with Parabolic Blends & with Via Points

- First Segment

To find the blend time t_1 at the initial point :

$$\frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1} = \ddot{\theta}_1 t_1$$ = velocity at 1st line segment ($\dot{\theta}_{12}$)

where $\ddot{\theta}_1 = \operatorname{sgn}(\theta_2 - \theta_1)\left|\ddot{\theta}_1\right|$



$$\therefore\ t_1 = t_{d_{12}} - \sqrt{t_{d_{12}}^{\ 2} - \frac{2(\theta_2 - \theta_1)}{\ddot{\theta}_1}}$$

Hence,

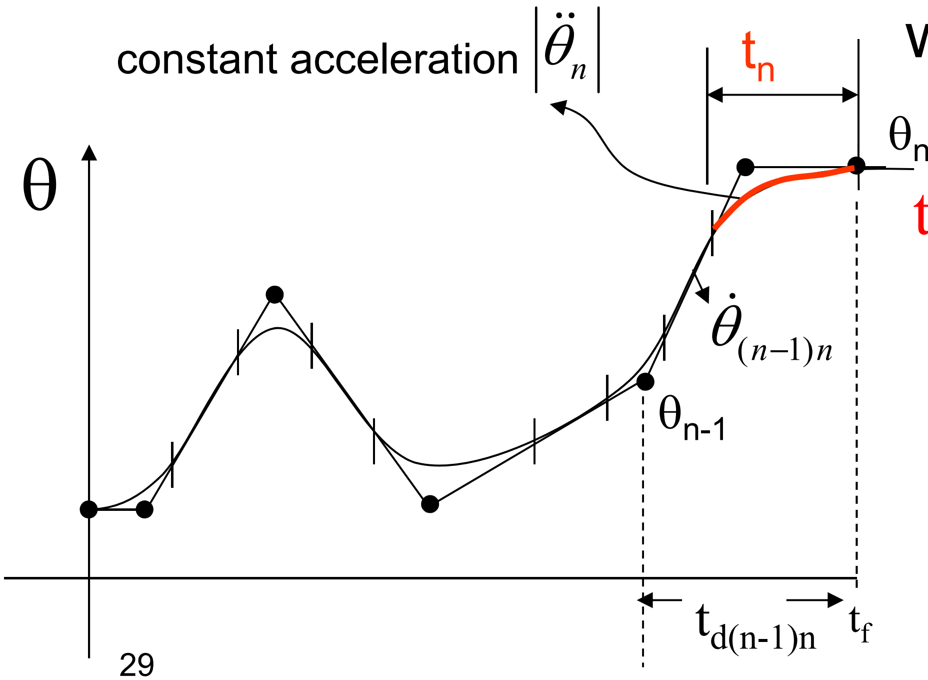
$$t_{12} = t_{d12} - t_1 - ½\, t_2$$

Linear Segments with Parabolic Blends & with Via Points

- Last segment connecting θ_{n-1} & θ_n

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2} t_n} = -\ddot{\theta}_n t_n$$

where $\ddot{\theta}_n = \mathrm{sgn}(\theta_{n-1} - \theta_n)\left|\ddot{\theta}_n\right|$



$$t_n = t_{d(n-1)n} - \sqrt{t_{d(n-1)n}^{\;2} + \frac{2(\theta_n - \theta_{n-1})}{\ddot{\theta}_n}}$$

Hence,

$$t_{(n-1)\,n} = t_{d(n-1)n} - t_n - \tfrac{1}{2}\, t_{n-1}$$

Summary

- Plan point-to-point trajectories in joint space
- Plan trajectories with via points
- Plan trajectories with *velocity* and *acceleration* constraints