National University of Singapore
Faculty of Engineering

ME5402/EE5016R Advanced Robotics

Sample Solution to Exercise 1

There can be more than one correct answer/solution to some of the questions. Email me at mpecck@nus.edu.sg if you have found any errors in the solution. – CK Chui

1. Define kinematics, workspace, and trajectory.

Answer

Kinematics is the study of position and derivatives of position without regard to forces, which cause the motion.

Workspace is the locus of positions and orientations achievable by the end-effector of a manipulator.

Trajectory is a time based function which specifies the position (and higher derivatives) of the robot mechanism for any value of time.

2. Define 'frame' and "degrees of freedom".

Answer

Frame is a coordinate system, usually specified in position and orientation relative to another (reference) frame.

Degrees of freedom is the number of independent variables, which must be specified in order to completely locate all members of a (rigid-body) mechanism.

3. A vector $^{A}\mathbf{P}$ is rotated about $\mathbf{z_A}$ by θ degrees and is subsequently rotated about $\mathbf{x_A}$ by ϕ degrees. Give the rotation matrix that accomplishes these rotations in the given order.

Solution

$$
\begin{aligned}
R &= rot(x,\phi)rot(z,\theta) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta & -s\theta & 0 \\ c\phi s\theta & c\phi c\theta & -s\phi \\ s\phi s\theta & s\phi c\theta & c\phi \end{bmatrix}
\end{aligned}
$$

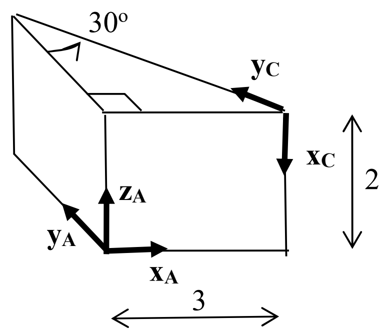
4. A frame {B} is located initially coincident with a frame {A}. We rotate {B} about $\mathbf{z_B}$ by θ degrees, and then we rotate the resulting frame about $\mathbf{x_B}$ by ϕ degrees. Give the rotation matrix that will change the descriptions of vectors from $^{B}\mathbf{P}$ to $^{A}\mathbf{P}$.

Solution

Since rotations are performed about axes of the frame being rotated, these are Euler-Angle style rotations:

$$
\begin{aligned}
{}^{A}_{B}R &= rot(z,\theta)rot(x,\phi) \\
&= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \\
&= \begin{bmatrix} c\theta & -s\theta c\phi & s\theta s\phi \\ s\theta & c\theta c\phi & -c\theta s\phi \\ 0 & s\phi & c\phi \end{bmatrix}
\end{aligned}
$$

5. Referring to the following figure, determine the homogeneous transformation matrix that describes frame {C} in frame {A}. Also determine the homogeneous transformation matrix that describes frame {A} in frame {C}.



Solution

$^{A}\mathbf{x}_{C} = [0\ 0\ \text{-}1]^{T}$ $\quad ^{A}\mathbf{y}_{C} = [\text{-}\cos 60^{\circ}\ \sin 60^{\circ}\ 0]^{T}$ $\quad ^{A}\mathbf{z}_{C} = [\sin 60^{\circ}\ \cos 60^{\circ}\ 0]^{T}$

$^{A}\mathbf{p}_{CORG} = [\ 3\ 0\ 2]^{T}$

$$
{}^{A}_{C}T = \begin{bmatrix} 0 & -0.5 & 0.866 & 3 \\ 0 & 0.866 & 0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{C}_{A}T = \begin{bmatrix} {}^{A}_{C}R^{T} & -{}^{A}_{C}R^{T}\,{}^{A}P_{CORG} \\ 000 & 1 \end{bmatrix}
$$

$$
= \cdots
$$

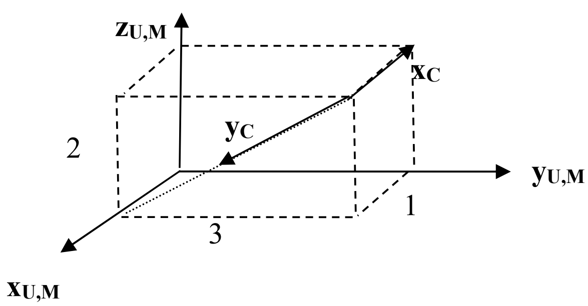
$$
= \begin{bmatrix} 0 & 0 & -1 & 2 \\ -0.5 & 0.866 & 0 & 1.5 \\ 0.866 & 0.5 & 0 & -2.598 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

6. Frames {M} and {C} are attached rigidly to a cuboid as shown in the following figure. Frame U is fixed and serves as the universal frame of reference. The cube undergoes the following motions in the indicated sequence:

   1) Rotation about the z axis of frame {C} by 30^o , then
   2) Translation of (1,2,3) along frame {C}, then
   3) Rotation about the x axis of frame {M} by 45^o, and then
   4) Rotation about the y axis of frame {U} by 60^o.

   Let ${}^{U}_{C_i}T$ and ${}^{U}_{M_i}T$ be the 4x4 homogeneous transformation matrices that describe the position and orientation of frame C and M, respectively, in U after motion *i*.

   Find ${}^{U}_{C_1}T$, ${}^{U}_{C_2}T$, ${}^{U}_{C_3}T$, ${}^{U}_{C_4}T$, ${}^{U}_{M_4}T$.



Solution

First find ${}^{U}_{C}T$ before the motions:

${}^{U}\mathbf{x}_C = [-1\ 0\ 0]^T$, ${}^{U}\mathbf{y}_C = \dfrac{1}{\sqrt{3^2+2^2}}[0\ {-3}\ {-2}]^T$, ${}^{U}\mathbf{z}_C = {}^{U}\mathbf{x}_C \times {}^{U}\mathbf{y}_C$

${}^{U}\mathbf{p}_C = [1\ 3\ 2]^T$

$$ {}^{M}_{C}T = {}^{U}_{C}T = \left[\begin{array}{ccc:c} & & & 1 \\ {}^{U}\mathbf{x}_C & {}^{U}\mathbf{y}_C & {}^{U}\mathbf{z}_C & 3 \\ & & & 2 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] $$

After motion 1:

$$ {}^{U}_{C_1}T = {}^{U}_{C}T\,{}^{C}_{C_1}T = {}^{U}_{C}T.Rot(\mathbf{z},30^o) = \cdots = \begin{bmatrix} -0.866 & 0.5 & 0 & 1 \\ -0.416 & -0.721 & -0.555 & 3 \\ -0.277 & -0.48 & 0.832 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

After motion 2:

$${}^{U}_{C_2}T = {}^{U}_{C_1}T\,{}^{C_1}_{C_2}T = {}^{U}_{C_1}T.Tran(1,2,3) = {}^{U}_{C_1}T.\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \cdots = \begin{bmatrix} -0.866 & 0.5 & 0 & 1.134 \\ -0.416 & -0.721 & -0.555 & -0.521 \\ -0.277 & -0.48 & 0.832 & 3.258 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After motion 3:

$${}^{U}_{M_3}T = {}^{U}_{M_2}T.Rot(\mathbf{x}, 45^o)$$

$$\begin{aligned}
{}^{U}_{C_3}T &= {}^{U}_{M_3}T\,{}^{M_3}_{C_3}T \\
&= {}^{U}_{M_2}T.Rot(\mathbf{x}, 45^o)\,{}^{M}_{C}T \quad \left(\text{Note: } {}^{M_3}_{C_3}T = {}^{M}_{C}T\right) \\
&= {}^{U}_{C_2}T.{}^{C_2}_{M_2}T.Rot(\mathbf{x}, 45^o)\,{}^{M}_{C}T \quad \left(\text{Note: } {}^{U}_{M_2}T = {}^{U}_{C_2}T.{}^{C_2}_{M_2}T\right) \\
&= {}^{U}_{C_2}T.{}^{C}_{M}T.Rot(\mathbf{x}, 45^o)\,{}^{M}_{C}T \quad \left(\text{Note: } {}^{C_2}_{M_2}T = {}^{C}_{M}T\right) \\
&= \cdots \\
&= \begin{bmatrix} -0.866 & 0.354 & 0.354 & 1.662 \\ -0.416 & -0.117 & -0.902 & -2.696 \\ -0.277 & -0.928 & 0.249 & 4.872 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

After motion 4:

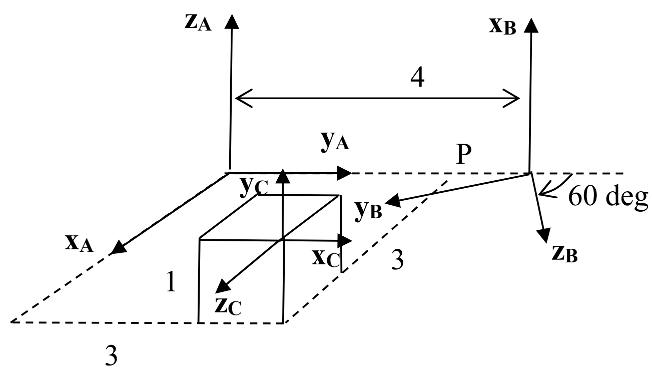
$${}^{U}_{C_4}T = Rot(\mathbf{y}, 60^o).{}^{U}_{C_3}T = \cdots = \begin{bmatrix} -0.6732 & -0.627 & 0.392 & 5.05 \\ -0.416 & -0.117 & -0.902 & -2.696 \\ 0.611 & -0.77 & -0.182 & 0.997 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{U}_{M_4}T = {}^{U}_{C_4}T\,{}^{C}_{M}T = \cdots = \begin{bmatrix} 0.6732 & 0.304 & 0.674 & 2.117 \\ 0.416 & 0.598 & -0.685 & -3.535 \\ -0.611 & 0.742 & 0.276 & -1.169 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. Frame {C} is firmly attached to a corner of the rigid cube with $\mathbf{z_c}$ parallel to $\mathbf{x_A}$ and $\mathbf{y_C}$ parallel to $\mathbf{z_A}$, as shown in the figure below. Frame {B} is located at a fixed position and orientation with respect to Frame {A} with $\mathbf{x_B}$ parallel to $\mathbf{z_A}$ and the angle 60 degrees represents a rotation about $\mathbf{x_B}$. The following ordered sequence of motions is applied to the cube:

   I) rotation about $\mathbf{y}_B$ by 45 degrees, followed by
   II) rotation about $\mathbf{x}_C$ by 30 degrees.

   Find the new position and orientation of Frame {C} expressed in Frame {A}.



<u>Solution</u>

${}^{A}\mathbf{x}_B = [0\ 0\ 1]^T$, ${}^{A}\mathbf{y}_B = [\cos60^o\ {-\sin60^o}\ \ 0]^T$, ${}^{A}\mathbf{z}_B = [\sin60^o\ \cos60^o\ \ 0]^T$
${}^{A}\mathbf{p}_B = [0\ 4\ 0]^T$
$\Rightarrow {}^{A}_{B}T$

${}^{A}\mathbf{x}_C = [0\ 1\ 0]^T$, ${}^{A}\mathbf{y}_C = [0\ 0\ 1]^T$, ${}^{A}\mathbf{z}_C = [1\ 0\ 0]^T$
${}^{A}\mathbf{p}_C = [3\ 3\ 1]^T$
$\Rightarrow {}^{A}_{C}T$

${}^{B}_{C}T = {}^{A}_{B}T^{-1}.{}^{A}_{C}T$ (Before the sequence of motion)

After motion I:

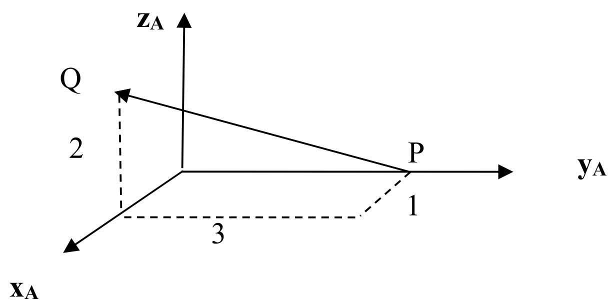
$${}^{B}_{C_1}T = Rot(\mathbf{y},45^o).{}^{B}_{C}T$$

After motion II:

$${}^{B}_{C_2}T = {}^{B}_{C_1}T.Rot(\mathbf{x},30^o) = Rot(\mathbf{y},45^o).{}^{B}_{C}T.Rot(\mathbf{x},30^o)$$

$${}^{A}_{C_2}T = {}^{A}_{B}T\,{}^{B}_{C_2}T = {}^{A}_{B}T.Rot(\mathbf{y},45^o).{}^{B}_{C}T.Rot(\mathbf{x},30^o) = \begin{bmatrix} -0.127 & -0.14 & 0.982 & 1.855 \\ 0.927 & -0.370 & 0.067 & 2.339 \\ 0.354 & 0.918 & 0.177 & 2.190 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8. Frame {B} is initially coincident to frame {A} in the following figure. Frame {B} is then rotated 30 degrees about the vector described by the directed line segment from P to Q (following the right-hand rule). Determine the position and orientation of the new frame {B} with respect to frame {A}. Express your answer in the form of a homogeneous transformation matrix.



Solution

Several ways to solve this problem!

Let's use the approach in Example 1-9. That is, we define two new frames, {A'} and {B'} which are obtained by translating {A} and {B}, respectively, to point P before performing the rotation. {B} and {B'} can be treated as if they are mounted on the same rigid body, whereas {A} and {A'} are fixed in the space.

$$ {}^{A}_{A'}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B'}_{B}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Note that the point P is on the axis of rotation. Now, let's rotate {B'} relative to {A'}. This is a rotation about an axis which passes through the origin, so we may use

$$ {}^{A'}R_{\mathbf{k}}(\theta) = \begin{pmatrix} k_x k_x v\theta + c\theta & k_y k_x v\theta - k_z\, s\theta & k_z k_x\, v\theta + k_y\, s\theta \\ k_x k_y v\theta + k_z\, s\theta & k_y k_y v\theta + c\theta & k_z k_y v\theta - k_x s\theta \\ k_x k_z v\theta - k_y\, s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{pmatrix} $$

to compute {B'} relative to {A'} after the rotation.

Here, ${}^{A'}\mathbf{k} = \dfrac{1}{\sqrt{1+9+4}}$ [1 -3 2]^T = [0.2673 -0.8018 0.5345]^T

$$ {}^{A'}_{B'}T = \left[\begin{array}{ccc|c} & & & 0 \\ & {}^{A'}R_{\mathbf{k}}\left(30^o\right) & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = $$

$$\begin{bmatrix} 0.8756 & -0.2960 & -0.3818 & 0 \\ 0.2386 & 0.9522 & -0.1910 & 0 \\ 0.4200 & 0.0762 & 0.9043 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, ${}^{A}_{B}T = {}^{A}_{A'}T\,{}^{A'}_{B'}T\,{}^{B'}_{B}T =$

$$\begin{bmatrix} 0.8756 & -0.2960 & -0.3818 & 0.8879 \\ 0.2386 & 0.9522 & -0.1910 & 0.1435 \\ 0.4200 & 0.0762 & 0.9043 & -0.2286 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}$$