Chapter 9 Decoupling Control

§9.1 Introduction

Consider an MIMO system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1}$$

or

$$Y(s) = C(sI - A)^{-1} BU(s) =: G(s)U(s), \tag{2}$$

where G is supposed to be an *m by m square* matrix.

What is a coupled system?

Consider a boiler shown in Figure 1, which has couplings or interactions between loops and causes difficulty in operating it.

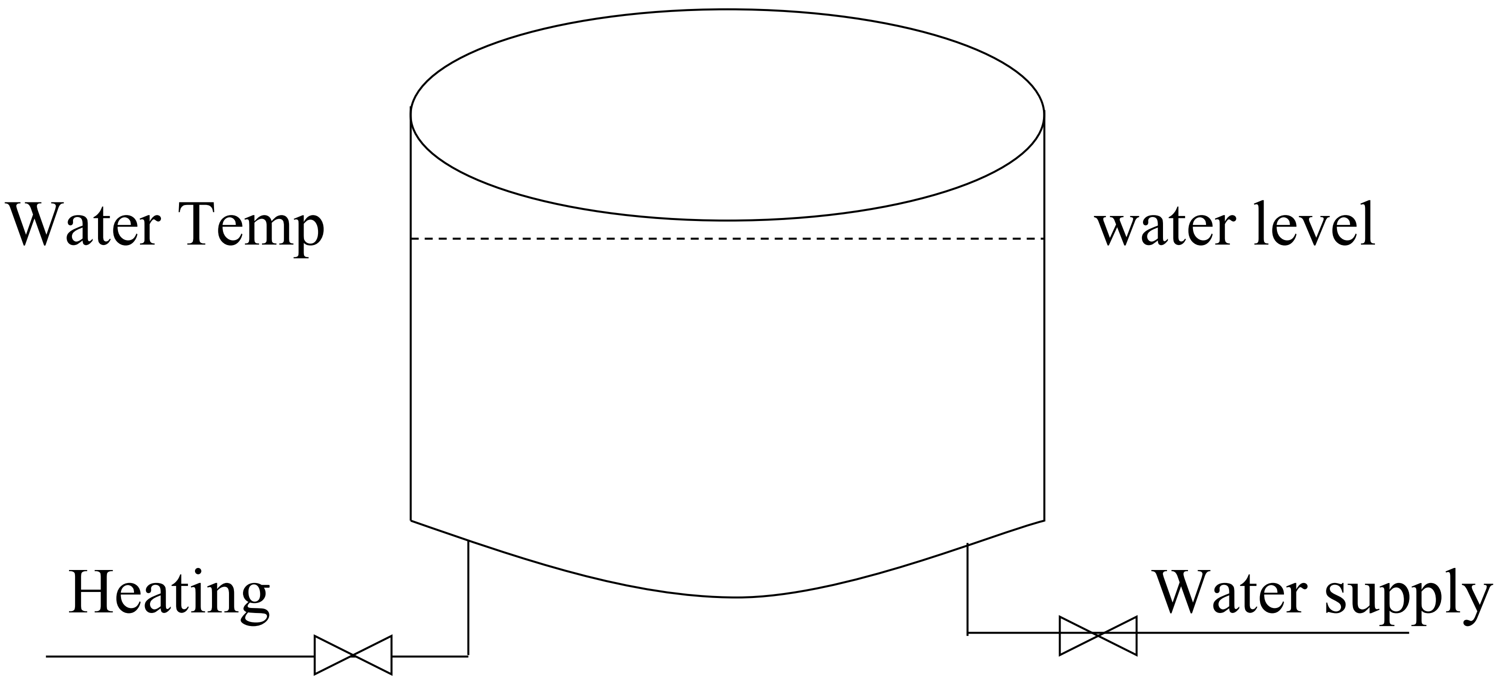

Figure 1 Boiler system.

It follows from (2) that

$$Y_1(s) = g_{11}(s)U_1(s) + g_{12}(s)U_2(s) + \cdots + g_{1m}(s)U_m(s) \ ,$$

$$Y_2(s) = g_{21}(s)U_1(s) + g_{22}(s)U_2(s) + \cdots + g_{2m}(s)U_m(s) \;,$$
$$\vdots \qquad (3)$$
$$Y_m(s) = g_{m1}(s)U_1(s) + g_{m2}(s)U_2(s) + \cdots + g_{mm}(s)U_m(s) \;.$$

If $g_{ij} \neq 0$, for some $i \neq j$, the system is called coupled since U_j effects not only Y_j but some Y_i, $i \neq j$.

What is a decoupled system?

If $g_{ii} \neq 0$ and $g_{ij} = 0$ for all $i \neq j$, the system is called decoupled. A decoupled system has a diagonal and non-singular *G(s)*:

$$y_1(s) = g_{11}(s)u_1(s),$$
$$y_2(s) = g_{22}(s)u_2(s), \qquad (4)$$
$$\vdots$$
$$y_m(s) = g_{mm}(s)u_m(s).$$

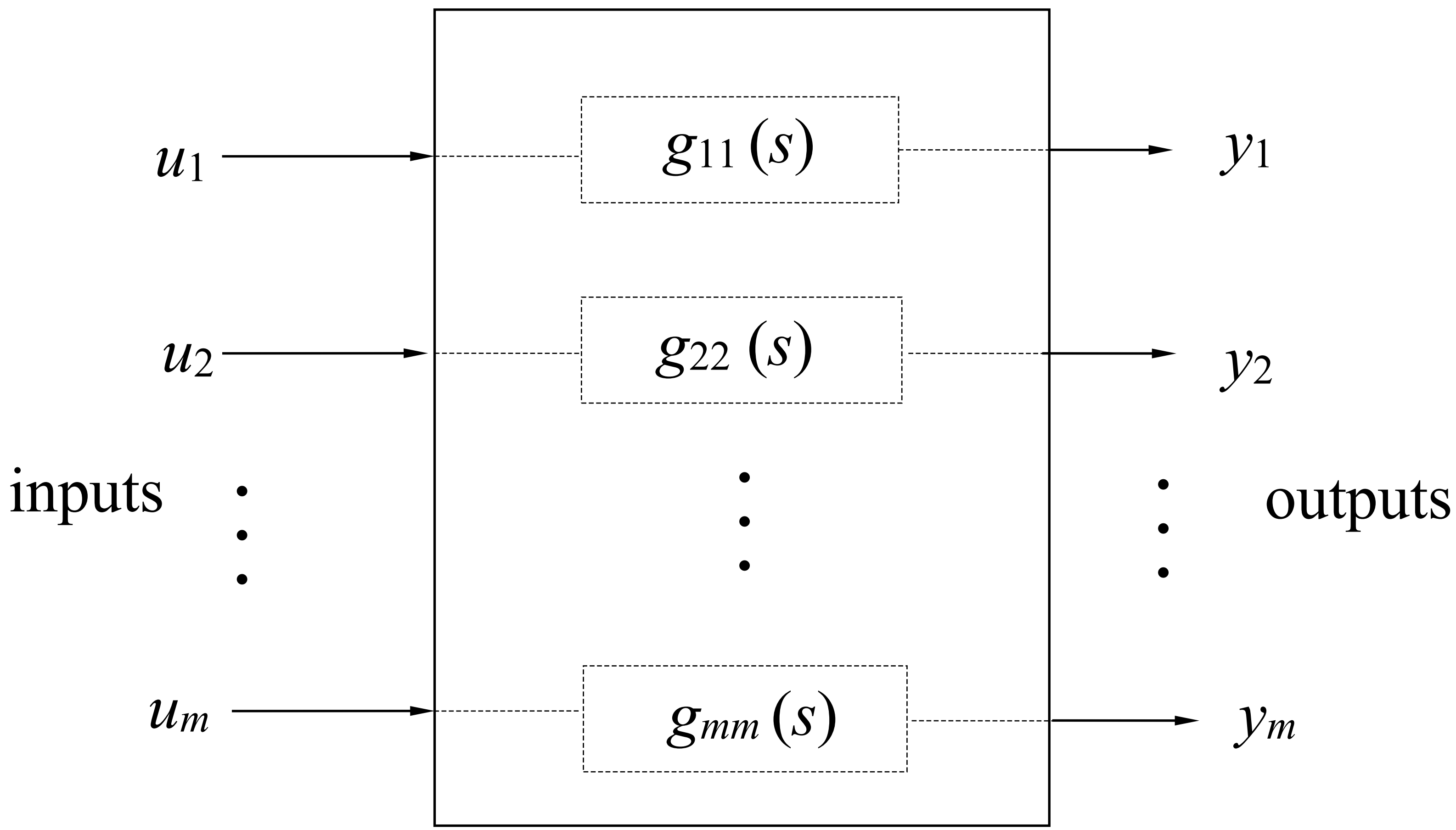

Figure 2 Decoupled system.

An Industrial Motivation: distillation column control

Process:

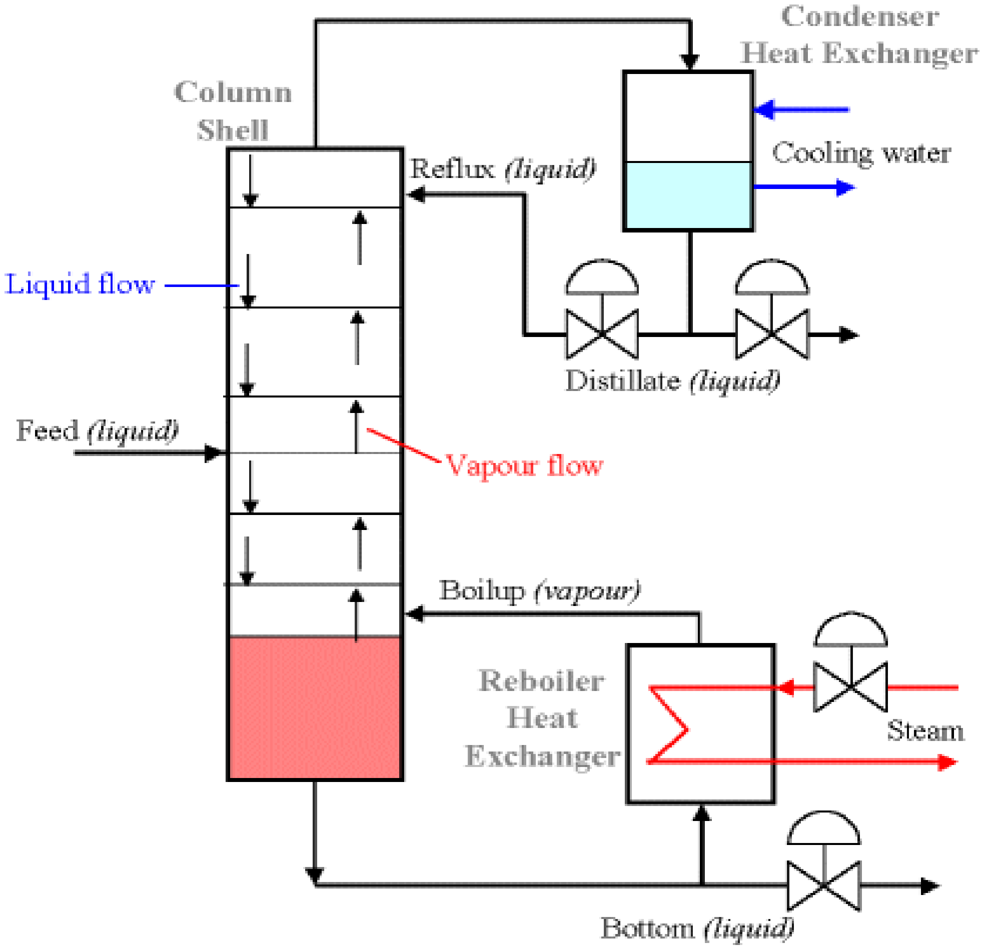

Figure 3 Distillation column.

Model: Wood and Berry (1973) transfer function model of a methanol-water distillation column is given by

$$\begin{bmatrix} X_D(s) \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{12.8}{16.7s+1}e^{-s} & \dfrac{-18.9}{21s+1}e^{-3s} \\ \dfrac{6.6}{10.9s+1}e^{-7s} & \dfrac{-19.4}{14.4s+1}e^{-3s} \end{bmatrix} \begin{bmatrix} R(s) \\ F(s) \end{bmatrix},$$

where $x_D(t)$ is the mole fraction of methanol in the top; $x_B(t)$ is the mole fraction of methanol in the bottom; $r(t)$ is the reflux flow rate; $f(t)$ is the steam flow rate.

Control requirements: decoupling, zero steady state error to step inputs and good dynamic performance.

What is decoupling?

- Decoupling control is to decouple a coupled plant such that feedback system is decoupled.

Why is decoupling?

- Decoupling is usually required in practice for easy operations.

How to decouple? *Two Schemes:*

1) State Feedback
2) Output Feedback

§9.2 Decoupling by State Feedback

Consider the system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$
$$y = Cx, \tag{5}$$

where $u, y \in R^m$, $x \in R^n$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, to be decoupled by the state feedback

$$u = -Kx + Fr. \tag{6}$$

The resultant system is

$$\dot{x} = (A - BK)x + BFr, \tag{7}$$

$$y = Cx, \tag{8}$$

and the transfer function matrix of the feedback system is

$$H(s) = C(sI - A + BK)^{-1} BF. \tag{9}$$

Let

$$G(s) = C(sI - A)^{-1} B. \tag{10}$$

Proposition 1 The transfer function matrix $H(s)$ of (9) is related to $G(s)$ of (10) by

$$\begin{aligned} H(s) &= G(s)[I - K(sI - A + BK)^{-1} B]F \\ &= G(s)\left[I + K(sI - A)^{-1} B\right]^{-1} F. \end{aligned} \tag{11}$$

Proof:

$$\begin{aligned} H(s) &= C(sI - A + BK)^{-1} BF = C[I](sI - A + BK)^{-1} BF \\ &= C\left[(sI - A)^{-1}(sI - A)\right](sI - A + BK)^{-1} BF \\ &= C(sI - A)^{-1}\left[(sI - A + BK) - BK\right][(sI - A + BK)^{-1}]BF \\ &= C(sI - A)^{-1}\left[I - BK(sI - A + BK)^{-1}\right][B]F \\ &= G(s)\left[I - K(sI - A + BK)^{-1} B\right]F. \end{aligned}$$

One yet needs to show

$$I - K(sI - A + BK)^{-1} B = \left[I + K(sI - A)^{-1} B\right]^{-1}. \qquad (12)$$

Or equivalently

$$\left[I - K\left(sI - A + BK\right)^{-1} B\right]\left[I + K\left(sI - A\right)^{-1} B\right] = I$$

$$LHS = I - K\left(sI - A + BK\right)^{-1} \{([sI - A + BK] - [BK])\left(sI - A\right)^{-1}\}B$$

$$+K(sI - A)^{-1} B - K(sI - A + BK)^{-1} BK(sI - A)^{-1} B$$

$$= I = RHS,$$

implies (12).

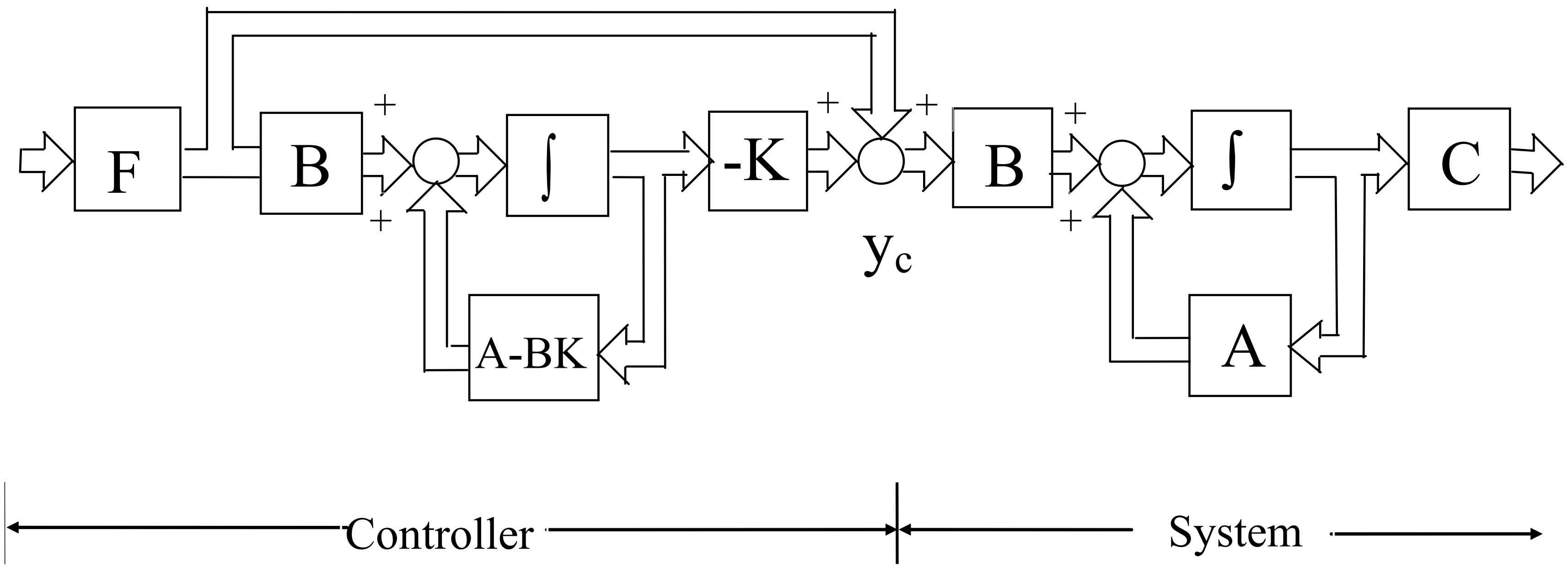

Figure 4 Series compensation.

Revision Notes

(i) Zero matrix / column / row

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = 0 \Leftrightarrow M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ all elements are zeros.}$$

If $M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $M \neq 0$.

(ii) Matrix to power j

$$A^j = \underbrace{AA\cdots A}_{j}$$

Say

$A^3 = AAA$

$A^0 = I$ by convention

A^{-1} is the inverse of A if A is non-singular.

(iii) Min

- $\min(2,4,1,5,12) = 1$

  ⌫ taking the minimum member from the set in the bracket

- $\min(s \mid (s-1)(s-2)(s-3) = 0) = \min(1,2,3) = 1$

  ⌫ the condition which s should meet to be in the set of s over which minimization is taken.

- $\min(s \mid (s-1)(s-2)(s-3) = 0, s \geq 2)$

  $= \min(s \mid s = 1,2,3, s \geq 2)$

  $= \min(2,3) = 2$

$$j_{\min} = \min(j \mid M^j = 0, j = 1,2,\cdots,), M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\because M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0 \ ,$$

so that $j = 1$ is not an eligible member.

$M^2 = M \cdot M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$, so that $j = 2$ is in the set.

$M^3 = M^2 \cdot M = 0 \cdot M = 0$, $M^j = 0,\ j \geq 2$, so that $j \geq 2$ are in the set.

$$\therefore j_{\min} = \min(2,3,\cdots) = 2.$$

End of Revision Notes

Partition

$$C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_m^T \end{bmatrix}.$$

Define $\sigma_i, i = 1,2,\cdots,m$, *as an integer by*

$$\sigma_i = \begin{cases} \min\left(j \middle| c_i^T A^{j-1} B \neq 0^T, \ j = 1,\ 2,\ \cdots,\ n \right); \\ \\ n, \quad if \quad c_i^T A^{j-1} B = 0^T, \ j = 1,\ 2,\ \cdots,\ n. \end{cases} \tag{13}$$

Calculation of σ_i

Let $i = 1$:

start with $j = 1,\ c_1^T A^{j-1} B = c_1^T A^0 B = c_1^T B \begin{cases} \neq 0 \rightarrow \sigma_1 = 1;\ \text{go to } i = 2. \\ = 0, \rightarrow j = 2 \end{cases}$

$$j = 2,\ \ c_1^T A^{j-1} B = c_1^T AB \begin{cases} \neq 0 \rightarrow \sigma_1 = 2,\ go \text{ to } i = 2 \\ = 0 \rightarrow j = 3, \ldots.. \end{cases}$$

Let $i = 2$:

$\vdots$

Let $i = m$:

Example 1 Let

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} x.$$

One proceeds as follows:

$$i = 1, \quad c_1^T = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$$

$$c_1^T B = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \neq 0 \rightarrow \sigma_1 = 1$$

$$i = 2, \quad c_2^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$c_2^T B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \neq 0 \rightarrow \sigma_2 = 1.$$

Example 2 Let

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x.$$

One proceeds as follows:

$$i = 1, \quad c_1^T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

$$c_1^T B = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix};$$

$$c_1^T AB = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \neq \mathbf{0} \to \sigma_1 = 2;$$

$i = 2,\quad c_2^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

$$c_2^T B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \to \sigma_2 = 1.$$

Theorem 1 *There exists a control law of the form (6) to decouple the system (5) if and only if the matrix*

$$B^* = \begin{pmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1} B \end{pmatrix} \tag{14}$$

is non-singular. If this is the case, let

$$C^* = \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \\ \vdots \\ c_m^T A^{\sigma_m} \end{bmatrix}, \ F = B^{*-1}, \ \textit{and} \ K = B^{*-1} C^* .$$

Then, the closed-loop system transfer function matrix is given by

$$H(s) = diag\left(s^{-\sigma_1},\ s^{-\sigma_2},\ \cdots,\ s^{-\sigma_m}\right),$$

which is called an integrator decoupled system.

Proof: Necessity. Write

$$G(s) = C(sI - A)^{-1} B$$

$$= \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_m^T \end{bmatrix} (sI - A)^{-1} B = \begin{bmatrix} c_1^T (sI - A)^{-1} B \\ c_2^T (sI - A)^{-1} B \\ \vdots \\ c_m^T (sI - A)^{-1} B \end{bmatrix} = \begin{bmatrix} g_1^{\;T}(s) \\ g_2^{\;T}(s) \\ \vdots \\ g_m^{\;T}(s) \end{bmatrix}.$$

Recall that

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1.$$

$$(y-x)^{-1} = y^{-1}(1-\frac{x}{y})^{-1} = y^{-1}\left\{1 + \frac{x}{y} + (\frac{x}{y})^2 + \cdots\right\}, \qquad \left|\frac{x}{y}\right| < 1.$$

Similarly for a matrix A, one has

$$(sI - A)^{-1} = s^{-1}\left[I - \frac{A}{s}\right]^{-1} = s^{-1}\left[I + As^{-1} + A^2 s^{-2} + \cdots\right]$$

$$= s^{-1} I + As^{-2} + A^2 s^{-3} + \cdots, \qquad \left\|\frac{A}{s}\right\| < 1.$$

The *i*th row of the transfer function $G(s)$ can be expanded in polynomials of s^{-i} as

$$g_i(s)^T = c_i^T (sI - A)^{-1} B = c_i^T B s^{-1} + c_i^T A B s^{-2} + \cdots$$

Note from definition of σ_i that

$$c_i^T B = c_i^T AB = \cdots = c_i^T A^{\sigma_i - 2} B = 0,\ c_i^T A^{\sigma_i - 1} B \neq 0.$$

One sees

$$\begin{aligned} g_i^T(s) &= c_i^T (sI - A)^{-1} B = c_i^T B s^{-1} + c_i^T A B s^{-2} + c_i^T A^2 B s^{-2} + \cdots \\ &= c_i^T A^{\sigma_i - 1} B s^{-\sigma_i} + c_i^T A^{\sigma_i} B s^{-(\sigma_i + 1)} + \cdots \\ &= s^{-\sigma_i} \left[c_i^T A^{\sigma_i - 1} B + c_i^T A^{\sigma_i} B s^{-1} + \cdots \right] \\ &\qquad c_i^T A^{\sigma_i} B s^{-1} + \cdots = c_i^T A^{\sigma_i} \left(s^{-1} I + A s^{-2} + A^2 s^{-3} + \cdots \right) B \\ &\qquad = c_i^T A^{\sigma_i} (sI - A)^{-1} B \end{aligned} \tag{15}$$

Using B^* of (14), $G(s)$ can be written as

$$G(s)=\begin{pmatrix} s^{-\sigma_1} & 0 & \cdots & 0 \\ 0 & s^{-\sigma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-\sigma_m} \end{pmatrix}\left[B^* + C^*\left(sI-A\right)^{-1}B\right], \tag{16}$$

where C^* is

$$C^* = \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \\ \vdots \\ c_m^T A^{\sigma_m} \end{bmatrix}, \qquad B^* = \begin{bmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1} B \end{bmatrix}. \tag{17}$$

Thus, it follows from Proposition 1 that

$$
\begin{aligned}
H(s) &= G(s)\left[I - K(sI - A + BK)^{-1}B\right]F \\
&= diag\left(s^{-\sigma_1}, s^{-\sigma_2}, \ldots, s^{-\sigma_m}\right)\left[B^* + C^*(sI - A)^{-1}B\right] \\
&\quad \times\left[I - K(sI - A + BK)^{-1}B\right]F \,.
\end{aligned}
\tag{18}
$$

For decoupling, $H(s)$ is diagonal and non-singular, so is

$$
\begin{bmatrix}
s^{\sigma_1} & 0 & \cdots & 0 \\
0 & s^{\sigma_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s^{\sigma_m}
\end{bmatrix} H(s)
$$

$$
\begin{aligned}
&= \left[B^* + C^*(sI - A)^{-1}B\right]\left[I - K(sI - A + BK)^{-1}B\right]F \\
&= B^*F + (?)s^{-1} + (?)s^{-2} + \cdots.
\end{aligned}
$$

B^*F must be diagonal and non-singular, and thus B^* is non-singular.

The sufficiency: Let B^* be non-singular. It follows from Proposition 1 that

$$H(s) = G(s)\left[I + K(sI - A)^{-1} B\right]^{-1} F,$$

$$H(s) = diag\left(s^{-\sigma_1}, s^{-\sigma_2}, \ldots, s^{-\sigma_m}\right)\left[B^* + C^*(sI - A)^{-1} B\right]$$
$$\times\left[F^{-1} + F^{-1}K(sI - A)^{-1} B\right]^{-1}. \tag{19}$$

Since B^* is non-singular, one can set K and F as

$$F = B^{*-1}, \tag{20}$$
$$K = B^{*-1}C^*. \tag{21}$$

Then, (19) becomes

$$H(s) = diag\left(s^{-\sigma_1}, s^{-\sigma_2}, \cdots, s^{-\sigma_m}\right)\left[B^* + C^*(sI - A)^{-1} B\right]$$
$$\times\left[B^* + C^*(sI - A)^{-1} B\right]^{-1}$$
$$= diag\left(s^{-\sigma}, s^{-\sigma_2}, \cdots, s^{-\sigma_m}\right), \tag{22}$$

which is decoupled and called an integrator decoupled system.

Example 1 (continued) The system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u,$$
$$y = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} x,$$

is required to be integrator-decoupled by state feedback. One sees that

$$G(s) = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$= \begin{pmatrix} \dfrac{s+1.5}{(s+1)^2} & \dfrac{0.5s+1}{(s+1)^2} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{pmatrix},$$

is coupled. We have calculated before:

$$\sigma_1 = 1,\ c_1^T B = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$$
$$\sigma_2 = 1,\ c_2^T B = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

One sees that

$$B^* = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix}$$

is non-singular and the system can be decoupled. One proceeds:

$$c_1^T A^{\sigma_1} = c_1^T A = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \end{bmatrix},$$

$$c_2^T A^{\sigma_2} = c_2^T A = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

so that

$$C^* = \begin{pmatrix} -0.5 & 0 \\ -1 & -1 \end{pmatrix}.$$

From (19) and (20), the control law has

$$F = B^{*-1} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix},$$

$$K = B^{*-1} C^* = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}.$$

The resultant feedback system becomes

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} r,$$

$$y = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} x,$$

with the transfer function:

$$H(s) = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{pmatrix}.$$

which is indeed integrator decoupled as expected from Theorem 1.

Example 2 (continued) The system,

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x,$$

needs to be decoupled. We have known before that

$$\sigma_1 = 2,\ c_1^T AB = \begin{bmatrix} 1 & 1 \end{bmatrix};$$
$$\sigma_2 = 1,\ c_2^T B = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

One calculates

$$c_1^T A^{\sigma_1} = c_1^T A^2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix},$$

$$c_2^T A^{\sigma_2} = c_2^T A = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Therefore, we have

$$B^* = \begin{bmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C^* = \begin{bmatrix} c_1 A^{\sigma_1} \\ c_2 A^{\sigma_2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$F = \left(B^*\right)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad K = \left(B^*\right)^{-1} C^* = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Decoupling with pole placement? One wishes to have the closed-loop transfer function matrix as

$$H(s)=\begin{pmatrix} \dfrac{1}{\phi_{f_1}} & 0 & \cdots & 0 \\ 0 & \dfrac{1}{\phi_{f_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{\phi_{f_m}} \end{pmatrix}$$

$$=\begin{pmatrix} \dfrac{1}{s^{\sigma_1}+\gamma_{11}s^{\sigma_1-1}+\cdots+\gamma_{1\sigma_1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{1}{s^{\sigma_m}+\gamma_{m1}s^{\sigma_m-1}+\cdots+\gamma_{m\sigma_m}} \end{pmatrix} \qquad (23)$$

instead of *so-called integrator decoupled system* in (22). Choose the desired poles as λ_i, $i = 1,2,\cdots,\sum_{i=1}^{m}\sigma_i$. Spread them into m sets λ_{ij}, $j = 1,2,\cdots\sigma_i$, $i = 1,2,\cdots m$. Form

$$\phi_{fi} = \prod_{j=1}^{\sigma_i}(s-\lambda_{ij}) = s^{\sigma_i} + \gamma_{i1}s^{\sigma_i - 1} + \cdots + \gamma_{i\sigma_i}.$$

Theorem 2 *When the system (5) can be decoupled by state feedback with*

$$F = \left(B^*\right)^{-1}, \quad K = \left(B^*\right)^{-1}\begin{bmatrix} c_1^T\phi_{f1}(A) \\ c_2^T\phi_{f2}(A) \\ \vdots \\ c_m^T\phi_{fm}(A) \end{bmatrix}, \tag{24}$$

where

$$\phi_{f_i}(A) = A^{\sigma_i} + \gamma_{i1}A^{\sigma_{i-1}} + \cdots + \gamma_{i\sigma_i}I,$$

then the resultant feedback system has the transfer function given by (23).

Proof: Note (15) for the *ith* row vector of the transfer function matrix:

$$g_i^T(s) = s^{-\sigma_i}\left[c_i^T A^{\sigma_i -1}B + c_i^T A^{\sigma_i} Bs^{-1} + \cdots\right]$$

Multiplying it by $\left(s^{\sigma_i} + \cdots + \gamma_{i\sigma_i}\right)$ yields

$$(s^{\sigma_i} + \gamma_{i1}s^{\sigma_i -1} + \cdots + \gamma_{i\sigma_1})g_i^T(s)$$

$$= (1 + \gamma_{i1}s^{-1} + \gamma_{i2}s^{-2} + \cdots + \gamma_{i\sigma_i}s^{-\sigma_i})$$

$$(c_i^T A^{\sigma_i -1}B + c_i^T A^{\sigma_i}Bs^{-1} + c_i^T A^{\sigma_i +1}Bs^{-2} + \cdots)$$

$$= c_i^T A^{\sigma_i -1}B + (c_i^T A^{\sigma_i}B + \gamma_{i1}c_i^T A^{\sigma_i -1}B + \gamma_{i2}c_i^T A^{\sigma_i -2}B + \cdots)s^{-1}$$

$$+ (c_i^T A^{\sigma_i +1}B + \gamma_{i1}c_i^T A^{\sigma_i}B + \gamma_{i2}c_i^T A^{\sigma_i -1}B + \cdots)s^{-2}$$

$$+ \cdots$$

$$= c_i^T A^{\sigma_i -1}B + (c_i^T A^{\sigma_i} + \gamma_{i1}c_i^T A^{\sigma_i -1} + \gamma_{i2}c_i^T A^{\sigma_i -2} + \cdots)[s^{-1}](B)$$

$$+ (c_i^T A^{\sigma_i} + \gamma_{i1}c_i^T A^{\sigma_i -1} + \gamma_{i2}c_i^T A^{\sigma_i -2} + \cdots)[As^{-2}](B)$$

$$+ \cdots$$

$$= c_i^T A^{\sigma_i -1} B + c_i^{**T} [s^{-1}] B + c_i^{**T} [A s^{-2}] B + \cdots$$
$$= c_i^T A^{\sigma_i -1} B + c_i^{**T} (I s^{-1} + A s^{-2} + A^2 s^{-3} + \cdots) B$$
$$= c_i^T A^{\sigma_i -1} B + c_i^{**T} (sI - A)^{-1} B \tag{25}$$

where

$$c_i^{**T} = c_i^T A^{\sigma_i} + \gamma_{i1} c_i^T A^{\sigma_i -1} + \cdots + \gamma_{i\sigma_i} c_i^T \tag{26}$$
$$= c_i^T [A^{\sigma_i} + \gamma_{i1} A^{\sigma_{i-1}} + \cdots + \gamma_{i\sigma_i} I] = c_i^T \phi_{f_i}(A).$$

Dividing both sides of (25) by $\left(s^{\sigma_i} + \gamma_{i1} s^{\sigma_i -1} + \cdots + \gamma_{i\sigma_i}\right)$ gives

$$G(s) = diag[\left(s^{\sigma_1} + \gamma_{11} s^{\sigma_1 -1} + \cdots + \gamma_{1\sigma_1}\right)^{-1}, \left(s^{\sigma_2} + \gamma_{21} s^{\sigma_2 -1} + \cdots + \gamma_{2\sigma_2}\right)^{-1},$$
$$\cdots, \left(s^{\sigma_m} + \gamma_{m1} s^{\sigma_m -1} + \cdots + \gamma_{m\sigma_m}\right)^{-1}] \times [B^* + C^{**}(sI - A)^{-1} B], \tag{27}$$

where

$$C^{**} = \begin{pmatrix} c_1^{**T} \\ c_2^{**T} \\ \vdots \\ c_m^{**T} \end{pmatrix} = \begin{bmatrix} c_1^{\,T}\phi_{f1}(A) \\ c_2^{\,T}\phi_{f2}(A) \\ \vdots \\ c_m^{\,T}\phi_{fm}(A) \end{bmatrix},$$

so that

$$H(s) = diag[\left(s^{\sigma_1} + \gamma_{11}s^{\sigma_1 - 1} + \cdots + \gamma_{1\sigma_1}\right)^{-1}, \ldots, \left(s^{\sigma_m} + \gamma_{m1}s^{\sigma_m - 1} + \cdots + \gamma_{m\sigma_m}\right)^{-1}]$$
$$\times [B^* + C^{**}(sI - A)^{-1}B]\left[F^{-1} + F^{-1}K(sI - A)^{-1}B\right]^{-1} \qquad (28)$$

Thus by choosing K and F as (24), the transfer function matrix of the closed loop system is given by (23).

In Example 1, the integrator decoupled system is obtained. Now, suppose that the same system is required to be decoupled to give

$$H(s) = diag\left[(s+1)^{-1}, (s+2)^{-1}\right].$$

One knows already $B^* = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix}$. Then, there remains $F = (B^*)^{-1} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$. Since $\phi_{f_1} = s+1,\ \phi_{f_2} = s+2,$ (24) becomes

$$K = (B^*)^{-1} \begin{pmatrix} c_1{}^T \phi_{f_1}(A) \\ c_2{}^T \phi_{f_2}(A) \end{pmatrix} = (B^*)^{-1} \left[\begin{pmatrix} c_1^T A \\ c_2^T A \end{pmatrix} + \begin{pmatrix} c_1^T \\ 2c_2^T \end{pmatrix} \right]$$

$$= \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \left[\begin{pmatrix} -0.5 & 0 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0.5 \\ 2 & 2 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

One can verify

$$A - BK = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}.$$

So

$$H(s) = diag\left[(s+1)^{-1}, (s+2)^{-1}\right].$$

§9.3 Decoupling Control by Output Feedback

In this section, we consider a unity output feedback system in Figure 5.

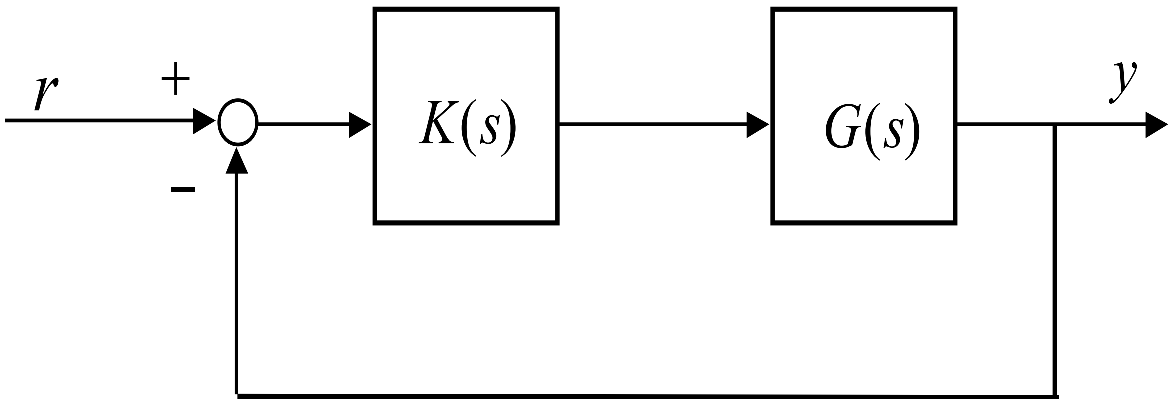

Figure 5 Unity output feedback system.

Our task here is to design $K(s)$ such that the feedback system is internally stable and the closed-loop transcfer function matrix,

$$H(s)=\left[I+G(s)K(s)\right]^{-1}G(s)K(s), \tag{29}$$

is decoupled, or diagonal and non-singular. Note

$$H^{-1} = (GK)^{-1}[I + GK]$$
$$= (GK)^{-1} + I\,, \tag{30}$$

and H is decoupled iff GK is so.

Issues with Unstable pole-zero cancellation and stabilizability

Example 3. Let $G(s) = \frac{1}{s-1}$ and $K(s) = \frac{s-1}{s+2}$. Then $\det(I + GK) = \frac{s+3}{s+2}$ has stable roots only, so that the Nyquist test cannot detect any instability. Now, we have

$$p_G = (s-1), \quad p_K = (s+2),$$

$$\det(I + GK) = 1 + \frac{1}{s+2} = \frac{s+3}{s+2}.$$

It can be shown that the closed-loop has the characteristic polynomial:

$$p_C(s) = p_G p_K \det(I + GK) = (s-1)(s+3),$$

which has a unstable pole at s=1 and is unstable. This is due to unstable pole-zero cancellation between G and K at $s=1$.

When the open loop system G(s) is unstable, it is easy to encounter the issue of unstable pole-zero cancellation. We need to be very careful in order to avoid this tricky issue.

When a $K(s)$ decouples $G(s)$ with no unstable pole-zero cancellation, the decoupled loop $G(s)K(s)$ can be stabilized loop by loop, so that the resultant system is decoupled and stable.

Theorem 3. *The plant* $G(s)$ *is decouplable and stablilizable by output feedback if and only if there is a controller* $K(s)$ *such that* $G(s)K(s)$ *is* *decoupled and there is no unstable pole-zero cancellation between G(s) and K(s).*

Design procedure. Theorem 3 tells that the decoupling problem with stability by output feedback can be solved by designing $K(s)$ in two stages, i.e., $K(s) = K_d(s)K_s(s)$. $K_d(s)$ is to make $G(s)K_d(s)$ diagonal and non-singular. Whenever such a $K_d(s)$ is found, it is always possible to design a diagonal controller $K_s(s)$ to stabilize the resultant $G(s)K_d(s)$ with SISO methods or pole placement technique loop by loop so that decoupling is not affected, and to make $K(s) = K_d(s)K_s(s)$ proper and make sure that there are unstable pole-zero cancellations between K(s) and G(s).

Design For stabilizer $K_s(s)$. A decoupled plant has m independent SISO loops. For each SISO loop, you can use SISO methods to design a SISO controller k_{si}. SISO design methods are well covered in undergraduate control courses. Alternatively, you may use pole placement or LQR control to stabilize each loop. Once all loops are done, set $K_s(s) = diag\{k_{si}\}$.

It is seen from the above that the key step in decoupling control is to find $K_d(s)$.

Design for decoupler $K_d(s)$: Suppose that $G(s)$ is non-singular. Write

$$G(s) = \frac{N(s)}{d(s)},$$

where $d(s)$ is a least common denominator of $G(s)$. If $\det(N)$ and $d(s)$ has no common unstable roots, then choose

$$K_d(s) = \mathrm{adj}(N(s))$$

so that

$$G(s)K_d(s) = \frac{N(s) \bullet adj(N(s))}{d(s)} = \frac{\det(N(s))}{d(s)} I_m \text{ is decoupled}$$

Since the system G(s) is stable, the least common denominator *d* (*s*) is of course stable. Therefore, there will be no unstable pole-zero cancellation in this step.

Example 4 Consider

$$G(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{-2}{(s+2)} \\ \frac{2}{(s+1)} & \frac{1}{s+1} \end{bmatrix}.$$

It follows that

$$G(s) = \frac{N(s)}{d(s)} = \frac{1}{(\mathrm{s}+2)(s+1)} \begin{bmatrix} s+1 & -2(s+1) \\ 2(s+2) & s+2 \end{bmatrix},$$

and

$$\det N(s) = 5(s+1)(s+2).$$

As $\det N(s)$ and $d(s)$ has no common unstable roots, we can choose

$$K_d = \text{adj } N(s) = \begin{bmatrix} s+2 & 2(s+1) \\ -2(s+2) & s+1 \end{bmatrix},$$

and

$$GK_d = \frac{\det N}{d(s)} I_2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

It is now easy to use a diagonal controller

$$K_s = \begin{bmatrix} k_{s1} & 0 \\ 0 & k_{s2} \end{bmatrix},$$

to stabilize the decoupled system and make $K(s) = K_d(s)K_s(s)$ proper using some SISO design method.

The overall controller is

$$K = K_d K_s = \begin{bmatrix} s+2 & 2(s+1) \\ -2(s+2) & s+1 \end{bmatrix} \begin{bmatrix} k_{s1} & 0 \\ 0 & k_{s2} \end{bmatrix}.$$

There are many possible solutions. One way is to design

$$K_s = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+1} \end{bmatrix}$$

Is $K(s)$ proper now? Yes.

$$K = K_d K_s = \begin{bmatrix} s+2 & 2(s+1) \\ -2(s+2) & s+1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{s+1} & 2 \\ \dfrac{-2(s+2)}{s+1} & 1 \end{bmatrix}$$

$$G(s)K(s)=\begin{bmatrix}\dfrac{1}{s+2} & \dfrac{-2}{s+2}\\ \dfrac{2}{s+1} & \dfrac{1}{s+1}\end{bmatrix}\begin{bmatrix}\dfrac{s+2}{s+1} & 2\\ \dfrac{-2(s+2)}{s+1} & 1\end{bmatrix}=\begin{bmatrix}\dfrac{5}{s+1} & 0\\ 0 & \dfrac{5}{s+1}\end{bmatrix}$$

Since it is decoupled system, the closed loop stability can be checked loop by loop.

For example, for the first open loop $H(s)=\dfrac{5}{s+1}$

The closed loop TF is

$$H_{cl}(s)=\frac{H(s)}{1+H(s)}=\frac{\dfrac{5}{s+1}}{1+\dfrac{5}{s+1}}=\frac{5}{s+6}$$

Stable!

Refinement. One way to reduce the decoupler order is to elaborate the above method as follows. Express $G(s)$ as

$$G(s) = diag\left\{\frac{1}{d_1}, \quad \frac{1}{d_1}, \quad \cdots, \quad \frac{1}{d_m}\right\} N_r(s),$$

where d_i is a least common denominator of i-th row of $G(s)$ so that $N_r(s)$ is a polynomial matrix. Choose

$$K_d = adj(N_r(s)).$$

Then, If $\det\left(N_r\right)$ and $d_i(s)$ have no common unstable roots for each i, there will be no unstable pole-zero cancellations in $G(s)K_d(s)$. And

$$G(s)K_d(s) = diag\left\{\frac{\det N_r(s)}{d_1}, \quad \frac{\det N_r(s)}{d_1}, \quad \cdots, \quad \frac{\det N_r(s)}{d_m}\right\}$$

is decoupled.

***************** Revision Notes********************

Least Common Denominator (LCD)

By definition, LCD, $d(s)$ of $\dfrac{\beta_i(s)}{\alpha_i(s)}$, $i = 1,2,\cdots,n$ is a least multipler of $\alpha_i(s)$, i.e. $\dfrac{d(s)}{\alpha_i(s)}$ is a polynomial and thus

$$d\begin{bmatrix} \dfrac{\beta_1(s)}{\alpha_1(s)} & \dfrac{\beta_2(s)}{\alpha_2(s)} & \cdots & \dfrac{\beta_n(s)}{\alpha_n(s)} \end{bmatrix}$$

is a polynomial row, where $\alpha_i(s)$ and $\beta_i(s)$ are polynomials.

- LCD of $\dfrac{1}{s+1}$ and $\dfrac{1}{s+2}$ is $(s+1)(s+2)$.
- LCD of $\dfrac{1}{s}$ and $\dfrac{1}{s^2}$ is (s^2)

For a $n \times n$ nonsingular matrix M, one has

$$M^{-1} = \frac{adj(M)}{\det(M)} \qquad \Rightarrow M \cdot adj(M) = \det(M) \cdot I$$

$$Adj(M) = \{c_{ij}\}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots \\ c_{12} & c_{22} & c_{32} & \\ \vdots & & & \end{bmatrix}, \qquad c_{ij}\text{: cofactor of } m_{ij}$$

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$adj(M) = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}$$

$$M^{-1} = \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}$$

************** **End of Revisions**************

Example 5. Let

$$G(s)=\begin{bmatrix}\frac{1}{s+2} & \frac{-1}{(s+1)(s+2)}\\ \frac{5}{s+3} & \frac{1}{s+3}\end{bmatrix}.$$

For this case, d(s)=(s+1)(s+2)(s+3), the non-refined method will give a decoupled system of order 3x2=6. With the refined method, one sees that

$$G(s)=\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}^{-1}N_r=\begin{bmatrix}(s+1)(s+2) & 0\\ 0 & s+3\end{bmatrix}^{-1}\begin{bmatrix}s+1 & -1\\ 5 & 1\end{bmatrix}$$

And $\det\left(N_r\right)=s+6$ and $d_i(s)$ have no common unstable roots for *i=1,2*. We can then take

$$K_d=adj(N_r)=\begin{bmatrix}1 & 1\\ -5 & s+1\end{bmatrix}.$$

It follows that

$$GK_d=diag\left\{\frac{s+6}{(s+1)(s+2)},\ \frac{s+6}{s+3}\right\},$$

which is of order 3. It is easy to verify that

$$K_s = \begin{bmatrix} \dfrac{1}{s+6} & 0 \\ 0 & \dfrac{1}{s+6} \end{bmatrix}$$

will stabilize GK_d and make the overall controller $K(s) = K_d(s)K_s(s)$ proper:

$$K = K_d K_s = \begin{bmatrix} \dfrac{1}{s+6} & \dfrac{1}{s+6} \\ \dfrac{-5}{s+6} & \dfrac{s+1}{s+6} \end{bmatrix}.$$

The resultant closed-loop system transfer function matrix is given by

$$\begin{aligned} H(s) &= \left[I + G(s)K(s)\right]^{-1} G(s)K(s) \\ &= diag\left\{\frac{1}{s^2+3s+3}, \frac{1}{s+4}\right\}, \end{aligned}$$

and it is stable and decoupled.

An Industrial Application: A distillation column control revisited

Model:

$$G(s)=\begin{bmatrix} \dfrac{12.8}{16.7s+1}e^{-s} & \dfrac{-18.9}{21s+1}e^{-3s} \\ \dfrac{6.6}{10.9s+1}e^{-7s} & \dfrac{-19.4}{14.4s+1}e^{-3s} \end{bmatrix}.$$

Control requirements: decoupling, zero steady state error to step inputs and good dynamic performance.

Solution: With the refined method, one sees that

$$G(s)=\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^{-1} N_r$$

$$=\begin{bmatrix} (16.7s+1)(21s+1)\,\mathrm{e}^{3s} & 0 \\ 0 & (10.9s+1)(14.4s+1)\,\mathrm{e}^{7s} \end{bmatrix}^{-1} \begin{bmatrix} 12.8(21s+1)\,\mathrm{e}^{2s} & -18.9(16.7s+1) \\ 6.6(14.4s+1) & -19.4(10.9s+1)\,\mathrm{e}^{4s} \end{bmatrix}$$

And

$$\det\left(N_r\right)=-248.32(21s+1)(10.9s+1)e^{6s}+124.74(16.7s+1)(14.4s+1)$$

Since $\det(N_r)$ and $d_i(s)$ have no common unstable roots for *i=1,2,* we can then take

$$K_d=adj(N_r)=\begin{bmatrix} -19.4(10.9s+1)e^{4s} & 18.9(16.7s+1) \\ -6.6(14s+1) & 12.8(21\mathrm{s}+1)e^{2s} \end{bmatrix}$$

The decoupled system is

$$G(s)\,\mathrm{K}_d(s) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^{-1} N_r \times adj(N_r) = \det(N_r)\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{-248.32(10.9s+1)\,\mathrm{e}^{3s}}{(16.7s+1)} + \dfrac{124.74(14.4s+1)\,\mathrm{e}^{-3s}}{(21s+1)} & 0 \\ 0 & \dfrac{-248.32(21s+1)\,\mathrm{e}^{-s}}{(14.4s+1)} + \dfrac{124.74(16.7s+1)\,\mathrm{e}^{-7s}}{(10.9s+1)} \end{bmatrix}$$

which is indeed decoupled but complicated. We can design a diagonal controller $K_s(s)$ loop by loop. $K_s(s)$ will stabilize $G\,K_d(s)$ and make the overall controller $K(s) = K_d(s)\,K_s(s)$ proper.

As the plant has time delays, the design of the stabilizer is quite involved, which is out of the scope of this module.