Task1 Introduction and Literature Review

With the advancement of industrialization and the advent of the information age, intelligent robots play an increasingly important role in intelligent manufacturing, intelligent transportation system, the Internet of things, medical health, and intelligent services.

From the perspective of the application environment, intelligent robots can be divided into industrial robots, service robots, and specialized robots.

Industrial robot

Industrial robots are important automation equipment of indthe ern manufacturing industry, which integrate the advanced technology of multi-disciplinary, such as machinery, electronics, control, computer, sensor, artificial intelligence, etc.

They are widely used in several industrial production activities, such as spraying, welding, handling, and have a great role in these sectors. Recently, robotic technology is developing towards high precision, high intelligence. Robot calibration technology has a great significance to improve the accuracy of the robot.

At present, one of the development directions of industrial robots is focused on how to improve positioning accuracy, which has also become one of the key technologies for the practical application of offline programming methods in advanced robotic manufacturing systems. The pose error of the robot can be greatly reduced by calibration, the absolute accuracy of the robot can be improved for the level of repeat accuracy.

There are many calibration algorithms, such as 1) Least Squares Algorithm; 2) Levenberg-Marquardt Algorithm; 3) Extended Kalman and Particle Filter Algorithm; 4) Maximum Likelihood Estimation Algorithm. However, **non-parametric calibration** uses **intelligent algorithms** to solve the problem of nonlinear calibration. Non-parametric calibration gives a new research idea for researchers, which has a close relationship with artificial intelligence.

Service robot

According to the definition of International Federation of Robotics (IFR), service robot is a “robot that performs useful tasks for humans or equipment excluding industrial automation applications”.

For example, the servant robots, such as intelligent sweeping robots and window cleaning robots, can work as human assistants to do housework. They can also provide navigation services, automatically making path planning and avoiding obstacles. Other kinds of service robots, like family socialization robots, companion robots, mobile assistant robots and pet exercising robots, are able to interact with people, as well as completing delivering missions, caring for the elderly and children, reminding about events and patrolling houses.

Rapidly improving technologies become smarter, more powerful, smaller, lighter and cheaper. These include sensors, cameras, speech processing, image processing, biometrics, analytics, mobile and cloud technologies, geo-tagging and more, and they are increasingly powered by **artificial intelligence (AI).**

Specialized robots

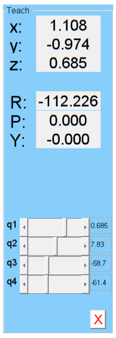
Specialized robots are robots that apply to special environments. They can assist in completing tasks in dangerous and harsh environment or tasks requiring high precision.

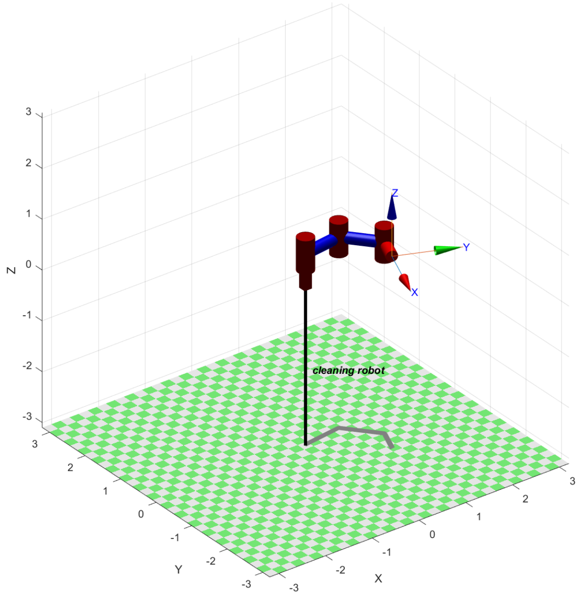
For example, the medical robots provide advanced solutions to surgical treatment and rehabilitation which is significant for reducing the difficulty of surgery and treatment, and shortening the recovery time, such robots include master-slave surgical robot, orthopedics robot, capsule endoscopy robot, rehabilitation robot, intelligent prostheses, aged service robot and nursing robot.

The military robots, including reconnaissance robots, battlefield robots, minesweeper robots and military UAVs, have a long history and have already been put into the battlefield, which make contributions to material transportation, search and exploration, anti-terrorist rescue and military attack. The anti-terrorist and rescue robots consist of EOD robots, fire-fighting robots, life detection and rescue robots, etc.

Concerning with the exploration cause, space robots, underwater robots, and pipeline robots have presented impressive performance. In addition, there are some robots for scientific researches and cutting-edge applications, including nanorobot, bionic robots, swarm robots, etc.

Task2 Kinematics and Computing

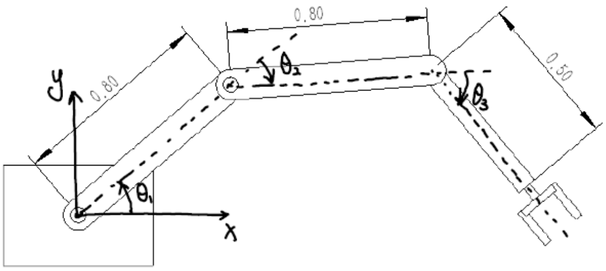


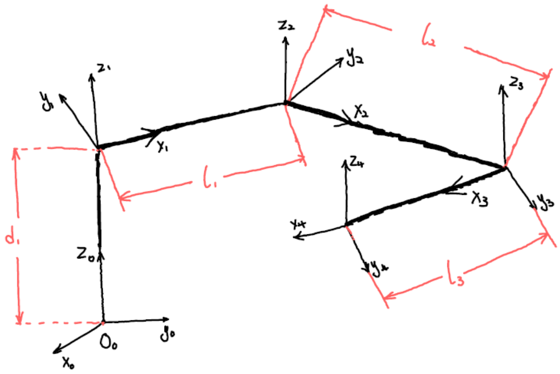


D-H table

Link	θ_i	d_i	a_i	α_i
1	0	d_1	0	0
2	θ_1	0	0.8	0
3	θ_2	0	0.8	0
4	θ_3	0	0.5	0

Forward (Direct) Kinematic Matrix





We first compute the 4x4 homogeneous matrices ${}^{i}_{i-1}A(q_i)$, and get the forward kinematics matrix result ${}^{4}_{0}A(q)$ as follow,

$$
\begin{aligned}
{}^{4}_{0}A(q) &= {}^{0}_{1}A(q_1){}^{1}_{2}A(q_2){}^{2}_{3}A(q_3){}^{3}_{4}A(q_4) \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & 0.5\cos(\theta_1+\theta_2+\theta_3)+0.8\cos(\theta_1+\theta_2)+0.8\cos\theta_1 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & 0.5\sin(\theta_1+\theta_2+\theta_3)+0.8\sin(\theta_1+\theta_2)+0.8\sin\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Jacobian Matrix

The let b_{i-1} denotes as the unit vector along z-axis of the frame $\{i-1\}$. we first determine the joint axes directions, b_{i-1} is the third column of the rotation matrix ${}^{0}_{i-1}R$.

$$
\bar{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
b_1 = {}^{0}_{1}R\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
b_2 = {}^{0}_{1}R^{1}_{2}R\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
b_3 = {}^{0}_{1}R^{1}_{2}R^{2}_{3}R\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

Let $r_{i-1,e}$ is the position vector from O_{i-1} to end-effector. $r_{i-1,e}$, can be computed using 4×4 homogeneous matrices ${}^{i-1}_{i}A$.

Let $X_{i-1,e}$ be 4×1 augmented vector of $r_{i-1,e}$ and $\bar{X} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$, we can get that $X_{i-1,e} = {}^{0}_{1}A \ldots {}^{n-1}_{n}A\bar{X} - {}^{0}_{1}A \ldots {}^{i-2}_{i-1}A\bar{X}$

$$
{}^{0}_{4}A = {}^{0}_{1}A^{1}_{2}A^{2}_{3}A^{3}_{4}A = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & 0.5\cos(\theta_1+\theta_2+\theta_3)+0.8\cos(\theta_1+\theta_2)+0.8\cos\theta_1 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & 0.5\sin(\theta_1+\theta_2+\theta_3)+0.8\sin(\theta_1+\theta_2)+0.8\sin\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
\bar{X}_{1,e} &= {}^{0}_{1}A^{1}_{2}A^{2}_{3}A^{3}_{4}A\bar{X} - {}^{0}_{1}A\bar{X} \\
\bar{X}_{2,e} &= {}^{0}_{1}A^{1}_{2}A^{2}_{3}A^{3}_{4}A\bar{X} - {}^{0}_{1}A^{1}_{2}A\bar{X} \\
\bar{X}_{3,e} &= {}^{0}_{1}A^{1}_{2}A^{2}_{3}A^{3}_{4}A\bar{X} - {}^{0}_{1}A^{1}_{2}A^{2}_{3}A\bar{X}
\end{aligned}
$$

We can get $r_{i-1,e}$, which is the position vector from O_{i-1} to end-effector (expressed in $O_o x_o y_o z_o$)

$$
\begin{aligned}
r_{1,e} &= X_{1,e}(1:3) \\
r_{2,e} &= X_{2,e}(1:3) \\
r_{3,e} &= X_{3,e}(1:3)
\end{aligned}
$$

Since first joint is the prismatic joint and the rest three joint are revolute joints, we can get the Jacobian matrix as follow,

$$
J = \begin{bmatrix} J_{L1} & J_{L2} & J_{L3} & J_{L4} \\ J_{A1} & J_{A2} & J_{A3} & J_{A4} \end{bmatrix}
$$

where

$$
\begin{bmatrix} J_{L1} \\ J_{A1} \end{bmatrix} = \begin{bmatrix} b_0 \\ 0 \end{bmatrix}, \begin{bmatrix} J_{L2} \\ J_{A2} \end{bmatrix} = \begin{bmatrix} b_1 \times r_{1,e} \\ b_1 \end{bmatrix}, \begin{bmatrix} J_{L3} \\ J_{A3} \end{bmatrix} = \begin{bmatrix} b_2 \times r_{2,e} \\ b_2 \end{bmatrix}, \begin{bmatrix} J_{L4} \\ J_{A4} \end{bmatrix} = \begin{bmatrix} b_3 \times r_{3,e} \\ b_3 \end{bmatrix}
$$

Therefore, we can get the Jacobian matrix as follow,

$$J =$$

$$\begin{bmatrix} 0 & -0.5\sin(\theta_1+\theta_2+\theta_3) - 0.8\sin(\theta_1+\theta_2) + 0.8\sin(\theta_1) & -0.5\sin(\theta_1+\theta_2+\theta_3) - 0.8\sin(\theta_1+\theta_2) & -0.5\sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0.5\cos(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) + 0.8\cos(\theta_1) & 0.8\sin(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) & 0.5\sin(\theta_1+\theta_2+\theta_3) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

For inverse kinematics,

with a given endpoint position p_x, p_y, p_z,

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 0.5\cos(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) + 0.8\cos\theta_1 \\ 0.5\sin(\theta_1+\theta_2+\theta_3) + 0.8\sin(\theta_1+\theta_2) + 0.8\sin\theta_1 \\ d_1 \end{bmatrix}$$

$$p_x = 0.5\cos(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) + 0.8\cos\theta_1 \tag{1}$$

$$p_y = 0.5\sin(\theta_1+\theta_2+\theta_3) + 0.8\sin(\theta_1+\theta_2) + 0.8\sin\theta_1 \tag{2}$$

$(1)^2 + (2)^2$, we can get that,

$$p_x^2 + p_y^2 = 1.53 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3) + 1.28\cos(\theta_2)$$

Since we cannot find a close form solution for θ_1, θ_2, θ_3, so assume we already know θ_2,

$$p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53 = 0.8\cos(\theta_2+\theta_3) + 0.8\cos(\theta_3)$$

Using sum-to-product formulas,

$$p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53 = 1.6\cos\left(\frac{\theta_2+2\theta_3}{2}\right)\cos\left(\frac{\theta_2}{2}\right)$$

$\therefore$ we can get θ_3 as follow,

$$\cos\left(\frac{\theta_2+2\theta_3}{2}\right) = \frac{p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53}{1.6\cos\left(\frac{\theta_2}{2}\right)}$$

$$\frac{\theta_2+2\theta_3}{2} = \pm\arccos\left(\frac{p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53}{1.6\cos\left(\frac{\theta_2}{2}\right)}\right)$$

$$\theta_3 = \pm\arccos\left(\frac{p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53}{1.6\cos\left(\frac{\theta_2}{2}\right)}\right) - \frac{\theta_2}{2}$$

Since we already get θ_2 and θ_3, to get θ_1 using (1) and (2),

$$\cos(\theta_1+\theta_2+\theta_3)p_x = 0.5\cos(\theta_1+\theta_2+\theta_3)^2 + 0.8\cos(\theta_1+\theta_2+\theta_3)\cos(\theta_1+\theta_2) + 0.8\cos(\theta_1+\theta_2+\theta_3)\cos\theta_1 \tag{3}$$

$$\sin(\theta_1+\theta_2+\theta_3)p_y = 0.5\sin(\theta_1+\theta_2+\theta_3)^2 + 0.8\sin(\theta_1+\theta_2+\theta_3)\sin(\theta_1+\theta_2) + 0.8\sin(\theta_1+\theta_2+\theta_3)\sin\theta_1 \tag{4}$$

$(3) + (4)$ we can get,

$$\cos(\theta_1+\theta_2+\theta_3)p_x + \sin(\theta_1+\theta_2+\theta_3)p_y = 0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3)$$

Let ϕ denotes a new angle, which $\cos(\phi) = \frac{p_x}{\sqrt{p_x^2+p_y^2}}$, $\sin(\phi) = \frac{p_y}{\sqrt{p_x^2+p_y^2}}$,

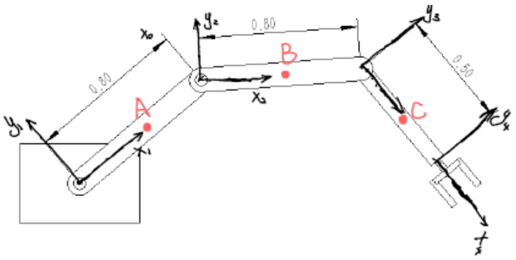
$$\sqrt{p_x^2+p_y^2}\cos(\theta_1+\theta_2+\theta_3-\phi) = 0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3)$$

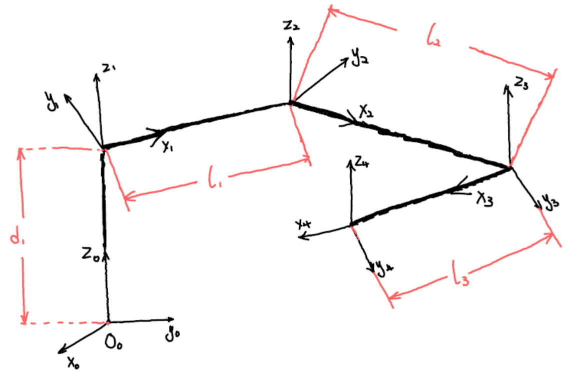
$$\theta_1+\theta_2+\theta_3-\phi = \pm\arccos\left(\frac{0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3)}{\sqrt{p_x^2+p_y^2}}\right)$$

$\therefore$ we can get θ_1 as follow,

$$\theta_1 = \pm \arccos\left(\frac{0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2 + \theta_3)}{\sqrt{p_x^2 + p_y^2}}\right) - \theta_2 - \theta_3 + \phi$$

Task3 Dynamics and Computing





Assume that the mass of each link is lumped at end of the link, so the center of mass of each link is represented in its coordinate system as,

$$P_{C1}^{1}=\begin{bmatrix}0\\0\\d_1\end{bmatrix},\quad P_{C2}^{2}=\begin{bmatrix}0.4\\0\\0\end{bmatrix},\quad P_{C3}^{3}=\begin{bmatrix}0.4\\0\\0\end{bmatrix},\quad P_{C4}^{4}=\begin{bmatrix}0.25\\0\\0\end{bmatrix}$$

Each coordinate system is represented under its previous coordinate system as

$$P_{1}^{0}=\begin{bmatrix}0\\0\\d_1\end{bmatrix},\quad P_{2}^{1}=\begin{bmatrix}0.8\\0\\0\end{bmatrix},\quad P_{3}^{2}=\begin{bmatrix}0.8\\0\\0\end{bmatrix},\quad P_{4}^{3}=\begin{bmatrix}0.5\\0\\0\end{bmatrix}$$

The xy position are as follow,

$$\begin{cases}x_A=0.4\cos\theta_1\\y_A=0.4\sin\theta_1\end{cases},\quad \begin{cases}x_B=0.4\cos(\theta_1+\theta_2)+0.8\cos\theta_1\\y_B=0.4\sin(\theta_1+\theta_2)+0.8\sin\theta_1\end{cases},\quad \begin{cases}x_C=0.25\cos(\theta_1+\theta_2+\theta_3)+0.8\cos(\theta_1+\theta_2)+0.8\cos\theta_1\\y_C=0.25\sin(\theta_1+\theta_2+\theta_3)+0.8\sin(\theta_1+\theta_2)+0.8\sin\theta_1\end{cases}$$

We can get the xy velocity as follow,

$$\begin{cases}\dot{x}_A=-0.4\sin\theta_1\dot{\theta}_1\\\dot{y}_A=0.4\cos\theta_1\dot{\theta}_1\end{cases}$$

$$\begin{cases}\dot{x}_B=-0.4\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)-0.8\sin\theta_1\dot{\theta}_1\\\dot{y}_B=0.4\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)+0.8\cos\theta_1\dot{\theta}_1\end{cases}$$

$$\begin{cases}\dot{x}_C=-0.25\sin(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)-0.8\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)-0.8\sin\theta_1\dot{\theta}_1\\\dot{y}_C=0.25\cos(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)+0.8\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)+0.8\cos\theta_1\dot{\theta}_1\end{cases}$$

So we can get v_A, v_B, v_C

$$v_A^2=\dot{x}_A^2+\dot{y}_A^2=0.16\dot{\theta}_1^2$$

$$v_B^2=\dot{x}_B^2+\dot{y}_B^2=0.64\dot{\theta}_1^2+0.16(\dot{\theta}_1+\dot{\theta}_2)^2+0.64\cos\theta_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)$$

$$\begin{aligned}v_C^2=\dot{x}_C^2+\dot{y}_C^2=&\,0.64\dot{\theta}_1^2+0.64(\dot{\theta}_1+\dot{\theta}_2)^2+0.0625(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2+0.4\cos\theta_3(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)(\dot{\theta}_1+\dot{\theta}_2)\\&+0.4\cos(\theta_2+\theta_3)\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)+1.28\cos\theta_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\end{aligned}$$

N-E Equations

So, for the Newton equation, $\dot{v}$ denotes acceleration of center of mass and F denotes the force at center of mass

$$m\dot{v} = F$$

For the Euler equation, ω---angular velocity, $\dot{\omega}$---angular acceleration, N---the torque acting on the body, I---the inertia tensor in a frame with its origin located at the center of mass

$$I\dot{\omega} + \omega \times I\omega = N$$

For prismatic joint 1,

Linear acceleration of the origin of frame 1 is

$$\dot{v}_1^1 = R_0^1[\dot{\omega}_0^0 P_1^0 + \omega_0^0 \times (\omega_0^0 \times P_1^0) + \dot{v}_0^0] + 2\omega_1^1 \times \dot{d}_1 Z_1^1 + \ddot{d}_1 Z_1^1$$

Linear acceleration of the center of mass

$$\dot{v}_{C1}^1 = \dot{\omega}_1^1 \times P_{C1}^1 + \omega_1^1 \times (\omega_1^1 \times P_{C1}^1) + \dot{v}_1^1$$

$$F_1^1 = m_1 \dot{v}_{C1}^1$$

$$N_1^1 = I_1^1 \dot{\omega}_1^1 + \dot{\omega}_1^1 \times I_1^1 \dot{\omega}_1^1$$

For revolute joint 2,

Linear acceleration of the origin of frame 2 is

$$\dot{v}_2^2 = R_1^2[\dot{\omega}_1^1 P_2^1 + \omega_1^1 \times (\omega_1^1 \times P_2^1) + \dot{v}_1^1]$$

Linear acceleration of the center of mass

$$\dot{v}_{C2}^2 = \dot{\omega}_2^2 \times P_{C2}^2 + \omega_2^2 \times (\omega_2^2 \times P_{C2}^2) + \dot{v}_2^2$$

$$F_2^2 = m_2 \dot{v}_{C2}^2$$

$$N_2^2 = I_2^2 \dot{\omega}_2^2 + \dot{\omega}_2^2 \times I_2^2 \dot{\omega}_2^2$$

For revolute joint 3,

Linear acceleration of the origin of frame 3 is

$$\dot{v}_3^3 = R_2^3[\dot{\omega}_2^2 P_3^2 + \omega_2^2 \times (\omega_2^2 \times P_3^2) + \dot{v}_2^2]$$

Linear acceleration of the center of mass

$$\dot{v}_{C3}^3 = \dot{\omega}_3^3 \times P_{C3}^3 + \omega_3^3 \times (\omega_3^3 \times P_{C3}^3) + \dot{v}_3^3$$

$$F_3^3 = m_3 \dot{v}_{C3}^3$$

$$N_3^3 = I_3^3 \dot{\omega}_3^3 + \dot{\omega}_3^3 \times I_3^3 \dot{\omega}_3^3$$

For revolute joint 4,

Linear acceleration of the origin of frame 4 is

$$\dot{v}_4^4 = R_3^4[\dot{\omega}_3^3 P_4^3 + \omega_3^3 \times (\omega_3^3 \times P_4^3) + \dot{v}_3^3]$$

Linear acceleration of the center of mass

$$\dot{v}_{C4}^4 = \dot{\omega}_4^4 \times P_{C4}^4 + \omega_4^4 \times (\omega_4^4 \times P_{C4}^4) + \dot{v}_3^3$$

$$F_4^4 = m_4 \dot{v}_{C4}^4$$

$$N_4^4 = I_4^4 \dot{\omega}_4^4 + \dot{\omega}_4^4 \times I_4^4 \dot{\omega}_4^4$$

L-E Equations

We denote the prismatic joint speed as v_1

$$v_1 = \dot{d}_1$$

Considering the kinetic energy of the system consists of the Kinetic Energy of all link1, link2, link3, link4 respectively, we can obtain that

$$\begin{aligned} K &= K_1 + K_2 + K_3 + K_4 \\ &= \frac{1}{2}m_1\dot{d_1}^2 + [\frac{1}{2}m_2 v_A^2 + \frac{1}{2}I_1\dot{\theta_1}^2] + [\frac{1}{2}m_3 v_B^2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2] + [\frac{1}{2}m_4 v_C^2 + \frac{1}{2}I_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2] \end{aligned}$$

The potential energy of this system can be shown as follow,

$$P = (m_1 + m_2 + m_3 + m_4)gd_1$$

Since v_A, v_B, v_C as follow,

$$v_A^2 = 0.16\dot{\theta}_1^2$$

$$v_B^2 = 0.64\dot{\theta}_1^2 + 0.16(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.64\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)$$

$$\begin{gathered} v_C^2 = 0.64\dot{\theta}_1^2 + 0.64(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.0625(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + 0.4\cos\theta_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)(\dot{\theta}_1 + \dot{\theta}_2) \\ +0.4\cos(\theta_2 + \theta_3)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) + 1.28\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) \end{gathered}$$

$$\begin{aligned} K &= K_1 + K_2 + K_3 + K_4 \\ &= \frac{1}{2}m_1\dot{d_1}^2 + [\frac{1}{2}m_2 v_A^2 + \frac{1}{2}I_1\dot{\theta_1}^2] + [\frac{1}{2}m_3 v_B^2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2] + [\frac{1}{2}m_4 v_C^2 + \frac{1}{2}I_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2] \end{aligned}$$

$$\begin{gathered} K = \frac{1}{2}m_1\dot{d_1}^2 + \frac{1}{2}m_2(0.16\dot{\theta}_1^2) + \frac{1}{2}I_1\dot{\theta_1}^2 + \frac{1}{2}m_3[0.64\dot{\theta}_1^2 + 0.16(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.64\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)] + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \\ \frac{1}{2}m_4[0.64\dot{\theta}_1^2 + 0.64(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.0625(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + 0.4\cos\theta_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)(\dot{\theta}_1 + \dot{\theta}_2) \\ +0.4\cos(\theta_2 + \theta_3)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) + 1.28\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)] + \frac{1}{2}I_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 \end{gathered}$$

$$\begin{gathered} => K = \frac{1}{2}m_1\dot{d_1}^2 + \frac{1}{2}(0.16m_2 + 0.64m_3 + 0.64m_4 + I_1)\dot{\theta}_1^2 + \frac{1}{2}(0.16m_3 + 0.64m_4 + I_2)(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ +\frac{1}{2}(0.0625m_3 + I_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + \frac{1}{2}(0.64\cos\theta_2 m_3 + 1.28\cos\theta_2 m_4)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) \\ +\frac{1}{2}(0.4\cos\theta_3 m_4)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)(\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2}(0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \end{gathered}$$

$$\begin{gathered} => K = \frac{1}{2}m_1\dot{d_1}^2 + \frac{1}{2}(0.16m_2 + 0.64m_3 + 0.64m_4 + I_1)\dot{\theta}_1^2 + \frac{1}{2}(0.16m_3 + 0.64m_4 + I_2)(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) \\ +\frac{1}{2}(0.0625m_3 + I_3)(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + 2\dot{\theta}_1\dot{\theta}_2 + 2\dot{\theta}_2\dot{\theta}_3 + 2\dot{\theta}_1\dot{\theta}_3) + \frac{1}{2}(0.64\cos\theta_2 m_3 + 1.28\cos\theta_2 m_4)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \\ +\frac{1}{2}(0.4\cos\theta_3 m_4)(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2\dot{\theta}_3 + \dot{\theta}_1\dot{\theta}_3) + \frac{1}{2}(0.4\cos(\theta_2 + \theta_3)m_4)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_1\dot{\theta}_3) \end{gathered}$$

$$\begin{gathered} => K = \frac{1}{2}m_1\dot{d_1}^2 + \frac{1}{2}(0.16m_2 + 0.64m_3 + 0.64m_4 + I_1 + 0.16m_3 + 0.64m_4 + I_2 + 0.0625m_3 + I_3 + 0.64\cos\theta_2 m_3 + 1.28\cos\theta_2 m_4 \\ +0.4\cos\theta_3 m_4 + 0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1^2 + \frac{1}{2}(0.16m_3 + 0.64m_4 + I_2 + 0.0625m_3 + I_3 + 0.4\cos\theta_3 m_4)\dot{\theta}_2^2 \\ +\frac{1}{2}(0.0625m_3 + I_3)\dot{\theta}_3^2 \\ +\frac{1}{2}(0.32m_3 + 1.28m_4 + 2I_2 + 0.125m_3 + 2I_3 + 0.64\cos\theta_2 m_3 + 1.28\cos\theta_2 m_4 + 0.8\cos\theta_3 m_4 + 0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1\dot{\theta}_2 \\ +\frac{1}{2}(0.125m_3 + 2I_3 + 0.4\cos\theta_3 m_4)\dot{\theta}_2\dot{\theta}_3 + \frac{1}{2}(0.125m_3 + 2I_3 + 0.4\cos\theta_3 m_4 + 0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1\dot{\theta}_3 \end{gathered}$$

$$=> K = \frac{1}{2}m_1\dot{d_1}^2 + \frac{1}{2}[0.16m_2 + (0.8625 + 0.64\cos\theta_2)m_3 + (1.28\cos\theta_2 + 0.4\cos\theta_3 + 0.4\cos(\theta_2+\theta_3))m_4 + I_1 + I_2 + I_3]\dot{\theta}_1^2$$
$$+\frac{1}{2}[0.2225m_3 + (0.64 + 0.4\cos\theta_3)m_4 + I_2 + I_3]\dot{\theta}_2^2$$
$$+\frac{1}{2}(0.0625m_3 + I_3)\dot{\theta}_3^2$$
$$+\frac{1}{2}[(0.445 + 0.64\cos\theta_2)m_3 + (1.28 + 1.28\cos\theta_2 + 0.8\cos\theta_3 + 0.4\cos(\theta_2+\theta_3))m_4 + 2I_2 + 2I_3]\dot{\theta}_1\dot{\theta}_2$$
$$+\frac{1}{2}(0.125m_3 + 2I_3 + 0.4\cos\theta_3 m_4)\dot{\theta}_2\dot{\theta}_3$$
$$+\frac{1}{2}[0.125m_3 + 2I_3 + (0.4\cos\theta_3 + 0.4\cos(\theta_2+\theta_3)m_4]\dot{\theta}_1\dot{\theta}_3$$

Since,

$$K = \frac{1}{2}\dot{q}^T D q$$

$$q = \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

From the form that, $m_1 = 1.0kg$, $m_2 = 1.2kg$, $m_3 = 1.0kg$, $m_4 = 0.6kg$, $I_1 = 0.256kg \cdot m^2$, $I_2 = 0.213kg \cdot m^2$, $I_3 = 0.05kg \cdot m^2$, now we can get

$$K = \frac{1}{2}\dot{d_1}^2 + \frac{1}{2}[1.408\cos\theta_2 + 0.24\cos\theta_3 + 0.24\cos(\theta_2+\theta_3) + 1.5735]\dot{\theta}_1^2$$
$$+\frac{1}{2}(0.24\cos\theta_3 + 0.8695)\dot{\theta}_2^2$$
$$+\frac{1}{2}(0.1125)\dot{\theta}_3^2$$
$$+[0.96\cos\theta_2 + 0.4\cos\theta_3 + 0.12(\theta_2+\theta_3) + 1.1255]\dot{\theta}_1\dot{\theta}_2$$
$$+(0.12\cos\theta_3 + 0.1125)\dot{\theta}_2\dot{\theta}_3$$
$$+[0.12\cos\theta_3 + 0.12(\theta_2+\theta_3) + 0.1125]\dot{\theta}_1\dot{\theta}_3$$

Let c_i denotes $\cos\theta_i$, s_i denotes $\sin\theta_i$, c_{ij} denotes $\cos(\theta_i+\theta_j)$

We can get D as follow,

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.408c_2 + 0.24c_3 + 0.24c_{23} + 1.5735 & 0.96c_2 + 0.4c_3 + 0.12c_{23} + 1.1255 & 0.12c_3 + 0.12c_{23} + 0.1125 \\ 0 & 0.96c_2 + 0.4c_3 + 0.12c_{23} + 1.1255 & 0.24c_3 + 0.8695 & 0.12c_3 + 0.1125 \\ 0 & 0.12c_3 + 0.12c_{23} + 0.1125 & 0.12c_3 + 0.1125 & 0.1125 \end{bmatrix}$$

Now we find the Christoffel Symbol,

$$C_{ijk} = \frac{1}{2}[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}]$$

$$c_{22} = -[0.12\sin(\theta_2+\theta_3) + 0.704\sin\theta_3]\dot{\theta}_2 - [0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_3$$
$$c_{23} = -[0.12\sin(\theta_2+\theta_3) + 0.704\sin\theta_2]\dot{\theta}_1 - [0.12\sin(\theta_2+\theta_3) + 0.96\sin\theta_2]\dot{\theta}_2 - [0.12\sin(\theta_2+\theta_3) + 0.2\sin\theta_3]\dot{\theta}_3$$
$$c_{24} = -[0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_1 - [0.12\sin(\theta_2+\theta_3)) + 0.2\sin\theta_3]\dot{\theta}_2 - [0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_3$$
$$c_{32} = [0.12\sin(\theta_2+\theta_3) + 0.704\sin\theta_3]\dot{\theta}_1 - 0.2\sin\theta_3\dot{\theta}_3$$
$$c_{33} = -0.12\sin\theta_3\dot{\theta}_3$$
$$c_{34} = -0.2\sin\theta_3\dot{\theta}_1 - 0.12\sin\theta_3\dot{\theta}_2 - 0.12\sin\theta_3\dot{\theta}_3$$
$$c_{42} = [0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_1 + 0.2\sin\theta_3\dot{\theta}_2$$
$$c_{43} = 0.2\sin\theta_3\dot{\theta}_1 + 0.12\sin\theta_3\dot{\theta}_2$$

So, we can get C as follow,

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & c_{32} & c_{33} & c_{34} \\ 0 & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

And we can get G as follow,

$$G = \frac{\partial P}{\partial q} = \begin{bmatrix} (m_1 + m_2 + m_3 + m_4)g \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After get D, C, G, The Lagrange-Euler Equation are as follow,

$$D\ddot{q} + C\dot{q} + G = \tau$$

Time-varying torque

The time-varying torque is at the right hand side of this equation

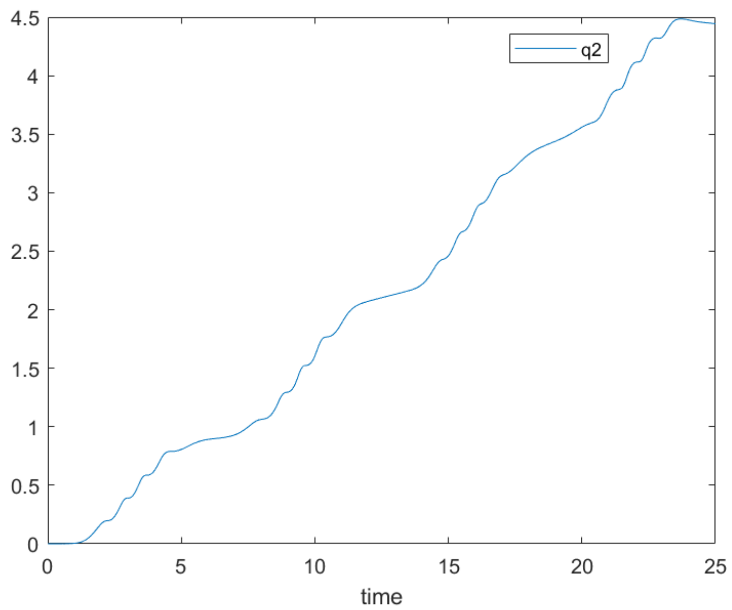
$$D\ddot{q} + C\dot{q} + G = \tau$$

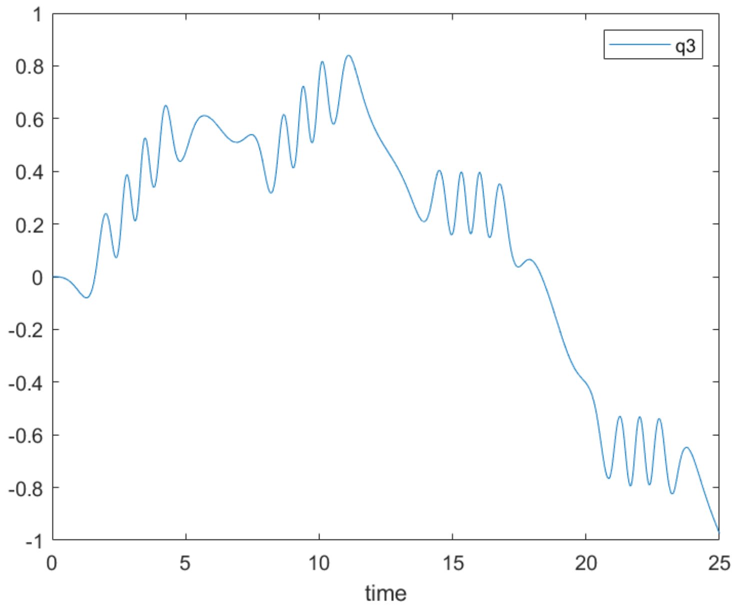
We design the time-varying preference torque as follow,

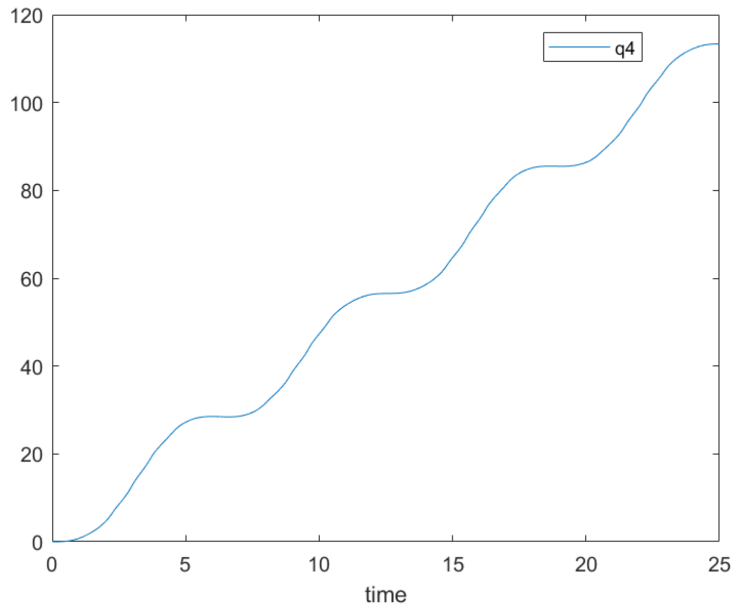
$$\tau = \begin{bmatrix} 0 \\ 0.8\sin(t) \\ 0.8\sin(t) \\ 0.5\sin(t) \end{bmatrix}$$

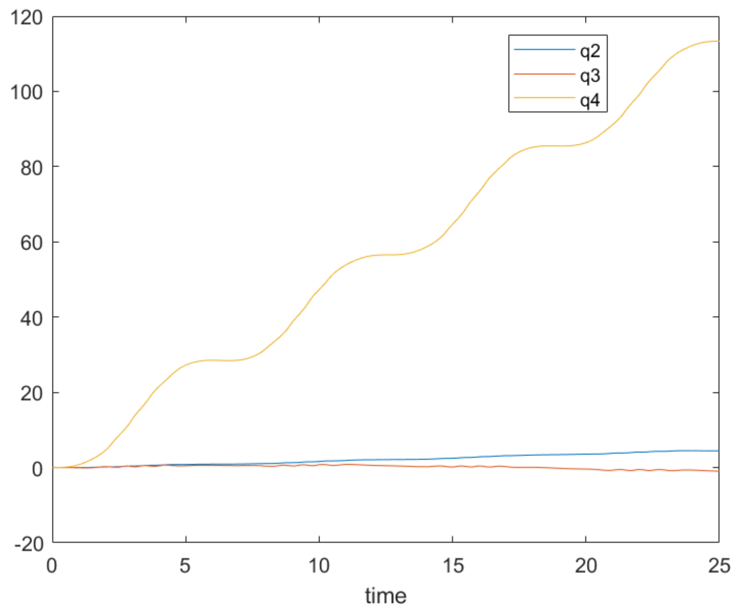
After simulate with MATLAB, qi denotes the positions, dqi denotes the velocity and $ddqi$ denotes the velocity in the figure,

We get positions $q2, q3, q4$ as follow,

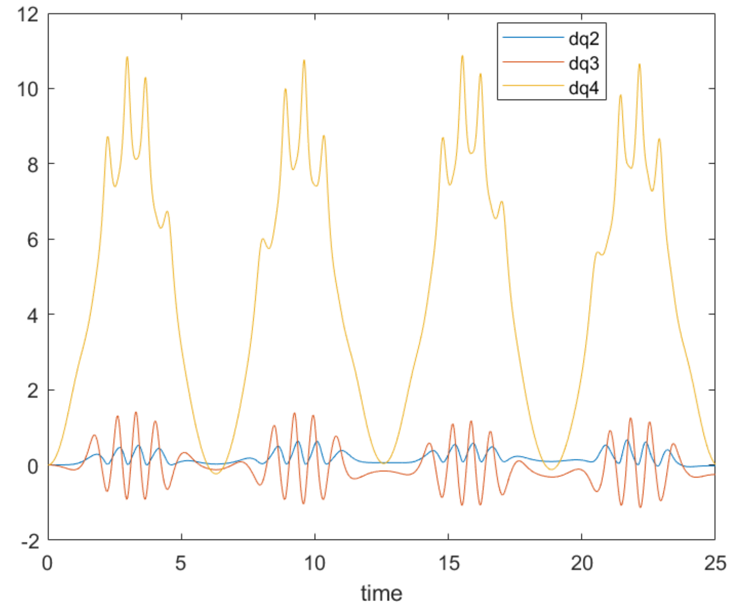




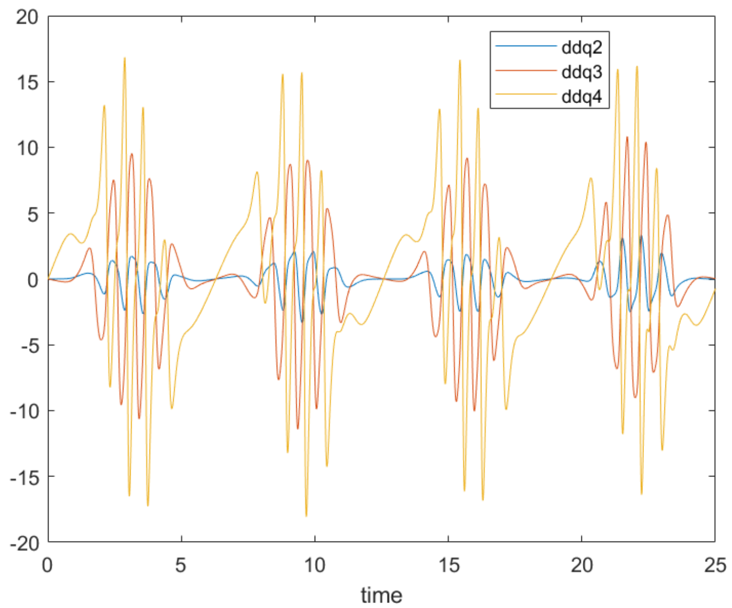




We get velocities $dq2, dq3, dq4$ as follow,



We get accelerations $ddq2$, $ddq3$, $ddq4$ as follow,



Task4 Control Design and Simulation

$$D(q)\ddot{q} + C(q,\dot{q})q + G(q) = \tau$$

Rewrite the manipulator's equations of motion as,

$$\ddot{q} = D(q)^{-1}(\tau - C(q,\dot{q})\dot{q} - G(q))$$

Choose $\ddot{q}$,

$$\ddot{q} = \ddot{q}_d + K_d\dot{e} + K_p e + K_i \int edt$$

The PID computed-torque control law is given as, where $e = q_d - q$,

$$D(q)(\ddot{q}_d + K_d\dot{e} + K_p e + K_i \int edt) + C(q,\dot{q})q + G(q) = \tau$$

Let the desired q,

$$q = \begin{bmatrix} 0.5\sin(t) \\ 0.2\sin(t) \\ 0.3\cos(t) \\ 0.4\sin(t) \end{bmatrix}$$

So the desired $\dot{q}$,

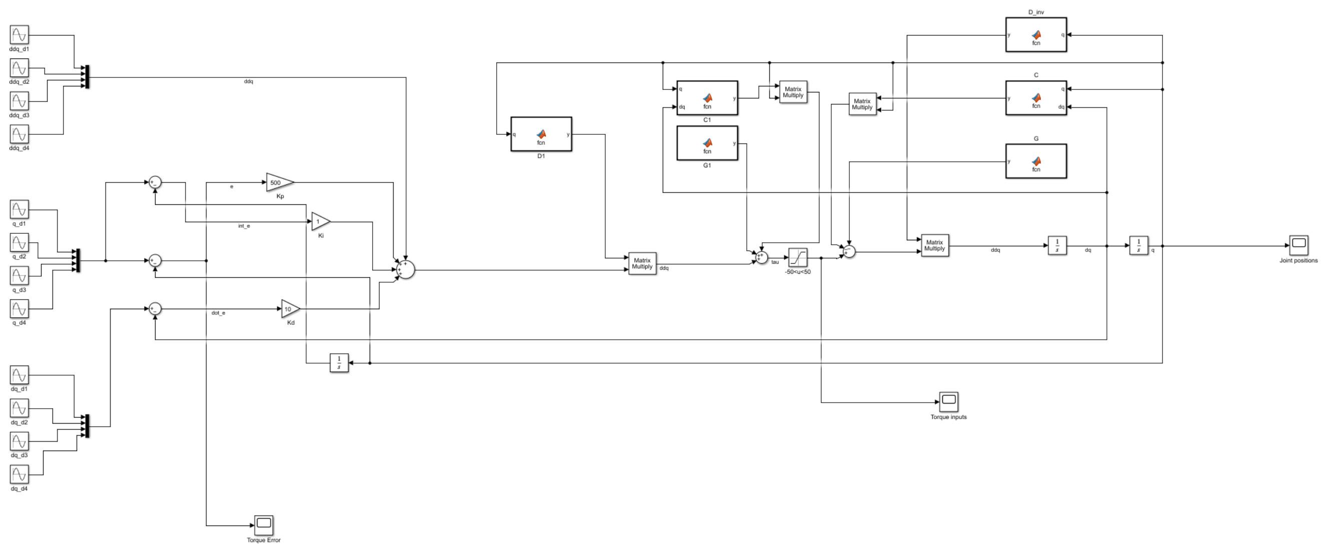
$$\dot{q} = \begin{bmatrix} 0.5\cos(t) \\ 0.2\cos(t) \\ -0.3\sin(t) \\ 0.4\cos(t) \end{bmatrix}$$

So the desired $\ddot{q}$,

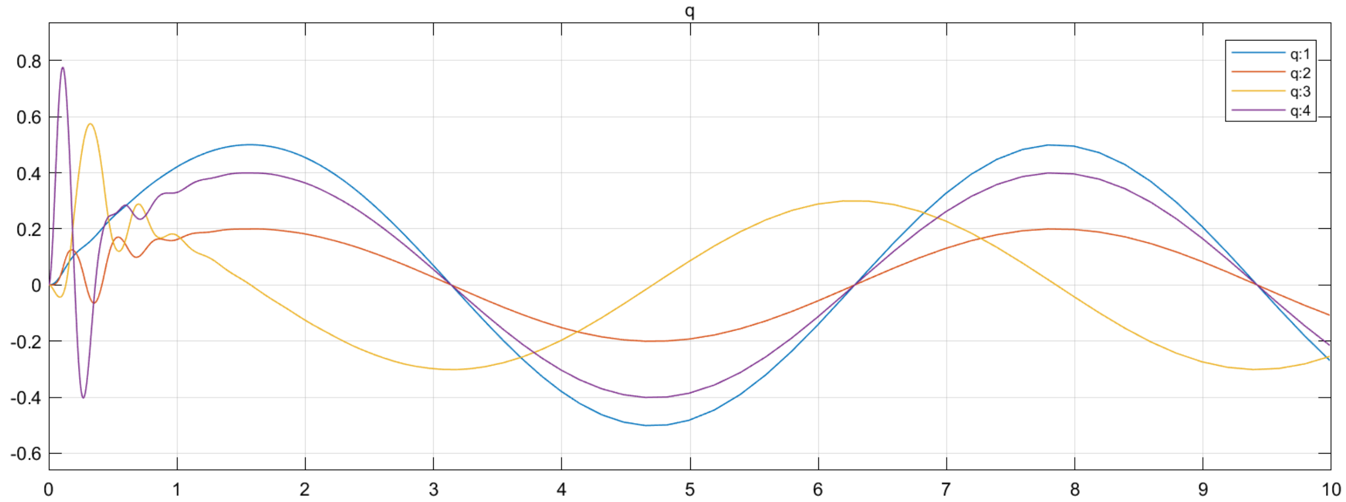
$$\ddot{q} = \begin{bmatrix} -0.5\sin(t) \\ -0.2\sin(t) \\ -0.3\cos(t) \\ -0.4\sin(t) \end{bmatrix}$$

Controller design and result

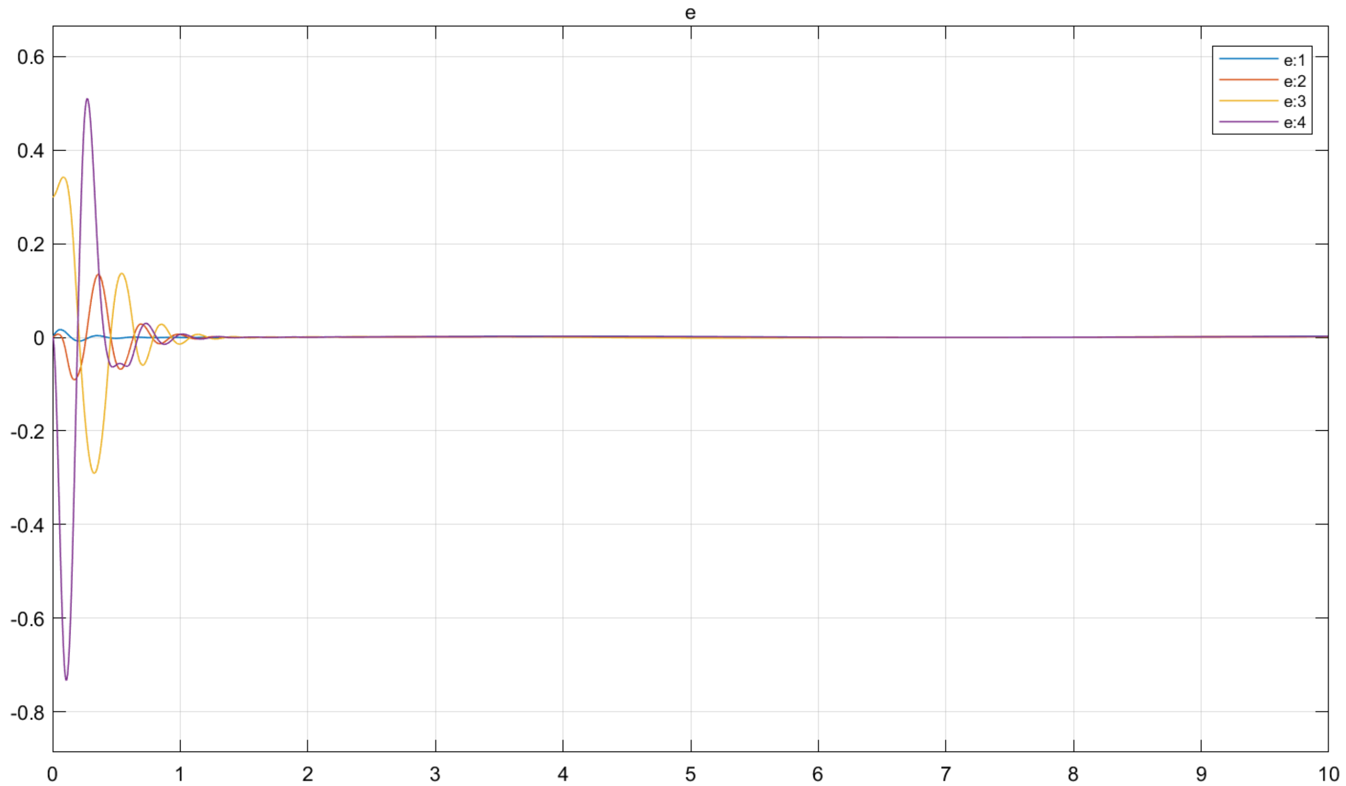
We build a PID controller using Simulink as follow, and set the simulation time to 10s.



The left of this system are the refererences q, $\dot{q}$, $\ddot{q}$, we set choose the PID parameters as $K_d = 10$, $K_i = 1$, $K_p = 500$, and we use a saturator to constraint the torque input in case the inputs overshot are too large after the amplification of K_p. We can get the stable joint positions q as following figure, and it can perfectly follow the references.



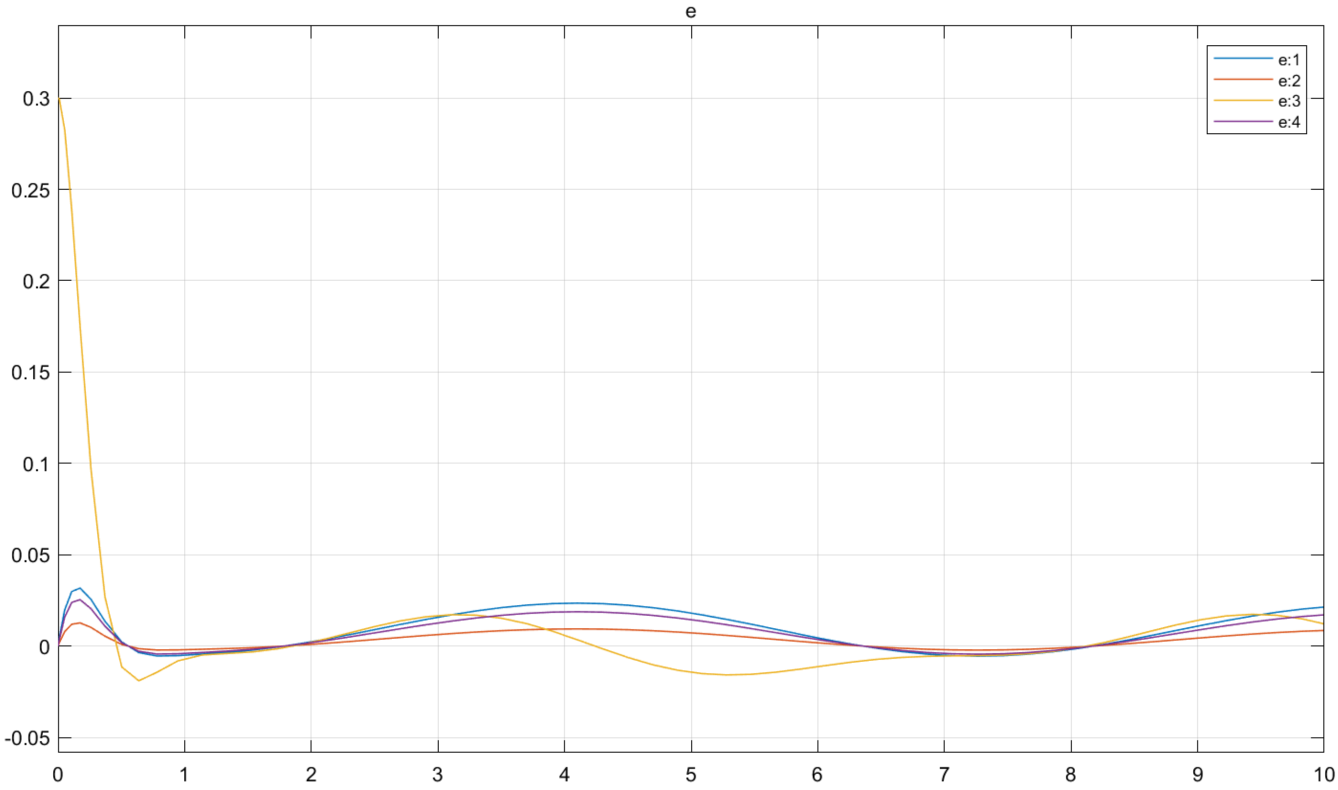
We now observe the closed-loop error $e = q_d - q$ as the following figure,



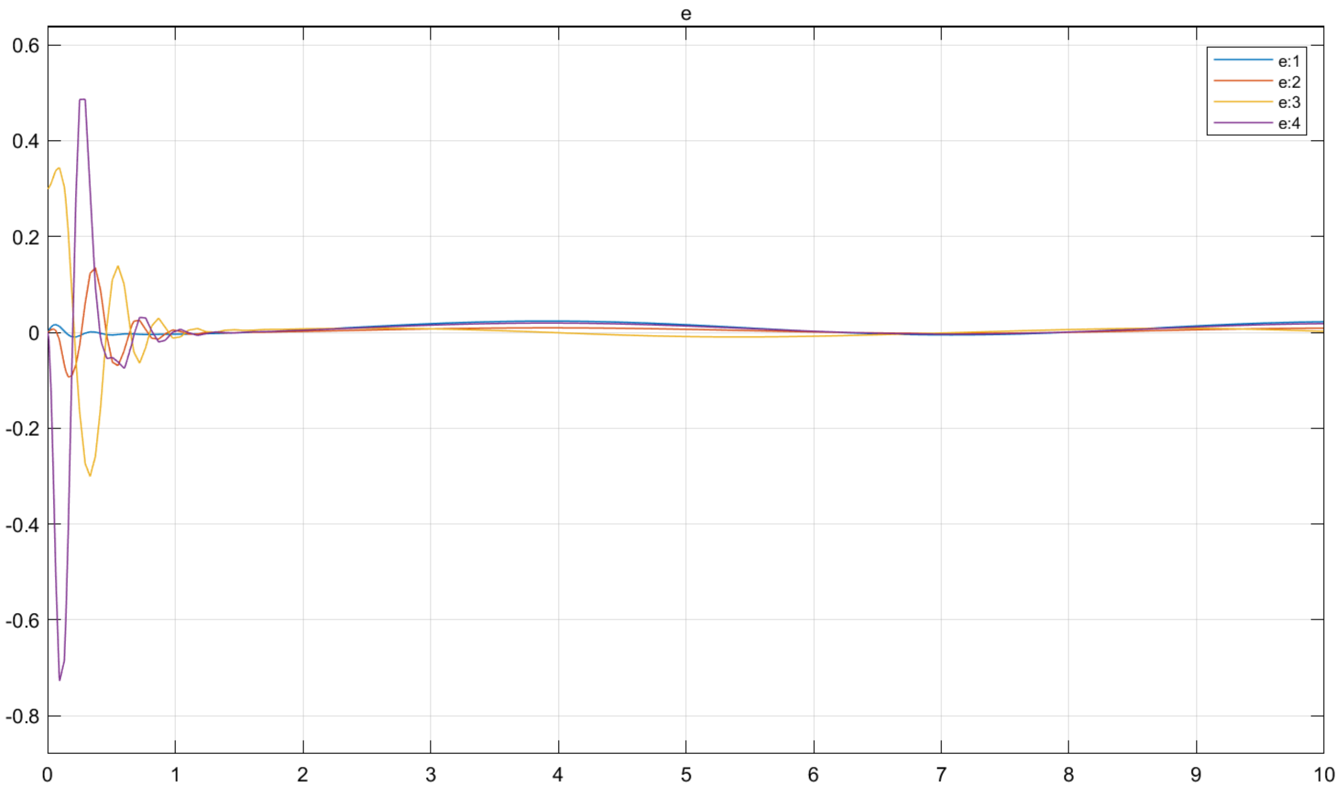
We can see that the closed-loop errors of 4 q_i are all converged to 0.

PID parameters influence analysis

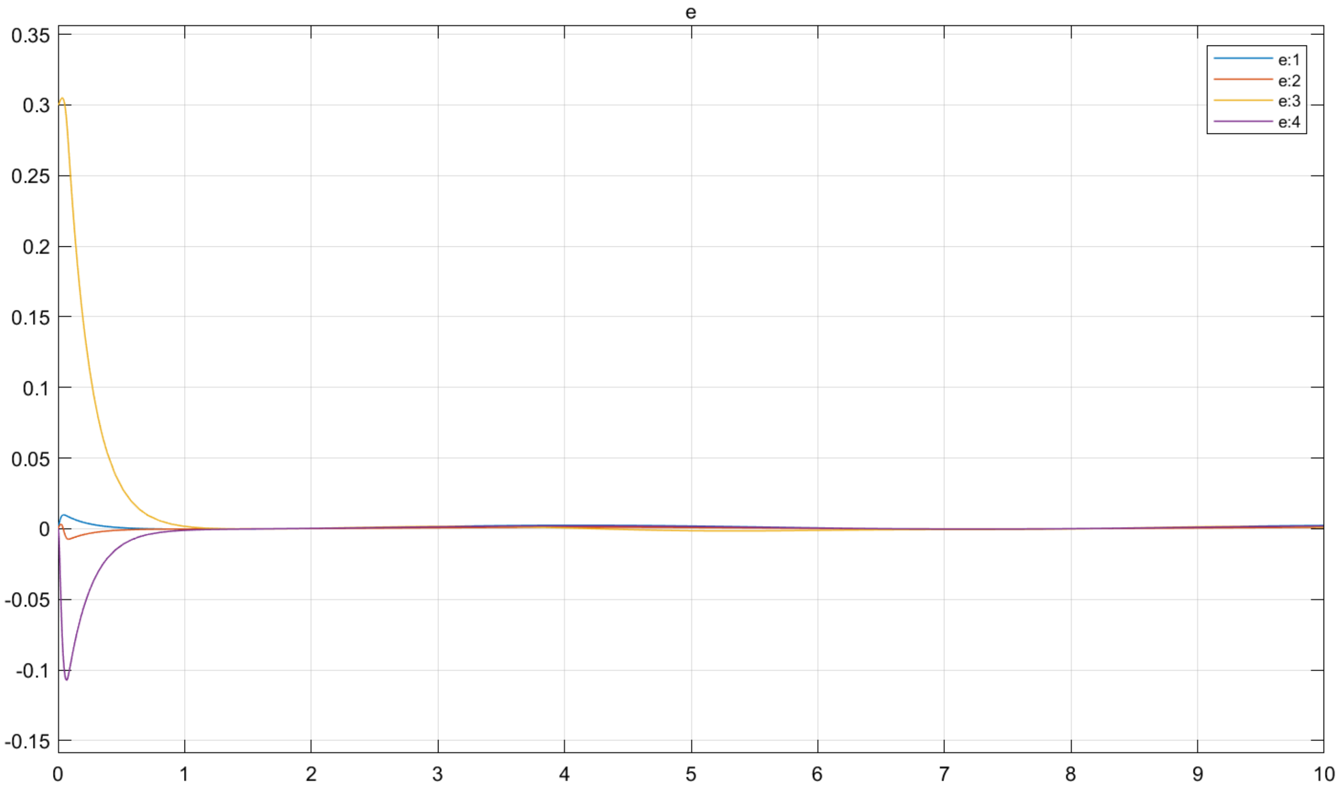
In this system, we find that K_p has played a great role in influencing the close loop error, if we set $K_p = 50$, we can find that the error performance are oscillating around 0 and the curves are not smooth. So we choose to set a large K_p to shrink the error e.



If we set $K_i = 10$ (a larger K_i), it will reduce the influence of Integral on error e, so we can find that the error performance are oscillating around 0 and the curves are not smooth. So we choose to set a smaller K_i.



If we set $K_d = 100$ (a larger K_d), it can suppress the overshot but to make the system less sensitive to response. So we choose to set a smaller K_i.



Simple trajectory planning task

Since we can not find a close form solution for inverse kinematics, we now consider to do the trajectory planning as following table,

joint angle	initial position	final position
θ_1	$-\frac{\pi}{2}$	0
θ_2	$\frac{\pi}{4}$	$\frac{\pi}{4}$
θ_3	0	$\frac{\pi}{2}$

we use in the fifth degree polynomial to do this trajectory planning,

$$\begin{aligned}\theta_1(t) &= 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5\\ \theta_2(t) &= 1 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5\\ \theta_3(t) &= 1 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5\end{aligned}$$

Let the desired q,

$$q = \begin{bmatrix} 0.5 \\ 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \\ 1 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 \\ 1 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 \end{bmatrix}$$

So the desired $\dot{q}$,

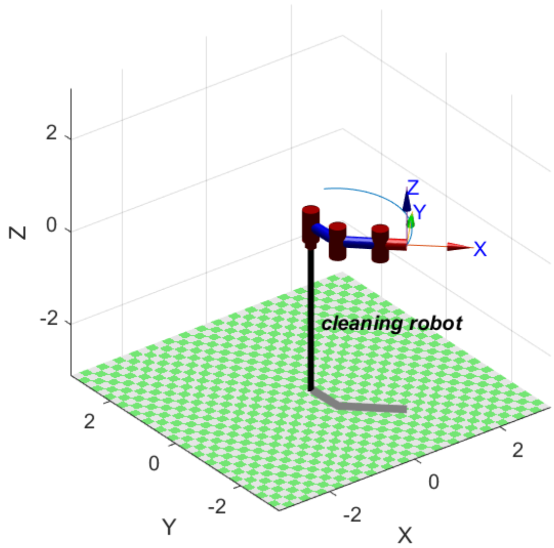
$$\dot{q} = \begin{bmatrix} 0 \\ a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 \\ b_1 + 2b_2 t + 3b_3 t^2 + 4b_4 t^3 + 5b_5 t^4 \\ c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 \end{bmatrix}$$

So the desired $\ddot{q}$,

$$\ddot{q} = \begin{bmatrix} 0 \\ 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 \\ 2b_2 + 6b_3 t + 12b_4 t^2 + 20b_5 t^3 \\ 2c_2 + 6c_3 t + 12c_4 t^2 + 20c_5 t^3 \end{bmatrix}$$

We can get the trajectory as following, the blue curve is the planned tracjectory,

Start



End