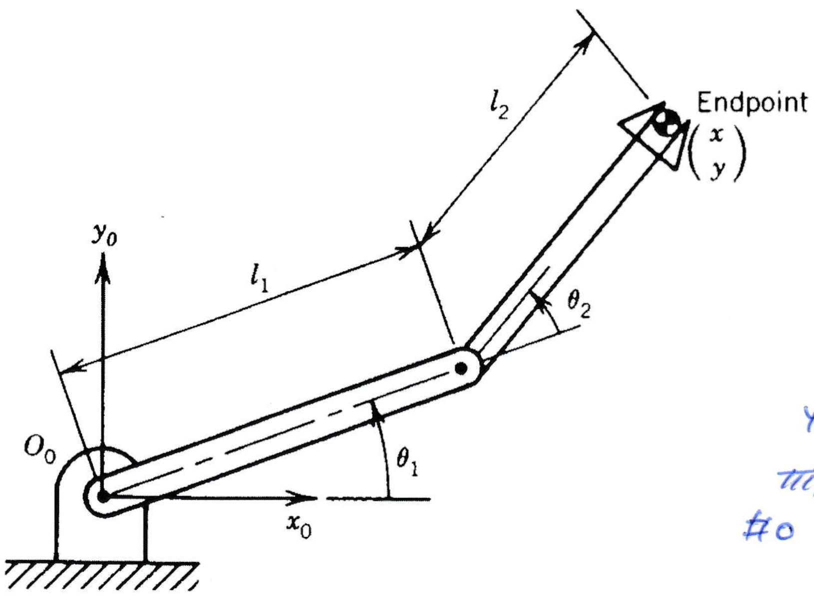
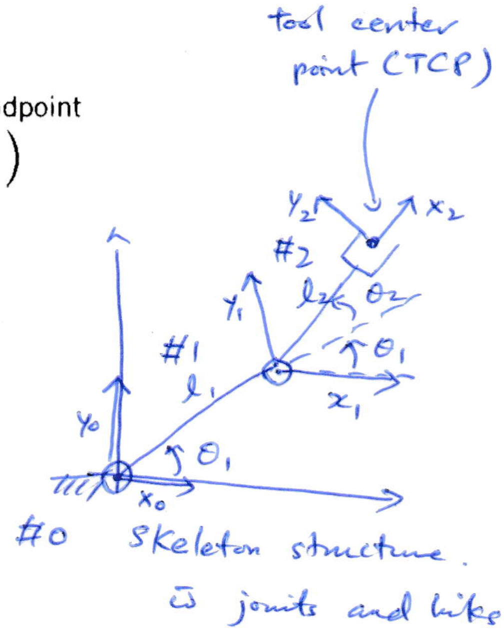
Planar Robot Kinematics

Most of the robot mechanisms of practical importance can be treated as planar mechanisms or reduced to planar problems. Planar mechanisms is generally simpler to analyse, and its underlying definitions and ideas can be extended to the 3D and general spatial mechanism.





2-Link RR Planar Manipulator

DH representation ("standard", classic):

(or ${}^0_1T$ or T^0_1 or A^0_1)

Link	d_i	θ_i	a_i	α_i
1	0	θ_1	l_1	0
2	0	θ_2	l_2 *	0

$$ {}^0_1A = \begin{bmatrix} c_1 & -s_1 & 0 & c_1 l_1 \\ s_1 & c_1 & 0 & s_1 l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2A = \begin{bmatrix} c_2 & -s_2 & 0 & c_2 l_2 \\ s_2 & c_2 & 0 & s_2 l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Kinematic equation of the manipulator

$$ {}^0_2T = {}^0_1A\,{}^1_2A = \begin{bmatrix} c_1 & -s_1 & 0 & c_1 l_1 \\ s_1 & c_1 & 0 & s_1 l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 & c_2 l_2 \\ s_2 & c_2 & 0 & s_2 l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(or ${}^0_2T(\theta_1, \theta_2)$)

$$ = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & 0 & c_1c_2l_2 - s_1s_2l_2 + c_1l_1 \\ s_1c_2 + c_1s_2 & -s_1s_2 + c_1c_2 & 0 & s_1c_2l_2 + c_1s_2l_2 + s_1l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

* we define the link length of the most distal link from most distal joint axis to a reference point (TCP)

Trigonometric identities:

sine and cosine for the sum ~~or~~ difference of angles θ_1 and θ_2

$$\cos(\theta_1 + \theta_2) = c_{12} = c_1 c_2 - s_1 s_2$$

$$\sin(\theta_1 + \theta_2) = s_{12} = c_1 s_2 + s_1 c_2$$

$$\cos(\theta_1 - \theta_2) = c_1 c_2 + s_1 s_2$$

$$\sin(\theta_1 - \theta_2) = s_1 c_2 - c_1 s_2$$

$$c_1 c_2 - s_1 s_2 = c_{12}$$

$$-c_1 s_2 - s_1 c_2 = -(s_2 c_1 + s_1 c_2) = -s_{12}$$

$$c_1 c_2 l_2 - s_1 s_2 l_2 + c_1 l_1 = l_2(c_1 c_2 - s_1 s_2) + c_1 l_1$$

$$= l_2 c_{12} + c_1 l_1$$

$$s_1 c_2 + c_1 s_2 = s_{12}$$

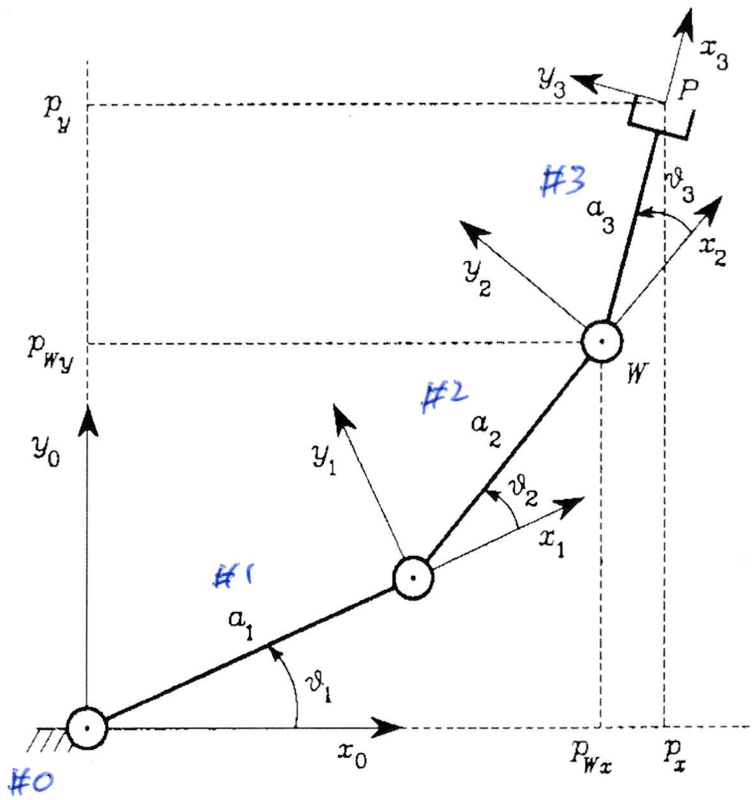
$$-s_1 s_2 + c_1 c_2 = c_1 c_2 - s_1 s_2 = c_{12}$$

$$s_1 c_1 l_2 + c_1 s_2 l_2 + s_1 l_1 = l_2(s_1 c_1 + c_1 s_2) + s_1 l_1$$

$$= l_2 s_{12} + s_1 l_1$$

$$\therefore \; {}^{0}_{2}T(\theta_1, \theta_2) = \begin{bmatrix} c_{12} & -s_{12} & 0 & c_1 l_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & s_1 l_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2-3 of Chapter 1 Kinematics



Planar 3R (or RRR) manipulator

Example :
shoulder swivel, elbow extension and pitch of Cincinnati Milacron T3 robot ;
Three revolve joints of 4 DOF SCARA manipulator ignoring the prismatic joint for gripper.

DH representation :

Link	d_i	θ_i	a_i	α_i
1	0	θ_1	a_1	0
2	0	θ_2	a_2	0
3	0	θ_3	a_3	0

Direct kinematics :

$$ {}^0_3T(\theta_1, \theta_2, \theta_3) = {}^0_1A \, {}^1_2A \, {}^2_3A $$

$$ = \begin{bmatrix} C_{123} & -S_{123} & 0 & a_1C_1 + a_2C_{12} + a_3C_{123} \\ S_{123} & C_{123} & 0 & a_1S_1 + a_2S_{12} + a_3S_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

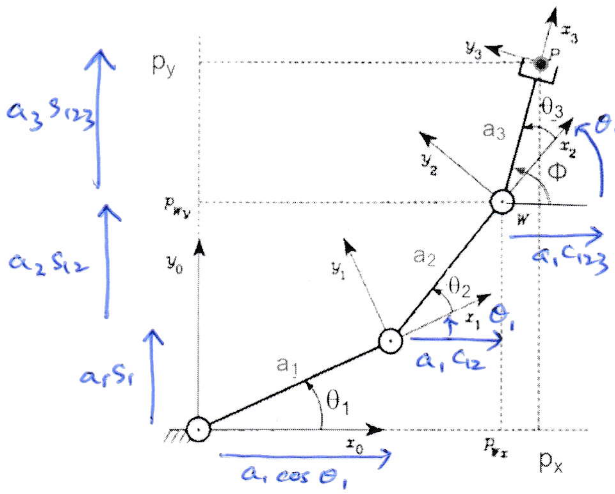
To compute the joint coordinates for a given set of end effector coordinates.

Inverse Kinematics

...Example 2-3:

- Solution:

The direct kinematics equation can be written in the following form:



$$\mathbf{x} = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix} \quad (2\text{-}3)$$

Cartesian coordinates of $P = (p_x, p_y)$ ← position

$$\left.\begin{aligned} p_x &= a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ p_y &= a_1 s_1 + a_2 s_{12} + a_3 s_{123} \end{aligned}\right\} \text{non linear}$$

orientation

$$\phi = \theta_1 + \theta_2 + \theta_3 \quad) \text{ linear}$$

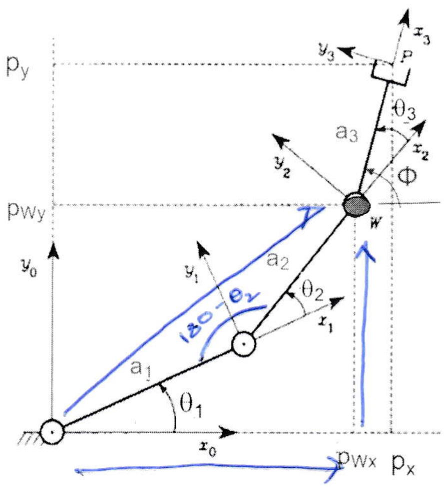
note: all angles measured counter clockwise and link lengths are assumed +ve going from one joint axis to the next joint axis.

Analytical method

Given the Cartesian coordinates of x, y and ϕ,
find the analytical expressions for joint angles θ_1, θ_2 and θ_3 in terms of Cartesian coordinates.

Inverse Kinematics

…Example 2-3:



To find θ_2:

wrist position

From (2-3), position of point W (origin of Frame 2):

$$p_{Wx} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12}$$
$$p_{Wy} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12} \quad \text{(2-4)}$$

=>

$$p_{Wx}^{\,2} + p_{Wy}^{\,2} = a_1^{\,2} + a_2^{\,2} + 2a_1 a_2 c_2$$ ← law of cosines

=>

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

Existence of a solution $\Leftrightarrow$ $-1 \le \dfrac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2} \le 1$

Set $s_2 = \pm\sqrt{1 - c_2^2}$

And $\theta_2 = A\tan 2(s_2, c_2)$

Two-step approach:

① Find the position of the wrist, W from p_x, p_y, and ϕ

② Find θ_1, θ_2 from W position, θ_3 can be determined from W.

From Eq (2-4) and trigonometric identities

If a triangle's angles are labeled a, b, and c where a is opposite side A, and so on, "law of cosines" is

$$A^2 = B^2 + C^2 - 2BC\cos a$$

$$-1 \le \cos\theta_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2} \le 1$$

$\cos(-180°)$ — $\cos 180°$

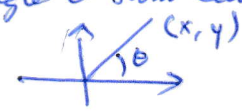
trigonometric identities: $c^2\theta + s^2\theta = 1$

$$c_2^{\,2} + s_2^{\,2} = 1 \Rightarrow s_2 = \pm\sqrt{1 - c_2^2}$$

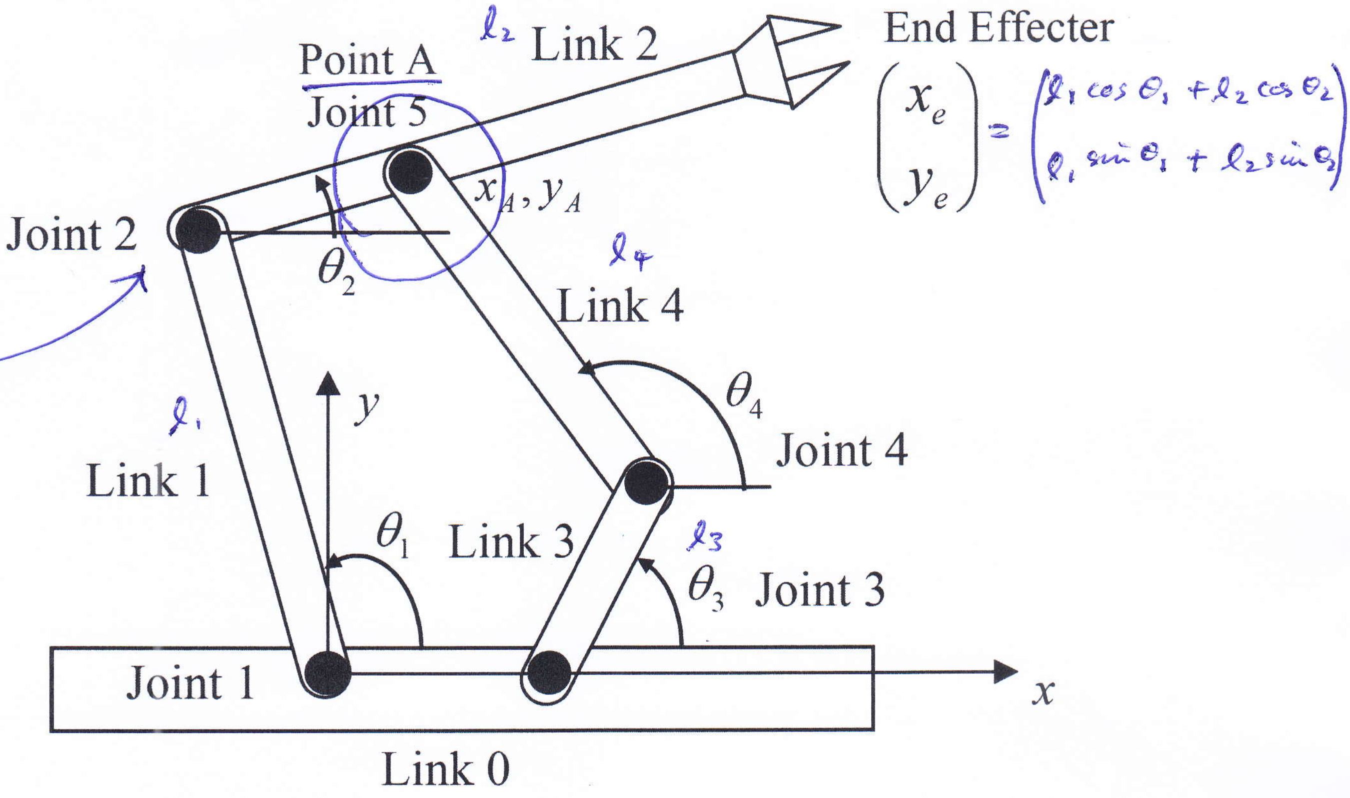
trigonometric

$\theta = A\tan 2(y, x)$ or $\arctan 2(y, x)$ converts angle θ from Cartesian to polar coordinate (r, θ)

and $x = r\cos\theta$, $y = r\sin\theta$



(Example: 5-bar-link planar robot (parallel link mechanism)



Inverse kinematics problem: to find θ_1, θ_3 for desired endpoint position (x_e, y_e)

Algorithm:

Step 1: Given x_e, y_e, find θ_1, θ_2 by solving the two-link inverse kinematics problem.

Step 2: Given θ_1, θ_2, obtain x_A, y_A — forward kinematics problem

Step 3: Given x_A, y_A, find θ_3, θ_4 by solving another two-link inverse kinematics problem.

(Source: Introduction to Robotics, H. Harry Asada).

2-Link RR Planar Manipulator

Jacobian matrix relating (x_e, y_e) w/ (θ_1, θ_2)

== Differential Relationship

endpoint (x, y)

$$x_e(\theta_1, \theta_2) = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$

$$y_e(\theta_1, \theta_2) = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

Small movements :

$$dx_e = \frac{\partial x_e(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial x_e(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

$$dy_e = \frac{\partial y_e(\theta_1, \theta_2)}{\partial \theta_1} d\theta_1 + \frac{\partial y_e(\theta_1, \theta_2)}{\partial \theta_2} d\theta_2$$

$$\underbrace{\begin{pmatrix} dx_e \\ dy_e \end{pmatrix}}_{dx} = J \cdot \underbrace{\begin{pmatrix} d\theta_1 \\ d\theta_2 \end{pmatrix}}_{dq}$$

$$dx = J \cdot dq$$

$$J = \begin{pmatrix} \frac{\partial x_e(\theta_1, \theta_2)}{\partial \theta_1} & \frac{\partial x_e(\theta_1, \theta_2)}{\partial \theta_2} \\ \frac{\partial y_e(\theta_1, \theta_2)}{\partial \theta_1} & \frac{\partial y_e(\theta_1, \theta_2)}{\partial \theta_2} \end{pmatrix}$$

$$= \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

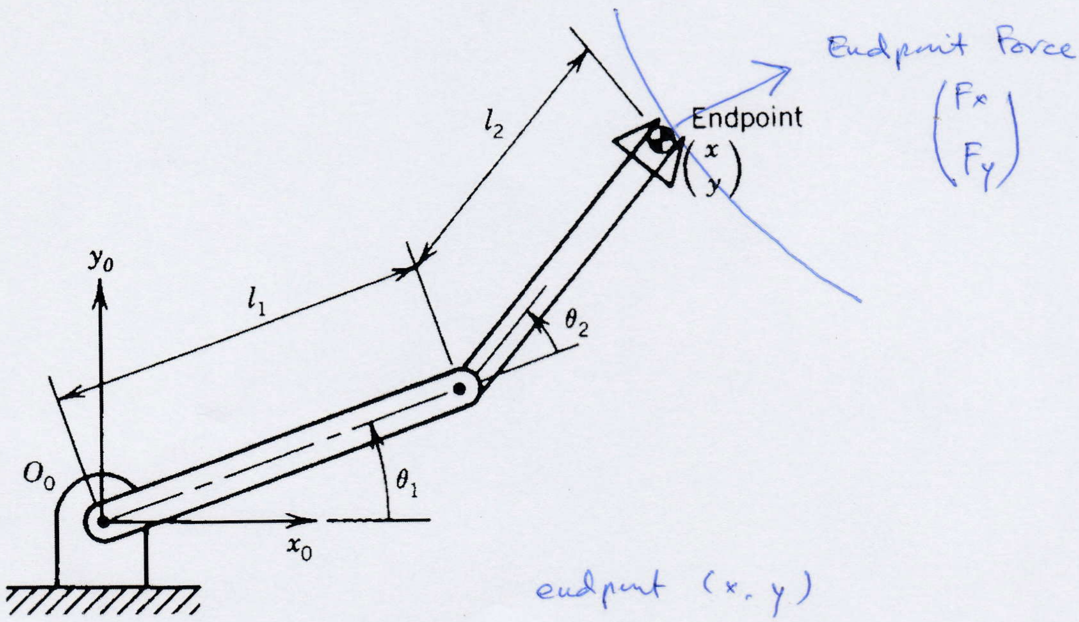
At the instant, $\dot{q} = (\dot{\theta}_1, \dot{\theta}_2)^T$

$\dot{v}_e = (\dot{x}_e, \dot{y}_e)$ is the resultant end-effector velocity vector.

$$\frac{dx}{dt} = J \frac{dq}{dt} \Rightarrow v_e = J \cdot \dot{q}$$

Jacobian determines the velocity relationship between the joints and end-effector.

The robot is interacting with a surface in a horizontal plane.
Obtain the equivalent joint torques $\tau = (\tau_1, \tau_2)^T$ for pushing the surface with endpoint force $F = (F_x, F_y)^T$.



Jacobian matrix relating (x_e, y_e) with (θ_1, θ_2)

$$J = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = J^T \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

$$= \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ -l_2\sin(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

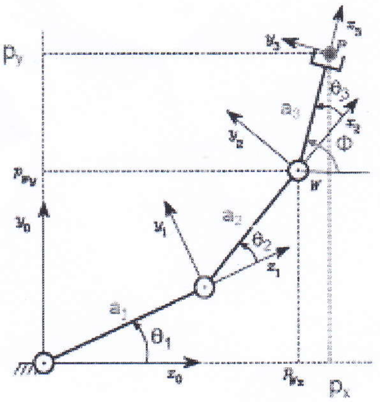
Differential Relationships of 3-link planar robot.

Inverse Kinematics

...Example 2-3:

- Solution:

The direct kinematics equation can be written in the following form:



$$\mathbf{x} = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix} \quad (2\text{-}3) \quad = \begin{bmatrix} x \\ y \\ \phi \end{bmatrix}$$

$$\dot{x} = -l_1\dot{\theta}_1 s_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2)s_{12} - l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)s_{123}$$

$$\dot{y} = l_1\dot{\theta}_1 c_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2)c_{12} + l_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)c_{123}$$

$$\dot{\phi} = (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -(l_1s_1 + l_2s_{12} + l_3s_{123}) & -(l_2s_{12} + l_3s_{123}) & -l_3s_{123} \\ (l_1c_1 + l_2c_{12} + l_3c_{123}) & (l_2c_{12} + l_3c_{123}) & l_3c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial \phi}{\partial \theta_1} & \frac{\partial \phi}{\partial \theta_2} & \frac{\partial \phi}{\partial \theta_3} \end{bmatrix}$$

Given $\dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$; $\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}$

If J is non singular, $\dot{q} = J^{-1}\dot{x}$

J is singular when determinant of $J = l_1 l_2 \sin\theta_2 = 0$

$\Rightarrow \theta_2$ is either 0 or 180°

APPENDIX A

Trigonometric identities

Formulas for rotation about the principal axes by θ:

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad (A.1)$$

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad (A.2)$$

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (A.3)$$

Identities having to do with the periodic nature of sine and cosine:

$$\begin{aligned} \sin\theta &= -\sin(-\theta) = -\cos(\theta + 90°) = \cos(\theta - 90°), \\ \cos\theta &= \cos(-\theta) = \sin(\theta + 90°) = -\sin(\theta - 90°). \end{aligned} \quad (A.4)$$

The sine and cosine for the sum or difference of angles θ_1 and θ_2:

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= c_{12} = c_1c_2 - s_1s_2, \\ \sin(\theta_1 + \theta_2) &= s_{12} = c_1s_2 + s_1c_2, \\ \cos(\theta_1 - \theta_2) &= c_1c_2 + s_1s_2, \\ \sin(\theta_1 - \theta_2) &= s_1c_2 - c_1s_2. \end{aligned} \quad (A.5)$$

The sum of the squares of the sine and cosine of the same angle is unity:

$$c^2\theta + s^2\theta = 1. \quad (A.6)$$

If a triangle's angles are labeled a, b, and c, where angle a is opposite side A, and so on, then the "law of cosines" is

$$A^2 = B^2 + C^2 - 2BC\cos a. \quad (A.7)$$

The "tangent of the half angle" substitution:

$$\begin{aligned} u &= \tan\frac{\theta}{2}, \\ \cos\theta &= \frac{1-u^2}{1+u^2}, \\ \sin\theta &= \frac{2u}{1+u^2}. \end{aligned} \quad (A.8)$$