EE5101/ME5401:
Linear Systems: Part II
Servo Control

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Chapter 10 Servo Control

What has been discussed? What is missing?

Do you know how to drive the state to zero (equilibrium point)?

Yes. Both pole placement and LQR can do it.

For pole placement, how to design the positions of the desired poles?

We can design a reference model to meet the time domain specifications such as settling time and overshoot etc.

Do we need to design the desired poles directly for LQR?

No. We need to design the weight matrices Q and R instead.

The weight matrices are designed to strike a balance between speed and cost.

Why is decoupling useful?

If the system is decoupled, we can use SISO design techniques for each channel separately without worrying about the coupling effects.

What has been discussed? What is missing?

If the goal is to keep the system at the equilibrium point (zero), then we have a pretty good idea on how to achieve this.

What if the objective is keeping the output at some nonzero constant or making the output track a time-varying reference signal?

Both pole placement and LQR focus upon how fast we want the system to behave, or in other words, the transient response.

Another important factor we need to consider is: accuracy.

Control Specifications
$$\begin{cases} Transient: \quad speed, \quad overshoot, \quad etc. \\ Accuracy: \quad e(t) = r(t) - y(t) \quad 0 \le t < \infty \\ Steady \quad State \quad Accuracy: \quad e(\infty) \end{cases}$$

How to achieve $e(\infty) = 0$

in face of disturbance w(t) and set-point change r(t)?

Asymptotic Tracking

We wish $y(t) = r(t),\ t \geq 0$, but this is impossible most of the time. What can be achieved is that for $r(t) \neq 0$ and $w(t) = 0$, there holds

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(r(t) - y(t)) = 0.$$

This is called the asymptotic tracking.

Asymptotic Regulation (Disturbance Rejection)

For $r(t) = 0$, $w(t) \neq 0$, if there holds

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(r(t) - y(t)) = \lim_{t\to\infty}(-y(t)) = 0,$$

it is called the asymptotic regulation.

If both asymptotic tracking and regulation are required, it is called the **servo control** problem.

How did we do in the classical SISO servo control with step reference inputs?

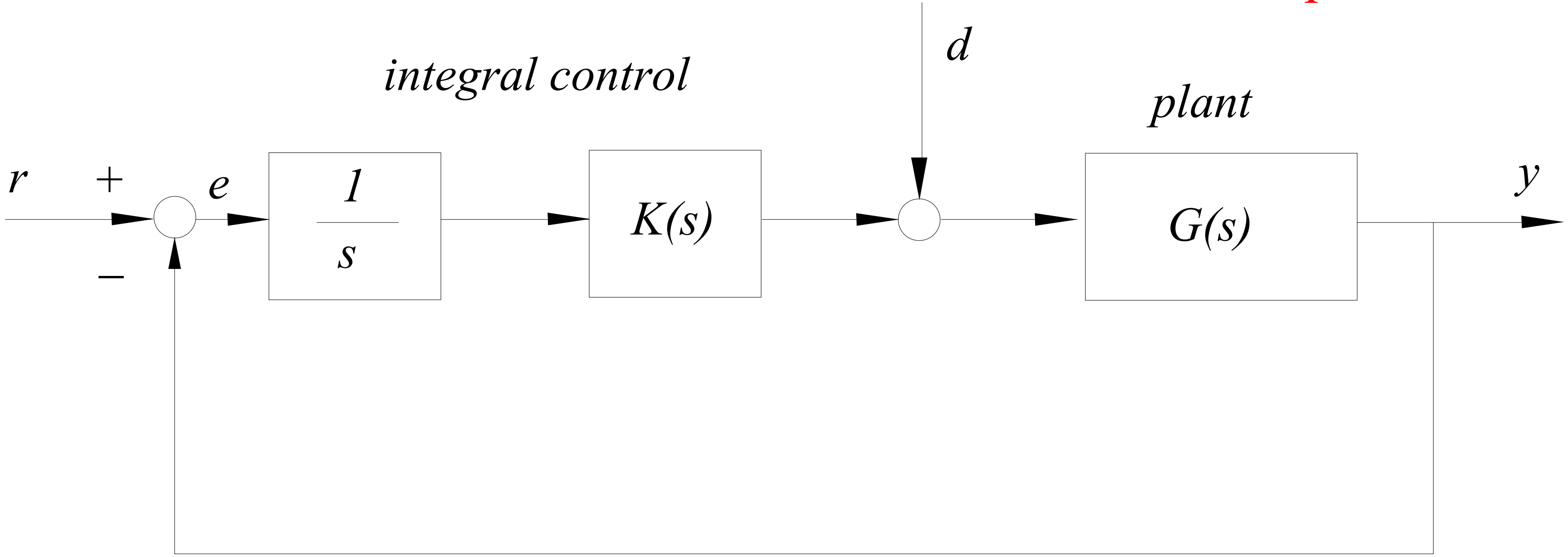


Figure 1 Classical integral control for SISO system.

An integral control can achieve a zero steady-state error in response to step inputs, as long as the closed-loop system is stable. Why?

If the input is a constant, what is the output for stable system at steady state?

The input and output of all the blocks should be constant at steady state.

Therefore, the input and output of the integrator must be constants at steady state.

What is the constant input to the integrator at steady state?

It must be ZERO! Otherwise the output cannot be a constant!

Two key elements to achieve zero steady-state error in this case are

- an integrator: being a right servo mechanism for step inputs.
- stabilization: enabling the servo mechanism to function properly.

The objective of this chapter is to extend this idea to

- General type of reference inputs
- Multivariable systems

It is possible only if we can have

- a right servo mechanism for any given input type;
- stabilization technique for a general plant.

That is,

Design = Servo mechanism + Stabilization!

We present

- Output feedback
- State feedback

Industrial Motivation: Satellite Antenna Servo Control

Process:

Model: It is desired to control the elevation of an antenna designed to track a satellite. Let the antenna and drive parts have a moment of inertia J and damping B. The equation of motion is given by

$$J\ddot{\theta} + B\dot{\theta} = T_c + T_d,$$

where θ is the angle, T_c is the net torque from the drive motor and T_d is the disturbance torque due to wind. By defining

$$a = B/J, \quad u = T_c/J, \quad d = T_d/J,$$

the equation reduces to

$$\theta(s) = \frac{1}{s(s/a+1)}u(s) + \frac{1}{s(s/a+1)}d(s), \ \text{a=0.1.}$$

Control: The design objective is to achieve servo control when both the reference and disturbance are step signals.

We are going to give the solution to this example later.

§10.2 Polynomial Approach to General SISO Servo Problem

Since we know how to solve the problem for step input, with transfer functions. We will first extend the solution to general servo problem using transfer functions. Then we will do it for state-space model.

Consider the unity output feedback system.

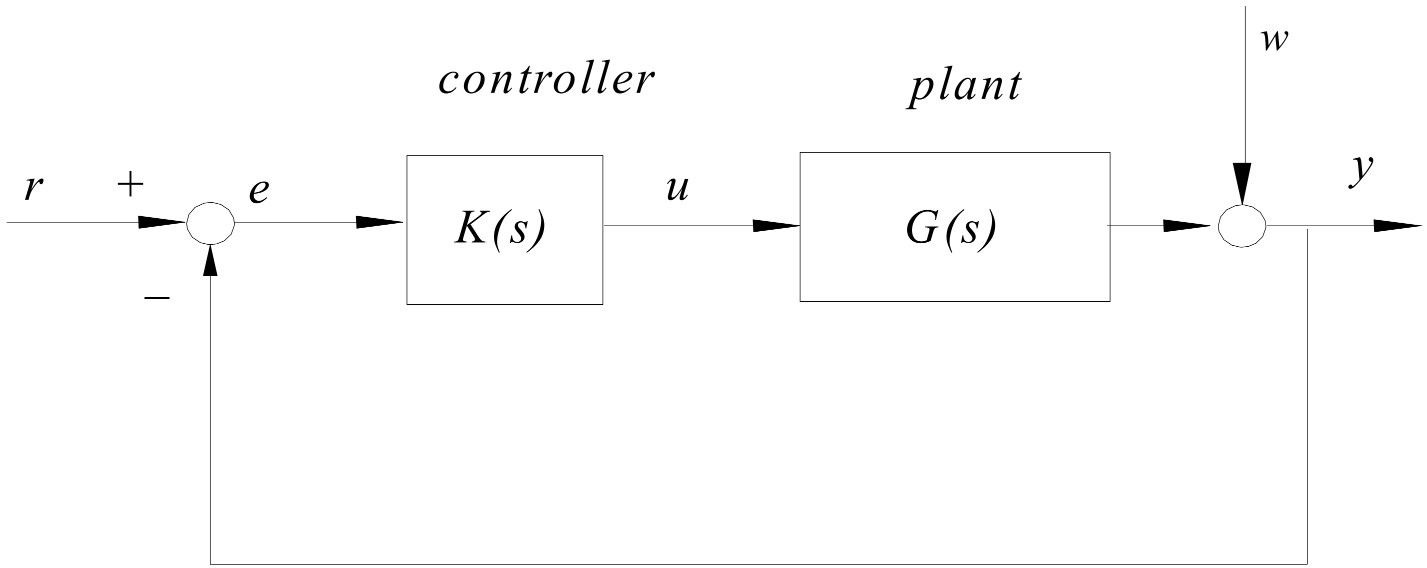


Figure 2 Unity output feedback system.

Objective: $e(\infty) = 0$ ***in face of r and w***

Let

$$G(s)=\frac{N(s)}{D(s)} \qquad \text{and} \qquad K(s)=\frac{B(s)}{A(s)}$$

The closed-loop transfer function between y and r is

$$H=\frac{GK}{1+GK}=\frac{\frac{N}{D}\frac{B}{A}}{1+\frac{N}{D}\frac{B}{A}}=\frac{NB}{DA+NB}$$

The feedback system is stable if and only if all the roots of its closed loop characteristic polynomial,

$$D_{cl}(s)=D(s)A(s)+N(s)B(s),$$

have negative real parts.

Is it a pole placement problem?

We solved the pole placement with state space model.
Now we need to solve it with transfer functions.

Do not worry. The solution is also as simple as that for the state space model approach.

Pole Placement

Example 1. Let $$G(s) = \frac{1}{s(s+2)}$$

When we choose the controller, we should always start with the simplest.

What is the simplest controller you can imagine?

If $K(s)$ is a gain of k (proportional control),

$$H(s) = \frac{kG(s)}{1+kG(s)} = \frac{\dfrac{k}{s(s+2)}}{1+\dfrac{k}{s(s+2)}} = \frac{k}{s^2+2s+k}$$

Is it possible to stabilize the closed loop by choosing proper value of k?

Certainly. Any positive k will do. However, can we place the poles to any desired positions?

For example, let the desired poles are -2 and -3, then we require

$$s^2 + 2s + k = (s+2)(s+3) = s^2 + 5s + 6$$

Is it possible?

Mission Impossible! Clearly, there is no k to meet the requirement.

It does not work because there is only one parameter to tune.

The controller needs more degree of freedom. How to add more design parameters to the controller?

We can increase the complexity, or the order of the controller!

Next increase the complexity from a constant gain to first order:

$$K(s) = \frac{B_0 + B_1 s}{A_0 + A_1 s},$$

$$H(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

$$= \frac{B_0 + B_1 s}{s(s+2)(A_1 s + A_0) + B_1 s + B_0}$$

$$= \frac{B_1 s + B_0}{A_1 s^3 + (2A_1 + A_0)s^2 + (2A_0 + B_1)s + B_0}.$$

The goal is to make the denominator of $H(s)$ equal to a desired one

$$P_c(s) = s^3 + F_2 s^2 + F_1 s + F_0,$$

where F_i are entirely arbitrary to have any desired roots. This leads to

$$A_1 s^3 + (2A_1 + A_0)s^2 + (2A_0 + B_1)s + B_0$$
$$= s^3 + F_2 s^2 + F_1 s + F_0$$

giving

$$A_1 = 1, \qquad 2A_1 + A_0 = F_2,$$

$$2A_0 + B_1 = F_1 \ \ , B_0 = F_0,$$

or

$$A_1 = 1, \ A_0 = F_2 - 2A_1,$$
$$B_1 = F_1 - 2F_2 + 4A_1, \qquad B_0 = F_0.$$

A solution always exists for arbitrary $P_c(s)$.

For example, if we assign the three poles of $H(s)$ as *-2* and $-2 \pm 2j$, then $P_c(s)$ becomes

$$P_c(s) = s^3 + F_2 s^2 + F_1 s + F_0$$
$$= (s+2)(s+2+2j)(s+2-2j)$$
$$= s^3 + 6s^2 + 16s + 16.$$

For this set of poles, we have

$$A_1 = 1, \qquad A_0 = 6 - 2 = 4$$

$$B_1 = 16 - 2 \cdot 6 + 4 = 8, \quad B_0 = 16,$$

and

$$K(s) = \frac{8s + 16}{s + 4}$$

What is the lesson learnt here?

Higher order of controller

>>> more free parameters

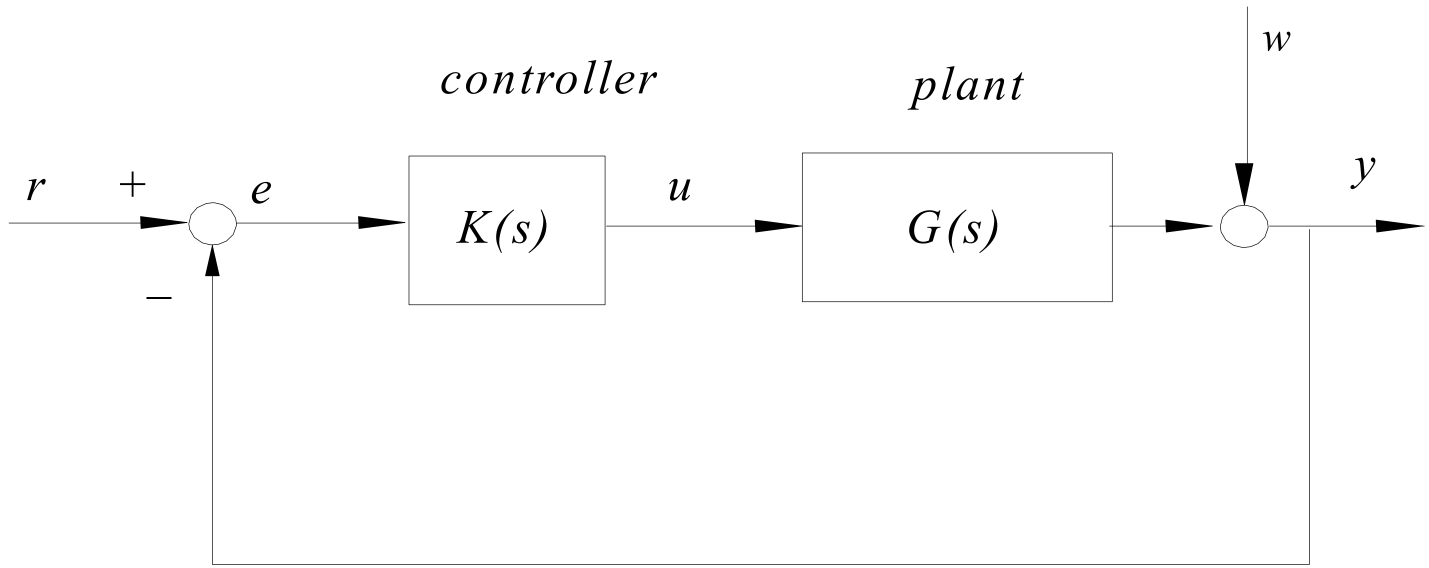
>>> better chance for pole placement!

However, the higher the order, the more complexity, and the more cost!

But further Questions arise:

- Is pole placement always possible with a sufficiently high-order controller?
- What is the suitable order of the controller?

We now look at the general case and establish the condition for achieving pole placement with transfer function approach.



Let
$$G(s) = \frac{N(s)}{D(s)}, \qquad K(s) = \frac{B(s)}{A(s)},$$

The order of the process is n. $\deg N(s) \le \deg D(s) = n$.

The closed loop transfer function H(s) is given by

$$H(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{\frac{N(s)}{D(s)}\frac{B(s)}{A(s)}}{1+\frac{N(s)}{D(s)}\frac{B(s)}{A(s)}} = \frac{N(s)B(s)}{D(s)A(s)+N(s)B(s)}. \qquad (1)$$

The pole-placement problem is equivalent to solving

$$D(s)A(s)+N(s)B(s)=P_c(s)\,, \qquad (2)$$

where $D(s)$ and $N(s)$ are known, the roots of $P_c(s)$ are the desired poles of the overall system to achieve, and $A(s)$ and $B(s)$ are unknown polynomials to be determined.

Q1: Are there solutions? What are the conditions?

Q2: What is the order of the controller? How to find out the solutions?

Q3: Are solutions realizable (proper)?

@@@@@@@@ Revision Notes @@@@@@@@

Coprimeness of two polynomials

Definition:

$\alpha(s)$ and $\beta(s)$ are coprime if they have no common factors, or roots.

Examples:

- $p_1 = (s+1)(s+2)$ and $p_2 = (s+3)(s+4)$ are coprime since they have no common factors or roots.

- $p_1 = (s+1)(s+2)$ and $p_2 = (s+2)(s+3)$ are NOT coprime since they have a common factor $(s+2)$.

@@@@@@@@ End of Revision Notes @@@@@@@@

Q1: Are there solutions? What are the conditions?

Let's take a look at one simple example.

If $D(s)$ and $N(s)$ both contain the factor $(s-2)$ or $D(s)=(s-2)\bar{D}(s)$ and $N(s)=(s-2)\bar{N}(s)$, then (2) becomes

$$D(s)A(s)+N(s)B(s)=(s-2)[\bar{D}(s)A(s)+\bar{N}(s)B(s)]$$

$$(s-2)[\bar{D}(s)A(s)+\bar{N}(s)B(s)]=P_c(s)$$

Is it possible to choose A and B to match any desired polynomial?

For example let $P_c(s)=(s+1)(s+2)(s+3)$.

$$(s-2)[\bar{D}(s)A(s)+\bar{N}(s)B(s)]=(s+1)(s+2)(s+3)$$

Is it impossible to find A and B to satisfy this equation?

$P_c(s)$ must contain the same common factor $(s-2)$. Thus, if $N(s)$ and $D(s)$ have a common factor, then not every root of $P_c(s)$ can be arbitrarily assigned. Therefore we assume from now on that $D(s)$ and $N(s)$ are coprime.

$\boxed{\text{Coprimeness: necessary and sufficient condition for Q1!}}$

Q2: What is the order of the controller? How to find out the solutions?

For Q2, let the order of the system be n, and the order of the controller be m,

$$D(s) := D_0 + D_1 s + D_2 s^2 + \cdots + D_n s^n, \qquad D_n \neq 0 \tag{3a}$$

$$N(s) := N_0 + N_1 s + N_2 s^2 + \cdots + N_n s^n, \tag{3b}$$

$$A(s) := A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m, \tag{4a}$$

$$B(s) := B_0 + B_1 s + B_2 s^2 + \cdots + B_m s^m. \tag{4b}$$

where D_i, N_i, A_i, B_i are all real numbers.

What is the order of the closed loop polynomial DA+NB?

The order of the closed loop is n+m.

Therefore the desired polynomial should be of order n+m

$$P_c(s) := F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m} \tag{5}$$

The substitution of these into (2) yields

$$
\begin{aligned}
&(D_0 + D_1 s + \cdots + D_n s^n)(A_0 + A_1 s + \cdots + A_m s^m) \\
&+(N_0 + N_1 s + \cdots + N_n s^n)(B_0 + B_1 s + \cdots + B_m s^m) \\
&= F_0 + F_1 s + \cdots + F_{n+m} s^{n+m},
\end{aligned}
$$

or

$$
\begin{aligned}
&(D_0 A_0 + N_0 B_0) + (D_1 A_0 + N_1 B_0 + D_0 A_1 + N_0 B_1)s \\
&+ \cdots + (D_n A_m + N_n B_m)s^{n+m} = F_0 + F_1 s + \cdots + F_{n+m} s^{n+m}.
\end{aligned}
$$

How many equations can you get by matching the coefficients of the two polynomials?

n+m+1!

Compare the two polynomials, we have

$$
\begin{aligned}
D_0 A_0 + N_0 B_0 &= F_0 \\
D_1 A_0 + N_1 B_0 + D_0 A_1 + N_0 B_1 &= F_1 \\
&\vdots \\
D_n A_m + N_n B_m &= F_{n+m}
\end{aligned}
$$

There are a total of $n + m + 1$ equations, which can be re-written as

$$\underbrace{\begin{bmatrix} D_0 & N_0 & 0 & 0 & & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & 0 \\ D_n & N_n & D_{n-1} & N_{n-1} & \cdots & D_0 & N_0 \\ 0 & 0 & D_n & N_n & & D_1 & N_1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & D_n & N_n \end{bmatrix}}_{S_m} \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \\ \vdots \\ A_m \\ B_m \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{m+n} \end{bmatrix} \qquad (6)$$

where the matrix S_m has

- $(m+n+1)$ rows, being the number of equations; and
- $2(m+1)$ columns, being number of unknowns

What is the maximal number of equations allowed for existence of solution?

The number of equations cannot be bigger than the number of design parameters!

$$n + m + 1 \le 2(m+1) \Rightarrow \mathrm{m} \ge n - 1$$

So we know that the order of the controller has to be

$$m \geq n - 1 \tag{7}$$

If $D(s)$ and $N(s)$ are **coprime** or have no common factors, it can be shown that the matrix S_m has a full row rank, and the solution always exist.

In this case, if $m = n - 1$, the matrix S_m is a non-singular square matrix of order $2n$, and for every $P_c(s)$, the solution of (6) is unique.

If $m \geq n$, then (6) has more unknowns than equations, how many solutions can you find? For instance, you have two unknowns but one equation like

$$A + B = 1$$

There are infinity number of solutions. To select the best one, you need to use other performance specifications.

Q3: Are solutions realizable (proper)?

Properness of Rational Functions

- **A *rational function*** is in form of $G(s) = \dfrac{\beta(s)}{\alpha(s)}$,where α, β are both polynomials.

- A rational function $G(s) = \dfrac{\beta(s)}{\alpha(s)}$ is called **proper** if $\deg(\alpha) \geq \deg(\beta)$.

For example, $G_1(s) = \dfrac{s+2}{s+1}$ is proper while $G_2(s) = s$ is not proper.

- A rational function $G(s)$ is called **strictly proper** if $\deg(\alpha) > \deg(\beta)$.

For example, $G_1(s) = \dfrac{s+2}{s+1}$ is proper but not strictly proper.

$G_3(s) = \dfrac{1}{s+1}$ is strictly proper.

The relative degree of proper TF is not negative!

Realizability of Controllers. $K(s)$ is physically realizable iff $K(s)$ is proper. Thus, an improper function is not realizable. This is best illustrated in the discrete time domain.

- Consider an improper controller:

$$K(z) = z$$

$$U(z) = K(z)E(z) = zE(z)$$

$$u(k) = e(k+1)$$

  Is it possible to implement this controller?

  The present output $u(k)$ of such a controller makes use of the future output tracking error $e(k+1)$ which is not available yet at the present. It is thus not possible to implement such a controller in real world.

- With a similar argument, the present output of a proper controller will make use of the present and past inputs, so available and realizable.

Theorem 1 *Consider the unity-feedback system shown in Figure 2 with a proper plant transfer function* $G(s) = N(s)/D(s)$ *with* $\deg D(s) = n$, *and* $N(s)$ *and* $D(s)$ *coprime. If* $m \geq n-1$, *then for any polynomial* $P_c(s)$ *of degree* $(n+m)$, *a proper compensator* $K(s) = B(s)/A(s)$ *of order m exists to achieve arbitrary pole placement. If* $m = n - 1$, *the compensator is unique. If* $m \geq n$, *the compensators are not unique.*

Has the theorem given answers to three Qs raised before?

- Q1: Are there solutions? What is the condition?

  Yes if $\mathrm{N}(s)$ and $\mathrm{D}(s)$ are coprime.

- Q2: What are the resulting controller and its order?

  It should be at least the plant order minus one: $m \geq n-1$

- Q3: Are there realizable (proper) solutions?

  Yes.

Example 2. Let

$$G(s)=\frac{N(s)}{D(s)}=\frac{s-2}{(s+1)(s-1)}=\frac{\overset{N_0}{-2}+\overset{N_1}{s}+\overset{N_2}{0\cdot s^2}}{\underset{D_0}{-1}+\underset{D_1}{0\cdot s}+\underset{D_2}{s^2}} \tag{8}$$

$D(s)$ and $N(s)$ have no common factor and $n=2$.

What is the order of the controller if we just want one solution?

m=n-1=1. So let

$$K(s)=\frac{B_0+B_1 s}{A_0+A_1 s}$$

For instance, let the three desired poles as -3, $-2\pm j$, so that

$$P_c(s):=(s+3)(s+2-j)(s+2+j)$$

$$=\underset{F_0}{15}+\underset{F_1}{17s}+\underset{F_2}{7s^2}+\underset{F_3}{s^3}$$

Compare the coefficients of the two polynomials

$$D(s)A(s)+N(s)B(s)=P_c(s)$$

We have

$$\begin{bmatrix} D_0 & N_0 & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 \\ D_2 & N_2 & D_1 & N_1 \\ 0 & 0 & D_2 & N_2 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 7 \\ 1 \end{bmatrix} \leftarrow \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad (9)$$

Its solution is

$$A_0=\frac{79}{3},\ A_1=1,\ B_0=-\frac{62}{3},\ B_1=-\frac{58}{3}.$$

$$K(s) = \frac{-\frac{62}{3} - \frac{58}{3}s}{\frac{79}{3} + s} = \frac{-(58s + 62)}{3s + 79}$$

$$= \frac{-(19.3s + 20.7)}{s + 26.3} \tag{10}$$

and the resulting overall closed loop T.F. is

$$H(s) = \frac{-(58s + 62)(s - 2)}{3(s^3 + 7s^2 + 17s + 15)}$$

Note that Theorem 1 is for pole placement, that is, we want to place closed-loop poles to any desired positions.

But if we simply want to stabilize the plant, that is, to make

$$D(s)A(s) + N(s)B(s) = P_c(s) \qquad (2)$$

stable, we can allow $D(s)$ and $N(s)$ have some stable common roots but no unstable common roots.

if $D(s)$ and $N(s)$ both contain the factor $(s-2)$ or $D(s) = (s-2)\overline{D}(s)$ and $N(s) = (s-2)\overline{N}(s)$, then (2) becomes

$$\begin{aligned} &D(s)A(s) + N(s)B(s) \\ &= (s-2)[\overline{D}(s)A(s) + \overline{N}(s)B(s)] \\ &= P_c(s) \end{aligned}$$

$P_c(s)$ must contain the same unstable common factor $(s-2)$ and thus always unstable.

if $D(s)$ and $N(s)$ both contain the factor $(s+2)$, or $D(s)=(s+2)\bar{D}(s)$ and $N(s)=(s+2)\bar{N}(s)$ then (2) becomes

$$
\begin{aligned}
& D(s)A(s)+N(s)B(s) \\
& =(s+2)[\bar{D}(s)A(s)+\bar{N}(s)B(s)] \\
& =P_c(s)
\end{aligned}
$$

$P_c(s)$ must contain the same common stable factor $(s+2)$ but it can still be stable as long as $[A(s)\bar{D}(s)+B(s)\bar{N}(s)]$ is made stable by proper choice of A(s) and B(s), which is always possible if $\bar{D}(s)$ and $\bar{N}(s)$ are coprime.

Corollary 1 *Consider the unity-feedback system shown in Figure 2 with a proper plant transfer function* $G(s)=N(s)/D(s)$ *and* $N(s)$ *and* $D(s)$ *have no common unstable root. Then, a proper compensator* $K(s)=B(s)/A(s)$ *exists to stabilize the plant.*

Break

State-of-the-art control systems

Future Robots (30:40)

Have we achieved zero steady-state error to step input?

Let's go back to example 2 and find out. In example 2, the closed loop TF is

$$H(s) = \frac{-(58s+62)(s-2)}{3(s^3+7s^2+17s+15)}$$

How to compute the steady state gain, i.e. DC gain, given the T. F. H(s)?

$$H(0) = 124/45 = 2.76.$$

Therefore, the output of this overall system will not track asymptotically the step reference input.

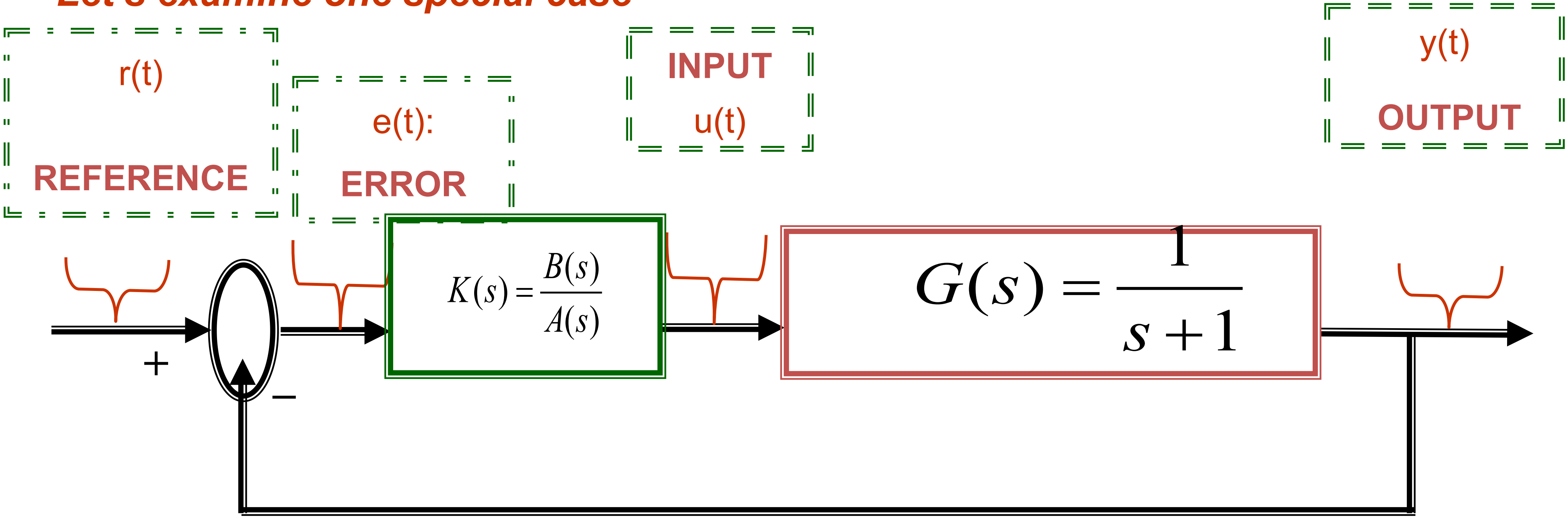
Where are we now?

Recall that servo control design consists of

- servo mechanism determination and
- overall system stabilization.

We have just developed a solution for stabilization. We now need to determine a suitable servo mechanism to enable asymptotic tracking and disturbance rejection.

Let's examine one special case



What is the closed loop transfer function?

$$\frac{Y(s)}{R(s)}=\frac{\frac{B(s)}{A(s)}\frac{1}{s+1}}{1+\frac{B(s)}{A(s)}\frac{1}{s+1}}=\frac{B(s)}{A(s)(s+1)+B(s)}$$

What is the error signal?

$$E(s)=R(s)-Y(s)=(1-\frac{B(s)}{A(s)(s+1)+B(s)})R(s)=\frac{A(s)(s+1)}{A(s)(s+1)+B(s)}R(s)$$

When the reference signal is a step signal $R(s)=\frac{1}{s},\qquad E(s)=\frac{A(s)(s+1)}{A(s)(s+1)+B(s)}\frac{1}{s}$

Check out the denominator, if s remains in the denominator, is it possible to make e(t)→ 0?

No! The only way to make it go to zero is to get rid of the unstable factor s.

Is it possible to design A(s) to make s disappear? Include s inside A(s) to cancel out this unstable factor!

Including s in A(s) is the same as putting integrator in the controller --- the classical method!

Servo mechanism determination

Suppose that the feedback system is stable.

Let the reference r(t) and disturbance w(t) be

$$R(s) = \boldsymbol{L}[r(t)] = N_r(s) / D_r(s)$$

and

$$W(s) = \boldsymbol{L}[w(t)] = N_w(s) / D_w(s)\,.$$

It is noted that the parts of $r(t)$ and $w(t)$ which **<u>go to zero</u>** as $t \to \infty$ have no effect on $y(t)$ as $t \to \infty$.

Hence, assume that some roots of $D_r(s)$ and $D_w(s)$ have zero or positive real parts. Take the least common denominator of the unstable modes of $R(s)$ and $W(s)$, and assign it as a **<u>polynomial</u>** $Q(s)$ of degree q.

As shown in the special case, the key of the servo mechanism is to design the factors of A(s) properly to cancel out all the unstable modes in the reference signal!

Finding Q

Eg.1

$$R(s)=\frac{1}{s},\qquad W(s)=\frac{1}{s(s+1)},\qquad \Rightarrow \qquad Q=s$$

Eg.2

$$R(s)=\frac{1}{s^2},\qquad W(s)=\frac{1}{s},\qquad \Rightarrow \qquad Q=s^2$$

Eg.3 $\quad r(t)=1+2t:$

$$R(s)=\frac{1}{s}+\frac{2}{s^2}=\frac{s+2}{s^2},\quad W(s)=\frac{1}{s}\quad \Rightarrow \quad Q=s^2$$

Eg.4 $\quad r = 1(t), \quad w(t) = \sin(\omega t):$

$$R(s) = \frac{1}{s},\ W(s) = \frac{\omega}{s^2 + \omega^2} \quad \Rightarrow \quad Q(s) = s\left(s^2 + \omega^2\right)$$

Eg.5 $\quad r = 0, \ w(t) = \sin(\omega t):$

$$R(s) = 0 = 0/1, \quad W(s) = \frac{\omega}{s^2 + \omega^2} \quad \Rightarrow \quad Q(s) = s^2 + \omega^2$$

Eg.6 $\quad r = \cos(\omega t), \ w(t) = 0:$

$$R(s) = \frac{s}{s^2 + \omega^2}, \quad \Rightarrow \quad Q(s) = s^2 + \omega^2$$

@@@@@@@@ Revision Notes @@@@@@@@

Motivated from the special cases:

- Step input, $\frac{1}{s}$, $\quad Q(s)=s$, $\quad \Rightarrow \frac{1}{s}$ or $\frac{1}{Q(s)}$
- Ramp input, $\frac{1}{s^2}$, $\quad Q(s)=s^2$, $\quad \Rightarrow \frac{1}{s^2}$ or $\frac{1}{Q(s)}$
- $\sin(\omega t)$, $\frac{1}{s^2+\omega^2}$, $Q(s)=s^2+\omega^2$, $\quad \Rightarrow \frac{1}{s^2+\omega^2}$ or $\frac{1}{Q(s)}$

When the reference input is step, the controller must have the integrator 1/s, i.e. 1/Q

A reasonable guess of the solution for general input is to

$$\textit{Include} \quad \frac{1}{Q} \quad \text{inside} \quad \textit{controller} \quad K(s)=\frac{B(s)}{A(s)}$$

In other words, simply include Q(s) as a part of A(s).

To simplify the derivation, we cascade it with a given proper plant

$$G(s)=\frac{N(s)}{D(s)},$$

to form a generalized plant:

$$\frac{1}{Q(s)}G(s)=\frac{N(s)}{Q(s)D(s)}$$

$$\frac{1}{Q(s)}G(s) = \frac{N(s)}{Q(s)D(s)}$$

The generalized plant is strictly proper as long as the plant is proper and $Q(s)$ is a polynomial of degree 1 or above. It has to be stabilized, in order for the servo mechanism to work and eliminate the steady state error.

What is the condition on the generalized plant transfer function such that its poles can be placed anywhere?

The generalized plant can be stabilized if $N(s)$ and $D(s)Q(s)$ have no common factors. ***This requires that no root of*** $Q(s)$ ***is a zero of the plant*** $G(s)$.

For example. Let

$$G(s) = \frac{s}{s+1}, \qquad Q(s) = s$$

Then $N(s) = s$, and $Q(s) = s$ are not coprime.
In this case, we cannot include 1/s in the controller!

Under coprimeness of polynomials $N(s)$ and $D(s)Q(s)$, a compensator $\widetilde{K}(s) = \dfrac{B(s)}{A(s)}$ of degree $l = n + q - 1$ will achieve pole-placement by considering $N/(DQ)$ of order $n+q$ as the plant.

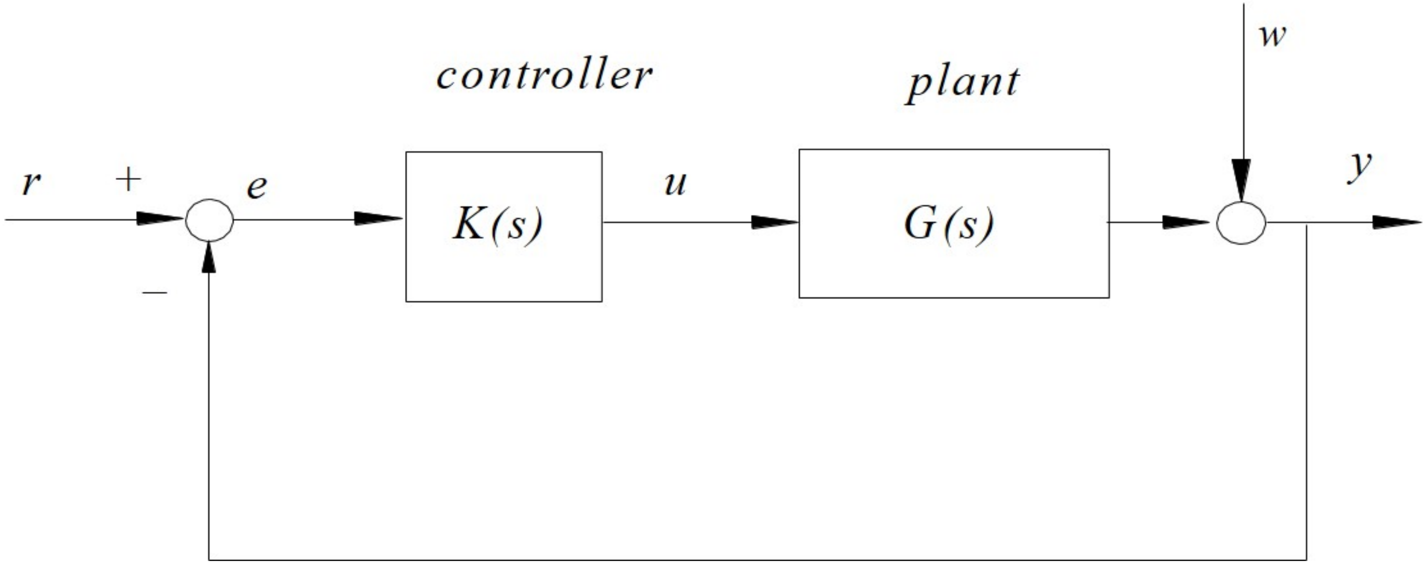
Practically, specify a desired stable $P_c(s)$ of degree $n+q+l$ with $l \geq n+q-1$ and solve the polynomial equation:

$$D(s)Q(s)A(s) + N(s)B(s) = P_c(s), \tag{12}$$

to get compensator $\widetilde{K}(s) = \dfrac{B(s)}{A(s)}$.

So we can see that a servo problem can be converted into pole placement problem.
If we know how to solve pole placement, then we can solve the servo problem.
But we need to show whether the tracking error converges to error or not.

Theorem 2 *Consider the unity-feedback system shown in Figure 2 with a proper plant transfer function* $G(s) = N(s)/D(s)$ *with* $N(s)$ *and* $D(s)$ *coprime. Let* $Q(s)$ *be the least common denominator of the unstable modes of the set point* $R(s)$ *and disturbance* $W(s)$. *If N(s) and D(s)Q(s) have no common unstable root, then there is a proper compensator* $\tilde{K}(s)$ *which stabilizes the generalized plant,* $N/(DQ)$, *and the overall controller* $K = \tilde{K}/Q$ *is proper and achieves asymptotic tracking and regulation robustly.*



Proof. The stabilization part has been shown before. We now prove zero steady state error.

Let $y_r(t)$ denote the output excited by r(t) when $w(t)=0$. Then

$$
\begin{aligned}
E_r(s) &= R(s) - Y_r(s) \\
&= R(s) - \frac{G(s)K(s)}{1+G(s)K(s)} R(s) \\
&= \frac{R(s)}{1+\dfrac{N(s)}{D(s)Q(s)}\dfrac{B(s)}{A(s)}} \\
&= \frac{D(s)A(s)}{D(s)Q(s)A(s)+N(s)B(s)} \cdot Q(s)\frac{N_r(s)}{D_r(s)}
\end{aligned}
\tag{13}
$$

Since all unstable roots of $D_r(s)$ are canceled by $Q(s)$, and all the poles of

$P_c(s) = A(s)D(s)Q(s) + B(s)N(s)$ are stable

so we have $e_r(t) \to 0$ as $t \to \infty$.

Similarly, let $y_W(t)$ be the output excited by $w(t)$ when $r(t) \equiv 0$. Then,

$$
\begin{aligned}
E_w(s) = R(s) - Y_w(s) &= 0 - \frac{1}{1+G(s)K(s)} W(s) \\
&= -\frac{A(s)D(s)Q(s)}{A(s)D(s)Q(s) + B(s)N(s)} W(s) \\
&= -\frac{A(s)D(s)}{A(s)D(s)Q(s) + B(s)N(s)} \cdot Q(s) \frac{N_w(s)}{D_w(s)}.
\end{aligned}
\tag{14}
$$

Since all unstable roots of $D_w(s)$ are canceled by $Q(s)$, all the poles of $E_w(s)$ have negative real parts, and we have $e_w(t) \to 0$ as $t \to \infty$. Thus, the disturbance rejection is also guaranteed.

Servo control design procedure:

(i) Obtain plant coprime polynomial fraction as $G = N(s) / D(s)$.

(ii) Determine Q from the given types of disturbance and reference, and include Q into the poles of the controller A(s). This is the servo mechanism. And the rest is to make the whole system stable.

(iii)Design $\tilde{K}$ to stabilize the generalized plant, $N / (DQ)$.

(iv) Form the servo controller as

$$K = \tilde{K} \frac{1}{Q}$$

Design = Servo mechanism + Stabilization!

Example 3 Let

$$G(s)=\frac{s+3}{s-1}:=\frac{N(s)}{D(s)}.$$

The reference is a sinusoid signal $r(t)=\sin(t)$, and the disturbance is of step type.

$$R(s)=\frac{1}{s^2+1}, W(s)=\frac{1}{s}$$

What is Q(s)?

$$Q=s(s^2+1)$$

Form the generalized plant:

$$\tilde{G}=\frac{1}{Q}G(s)=\frac{s+3}{s^4-s^3+s^2-s},$$

What is the order of the generalized plant? n=4

What is the order of the controller $\tilde{K}$? n-1=3

A controller $\tilde{K}$ of order 3 can be determined from

$$DQA+NB=P_c \tag{15}$$

If $P_c(s)$ is chosen as

$$P_c(s) = (s+3)^6 (s+1)$$
$$= s^7 + 19s^6 + 153s^5 + 675s^4 + 1755s^3 + 2673s^2 + 2187s + 729,$$

then (15) is solved to get the solution as

$$A(s) = A_0 + A_1 s + A_2 s^2 + A_3 s^3 = 363+172\mathrm{s}+20\mathrm{s}^2+s^3$$
$$B(s) = B_0 + B_1 s + B_2 s^2 + B_3 s^3 = 243+769\mathrm{s}+571s^2+465s^3.$$

Thus $\tilde{K}$ is

$$\tilde{K} = \frac{465s^3 + 571s^2 + 769s + 243}{\mathrm{s}^3+20\mathrm{s}^2+172\mathrm{s}+363} \quad (16)$$

The complete servo controller is

$$K(s) = \frac{1}{s(s^2+1)}\tilde{K}.$$

What is the order of the overall controller?

3+3=6.

Is it possible to reduce the order of the controller?

Simplified design procedure for Servo Control :

(i) Obtain plant coprime polynomial fraction as $G = N(s)/D(s)$.

(ii) Determine Q from the given types of disturbance and reference.

(iii) Skip the step of generalized plant. Directly include Q(s) as a factor of A(s) or in other words, design

$$A(s) = Q(s)A'(s)$$

(iv) Solve the pole placement problem:

$$D(s)Q(s)A'(s) + N(s)B(s) = P_c(s)$$

Example 3 Let

$$G(s) = \frac{s+3}{s-1} := \frac{N(s)}{D(s)}.$$

The reference is a sinusoid signal $r(t) = \sin(t)$, and the disturbance is of step type.

$$R(s) = \frac{1}{s^2+1}, \mathrm{W(s)} = \frac{1}{s}$$

What is Q(s)?

$$Q = s(s^2 + 1)$$

Design A(s) as

$$A(s) = Q(s)(s + A_0) = s(s^2 + 1)(s + A_0)$$

Since the order of A is 4, let

$$B(s) = B_0 + B_1 s + B_2 s^2 + B_3 s^3$$

such that the controller is strictly proper.

$$A(s)D(s)+B(s)N(s)=s(s^2+1)(s+A_0)(s-1)+(B_0+B_1s+B_2s^2+B_3s^3)(s+3)=P_c(s)$$

Notice that (s+3) is a stable factor, which can be cancelled out. Let

$$P_c(s)=(s+1)^4(s+3)=(s^4+4s^3+6s^2+4s+1)(s+3)$$

$$s+A_0=s+3$$

then

$$A(s)D(s)+B(s)N(s)=(s+3)s(s^2+1)(s-1)+(B_0+B_1s+B_2s^2+B_3s^3)(s+3)$$
$$=(s+3)(B_0+(B_1-1)s+(B_2+1)s^2+(B_3-1)s^3+s^4)$$
$$=(s^4+4s^3+6s^2+4s+1)(s+3)$$

$$B_0+(B_1-1)s+(B_2+1)s^2+(B_3-1)s^3+s^4=1+4s+6s^2+4s^3+s^4$$

Comparing the coefficients, we have

$$B(s)=1+5\text{s}+5s^2+5s^3$$

$$A(s)=s(s^2+1)(s+3)$$

What is the order of the controller?

The order is 4. So the controller is much simpler than the previous one with the order of 6.

Industrial application: Antenna servo control re-visited

Model: The system is described by

$$\theta(s) = \frac{1}{s(s/a+1)} u(s) + \frac{1}{s(s/a+1)} d(s).$$

Design: For $a = 0.1$, $R(s) = 1/s$, $d(s) = 1/s$, we have

$$W(s) = \frac{1}{s^2(10s+1)}$$

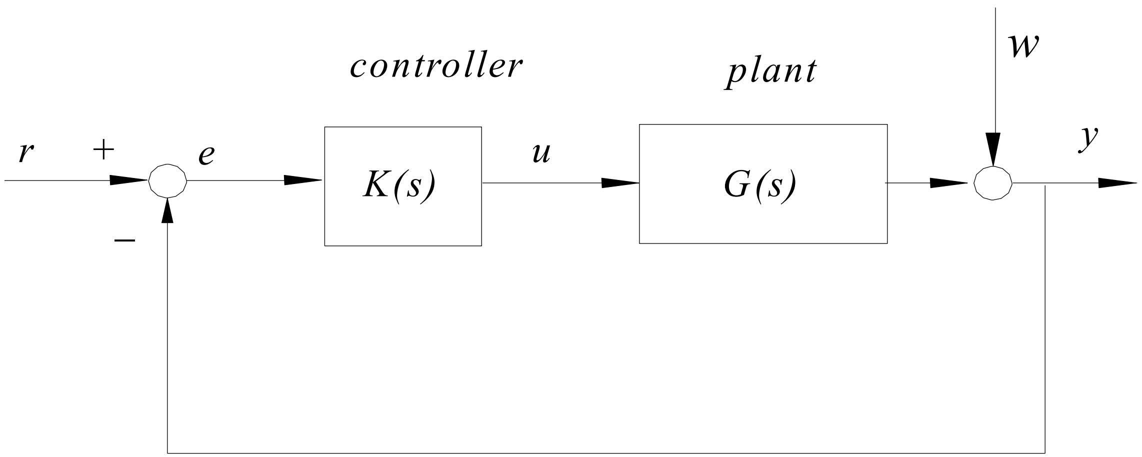


Figure 2 Unity output feedback system.

What is Q(s)?

$$Q(s) = s^2$$

Form the generalized plant

$$\tilde{G}(s) = \frac{1}{Q(s)} G(s) = \frac{1}{s^2} \frac{1}{s(10s+1)} = \frac{1}{s^3(10s+1)}$$

of order 4.

A controller $\tilde{K}$ of order 3 can be determined from

$$DQA + NB = P_c$$

If $P_c(s)$ is chosen to be

$$\begin{aligned} P_c(s) &= (s+2)^6 (s+1) \\ &= s^7 + 13s^6 + 72s^5 + 220s^4 + 400s^3 + 432s^2 + 256s + 64, \end{aligned}$$

then the solution is

$$A(s) = A_0 + A_1 s + A_2 s^2 + A_3 s^3 = 21.29 + 7.07\mathrm{s} + 1.29\mathrm{s}^2 + 0.1 s^3$$

$$B(s) = B_0 + B_1 s + B_2 s^2 + B_3 s^3 = 64 + 256\mathrm{s} + 432 s^2 + 378 s^3.$$

Thus $\tilde{K}$ is

$$\tilde{K} = \frac{378 s^3 + 432 s^2 + 256\mathrm{s} + 64}{0.1 s^3 + 1.29\mathrm{s}^2 + 7.07\mathrm{s} + 21.29},$$

And the complete controller is

$$\begin{aligned} K(s) &= \frac{1}{s^2}\tilde{K} \\ &= \frac{378 s^3 + 432 s^2 + 256\mathrm{s} + 64}{\mathrm{s}^2\left(0.1 s^3 + 1.29\mathrm{s}^2 + 7.07\mathrm{s} + 21.29\right)}. \end{aligned}$$

The closed-loop response is given in Figure 4 and has a large overshoot.

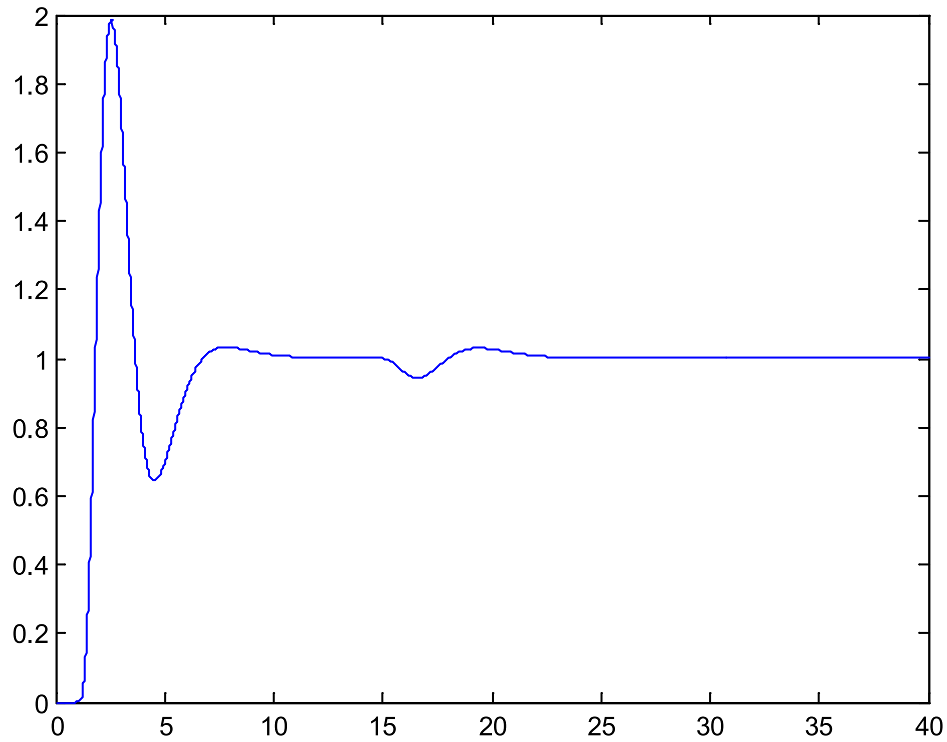


Figure 4 System step response (Design 1)

A simpler design could be achieved as follows.

- Servo mechanism. Since $Q(s) = s^2$, the open-loop GK should have the factor $1/s^2$. Noting that G(s) already has $1/s$, thus the controller only needs an integrator $1/s$, and can be given by

$$K(s) = \frac{\tilde{K}(s)}{s}$$

as long as $\tilde{K}$ stabilizes the generalized plant.

$$\tilde{G}(s) = \frac{1}{s^2(10s+1)}$$

- Stabilizer. Stable pole-zero cancellation may be used to simplify design and reduce controller order. Let $\tilde{K}(s)$ cancel the stable pole of G(s) so that

$$\tilde{K}(s) = (10s+1)(k_1 s + k_2)$$

and the open-loop becomes

$$GK = \tilde{G}\tilde{K} = \frac{k_1 s + k_2}{s^2}$$

giving the characteristic equation of the closed-loop as

$$P_c(s) = s^2 + k_1 s + k_2 = 0.$$

Choose $k_1 = k_2 = 4$ to assign two closed-loop poles both at -2. This leads to the following servo controller,

$$K(s) = \frac{(10s+1)(4s+4)}{s} = \frac{40s^2 + 44s + 4}{s}$$

which is of PID type.

The response from this new design is displayed in Figure 5 and has much smaller overshoot.

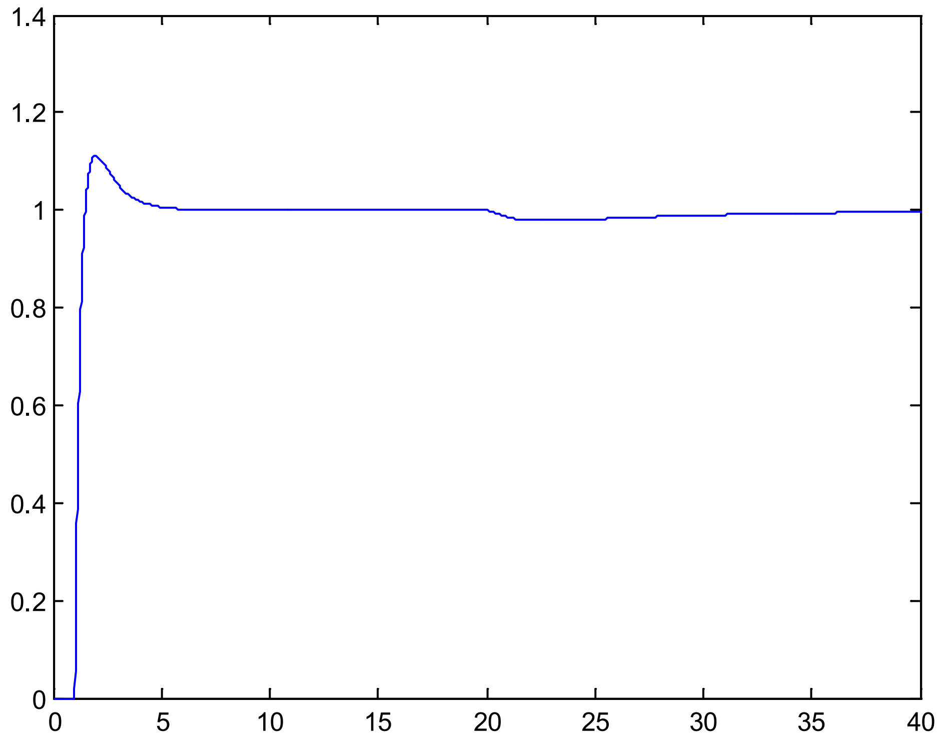


Figure 5 System step response (Design 2)

It is better than the design 1 since the overshoot is much smaller now.

Remark on realizability of PID controller.

The PID in its ideal/theoretical form,

$$K_p(1+\frac{K_I}{s}+K_D s),$$

Is it proper?

It is not proper, thus not realizable. For implementation, so-called industrial D control is used to approximate the derivative term by

$$K_D s \approx \frac{K_D s}{1+(K_D / N)s}$$

where N is usually 3 to 20. Then, it becomes proper and realizable.

§10.3 Multivariable Integral Control

We have already given a complete solution to servo control for SISO system.

Can we apply the same design directly to MIMO system?

No. The output is affected by all the inputs, not just one input. We need to use state space approach to deal with MIMO system unless the system can be decoupled!

Consider an $m \times m$ plant with m inputs and m outputs:

$$\dot{x} = Ax + Bu + B_w w, \tag{17}$$

$$y = Cx, \tag{18}$$

The error vector e is defined as

$$e = r - y. \tag{19}$$

The objective is to achieve zero steady state error for all the outputs.

$$\dot{x} = Ax + Bu + B_w w,$$
$$y = Cx,$$

What happens if we just use a simple state feedback controller?

Let the state feedback controller be

$$u = -Kx + Fr,$$

$$\dot{x} = Ax + Bu + B_w w = (A - BK)x + BFr + B_w w$$

$$y = Cx$$

If there is no disturbance (w=0), it is possible to design K and F properly such that the output y will follow the reference input r.

Due to the presence of unknown disturbance, the simple state feedback controller does not work anymore.

We need to have the proper servo mechanism!

Suppose that the disturbance w and reference r are **both of step type**. To achieve zero steady state error, like SISO integral control, we introduce one integrator to each channel:

$$v(t) = \int_0^t e\,(\tau) d\tau$$

Then, it follows that

$$\dot{v}(t) = e(t) = r - y(t) = r - Cx(t). \tag{20}$$

If the controller contains v(t), then the dynamics of the state vector x(t) will also depend upon v(t). Therefore, we should put x and v together to form a new dynamic system.

Equations (17), (18) and (20) form an augmented system:

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} A & O \\ -C & O \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B \\ O \end{pmatrix} u + \begin{pmatrix} B_w \\ O \end{pmatrix} w + \begin{pmatrix} 0 \\ I \end{pmatrix} r \tag{21}$$

$$\dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u + \overline{B}_w w + \overline{B}_r r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \overline{C}\overline{x} \tag{22}$$

We need to check whether the augmented system is controllable or not.

The controllability matrix is

$$Q_c = \begin{pmatrix} B & AB & A^2B & \cdots \\ 0 & -CB & -CAB & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} A & B \\ -C & 0 \end{pmatrix} \begin{pmatrix} 0 & B & AB & \cdots \\ I & 0 & 0 & \cdots \end{pmatrix}. \tag{23}$$

The augmented system is controllable if and only if

(i) the plant is controllable and

(ii)

$$rank \begin{pmatrix} A & B \\ -C & 0 \end{pmatrix} = n+m.$$

or equivalently,

$$rank \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n+m. \tag{24}$$

If the augmented system is controllable, it can be stabilized by the state feedback control law:

$$u = -K\bar{x} = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

or

$$u(t) = -\underset{\substack{\uparrow \\ P}}{K_1} x - \underset{\substack{\uparrow \\ I}}{K_2} \int_0^t e \, d\tau, \tag{26}$$

which contains integral control. The feedback gain K may be determined by pole placement procedure or LQR.

The resultant feedback system is

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} A - BK_1 & -BK_2 \\ -C & O \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B_w \\ O \end{pmatrix} w + \begin{pmatrix} 0 \\ I \end{pmatrix} r \tag{27}$$

$$\dot{\bar{x}} = \bar{A}_C \bar{x} + \bar{B}_w w + \bar{B}_r r$$

$$y = [C \quad 0]\begin{bmatrix} x \\ v \end{bmatrix}$$

$$y = \bar{C} \quad \bar{x} \tag{28}$$

Once the feedback system is stable, each signal in the system in response to step inputs will be constant in the steady state, and so is $v(t)$.

It follows that

$$e(t) = \dot{v}(t) = 0, \qquad as \quad t \to \infty.$$

To track a constant step signal, the servo mechanism relies on the magic power of integrator:

The input to the integrator must be ZERO at steady state if all the signals are constant at steady state.

Can we use this magic weapon to deal with other types of reference signals?

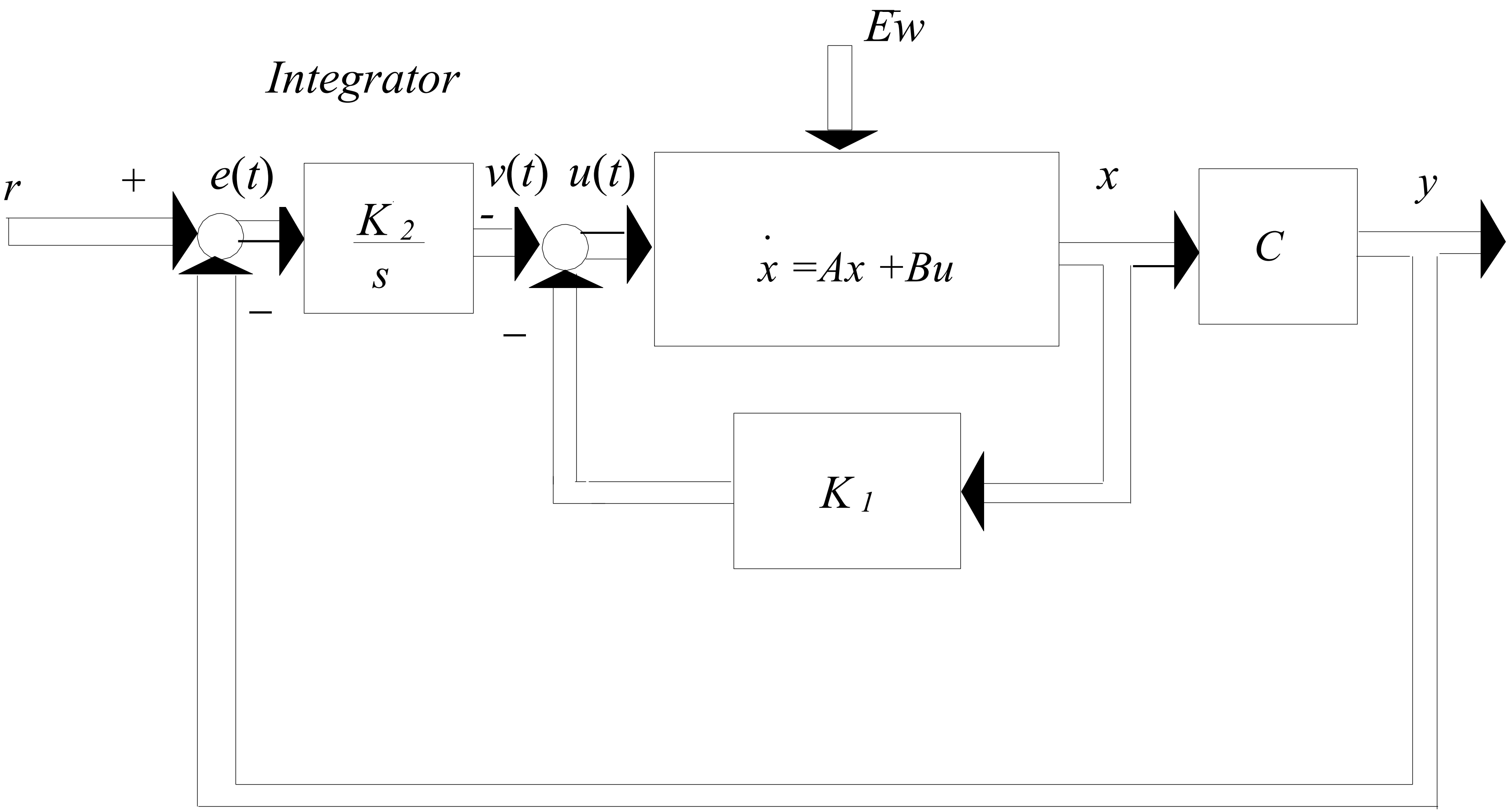


Figure 6 MIMO integral control system.

If the reference signal is sinusoid, would this integral control solve the tracking problem?

The output will be a sinusoid, so there is no guarantee that the input to the integrator is ZERO!

This problem has also been completely solved by building an augmented system. But we do not have time to cover it.

Example 5. Design an integral controller for

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w, \quad x(0) = 0,$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x.$$

One may readily check that it is controllable and meets (24). The augmented system is

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{\bar{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{\bar{x}} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\bar{B}} u + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{B}_w} w + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\bar{B}_r} r.$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{\bar{C}} \begin{pmatrix} x \\ v \end{pmatrix}.$$

It can be checked that this augmented system is controllable. To stabilize the closed-loop augmented system, we can use pole placement or LQR method. Take the LQR optimal control for an example.

The optimal control minimizing $J=\int_0^\infty \left(\bar{x}^T Q\bar{x}+u^T Ru\right)dt$

yields the feedback gain for state feedback $u=-K\bar{x}$

$$K=R^{-1}\bar{B}^T P,$$

where P is the positive definite solution of the following Riccati equation

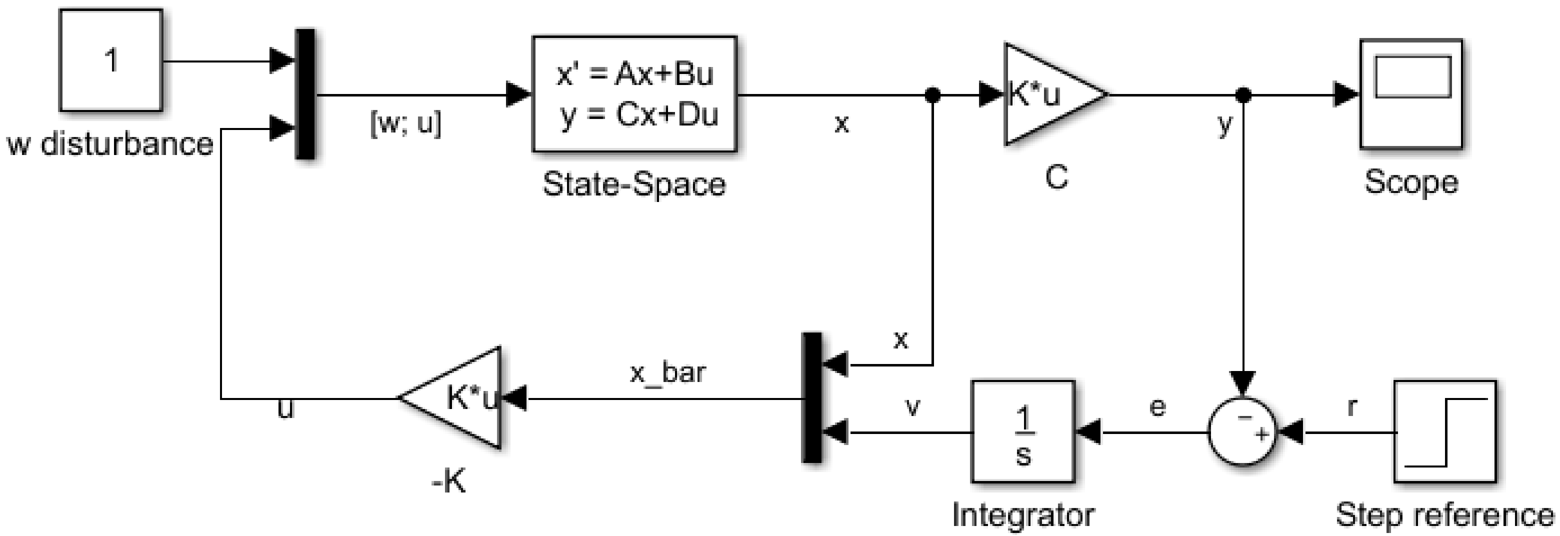
$$\bar{A}^T P+P\bar{A}+Q-P\bar{B}\bar{B}^T P=0$$

For $Q=I, R=I$, the solution can be solved as

$$K=\begin{bmatrix} 1.6464 & 0.4633 & -0.9626 & 0.2709 \\ 0.4633 & 1.9071 & -0.2709 & -0.9626 \end{bmatrix}$$

(Hint: use “care” function of MATLAB to solve this high-dimensional Riccati equation.)

A Simulink model can be built to verify the integral control when there is constant disturbance involved.



The result response of y to track the unit setting point $r = [1, 1]^T$ is

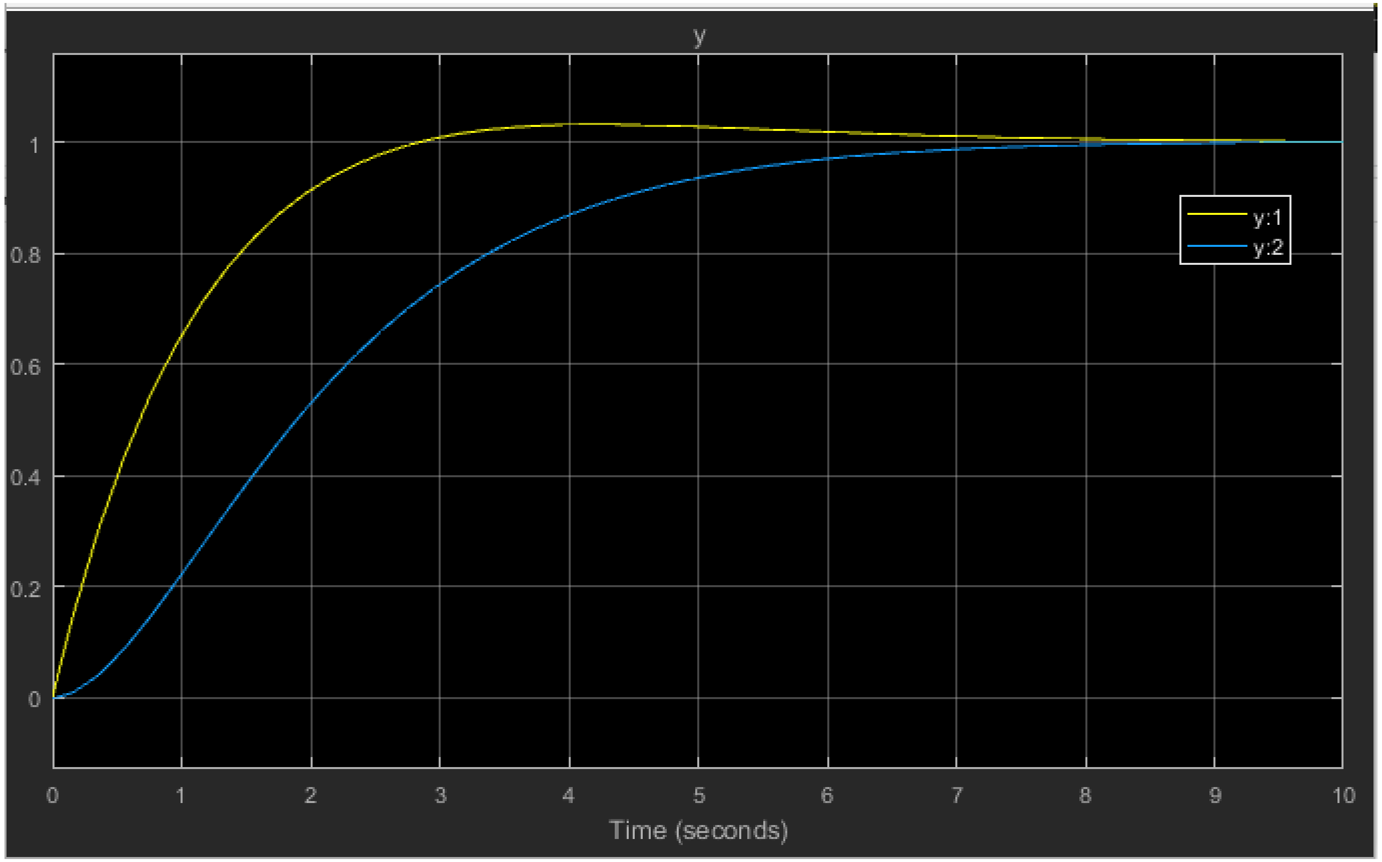
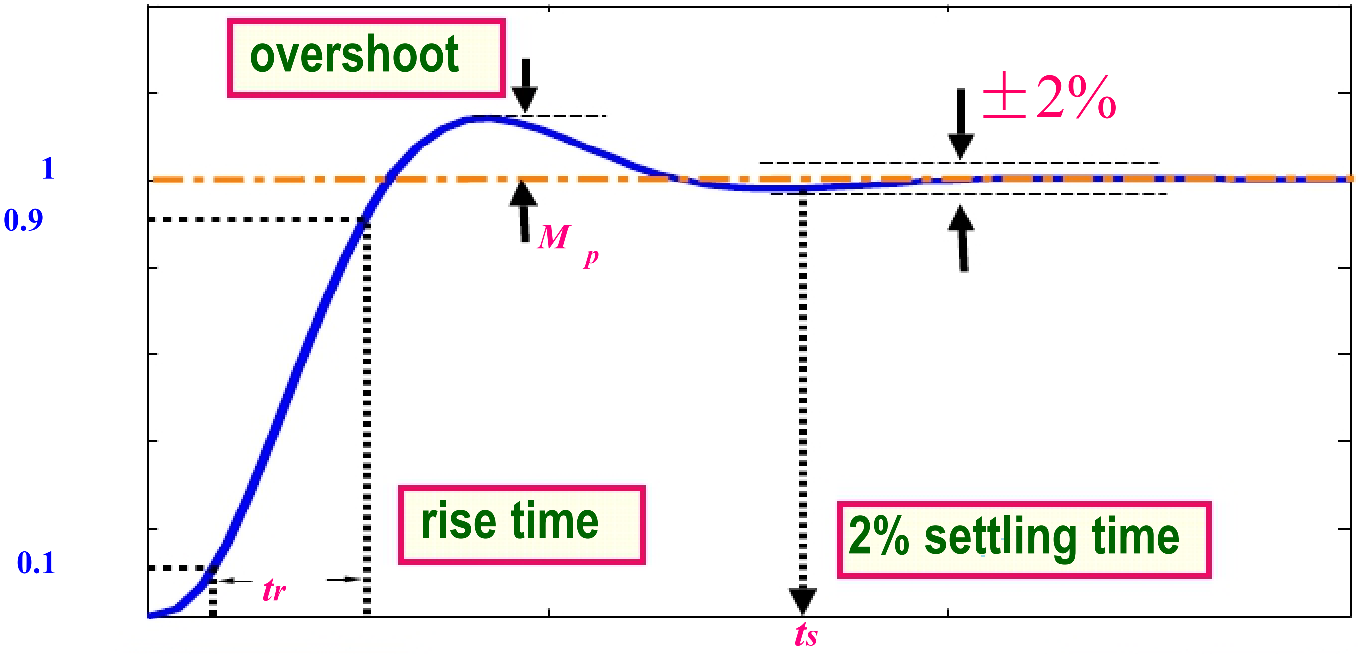


Figure 8 System step response for Example 5.

Q & A…

THANK YOU !

Settling time, overshoot and rise time — time domain specifications



$$t_r \cong \frac{1.8}{\omega_n}$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_s \cong \frac{4.0}{\zeta\omega_n}$$

$$(t_s,\ M_p,\ t_r) \Leftrightarrow (\zeta, \omega_n) \Leftrightarrow \text{Reference Model}$$

return