$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i \qquad \forall n \geq 2.$$

Is it true that $\boxed{\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i \;?}$

Allowed to take limits for numbers.

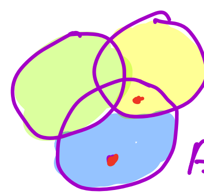
$\lim_{n\to\infty} \frac{1}{n} = 0.$ Limit of sets. limsup / liminf $\subseteq$

nested. $A_1 \supset A_2 \supset A_3 \supset \cdots$ $a_n = (-1)^n$

$\lim_{i\to\infty} A_i$

Take $x \in \bigcup_i A_i \Rightarrow \exists\, i \in \mathbb{N}$ s.t. $x \in A_i$.

$\Rightarrow \boxed{x \in B_j \text{ for some } j\ ??}$

WTP: $\bigcup_{n=1}^{\infty\, m} A_n = \bigsqcup_{n=1}^{\infty\, m} B_n$ A_1  A_2 A_3.

$B_j \subseteq A_j$

$B_1 = A_1,\ B_2 = A_2 \setminus A_1,\ B_3 = A_3 \setminus (A_1 \cup A_2)$

Pf: $\supseteq$ obvious.

$\subseteq$ Let $z \in \bigcup_{n=1}^{\infty\, m} A_n \Rightarrow z \in A_n$ for some $1 \leq n \leq m$. $n \geq 1$

By construction, $z \in B_j$ for some $j \leq n \leq m$

$\Rightarrow z \in \bigcup_{j=1}^{m} B_j$ $j \leq n$ //// $z \in \bigcup_{j=1}^{\infty} B_j$

$$\bigcup_{n=1}^{m\ \infty} A_n \subseteq \bigcup_{n=1}^{m\ \infty} B_n \quad z \in \bigcup_{j=1}^{n} B_j \subseteq \bigcup_{j=1}^{\infty} B_j$$

$$=$$

$\{\mathcal{F}_\alpha\}_{\alpha \in I}$ family of σ-alg.

NTP: $\mathcal{F} = \bigcap_{\alpha \in I} \mathcal{F}_\alpha$ is a σ-alg.

Each $\mathcal{F}_\alpha$ is a σ-alg.

i) $\Omega \in \mathcal{F}_\alpha \quad \forall \alpha \in I.$

$\Rightarrow \Omega \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha = \mathcal{F}$ //.

ii) $A_1, A_2, A_3, \dots \in \mathcal{F} = \bigcap_{\alpha \in I} \mathcal{F}_\alpha.$ arbitrary intersection

$\Rightarrow A_1, A_2, A_3, \dots \in \mathcal{F}_\alpha \quad \forall \alpha \in I.$

(This comes from the def. of $\bigcap$)

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\alpha, \ \forall \alpha \in I.$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha \in I} \mathcal{F}_\alpha = \mathcal{F}.$

$\boxed{\sigma}$- algebra
$\Updownarrow$
Σ
$\downarrow$
countable

$\Rightarrow \mathcal{F}$ is closed under countable union.

iii) If $A \in \mathcal{F} \Rightarrow A^c = \Omega \setminus A \in \mathcal{F}$.

$\mathcal{F}_1$ & $\mathcal{F}_2$ then $\mathcal{F}_1 \cup \mathcal{F}_2$ need not be a σ-alg.

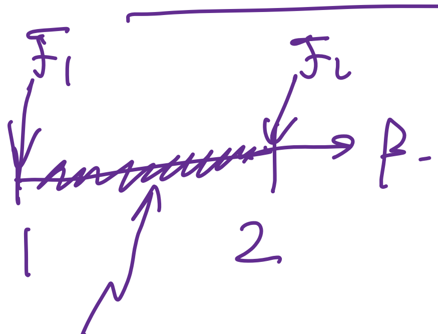
$\left(\mathcal{F}_1 , \mathcal{F}_2 \qquad \underset{\alpha \in (1,2)}{\mathcal{F}_\alpha = \{ \phi, \Omega \}} \right)$

$[1,2]$ σ-alg.

$\{ \mathcal{F}_\beta \}_{\beta \in [1,2]}$

violate the countable union property.

$\bigcup_{\beta \in [1,2]} \mathcal{F}_\beta$ is not a σ-alg.



$\mathcal{F}_\alpha$: trivial σ-alg.

Claim:

$$P\left(\bigcup_{j=1}^{n} A_j \right) \le \min \left\{ \sum_{j=1}^{n} P(A_j), 1 \right\}.$$

$$\wedge|$$

Claim:

$$P\left(\bigcup_{j=1}^{n} A_j \right) \le \underbrace{\left(\sum_{j=1}^{n} P(A_j) \right)}_{B.}^{\ell}, \quad \forall\, 0 \le \ell \le 1.$$

n large

Pf: i) $B \le 1$. $B^p \geq B. \Rightarrow P(\bigcup^n A_j) \le \sum P(A_j)$

$\le (\sum P(A_j))^p$

ii) $B > 1$

$$P(\bigcup_{j=1}^n A_j) \le \underbrace{B^p}_{>1} \quad \text{union bd}$$

$B > 1 \quad B^p > 1$

$$\underbrace{\mathcal{F}_1, \quad \mathcal{F}_2}_{\in \sigma\text{-alg} \in} \qquad \mathcal{F}_1 \cup \mathcal{F}_2$$

not σ-algebra.

$A_1 \quad A_2$

Counterexample.

$\underline{A_1 \cup A_2} \notin \underline{\mathcal{F}_1 \cup \mathcal{F}_2}$.

$$\mathcal{F}_1 = \{\phi, \overset{A_1}{\{a\}}, \{b,c\}, \underset{\{a,b,c\}}{\underset{\parallel}{\Omega}}\}$$

$$\mathcal{F}_2 = \{\phi, \overset{A_2}{\{b\}}, \{a,c\}, \Omega\}.$$

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \underbrace{\{a\}, \{b\}}, \{b,c\}, \{a,c\}, \Omega\}.$$