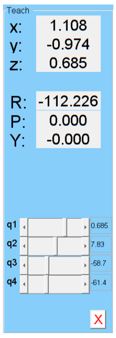
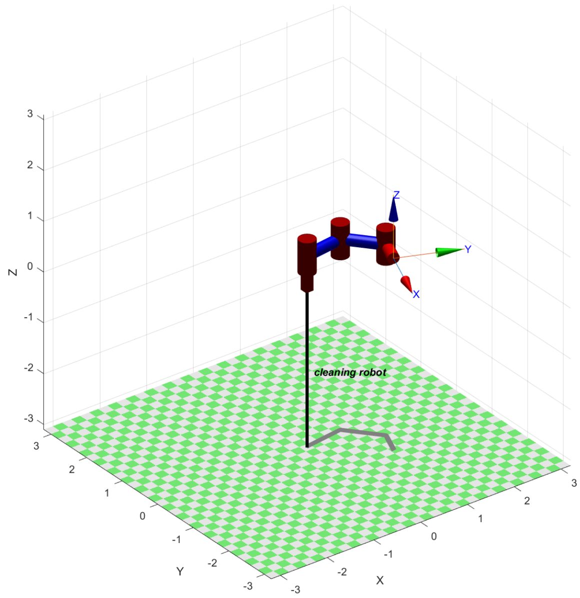
Task1 Introduction and Literature Review

Task2 Kinematics and Computing

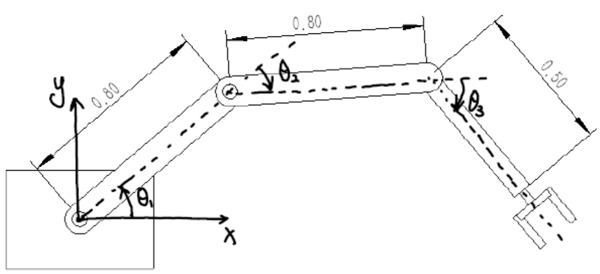


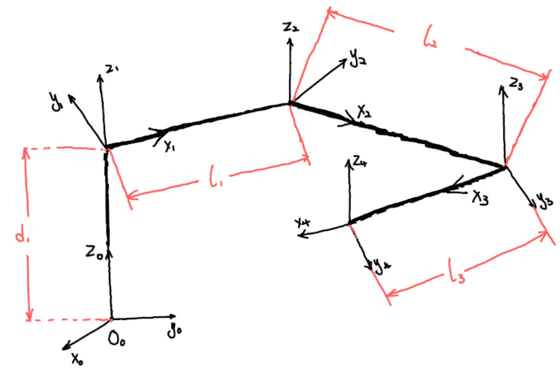


D-H table

Link	θ_i	d_i	a_i	α_i
1	0	d_1	0	0
2	θ_1	0	0.8	0
3	θ_2	0	0.8	0
4	θ_3	0	0.5	0

Forward (Direct) Kinematic Matrix





We first compute the 4x4 homogeneous matrices ${}^{i}_{i-1}A(q_i)$, and get the forward kinematics matrix result ${}^4_0A(q)$ as follow,

$$
\begin{aligned}
{}^4_0A(q) &={}^0_1 A(q_1){}^1_2A(q_2){}^2_3A(q_3){}^3_4A(q_4) \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & 0.5\cos(\theta_1+\theta_2+\theta_3)+0.8\cos(\theta_1+\theta_2)+0.8\cos\theta_1 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & 0.5\sin(\theta_1+\theta_2+\theta_3)+0.8\sin(\theta_1+\theta_2)+0.8\sin\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Jacobian Matrix

The let b_{i-1} denotes as the unit vector along z-axis of the frame $\{i-1\}$. we first determine the joint axes directions, b_{i-1} is the third column of the rotation matrix ${}^{0}_{i-1}R$.

$$\bar{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$b_1 ={}^0_1 R\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$b_2 ={}^0_1 R^1_2R\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$b_3 ={}^0_1 R^1_2R^2_3R\bar{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $r_{i-1,e}$ is the position vector from O_{i-1} to end-effector. $r_{i-1,e}$, can be computed using 4×4 homogeneous matrices ${}^{i-1}_{i}A$.

Let $X_{i-1,e}$ be 4×1 augmented vector of $r_{i-1,e}$ and $\bar{X} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$, we can get that $X_{i-1,e} ={}^0_1 A\ldots{}^{n-1}_{n} A\bar{X} -{}^0_1 A\ldots{}^{i-2}_{i-1} A\bar{X}$

$${}^0_4A ={}^0_1 A^1_2A^2_3A^3_4A = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & 0.5\cos(\theta_1+\theta_2+\theta_3)+0.8\cos(\theta_1+\theta_2)+0.8\cos\theta_1 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & 0.5\sin(\theta_1+\theta_2+\theta_3)+0.8\sin(\theta_1+\theta_2)+0.8\sin\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
\bar{X}_{1,e} &={}^0_1 A^1_2A^2_3A^3_4A\bar{X} -{}^0_1 A\bar{X} \\
\bar{X}_{2,e} &={}^0_1 A^1_2A^2_3A^3_4A\bar{X} -{}^0_1 A^1_2A\bar{X} \\
\bar{X}_{3,e} &={}^0_1 A^1_2A^2_3A^3_4A\bar{X} -{}^0_1 A^1_2A^2_3A\bar{X}
\end{aligned}
$$

We can get $r_{i-1,e}$, which is the position vector from O_{i-1} to end-effector (expressed in $O_o x_o y_o z_o$)

$$
\begin{aligned}
r_{1,e} &= X_{1,e}(1:3) \\
r_{2,e} &= X_{2,e}(1:3) \\
r_{3,e} &= X_{3,e}(1:3)
\end{aligned}
$$

Since first joint is the prismatic joint and the rest three joint are revolute joints, we can get the Jacobian matrix as follow,

$$J = \begin{bmatrix} J_{L1} & J_{L2} & J_{L3} & J_{L4} \\ J_{A1} & J_{A2} & J_{A3} & J_{A4} \end{bmatrix}$$

where

$$\begin{bmatrix} J_{L1} \\ J_{A1} \end{bmatrix} = \begin{bmatrix} b_0 \\ 0 \end{bmatrix}, \begin{bmatrix} J_{L2} \\ J_{A2} \end{bmatrix} = \begin{bmatrix} b_1 \times r_{1,e} \\ b_1 \end{bmatrix}, \begin{bmatrix} J_{L3} \\ J_{A3} \end{bmatrix} = \begin{bmatrix} b_2 \times r_{2,e} \\ b_2 \end{bmatrix}, \begin{bmatrix} J_{L4} \\ J_{A4} \end{bmatrix} = \begin{bmatrix} b_3 \times r_{3,e} \\ b_3 \end{bmatrix}$$

Therefore, we can get the Jacobian matrix as follow,

$$J = \begin{bmatrix} 0 & -0.5\sin(\theta_1+\theta_2+\theta_3) - 0.8\sin(\theta_1+\theta_2) + 0.8\sin(\theta_1) & -0.5\sin(\theta_1+\theta_2+\theta_3) - 0.8\sin(\theta_1+\theta_2) & -0.5\sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0.5\cos(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) + 0.8\cos(\theta_1) & 0.8\sin(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) & 0.5\sin(\theta_1+\theta_2+\theta_3) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

For inverse kinematics,

with a given endpoint position p_x, p_y, p_z,

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 0.5\cos(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) + 0.8\cos\theta_1 \\ 0.5\sin(\theta_1+\theta_2+\theta_3) + 0.8\sin(\theta_1+\theta_2) + 0.8\sin\theta_1 \\ d_1 \end{bmatrix}$$

$$p_x = 0.5\cos(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) + 0.8\cos\theta_1 \tag{1}$$

$$p_y = 0.5\sin(\theta_1+\theta_2+\theta_3) + 0.8\sin(\theta_1+\theta_2) + 0.8\sin\theta_1 \tag{2}$$

$(1)^2 + (2)^2$, we can get that,

$$p_x^2 + p_y^2 = 1.53 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3) + 1.28\cos(\theta_2)$$

Since we cannot find a close form solution for θ_1, θ_2, θ_3, so assume we already know θ_3,

$$p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53 = 0.8\cos(\theta_2+\theta_3) + 0.8\cos(\theta_3)$$

Using sum-to-product formulas,

$$p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53 = 1.6\cos\left(\frac{\theta_2+2\theta_3}{2}\right)\cos\left(\frac{\theta_2}{2}\right)$$

$\therefore$ we can get θ_3 as follow,

$$\cos\left(\frac{\theta_2+2\theta_3}{2}\right) = \frac{p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53}{1.6\cos\left(\frac{\theta_2}{2}\right)}$$

$$\frac{\theta_2+2\theta_3}{2} = \pm\arccos\left(\frac{p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53}{1.6\cos\left(\frac{\theta_2}{2}\right)}\right)$$

$$\theta_3 = \pm\arccos\left(\frac{p_x^2 + p_y^2 - 1.28\cos(\theta_2) - 1.53}{1.6\cos\left(\frac{\theta_2}{2}\right)}\right) - \frac{\theta_2}{2}$$

Since we already get θ_2 and θ_3, to get θ_1 using (1) and (2),

$$\cos(\theta_1+\theta_2+\theta_3)p_x = 0.5\cos(\theta_1+\theta_2+\theta_3)^2 + 0.8\cos(\theta_1+\theta_2+\theta_3)\cos(\theta_1+\theta_2) + 0.8\cos(\theta_1+\theta_2+\theta_3)\cos\theta_1 \tag{3}$$

$$\sin(\theta_1+\theta_2+\theta_3)p_y = 0.5\sin(\theta_1+\theta_2+\theta_3)^2 + 0.8\sin(\theta_1+\theta_2+\theta_3)\sin(\theta_1+\theta_2) + 0.8\sin(\theta_1+\theta_2+\theta_3)\sin\theta_1 \tag{4}$$

$(3) + (4)$ we can get,

$$\cos(\theta_1+\theta_2+\theta_3)p_x + \sin(\theta_1+\theta_2+\theta_3)p_y = 0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3)$$

Let ϕ denotes a new angle, which $\cos(\phi) = \frac{p_x}{\sqrt{p_x^2+p_y^2}}$, $\sin(\phi) = \frac{p_y}{\sqrt{p_x^2+p_y^2}}$,

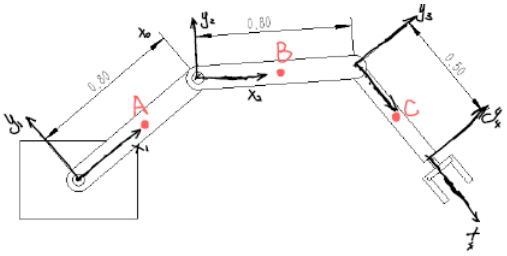
$$\sqrt{p_x^2+p_y^2}\cos(\theta_1+\theta_2+\theta_3-\phi) = 0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3)$$

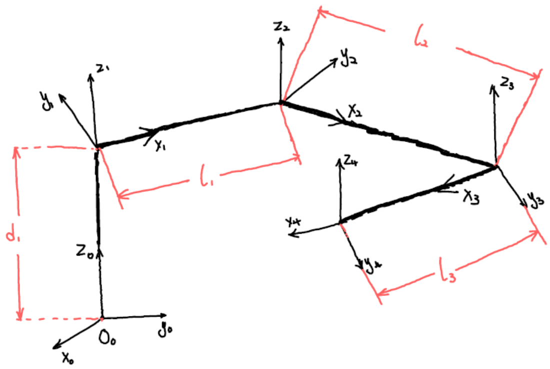
$$\theta_1+\theta_2+\theta_3-\phi = \pm\arccos\left(\frac{0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3)}{\sqrt{p_x^2+p_y^2}}\right)$$

$\therefore$ we can get θ_1 as follow,

$$\theta_1 = \pm \arccos\left(\frac{0.5 + 0.8\cos(\theta_3) + 0.8\cos(\theta_2+\theta_3)}{\sqrt{p_x^2+p_y^2}}\right) - \theta_2 - \theta_3 + \phi$$

Task3 Dynamics and Computing





Assume that the mass of each link is lumped at end of the link, so the center of mass of each link is represented in its coordinate system as,

$$P_{C1}^1 = \begin{bmatrix}0\\0\\d_1\end{bmatrix},\quad P_{C2}^2 = \begin{bmatrix}0.4\\0\\0\end{bmatrix},\quad P_{C3}^3 = \begin{bmatrix}0.4\\0\\0\end{bmatrix},\quad P_{C4}^4 = \begin{bmatrix}0.25\\0\\0\end{bmatrix}$$

Each coordinate system is represented under its previous coordinate system as

$$P_1^0 = \begin{bmatrix}0\\0\\d_1\end{bmatrix},\quad P_2^1 = \begin{bmatrix}0.8\\0\\0\end{bmatrix},\quad P_3^2 = \begin{bmatrix}0.8\\0\\0\end{bmatrix},\quad P_4^3 = \begin{bmatrix}0.5\\0\\0\end{bmatrix}$$

The xy position are as follow,

$$\begin{cases}x_A = 0.4\cos\theta_1\\ y_A = 0.4\sin\theta_1\end{cases},\quad \begin{cases}x_B = 0.4\cos(\theta_1+\theta_2) + 0.8\cos\theta_1\\ y_B = 0.4\sin(\theta_1+\theta_2) + 0.8\sin\theta_1\end{cases},\quad \begin{cases}x_C = 0.25\cos(\theta_1+\theta_2+\theta_3) + 0.8\cos(\theta_1+\theta_2) + 0.8\cos\theta_1\\ y_C = 0.25\sin(\theta_1+\theta_2+\theta_3) + 0.8\sin(\theta_1+\theta_2) + 0.8\sin\theta_1\end{cases}$$

We can get the xy velocity as follow,

$$\begin{cases}\dot{x}_A = -0.4\sin\theta_1\dot{\theta}_1\\ \dot{y}_A = 0.4\cos\theta_1\dot{\theta}_1\end{cases}$$

$$\begin{cases}\dot{x}_B = -0.4\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) - 0.8\sin\theta_1\dot{\theta}_1\\ \dot{y}_B = 0.4\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + 0.8\cos\theta_1\dot{\theta}_1\end{cases}$$

$$\begin{cases}\dot{x}_C = -0.25\sin(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3) - 0.8\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) - 0.8\sin\theta_1\dot{\theta}_1\\ \dot{y}_C = 0.25\cos(\theta_1+\theta_2+\theta_3)(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3) + 0.8\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) + 0.8\cos\theta_1\dot{\theta}_1\end{cases}$$

So we can get v_A, v_B, v_C

$$v_A^2 = \dot{x}_A^2 + \dot{y}_A^2 = 0.16\dot{\theta}_1^2$$

$$v_B^2 = \dot{x}_B^2 + \dot{y}_B^2 = 0.64\dot{\theta}_1^2 + 0.16(\dot{\theta}_1+\dot{\theta}_2)^2 + 0.64\cos\theta_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)$$

$$v_C^2 = \dot{x}_C^2 + \dot{y}_C^2 = 0.64\dot{\theta}_1^2 + 0.64(\dot{\theta}_1+\dot{\theta}_2)^2 + 0.0625(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)^2 + 0.4\cos\theta_3(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3)(\dot{\theta}_1+\dot{\theta}_2)$$
$$+0.4\cos(\theta_2+\theta_3)\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3) + 1.28\cos\theta_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)$$

N-E Equations

So, for the Newton equation, $\dot{v}$ denotes acceleration of center of mass and F denotes the force at center of mass

$$m\dot{v} = F$$

For the Euler equation, ω---angular velocity, $\dot{\omega}$---angular acceleration, N---the torque acting on the body, I---the inertia tensor in a frame with its origin located at the center of mass

$$I\dot{\omega} + \omega \times I\omega = N$$

For prismatic joint 1,

Linear acceleration of the origin of frame 1 is

$$\dot{v}_1^1 = R_0^1[\dot{\omega}_0^0 P_1^0 + \omega_0^0 \times (\omega_0^0 \times P_1^0) + \dot{v}_0^0] + 2\omega_1^1 \times \dot{d}_1 Z_1^1 + \ddot{d}_1 Z_1^1$$

Linear acceleration of the center of mass

$$\dot{v}_{C1}^1 = \dot{\omega}_1^1 \times P_{C1}^1 + \omega_1^1 \times (\omega_1^1 \times P_{C1}^1) + \dot{v}_1^1$$

$$F_1^1 = m_1 \dot{v}_{C1}^1$$

$$N_1^1 = I_1^1 \dot{\omega}_1^1 + \dot{\omega}_1^1 \times I_1^1 \dot{\omega}_1^1$$

For revolute joint 2,

Linear acceleration of the origin of frame 2 is

$$\dot{v}_2^2 = R_1^2[\dot{\omega}_1^1 P_2^1 + \omega_1^1 \times (\omega_1^1 \times P_2^1) + \dot{v}_1^1]$$

Linear acceleration of the center of mass

$$\dot{v}_{C2}^2 = \dot{\omega}_2^2 \times P_{C2}^2 + \omega_2^2 \times (\omega_2^2 \times P_{C2}^2) + \dot{v}_2^2$$

$$F_2^2 = m_2 \dot{v}_{C2}^2$$

$$N_2^2 = I_2^2 \dot{\omega}_2^2 + \dot{\omega}_2^2 \times I_2^2 \dot{\omega}_2^2$$

For revolute joint 3,

Linear acceleration of the origin of frame 3 is

$$\dot{v}_3^3 = R_2^3[\dot{\omega}_2^2 P_3^2 + \omega_2^2 \times (\omega_2^2 \times P_3^2) + \dot{v}_2^2]$$

Linear acceleration of the center of mass

$$\dot{v}_{C3}^3 = \dot{\omega}_3^3 \times P_{C3}^3 + \omega_3^3 \times (\omega_3^3 \times P_{C3}^3) + \dot{v}_3^3$$

$$F_3^3 = m_3 \dot{v}_{C3}^3$$

$$N_3^3 = I_3^3 \dot{\omega}_3^3 + \dot{\omega}_3^3 \times I_3^3 \dot{\omega}_3^3$$

For revolute joint 4,

Linear acceleration of the origin of frame 4 is

$$\dot{v}_4^4 = R_3^4[\dot{\omega}_3^3 P_4^3 + \omega_3^3 \times (\omega_3^3 \times P_4^3) + \dot{v}_3^3]$$

Linear acceleration of the center of mass

$$\dot{v}_{C4}^4 = \dot{\omega}_4^4 \times P_{C4}^4 + \omega_4^4 \times (\omega_4^4 \times P_{C4}^4) + \dot{v}_3^3$$

$$F_4^4 = m_4 \dot{v}_{C4}^4$$

$$N_4^4 = I_4^4 \dot{\omega}_4^4 + \dot{\omega}_4^4 \times I_4^4 \dot{\omega}_4^4$$

L-E Equations

We denote the prismatic joint speed as v_1

$$v_1 = \dot{d}_1$$

Considering the kinetic energy of the system consists of the Kinetic Energy of all link1, link2, link3, link4 respectively, we can obtain that

$$\begin{aligned} K &= K_1 + K_2 + K_3 + K_4 \\ &= \frac{1}{2}m_1\dot{d}_1^{\ 2} + [\frac{1}{2}m_2v_A^2 + \frac{1}{2}I_1\dot{\theta}_1^{\ 2}] + [\frac{1}{2}m_3v_B^2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2] + [\frac{1}{2}m_4v_C^2 + \frac{1}{2}I_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2] \end{aligned}$$

The potential energy of this system can be shown as follow,

$$P = (m_1 + m_2 + m_3 + m_4)gd_1$$

Since v_A, v_B, v_C as follow,

$$v_A^2 = 0.16\dot{\theta}_1^2$$

$$v_B^2 = 0.64\dot{\theta}_1^2 + 0.16(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.64\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)$$

$$\begin{aligned} v_C^2 = 0.64\dot{\theta}_1^2 + 0.64(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.0625(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + 0.4\cos\theta_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)(\dot{\theta}_1 + \dot{\theta}_2) \\ +0.4\cos(\theta_2 + \theta_3)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) + 1.28\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) \end{aligned}$$

$$\begin{aligned} K &= K_1 + K_2 + K_3 + K_4 \\ &= \frac{1}{2}m_1\dot{d}_1^{\ 2} + [\frac{1}{2}m_2v_A^2 + \frac{1}{2}I_1\dot{\theta}_1^{\ 2}] + [\frac{1}{2}m_3v_B^2 + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2] + [\frac{1}{2}m_4v_C^2 + \frac{1}{2}I_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2] \end{aligned}$$

$$\begin{aligned} K = \frac{1}{2}m_1\dot{d}_1^{\ 2} + \frac{1}{2}m_2(0.16\dot{\theta}_1^2) + \frac{1}{2}I_1\dot{\theta}_1^{\ 2} + \frac{1}{2}m_3[0.64\dot{\theta}_1^2 + 0.16(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.64\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)] + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \\ \frac{1}{2}m_4[0.64\dot{\theta}_1^2 + 0.64(\dot{\theta}_1 + \dot{\theta}_2)^2 + 0.0625(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + 0.4\cos\theta_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)(\dot{\theta}_1 + \dot{\theta}_2) \\ +0.4\cos(\theta_2 + \theta_3)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) + 1.28\cos\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)] + \frac{1}{2}I_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 \end{aligned}$$

$$\begin{aligned} => K = \frac{1}{2}m_1\dot{d}_1^{\ 2} + \frac{1}{2}(0.16m_2 + 0.64m_3 + 0.64m_4 + I_1)\dot{\theta}_1^2 + \frac{1}{2}(0.16m_3 + 0.64m_4 + I_2)(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ +\frac{1}{2}(0.0625m_3 + I_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 + \frac{1}{2}(0.64\cos\theta_2m_3 + 1.28\cos\theta_2m_4)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) \\ +\frac{1}{2}(0.4\cos\theta_3m_4)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)(\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2}(0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \end{aligned}$$

$$\begin{aligned} => K = \frac{1}{2}m_1\dot{d}_1^{\ 2} + \frac{1}{2}(0.16m_2 + 0.64m_3 + 0.64m_4 + I_1)\dot{\theta}_1^2 + \frac{1}{2}(0.16m_3 + 0.64m_4 + I_2)(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) \\ +\frac{1}{2}(0.0625m_3 + I_3)(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + 2\dot{\theta}_1\dot{\theta}_2 + 2\dot{\theta}_2\dot{\theta}_3 + 2\dot{\theta}_1\dot{\theta}_3) + \frac{1}{2}(0.64\cos\theta_2m_3 + 1.28\cos\theta_2m_4)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \\ +\frac{1}{2}(0.4\cos\theta_3m_4)(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2\dot{\theta}_3 + \dot{\theta}_1\dot{\theta}_3) + \frac{1}{2}(0.4\cos(\theta_2 + \theta_3)m_4)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_1\dot{\theta}_3) \end{aligned}$$

$$\begin{aligned} => K = \frac{1}{2}m_1\dot{d}_1^{\ 2} + \frac{1}{2}(0.16m_2 + 0.64m_3 + 0.64m_4 + I_1 + 0.16m_3 + 0.64m_4 + I_2 + 0.0625m_3 + I_3 + 0.64\cos\theta_2m_3 + 1.28\cos\theta_2m_4 \\ +0.4\cos\theta_3m_4 + 0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1^2 + \frac{1}{2}(0.16m_3 + 0.64m_4 + I_2 + 0.0625m_3 + I_3 + 0.4\cos\theta_3m_4)\dot{\theta}_2^2 \\ +\frac{1}{2}(0.0625m_3 + I_3)\dot{\theta}_3^2 \\ +\frac{1}{2}(0.32m_3 + 1.28m_4 + 2I_2 + 0.125m_3 + 2I_3 + 0.64\cos\theta_2m_3 + 1.28\cos\theta_2m_4 + 0.8\cos\theta_3m_4 + 0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1\dot{\theta}_2 \\ +\frac{1}{2}(0.125m_3 + 2I_3 + 0.4\cos\theta_3m_4)\dot{\theta}_2\dot{\theta}_3 + \frac{1}{2}(0.125m_3 + 2I_3 + 0.4\cos\theta_3m_4 + 0.4\cos(\theta_2 + \theta_3)m_4)\dot{\theta}_1\dot{\theta}_3 \end{aligned}$$

$$=> K = \frac{1}{2}m_1\dot{d_1}^2 + \frac{1}{2}[0.16m_2 + (0.8625 + 0.64\cos\theta_2)m_3 + (1.28\cos\theta_2 + 0.4\cos\theta_3 + 0.4\cos(\theta_2+\theta_3))m_4 + I_1 + I_2 + I_3]\dot{\theta}_1^2$$
$$+\frac{1}{2}[0.2225m_3 + (0.64 + 0.4\cos\theta_3)m_4 + I_2 + I_3]\dot{\theta}_2^2$$
$$+\frac{1}{2}(0.0625m_3 + I_3)\dot{\theta}_3^2$$
$$+\frac{1}{2}[(0.445 + 0.64\cos\theta_2)m_3 + (1.28 + 1.28\cos\theta_2 + 0.8\cos\theta_3 + 0.4\cos(\theta_2+\theta_3))m_4 + 2I_2 + 2I_3]\dot{\theta}_1\dot{\theta}_2$$
$$+\frac{1}{2}(0.125m_3 + 2I_3 + 0.4\cos\theta_3 m_4)\dot{\theta}_2\dot{\theta}_3$$
$$+\frac{1}{2}[0.125m_3 + 2I_3 + (0.4\cos\theta_3 + 0.4\cos(\theta_2+\theta_3)m_4]\dot{\theta}_1\dot{\theta}_3$$

Since,

$$K = \frac{1}{2}\dot{q}^T Dq$$

$$q = \begin{bmatrix} \dot{d}_1 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

From the form that, $m_1 = 1.0kg$, $m_2 = 1.2kg$, $m_3 = 1.0kg$, $m_4 = 0.6kg$, $I_1 = 0.256kg\cdot m^2$, $I_2 = 0.213kg\cdot m^2$, $I_3 = 0.05kg\cdot m^2$, now we can get

$$K = \frac{1}{2}\dot{d_1}^2 + \frac{1}{2}[1.408\cos\theta_2 + 0.24\cos\theta_3 + 0.24\cos(\theta_2+\theta_3) + 1.5735]\dot{\theta}_1^2$$
$$+\frac{1}{2}(0.24\cos\theta_3 + 0.8695)\dot{\theta}_2^2$$
$$+\frac{1}{2}(0.1125)\dot{\theta}_3^2$$
$$+[0.96\cos\theta_2 + 0.4\cos\theta_3 + 0.12(\theta_2+\theta_3) + 1.1255]\dot{\theta}_1\dot{\theta}_2$$
$$+(0.12\cos\theta_3 + 0.1125)\dot{\theta}_2\dot{\theta}_3$$
$$+[0.12\cos\theta_3 + 0.12(\theta_2+\theta_3) + 0.1125]\dot{\theta}_1\dot{\theta}_3$$

Let c_i denotes $\cos\theta_i$, s_i denotes $\sin\theta_i$, c_{ij} denotes $\cos(\theta_i+\theta_j)$

We can get D as follow,

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.408c_2 + 0.24c_3 + 0.24c_{23} + 1.5735 & 0.96c_2 + 0.4c_3 + 0.12c_{23} + 1.1255 & 0.12c_3 + 0.12c_{23} + 0.1125 \\ 0 & 0.96c_2 + 0.4c_3 + 0.12c_{23} + 1.1255 & 0.24c_3 + 0.8695 & 0.12c_3 + 0.1125 \\ 0 & 0.12c_3 + 0.12c_{23} + 0.1125 & 0.12c_3 + 0.1125 & 0.1125 \end{bmatrix}$$

Now we find the Christoffel Symbol,

$$C_{ijk} = \frac{1}{2}[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}]$$

$$c_{22} = -[0.12\sin(\theta_2+\theta_3) + 0.704\sin\theta_3]\dot{\theta}_2 - [0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_3$$
$$c_{23} = -[0.12\sin(\theta_2+\theta_3) + 0.704\sin\theta_2]\dot{\theta}_1 - [0.12\sin(\theta_2+\theta_3) + 0.96\sin\theta_2]\dot{\theta}_2 - [0.12\sin(\theta_2+\theta_3) + 0.2\sin\theta_3]\dot{\theta}_3$$
$$c_{24} = -[0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_1 - [0.12\sin(\theta_2+\theta_3)) + 0.2\sin\theta_3]\dot{\theta}_2 - [0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_3$$
$$c_{32} = [0.12\sin(\theta_2+\theta_3) + 0.704\sin\theta_3]\dot{\theta}_1 - 0.2\sin\theta_3\dot{\theta}_3$$
$$c_{33} = -0.12\sin\theta_3\dot{\theta}_3$$
$$c_{34} = -0.2\sin\theta_3\dot{\theta}_1 - 0.12\sin\theta_3\dot{\theta}_2 - 0.12\sin\theta_3\dot{\theta}_3$$
$$c_{42} = [0.12\sin(\theta_2+\theta_3) + 0.12\sin\theta_3]\dot{\theta}_1 + 0.2\sin\theta_3\dot{\theta}_2$$
$$c_{43} = 0.2\sin\theta_3\dot{\theta}_1 + 0.12\sin\theta_3\dot{\theta}_2$$

So, we can get C as follow,

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & c_{32} & c_{33} & c_{34} \\ 0 & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

And we can get G as follow,

$$G = \frac{\partial P}{\partial q} = \begin{bmatrix} (m_1 + m_2 + m_3 + m_4)g \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

After get D, C, G, The Lagrange-Euler Equation are as follow,

$$D\ddot{\theta} + C\dot{\theta} + G = \tau$$

Time-varying torque

The time-varying torque is at the right hand side of this equation

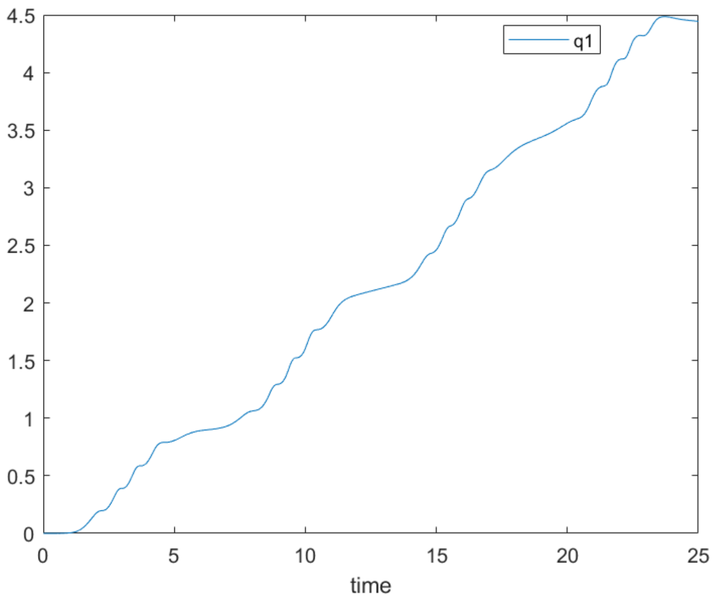
$$D\ddot{\theta} + C\dot{\theta} + G = \tau$$

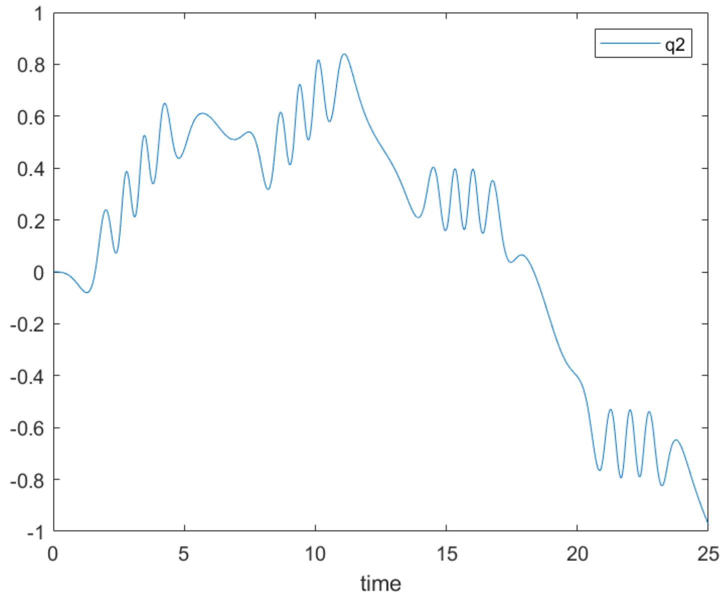
We design the time-varying preference torque as follow,

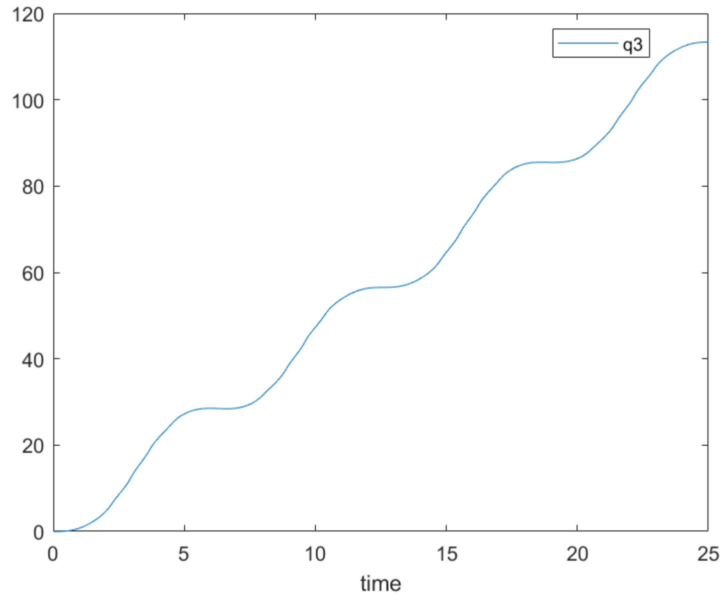
$$\tau = \begin{bmatrix} 0 \\ 0.8\sin(t) \\ 0.8\sin(t) \\ 0.5\sin(t) \end{bmatrix}$$

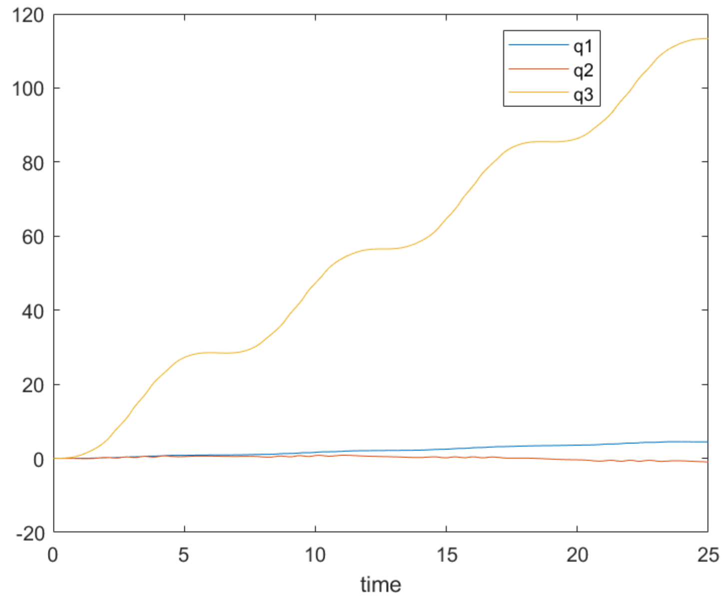
After simulate with MATLAB, qi denotes the positions, dqi denotes the velocity and $ddqi$ denotes the velocity in the figure,

We get positions $q1, q2, q3$ as follow,

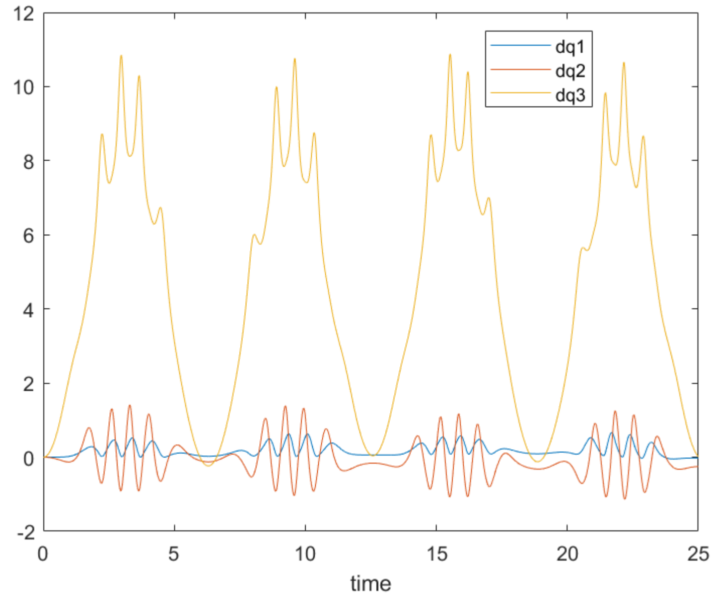




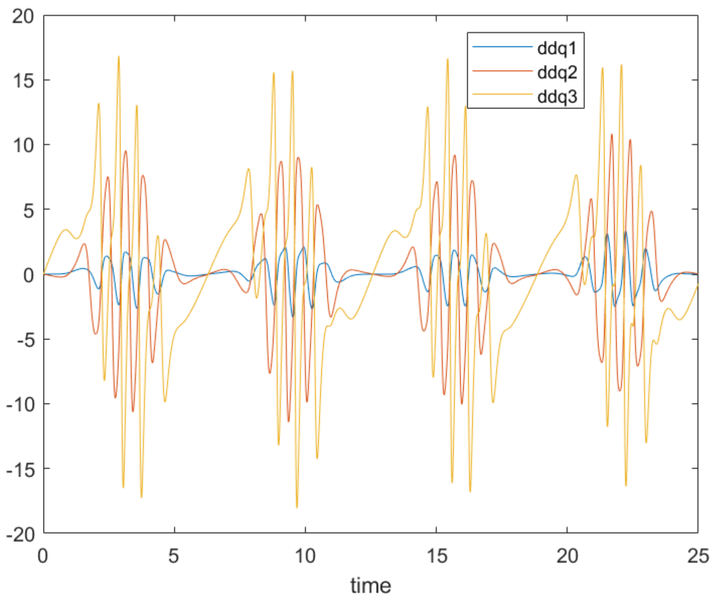




We get velocities $dq1, dq2, dq3$ as follow,



We get accelerations $ddq1, ddq2, ddq3$ as follow,



Task4 Control Design and Simulation

$$D(q)\ddot{q} + C(q,\dot{q})q + G(q) = \tau$$

Rewrite the manipulator's equations of motion as,

$$\ddot{q} = D(q)^{-1}(\tau - C(q,\dot{q})\dot{q} - G(q))$$

Choose $\ddot{q}$,

$$\ddot{q} = \ddot{q}_d + K_v\dot{e} + K_p e + K_i \int edt$$

The PID computed-torque control law is given as, where $e = q_d - q$,

$$D(q)(\ddot{q}_d + K_v\dot{e} + K_p e + K_i \int edt) + C(q,\dot{q})q + G(q) = \tau$$

Let the desired q,

$$q = \begin{bmatrix} 0 \\ 0.2\sin(t) \\ 0.3\cos(t) \\ 0.4\sin(t) \end{bmatrix}$$

So the desired $\dot{q}$,

$$\dot{q} = \begin{bmatrix} 0 \\ 0.2\cos(t) \\ -0.3\sin(t) \\ 0.4\cos(t) \end{bmatrix}$$

So the desired $\ddot{q}$,

$$\ddot{q} = \begin{bmatrix} 0 \\ -0.2\sin(t) \\ -0.3\cos(t) \\ -0.4\sin(t) \end{bmatrix}$$