



Q.1

$$(a). |\lambda I - A| = \begin{vmatrix} \lambda+2 & 1 \\ -1 & \lambda+4 \end{vmatrix} = \lambda^2 + 6\lambda + 9 = 0$$

The eigenvalue of  $A$  are  $-3$  and  $-3$

$$\text{Let } h(\lambda) = d_0(t) + \lambda d_1(t)$$

When  $\lambda = -3$ , we have

$$e^{-3t} = d_0(t) - 3d_1(t) \quad (1)$$

Then,  $\frac{dh(\lambda)}{d\lambda} \big|_{\lambda=-3} = \frac{dh(\lambda)}{d\lambda} \big|_{\lambda=-3}$ , we can get

$$t \cdot e^{-3t} = d_1(t) \quad (2)$$

Solving, we get  $d_0(t) = e^{-3t} + 3te^{-3t}$ , and

$$e^{At} = d_0 I + d_1 \cdot A = \begin{pmatrix} e^{-3t} + te^{-3t} & -te^{-3t} \\ te^{-3t} & e^{-3t} - te^{-3t} \end{pmatrix}$$

(b). Assume that,  $P = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix}$ , we can get.

$$A \begin{cases} x = P \bar{x} \\ \bar{A} = P^{-1} A P \\ \bar{B} = P^{-1} B \\ \bar{C} = C P \end{cases} \Rightarrow \begin{matrix} \cancel{4p_1 + 6p_2 = p_1 + 2p_3} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{matrix}$$

$$\Rightarrow \begin{cases} P \bar{A} = A P \\ \bar{C} = C P \end{cases} \Rightarrow \begin{cases} \begin{pmatrix} 4p_1 + 6p_2 & p_1 + p_2 \\ 4p_3 + 6p_4 & p_3 + p_4 \end{pmatrix} = \begin{pmatrix} p_1 + 2p_3 & p_2 + 2p_4 \\ 3p_1 + 4p_3 & 3p_2 + 4p_4 \end{pmatrix} \\ [1 \quad -0.5] = [p_1 \quad p_2] \end{cases}$$

$$\Rightarrow \begin{cases} p_1 = 1 \\ p_2 = -0.5 \\ p_3 = 0 \\ p_4 = 0.5 \end{cases} \Rightarrow P = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix}$$

Q.1 (c). We can use Lyapunov Equation

Let  $P = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}$ ,  $Q = I$

$$A^T P + P A = -Q$$

$$\begin{pmatrix} -2 & -5 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} P_1 = \frac{19}{6} \\ P_2 = -\frac{7}{6} \\ P_3 = \frac{1}{6} \end{cases} \Rightarrow P = \begin{pmatrix} \frac{19}{6} & -\frac{7}{6} \\ -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$\det P = \begin{vmatrix} \frac{19}{6} & -\frac{7}{6} \\ -\frac{7}{6} & \frac{1}{6} \end{vmatrix} = -\frac{5}{6} < 0$$

So, the system is not asymptotically stable.

Let's calculate the eigenvalue of  $A$ .

$$|\lambda I - A| = \begin{vmatrix} \lambda + 2 & -1 \\ 5 & \lambda - 4 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0, \lambda_1 = 3, \lambda_2 = -1$$

Let  $h(\lambda) = d_0(t) + \lambda d_1(t)$

$$\begin{cases} e^{-t} = d_0 - d_1 \\ e^{3t} = d_0 + 3d_1 \end{cases} \Rightarrow \begin{cases} d_0 = \frac{3e^{-t} + e^{3t}}{4} \\ d_1 = \frac{e^{3t} - e^{-t}}{4} \end{cases}$$

$$e^{At} = d_0(t)I + d_1(t)A = \begin{pmatrix} \frac{3e^{-t} + e^{3t}}{4} & \frac{e^{3t} - e^{-t}}{4} \\ \frac{5}{4}(e^{-t} - e^{3t}) & \frac{3e^{3t} - e^{-t}}{4} \end{pmatrix}$$

$$X(t) = e^{At} \cdot X_0 = \begin{pmatrix} \frac{1}{4}(3e^{-t} - e^{3t}) & \frac{1}{4}(e^{3t} - e^{-t}) \\ \frac{5}{4}(e^{-t} - e^{3t}) & \frac{1}{4}(3e^{3t} - e^{-t}) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

Q. 2 (i) The eigenvalue of  $A = \begin{bmatrix} -1 & \alpha_1 & 0 \\ 0 & -1 & \alpha_2 \\ 0 & 0 & -1 \end{bmatrix}$  is  $\lambda = -1$

The matrix  $[A - \lambda I, B]$  can be expressed as:

$\begin{bmatrix} 0 & \alpha_1 & 0 & 1 \\ 0 & 0 & \alpha_2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ , No matter what  $\alpha_1, \alpha_2$  is, this matrix won't be full row rank.

So, the system won't be controllable

(ii) The T.F is

$$y(s) = C \cdot (sI - A)^{-1} B = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & \alpha_1 & 0 \\ 0 & s+1 & \alpha_2 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{(s+1) + \alpha_1}{(s+1)^2}$$

Q2 (b). Let  $P = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}$ ,  $Q = I$

$$A^T P + P A = -Q$$

$$\begin{bmatrix} -1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2P_1 - 10P_3 = -1 \\ 2P_2 + 2P_3 = -1 \\ P_1 - 5P_3 = 0 \end{cases}$$

, There is no solution, so the system is not stable.

(c).

Q.3

(a). First, combine controllability matrix

$$W_c = \{B \quad AB \quad A^2B\} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} = \{b_1, b_2, Ab_1, Ab_2, A^2b_1, A^2b_2\}.$$

Let's choose first 3 independent vector  $b_1, b_2, Ab_1$ .

$$C = \{b_1, Ab_1, b_2\} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } d_1 = 2, d_2 = 1$$

$$C^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ we should choose second and third rows}$$

$$T = \begin{bmatrix} q_{12}^T \\ q_{21}^T \\ q_{33}^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \bar{B} = TB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let  $\bar{K} = \begin{bmatrix} \bar{K}_{11} & \bar{K}_{12} & \bar{K}_{13} \\ \bar{K}_{21} & \bar{K}_{22} & \bar{K}_{23} \end{bmatrix}$ , from the closed loop matrix:

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 \\ -\bar{K}_{11} & 1-\bar{K}_{12} & -\bar{K}_{13} \\ 1-\bar{K}_{21} & -\bar{K}_{22} & -\bar{K}_{23} \end{bmatrix}$$

The desired eigenvalues are  $-1$ .

$$\det(sI - A_d) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

$$A_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \quad \text{Compare } \bar{A} - \bar{B}\bar{K} \text{ and } A_d.$$

$$\bar{K}_{11} = 0, \bar{K}_{12} = \bar{K}_{13} = 1, \bar{K}_{21} = 2, \bar{K}_{22} = \bar{K}_{23} = 3$$

$$\bar{K} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 3 \end{bmatrix}, \quad K = \bar{K} \cdot T = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix}$$

The controller is

$$u = -KX = -\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix} \cdot X.$$

Q.3 (b).  $\dot{x} = x + u$ , let  $u = -kx$ .

$$\dot{x} = (1-k) \cdot x, \quad x(t) = x(0) \cdot e^{(k-1)t} = c \cdot e^{-(k-1)t}$$

The function will change to.

$$\begin{aligned} J &= \int_0^{\infty} (x^2 + x \cdot (-kx) + (-kx)^2) dt \\ &= (k^2 - k + 1) \int_0^{\infty} x^2 dt \\ &= (k^2 - k + 1) c^2 \int_0^{\infty} e^{-2(k-1)t} dt \\ &= (k^2 - k + 1) \cdot c^2 \cdot \frac{1}{2(k-1)} \end{aligned}$$

$$\frac{dJ}{dk} = 0, \quad \text{We can get.}$$

$$(2k-1) \cdot 2(k-1) - 2(k^2 - k + 1) = 0$$

$$k(k-2) = 0.$$

$$\text{Hence } k_1 = 0, \quad k_2 = 2.$$

To assure the system is stable, we need  $k > 1$ .

$$\text{So, } k = 2.$$

Q.4 (a)

First, we need to get the T.F.

$$G(s) = C \cdot (sI - A)^{-1} \cdot B = C \cdot \frac{\text{adj}(sI - A)}{\det(sI - A)} \cdot B \Rightarrow$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & \frac{1}{s^2(s-1)} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{s-1} & \frac{s+1}{s(s-1)} \\ 0 & 0 \end{bmatrix}.$$

$$D(s) = s(s-1), \quad N(s) = \begin{bmatrix} s & s+1 \\ 0 & 0 \end{bmatrix}.$$

~~$N(s) = \begin{bmatrix} s & s+1 \\ 0 & 0 \end{bmatrix}$~~

$$K_d = \text{adj } N(s) = \begin{bmatrix} 0 & -(s+1) \\ 0 & s \end{bmatrix}.$$

Q.4 (b)

(i):  $\zeta = 0.5$ ,  $\omega_n = 1$ .  $P_c' = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + s + 1$ .

(ii):  $r = c + a \sin(t)$   $R(s) = \frac{c}{s} + \frac{a}{s^2 + 1}$

(iii)  $d = c'$   $D(s) = \frac{c'}{s}$

So,  ~~$Q(s) = s(s+1)$~~ . Because  $Y(s) = \frac{1}{2s+1} U(s) + \frac{1}{s} D(s)$ ,  $Q(s) = s^2 \cdot (s^2 + 1)$

$G(s) = \frac{1}{2s+1}$ ,  $N(s) = 1$ ,  $D(s) = 2s+1$

Let  $A(s) = Q(s) \cdot \cancel{A(s)} = s^2(s^2 + 1)$

$B(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4$ .

$P_c = P_c' \cdot (s+1)^2 (2s+1)$

We can get

$D(s) A(s) + N(s) B(s) = P_c$ .

$(2s+1)s^2(s^2+1) + (b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4) = (s^2 + s + 1)(s+1)^2(2s+1)$

$\Rightarrow 2s^5 + (1+b_4)s^4 + (2+b_3)s^3 + (1+b_2)s^2 + b_1 s + b_0 = 2s^5 + 7s^4 + 11s^3 + 10s^2 + 5s + 1$

$\Rightarrow \begin{cases} b_0 = 1 \\ b_1 = 5 \\ b_2 = 9 \\ b_3 = 9 \\ b_4 = 8 \end{cases}$ , So:  $A(s) = s^2(s^2 + 1)$   
 $B(s) = 8s^4 + 9s^3 + 9s^2 + 5s + 1$

$K(s) = \frac{B(s)}{A(s)} = \frac{8s^4 + 9s^3 + 9s^2 + 5s + 1}{s^2(s^2 + 1)}$

Because  $P_s$  contain the two desire poles in  $(s^2 + s + 1)$ ,

So, ii) is met

and  $A(s)$  contain  $Q(s)$  which can ~~be~~ contain unstable poles from  $r(t)$  and  $\frac{1}{s} D(s)$ , so ii) and iii) are met.



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