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Q1:

a)

$$\frac{1}{C}\int idt = y$$
$$i = C\frac{dy}{dt} = C\cdot\dot{y}$$

So, the first equation express as:

$$L\dot{i} + Ri + y = u$$
$$LC\ddot{y} + RC\dot{y} + y = u$$

Laplace transform (assume the initial conditions is 0):

$$s^2LCY(s) + sRCY(s) + Y(S) = U(S)$$
$$\frac{Y(S)}{U(S)} = \frac{1}{s^2LC + sRC + 1} = \frac{1}{s^2 + 2s + 1}$$

b)

Because $\begin{cases} x_1 = e_o \\ x_2 = \dot{e}_o \end{cases}$, we can get:

$$LC\dot{x}_2 + RCx_2 + x_1 = u$$

Then, we can get two equations as follow:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + u$$

The state-space representation of the system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}e_i = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

($A = \begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$, $B = \begin{bmatrix}0\\ 1\end{bmatrix}$, $C = \begin{bmatrix}1 & 0\end{bmatrix}$)

c)

First, let's calculate Φ and Γ

$$
\begin{aligned}
e^{At} &= \mathcal{L}^{-1}(sI-A)^{-1} \\
&= \mathcal{L}^{-1}\begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1} \\
&= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ -\frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}+te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{bmatrix}
\end{aligned}
$$

$$
\Phi = e^{Ah} = \begin{bmatrix} 2e^{-1} & e^{-1} \\ -e^{-1} & 0 \end{bmatrix}
$$

$$
\begin{aligned}
\Gamma &= \left(\int_0^h e^{Av}dv\right)B \\
&= A^{-1}(e^{Ah}-e^{A0})B \\
&= \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2\mathrm{e}^{-1}\text{-}1 & e^{-1} \\ -e^{-1} & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -2e^{-1}+1 \\ e^{-1} \end{bmatrix}
\end{aligned}
$$

Then, we can get the state-space representation of the sampled system.

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 2e^{-1} & e^{-1} \\ -e^{-1} & 0 \end{bmatrix}x(k) + \begin{bmatrix} -2e^{-1}+1 \\ e^{-1} \end{bmatrix}u(k) \\
y(k) &= [1 \quad 0]x(k)
\end{aligned}
$$

d)

Assuming the initial conditions is 0

$$
\begin{aligned}
zX(z) &= \Phi X(z) + \Gamma U(z) \\
X(z) &= (zI-\Phi)^{-1}\Gamma U(z)
\end{aligned}
$$

Then, we can get $Y(z)$

$$Y(z) = C(zI - \Phi)^{-1}\Gamma U(z)$$
$$\frac{Y(z)}{U(z)} = C(zI - \Phi)^{-1}\Gamma$$
$$= \begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 2e^{-1} & e^{-1} \\ -e^{-1} & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} -2e^{-1}+1 \\ e^{-1} \end{bmatrix}$$
$$= \frac{z(-2e^{-1}+1)+e^{-2}}{(z-e^{-1})^2}$$

e)

$$zX(z) - zx(0) = \Phi X(z) + \Gamma U(z)$$

$$Y(z) = C(zI - \Phi)^{-1}(\Gamma U(z) + zx(0))$$
$$= \begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 2e^{-1} & e^{-1} \\ -e^{-1} & 0 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} -2e^{-1}+1 \\ e^{-1} \end{bmatrix}\frac{z}{z-1} + \begin{bmatrix} z \\ 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} \frac{z}{(z-e^{-1})^2} & \frac{e^{-1}}{(z-e^{-1})^2} \\ -\frac{e^{-1}}{(z-e^{-1})^2} & \frac{z-2e^{-1}}{(z-e^{-1})^2} \end{bmatrix}\begin{bmatrix} \frac{z^2-2e^{-1}z}{z-1} \\ \frac{e^{-1}z}{z-1} \end{bmatrix}$$
$$= \frac{z}{z-1}$$

$y(k) = Z^{-1}\{Y(z)\} = u(k)$, $y(k)$ is a unit step

Q2:

a)

1. This system is not stable, because it has a multiple pole ‘0’.
2. No. Because the zero of the transfer function is ‘1’.

b)

No.

If the pole is $\lambda = \sigma + j\omega$ before sampling, after sampling, the pole will become

$e^{h\lambda} = e^{h\sigma}e^{jh\omega}$. So, if λ is not stable, $e^{h\lambda}$ won’t stable.

c)

Yes, after sampling, there may be a stable inverse.

$$G(s) = \frac{s-1}{s^2(s+2)} = -\frac{3}{8}\frac{2}{s+2} + \frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2}$$

Z domain:

$$G(z) = -\frac{3}{8}\frac{1-e^{-2h}}{z-e^{-2h}} + \frac{3}{4}\frac{h}{z-1} - \frac{1}{2}\frac{h^2(z+1)}{2(z-1)^2}$$

$$= \frac{[-\frac{3}{8}(1-e^{-2h}) + \frac{3}{4}h - \frac{1}{4}h^2]z^2 + [\frac{3}{4}(1-e^{-2h}) - \frac{3}{4}h(1+e^{-2h}) - \frac{1}{4}h^2(1-e^{-2h})]z + [-\frac{3}{8}(1-e^{-2h}) + \frac{3}{4}he^{-2h} + \frac{1}{4}h^2e^{-2h}]}{(z-e^{-2h})(z-1)^2}$$

There are two zeros:

z_1, z_2

Q3:

a)

The eigenvalues of $\begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$ are: $\lambda = 3,\ \lambda = -2$. Not all the poles are negative, the system is not stable.

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

W_c is non-singular, so the system is controllable.

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

W_o is non-singular, so the system is observable.

b)

Assuming all initial conditions are zero.

$$zX(z) = \Phi X(z) + \Gamma U(z)$$
$$X(z) = \left(zI - \Phi\right)^{-1}\Gamma U(z)$$

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{U(z)} \\
&= C\left(zI - \Phi\right)^{-1}\Gamma \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{z+1}{(z-3)(z+2)}
\end{aligned}
$$

Input-output difference equation:

$$
y(k+2) - y(k+1) - 6y(k) = u(k+1) + u(k)
$$

c)

$$
\begin{aligned}
&Y(z) = H(z)U(z) = H(z)K(U_c(z) - Y(z)) \\
&(1 + H(z)K)Y(z) = H(z)KU_c(z)
\end{aligned}
$$

$$
\begin{aligned}
Y(z) &= (1 + H(z)K)^{-1}H(z)KU_c(z) \\
&= \frac{(z-3)(z+2)}{z^2 + (K-1)z + (K-6)} \frac{K(z+1)}{(z-3)(z+2)} U_c(z) \\
&= \frac{K(z+1)}{z^2 + (K-1)z + (K-6)} U_c(z)
\end{aligned}
$$

d)

$$
\begin{aligned}
&a_1 = K-1,\ a_2 = K-6 \\
&\begin{cases} 1 - a_2^2 > 0 \\ \dfrac{1-a_2}{1+a_2}((1+\alpha_2)^2 - \alpha_1^2) > 0 \end{cases} \\
&\Rightarrow \begin{cases} (K-7)(K-5) > 0 \\ \dfrac{(K-3)(K-7)}{K-5} > 0 \end{cases} \Rightarrow \begin{cases} 5 < K < 7 \\ 3 < K < 5 \ or \ K > 7 \end{cases}
\end{aligned}
$$

This equation has no solution. No matter what K is, this system is unstable.

e)

$$
Y(z) = \frac{K(z+1)}{z^2 + (K-1)z + (K-6)} U(z)
$$

$$
\begin{aligned}
E(z) &= U(z) - Y(z) \\
&= \frac{(z-3)(z+2)}{z^2 + (K-1)z + (K-6)} \frac{z}{z-1}
\end{aligned}
$$

If this is a stable system, according to the final value theorem, the final state is:

$$e = \lim_{k\to\infty}(e(k)) = \lim_{z\to 1}(z-1)E(z) = -\frac{3}{K-3}$$

But the system is unstable, so we try to calculate $e(k)$

$$\begin{aligned} E(z) &= \frac{(z-3)(z+2)}{z^2+(K-1)z+(K-6)}\frac{z}{z-1} \\ &= z\left(\frac{A}{z-p_1}+\frac{B}{z-p_2}+\frac{C}{z-1}\right) \end{aligned}$$

$(p_1+p_2=-K+1,\ p_1\cdot p_2=K-6)$

Then, we can calculate the value of A, B, C

$$\begin{cases} A+B+C=1 \\ Ap_2+Bp_1+C(p_1+p_2)+A+B=1 \\ Ap_2+Bp_1+Cp_1p_2=-6 \end{cases}$$

$$\Rightarrow \begin{cases} A=\dfrac{K}{K-3}\dfrac{p_1+3}{p_1-p_2} \\ B=\dfrac{K}{K-3}\dfrac{p_2+3}{p_2-p_1} \\ C=-\dfrac{3}{K-3} \end{cases}$$

$$\begin{aligned} e(k) &= Z^{-1}\{E(z)\} \\ &= Z^{-1}\left\{\frac{Az}{z-p_1}+\frac{Bz}{z-p_2}+\frac{Cz}{z-1}\right\} \\ &= Ap_1^k+Bp_2^k+C \\ &= \frac{K}{K-3}\frac{p_1+3}{p_1-p_2}p_1^k+\frac{K}{K-3}\frac{p_2+3}{p_2-p_1}p_2^k-\frac{3}{K-3} \end{aligned}$$

$$\left(p_1=\frac{(1-K)+\sqrt{K^2-6K+25}}{2},\ p_2=\frac{(1-K)-\sqrt{K^2-6K+25}}{2}\right)$$

No matter what K is, p_1 and p_2 can not be the stable poles at the same time.

$$\lim_{k\to\infty} e(k)=\infty$$

Q4:

a)

$$y(k+1)=3y(k-1)-2y(k-2)+u(k)-2u(k-1)+u(k-2)$$
$$y(k+3)-3y(k+1)+2y(k)=u(k+2)-2u(k+1)+u(k)$$
$$z^3Y(z)-3zY(z)+2Y(z)=z^2U(z)-2zU(z)+U(z)$$

$$H(z)=\frac{Y(z)}{U(z)}=\frac{z^2-2z+1}{z^3-3z+2}=\frac{(z-1)^2}{(z-1)^2(z+2)}$$

There is a multiple positive pole ‘z=1’, the system is not stable.

There is a multiple positive zero ‘z=1’, the inverse system is not stable

b)

Observable Canonical Form:

$$z(k+1)=\begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} z(k)+\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} u(k)$$
$$y(k)=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} z(k)$$

Then we can get controllability matrix and observability matrix:

$$W_c=\begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma \end{bmatrix}=\begin{bmatrix} 1 & -2 & 4 \\ -2 & 4 & -8 \\ 1 & -2 & 4 \end{bmatrix}$$

$$W_o=\begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$rank(W_c)=1,\ rank(W_o)=3$. This is an observable but not controllable form.

c)

No, we can’t realize this system controllable but not observable.

Controllable Canonical Form:

$$z(k+1)=\begin{bmatrix} 0 & 3 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}+\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k)$$
$$y(k)=\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} z(k)$$

Then we can get controllability matrix and observability matrix:

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi^2\Gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W_o = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 4 & -8 & 4 \end{bmatrix}$$

$rank(W_c) = 3,\ rank(W_o) = 2$.

But the in controllable canonical form, the transfer function is change. We can get three equations from the above state-space.

$$\begin{aligned} z_1(k+1) &= 3z_2(k) - 2z_3(k) + u(k) \\ z_2(k+1) &= z_1(k) \\ z_3(k+1) &= z_2(k) \end{aligned}$$

Since $y(k) = z_1(k) - 2z_2(k) + z_3(k)$, it follows that

$$\begin{aligned} y(k+1) &= z_1(k+1) - 2z_2(k+1) + z_3(k+1) \\ &= -2(z_1(k) - 2z_2(k) + z_3(k)) + u(k) \\ &= -2y(k) + u(k) \end{aligned}$$

$$G(z) = \frac{1}{z+2}$$

Therefore, the state-space model is corresponding to the transfer function $\frac{1}{z+2}$,

instead of $\frac{(z-1)^2}{(z-1)^2(z+2)}$

d)

No, the reason is same as c). Although $\frac{1}{z+2}$ is both controllable and observable, we can't get the conclusion that the original system is controllable and observable.