EE5137 : Stochastic Processes (Spring 2022)
Events and the Bernoulli Process

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1 Events

Let Ω be the sample space. A *σ-algebra* on Ω is a collection $\mathfrak{F}$ of subsets of Ω satisfying

1. $\Omega \in \mathfrak{F}$;
2. For any countable collection $\{A_i\}_{i=1}^{\infty}$ such that $A_i \in \mathfrak{F}$ for all $i \in \mathbb{N} = \{1, 2, \ldots\}$,
$$\bigcup_{i=1}^{\infty} A_i \in \mathfrak{F}. \tag{1}$$
3. For any $A \in \mathfrak{F}$, $A^{\mathrm{c}} := \Omega \setminus A \in \mathfrak{F}$.

Each $A \in \mathfrak{F}$ is called an *event*. The condition in the second point says that σ-algebras are *closed under countable unions*. The condition in the third point says that σ-algebras are *closed under complementation*.

The pair $(\Omega, \mathfrak{F})$, in which $\mathfrak{F}$ is a σ-algebra on Ω, is called a *measurable space*.

We can check that the *intersection* of (an arbitrary number of) σ-algebras is still a σ-algebra. But an *arbitrary union* of σ-algebras need not be a σ-algebra (Problem Set 1).

2 Random variables

Given a measurable space $(\Omega, \mathfrak{F})$, a *random variable* (more precisely a *measurable function*) is a function $X : \Omega \to \mathbb{R}$ such that the set
$$\{\omega \in \Omega : X(\omega) \le x\} \in \mathfrak{F}, \quad \forall\, x \in \mathbb{R}. \tag{2}$$
Hence, it makes sense to define the cumulative distribution function
$$F_X(x) := \Pr(\{\omega \in \Omega : X(\omega) \le x\}), \tag{3}$$
which is often more succinctly written as $\Pr(X \le x)$.

Does it makes sense to write $\Pr(X = x)$, which presumably means $\Pr(\{\omega \in \Omega : X(\omega) = x\})$? It suffices to prove that the set $\{\omega \in \Omega : X(\omega) = x\} \in \mathfrak{F}$, i.e., it is an event. To this end, define
$$A_n := \{\omega \in \Omega : X(\omega) \le x - 1/n\}, \quad \forall\, n \in \mathbb{N}. \tag{4}$$
Note that each A_n is indeed an event because X is a random variable and $x - 1/n \in \mathbb{R}$ for all n. Convince yourself now that
$$A := \bigcup_{n=1}^{\infty} A_n = \{\omega \in \Omega : X(\omega) < x\}. \tag{5}$$

Because σ-algebras are closed under countable unions, $A \in \mathfrak{F}$. Now, $\{\omega \in \Omega : X(\omega) \le x\} \in \mathfrak{F}$ because X is a random variable and $x \in \mathbb{R}$. So its complement

$$B := \{\omega \in \Omega : X(\omega) > x\} \in \mathfrak{F}. \tag{6}$$

Clearly,

$$\{\omega \in \Omega : X(\omega) = x\} = \Omega \setminus (A \cup B). \tag{7}$$

Since $A \cup B \in \mathfrak{F}$, so is its complement $\Omega \setminus (A \cup B)$ which means that $\{\omega \in \Omega : X(\omega) = x\} \in \mathfrak{F}$, so we can measure its probability and so $\Pr(X = x)$ is legitimate.

What we have concluded is that all sets of the form

$$\begin{array}{ll} \{\omega \in \Omega : X(\omega) \le x\}, & \{\omega \in \Omega : X(\omega) < x\}, \\ \{\omega \in \Omega : X(\omega) \ge x\}, & \{\omega \in \Omega : X(\omega) > x\}, \\ \{\omega \in \Omega : X(\omega) = x\} & \end{array}$$

are legitimate events.

3 Some Notes on the Bernoulli Process

Recall that the Bernoulli process can be defined in terms of an i.i.d. sequence of Bernoullli random variables $\{Z_i\}_{i=1}^{\infty}$ in which $p = \Pr(Z_1 = 1)$. To this process, we can define several other stochastic processes, such as the cumulative sum (aggregate number of arrivals) $S_n = \sum_{i=1}^{n} Z_i$. The time of the k-th arrival is Y_k and the interarrival times can be expressed as $X_1 = Y_1$ and $X_k = Y_k - Y_{k-1}$ for $k \ge 2$. In other words, $Y_k = \sum_{j=1}^{k} X_j$.

It is clear that X_1 is a Geometric random variable with parameter p, i.e.,

$$p_{X_1}(k) = \Pr(X_1 = k) = (1-p)^{k-1} p, \quad k = 1, 2, 3, \ldots. \tag{8}$$

We show that X_2 is independent of X_1 and X_2 is also a Geometric random variable with parameter p. Consider $\Pr(X_1 = k, X_2 = l)$. The event therein refers to the event that the first interarrival time is k and the second is l. Thus, equivalently, this means that there are exactly two arrivals in the first $k + l$ times slots, exactly at times k and $k + l$ respectively. Thus,

$$\Pr(X_1 = k, X_2 = l) = \underbrace{(1-p)^{k-1} p}_{k-1 \text{ failures and 1 success at } k} \times \underbrace{(1-p)^{l-1} p}_{l-1 \text{ failures and 1 success at } k+l} \tag{9}$$

Thus,

$$\Pr(X_2 = l \mid X_1 = k) = \frac{\Pr(X_1 = k, X_2 = l)}{\Pr(X_1 = k)} = \frac{(1-p)^{k-1} p (1-p)^{l-1} p}{(1-p)^{k-1} p} = (1-p)^{l-1} p, \tag{10}$$

which is independent of k. Thus

$$\Pr(X_2 = l \mid X_1 = k) = \Pr(X_2 = l) = (1-p)^{l-1} p, \tag{11}$$

which shows that X_1 is independent of X_2 and follows a Geometric distribution with parameter p as desired.