EE5137 Stochastic Processes: Problem Set 5
Assigned: 11/02/22, Due: 18/02/22

There are six (6) non-optional problems in this problem set.

1. Exercise 2.1(b) (Gallager's book)
2. Exercise 2.2(a) and 2.2(b) (Gallager's book)
3. Exercise 2.4 (Gallager's book)
4. Exercise 2.7 (Gallager's book)

   Hint for Part(a): Recall that the derivative of a function $f(t)$ at the point τ is

$$\left.\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right|_{t=\tau} = \lim_{\delta\downarrow 0}\frac{f(\tau+\delta)-f(\tau)}{\delta}.$$

5. Transmitters A and B independently send messages to a single receiver in a Poisson manner with rates λ_{A} and λ_{B} respectively. All the messages are so brief that we may assume that they occupy single points in time. The number of words in a message, regardless of the source that is transmitting it, is a random variable with PMF

$$p_W(w) = \begin{cases} 2/6 & w=1 \\ 3/6 & w=2 \\ 1/6 & w=3 \\ 0 & \text{otherwise} \end{cases}$$

   and is independent of everything else.

   (a) What is the probability that during an interval of duration t, a total of exactly 9 messages will be received?

   (b) Let N be the total number of words received during an interval of duration t. Determine the expected value of N.

   (c) Determine the PDF of the time from $t=0$ until the receiver has received exactly eight three-word messages from transmitter A.

   (d) What is the probability that exactly 8 out of the next 12 messages received will be from transmitter A?

6. Consider a Poisson process of rate $\lambda > 0$. Let t^* be a fixed time instant and consider the length of the interarrival interval $[U, V]$ that contains t^*. In this question, we would like to determine the distribution of

$$L = (t^* - U) + (V - t^*).$$

   (i) Give a one sentence answer as to why $V - t^*$ is independent of $t^* - U$.

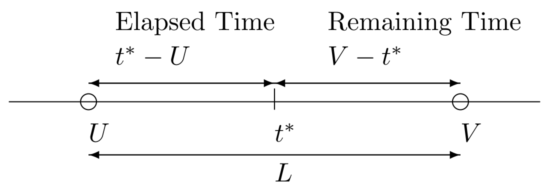

Figure 1: Here U and V are successive arrival epochs and t^* is a fixed time instance between U and V.

(ii) In class, we determined the distribution of $V - t^*$. What is this distribution?

(iii) Consider the event
$$\{t^* - U > x\}.$$
This event is the same as
$$\{\text{there are } k \text{ arrivals in the interval } [t^* - x, t^*]\}.$$
Find the integer k. No explanation is needed.

(iv) Hence, find the distribution of $t^* - U$.

(v) By using the preceding parts, find the distribution of L.

(vi) What is the distribution of an interarrival time of a Poisson process? Why is this the same or different from that of L in part (v)?

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7. (Optional) Exercise 2.8 (Gallager's book)
8. (Optional) Exercise 2.9 (Gallager's book)
9. (Optional) Exercise 2.12 (Gallager's book)
10. (Optional) Customers depart from a bookstore according to a Poisson process with rate λ per hour. Each customer buys a book with probability p, independent of everything else.

    (a) Find the distribution of the time until the first sale of a book.

    (b) Find the probability that there are no books sold during a particular hour.

    (c) Find the expected number of customers who buy a book during a particular hour.

11. (Optional) Let S_1 and S_2 be independent and exponentially distributed with parameters λ_1 and λ_2, respectively. Show that the expected value of $\max\{S_1, S_2\}$ is
$$\mathbb{E}[\max\{S_1, S_2\}] = \frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right)$$
using Poisson Processes.

    Hint: Consider two independent Poisson processes with rates λ_1 and λ_2, respectively. We interpret S_1 as the first arrival time in the first process, and S_2 the first arrival time in the second process. Let $V = \min\{S_1, S_2\}$ be the first time when one of the processes registers an arrival. Let $W = \max\{X_1, X_2\} - V$ be the additional time until both have registered an arrival. Now calculate the expectations of V and W to find the expectation of the desired $\max\{S_1, S_2\}$.