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1. (a) Show that $E(X_1+\cdots+X_n)=\bar{X}_1+\cdots+\bar{X}_n$

Pf: We can use induction to prove that;

Assume $E(X_1+\cdots+X_n)=\bar{X}_1+\cdots+\bar{X}_n$ ①

Let $X_1+\cdots+X_n=Z$, ① can be expressed as: $E(Z)=\bar{Z}$

$$E(Z+X_{n+1})=\iint (Z+X_{n+1}) f_{Z+X_{n+1}}(Z,X_{n+1})\,dX_{n+1}\,dZ$$

$$=\iint Z\cdot f_{Z+X_{n+1}}(Z,X_{n+1})\,dX_{n+1}\,dZ+\iint X_{n+1} f_{Z+X_{n+1}}(Z,X_{n+1})\,dX_{n+1}\,dZ.$$

$$=\int Z\underbrace{\int f_{Z+X_{n+1}}(Z,X_{n+1})\,dX_{n+1}}_{f_Z(Z)}\,dZ+\int X_{n+1}\underbrace{\int f_{Z+X_{n+1}}(Z,X_{n+1})\,dZ}_{f_{X_{n+1}}(X_{n+1})}\,dX_{n+1}$$

$$=E(Z)+E(X_{n+1})$$

$$=\bar{Z}+\bar{X}_{n+1}$$

$$=\bar{X}_1+\cdots+\bar{X}_n+\bar{X}_{n+1}$$

(b) When $X_1, \dots X_n$ are statistically indepent, show that:

$E(X_1 \cdot X_2 \cdot \dots \cdot X_n) = E(X_1) \cdot E(X_2) \cdot \dots \cdot E(X_n)$

Pf: We can use induction.

Assume that $E(X_1 \cdots X_n) = E(X_1) \cdot \dots \cdot E(X_n)$

Let $X_1 \cdot X_2 \cdot \dots X_n = z$, so $E(z) = E(X_1) \cdot \dots \cdot E(X_n)$

~~$E(z \cdot X_{n+1})$~~ $E(X_1 \cdot X_2 \cdot \dots \cdot X_n \cdot X_{n+1}) = E(z \cdot X_{n+1}) = \iint z \cdot X_{n+1} f_{zX_{n+1}}(z, X_{n+1})\, dz\, dX_{n+1}$ ①

Because $X_1 \cdots X_n$ are indepent, z and X_{n+1} are indepent.

We can get

$f_{zX_{n+1}}(z, X_{n+1}) = f_z(z) \cdot f_{X_{n+1}}(X_{n+1})$

Take into equation ①,

$$E(z \cdot X_{n+1}) = \iint z \cdot X_{n+1} \cdot f_z(z) \cdot f_{X_{n+1}}(X_{n+1})\, dz\, dX_{n+1}$$

$$= \left[\int z f_z(z) dz\right] \cdot \left[\int X_{n+1} f_{X_{n+1}}(X_{n+1})\, dX_{n+1}\right]$$

$$= E(z) \cdot E(X_{n+1})$$

$$= E(X_1) \cdot E(X_2) \cdot \dots \cdot E(X_n) \cdot E(X_{n+1}).$$

(c) When $X_1, X_2, \dots X_n$ are statistically independent, show that

$$Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n)$$

Pf: $Var(\sum_{i=1}^{n} X_i) = E[(\sum_{i=1}^{n} X_i)^2] - [E(\sum_{i=1}^{n} X_i)]^2$ ①

We know that:

$$E[(\sum_{i=1}^{n} X_i)^2] = E(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i \cdot X_j) \overset{1.(b)}{=\!=} \sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i \cdot X_j)$$

Similarly:

$$[E(\sum_{i=1}^{n} X_i)]^2 \overset{1.(a)}{=\!=} [\sum_{i=1}^{n} E(X_i)]^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i) \cdot E(X_j)$$

So, ① can be expressed as:

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} [E(X_i \cdot X_j) - E(X_i) \cdot E(X_j)] = \sum_{i=1}^{n}\sum_{j=1}^{n} cov(X_i, X_j).$$

Because variables are uncorrelated, when $i \neq j$, $coV(X_i, X_j) = 0$ (1.(b). $E(XY) = E(X) \cdot E(Y)$)

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} coV(X_i, X_i) = \sum_{i=1}^{n} Var(X_i).$$

2. Assume that X is a non-negative discrete rv, let $Y=h(X)$ for some non-negative function h. Let $b_i=h(a_i)$, $i\geq 1$ be the ith value taken on by Y.

Show that:

$$E(Y)=\sum_i b_i P_Y(b_i)=\sum_i h(a_i)P_X(a_i)$$

Pf: Because $b_i=h(a_i) \Rightarrow a_i=h^{-1}(b_i)$

$$P_Y(b_i)=P_r(Y=b_i)=P_r[h(X)=b_i]=P_r[X=h^{-1}(b_i)]=P_r[X=a_i]=P_X(a_i)$$

So:

$$E(Y)=\sum_i b_i\cdot P_Y(b_i)=\sum_i h(a_i)\cdot P_X(a_i)$$

3. (a)

From 1.30, we know $E(x) = \sum_i F_X^c(x)$

$$F_Y^c(y) = 1 - F_Y(y) = \frac{2}{(y+1)(y+2)}$$

$$E(Y) = \sum_{i=0}^{\infty} F_Y^c(y) = \sum_{i=0}^{\infty} \frac{2}{(y+1)(y+2)} = \sum_{i=0}^{\infty} \frac{2}{y+1} - \frac{2}{y+2} = 2$$

(b) PMF: $P_Y(0) = F_Y(0) = 1 - \frac{2}{(0+1)(0+2)} = 0.$

for each $y \geq 1$, the PMF given by:

$$P_Y(y) = F_Y(y) - F_Y(y-1) = [1 - \frac{2}{(y+1)(y+2)}][1 - \frac{2}{y(y+1)}] = \frac{4}{y(y+1)(y+2)}$$

$$\cancel{E(Y) = \sum_{i=1}^{\infty}} \; E(Y) = 0 \cdot P_Y(0) + \sum_{i=1}^{\infty} y \cdot P_Y(y)$$

$$= \sum_{i=1}^{\infty} \frac{4}{y+1} - \frac{4}{y+2}$$

$$= 2.$$

(c) Because $P_{X|Y}(x|Y) = \frac{1}{y}$ for $1 \leq x \leq y$, that means x is uniformly distributed over the 1 to y. so:

$$\cancel{E[X|Y=y] = \frac{1+y}{2}} \; E(X|Y=y) = \sum_{x=1}^{y} x \cdot \frac{1}{y} = \frac{1}{y} \cdot \sum_{x=1}^{y} x = \frac{1}{y} \cdot \frac{y \cdot (1+y)}{2} = \frac{1+y}{2}$$

$$E(X) = E_Y[E(X|Y=y)] = E_Y\left(\frac{1+y}{2}\right) = E\left(\frac{1}{2}\right) + \frac{1}{2} \cdot E(y) = \frac{1}{2} + \frac{1}{2} \times 2 = \frac{3}{2}$$

$$P_X(x) = \sum_{y=1}^{\infty} P_{X|Y}(x|Y) \cdot P_Y(y) = \sum_{y=1}^{\infty} \frac{4}{y^2(y+1)(y+2)} = ?$$

(d) Similar with 3.(c).

$$E(Z|Y=y) = \sum_{z=1}^{y^2} z \cdot \frac{1}{y^2} = \frac{1}{y^2} \cdot \frac{y^2(1+y^2)}{2} = \frac{1+y^2}{2}$$

$$E(Z) = E_Y[E(Z|Y=y)] = E\left(\frac{1+y^2}{2}\right) = \frac{1}{2} + \frac{1}{2} E(Y^2) = \frac{1}{2} + \frac{1}{2} \sum_{y=1}^{\infty} y^2 \cdot \frac{4}{y(y+1)(y+2)}$$

$$= \frac{1}{2} + 2 \cdot \sum_{y=1}^{\infty} \frac{2}{y+2} - \frac{1}{y+1}$$

$$= \frac{1}{2} + 2 \cdot \sum_{y=1}^{\infty} \frac{1}{y+2} - 1 \Rightarrow \infty$$

4.

p.f: $P_Z(z) = P_r(Z=z)$

$\ast$ Because $Z = X+Y$, assume $X = j$, $Y = z-j$.

$$\Rightarrow P_Z(z) = \sum_{j=0}^{z} P_r(X=j)\cdot P_r(Y=z-j)$$

$$= \sum_{j=0}^{z} \frac{\lambda^j \cdot e^{-\lambda}}{j!} \cdot \frac{\mu^{z-j}\, e^{-\mu}}{(z-j)!}$$

$$= \sum_{j=0}^{z} \frac{1}{j!(z-j)!} \cdot \lambda^j \cdot \mu^{z-j} \cdot e^{-\lambda-\mu}$$

$$= e^{-\lambda-\mu} \cdot \frac{1}{z!} \cdot \sum_{j=0}^{z} \frac{z!}{j!(z-j)!} \lambda^j \cdot \mu^{z-j}$$

Because $(a+b)^k = \sum_{\cancel{A}=0}^{k} \frac{k!}{n!\cdot(k-n)!} \cdot a^n \cdot b^{k-n}$, We can get:

$$P_z(z) = e^{-\lambda-\mu} \cdot \frac{1}{z!} \cdot (\lambda+\mu)^z$$

Let $\lambda + \mu = \lambda_1$

$$P_Z(z) = \frac{\lambda_1^z \cdot e^{-\lambda_1}}{z!}$$, is a Possion rv.

conditional distribution:

$$P_{X|Z}(X|Z=n) = \frac{P_r(X=k, Z=n)}{P_r(Z=n)} = \frac{P_r(X=k)\cdot P_r(Y=n-k)}{P_r(Z=n)}$$

$$= \frac{\frac{\lambda^k e^{-\lambda}}{k!} \cdot \frac{\mu^{n-k}\cdot e^{-\mu}}{(n-k)!}}{\frac{(\lambda+\mu)^n \cdot e^{-(\lambda+\mu)}}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{\lambda^k \cdot \mu^{n-k}}{(\lambda+\mu)^n}$$

$$= \frac{n!}{k!\cdot(n-k)!} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{n-k}$$

5. The ~~proper~~ probability of collect ith capon is:

$$P_i = \frac{1}{n} \cdot [n-(i-1)] = \frac{n-i+1}{n}$$

$$E(X_i) = \frac{1}{P_{i+1}} = \frac{n}{n-i}$$

From 1. (a), we know $E(X_1+X_2+\dots X_n) = E(X_1) + E(X_2) + \dots E(X_n)$

So,
$$C_n = E\left[\sum_{i=0}^{n-1} X_i\right] = \sum_{i=0}^{n-1} E(X_i) = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{i=0}^{n-1} \frac{1}{n-i}$$

~~Because~~ From hint, we know that:

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + O(1)$$

So:

$$C_n = n \cdot \sum_{i=0}^{n-1} \frac{1}{n-i}$$

$$= n \cdot \sum_{i=1}^{n} \frac{1}{i}$$

$$= n \cdot \ln n + n\gamma + nO(1)$$

When n is large $n\ln n \gg n\gamma$

So. $C_n \approx n \ln n$.

6.

(a) $E(X_i) = E[1 \cdot Y + 0 \cdot (1-Y)] = E(Y) = \mu$, for each $1 \le i \le n$

From 1. (a), we know that $E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$

So:

$E(S_n) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots E(X_n) = n \cdot E(Y) = n \cdot \mu$

(b) $Var(X_i) = E[X_i^2] - [E(X_i)]^2$

Because $X_i^2 = X_i$

$Var(X_i) = E(X_i) - [E(X_i)]^2$

$= \mu - \mu^2$

(c) $Cov(X_i, X_j) = E[(X_i - E[X_i]) \cdot (X_j - E[X_j])]$

$= E[X_i \cdot X_j - X_i \cdot E[X_j] - X_j \cdot E[X_i] + E[X_i] \cdot E[X_j]]$

$= E[X_j \cdot X_j] - E[X_i] \cdot E[X_j] - E[X_j] \cdot E[X_i] + E[X_j] \cdot E[X_j]$

$= E[X_i \cdot X_j] - E[X_i] \cdot E[X_j]$.

We know that $X_1, X_2, \dots X_n$ are conditionally indepent given the event $\{Y=y\}$.

$E[X_i \cdot X_j] = E_Y[E(X_i \cdot X_j | Y=y)]$

$= E_Y[E(X_i | Y=y) \cdot E(X_j | Y=y)]$

$= E_{\cancel{X}}[Y^2]$

Because $Var(Y) = E[Y^2] - [E[Y]]^2$

$E[X_i \cdot X_j] = E[Y^2] = Var(Y) + [E[Y]]^2 = \sigma^2 + \mu^2 \ne \mu^2 = E[X_i] \cdot E[X_j]$

$Cov(X_i, X_j) \ne 0$, X_i and X_j are not independent.

(d) We know that $X_1, \dots X_n$ are conditionally independent given the event $\{Y=y\}$.

$$\cancel{E}\ Var(S_n|Y) = E[S_n^2|Y] - (E[S_n|Y])^2$$

$$E[Var(S_n|Y)] = E[E[S_n^2|Y]] - E[(E[S_n|Y])^2]$$
$$= E[S_n^2] - E[(E[S_n|Y])^2] \quad ①$$

$$Var(E[S_n|Y]) = E[(E[S_n|Y])^2] - (E[E[S_n|Y]])^2$$
$$= E[(E[S_n|Y])^2] - (E[S_n])^2 \quad ②$$

combine ① and ②, we can get:

$$RHS = ① + ②$$
$$= E[S_n^2] - E[(E[S_n|Y])^2] + E[(E[S_n|Y])^2] - (E[S_n])^2$$
$$= E[S_n^2] - (E[S_n])^2$$
$$= Var(S_n) = LHS.$$

(e) $E[S_n|Y] = \sum_{i=1}^{n} E[X_i|Y] = nY$

$$Var(S_n|Y) = E[S_n^2|Y] - (E[S_n|Y])^2$$
$$= E\Big[\sum_{i=1}^{n}\sum_{j=1: i\neq j}^{n} X_i \cdot X_j + \sum_{i=1}^{n} X_i^2 \,\Big|\, Y\Big] - n^2Y^2$$
$$= \sum_{i=1}^{n}\sum_{j=1: i\neq j}^{n} E(X_i|Y)E(X_j|Y) + nY - n^2Y^2$$
$$= n^2 \cdot Y^2 + nY - n^2Y^2$$
$$= n \cdot Y$$

$$Var(S_n) = E[nY] + Var(nY)$$
$$= n \cdot E[Y] + n^2\, Var(Y)$$
$$= n\mu + n^2\sigma^2$$