Q.1

(a)
$$H(z) = \frac{Y(z)}{U(z)} = C(zI-\phi)^{-1}\cdot\Gamma = \frac{\alpha\cdot z+1}{z^2-\alpha z-1}$$

We can use Jury's stability criterion

$a_1 = -\alpha;\quad a_2 = -1$

$$\begin{cases} 1-\alpha_2^2>0 \\ \frac{1-\alpha_2}{1+\alpha_2}\left[(1+\alpha_2)^2-\alpha_1^2\right]>0 \end{cases} \Rightarrow \emptyset$$

No mater what α is, the open loop system is unstable.

(b) The desired poles are z^2, so $A_c(z) = z^2$

$$W_c = [\Gamma \quad \phi\Gamma] = \begin{bmatrix} 1 & 1 \\ 1 & 1+\alpha \end{bmatrix}$$

$$L = [0 \quad 1]\cdot W_c^{-1} A_c(\phi)$$

$$= [0 \quad 1]\begin{bmatrix} 1 & 1 \\ 1 & 1+\alpha \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix}^2$$

$$= [1-\frac{1}{\alpha} \quad \alpha-1+\frac{1}{\alpha}]$$

The controller can be expresged as:

$$u(k) = -Lx(k) = [\frac{1}{\alpha}-1 \quad 1-\frac{1}{\alpha}-\alpha]\cdot x(k)$$

(c) The observer can be expresed as:

$$\hat{x}(k+1) = \phi\hat{x}(k) + \Gamma u(k) + K[y(k)-\hat{y}(k)]$$

$$\hat{y}(k) = C\hat{x}(k)$$

$$W_o = \begin{bmatrix} C \\ C\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad f(z) = z^2$$

$$K = f(\phi)\cdot W_o^{-1}\cdot [0 \quad 1]^T$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix}^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ 1+\alpha^2 \end{bmatrix}$$

The controller can be expressed as:

$u(k) = -L\cdot\hat{x}(k)$, where L is from Q.1 (b)

Q.1

(d) Let $z(k)=\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$

$$z(k+1)=\phi_z z(k)+\Gamma_z u(k)$$
$$y(k)=C_z z(k)$$

$$\phi_z=\begin{bmatrix} \phi & \phi_{xw} \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix},\quad \Gamma_z=\begin{bmatrix} \Gamma \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\quad C_z=[C \quad 0]=[1 \quad 0 \quad 0].$$

The observer can be expressed as:

$$\hat{z}(k+1)=\phi_z \hat{z}(k)+\Gamma_z u(k)+\cancel{K(y(k)-\hat{y}(k))}\; K[y(k)-\hat{y}(k)]$$
$$\hat{y}(k)=C_z\hat{z}(k)$$

$$W_o=\begin{bmatrix} C_z \\ C_z\phi_z \\ C_z\phi_z^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The charateristic polynomial is $f(z)=z^3$.

$$K=f(\phi_z)W_o^{-1}[0 \quad 0 \quad 1]^T$$
$$=\begin{bmatrix} 0 & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}^3\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$=\begin{bmatrix} \alpha+1 \\ \alpha^2+\alpha+2 \\ 1 \end{bmatrix}.$$

Then, Because we want eliminate the disturbance.

$$u(k)=-[L \quad L_w]\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}=-Lx(k)-L_w w(k)$$

$$H_w(z)=\frac{Y(z)}{W(z)}=C[zI-(\phi-\Gamma L)]^{-1}(\phi_{xw}-\Gamma L_w).$$

When, $H_w(1)=0$, the disturbance can be eliminated.

$\Rightarrow L_w=$

$u(k)=-Lx(k)-L_w w(k)$

Q. 1

(e) ~~x(k)=~~ Let $\phi = \begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix}$, $\Gamma = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$x(k) = \phi x(k-2) + \Gamma u(k-4)$ ①

$x(k-2) = \phi x(k-4) + \Gamma u(k-6)$ ②

take ② into ①.

$x(k) = \phi[\phi x(k-4) + \Gamma u(k-6)] + \Gamma u(k-4)$

$= \phi^2 x(k-4) + \phi \cdot \Gamma u(k-6) + \Gamma u(k-4)$.

~~So:~~ $k = 1, 2, \cdots$

~~$x(k) = x(2k) = \phi^k x(0) + \phi^{k-1}\Gamma u(k-2) + \cdots + \phi \Gamma u(2k-4)$~~

So:

$x(2k) = \phi^k x(0) + \phi^{k-1} \cdot \Gamma u(-2) + \cdots + \Gamma u(2k-4)$

~~$x(2k+1) = \phi^k x(0) +$~~

$x(2k-1) = \phi^k \Gamma u(-3) + \phi^{k-1} \Gamma u(-1) + \cdots + \Gamma u(2k-5)$

Q. 2

(a) $q \cdot y(k) = q^{-1} y(k) + 2u(k) + q^{-1} \cdot u(k) + q \cdot v(k) + v(k)$

$(q^2 - 1) y(k) = (2q + 1) u(k) + (q^2 + q) \cdot v(k)$

Open loop T.F. from u to y.

$G(z) = \frac{Y(z)}{U(z)} = \frac{2z+1}{z^2-1}$

T.F from v to y:

$G_v(z) = \frac{Y(z)}{V(z)} = \frac{z^2+z}{z^2-1}$.

(b) $U(z) = \frac{T(z)}{R(z)} U_c(z) - \frac{S(z)}{R(z)} \cdot Y(z)$.

$A_m(z) = z^2$.

We want to reject ~~v(k)~~ disturbance,

so, $H_v(1) = 0$, $R(z)$ contain : $z-1$.

From Q.2 (a) we know

$A(z) = z^2 - 1$, $B(z) = 2z+1$, $B_v(z) = z^2 + z$.

Because $B(z)$ is stable, Let $A_{cl}(z) = A_o(z) \cdot A_m(z) = z^3 B(z)$

$R(z) = (z-1)(r_0 + r_1 z)$, $S(z) = s_0 + s_1 z + s_2 z^2$

$A_{cl}(z) = A(z) R(z) + B(z) S(z)$

~~$z^3 B(z) = A(z) R(z)$~~

$z^3 \cdot B(z) = (z^2 - 1)(z-1)(r_0 + r_1 z) + B(z)(s_0 + s_1 z + s_2 z^2)$

$\Rightarrow \begin{cases} r_0 = 1 \\ r_1 = 2 \\ s_0 = -1 \\ s_1 = 1 \\ s_2 = 1 \end{cases} \Rightarrow S_z = z^2 + z - 1, \quad R(z) = (z-1)(2z+1)$

Close T.F:

$H_{cl}(z) = \frac{B_z T(z)}{A_{cl}(z)} = \frac{T(z)}{z^3}$. We want $y(k)$ follow $\frac{1}{z^3}$

so, $T(z) = z$.

Q.2

(c) Yes.

Q.3

(a) $K_f(2) = P(2|1) \cdot C^T [C \cdot P(2|1) \cdot C^T + R_2]^{-1}$

$= \begin{bmatrix} 0.5454 \\ 0.8376 \end{bmatrix}$

$K(2) = (A P(2|1) C^T)(C P(2|1) C^T + R_2)^{-1} = \begin{bmatrix} 0.4363 \\ 1.274 \end{bmatrix}.$

(b) ~~$\hat{x}(2|2) = A$~~

$\hat{x}(2|2) = \hat{x}(2|1) + K_f(2)(y(2) - C x(2|1))$

$= \begin{bmatrix} 0.8 \\ 1.8 \end{bmatrix} + \begin{bmatrix} 0.5454 \\ 0.8376 \end{bmatrix} \left(2 - [0 \ \ 1] \begin{bmatrix} 0.8 \\ 1.8 \end{bmatrix} \right)$

$= \begin{bmatrix} 0.909 \\ 1.9675 \end{bmatrix}$

because w(k) and v(k) are zero-mean.

y~~(2) = Cx(2)~~ , ~~x(2) =~~

$E(x(2) = E A x(1) = E A^2 x(0)]$

$E[x(2) - \hat{x}(2|2)] = E[A^2 x(0) - \hat{x}(2|2)]$

$E\{[x(2) - \hat{x}(2|2)][x(2) - \hat{x}(2|2)]^T\} = P(2|2) = \cancel{\frac{P(2|1) R_2}{C P(2|1) C^T + R_2}} =$

$= P(2|1) - P(2|1) C^T (C P(2|1) C^T + R_2)^{-1} C P(2|1)$

$= \begin{bmatrix} 1.3636 & 0.2727 \\ 0.2727 & 0.4188 \end{bmatrix}$

Q.3 (c)

$$\hat{x}(3|2) = A\hat{x}(2|1) + K(2)[y(2) - C\hat{x}(2|1)]$$

$$= \begin{bmatrix} 0.7272 \\ 2.6948 \end{bmatrix}$$

$$E\{[x(3) - \hat{x}(3|2)][x(3) - \hat{x}(3|2)]^T\} = P(3|2) \approx$$

$$= AP(2|1) - K(2)[CP(2|1)C^T + R_2]K^T(2) + R_1$$

$$= \begin{bmatrix} 2.2377 & -0.368 \\ 1.792 & -1.075 \end{bmatrix}$$

(d). $A_m(z) = [z-(0.32+j0.29)][z-(0.32-j0.29)] = z^2 - 0.64z + 0.1865.$

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.8 & 1 \end{bmatrix}.$$

$$K_{ob} = A_m(A) \cdot W_o^{-1} \cdot [0 \quad 1]^T \approx$$

$$= (A^2 + 0.64A + 0.1865I)(W_o^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.6731 \\ 2.44 \end{bmatrix}.$$

Q.4

(a) $\cancel{\frac{Y(z)}{U(z)} = z}$ Apply z-trans.

$$z X_p(z) = a_p X_p(z) + (1-a_p) U(z)$$

$$X_p(z) = \frac{1-a_p}{z-a} U(z)$$

$$Y(z) = X_p(z) = \frac{1-a_p}{z-a} U(z)$$

so, open loop T.F.

$$\frac{Y(z)}{U(z)} = \frac{1-a_p}{z-a}$$

(b) Let $x(k) = \begin{Bmatrix} \Delta X_p(k) \\ y(k) \end{Bmatrix}$.

$$\cancel{\Delta} X_p(k+1) - X_p(k) = a_p[X_p(k) - X_p(k-1)] + (1-a_p)\cdot[u(k) - u(k-1)]$$

$$\Delta X_p(k+1) = a_p \Delta X_p(k) + \overset{(1-a_p)}{\cancel{}} \Delta u(k)$$

$$y(k+1) - y(k) = X_p(k+1) - X_p(k) = \Delta X_p(k+1)$$

$$y(k+1) = y(k) + a_p \Delta X_p(k) + (1-a_p)\Delta u(k).$$

So: $A = \begin{bmatrix} a_p & 0 \\ a_p & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1-a_p \\ 1-a_p \end{bmatrix}$, $C = [0 \quad 1]$

$$x(k) = \begin{Bmatrix} \Delta X_p(x) \\ y(k) \end{Bmatrix} = \begin{Bmatrix} X_p(k) - X_p(k-1) \\ y(k) \end{Bmatrix}.$$

(c) $\bar{R}_s = \overbrace{[1 \quad 1 \dots 1]}^{N_p = n} \cdot r(k)$

$$\bar{F} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N_p} \end{bmatrix} = \begin{bmatrix} a_p & 1 \\ a_p^2 + a_p & 1 \\ \vdots & \vdots \\ a_p^n + a_p^{n-1} + \dots + a_p & 1 \end{bmatrix}$$

$$\bar{R} = r_w \cdot I_{N_C \times N_C} = 0.$$

$$\cancel{\phi = \begin{bmatrix} CB & 0 & \dots \\ CAB & CB & 0 \dots \\ \vdots \\ CA^{n-1}B & CA^{n-2}B & \dots CA \end{bmatrix}}$$

$$\phi = \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{n-1}B \end{bmatrix} = \begin{bmatrix} (1-a_p)(a_p+1) \\ (1-a_p)(a_p^2 + a_p + 1) \\ \vdots \\ (1-a_p)(a_p^n + a_p^{n-1} + \dots + a_p + 1) \end{bmatrix}$$

Q.4

(d) $\lim_{n\to\infty} \frac{1}{n} \phi^T \phi$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \left[(1-a_p)(a_p+1), \ \cdots \ (1-a_p)(a_p^n + a_p^{n-1} + \cdots + 1)\right] \cdot \begin{bmatrix} (1-a_p)(a_p+1) \\ \vdots \\ (1-a_p)(a_p^n + a_n^{n-1} + \cdots + 1) \end{bmatrix}$$

$$= \cancel{\lim_{n\to\infty} \frac{1}{n} (1-a_p)^2 (1-a_p)(}$$

(e) $\lim_{n\to\infty} \frac{1}{n} \phi^T \bar{R}_s$

(f) $\lim_{n\to\infty} \phi^T F$

(g) $K_r = [1 \ 0 \ \cdots \ ](\phi^T \phi + \bar{R})^{-1} \phi^T \bar{R}_s$

$K_{mpc} = [1 \ 0 \ \cdots \ ](\phi^T \phi + \bar{R})^{-1} \phi^T F$.