Linear System (ME5401/EE5101)

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Outline

Preliminaries

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References:

1 Linear System Theory and Design by C.T. Chen, 3rd Edition, Published by Oxford University Press.

2 Linear Systems by T. Kailath. Published by Prentice Hall 1980.

3 Modern Control Theory by W. Brogen, 3rd Edition, Prentice-Hall, 1991.

4 Fundamental of State Space Linear Systems, by John Bay, First Edition, McGraw Hill, 1999.

5 Modern Control Engineering by K. Ogata, 2nd Edition, Prentice Hall, 1990.

6 Control System Design by B. Friedland. McGraw Hill 1987.

Preliminaries

Outline of course:

1. Introduction to Linear Control System
2. Review of Linear Algebra
3. State space solutions and properties
4. Controllability and Observability
5. Canonical Forms and Realization
6. Stability
7. Pole Placement
8. Quadratic Optimal Control
9. Decoupling Control
10. State Estimation
11. Servo Control

Additional Requirement: Project (30%)

Introduction

Review of Classical Control Systems
Definition:

- A Control System is a system in which some physical quantities are controlled by regulating certain energy inputs.
- A system is a group of physical components assembled to perform a specific function. The components may be electrical, mechanical, thermal, biomedical, chemical, pneumatic or any combination of the above.
- A physical quantity may be temperature, pressure, electrical voltage, mechanical position, velocity, pH value, liquid level etc.
- Applications of Control System include
  Space vehicle, missile guidance, autopilot, robotic systems, Power system, assembly line, Chemical processes, Quality Control etc.

Introduction

Features of Classical and Modern Control Theories

Classical:

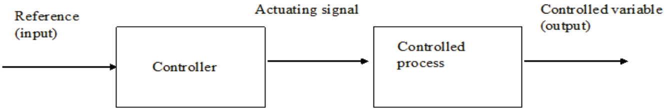
- Uses extensively the Laplace Operator in the description of a system.
- Based on Input/Output relationship called Transfer Function. All initial conditions are assumed zero in T.F. representation.
- First developed for single-input-single-output system, extension to multi-input-multi-output systems.
- Only for systems described by linear differential equations with constant coefficients.
- Uses heavily concepts like frequency response, root-locus, phase and gain margin.
- Formulation is not well-suited for Digital computer simulation.

Modern Control Theory:

- Uses time domain description known as state space representation.
- Naturally adapted for MIMO system and easily extended to linear time-varying, nonlinear and distributed systems.
- Incorporates initial conditions into its descriptions.
- Well suited for computer simulation.
- Provides a complete description of the system.

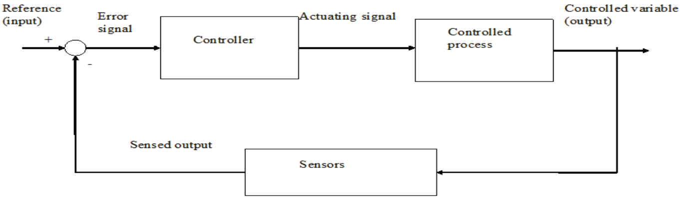
Review of Classical Control Systems

Open-loop control system



- Output has no effect on control action. Examples: Traffic light, microwave oven, dryers, toasters.

Closed-loop control system



- Actuating signal and hence output, depends on the input and the current state of the output. Examples: Servo Controlled motor, missile, robots etc.

Review of Classical Control Systems

Laplace Transformation

- Given a function $f(t)$ where $t \geq 0$.
- Let $s = \sigma + j\omega$ be a complex variable where σ, ω are both variables.
- The Laplace transform of $f(t)$ denoted by $F(s)$ is

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

- Existence; $F(s)$ exists if the integral converges.

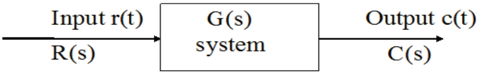
Properties of Laplace Transform

1. $\mathcal{L}[Af(t)] = A\mathcal{L}[f(t)]$ where A is a constant.
2. $\mathcal{L}[A_1 f_1(t) + A_2 f_2(t)] = A_1\mathcal{L}[f_1(t)] + A_2\mathcal{L}[f_2(t)]$

Review of Classical Control Systems

Transfer Function:

Consider the system given by



where

$$a_0 \frac{d^n}{dt^n} c(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} c(t) + \cdots + a_n c(t) = b_0 \frac{d^m}{dt^m} r(t) + b_1 \frac{d^{m-1}}{dt^{m-1}} r(t) + \cdots + b_m r(t)$$

with $n \geq m$.

- The transfer function of a linear, time-invariant, differential equation is defined as

$$G(s) = \frac{\mathcal{L}[output]}{\mathcal{L}[input]} = \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots a_n}$$

- In deriving a transfer function, all initial conditions are assumed zero.

Review of Classical Control Systems

Other Definitions:

- Proper Transfer Function: The transfer function $G(s)$ is called proper if $n \geq m$ $(i.e., G(s)|_{s=\infty} = \text{constant})$.
- Strictly proper transfer function: The transfer function $G(s)$ is called strictly proper if $n > m(i.e., G(s)|_{s=\infty} = 0)$.
- Order of a system: The highest power of "s'' in the denominator of $G(s)$ is called the order of the system.
- Poles of $G(s)$: The roots of the denominator polynomial of $G(s)$.
- Zeros of $G(s)$: The roots of the numerator polynomial of $G(s)$.
- When is $G(s)$ stable?

Example: Suppose

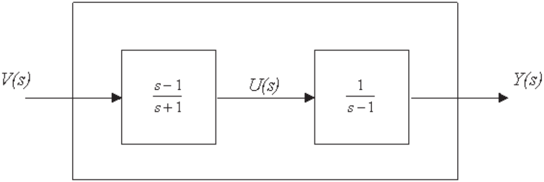
$$G(s) = \frac{s^2 + 2s + 2}{s(s^2 + 3s + 2)}$$

By definition, $n = 3, m = 2 \Rightarrow G(s)$ is strictly proper.

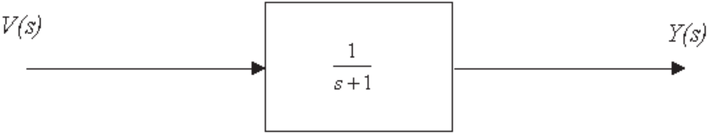
Order of system is 3, Poles of system are $s = 0, -2, -1$ and Zeros of system are $s = -1 \pm j1$.

Problem with Classical Control

Consider the following system



where the plant $H_f(s) = \frac{1}{s-1}$ is unstable and the compensator is chosen to be $H_c = \frac{s-1}{s+1}$. Overall transfer function is given by $\frac{Y(s)}{V(s)} = H_f(s)H_c(s) = \frac{1}{s+1}$.



Will this work in reality?

Problem with Classical Control

If we let $x_2(t) = y(t)$, then from Figure, we have

$$\dot{x}_2(t) = x_2(t) + u(t) \tag{1}$$

Similary, we have $\frac{U(s)}{V(s)} = \frac{s-1}{s+1}$ from the figure, or, $U(s) = V(s) - \frac{2}{s+1}V(s)$. If we let $x_1(s) = -\frac{2}{s+1}V(s)$, we have $u(s) = v(s) + x_1(s)$, then

$$\dot{x}_1(t) + x_1(t) = -2v(t). \tag{2}$$

If we include the initial conditions of (1) and (2), it can be shown that

$$x_1(t) = x_1(0)e^{-t} - 2\int_0^t e^{-\tau}v(t-\tau)d\tau$$

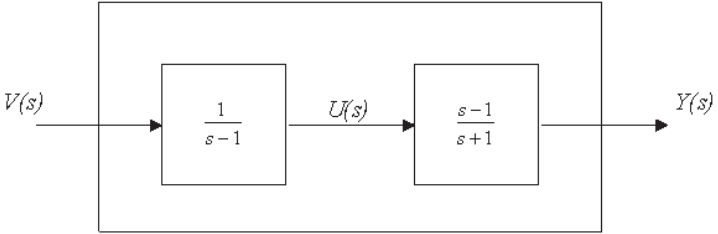
and

$$y(t) = x_2(t) = e^t x_2(0) + \frac{1}{2}(e^t - e^{-t})x_1(0) + \int_0^t e^{-\tau}v(t-\tau)d\tau$$

- Output will grow without bounds because of the e^t term (unless initial conds. can be guaranteed to be zero or $x_1(0) = -2x_2(0)$.

Problem with Classical Control

- Further insight into the problem can be obtained when the two blocks are interchanged. Consider the case



- Overall system $\frac{Y(s)}{V(s)} = \frac{1}{s+1}$ is the same BIBO stable system as before.

Let $x_1(s) = U(s)$ and $x_2(s) = -\frac{2}{s+1}x_1(s)$. Then, $y(t) = x_1(t) + x_2(t)$.

Problem with Classical Control

By incorporating the initial conditions, it can be shown that

$$x_1(t) = x_1(0)e^t + \int_0^t e^\tau v(t-\tau)d\tau$$

$$x_2(t) = x_2(0)e^{-t} + (e^{-t} - e^t)x_1(0) + \int_0^t e^{-\tau} v(t-\tau)d\tau - \int_0^t e^\tau v(t-\tau)d\tau$$

and

$$y(t) = (x_1(0) + x_2(0))e^{-t} + \int_0^t e^{-\tau} v(t-\tau)d\tau$$

- Now the system is stable as far as $y(t)$ goes, even if init cond. are non-zero.
- However, system is internally unstable because $x_1(t)$ and $x_2(t)$ have terms that grow as e^t; after awhile, the state will saturate or burn out.

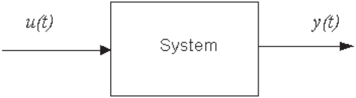
Conclusions: The internal behavior of a system is more complicated than indicated by the external behavior!

Mathematical Description of Systems

Class of systems:

The class of systems studied in this course has

- Some input terminals and some output terminals.
- If an excitation or input is applied, a unique response or output can be measured at the output terminals.



- A system with only one input terminal and one output terminal is termed a single-input-single-output (SISO) system.
- A system with two or more input terminals/or two or more output terminals is called a multi-variable system or multi-input-multi-output (MIMO) system.

Continuous time system:

- A system is called a continuous-time system if it accepts continuous-time signals as its input and generates continuous-time signals at its output.
- If the input has one terminal, $u(t)$ denotes a scalar quantity, otherwise, $u(t) = [u_1(t)\ u_2(t)\ \cdots\ u_p(t)]^T$ denotes a $p \times 1$ vector.

Mathematical Description of Systems

Causality:

- A system is called memoryless if its output $y(t_0)$ depends only on the input applied at time t_0; it is independent of the input applied before or after t_0.
- Most system have memory. By this, we meant that $y(t_0)$ depends on $u(t)$ for $t < t_0, t = t_0$ and $t > t_0$.
- A system is called causal or non-anticipatory system if its current output depends on past and current inputs but not on future input. If a system is non-causal then its current output will depend on future input. No physical system has such capability. (This corresponds to the $n < m$ condition in the transfer function representation.)
- Current output of a causal system is affected by its past inputs. How far back in time will the past input affect the current output?
- In general, the time should go back to $-\infty$. However, tracking $u(t), -\infty \leq t \leq t_0$ is not convenient.
- The concept of state deals with this problem.

Mathematical Description of Systems

States and associated definitions:

- The state $x(t_0)$ of a system is the information at t_0 that, together with input $u(t)$ for $t \geq t_0$ determines uniquely the output $y(t)$ for all $t \geq t_0$.
- State Variables: The state variables of a dynamical system are the smallest set of variables that determine the state of the system.
- State Vector: The n state variables can be considered as elements or components of the n-dimensional vector $x(t)$, called the state vector.
- State Space: State space is defined as the n-dimensional space in which the components of the state vector represent its' coordinate axes.
- State Trajectory: The path produced in the state space by the state vector as it changes with the passage of time.

Lumpedness:

- A system is said to be lumped if its number of state variables is finite or the state vector is finite.
- A system is said to be distributed if its state has infinitely many state variables.

Linear Systems

Based on the above, define the triplet

$$\{x(t_0), u(t) \text{ for } t \geq t_0, y(t) \text{ for } t \geq t_0\}$$

as the output $y(t)$ for $t \geq t_0$ using $u(t)$ for $t \geq t_0$ with state $x(t_0)$ at time t_0.

Definition: A system is called a linear system if for every t_0 and any two triplets $\{x_1(t_0), u_1(t) \text{ for } t \geq t_0, y_1(t) \text{ for } t \geq t_0\}, \{x_2(t_0), u_2(t) \text{ for } t \geq t_0, y_2(t) \text{ for } t \geq t_0\}$, we have for any real α the following two triplets:

$$\{x_1(t_0) + x_2(t_0), u_1(t) + u_2(t) \text{ for } t \geq t_0, y_1(t) + y_2(t) \text{ for } t \geq t_0\} \text{(additivity)}$$

$$\text{and } \{\alpha x(t_0), \alpha u(t) \text{ for } t \geq t_0, \alpha y(t) \text{ for } t \geq t_0\} \text{(homogeneity)}$$

The above two properties can be combined into one as

$$\{\alpha_1 x_1(t_0) + \alpha_2 x_2(t_0), \alpha_1 u_1(t) + \alpha_2 u_2(t) \text{ for } t \geq t_0, \alpha_1 y_1(t) + \alpha_2 y_2(t) \text{ for } t \geq t_0\}$$

for any real constants α_1 and α_2.

- The above property is known as superposition.
- A nonlinear system is one where the superposition property does not hold.

Linear Systems

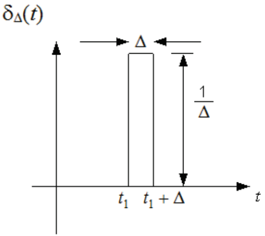
- If $u(t) = 0$ for all $t \geq t_0$, $y(t)$ is excited by $x(t_0)$ only and is known as zero-input response, $y_{zi}(t)$.
- The corresponding triplet is $\{x(t_0), 0, y_{zi}(t) \text{ for } t \geq t_0\}$.
- If $x(t_0) = 0$, $y(t)$ is excited by $u(t)$ for $t \geq t_0$ only and is known as zero-state response, $y_{zs}(t)$.
- The corresponding triplet is $\{0, u(t) \text{ for } t \geq t_0, y_{zs}(t) \text{ for } t \geq t_0\}$.
- Using the additivity property,

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

- The above shows that the response of every linear system can be decomposed into the sum of two responses.
- These two responses can be studied separately and their sum yields the complete response.

Input-Output Description

We now develop the mathematical equation that describes the zero-state response of a SISO system and assume that all its initial states are zero.
Let $\delta_\Delta(t - t_1)$ be the pulse function as shown below:



where the pulse is of width Δ and height $1/\Delta$ located at time t_1. Thus,

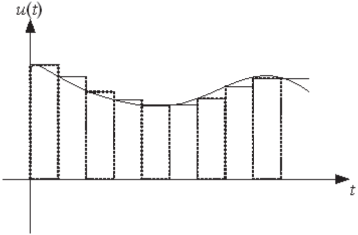
$$\delta_\Delta(t - t_1)\Delta = 1 \text{ for } t_1 \leq t < t_1 + \Delta$$
$$\delta_\Delta(t - t_1)\Delta = 0 \text{ otherwise.}$$

Then, any input function $u(t)$ can be approximated by

$$u(t) \approx \sum_i u(t_i)\delta_\Delta(t - t_i)\Delta \tag{3}$$

Input-Output Description

as shown below



Let $g_\Delta(t, t_i)$ be the output at time t excited by a pulse $u(t) = \delta_\Delta(t - t_i)$ applied at time t_i. Then, by linearity, we have

Input	Output
$\delta_\Delta(t - t_i)$	$g_\Delta(t, t_i)$
$\delta_\Delta(t - t_i)u(t_i)\Delta$	$g_\Delta(t, t_i)u(t_i)\Delta$ (homogeneity)
$\sum_i \delta_\Delta(t - t_i)u(t_i)\Delta$	$\sum_i g_\Delta(t, t_i)u(t_i)\Delta$ (additivity)

Thus, the output $y(t)$ excited by the input $u(t)$ can be approximated by

$$y(t) = \sum_i g_\Delta(t, t_i)u(t_i)\Delta \tag{4}$$

Input-Output Description

Let $\Delta \to 0$, and denote

$$\lim_{\Delta \to 0} \delta_\Delta(t - t_i) \text{ as } \delta(t - t_i)$$

$$\lim_{\Delta \to 0} g_\Delta(t, t_i) \text{ as } g(t, t_i)$$

- In (3), summation becomes integration and approximation becomes equality.
- Similarly, t_i becomes a continuum τ, Δ becomes $d\tau$ and (4) becomes

$$y(t) = \int_{-\infty}^{\infty} g(t, \tau) u(\tau) d\tau \tag{5}$$

- $g(t, \tau)$ refers to the output observed at time t due to impulse applied at time τ and hence, is known as the impulse response.
- By causality,

$$g(t, \tau) = 0 \text{ for all } t < \tau.$$

  This implies that the upper limit of the integral of (5) becomes t.

Input-Output Description

A system is said to be relaxed at t_0 if its initial states at t_0 is 0. Hence, $y(t), t \geq t_0$ is excited exclusively by $u(t), t \geq t_0$ for a relaxed system at t_0, or (5) becomes

$$y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau \tag{6}$$

- The condition of lumpedness is not used in the above. Hence, (6) is good for lumped or distributed system.

If the system has p inputs and q outputs, then (6) can be extended to

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

where

$$G(t,\tau) = \begin{pmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & \dots & g_{1p}(t,\tau) \\ g_{21}(t,\tau) & g_{22}(t,\tau) & \dots & g_{2p}(t,\tau) \\ \vdots & \vdots & & \vdots \\ g_{q1}(t,\tau) & g_{q2}(t,\tau) & \dots & g_{qp}(t,\tau) \end{pmatrix}$$

- $G(t,\tau)$ is known as the impulse response matrix of the system.

Input-Output Description

- Recall that for any function $f(t)$, $f(t+T)$ is its T-time shifted function.
- Recall also that $g(t, \tau + T)$ is the impulse response of system when impulse is applied at time $t + T$.
- Now, a system is time-invariant if it satisfies

$$g(t,\tau) = g(t+T, \tau+T) \text{ for any } T$$

- In particular, if $T = -\tau$, then

$$g(t+T, \tau+T) = g(t-\tau, 0) = g(t-\tau)$$

- Equation (6) becomes

$$y(t) = \int_{t_0}^{t} g(t-\tau)u(\tau)d\tau \tag{7}$$

- Note that $g(\cdot)$ now becomes a function of a single variable.
- (7) is used extensively in classical control analysis.
- Extension of (7) to MIMO case is direct resulting in impulse matrix $G(t-\tau)$.

State Space Representation

Every linear lumped system can be described by a set of equations of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{8}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{9}$$

where

$$x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^q$$

$$A(t) \in \mathbb{R}^{n\times n}, B(t) \in \mathbb{R}^{n\times p}, C(t) \in \mathbb{R}^{q\times n}, D(t) \in \mathbb{R}^{q\times p}$$

- Equation (8) is a set of N first-order differential equations with p inputs.
- Equation (9) consists of q algebraic equations relating $x(t)$ and $u(t)$ to the output $y(t)$.
- Collectively, (8) and (9) are known as the state space equations.
- When $A(t), B(t), C(t), D(t)$ are constants, the SS equations describe an LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{10}$$

$$y(t) = Cx(t) + Du(t) \tag{11}$$

Linearization

Most physical systems are nonlinear and time varying described by nonlinear differential equation of the general form

$$\dot{x} = f(x, u) = \begin{pmatrix} f_1(x, u) \\ \vdots \\ f_n(x, u) \end{pmatrix} \tag{12}$$

$$y = h(x, u) \tag{13}$$

where f and h are nonlinear functions. The behavior of system described by such equations can be very complicated.

- Some nonlinear systems can be approximated by linear equations under certain conditions.
- Suppose for some input function $u_0(t)$ and some initial states, $x_0(t)$ is the solution of (12). Then,
$$\dot{x}_0(t) = f(x_0(t), u_0(t))$$
- Consider a perturbed system arising from $u(t) = u_0(t) + \delta u(t)$ and the solution of (12) is also perturbed in the form of $x_0(t) + \delta x(t)$. Linearizing via Taylor's series expansion about $(x_0(t), u_0(t))$ yields

Linearization

$$
\begin{aligned}
\dot{x}_0 + \delta\dot{x} &= f(x_0(t) + \delta x(t), u_0(t) + \delta u(t)) \\
&= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}|_{x_0(t),u_0(t)}\delta x + \frac{\partial f}{\partial u}|_{x_0(t),u_0(t)}\delta u + \cdots
\end{aligned}
$$

- If we neglect higher-order terms, we have

$$
\delta\dot{x} = \frac{\partial f}{\partial x}|_{x_0(t),u_0(t)}\delta x + \frac{\partial f}{\partial u}|_{x_0(t),u_0(t)}\delta u
$$

  which is of the form $\delta\dot{x} = A(t)\delta x + B(t)\delta u$ with

$$
A(t) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}|_{x_0(t),u_0(t)}, \quad B(t) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \frac{\partial f_2}{\partial u_1} & \cdots & & \frac{\partial f_2}{\partial u_p} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & & \cdots & \frac{\partial f_n}{\partial u_p} \end{pmatrix}|_{x_0(t),u_0(t)}
$$

The nonlinear output equation

$$y_0(t) = h(x_0(t), u_0(t))$$

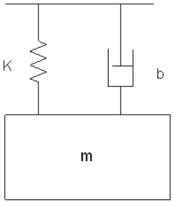
can be similarly linearized to give the approximate output

$$\delta y(t) = C(t)\delta x + D(t)\delta u$$

with $\delta y(t) = y(t) - y_0(t)$ and

$$C(t) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \cdots & & \frac{\partial h_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial h_q}{\partial x_1} & & \cdots & \frac{\partial h_q}{\partial x_n} \end{pmatrix} |_{x_0(t),u_0(t)}, D(t) = \begin{pmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} & \cdots & \frac{\partial h_1}{\partial u_p} \\ \frac{\partial h_2}{\partial u_1} & \cdots & & \frac{\partial h_2}{\partial u_p} \\ \vdots & & \ddots & \vdots \\ \frac{\partial h_q}{\partial u_1} & & \cdots & \frac{\partial h_q}{\partial u_p} \end{pmatrix} |_{x_0(t),u_0(t)}$$

Example 1: Mass-Spring-Damper System



From Newton's second law, we have

$$m\frac{d^2z(t)}{dt^2} = f(t) - b\frac{dz(t)}{dt} - kz(t)$$

Choose $x_1 = z$ and $x_2 = \dot{z}$. Then,

$$\begin{aligned}\dot{x}_1 &= \dot{z} = x_2 \\ \dot{x}_2 &= \ddot{z} = m^{-1}(f(t) - bx_2 - kx_1)\end{aligned}$$

or, in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} f$$

If we are interested in the position of mass, then

$$y(t) = [1 \quad 0] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If we are interested in the force acting on the spring, then

$$y(t) = kx_1 = [k \quad 0] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 \cdot f$$

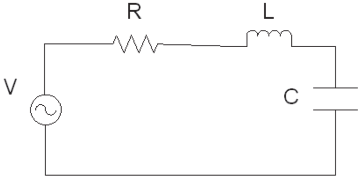
If we are interested in both the force acting on the dashpot and the force on the spring, then

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 0 & b \\ k & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 \cdot f$$

or if we want to know about the restoring forces acting on the mass, then

$$y(t) = [k \quad b] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 \cdot f$$

Example 2: RLC Electrical Circuit



Let $x_1(t)$ be the voltage across the capacitor, $x_2(t)$ be the current in the circuit. By Kirchoff's law, we have

$$v(t) = R \cdot x_2(t) + L \cdot \frac{dx_2(t)}{dt} + x_1(t)$$
$$\dot{x}_1 = \frac{1}{c} x_2(t)$$

Rearranging, we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{c} \\ -\frac{1}{L} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} v$$

Suppose we are interested in the voltage across the inductor, then we have

$$y(t) = L\frac{dx_2(t)}{dt} = L\dot{x}_2 = L(-\frac{1}{L}x_1 - \frac{R}{L}x_2) + L\frac{1}{L}v = -x_1 - Rx_2 + v$$

$$\text{or, } y(t) = [-1 \quad -R] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + v$$

If we denote the voltage across the resistor as x_3, then $x_3 = Rx_2$. The state equation can also be written in terms of x_1 and x_3 as

$$\begin{aligned}
\dot{x}_1(t) &= \frac{1}{c}x_2 = \frac{1}{cR}x_3(t) \\
\dot{x}_2(t) &= \frac{1}{R}\dot{x}_3(t) = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}v \\
\text{or, } \dot{x}_3(t) &= -\frac{R}{L}x_1 - \frac{R^2}{L}\cdot\frac{1}{R}x_3 + \frac{R}{L}v
\end{aligned}$$

which in matrix form is

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{cR} \\ -\frac{R}{L} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{R}{L} \end{pmatrix} v$$

Suppose we are still interested in the voltage across the inductor, then

$$\begin{aligned}
y(t) = L\frac{dx_2(t)}{dt} &= \frac{L}{R}\dot{x}_3 = \frac{L}{R}\left(-\frac{R}{L}x_1 - \frac{R}{L}x_3\right) + v \\
&= -x_1 - x_3 + v
\end{aligned}$$

$$y(t) = [-1 \quad -1] \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + v$$