EE5137 Stochastic Processes: Problem Set 4
Assigned: 04/02/22, Due: 11/02/22

There are eight (8) non-optional problems in this problem set. There are not many problems in Poisson processes as we have not covered enough, so I'm giving some practice problems on probability.

1. For a Poisson process, which of the following is/are true?
   (i) $\{N(t) \geq n\} = \{S_n \leq t\}$;
   (ii) $\{N(t) < n\} = \{S_n > t\}$;
   (iii) $\{N(t) \leq n\} = \{S_n \geq t\}$;
   (iv) $\{N(t) > n\} = \{S_n < t\}$.

   Solution: It is easy to see the only parts (i) and (ii) are true by drawing a timeline. More rigorously, first we show that (i) is true as follows. First, realize that if $S_n \leq t$ then we also have $S_1 \leq S_2 \leq \ldots S_n \leq t$. This means that there are at least n arrivals (at time $S_1, S_2, \ldots, S_n$) from time 0 to t, which leads to $N(t) \geq n$. Conversely, if $N(t) \geq n$, we have $S_n \leq t$. Combining these facts, we have $\{N(t) \geq n\} = \{S_n \leq t\}$. Since (i) is true, (ii) is true by taking complement.

   Now, it is also easy to see that a realization that $s_1 < s_2 < \ldots < s_{n-1} < s_n < t < s_{n+1}$ belongs to $\{N(t) \leq n\}$, but not belongs to $\{S_n \geq t\}$. It follows that (iii) is not true. Since (iii) is not true, (iv) is not true by taking complement.

2. An athletic facility has 5 tennis courts. Pairs of players arrive at the courts and use a court for an exponentially distributed time with mean 40 minutes. Suppose a pair of players arrives and finds all courts busy and k other pairs waiting in queue. What is the expected waiting time to get a court?

   Solution: As long as the pair of players is waiting, all five courts are occupied by other players. When all five courts are occupied, the time until a court is freed up is exponentially distributed with mean $40/5 = 8$. Four our pair of players to get a court, a court must be freed up $k + 1$ times. Thus, the expected waiting time is $8(k + 1)$.

3. (Optional) Exercise 2.3 (Gallager's book) The purpose of this exercise is to give an alternate derivation of the Poisson distribution for $N(t)$, the number of arrivals in Poisson process up to time t. Let λ be the rate of the process.

   (a) Find the conditional probability $\Pr\{N(t) = n | S_n = \tau\}$ for all $\tau \leq t$.
   (b) Using the Erlang density for S_n, use (a) to find $\Pr\{N(t) = n\}$.

   Solution:

   (a) The condition $S_n = \tau$ means that the epoch of the nth arrival is τ. Conditional on this, the event $\{N(t) = n\}$ for some $t > \tau$ means there have been no subsequent arrivals from τ to t. In other words, it means that the $(n + 1)$-st interarrival time, X_{n+1} exceeds $t - \tau$. This interarrival time is independent of S_n and thus,

   $$\Pr\{N(t) = n | S_n = \tau\} = \Pr\{X_{n+1} > t - \tau\} = e^{-\lambda(t-\tau)}, \quad \text{for} \quad t > \tau. \tag{1}$$

(b) We find $\Pr\{N(t)=n\}$ simply by averaging (1) over S_n.

$$\begin{aligned}\Pr\{N(t)=n\} &= \int_0^\infty \Pr\{N(t)=n|S_n=\tau\} f_{S_n}(\tau)d\tau &(2)\\ &= \int_0^t e^{-\lambda(t-\tau)} \frac{\lambda^n \tau^{n-1} e^{-\lambda\tau}}{(n-1)!} d\tau &(3)\\ &= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t \tau^{n-1} d\tau &(4)\\ &= \frac{(\lambda t)^n e^{-\lambda t}}{n!}. &(5)\end{aligned}$$

4. Prove that the Geometric distribution

$$p_X(k) = (1-p)^{k-1}p, \quad k \in \mathbb{N} = \{1, 2, \ldots\}$$

has the memoryless property.

In fact, the Geometric distribution is the only distribution supported on $\mathbb{N}$ that is memoryless. This is analogous to the fact that the Exponential distribution is the only distribution supported on $[0,\infty)$ that is memoryless.

Solution: To show that any distribution has the memoryless property, we need to show that

$$\Pr(X > x+t) = \Pr(X > x)\Pr(X > t). \tag{6}$$

For the Geometric distribution $p_X(k)$, we have

$$\Pr(X > x) = \sum_{k=x+1}^{\infty} (1-p)^{k-1}p = p \cdot \frac{(1-p)^x}{1-(1-p)} = (1-p)^x$$

Now clearly, the memoryless property in (6) holds true.

5. Let X_n denote a Binomial random variable with n trials and probability of success p_n. If $np_n \to \lambda$ as $n \to \infty$, show that for any fixed $i \in \mathbb{N} \cup \{0\}$,

$$\Pr(X_n = i) \to \frac{e^{-\lambda}\lambda^i}{i!}, \quad \text{as } n \to \infty.$$

Solution: Fix $i \in \{0, 1, \ldots, n\}$. We have

$$\begin{aligned}\Pr(X_n = i) &= \binom{n}{i} p_n^i (1-p_n)^{n-i}\\ &= \frac{n!}{i!(n-i)!} p_n^i (1-p_n)^{n-i}\\ &= \frac{n(n-1)(n-2)\ldots(n-i+1)}{i!} p_n^i (1-p_n)^{n-i}\\ &= \frac{(np_n)((n-1)p_n)((n-2)p_n)\ldots((n-i+1)p_n)}{i!}(1-p_n)^n(1-p_n)^{-i}\end{aligned}$$

We note that $np_n \to \lambda$ and so $(n-j)p_n \to \lambda$ for each $j = 0, 1, \ldots, i-1$. Furthermore, $(1-p_n)^n \to e^{-\lambda}$ because $np_n \to \lambda$. More precisely here, for fixed $\epsilon > 0$ and for n large enough, $|np_n - \lambda| < \epsilon$. Hence, $(1-p_n)^n \leq (1-(\lambda-\epsilon)/n)^n \to e^{-\lambda+\epsilon}$. Similarly, $(1-p_n)^n \geq e^{-\lambda-\epsilon}$. By the arbitrariness of $\epsilon > 0$, we have $(1-p_n)^n \to e^{-\lambda}$. Finally, since i is fixed $(1-p_n)^i \to 1$. Putting these facts together, we get

$$\Pr(X_n = i) \to \frac{\lambda^i e^{-\lambda}}{i!} \quad \text{as} \quad n \to \infty.$$

6. Let the sample space $\Omega = \{1, 2, \ldots, p\}$ for a *prime number* p, and A and B are subsets of Ω (events) and $\mathbb{P}(A) = |A|/p$ ($\mathbb{P}$ represents the uniform distribution on Ω). Prove that if A and B are independent, then either A or B is the empty set $\emptyset$ or the sample space Ω.

   Solution: . Let $|A| = a$ and $|B| = b$ and $c = |A \cap B|$. If A and B are independent, then

$$\frac{c}{p} = \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = \frac{ab}{p^2}.$$

   Hence $pc = ab$, so $p \mid ab$ (ab is divisible by p). Since p is prime, at least one of a and b is divisible by p (if both are not divisible by p, then ab will not be divisible by p). Since $0 \le a, b \le p$, at least one of a and b is equal to 0 or p, i.e., at least one of A and B is equal to $\emptyset$ or Ω.

7. If X is a random variable with the property that $\Pr(0 \le X \le a) = 1$, show that

$$\mathsf{Var}(X) \le a^2/4.$$

   Solution: Define the function $g(t) = \mathbb{E}[(X - t)^2]$. Then clearly, $g'(t^*) = 0$ implies that $t^* = \mathbb{E}X$. Furthermore, $g''(t^*) = 2 > 0$ so the function g achieves its unique minimum at $t^* = \mathbb{E}X$. This means that

$$\mathsf{Var}(X) = g(\mathbb{E}X) \le g\left(\frac{a}{2}\right) = \mathbb{E}\left[\left(X - \frac{a}{2}\right)^2\right] = \frac{1}{4}\mathbb{E}\big[(2X - a)^2\big]$$

   Since $X \in [0, a]$ almost surely, we know that $2X - a \in [-a, a]$ almost surely. This means that $\mathbb{E}\big[(2X - a)^2\big] \le a^2$. This proves the claim. It is also easy to see that the random variable that ensures that equality is met is X such that $X = 0$ and $X = a$ with probability 1/2.

8. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed exponential random variables with parameter λ. Let M_n denote $\max\{X_1, \ldots, X_n\}$. Show there exists a random variable Z such that $\{M_n - \frac{1}{\lambda}\log n\}_{n=1}^{\infty}$ converges in distribution to Z. This is the *Gumbel distribution*.

   Solution: The cumulative distribution function of M_n is evaluated at any $s \ge 0$

$$F_{M_n}(s) = \Pr(M_n \le s) = \Pr(\max\{X_1, \ldots, X_n\} \le s) = \prod_{i=1}^{n} \Pr(X_i \le s) = \prod_{i=1}^{n}(1 - e^{-\lambda s}) = (1 - e^{-\lambda s})^n$$

   Consequently, the cumulative distribution function of $K_n := M_n - \frac{1}{\lambda}\log n$ evaluated at t can be expressed as

$$F_{K_n}(t) = \Pr\left(M_n - \frac{1}{\lambda}\log n \le t\right) = \Pr\left(M_n \le t + \frac{1}{\lambda}\log n\right).$$

   Since $t + \frac{1}{\lambda}\log n \ge 0$ for n sufficiently large (t is fixed), we can take $s = t + \frac{1}{\lambda}\log n$ for all n sufficiently large and apply the formula for $F_{M_n}(s)$ to get

$$F_{K_n}(t) = \left(1 - \frac{1}{n}e^{-\lambda t}\right)^n.$$

   Now, we use the fact that $\lim_{n\to\infty}(1 + x/n)^n = e^x$ to get that

$$\lim_{n\to\infty} F_{K_n}(t) = e^{-e^{-\lambda t}} \qquad \forall\, t \in \mathbb{R}.$$

   Hence, the limiting distribution has CDF given by $e^{-e^{-\lambda t}}$ for all $t \in \mathbb{R}$.