National University of Singapore
Department of Mechanical Engineering

ME5401/MCH5201/EE5101 Linear System 2021/2022

Tutorial 3

---

1. Assume that a SISO system $\{A,B,C,D\}$ is expressed in controllable canonical form. Show that the controllability matrix has a simple form of

$$U = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 1 & e_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & 1 & e_1 & \cdots & e_{n-2} \\ 1 & e_1 & \cdots & \cdots & e_{n-1} \end{bmatrix}$$

where $e_k = -\sum_{i=0}^{k-1} \alpha_{i+1} e_{k-i-1}$, $k=1,2,\ldots,n-1$; $e_0 = 1$ and α_is are the coefficients of the characteristic equation $s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n = 0$ of A. What can you say about the controllability of the system. Verify also that

$$U^{-1} = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \vdots & \cdots & 1 & 0 \\ \vdots & \vdots & \cdots & 0 & \vdots \\ \alpha_1 & 1 & \cdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

2. Consider a SISO system expressed in controllable canonical form. Find the transfer function representation of the system.

3. Using the eigenvalue test, check if the following system is asymptotically stable

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x(t)$$

4. Using Lyapunov stability analysis, check if the following system is asymptotically stable.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x(t).$$

5. Consider the following systems. Is the realization minimal? Is it BIBO stable? Is it asymptotically stable? Explain your answer.

(i)
$$\dot{x} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

(ii)
$$\dot{x} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

6. This result is used in the section of uncontrollable system where the equivalence of the transfer functions of the original system and the controllable subsystem is established. Verify that if A,B,C,D are respectively, $n\times n$, $n\times m$, $m\times n$ and $m\times m$ matrices, then

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}.$$

7. Given that A and B are $n\times n$ and $n\times m$ matrices. Is it true that the rank of $[B \;\; AB \;\; A^2B \; \ldots \; A^{n-1}B]$ equals the rank of $[AB \;\; A^2B \;\; \ldots. \;\; A^nB]$? If so, state the reason. If not, state the condition under which it would.