Q.1

(a). $|\lambda I - A| = \begin{vmatrix} \lambda+2 & 1 \\ -1 & \lambda+4 \end{vmatrix} = \lambda^2 + 6\lambda + 9 = 0$

The eigenvalue of A are -3 and -3

Let $h(\lambda) = \alpha_0(t) + \lambda \alpha_1(t)$

When $\lambda = -3$, we have

$$e^{-3t} = \alpha_0(t) - 3\alpha_1(t) \quad ①$$

Then, $\frac{df(\lambda)}{d\lambda}\Big|_{\lambda=-3} = \frac{dh(\lambda)}{d\lambda}\Big|_{\lambda=-3}$, we can get

$$t \cdot e^{-3t} = \alpha_1(t) \quad ②$$

Solving, we get $\alpha_0(t) = e^{-3t} + 3te^{-3t}$, and

$e^{At} = \alpha_0 I + \alpha_1 \cdot A = \begin{pmatrix} e^{-3t} + te^{-3t} & -te^{-3t} \\ te^{-3t} & e^{-3t} - te^{-3t} \end{pmatrix}$

(b). Assume that, $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$, We can get.

$$\begin{cases} X = P\bar{X} \\ \bar{A} = P^{-1}AP \\ \bar{B} = P^{-1}B \\ \bar{C} = CP \end{cases} \Rightarrow \cancel{4P_1 + 6P_2 = P_1 + 2P_3}$$

$$\Rightarrow \begin{cases} P\bar{A} = AP \\ \bar{C} = CP \end{cases} \Rightarrow \begin{cases} \begin{pmatrix} 4P_1 + 6P_2 & P_1 + P_2 \\ 4P_3 + 6P_4 & P_3 + P_4 \end{pmatrix} = \begin{pmatrix} P_1 + 2P_3 & P_2 + 2P_4 \\ 3P_1 + 4P_3 & 3P_2 + 4P_4 \end{pmatrix} \\ [1 \quad -0.5] = [P_1 \quad P_2] \end{cases}$$

$$\Rightarrow \begin{cases} P_1 = 1 \\ P_2 = -0.5 \\ P_3 = 0 \\ P_4 = 0.5 \end{cases} \Rightarrow P = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix}$$

Q.1 (c). We can use Lyapunov Equation

Let $P = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}$, $Q = I$

$$A^T P + PA = -Q$$

$$\begin{pmatrix} -2 & -5 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} P_1 = \frac{19}{6} \\ P_2 = -\frac{7}{6} \\ P_3 = \frac{1}{6} \end{cases} \Rightarrow P = \begin{pmatrix} \frac{19}{6} & -\frac{7}{6} \\ -\frac{7}{6} & \frac{1}{6} \end{pmatrix}$$

$$\det P = \begin{vmatrix} \frac{19}{6} & -\frac{7}{6} \\ -\frac{7}{6} & \frac{1}{6} \end{vmatrix} = -\frac{5}{6} < 0$$

So, the system is not asymptotically stable.

Let's calcuate the eigenvalue of A.

$$|\lambda I - A| = \begin{vmatrix} \lambda+2 & -1 \\ 5 & \lambda-4 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0 \text{ , } \lambda_1 = 3,\ \lambda_2 = -1$$

Let $h(\lambda) = \alpha_0(t) + \lambda\alpha_1(t)$

$$\begin{cases} e^{-t} = \alpha_0 - \alpha_1 \\ e^{3t} = \alpha_0 + 3\alpha_1 \end{cases} \Rightarrow \begin{cases} \alpha_0 = \frac{3e^{-t} + e^{3t}}{4} \\ \alpha_1 = \frac{e^{3t} - e^{-t}}{4} \end{cases}$$

$$e^{At} = \alpha_0(t) I + \alpha_1(t) \cdot A = \begin{pmatrix} \frac{5e^{-t} - e^{3t}}{4} & \frac{e^{3t} - e^{-t}}{4} \\ \frac{5}{4}(e^{-t} - e^{3t}) & \frac{5e^{3t} - e^{-t}}{4} \end{pmatrix}$$

$$X(t) = e^{At} \cdot X_0 = \begin{pmatrix} \frac{1}{4}(5e^{-t} - e^{3t}) & \frac{1}{4}(e^{3t} - e^{-t}) \\ \frac{5}{4}(e^{-t} - e^{3t}) & \frac{1}{4}(5e^{3t} - e^{-t}) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

Q. 2 (i) The eigen value of $A = \begin{bmatrix} -1 & \alpha_1 & 0 \\ 0 & -1 & \alpha_2 \\ 0 & 0 & -1 \end{bmatrix}$ is $\lambda = -1$

The matrix $[A-\lambda I, B]$ can be expressed as:

$\begin{bmatrix} 0 & \alpha_1 & 0 & 1 \\ 0 & 0 & \alpha_2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. No matter what α_1, α_2 is, this matrix won't be full row rank.

So, the system won't controllable

(ii) The T.F is

$y(s) = C \cdot (sI-A)^{-1} B = [1 \quad 1 \quad 0] \begin{bmatrix} s+1 & \alpha_1 & 0 \\ 0 & s+1 & \alpha_2 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \frac{(s+1)+\alpha_1}{(s+1)^2} +$

Q2 (b). Let $P = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}$, $Q = I$

$A^T P + PA = -Q$.

$\begin{bmatrix} -1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow \begin{cases} -2P_1 - 10P_2 = -1 \\ 2P_2 + 2P_2 = -1 \\ P_1 - 5P_3 = 0 \end{cases}$, There is no solution, so, the systen is not stable.

(c).

Q.3

(a). First, combine controllability matrix

$$W_c = \{B \quad AB \quad A^2B\} = \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} = \{b_1 \quad b_2 \quad Ab_1 \quad Ab_2 \quad A^2b_1 \quad A^2b_2\}.$$

Let's choose first 3 independently vector b_1, b_2, Ab_1.

$$C = \{b_1, Ab_1, b_2\} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and } d_1 = 2, \; d_2 = 1$$

$$C^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{we should choose second an third rows}$$

$$T = \begin{bmatrix} q_2^T \\ q_2^T A \\ q_3^T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \bar{B} = TB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $\bar{K} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} \end{bmatrix}$, from the closed loo matrix:

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 \\ -\bar{k}_{11} & 1-\bar{k}_{12} & -\bar{k}_{13} \\ 1-\bar{k}_{21} & -\bar{k}_{22} & -\bar{k}_{23} \end{bmatrix}$$

The desired eigenvalues are -1.

$$\det(sI - A_d) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$$

$$A_d = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}, \quad \text{Compare } \bar{A} - \bar{B}\bar{K} \text{ and } A_d.$$

$$\bar{k}_{11} = 0, \quad \bar{k}_{12} = \bar{k}_{13} = 1, \quad \bar{k}_{21} = 2, \quad \bar{k}_{22} = \bar{k}_{23} = 3$$

$$\bar{K} = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 3 \end{bmatrix}, \quad K = \bar{K} \cdot T = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix}$$

The controller is

$$u = -KX = -\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix} \cdot X.$$

Q.3 (b). $\dot{x} = x + u$, let $u = -kx$.

$\dot{x} = (1-k)\cdot x$, $x(t) = x(0)\cdot e^{-(k-1)t} = c\cdot e^{-(k-1)t}$

The function will change to.

$$J = \int_0^{\infty} (x^2 + x\cdot(-kx) + (-kx)^2 \, dt$$

$$= (k^2-k+1) \int_0^{\infty} x^2 \, dt.$$

$$= (k^2-k+1)\, c^2 \cdot \int_0^{\infty} e^{-2(k-1)t} \, dt$$

$$= (k^2-k+1)\cdot c^2 \cdot \frac{1}{2(k-1)}$$

$\frac{dJ}{dk} = 0$, we can get.

$$(2k-1)\cdot 2(k-1) - 2(k^2-k+1) = 0$$

$$k(k-2) = 0.$$

为 $k_1 = 0$, $k_2 = 2$.

To assure the system is stable, we need $k > 1$.

So, $k = 2$.

Q.4 (a)

First, we need to get the T.F.

$$G(s) = C\cdot(sI - A)^{-1}\cdot B = C\cdot \frac{adj(sI-A)}{det(sI-A)}\cdot B$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s(s-1)} & \frac{1}{s^2(s-1)} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & \frac{s+1}{s(s-1)} \\ 0 & 0 \end{bmatrix}.$$

$$d(s) = s(s-1), \quad N(s) = \begin{bmatrix} s & s+1 \\ 0 & 0 \end{bmatrix}.$$

~~N(s)=~~

$$K_d = adj\,N(s) = \begin{bmatrix} 0 & -(s+1) \\ 0 & s \end{bmatrix}.$$

Q.4 (b)

(i): $\zeta = 0.5$, $W_n = 1$. $P_c' = s^2 + 2\zeta W_n s + W_n^2 = s^2 + s + 1$.

(ii): $r = c + a\sin(t)$ $\quad R(s) = \frac{c}{s} + \frac{a}{s^2+1}$

(iii) $d = c'$ $\quad D(s) = \frac{c'}{s}$

~~So, Q(s) = s(s^2+1)~~. Because $Y(s) = \frac{1}{2s+1} U(s) + \frac{1}{s} D(s)$, $Q(s) = s^2 \cdot (s^2+1)$

$G(s) = \frac{1}{2s+1}$, $N(s) = 1$, $D(s) = 2s+1$

Let $A(s) = Q(s) \cdot \cancel{\quad} = s^2(s^2+1)$

$B(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4$.

$P_c = P_c' \cdot (s+1)^2 (2s+1)$

We can get

$D(s) A(s) + N(s) B(s) = P_c$.

$(2s+1)s^2(s^2+1) + (b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4) = (s^2+s+1)(s+1)^2(2s+1)$

$\Rightarrow 2s^5 + (1+b_4)s^4 + (2+b_3)s^3 + (1+b_2)s^2 + b_1 s + b_0 = 2s^5 + 7s^4 + 11s^3 + 10s^2 + 5s + 1$

$\Rightarrow \begin{cases} b_0 = 1 \\ b_1 = 5 \\ b_2 = 9 \\ b_3 = 9 \\ b_4 = 8 \end{cases}$, So: $A(s) = s^2(s^2+1)$, $B(s) = 8s^4 + 9s^3 + 9s^2 + 5s + 1$

$K(s) = \frac{B(s)}{A(s)} = \frac{8s^4 + 9s^3 + 9s^2 + 5s + 1}{s^2(s^2+1)}$.

Because P_s cotain the two desire poles in (s^2+s+1),

So, (i) is met

and ~~A(s)~~ A(s) cotein Q(s) wihitch can ~~st~~ cotain unstable poles from r(t) and $\frac{1}{s}D(s)$, so (ii) and (iii) are met.