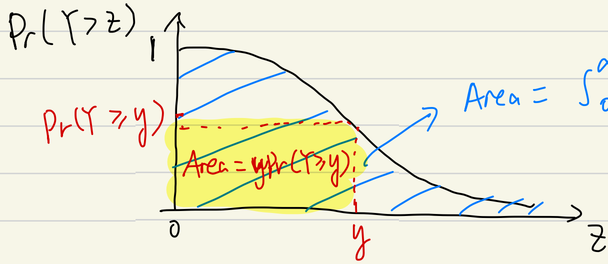
Section 1.6 of Gallager's book
- Markov's inequality
- Chebyshev's inequality
- Chernoff bounds

- Markov's Inequality.

Lem: Let Y be non-negative r.v. with mean of $\mathbb{E}Y$ then for any $y>0$
$$\Pr(Y \geq y) \leq \frac{\mathbb{E}Y}{y}.$$

Complementary CDF of Y:



Area $= \int_0^\infty \Pr(Y>z)\,dz = \mathbb{E}Y.$

(Fact: Layer-cake Representation in Lecture 2)

$$y\Pr(Y\geq y) \leq \mathbb{E}Y.$$

$\lim_{y\to\infty} y\Pr(Y\geq y) = 0$ if $\mathbb{E}Y<\infty$

Ex: If Y represents height of a random student in the class. $\mathbb{E}Y = 1.6$m.
what's the chance that Y exceeds $1.5\mathbb{E}Y$?
$$\Pr(Y \geq \underbrace{1.5\mathbb{E}Y}_{=\,2.4\text{m.}}) \leq \frac{\mathbb{E}Y}{1.5\mathbb{E}Y} = \frac{2}{3}$$

Ex: If $X\in\{0,5\}$, $\Pr(X=0)=\frac{24}{25}$, $\Pr(X=5)=\frac{1}{25}$
$\mathbb{E}X = \frac{1}{25}\cdot 5 = \frac{1}{5}$. What's the chance that $X\geq 5$?
$$\Pr(X\geq 5) \leq \frac{\mathbb{E}X}{5} = \frac{1}{25} = \Pr(X=5) \implies \text{tight}$$
(no better bound using only $\mathbb{E}X$)

Pf: $\mathbb{1}\{Y \geq y\} \leq \frac{Y}{y}$, $\forall y > 0$.

$\begin{cases} \text{if } Y \geq y, \ \frac{Y}{y} \geq 1 = \mathbb{1}\{Y \geq y\}. \\ \text{if } Y < y, 1 > \frac{Y}{y} \geq 0 = \mathbb{1}\{Y \geq y\}. \end{cases}$

Take Expectations on both sides $\Rightarrow \underbrace{\mathbb{E}\,\mathbb{1}\{Y \geq y\}}_{\parallel} \leq \frac{\mathbb{E}Y}{y}$.

$\Rightarrow \Pr(Y \geq y) \leq \frac{\mathbb{E}Y}{y}$.

— Chebyshev's Inequality.

Lem: Let Z is a r.v. with finite mean $\mathbb{E}Z$ & finite variance $\mathrm{Var}(Z) = \sigma_Z^2$

Then

$$\Pr((Z - \mathbb{E}Z)^2 \geq y) \leq \frac{\sigma_Z^2}{y}, \quad \forall y > 0.$$

Rmk: Chebyshev's ineq. applies to r.v. with finite mean & variance

first moment ‖ ; first and second moment ‖.

Markov's ineq applies to non-negative rv with finite mean

(Variance can be $+\infty$)

$\Rightarrow$ the more moments we use, the tighter the results we can get in general

Rmk: What's the prob that Z deviates from $\mathbb{E}Z$ by $\geq c\sigma_Z$ for some $c > 0$?

$$\Pr(|Z - \mathbb{E}Z| \geq c\sigma_Z) = \Pr((Z - \mathbb{E}Z)^2 \geq c^2\sigma_Z^2) \leq \frac{1}{c^2}$$

Pf: Apply Markov's ineq. to non-negative r.v. $(Z - \mathbb{E}Z)^2$

$$\Pr((Z - \mathbb{E}Z)^2 \geq y) \leq \frac{\mathbb{E}[(Z - \mathbb{E}Z)^2]}{y} = \frac{\mathrm{Var}Z}{y} = \frac{\sigma_Z^2}{y}. \quad \blacksquare$$

— Chernoff bounds

Let $X_1, X_2, \cdots, X_n$ be iid Bernoulli rv where $\Pr(X_i = 1) = p$, $\Pr(X_i = 0) = 1-p = q$.

Fix $\varepsilon > 0$, $\Pr\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - p\right| > \varepsilon\right) = \Pr\left(\left(\frac{1}{n}\sum_{i=1}^n X_i - p\right)^2 > \varepsilon^2\right)$

$$\leq \frac{\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n X_i - p\right)^2\right]}{\varepsilon^2} \quad \text{(Markov's ineq)}$$

$$= \frac{\frac{1}{n^2}\sum_{i=1}^n \mathbb{E}[X_i^2] + \frac{1}{n^2}\sum_{i \neq j}\mathbb{E}[X_i X_j] - 2p\frac{1}{n}\sum_{i=1}^n \mathbb{E}X_i + p^2}{\varepsilon^2}$$

$$= \frac{p - p^2}{n\varepsilon^2}$$

Let $p = q = \frac{1}{2}$ & $\varepsilon = 0.01$

Let $\Pr(|\frac{1}{n}\sum_{i=1}^{n} X_i - p| > \varepsilon) \le 0.01 \implies n \geq 250000$ samples

Do we need this # of samples to get $\Pr(\cdots) \le 0.01$?

$\begin{cases} \text{Yes if further moments don't exist} \\ \text{No, if more moments exist.} \end{cases}$

MGF of r.v. X. $g_X(r) = \mathbb{E}[e^{rX}]$, $\forall r \in \mathbb{R}$

Assume that $g_X(r)$ exists for all r in a neighborhood of 0

Note that $g_X(0) = 1$

$g_X'(r) = \mathbb{E}[Xe^{rX}]$, $g_X''(r) = \mathbb{E}[X^2 e^{rX}]$, $\cdots$, $g_X^{(k)}(r) = \mathbb{E}[X^k e^{rX}]$

$g_X^{(k)}(0) = \mathbb{E}[X^k]$

Thus, the MGF $g_X(r) \implies$ the moments $\mathbb{E}[X^k]$ of X.

For any r.v. Z, Markov's ineq. to get

$$\Pr(e^{rZ} \geq y) \le \frac{\mathbb{E}[e^{rZ}]}{y} = \frac{g_Z(r)}{y}. \qquad (e^{rZ} > 0)$$

Replace y be e^{rb} for some $b \in \mathbb{R} \implies \Pr(e^{rZ} \geq e^{rb}) \le \exp(-rb)\, g_Z(r)$ $(\forall r > 0)$

$\Pr(e^{rZ} \geq e^{rb}) = \Pr(Z \geq b)$

In the other direction, for $r < 0$.

$$\Pr(Z \le b) = \Pr(e^{rZ} \geq e^{rb}) \le \exp(-rb)\, g_Z(r), \quad \forall r < 0.$$

— Go back to the original problem, if consider X_i's iid with $\mathbb{E}X = 0$ and $g_X(r)$ exists

Fix $r > 0$, $\Pr(\frac{1}{n}\sum_{i=1}^{n} X_i > \varepsilon) = \Pr(\sum_{i=1}^{n} X_i > n\varepsilon)$

(one-sided version) $= \Pr(\exp(r\sum_{i=1}^{n} X_i) > \exp(rn\varepsilon))$

$\le \exp(-nr\varepsilon)\, \mathbb{E}[\exp(r\sum_{i=1}^{n} X_i)]$

$= (\mathbb{E}[\exp(rX)])^n \exp(-nr\varepsilon)$

$$= (g_X(r))^n \exp(-nr\varepsilon)$$

Define the semi-invariant MGF (also called cumulant generating function)

$$\gamma_X(r) := \ln g_X(r)$$

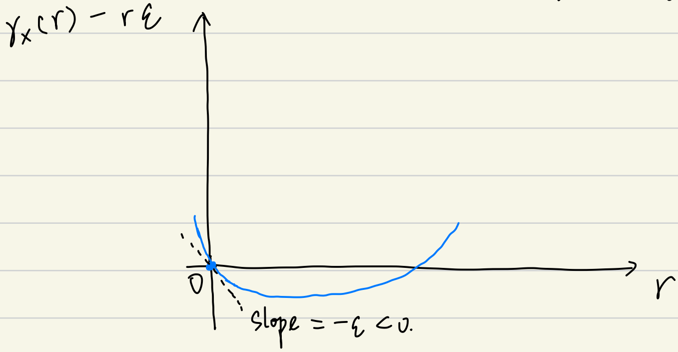
$$\Longrightarrow \Pr\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \varepsilon\right) \le \exp\left(n\left(\gamma_X(r) - r\varepsilon\right)\right)$$

This bound is exponential in n for fixed ε and fixed r.

To obtain the smallest upper bound, we minimize $\gamma_X(r) - r\varepsilon$ over all $r > 0$.

Note that $\gamma_X(r) - r\varepsilon \big|_{r=0} = 0$, $\frac{d}{dr}\gamma_X(r) - r\varepsilon \big|_{r=0} = \mathbb{E}X - \varepsilon = -\varepsilon < 0$.

This means that $\gamma_X(r) - r\varepsilon$ must be negative for sufficiently small r.



$$\gamma_X(r) - r\varepsilon \big|_{r=0} = 0.$$
$$(\gamma_X(r) - r\varepsilon)' \big|_{r=0} = -\varepsilon < 0$$
$$(\gamma_X(r) - r\varepsilon)'' \big|_{r=0} = \mathrm{Var}X > 0.$$

Thm: (Chernoff Bounds)

Let $\{X_i\}_{i=0}^{\infty}$ be iid r.v. & $S_n = X_1 + \cdots + X_n$.

Assume MGF $g_X(r)$ exists $\forall r \in \mathbb{R}$. Then

$$\Pr(S_n \ge n\varepsilon) \le \exp(n\mu_X(\varepsilon)) \quad \text{where}$$

$$\mu_X(\varepsilon) = \inf_{r>0}\{\gamma_X(r) - r\varepsilon\}$$

Furthermore, $\mu_X(\varepsilon) < 0$ for $\varepsilon < \mathbb{E}X$ & $\mu_X(\varepsilon) = 0$ for $\varepsilon \ge \mathbb{E}X$

Rmk: Compared with Markov's ineq & Chebyshev's ineq.

Chernoff bounds make use ALL moments since we assume $g_X(r) = \mathbb{E}[e^{rX}]$ exist.

By Taylor expansion: $\mathbb{E}[e^{rX}] = \sum_{k=0}^{\infty} \mathbb{E}\left[\frac{(rX)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{r^k}{k!}\mathbb{E}[X^k]$.

Ex : $X \in \{0,1\}$, $\Pr(X=1) = p$, $\Pr(X=0) = 1-p = q$.

$g_X(r) = q + pe^r$, $\forall r \in \mathbb{R}$. $\gamma_X(r) = \ln(q+pe^r)$

Consider $\Pr(S_n \geq n(p+\varepsilon)) = \Pr(\frac{1}{n}\sum_{i=1}^{n} X_i \geq p+\varepsilon)$

According to the Chernoff bounds, this prob $\leq \exp(n\mu_X(p+\varepsilon))$

where $\mu_X(p+\varepsilon) = \inf_{r>0} \{\gamma_X(r) - r(p+\varepsilon)\}$

$= \inf_{r>0} \{\ln(q+pe^r) - r(p+\varepsilon)\}$

After some differentiations: $r^* = \ln \frac{(p+\varepsilon)(1-p)}{(1-p-\varepsilon)p}$

For $\varepsilon > 0$, this $r^* > 0$. achieving the min over $r > 0$.

$\mu_X(p+\varepsilon) = (p+\varepsilon)\ln\frac{p}{p+\varepsilon} + (1-p-\varepsilon)\ln\frac{1-p}{1-p-\varepsilon} = -D(p+\varepsilon \| p) < 0.$

$\Rightarrow \Pr(\frac{1}{n}\sum_{i=1}^{n} X_i \geq p+\varepsilon) \leq \exp(-nD(p+\varepsilon\|p))$

If we want $\downarrow$ to be ≤ 0.005 & $\varepsilon = 0.01$, $p = q = \frac{1}{2}$

we need $n \geq \frac{\ln 200}{D(p+\varepsilon\|p)} \approx 263$ samples.

prob ≤ 0.01, we need 15000 samples.

Quiz 1: i) Next Friday from 8pm – 9pm.
ii) One sheet of handwritten notes (both sides)
iii) Everyone must show up in LT1 to take the quiz

Sec 1.7 Laws of large numbers.

Let $\{b_i\}_{i=1}^{\infty} \subset \mathbb{R}$ be a sequence of real numbers.
$b_i \to b$ as $i \to \infty$

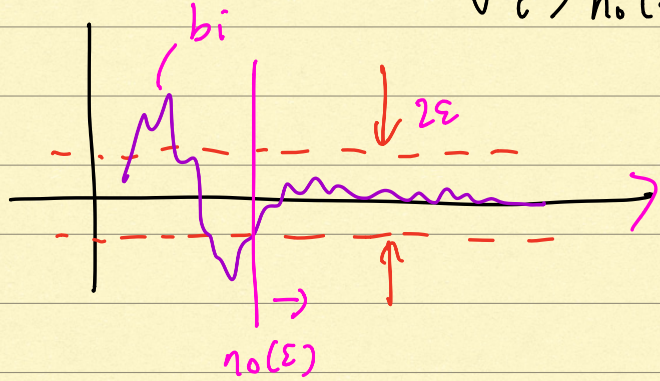
$\iff \forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t.

$|b_i - b| < \varepsilon \quad \forall i > n_0(\varepsilon)$.

e.g. $b_i = 1/i \quad \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right) \quad b_i \to 0 = b$

$\underline{\varepsilon = 10^{-6}}, \quad n_0(\varepsilon) = 10^6 + 1, \quad \frac{1}{i} < 10^{-6}$

$\forall i > n_0(\varepsilon) = 10^6 + 1$



Recall that rvs X_n are functions from the sample space Ω to $\mathbb{R}$ (i.e., $X_n(\omega) \in \mathbb{R}$).

Hence there are many ways a seq. of rvs $\{X_n\}$ can converge to a limiting rv.

<u>Weak law of large numbers</u> (WLLN)

<u>Thm</u>: $\forall n \geq 1$, let $S_n = X_1 + \cdots + X_n$ be the sum of n i.i.d. rvs each with finite variance $\sigma^2 < \infty$. Then $\forall \varepsilon > 0$

population av.

$$\lim_{n\to\infty} \Pr\left(\left| \frac{1}{n} S_n - \mathbb{E}X \right| > \varepsilon \right) = 0$$

empirical average

In other words, emp. av. $\frac{1}{n} S_n$ "converges in probability" to $\mathbb{E}X$.

$$\Leftrightarrow \qquad \frac{1}{n} S_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{p} \mathbb{E}X. \quad (n\to\infty)$$

$\left(\text{Ex: } \sigma^2 < \infty \Rightarrow \mathbb{E}X < \infty \right)$

Pf: By Chebyshev's Ineq.

$$\Pr\left(\left| \frac{1}{n} \sum_{i=1}^{n} (X_i - \mathbb{E}X) \right|^2 > \varepsilon^2 \right)$$

$$\leq \frac{\mathbb{E}\left[\frac{1}{n^2} \left| \sum_{i=1}^{n} (X_i - \mathbb{E}X) \right|^2 \right]}{\varepsilon^2}$$

$$= \frac{n\sigma^2}{n^2\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}.$$

$$\boxed{\begin{array}{c} Var\, Y = \mathbb{E}(Y^2) \\ Y \text{ z.m.} \end{array}}$$

$$\mathbb{E}\left[\left(\sum_{i=1}^{n}(X_i - \mathbb{E}X)\right)^2\right] = Var\left(\sum_{i=1}^{n}(X_i - \mathbb{E}X)\right)$$

$$= \sum_{i=1}^{n} Var(X_i - \cancel{\mathbb{E}X}) = n\sigma^2$$

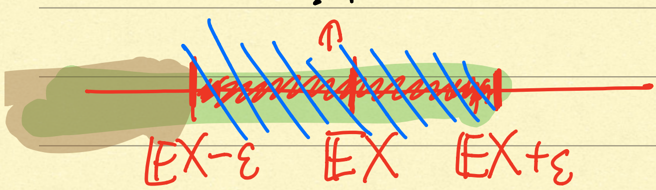
$$Pr\left(\left|\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mathbb{E}X\right| > \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2} = O\left(\frac{1}{n}\right) \to 0.$$

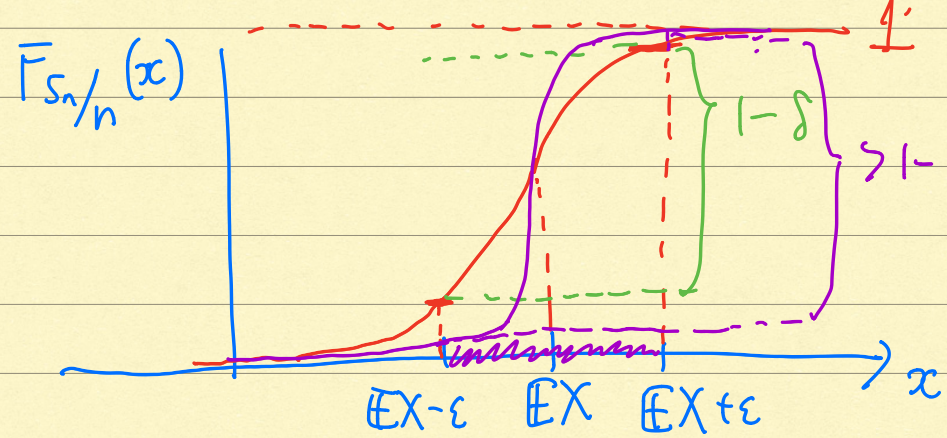
////

$$Pr\left(\left|\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}_{S_n/n} - \mathbb{E}X\right| \leq \varepsilon\right) = \boxed{F_{\frac{S_n}{n}}(\mathbb{E}X+\varepsilon) - F_{\frac{S_n}{n}}(\mathbb{E}X-\varepsilon)}$$

$$\| $$

$$Pr\left(\frac{S_n}{n} \leq \mathbb{E}X+\varepsilon\right)$$





Rmk: The bound $\underline{\sigma^2/n\varepsilon^2}$ is extremely loose in practice.
$\uparrow$ chebyshev

In fact for X s.t. MGF $g_X(r)$ exists.

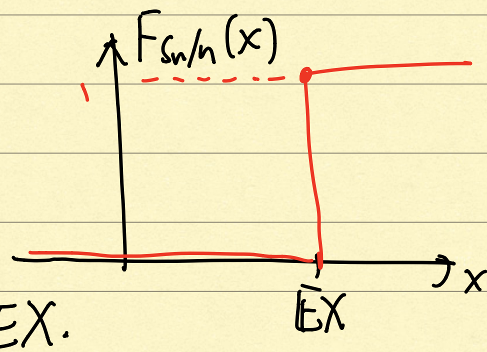
$$\Pr\left(\left|\frac{1}{n}S_n - \mathbb{E}X\right| > \varepsilon\right) \le \exp(n\mu_X(\mathbb{E}X+\varepsilon)) + \exp(n\mu_{-X}(\mathbb{E}X-\varepsilon))$$

$$\Pr\left(\frac{1}{n}S_n > \mathbb{E}X+\varepsilon\right) + \Pr\left(\frac{1}{n}S_n < \mathbb{E}X-\varepsilon\right)$$

$$\sim \exp(-nc) \qquad c > 0.$$

$\uparrow$ Chernoff. exponentially fast decay.

Central Limit Thm



What does $F_{S_n/n}$ look like when $n \to \infty$?

A step f^n that jumps at $\mathbb{E}X$.
Not interesting.

$$\{X_i\}_{i=1}^{\infty} \text{ i.i.d.} \qquad Z_n := \frac{1}{\sqrt{n}\,\sigma}\sum_{i=1}^{n}(X_i - \mathbb{E}X)$$

Rmk: $\mathbb{E}X = 0$, $\sigma = 1$, $Z_n = \sqrt{n}\left(\frac{S_n}{n}\right)$

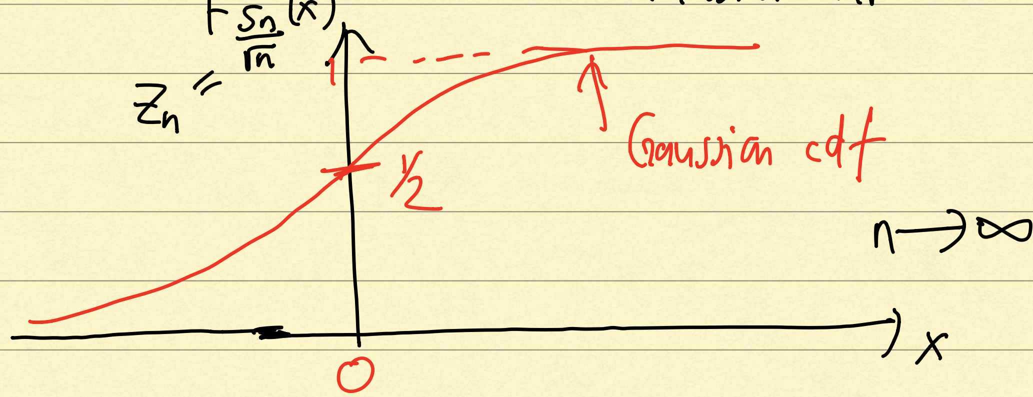
$$\mathbb{E}Z_n = \frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n}\mathbb{E}[X_i - \mathbb{E}X] = 0.$$

$$\mathrm{Var}\, Z_n = \frac{1}{n\sigma^2} \sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n\sigma^2}(n\sigma^2) = 1$$

<u>Thm</u>: (CLT) $\{X_i\}_{i=1}^{\infty}$ i.i.d. $\mathbb{E}X,\ \sigma^2 < \infty$.

$$\forall z \in \mathbb{R}, \quad \lim_{n\to\infty} \underbrace{\Pr(Z_n \le z)}_{F_{Z_n}(z)} = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du.$$

Gaussian cdf



$$\left(\text{Ex: } \gamma > 0,\ \frac{S_n}{n^{1/2+\gamma}} \to 0 \text{ in prob. if } \mathbb{E}X = 0 \right)$$

$Z_n \xrightarrow{d} N(0,1)$. "$Z_n$ converges to a standard Gaussian in dist^{n}".

<u>Pf sketch of CLT</u>: $\mathbb{E}X = 0$, $\mathrm{Var}\,X = 1$.

$$\mathbb{E}[e^{rZ_n}] = \mathbb{E}\left[e^{\frac{r}{\sqrt{n}}\sum_{i=1}^{n} X_i}\right] = \left(g_X\left(\tfrac{r}{\sqrt{n}}\right)\right)^n$$

$r \in \mathbb{R}$

$$\mathbb{E}\left[\prod_{i=1}^{n} e^{r/\sqrt{n}\, X_i}\right] = \prod_{i=1}^{n} \mathbb{E}\left[e^{r/\sqrt{n}\, X_i}\right]$$

$$\log \mathbb{E}[e^{rZ_n}] = \underbrace{\log g_X\left(\frac{r}{\sqrt{n}}\right)}_{\text{semi-invariant MGF}}$$

$$g_X(r/\sqrt{n}) \cong g_X(0) + \cancel{g_X'(0)}^{\mathbb{E}X=0 \to 0}\, r/\sqrt{n} + \frac{g_X''(0)}{2} \frac{r^2}{n} + \cancel{\cdots}, \qquad g_X''(0) = \mathbb{E}X^2 = 1$$

$$\cong 1 + \frac{1}{2}\frac{r^2}{n}.$$

$$\log \mathbb{E}[e^{rZ_n}] = \log\left(1 + \frac{1}{2}\frac{r^2}{n}\right) \simeq \frac{r^2}{2n}.$$

$$\log(1+\varepsilon) \approx \varepsilon$$

$$\mathbb{E}[e^{rZ_n}] \cong \left(\underbrace{e^{r^2/2\cancel{n}}}_{g_X(r/\sqrt{n})}\right)^{\cancel{n}} = e^{r^2/2}$$

$$\text{MGF of } Z_n \to \text{MGF } N(0,1) \qquad \text{MGF of } N(0,1).$$

pointwise

Lévi's continuity Theorem $\Downarrow$

$$F_{Z_n}(z) \to \Phi(z) \quad \forall z \in \mathbb{R}$$

Recall $\quad \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n) \xrightarrow{d} N(0,1)$

if $\mathbb{E}X_i = 0$ & $\mathrm{Var}\, X_i = 1$.

More generally. $\qquad X_i$ i.i.d.

Def: A sequence of rvs $\{Z_n\}_{n=1}^{\infty}$ converge in distribution Z if

$$\lim_{n\to\infty} F_{Z_n}(z) = F_Z(z)$$

for all points z of continuity of $F_Z(z)$

By CLT: $\left\{ Z_n = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (X_i - \mathbb{E}X) \right\}_{n=1}^{\infty}$

$$Z_n \xrightarrow{d} N(0,1)$$

Def: A sequence of rvs $\{Z_n\}_{n=1}^{\infty}$ <u>converges in probability</u> to Z if

$$\forall \varepsilon > 0, \quad \lim_{n\to\infty} \Pr\left(|Z_n - Z| > \varepsilon \right) = 0$$

Eg: $Z_n = \frac{1}{n} S_n \xrightarrow{P} \mathbb{E}X \qquad (\sigma^2 < \infty)$

$S_n = X_1 + \cdots + X_n$ i.i.d

Fact: Conv. in prob $\Rightarrow$ Conv. in distribution. (HW).

Def: A sequence of rvs $\{Z_n\}_{n=1}^{\infty}$ <u>converges in</u> <u>mean square</u> to Z if

$$\lim_{n\to\infty} \mathbb{E}\left[(Z_n - Z)^2 \right] = 0$$

Fact: $\frac{S_n}{n} \to \mathbb{E}X$ in ms. $(\sigma^2 < \infty)$

$S_n = X_1 + \cdots + X_n$ (X_i iid)

Pf: $\mathbb{E}\left[\left(\underbrace{\frac{S_n}{n} - \mathbb{E}X}_{\text{has zero mean}}\right)^2\right] = \mathrm{Var}\left(\frac{S_n}{n}\right) \to 0.$

$$\mathbb{E}\left[\frac{S_n}{n}\right] = \frac{1}{n}\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}X.$$

$$\mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i)$$

$$= \frac{\sigma^2}{n} \to 0.$$

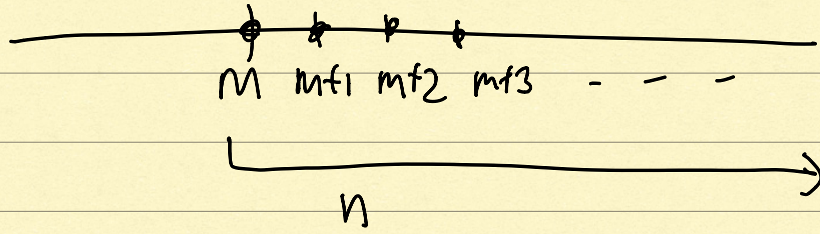
Def: $\{Z_n\}_{n=1}^{\infty}$ converges to Z with probability 1 (almost surely) if

$$\Pr\left(\left\{\omega \in \Omega : \lim_{n\to\infty} Z_n(\omega) = Z(\omega)\right\}\right) = 1.$$

$Z_n \xrightarrow{a.s.} Z$ $\qquad Z_n \to Z$ wp 1.

Fact: $Z_n \xrightarrow{a.s.} Z$ if $\forall \varepsilon > 0$,

$$\lim_{m\to\infty} \Pr\left(\underline{|Z_n - Z| < \varepsilon} \text{ for all } n \geq m\right) = 1$$



Rmk: Conv. wp 1 means that the set of sample paths $w \in \Omega$ that $Z_n(w)$ converges to $Z(w)$ (in the sense of a sequence of numbers converging to a limit) has prob. 1.

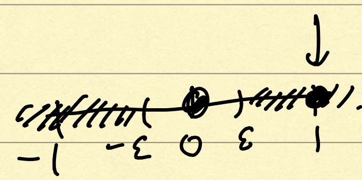
Ex: $$X_n = \begin{cases} 0 & \text{w.p. } 1-\frac{1}{n} \\ 1 & \text{w.p. } \frac{1}{n} \end{cases}$$

$\{X_n\}$ independent.

Claim: i) $X_n \xrightarrow{P} 0$.

ii) $X_n \xrightarrow{ms} 0$.

iii) $X_n \not\to 0$ a.s.



Pf of i) $\Pr(|X_n - 0| \geq \varepsilon)$ $\quad \varepsilon \in (0,1)$

$= \Pr(X_n = 1) = 1/n \to 0.$

$\therefore$ By the def^{n} of conv. in prob. $X_n \xrightarrow{P} 0$. ///.

ii) $\mathbb{E}[(X_n - 0)^2] = \mathbb{E} X_n = 0 \cdot (1-\frac{1}{n}) + 1 \cdot \frac{1}{n}$

$= \frac{1}{n} \to 0.$

By the def^{n} of conv. in ms. $X_n \xrightarrow{ms} 0$.

iii)

ε small $\quad Pr(|X_n - 0| < \varepsilon \text{ for all } n \geq m) =: a_m.$

$$a_m = Pr(X_n = 0 \text{ for all } n \geq m)$$

$$= Pr(X_m = 0, X_{m+1} = 0, X_{m+2} = 0, \dots)$$

$$a_m = \lim_{k\to\infty} \prod_{n=m}^{k} Pr(X_n = 0)$$

$$= \lim_{k\to\infty} \prod_{n=m}^{k} \left(1 - \frac{1}{n}\right) = \lim_{k\to\infty} \prod_{n=m}^{k} \frac{n-1}{n}$$

$$= \lim_{k\to\infty} \frac{m-1}{\cancel{m}} \; \frac{\cancel{m}}{\cancel{m+1}} \; \frac{\cancel{m+1}}{\cancel{m+2}} \; \cancel{\cdots} \; \frac{\cancel{k-1}}{k}$$

$$= \lim_{k\to\infty} \frac{m-1}{k} = 0 \quad \text{for every fixed } m \in \mathbb{N}.$$

$$a_m = Pr(X_n = 0 \text{ for all } n \geq m) = 0 \quad \forall m \in \mathbb{N}.$$

$$\lim_{m\to\infty} a_m = 0 \neq 1.$$

ms conv

$\Downarrow$

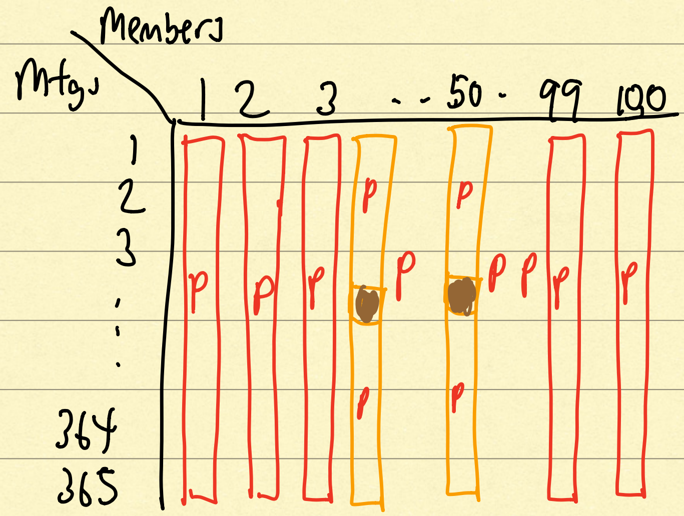
a.s conv (wp 1) $\Rightarrow$ conv in prob $\underset{HW}{\Rightarrow}$ conv dist^{n}.

Analogue: Committee of 100 members.
Each member is represented by $\omega \in \Omega$.

One mtg involving the committee everyday.
throughout the year.

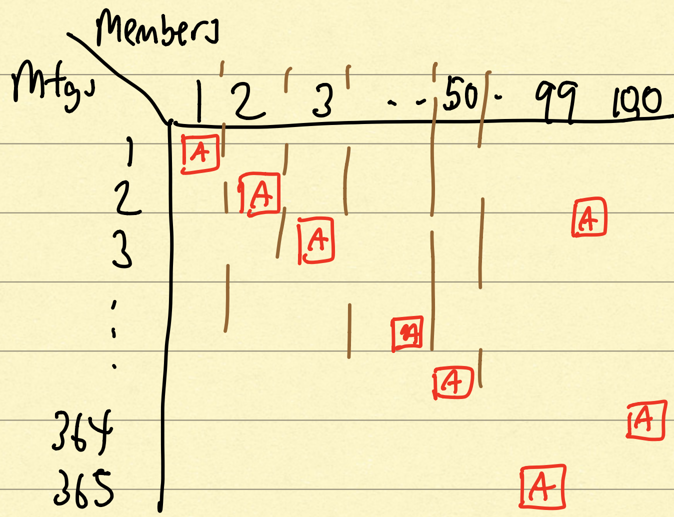
Interested in the attendance of the committee members

Conv. a.s. $\Rightarrow$ Almost all members have perfect attendance over the 365 days.



$$\Pr\left(\left\{\omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Conv. in prob: All meetings were almost all full.

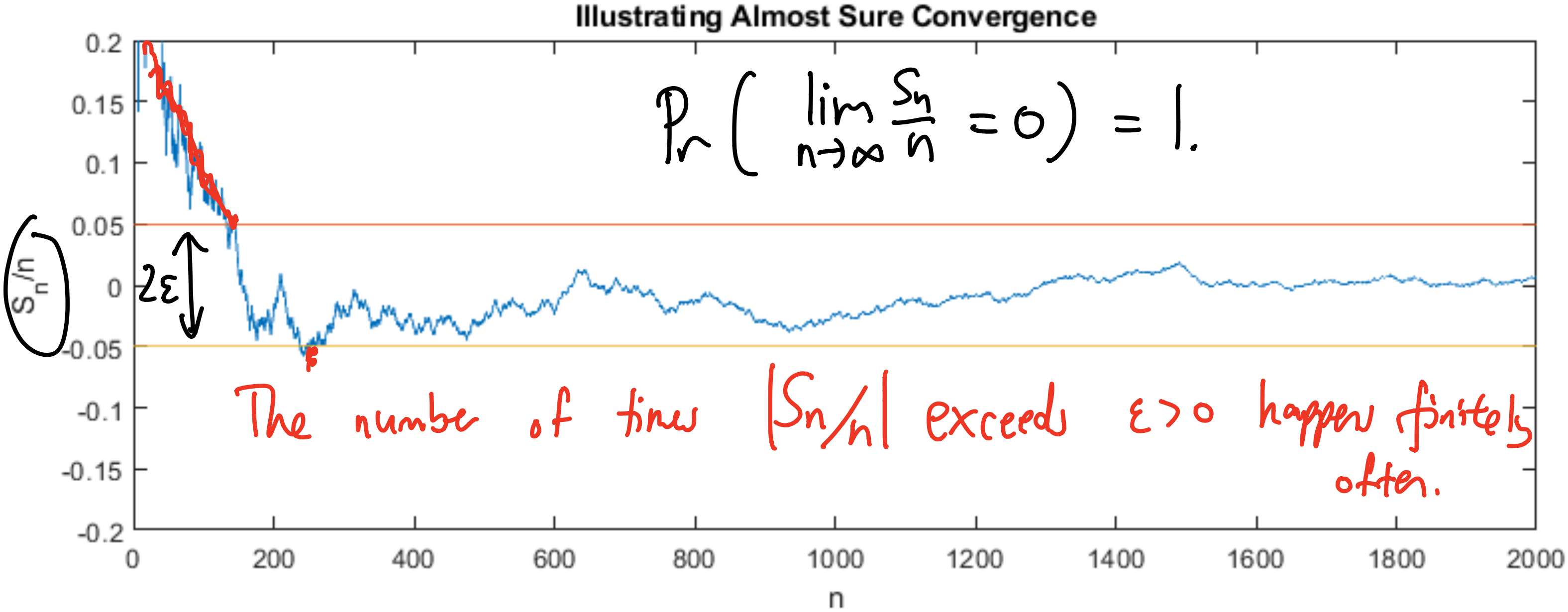


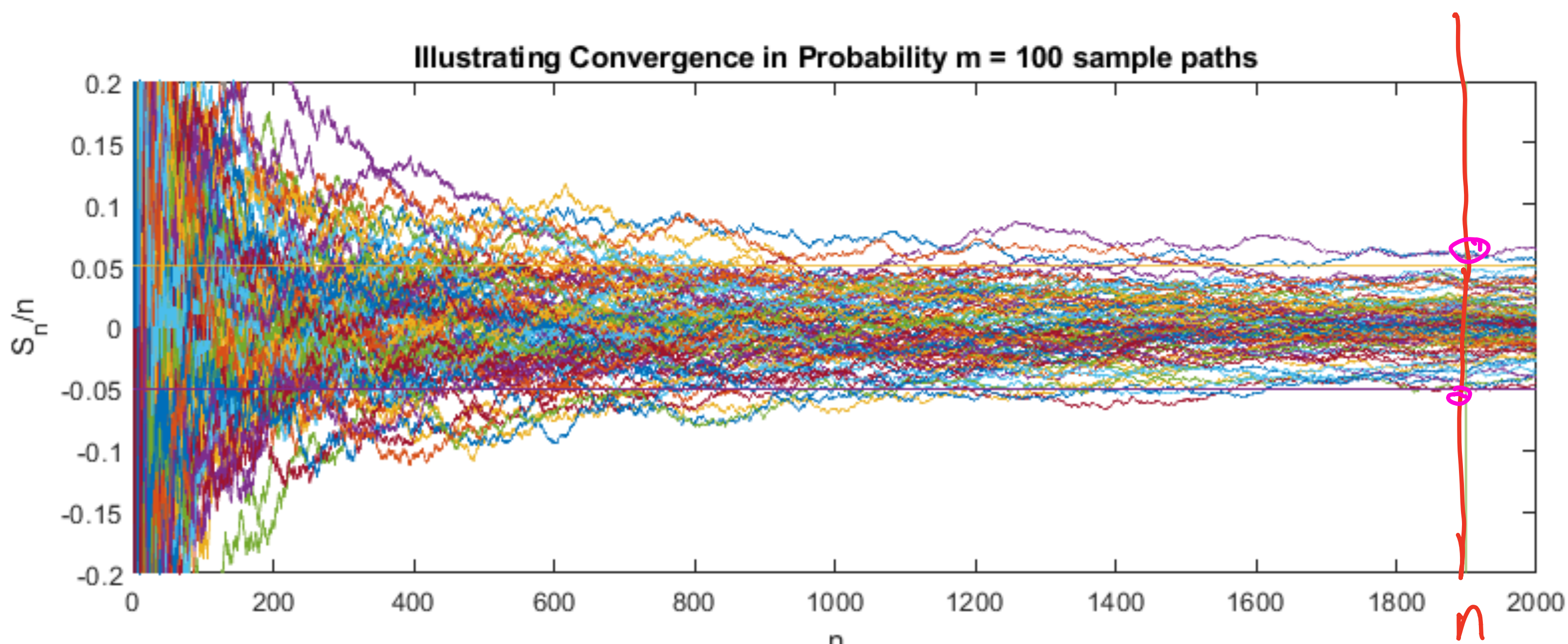
Set of elements $\omega \in \Omega$ for which $X_n(\omega) \approx X(\omega)$ has prob $\to 1$

Note that if all mtgs were nearly full, it may not be the case that any member has perfect attendance.

$$\boxed{\begin{aligned} \prod_{i=1}^{\infty} a_i &= \lim_{k \to \infty} \prod_{i=1}^{k} a_i \\ \sum_{i=1}^{\infty} a_i &= \lim_{k \to \infty} \sum_{i=1}^{k} a_i \end{aligned}}$$

$$X_n = \begin{cases} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases} \qquad S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$





$$\lim_{n\to\infty} \Pr\left(\left|\frac{S_n}{n} - 0\right| > \varepsilon\right) = 0.$$

The fraction of sample paths that exceed $\pm\varepsilon$ is small