National University of Singapore
ME5402/EE5106R

Linear Algebra Review

1) Mathematical notation

$\exists$ - there exist
$\forall$ - for all
$\Leftrightarrow$ - if and only if

2) Euclidean Space, R^n

Definition: Vector space
A vector space $\mathbf{X}$ is a collection of elements called vectors which includes the following vectors:
1. A vector $\mathbf{0}$ s.t. $\mathbf{x+0=x}$, for $\mathbf{x} \in \mathbf{X}$
2. A unique vector $\mathbf{y} \in \mathbf{X}$ for every vector $\mathbf{x} \in \mathbf{X}$ s.t. $\mathbf{x}+\mathbf{y}=\mathbf{0}$

And, for scalars a, b and $\mathbf{x,y,z} \in \mathbf{X}$, the following properties are satisfied:
I
1. $\mathbf{x}+\mathbf{y} \in \mathbf{X}$ (Closure under addition)
2. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ (commutativity of addition)
3. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ (associativity of addition)

II
1. $a\mathbf{x} \in \mathbf{X}$ (Closure under scalar multiplication)
2. $a(b\mathbf{x})=(ab)\mathbf{x}$ (associativity of scalar multiplication)
3. $(a+b)\mathbf{x}=a\mathbf{x}+b\mathbf{x}$ (Distributivity)
4. $a(\mathbf{x} + \mathbf{y})=a\mathbf{x}+a\mathbf{y}$ (Distributivity)
5. $1\mathbf{x}=\mathbf{x}$
6. $0\mathbf{x}=\mathbf{0}$

<u>Euclidean space (a vector space), **$\mathbf{R^n}$**:</u>

The set of all column vectors with n components (real numbers), $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

E.g. $\mathbf{R^2}$ is the 2D space, $\mathbf{R^3}$ is the 3D space.

Addition of two vectors: $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$

Remark: Addition operation on two vectors from a vector space, say R^n, will always yield a vector in the same vector space.

Multiplication of a vector with a scalar, λ, are defined as:

$$\lambda\mathbf{x} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$$

Remark: Scalar multiplication on a vector from a vector space, say R^n, will always yield a vector in the same vector space.

Inner product (dot product, scalar product) of two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ or $\mathbf{x} \cdot \mathbf{y}$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

A *norm* || . || on R^n is a mapping that assigns a scalar ||**x**|| to every $\mathbf{x} \in R^n$ and that has the following properties:

(a) $\|\mathbf{x}\| \geq 0 \ \forall \ \mathbf{x} \in R^n$
(b) $\|c\mathbf{x}\| = |c| \ \|\mathbf{x}\|$ for every $c \in R$ and every $\mathbf{x} \in R^n$
(c) $\|\mathbf{x}\| = 0 \Leftrightarrow$ if $\mathbf{x} = \mathbf{0}$
(d) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\| \ \forall \ \mathbf{x},\mathbf{y} \in R^n$ (Triangular inequality)

Euclidean norm (or L_2 norm) of **x**:

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}.\mathbf{x})} = \left(\sum_{i=1}^{n} x_i^{\ 2} \right)^{1/2}$$

Schwarz inequality (applies to the Euclidean norm):

$$|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\| \ \|\mathbf{y}\|$$

with equality holding if and only if $\mathbf{x} = \alpha\mathbf{y}$ for some scalar α

Orthogonal:
Two vectors $\mathbf{x_1}$ and $\mathbf{x_2}$ are *orthogonal* if their inner product is zero, i.e. $\mathbf{x_1} \cdot \mathbf{x_2} = 0$

Orthonormal:
Two vectors $\mathbf{x_1}$ and $\mathbf{x_2}$ are *orthonormal* if $\mathbf{x_1} \cdot \mathbf{x_2}=0$ and in addition, $\|\mathbf{x_1}\|=\|\mathbf{x_2}\|=1$
A set of vectors $\mathbf{x_i}$, i=1, 2,…,m, is said to be *orthogonal* if $\mathbf{x_i} \cdot \mathbf{x_j} =0$ if $i \neq j$ and $\mathbf{x_i} \cdot \mathbf{x_j} \neq 0$ if $i = j$

A set of vectors $\mathbf{x_i}$, i=1, 2,…,m, is said to be *orthonormal* if $\mathbf{x_i} \cdot \mathbf{x_j} = 0$ if $i \neq j$ and $\mathbf{x_i} \cdot \mathbf{x_j} = 1$ if $i = j$

3) Matrices

m x n matrix: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} = \mathbf{A} = [a_{ij}]$

Product of a matrix $\mathbf{A}$ and a scalar α:

$$\alpha\mathbf{A} \text{ or } \mathbf{A}\alpha = [\alpha a_{ij}]$$

Product of two matrices $\mathbf{A}$ and $\mathbf{B}$:
Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be a mxn, nxp and mxp matrices, respectively.

$\mathbf{AB} = \mathbf{C}=[c_{ij}]$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Transpose of a mxn matrix **A** is the nxm matrix:

$\mathbf{A}^T = [a'_{ij}] = [a_{ji}]$

Note that $(A+B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$.

A square matrix **A** is *symmetric* if $\mathbf{A}^T=\mathbf{A}$

Determinant of square matrix **A**, det(**A**) *(scalar-valued function of* **A***):*

$$\det(\mathbf{A}) = |\mathbf{A}| = \sum_{j=1}^{n} a_{ij} c_{ij} = \sum_{i=1}^{n} a_{ij} c_{ij}$$, for any fixed *i*(or any fixed *j*)

where c_{ij} is the cofactor corresponding to entry a_{ij} of **A;**
$c_{ij} = (-1)^{i+j} \det(M_{ij})$ where M_{ij} is the $(n-1)\times(n-1)$ submatrix of **A** that results when the i^{th} row and the j^{th} column of **A** are deleted.

Special cases:

1. n=2: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$
2. Triangular matrix

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & & & & \\ a_{n1} & a_{n2} & & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

Non-singular matrix

A square matrix (nxn) **A** is called *non-singular* or *invertible* if there is an nxn matrix called the inverse of **A** (denoted by $\mathbf{A}^{-1}$), such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$, where **I** is the nxn identity matrix.

> Remark: Determinant of non-singular matrix is nonzero.

Inverse of a nonsingular square matrix **A**, $\mathbf{A}^{-1}$:

- A square matrix $A \in R^{n\times n}$ has an inverse, written as $\mathbf{A}^{-1}$, if and only if det(A)≠0 or A is non-singular.
- Calculation of $\mathbf{A}^{-1}$:

  $$A^{-1} = \frac{adj(A)}{\det(A)} = \frac{1}{\det(\mathbf{A})}[c_{ij}]^T$$

  , where $[c_{ij}]$ is the cofactor matrix

  Special case: n=2

  $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Positive Definite and *Semidefinite* Matrices

A square nxn matrix **A** is said to be *positive semidefinite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0 \ \forall \mathbf{x} \in R^n$, $\mathbf{x} \neq \mathbf{0}$

A square nxn matrix **A** is said to be *positive definite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \ \forall \mathbf{x} \in R^n$, $\mathbf{x} \neq \mathbf{0}$

The matrix **A** is said to be *negative semidefinite* (*definite*) if (–**A**) is *positive semidefinite* (*definite*).

Definition: The *column space* of matrix **A** consists of all linear combinations of the columns. The combinations are the vectors **Ax**.

The system **Ax** = **b** is solvable if and only if **b** is in the column space of A.

Notation: The column space of A is denoted by **R**(**A**). **R** stands for “range”. Hence, the column space is also called the range space of **A**.

Definition: A *subspace* of a vector space is a set of vectors (including **0**) that satisfies two requirements:
If **v** and **w** are vectors in the subspace and *c* is any scalar, then

(i) **v** + **w** is in the subspace and
(ii) *c***v** is in the subspace.