EE5137 Stochastic Processes: Problem Set 9
Assigned: 25/03/22, Due: 01/04/22

There are six (6) non-optional problems in this problem set.

1. Exercise 5.11 (Gallager's book)

   Solutions:

   (a) From (5.17) in the book,

   $$\lim_{t\to\infty}\int_0^t Y(\tau)\,\mathrm{d}\tau = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} \qquad \text{w.p. } 1.$$

   For $f_X(x) = e^{-x}$, $\mathbb{E}[X] = 1$ and $\mathbb{E}[X^2] = 2$. Thus the time-average residual life is 1. This is something we already know, since if the interarrival time is exponential, the renewal process is Poisson.

   For the density $f_X(x) = 3/(x+1)^4\mathbb{1}\{x \ge 0\}$, $\mathbb{E}[X] = 1/2$ and $\mathbb{E}[X^2] = 1$, so the time-average residual life is again 1.

   (b) We regard $Y^2(t)$ as simply another reward function. Given an inter-renewal interval x_n, the reward at each residual life r from 0 to x_n is r^2, so the accumulated reward over that interval is $r_n = \int_0^{x_n} r^2\,\mathrm{d}r = x_n^3/3$. Thus the expected reward over an interval is $\mathbb{E}[R_n] = \mathbb{E}[X^3] = 3$. Using Theorem 5.4.5, then, the time-average second moment is

   $$\lim_{t\to\infty}\int_0^t Y(\tau)^2\,\mathrm{d}\tau = \frac{\mathbb{E}[X^3]}{3\mathbb{E}[X]} \qquad \text{w.p. } 1 \quad \text{if } \mathbb{E}[X^3] < \infty.$$

   For $f_X(x) = e^{-x}$, $\mathbb{E}[X^3] = 6$ so the time-average second moment is 2.

   For the density $f_X(x) = 3/(x+1)^4\mathbb{1}\{x \ge 0\}$, $\mathbb{E}[X^3] = \int_0^\infty 3x^3(1+x)^{-4}\,\mathrm{d}x = \infty$ so the time-average second moment is infinite. This is somewhat shocking, but the distribution of X has tails that go to 0 rather slowly, and the enormous accumulated reward over large sample values of X_n cause this infinite reward.

   The exponential density yields a Poisson process and the residual life at any time is exponential, independent of the previous arrival. Thus $\mathbb{E}[Y(t)] = 1$ for all t and $\mathbb{E}[Y^2(t)] = 2$ for all t.

2. Exercise 5.15 (Gallager's book)

   Solutions:

   (a) Given that $J = m$ for any given $m \ge 1$, we see that $\mathbb{1}\{J \ge n\}$ is 1 for each $n \le m$ and 0 for $n > m$. Thus $\sum_{n\ge 1}\mathbb{1}\{J \ge n\} = m$ given $J = m$. Since this is true for all sample values m of J, $J = \sum_{n\ge 1}\mathbb{1}\{J \ge n\}$.

   (b) Given $J = m$, $\mathbb{1}\{J \ge n\}$ is 1 for $n \le m$ and 0 for $n > m$. Thus $\mathbb{1}\{J \ge n\} \ge \mathbb{1}\{J \ge n+1\}$ with strict inequality when $n = m$. The statement $\mathbb{1}\{J \ge n\} \ge \mathbb{1}\{J \ge n+1\}$ is true for each event $\{J = m\}$, however and thus true in general. The strange exception of a set of probability 0 is because random variables can be undefined on a set of 0 probability, which is something we don't usually concern ourselves with and need not concern ourselves with here.

3. Exercise 5.16 (Gallager's book)

   Solutions:

   (a) In a Bernoulli process $\{X_n\}$, we call trial n a success if $X_n = 1$. Define a stopping trial J as the first trial n at which $\sum_{m=1}^{n} X_m = k$. This constitutes a stopping rule since $\mathbb{1}\{J = n\}$ is a function of $X_1, \ldots, X_n$. Given $J = n$, $S_J = X_1 + \ldots + X_n = k$, and since this is true for all n, $S_j = k$ unconditionally. Thus $\mathbb{E}[S_J] = k$. From Wald's equality, $E[S_J] = \bar{X}\mathbb{E}[J]$ so $\mathbb{E}[J] = k/\bar{X}$.

   We should have shown that $\mathbb{E}[J] < \infty$ to justify the use of Wald's equality, but we will show that in (b).

   (b) Let $\Pr(X_n = 1) = p$. Then the first success comes at trial 1 with probability p, and at trial n with probability $(1-p)^{n-1}p$. The expected time to the first success is then $1/p = 1/\bar{X}$. The expected time to the k-th success is then $k/\bar{X}$, which agrees with the result in (a).

   The reader might question the value of Wald's equality in this exercise, since the demonstration that $\mathbb{E}[J] < \infty$ was most easily accomplished by solving the entire problem by elementary means. In typical applications, however, the condition that $\mathbb{E}[J] < \infty$ is essentially trivial.

4. Exercise 5.17 (Gallager's book)

   Solutions:

   (a) We show below that the weak law is sufficient. The event $\{J > n\}$ is the event that $S_i > -d$ for all $1 \le i \le n$, i.e., $\{J > n\} = \cap_{1 \le i \le n}\{S_i > -d\}$. Thus, $\Pr(J > n) \le \Pr(S_n > -d)$. Since $\mathbb{E}[S_n] = n\bar{X}$ and $\bar{X} < 0$, we see that the event $\{S_n > -d\}$ for large n is an event in which S_n is very far above its mean. Putting this event in terms of distance from the sample average to the mean,

   $$\Pr(S_n > -d) = \Pr\left(\frac{S_n}{n} - \bar{X} > -\frac{d}{n} - \bar{X}\right)$$

   The WLLN says that $\lim_{n\to\infty} \Pr(|\frac{S_n}{n} - \bar{X}| > \epsilon) = 0$ for all $\epsilon > 0$, and this implies the same statement with the absolute value signs removed, i.e., $\lim_{n\to\infty} \Pr(\frac{S_n}{n} - \bar{X} > \epsilon) = 0$. If we choose $\epsilon = -\bar{X}/2$, in the equation above, it becomes

   $$\lim_{n\to\infty} \Pr\left(S_n > \frac{n\bar{X}}{2}\right) = 0.$$

   Since $\bar{X} < 0$, we see that $-d > n\bar{X}/2$ for $n > 2|d/\bar{X}|$ and thus $\lim_{n\to\infty} \Pr(S_n > -d) = 0$.

   (b) One stops playing on trial $J = n$ if one's capital reaches 0 for the first time on the n-th trial, i.e., if $S_n = -d$ for the first time at trial n. This is clearly a function of $\{X_1, \ldots, X_n\}$, so J is a stopping rule. Note that stopping occurs exactly on reaching $-d$ since S_n can decrease with n only in increments of -1 and S_n is always integer. Thus $S_J = -d$.

   Using Wald's equation, we then have

   $$\mathbb{E}[J] = -\frac{d}{\bar{X}}$$

   which is positive since $\bar{X}$ is negative. You should note from the exercises we have done with Wald's equality that it is often used to solve for $\mathbb{E}[J]$ after determining $\mathbb{E}[S_J]$.

5. Consider an i.i.d. sequence $\{X_n\}_{n\ge1}$ with a discrete distribution that is uniform over the integers $\{1, 2, \ldots, 10\}$, i.e., $\Pr(X = i) = 1/10$, for $1 \le i \le 10$. Imagine that these are bonuses that are given to you by your employer each year. Let $J = \min\{n \ge 1 : X_n = 6\}$, the first time that you receive a bonus of size 6. What is the expected total (cumulative) amount of bonus received up to time J?

   Solutions: We have

   $$\mathbb{E}\left[\sum_{n=1}^{J} S_n\right] = \mathbb{E}[J]\mathbb{E}[X] = 5.5\mathbb{E}[J]$$

if we can show that J is a stopping time with finite mean.

That J is a stopping time follows since it is a "first passage time": $\{J = 1\} = \{X_1 = 6\}$ and in general $\{J = n\} = \{X_1 \neq 6, \ldots, X_{n-1} \neq 6, X_n = 6\}$ only depends on $X_1, \ldots, X_n$.

We need to calculate $\mathbb{E}[J]$. Noting that $\Pr(J = 1) = \Pr(X_1 = 6) = 0.1$ and in general, from the i.i.d. assumption placed on $\{X_n\}$,

$$\Pr(J = n) = \Pr(X_1 \neq 6, \ldots, X_{n-1} \neq 6, X_n = 6) = 0.9^{n-1}0.1 \qquad n \geq 1,$$

we conclude that J has a geometric distribution with "success" probability $p = 0.1$, and hence $\mathbb{E}[J] = 1/p = 10$: And our final answer is $\mathbb{E}[J]\mathbb{E}[X] = 55$.

Note here that before time $J = n$ the random variables $\{X_1, \ldots, X_{n-1}\}$ no longer have the original uniform distribution; they are biased in that none of them takes on the value 6. So in fact they each have the conditional distribution $(X|X < 6)$ and thus an expected value different from 5.5. Moreover, the random variable at time $J = n$ has value 6; $X_n = 6$ and hence is not random at all. The point here is that even though all these random variables are biased, in the end, on average, Wald's equation let's us treat the sum as if they are not biased and are independent of J.

To see how interesting this is, note further that we would get the same answer 55 by using any of the stopping times $J = \min\{n \geq 1 : X_n = k\}$; nothing special about $k = 6$.

This should indicate to you why Wald's equation is so important and useful.

6. Consider a miner trapped in a room that contains three doors. Door 1 leads her to freedom after two-days' travel; door 2 returns her to her room after four-days' journey; and door 3 returns her to her room after eight-days' journey. Suppose at all times she is equally to choose any of the three doors, and let T denote the time it takes the miner to become free.

   (a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time J such that
   $$T = \sum_{i=1}^{J} X_i.$$
   Note: You may have to imagine that the miner continues to randomly choose doors even after she reaches safety.

   (b) Use Wald's equation to find $\mathbb{E}[T]$.

   (c) Compute $\mathbb{E}[\sum_{i=1}^{J} X_i | J = j]$ and note that it is not equal to $\mathbb{E}[\sum_{i=1}^{j} X_i]$.

   (d) Use part (c) for a second derivation of $\mathbb{E}[T]$.

   Solutions:

   (a) Define the random variable
   $$X = \begin{cases} 2 & \text{Door 1 w.p. } 1/3 \\ 4 & \text{Door 2 w.p. } 1/3 \\ 8 & \text{Door 3 w.p. } 1/3 \end{cases}$$
   and $\{X_i\}_{i \in \mathbb{N}}$ are i.i.d. copies of X. Let $J = \min\{n \geq 1 : X_n = 2\}$. Clearly J is a stopping time as the event $\{J = n\}$ is determined by the first n observations of $\{X_i\}$.

   (b) Using Wald's equation, $\mathbb{E}[T] = \mathbb{E}[J]\mathbb{E}[X]$. Further $\mathbb{E}[J] = 3$ since J follows a geometric distribution with parameter $p = 1/3$. Also $\mathbb{E}[X] = 14/3$. Thus $\mathbb{E}[T] = 14$.

   (c) We have
   $$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{J} X_i \,\middle|\, J = j\right] &= \mathbb{E}\left[\sum_{i=1}^{J} X_i \,\middle|\, X_1 \neq 2 \ldots, X_{j-1} \neq 2, X_j = 2\right] \\ &= 2 + (j-1)\mathbb{E}[X_i | X_i \neq 2] = 2 + (j-1)6 = 6j - 4. \end{aligned}$$

In contrast,

$$\mathbb{E}\left[\sum_{i=1}^{j} X_i\right] = j\mathbb{E}[X_i] = \frac{14j}{3}.$$

(d) By the law of iterated expectations

$$\mathbb{E}[T] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{J} X_i \,\middle|\, J\right]\right] = \mathbb{E}[6J-4] = 6\times 3 - 4 = 14.$$

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7. (Optional) Exercise 5.18 (Gallager's book)

Solutions:

We start with a brief discussion of how to find examples, since this is a major part of creative research, but is practically ignored in engineering, science, and mathematics curricula. One should start by trying to combine whatever insight one has about the problem with the simplest possible cases, where simplicity might involve very little choice (such as here with $J = 1$ or 2), or great symmetry, or very extreme cases. One then tries some simple examples which might not provide the desired result, and tries to understand what has gone wrong. One also, of course, uses any available hints. In this case, using the hint, we need only think about the four pairs of binary numbers and how to map each into J. Since we are looking for an example that is not a stopping rule, choosing $J = 1$ must sometimes depend on X_2. We also want our example to satisfy $\mathbb{E}[S_J] \neq \mathbb{E}[J]\mathbb{E}[X]$. We have no control over $\mathbb{E}[X]$, so we want to either make $\mathbb{E}[S_J]$ small or large for given $\mathbb{E}[J]$. If $\mathbb{E}[S_J]$ represents winnings and $\mathbb{E}[J]$ represents effort to get those winnings, we want to get large winnings with little effort. A good strategy is now almost self evident. When $X_2 = 1$, we choose $J = 2$ and when $X_2 = 0$, we choose $J = 1$. Thus, we get a win when $X_2 = 1$ and don't waste the effort when $X_2 = 0$. This requires clairvoyance, or peeking, etc., but this just means we are not using a stopping rule. With this rule $\mathbb{E}[S_J] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 1$. Also, $\mathbb{E}[J] = 3/2$ and $\mathbb{E}[J]\mathbb{E}[X] = 3/4$, so Wald's equality does not hold (which is not surprising since J is not a stopping rule).

8. (Optional) Players Jack and Jill will start with \$5 and \$10 respectively and play a game by making a series of \$1 bets until one of them loses all his/her money. We'll assume that in each bet, Jack wins with probability $p = 1/2$, Jill with probability $q = 1/2$, and tie (no money exchanged) with probability $r = 0$, so that

$$p + q + r = 1.$$

Let T be the number of bets made until the game ends. Calculate $\mathbb{E}[T]$ and the respective probabilities of Jack or Jill winning.

Solution: We let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables taking the values $1, -1, 0$ with probabilities p, q, rrespectively. Let Y_n be the amount of money that Jack has after n bets, so that

$$Y_n = 5 + \sum_{i=1}^{n} X_i.$$

Note that by the rules of the game, $0 \le Y_n \le 15$ for all n. Let J be the number of bets made until the game ends. Then J is a stopping time, and Y_T represents the amount of money that Jack has when the game ends. The random sum Y_T and take only two values, 0 (Jill wins) or 15 (Jack wins). We will use Wald's identities to calculate $\mathbb{E}[J]$ and the respective probabilities of Jack or Jill winning.

Suppose that $p = q = 1/2$, so that $\mathbb{E}[X] = 0$. By the first Wald's equation, it follows that

$$\mathbb{E}[Y_J] = 5.$$

Then

$$\mathbb{E}[Y_J] = 0 \cdot \Pr(Y_J = 0) + 15 \cdot \Pr(Y_J = 15) = 5$$

so that

$$\Pr(Y_J = 15) = 1/3.$$

This means that

$$\Pr(\text{Jack wins}) = 1/3 \qquad \Pr(\text{Jill wins}) = 2/3.$$

This also gives

$$\mathrm{Var}(Y_J) = \mathbb{E}[Y_J^2] - 5^2 = 0 \cdot \Pr(Y_J^2 = 0) + 15^2 \cdot \Pr(Y_J^2 = 15^2) - 25 = 15^2 \cdot 1/3 - 25 = 50.$$

By the "second Wald's identity" (which is that $\mathrm{Var}(Y_J) = \mathrm{Var}(X)\mathbb{E}[J]$), $\mathrm{Var}(Y_J) = \mathbb{E}[X^2]\mathbb{E}[J]$. Also,

$$X^2 = 1 \qquad \Longrightarrow \qquad \mathbb{E}[X^2] = 1$$

Hence $\mathbb{E}[J] = 50$.