# EE5137 Stochastic Processes: Problem Set 4
# Assigned: 04/02/22, Due: 11/02/22

There are eight (8) non-optional problems in this problem set. There are not many problems in Poisson processes as we have not covered enough, so I'm giving some practice problems on probability.

1. For a Poisson process, which of the following is/are true?
   (i) $\{N(t) \geq n\} = \{S_n \leq t\}$;
   (ii) $\{N(t) < n\} = \{S_n > t\}$;
   (iii) $\{N(t) \leq n\} = \{S_n \geq t\}$;
   (iv) $\{N(t) > n\} = \{S_n < t\}$.

2. An athletic facility has 5 tennis courts. Pairs of players arrive at the courts and use a court for an exponentially distributed time with mean 40 minutes. Suppose a pair of players arrives and finds all courts busy and $k$ other pairs waiting in queue. What is the expected waiting time to get a court?

3. Exercise 2.3 (Gallager's book)

4. Prove that the Geometric distribution

   $$p_X(k) = (1-p)^{k-1}p, \quad k \in \mathbb{N} = \{1, 2, \ldots\}$$

   has the memoryless property.

   *In fact, the Geometric distribution is the only distribution supported on* $\mathbb{N}$ *that is memoryless. Try proving this. This is analogous to the fact that the Exponential distribution is the only distribution supported on* $[0, \infty)$ *that is memoryless.*

5. Let $X_n$ denote a Binomial random variable with $n$ trials and probability of success $p_n$. If $np_n \to \lambda$ as $n \to \infty$, show that for any fixed $i \in \mathbb{N} \cup \{0\}$,

   $$\Pr(X_n = i) \to \frac{e^{-\lambda}\lambda^i}{i!}, \quad \text{as } n \to \infty.$$

6. Let the sample space $\Omega = \{1, 2, \ldots, p\}$ for a *prime number* $p$, and $A$ and $B$ are subsets of $\Omega$ (events) and $\mathbb{P}(A) = |A|/p$ ($\mathbb{P}$ represents the uniform distribution on $\Omega$). Prove that if $A$ and $B$ are independent, then either $A$ or $B$ is the empty set $\emptyset$ or the sample space $\Omega$.

7. If $X$ is a random variable with the property that $\Pr(0 \leq X \leq a) = 1$, show that

   $$\mathsf{Var}(X) \leq a^2/4.$$

8. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed exponential random variables with parameter $\lambda$. Let $M_n$ denote $\max\{X_1, \ldots, X_n\}$. Show there exists a random variable $Z$ such that $\{M_n - \frac{1}{\lambda}\log n\}_{n=1}^{\infty}$ converges in distribution to $Z$. This is the *Gumbel distribution.*