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1. (i) and (ii) are true; (iii) and (iv) are wrong.

P.f.: (i) $S_n \le t$, this mean that n^{th} arrival occured at some time $\tau \le t$

$\Rightarrow N(\tau) = n$, $N(t) \ge N(\tau) = n$.

$\Rightarrow \{S_n \le t\} \subseteq \{N(t) \ge n\}$ ①

$N(t) \ge n$, this mean there are m arrivals at time t, $N(t) = m$.

$m \ge n$.

$\Rightarrow S_m \le t$, because $m \ge n$

$\Rightarrow S_n \le S_m \le t$.

$\{N(t) \ge n\} \subseteq \{S_n \le t\}$ ②

Combine ① and ②, we canget $\{N(t) \ge n\} = \{S_n \le t\}$;

(ii) Since (i) is true, (ii) is true by taking complement.

(iii). When, $S_n < t < S_{n+1}$, $N(t) = n \subseteq \{N(t) \le n\}$,

But $S_n < t$, so (iii) is false.

(iv) ~~Since (iii) is not true, it~~

When $S_n < t < S_{n+1}$, $N(t) = n$, so (iv) is false

2. Because there are 5 courts, the time of #1 arrival is a exponentially distrivuted with mean:

$$E(X) = \frac{40 \text{ min}}{5} = 8 \text{ min}$$

If there ae k other pairs waiting in queue, the new pairs can get a court at $k+1$ times.

$$E(t) = E(X)\cdot(k+1) = 8(k+1) \text{ mins}$$

3.

a: $P_r\{N(t)=n \mid S_n=\tau\} = P_r\{X_{n+1} > t-\tau\} = 1 - \int_0^{t-\tau} \lambda e^{-\lambda x} dx = e^{-\lambda(t-\tau)}$

b. $P_r\{N(t)=n\} = \int_0^t P_r\{N(t)=n \mid S_n=\tau\}\cdot f_{S_n}(\tau)\, d\tau$

$$= \int_0^t e^{-\lambda(t-\tau)} \cdot \frac{\lambda^n \cdot \tau^{n-1} \cdot e^{-\lambda\tau}}{(n-1)!} d\tau$$

$$= \frac{\lambda^n \cdot e^{-\lambda t}}{(n-1)!} \cdot \int_0^t \tau^{n-1} d\tau$$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \cdot \frac{1}{n} \tau^n \Big|_0^t$$

$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

4. Prove that Geometric distribution has the memoryless property:

Pf: The memoryless defination:

$$P_r(X > t+x) = P_r(X>t) \cdot P_r(X>x)$$

$$LHS = \sum_{i=t+x+1}^{\infty} (1-p)^{i-1} p$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} p - \sum_{i=1}^{t+x} (1-p)^{i-1} p$$

$$= \frac{p}{1-(1-p)} - \frac{(1-p)^{t+x} \cdot p - p}{(1-p)-1}$$

$$= 1 + [(1-p)^{t+x} - 1]$$

$$= (1-p)^{t+x}$$

$$RHS = P_r(X>t) \cdot P_r(X>x)$$

$$= (1-p)^t (1-p)^x$$

$$= (1-p)^{x+t} = LHS.$$

3. Because X_n is a Bionmial random variable,

$$\cancel{P_r(X_n = i) = \binom{n}{k}(nP_n)^i(1-}$$

$$P_r(X_n = i) = \binom{n}{k} P_n^i (1-P_n)^{n-i}$$

$$= \frac{n!}{i!(n-i)!} P_n^i (1-P_n)^{n-i}$$

$$= \frac{n\cdot(n-1)\cdots(n-i+1)}{i!} \cdot \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{n^i} \cdot \frac{\lambda^i}{i!} \cdot \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

when $n \to \infty$, $n(n-1)\cdots(n-i+1) = \underbrace{n\cdot n\cdots n}_{i} = n^i$

$$\left(1-\frac{\lambda}{n}\right)^i \xrightarrow[n\to\infty]{} 1$$

$$\left(1-\frac{\lambda}{n}\right)^n \xrightarrow[n\to\infty]{} e^{\lambda}.$$

$$RHS = \frac{n^i}{n^i} \cdot \frac{\lambda^i}{i!} \cdot \frac{e^{\lambda}}{1} = \frac{\lambda^i e^{\lambda}}{i!}$$

6. ~~Lea~~

P.f: Let $a = |A|$, $b = |B|$, $c = |C| = |A \cap B|$

If A and B are independ, we can get:

$$Pr(C) = Pr(A) \cdot Pr(B)$$

$$\frac{c}{p} = \frac{a}{p} \cdot \frac{b}{p}.$$

$$pc = a \cdot b.$$

since p is prime, at least one of a and b is divisible by p

Because $0 \le a, b \le p$, ~~so~~ at least one of a and b is equal to 0 or p

7. Let $Y = \cancel{X} - \frac{a}{2}$. Because $Pr(0 \le X \le a) = 1$, so $0 \le X \le a$

$-\frac{a}{2} = 0 - \frac{a}{2} \le Y \le a - \frac{a}{2} = \frac{a}{2}$, $|Y| \le \frac{a}{2}$

Since X and Y only differ by a constant, $Var(X) = Var(Y)$

$$Var(X) = Var(Y) = E(Y^2) - (E[Y])^2 \le E[Y^2] \le \frac{a^2}{4}$$

8. We want to find the distribution of $M_n - \frac{1}{\lambda}\log n$.

$\cancel{P\{M_n - \frac{1}{\lambda}\log}$

$$F_X(x) = P_r\{M_n - \frac{1}{\lambda}\log n \le x\}$$

$$= P_r\{M_n \le x + \frac{1}{\lambda}\log n\}. \qquad ①$$

Because $M_n = \max\{X_1, \dots X_n\}$. If ① is real. that means every $X_i \le x + \frac{1}{\lambda}\log n$. Because $\{X_n\}_{n=1}^{\infty}$ is i.i.d. then:

$$RHS = \prod_{n=1}^{\infty} Pr\{X_n \le x + \frac{1}{\lambda}\log n\}.$$

Because $\{X_n\}_{n=1}^{\infty}$ is in exponetial distribution,

We know $Pr\{X_n \le x\} = 1 - e^{-\lambda x}$.

$$RHS = \prod_{n=1}^{\infty} \left[1 - e^{-\lambda\cdot(x + \frac{1}{\lambda}\log n)}\right] = \left[1 - e^{-\lambda\cdot(x + \frac{1}{\lambda}\log n)}\right]^n$$

$$= \left[1 - \frac{e^{-\lambda x}}{n}\right]^n$$

$$= e^{e^{-\lambda x}} \quad (n \to \infty).$$