EE5104/6104 Adaptive Control System (Part 2)

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TOPICS:

Sliding Control

Auto-Tuning

Self-Oscillating Adaptive Systems

Real-Time Parameter Estimation

Recommended Text

Åstrom Karl J and Bjorn Wittenmark, "Adaptive Control," 2nd Edition, 1995

Lennart Ljung, "System Identification: Theory for the User," 2nd Edition, 1999

Hassan K. Khalil, "Nonlinear Systems," 3rd Edition, 2002

SLIDING CONTROL

This is a special version of on-off control. The key idea is to apply strong control action when the system deviates from the desired behavior.

Assume that the system we want to control is described by the nonlinear equation

$$\frac{d^n y}{dt^n} = f_1(y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}) + g_1(y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}})u \tag{1}$$

Select the states

$$x = [\, x_1 \; x_2 \; \ldots \; x_{n-1} \; x_n \,]^T = [\, \tfrac{d^{n-1}y}{dt^{n-1}} \; \tfrac{d^{n-2}y}{dt^{n-2}} \; \ldots \; \tfrac{dy}{dt} \; y \,]^T \tag{2}$$

and rewrite the system as

$$
\begin{aligned}
\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 \; 0 \; \ldots \; 0 \; 0 \\ 1 \; 0 \; \ldots \; 0 \; 0 \\ \vdots \\ 0 \; 0 \; \ldots \; 1 \; 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(x) + g_1(x)u \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} f_1(x) \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} g_1(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \\
&= f(x) + g(x)u \qquad (3) \\
y &= [0 \; 0 \; \ldots \; 0 \; 1]x \qquad (4)
\end{aligned}
$$

where $f(x)$ and $g(x)$ are vectors.

Let the switching surface be

$$\sigma(x) = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = p^T x = 0$$

Using the definition of the state vector

$$\sigma(x) = p_1 \frac{d^{n-1}y}{dt^{n-1}} + p_2 \frac{d^{n-2}y}{dt^{n-2}} + \cdots + p_n y = 0 \tag{5}$$

The dynamic behavior on the sliding surface can be specified by a proper choice of the number p_i. It will be stable if the polynomial

$$P(s) = p_1 s^{n-1} + p_2 s^{n-2} + \cdots + p_n \tag{6}$$

has all its roots in the left-half plane.

To determine a control law that keeps the system on $\sigma(x) = 0$, we introduce the Lyapunov function

$$V(x) = \sigma^2(x)/2 \tag{7}$$

The time derivative of V is given by

$$\frac{dV}{dt} = \sigma(x)\dot{\sigma}(x) = \sigma p^T \dot{x} = \sigma(p^T f(x) + p^T g(x)u(t)) \tag{8}$$

Choose the control law

$$u(t) = -\frac{p^T f}{p^T g} - \frac{\mu}{p^T g}\mathrm{sign}(\sigma(x)) \tag{9}$$

so that

$$\frac{dV}{dt} = -\mu\sigma(x)\mathrm{sign}(\sigma(x)) \tag{10}$$

is always negative for $\sigma(x) \neq 0$.

Steady State Response

This must mean that at steady state ($t \to \infty$): (i) from Equations (7) and (10), $\sigma(x) = 0$, (ii) from Equation (5) y = 0.

Transient Response

Assume that the system has initial values $\sigma(x) = \sigma_0 > 0$, and let t_σ be the time when the switching surface is reached. From Equations (8) and (10) we find that

$$\dot{\sigma}(x) = -\mu$$

Integrating this equation

$$\int_{\sigma_0}^{0} d\sigma(x) = -\int_0^{t_\sigma} \mu dt$$

gives

$$0 - \sigma_0 = -\mu(t_\sigma - 0)$$

which gives $t_\sigma = \sigma_0/\mu$. Using the same arguments for $\sigma_0 < 0$ shows that $t_\sigma = |\sigma_0|/\mu$. The subspace $\sigma(x) = 0$ is asymptotically stable, and the state will stay on the switching surface once it is reached. The motion along the surface is determined by Equation (5).

Uncertainties in f and g can be handled if μ is sufficiently large. Assume that the design of the control law is based on the approximate values $\hat{f}$ and $\hat{g}$ instead of the true ones. Then

$$\frac{dV}{dt} = \sigma \left(\frac{p^T(f\hat{g}^T - \hat{f}g^T)p}{p^T\hat{g}} - \mu \frac{p^T g}{p^T \hat{g}} \operatorname{sign}(\sigma) \right)$$

The right-hand side is negative if μ is sufficiently large, provided that $p^T\hat{g}$ and $p^T g$ have the same sign. The system will thus be insensitive to uncertainties in the process model.

Smooth Control Laws

The control law (9) has the drawback that the relay chatters. One way to avoid this is to make the relay characteristics smoother. The sign function in Equation (9) is now replaced by the saturation function

$$\mathrm{sat}(\sigma, \varepsilon) = \begin{cases} 1 & \sigma > \varepsilon \\ \sigma/\varepsilon & -\varepsilon \leq \sigma \leq \varepsilon \\ -1 & \sigma < -\varepsilon \end{cases}$$

The control law is then

$$u(t) = \frac{p^T f}{p^T g} - \frac{\mu}{p^T g}\mathrm{sat}(\sigma(x), \varepsilon) \tag{11}$$

Example

Consider the unstable system

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = Ax + Bu \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}$$

which has the transfer function

$$G(s) = \frac{1}{s(s-1)}$$

Choose

$$\sigma(x) = p_1 x_1 + p_2 x_2 = x_1 + x_2 = \frac{dy}{dt} + y = 0$$

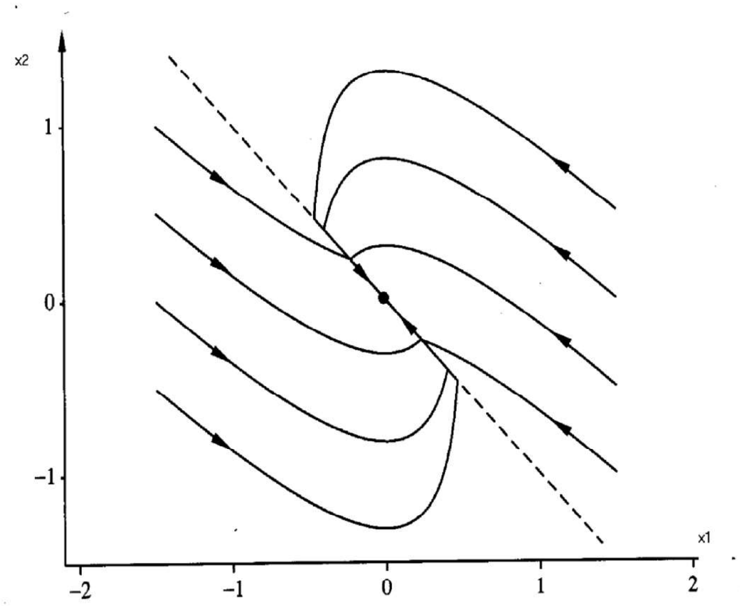
The controller from Equation (9) is now

$$\begin{aligned} u(t) &= -\frac{p^T A x}{p^T B} - \mu\,\mathrm{sign}(\sigma(x)) \\ &= -[\,2\;\;0\,]x(t) - \mu\,\mathrm{sign}(\sigma(x)) \end{aligned}$$

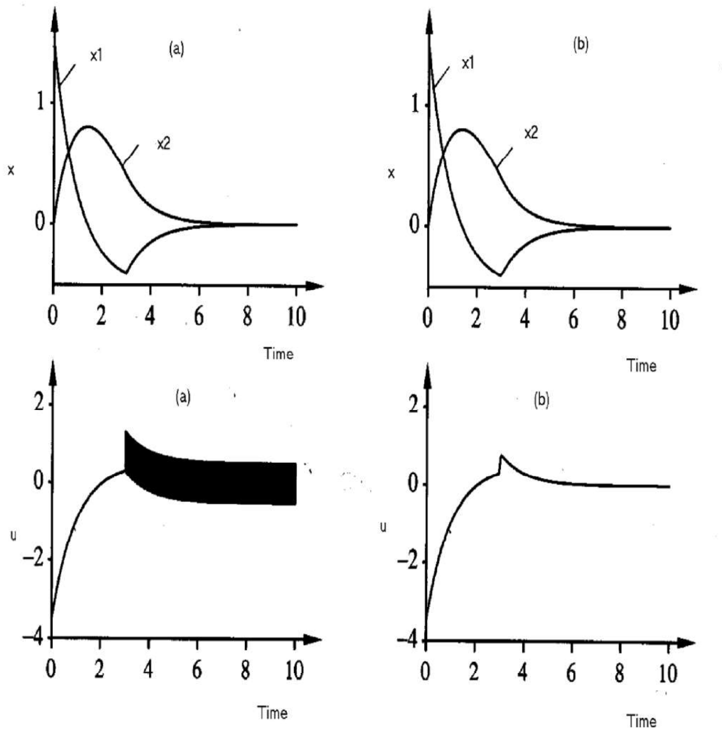
The smooth control law is

$$u(t) = -[\,2\;\;0\,]x(t) - \mu\,\mathrm{sat}(\sigma(x))$$

The phase plane when $\mu = 0.5$ is shown below. The input and output for both control laws are also shown below.



Phase portrait of the system in the Example. The dashed line shows $\sigma(x) = 0$.



The states and the output as a function of time in the Example. The initial conditions are $x_1(0) = 1.5$ and $x_2(0) = 0$. The controllers are (a) Eq. (9) with $\sigma = 0.5$; (b) Eq. (11) with $\sigma = 0.5$ and $\varepsilon = 0.01$.

DESCRIBING FUNCTION ANALYSIS

For some nonlinear systems and under certain conditions, an extended version of the frequency response method, called the describing function method can be used to approximately analyze and predict nonlinear behaviour. The main use of describing function method is for the prediction of limit cycles in nonlinear systems.

The applicability to limit cycle analysis is due to the fact that the form of the signals in a limit-cycling system is usually approximately sinusoidal. Consider a sinusoidal input to the nonlinear element, of amplitude A and frequency ω, i.e. $e(t) = A\sin(\omega t)$, as shown in the Figure below

The output of the nonlinear component $c(t)$ is often a periodic func-

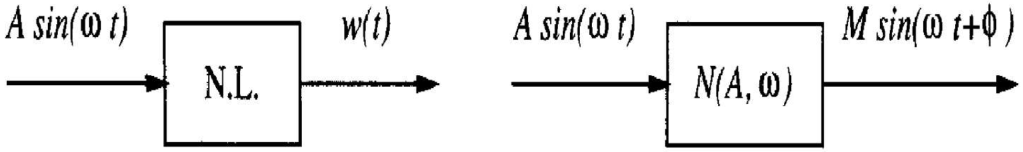


Figure 1:

tion. Using Fourier series, this periodic function can be expanded as

$$c(t) \approx a_1 \cos(\omega t) + b_1 \sin(\omega t) = M \sin(\omega t + \phi)$$

where

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} c(t) \cos(\omega t) d(\omega t) && (1) \\ b_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} c(t) \sin(\omega t) d(\omega t) && (2) \\ M &= \sqrt{a_1^2 + b_1^2} \\ \phi &= \arctan(\frac{a_1}{b_1}) \end{aligned}$$

In complex representation, this sinusoid can be written as

$$c \approx M\angle\phi$$

Similar to the concept of frequency response function, which is the frequency domain ratio of the sinusoidal input and the sinusoidal output of a system, the describing function of the nonlinear element is the complex ratio of the fundamental component of the input and output signals:

$$N(A,\omega) = \frac{M\angle\phi}{A\angle 0} = \frac{1}{A}(b_1 + ja_1)$$

The nonlinear element, in the presence of sinusoidal input, can be treated as if it were a linear element with a frequency response function $N(A,\omega)$. The describing function, unlike linear system frequency response transfer function, changes with input amplitude A.

Computing Describing Functions

Analytical Calculation

When the nonlinear characteristics $c = f(e)$ are given by an explicit function and the integration in Equations (1) and (2) can be carried out, then analytical evaluation of the describing function based on Equations (1) and (2) is desirable.

Numerical Integration

For nonlinearities whose input-output relationship $c = f(e)$ is given by graphs or tables, numerical integration can be used to evaluate the describing functions. The idea is, of course to approximate the integral in Equations (1) and (2) by discrete sums over small intervals.

Experimental Evaluation

When a system nonlinearity can be isolated and excited with sinusoidal inputs of known amplitude and frequency, experimental determination of the describing function can be obtained by using a harmonic analyzer on the output of the nonlinear element. This is quite similar to the experimental determination of frequency response functions for linear elements. The difference here is that not only the frequencies, but also the amplitudes of the input sinusoidal should be varied. The results of of the experiments are a set of curves on the complex plane representing the describing function $N(A, \omega)$, instead of analytical expressions.

Relay

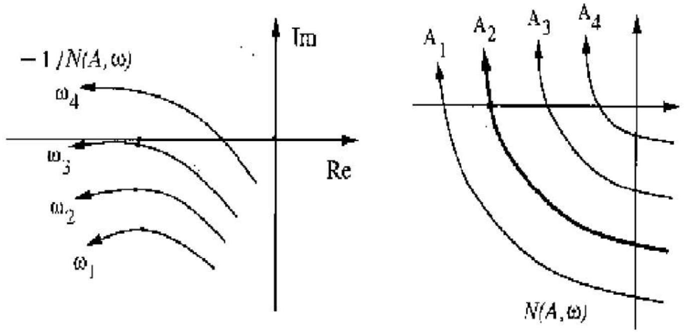


Figure 2:

$$c(t) = \begin{cases} M & 0 \le \omega t \le \pi \\ -M & \pi < \omega t \le 2\pi \end{cases}$$

$$b_1 = \frac{4}{\pi} \int_0^{\pi/2} M \sin(\omega t) d(\omega t) = \frac{4}{\pi} M$$

Describing function:

$$N(A) = \frac{4M}{\pi A}$$

Existence of Limit Cycle

Similar to Linear system condition for sustained oscillation, nonlinear system limit cycling is obtained when

$$G(j\omega)N(A,\omega) = -1$$

which can be written as

$$G(j\omega) = -\frac{1}{N(A,\omega)}$$

AUTO-TUNING

Adaptive schemes like Model Reference Adaptive Control and Self-Tuning Regulator require a priori information about the process dynamics. It is particularly important to know time scales, which are critical for determining suitable sampling intervals and filtering. From the user's point of view it would be ideal to have an auto-tuning function in which the regulator can be tuned simply by pushing a button. Special techniques for automatic tuning of simple regulators were therefore developed. These techniques are also useful for providing pre-tuning of more complicated adaptive systems. They can be characterized as crude robust methods that provide ball park information. They are thus ideal complements to the more sophisticated adaptive methods.

AUTO-TUNING TECHNIQUES

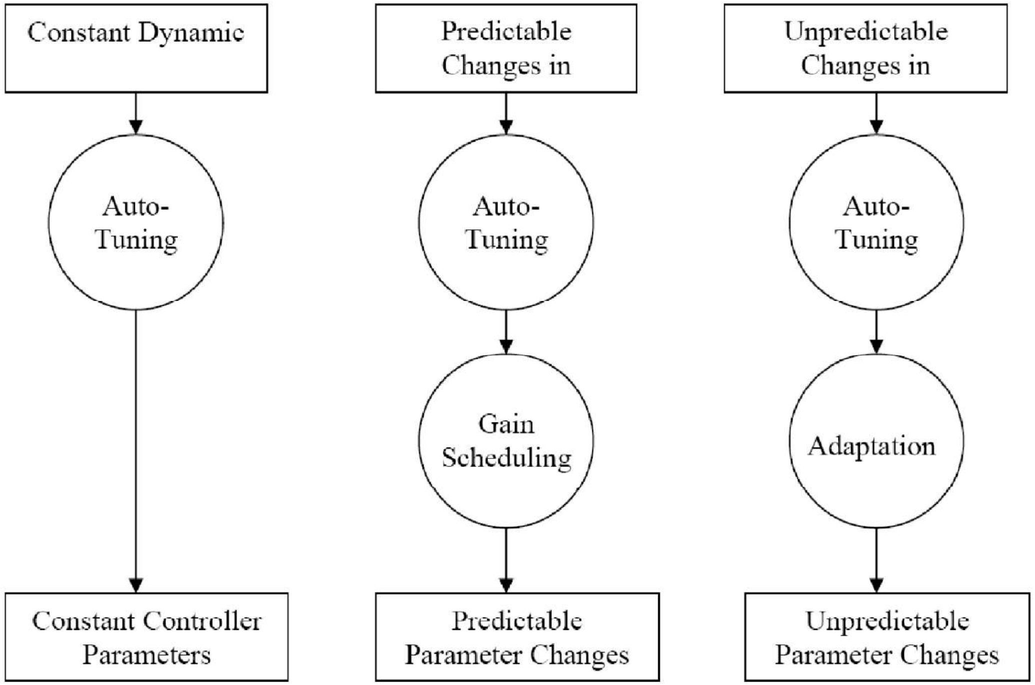


Figure 1: How to use different techniques

The PID controllers are the standard tool for industrial automation.

The textbook version of the algorithm is

$$u(t) = K_c\left(e(t) + \frac{1}{T_i}\int_0^t e(s)ds + T_d\frac{de}{dt}\right)$$

where u is the control variable, e is error defined as $e = u_c - y$ where u_c is the reference value, and y is the process output.

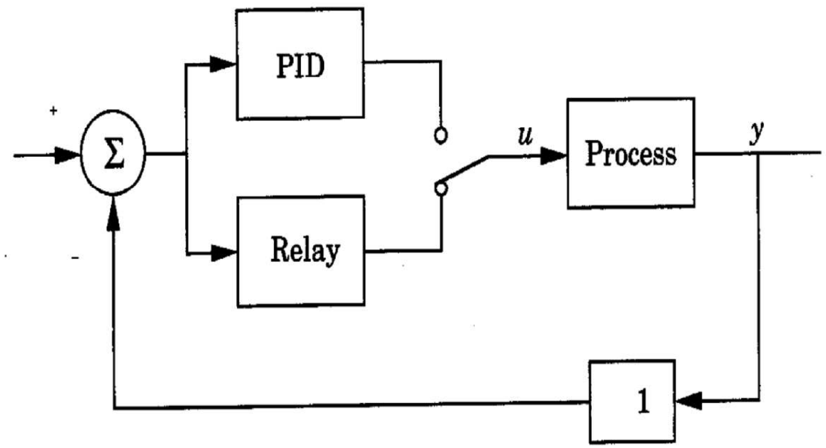
TRANSIENT RESPONSE METHOD

The Ziegler-Nichols Step Response Method

METHODS BASED ON RELAY FEEDBACK

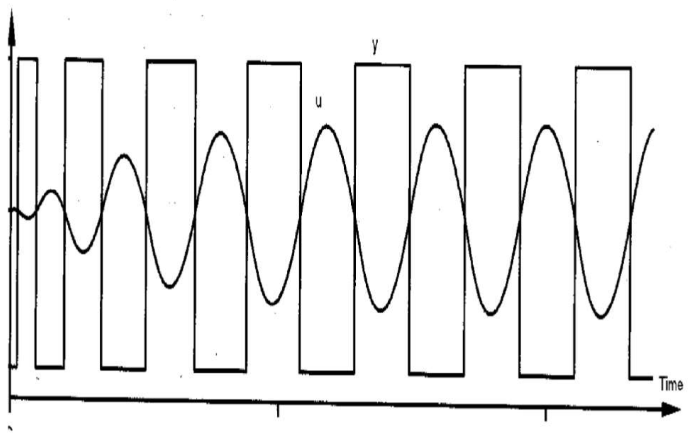
The basic idea is the observation that many process have limit cycle oscillations under relay feedback. A block diagram of such a system is shown in the Figure below.

The input and output signals obtained when the command signal, u_c



is zero are shown in the Figure below.

The figure shows that a limit cycle oscillation is established quite rapidly. We can intuitively understand what happens in the following way: the input to the process is a square wave with frequency ω_u. By a Fourier series expansion we can represent the input by a sum of sinusoids with frequencies ω_u, $3\omega_u$, and so on. The output is



approximately sinusoidal which means that the process attenuates the higher harmonics effectively. Let the amplitude of the square wave be d; then the fundamental component has the amplitude $4d/\pi$. Making the approximation that all higher harmonics can be neglected, we find that the process output is a sinusoid with frequency ω_u and amplitude

$$a = \frac{4d}{\pi}|G(i\omega_u)|$$

To have an oscillation, the output must also go through zero when the relay switches. Moreover, the fundamental component of the input and the output must have opposite phase. We can thus conclude that the frequency ω_u must be such that the process has a phase lag of $180°$. The conditions for oscillation are thus

$$\begin{aligned} \arg G(i\omega_u) &= -\pi \\ |G(i\omega_u)| &= \frac{a\pi}{4d} = \frac{1}{K_u} \end{aligned} \tag{1}$$

where K_u can be regarded as the equivalent gain of the relay for transmission of sinusoidal signals with amplitude a. For historical reasons this parameter is call the ultimate gain. It is the gain that brings a system with transfer function $G(s)$ to the stability boundary under pure proportional control. The period $T_u = 2\pi/\omega_u$ is similarly called the ultimate period. An experiment with relay feedback is thus a convenient way to determine the ultimate period and the ultimate gain. Notice also that an input signal whose energy content is concentrated at ω_u is generated automatically in the experiment. Ziegler and Nichols have devised a very simple heuristical method for determining the parameters of a PID controller based on critical gain and the critical period. The controller settings are given in the Table below.

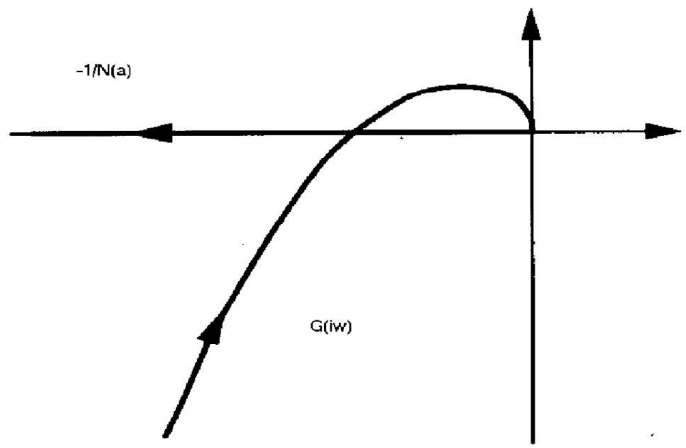
Controller	K_c	T_i	T_d
P	$0.5K_u$		
PI	$0.4K_u$	$0.8T_u$	
PID	$0.6K_u$	$0.5T_u$	$0.12T_u$

The Method of Describing Function

The approximate method used to derive the condition for relay oscillations given by Equation (1) is called the method of harmonic balance. We will now describe a slight variation of the method that can be used to obtain additional insight. This is called the describing function method. Consider a simple feedback system composed of a linear part with the transfer function $G(s)$ and feedback with an ideal relay as shown in the Figure. The conditions for limit cycle oscillations can be determined approximately by investigating the propagation of sinusoidal signals around the loop. There will be higher harmonics because of the relay, but they will be neglected. The propagation of a sine wave through the linear system is described by the complex number $G(i\omega)$. Similarly, the propagation of a sine wave through the nonlinearity can also be characterized by a complex number $N(a)$, which depends on the amplitude of the signal at the input of the nonlinearity. $N(a)$ is called the describing function of the nonlinearity. The condition for oscillation is then that the signal comes back with the same amplitude and phase as it passes the closed loop. This given the condition

$$G(i\omega)N(a) = -1$$

This condition can be represented graphically by also plotting the curve $N(a)$ in the Nyquist diagram (see Figure).



For the relay the nonlinearity is

$$N(a) = \frac{4d}{a\pi}$$

because a is the input signal amplitude and the fundamental component of the output has amplitude $4d/\pi$. A possible oscillation is at the intersection of the curves. The frequency is read from the Nyquist curve and the amplitude from the describing function.

Example: Relay oscillation

Consider a system

$$G(s) = \frac{K\alpha}{s(s+1)(s+\alpha)}$$

$K = 5$, $\alpha = 10$, $d = 1$, and $u_c = 0$. Simple calculations show that

$$\arg G(i\omega_u) = -\frac{\pi}{2} - \tan^{-1}\omega_u - \tan^{-1}\frac{\omega_u}{\alpha}$$

$$
\begin{aligned}
&= -\frac{\pi}{2} - \tan^{-1}\frac{\omega_u(\alpha+1)}{\alpha-\omega_u^2} = -\pi \\
\omega_u &= \sqrt{\alpha}
\end{aligned}
$$

The approximate analysis thus gives the following estimate of the period:

$$
T_u = \frac{2\pi}{\sqrt{\alpha}} = \frac{6.28}{\sqrt{\alpha}} = 1.99
$$

Using Equation (1) gives $a = 4d|G(i\omega_u)|/\pi = 0.58$. From the simulations it can be determined that the true values are $T_u = 2.07$ and $a = 0.62$, which show that the describing function method gives fair but not very accurate estimates in this example.

RELAY OSCILLATIONS

Introduce the following state space realization of the transfer function

$G(s)$:

$$\begin{aligned} \frac{dx}{dt} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2}$$

The relay can be described by

$$u = \begin{cases} d & \text{if } e > 0 \\ -d & \text{if } e < 0 \end{cases} \tag{3}$$

where $e = u_c - y$.

THEOREM Limit cycle period

Assume that the system defined in the Figure and by Equations (2) and (3) has a symmetric limit cycle with period T. The period T is then the smallest value of $T > 0$ that satisfies that equation

$$C(I + \Phi)^{-1}\Gamma = 0 \tag{4}$$

where

$$\Phi = e^{AT/2}$$

and

$$\Gamma = \int_0^{T/2} e^{As} ds B$$

Proof: Let t_k denote the times when the relay switches. Since the limit cycle is symmetric, it follows that

$$t_{k+1} - t_k = T/2$$

Assume that the control signal u is d over the interval (t_k, t_{k+1}). Integration of Equation (2) over the interval gives

$$x(t_{k+1}) = \Phi x(t_k) + \Gamma d$$

Since the limit cycle is symmetric, it also follows that

$$x(t_{k+1}) = -x(t_k) = \Phi x(t_k) + \Gamma d$$

Hence

$$x(t_k) = -(I + \Phi)^{-1}\Gamma d$$

Since the output $y(t)$ must be zero at t_k, it follows that

$$y(t_k) = Cx(t_k) = -C(I + \Phi)^{-1}\Gamma d = 0$$

which gives Equation (4) $\triangle$

Remark The condition of Equation (4) can also be written as

$$H_{T/2}(-1) = 0 \tag{5}$$

where $H_{T/2}(z)$ is the pulse transfer function obtained when sampling the system of Equation (2) with period $T/2$. (Note: $H(z) = C(zI - \Phi)^{-1}\Gamma$)

Example: Limit cycle period

Consider the same process as the last example. To apply the Limit Cycle Period Theorem, the system $G(s)$ is sampled with period h.

$$H_h(z) = \frac{Kh}{(z-1)} - \frac{K\alpha(1-e^{-h})}{(\alpha-1)(z-e^{-h})} + \frac{K(1-e^{-\alpha h})}{\alpha(\alpha-1)(z-e^{-\alpha h})}$$

Hence

$$H_h(-1) = -\frac{Kh}{2} + \frac{K\alpha(1-e^{-h})}{(\alpha-1)(1+e^{-h})} - \frac{K(1-e^{-\alpha h})}{\alpha(\alpha-1)(1+e^{-\alpha h})}$$

$$= -\frac{Kh}{2} + \frac{K\alpha}{\alpha - 1}\left(\frac{1-e^{-h}}{1+e^{-h}} - \frac{1}{\alpha^2}\frac{1-e^{-\alpha h}}{1+e^{-\alpha h}}\right) = 0$$

Numerical search for the value of h that satisfies this equation gives $h = 1.035$. This gives $T_u = 2.07$.

Root-Locus Analysis of Relay System

Example

The input, $u(t)$, and output signals, $y(t)$, obtained when the command signal, $r(t)$ is zero are shown in Figure 1 below.

$$G_p(s) = \frac{1-0.1s}{(s+1)(0.1s+1)}$$

$$u(t) = \begin{pmatrix} 1 & \text{if } e(t) = r(t) - y(t) \geq 0 \\ -1 & \text{if } e(t) = r(t) - y(t) < 0 \end{pmatrix}$$

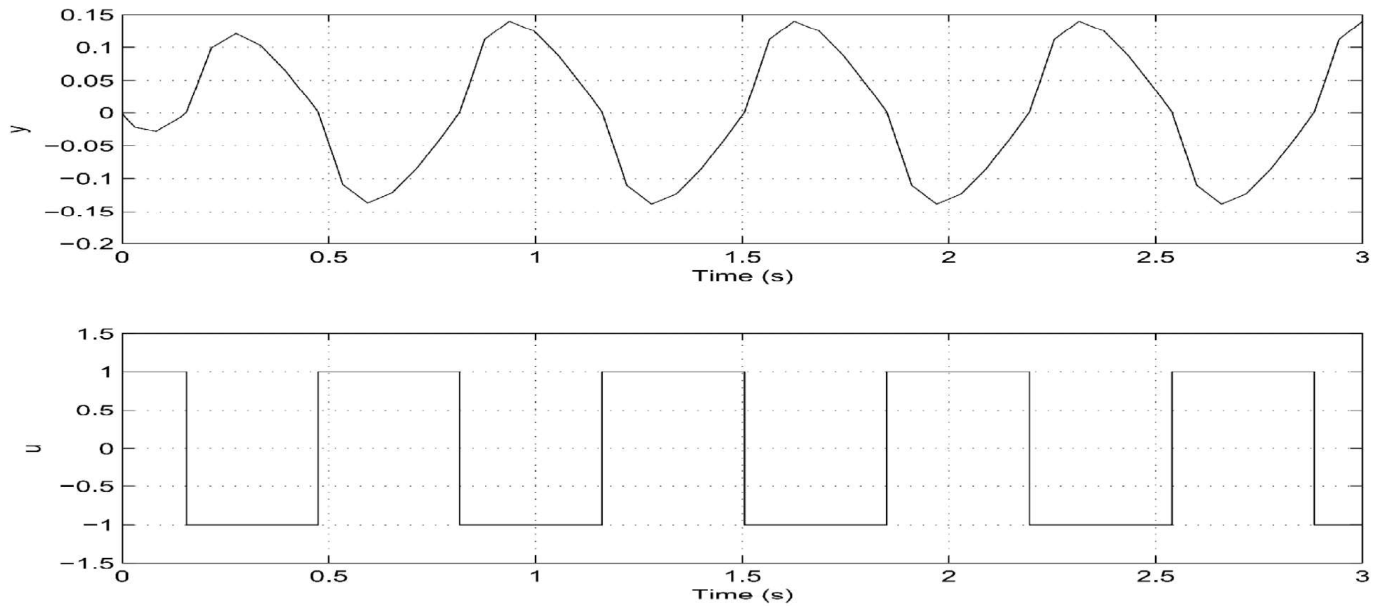


Figure 1: Sustained relay oscillation

Let the relay be approximated by a variable gain K

$$u(t) = Ke(t)$$

e(t)	u(t)	K
0	1	∞
0.05	1	20
0.1	1	10
0.2	1	5
1	1	1
∞	1	0

Closed-loop poles

$$
\begin{aligned}
1 + KG_p(s) &= 0 \\
1 + K\frac{1 - 0.1s}{(s+1)(0.1s+1)} &= 0 \\
0.1s^2 + (1.1 - 0.1K)s + 1 + K &= 0
\end{aligned}
$$

$$s = \frac{-(1.1 - 0.1K) \pm \sqrt{(1.1 - 0.1K)^2 - 0.4(1. + K)}}{2(0.1)}$$

K	s
0	$-1, -10$
$\vdots$	poles on left hand side
9	$-1 \pm 9.95j$
10	$-0.5 \pm 10.48j$
11	$\pm 10.95j$
12	$0.5 \pm 11.39j$
13	$1 \pm 11.79j$
$\vdots$	poles on right hand side

Figure 1 shows that a limit cycle oscillation is established quite rapidly.

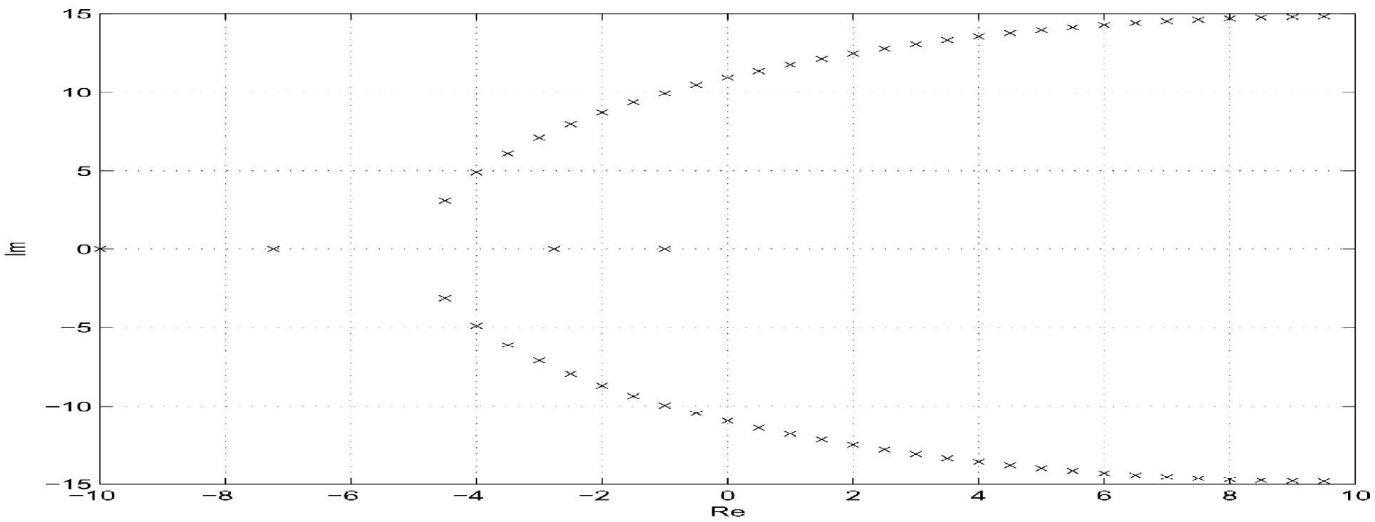
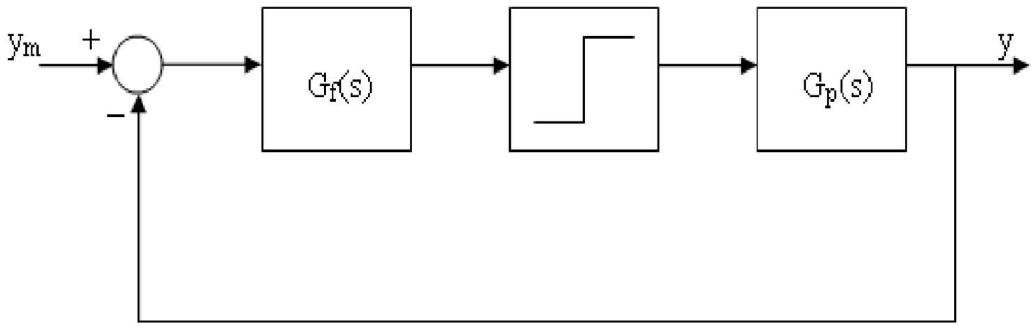


Figure 2: Root-Locus

SELF-OSCILLATING ADAPTIVE SYSTEMS

A system that is insensitive to parameter variations can be obtained by using a two-degree-of-freedom configuration with a high-gain feedback and a feedforward compensator. A characteristic feature of the SOAS is that there is a limit cycle oscillation. The system thus represents a type of adaptive control in which there are intentional perturbations, which excite the system all the time. The SOAS is based on three useful ideas: model-following, automatic generation of test signals, and use of a relay. The key result is that the loop gain is automatically adjusted to give an amplitude margin $A_m = 2$.

The relay is used to introduce a limit cycle oscillation in the system. When the reference signal is changed, or when there are disturbances, there will also be other signals in the system, which will be superim-



posed on the limit cycle oscillations. The signals that appear in the system will thus be of the form

$$s(t) = a \sin \omega t + b(t)$$

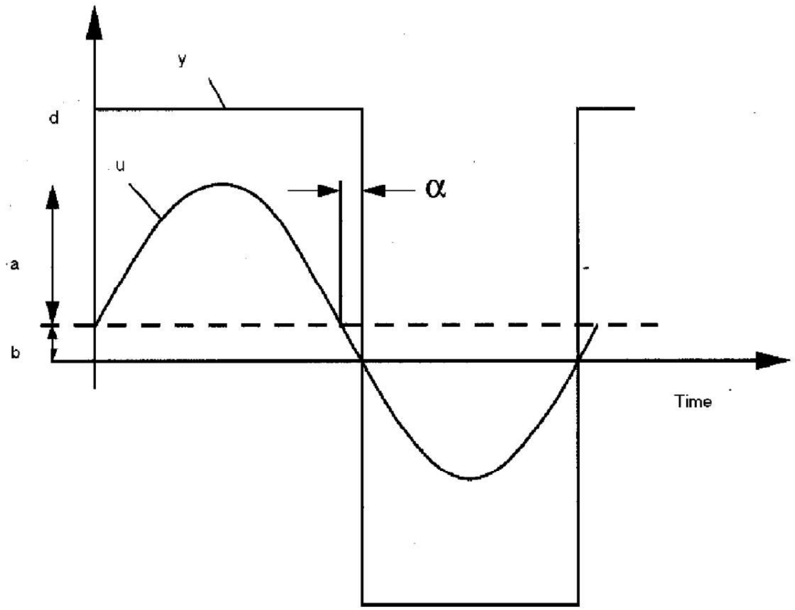
where $a \sin \omega t$ denotes that limit cycle oscillation. It is assumed that $b(t)$ varies much more slowly than $\sin \omega t$. Furthermore, assume that $b(t)$ is smaller than a.

The Dual-Input Describing Function

It is assumed the $b(t)$ varies so slowly that it can be approximated by a constant. The input to the relay is thus of the form

$$u(t) = a \sin \omega t + b$$

The relay input and output are shown in the figure below.



The relay output can be expanded in a Fourier series

$$y(t) = bN_B + aN_A \sin \omega t + aN_{A_2} \sin 2\omega t + \cdots \tag{1}$$

where the number N_A and N_B are given by

$$\begin{aligned}
N_B &= \frac{1}{2\pi b} \int_0^{2\pi} y(t)dt = \frac{d(\pi + \alpha) - d(\pi - 2\alpha) + \alpha d}{2\pi b} \\
&= \frac{4\alpha d}{2\pi b} = \frac{2\alpha d}{\pi b} = \frac{2d}{\pi b} \sin^{-1}\left(\frac{b}{a}\right) \\
N_A &= \frac{1}{\pi a} \int_0^{2\pi} y(t) \sin \omega t dt = \frac{2d}{\pi a} \int_\alpha^{\pi-\alpha} \sin \omega t dt \\
&= \frac{4d}{\pi a} \cos \alpha = \frac{4d}{\pi a} \sqrt{1 - (b/a)^2}
\end{aligned}$$

Small values of b/a give the approximations

$$N_A \approx \frac{4d}{\pi a} \qquad N_B \approx \frac{2d}{\pi a}$$

Notice that

$$N_A \approx 2N_B \tag{2}$$

The transmission of the constant level b and of the first harmonic $\sin \omega t$ are thus characterized by the equivalent gains N_B and N_A. Since the linear parts will normally attenuate high frequencies more than low frequencies, a reasonable approximation is often obtained by considering only the constant part and the first harmonic. The number N_B, which describes the propagation of a constant signal, is called the dual-input describing function. The dual-input describing function can be used to characterize the transmission of slowly varying signals. A detailed analysis of the accuracy of the approximation is fairly complicated. Let it therefore suffice to mention some rules of thumb for using the approximation. The ratio a/b should be greater than 3, and the ratio of the limit cycle frequency to the signal y_m frequency should also be greater than 3. It is strongly recommenced that the analysis be supplemented by simulation.

Main Result

The amplitude of the limit cycle at the relay input is given by

$$N_A|G(i\omega_u)| = 1 \tag{3}$$

If the signals vary slowly in comparison with the limit cycle oscillations, the propagation through the relay is approximately described by the dual-input describing function N_B. The propagation of slowly varying signals is thus approximately described by the loop transfer function

$$G_0(s) = N_B(a)G_p(s)$$

If follows from Eqs. (2) and (3) that

$$|G_0(i\omega_u)| = N_B(a)|G_p(i\omega_u)| = \frac{1}{2}N_A|G_p(i\omega)| = 0.5$$

RESULT 1 Amplitude margin of the SOAS

The SOAS automatically adjusts itself so that the response to reference signals is approximately described by the closed-loop transfer function

$$G_c(s) = \frac{kG_p(s)}{1 + kG_p(s)}$$

where the gain k is such that the amplitude margin is 2.

The result can also be stated in the following way: The relay acts as a variable gain. The magnitude of the gain depends on the amplitude of the sinusoidal signal at the relay input. This gain is automatically set by the limit cycle oscillation to such a value that the loop gain becomes 0.5 at the frequency of the limit cycle.

Example. A basic SOAS

Assume that the linear parts are characterized by the transfer function

$$G_p(s) = \frac{K\alpha}{s(s+1)(s+\alpha)}$$

From the (Example Relay Oscillation) the period of the limit cycle is approximately given by

$$\omega_u = \sqrt{\alpha}$$

The magnitude of the transfer function at this frequency is

$$\left|G_p(i\omega_u)\right| = \frac{K}{\alpha+1}$$

If the relay amplitude is d, it follows that the amplitude of the limit cycle oscillation at the relay input is approximately given by

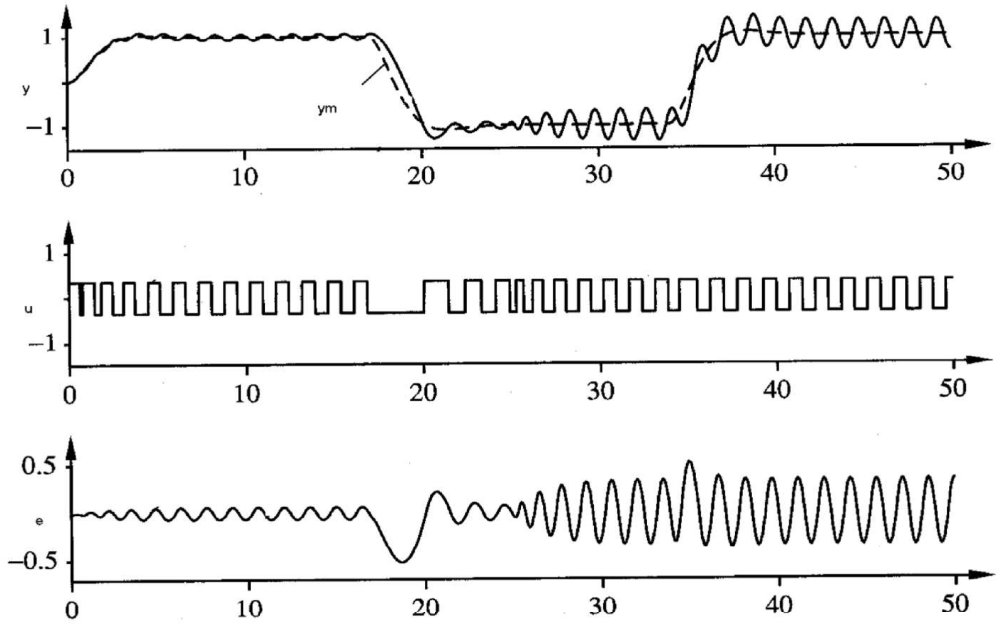
$$e_0 = \frac{Kd}{1+\alpha}$$

A simulation of the system is shown in the figure.

The feedforward transfer function is a second-order system with the damping 0.7 and the natural frequency 1 rad/s. The nominal values of the parameters are $K = 3$, $d = 0.35$, and $\alpha = 20$. The approximate analysis gives a limit cycle with period $T = 1.4$ and amplitude 0.05. The process gain is suddenly increase by a factor of 5 at $t = 25$.

Design of an SOAS

1. The relay amplitude is first determined such that the desired control authority is obtained. This can be estimated by analyzing the



response of the process to constant control signals.

2. When the relay amplitude is specified, the desired limit cycle frequency can be determined from the condition

$$d|G_p(i\omega_u)| = e_0$$

where e_0 is the tolerable limit cycle amplitude in the error signal and $G_p(s)$ is the transfer function of the process. It is necessary to check that the frequency obtained is reasonable.

3. The final step is to determine the transfer function $G_f(s)$ of the linear compensator such that

$$\arg G_f(i\omega_u) + \arg G_p(i\omega_u) = -\pi$$

   A large phase lead may be necessary, but this may not be realizable because of noise sensitivity.

4. Check that the linear closed-loop system with the loop gain $G_0(s)$ $= KG_f(s)G_p(s)$ will work well when the gain K is adjusted so that the amplitude margin is 2. If this is not the case, the compensator $G_f(s)$ must be modified.

Example SOAS with lead network

The relay control in (Example A basic SOAS) gave an error amplitude of about $e_0 = 0.05$. Assume that we want to decrease the amplitude by a factor of 5 while maintaining $d = 0.35$. This gives a new oscillation frequency ω_u' such that

$$d|G_p(i\omega_u')| = e_0 = 0.01$$

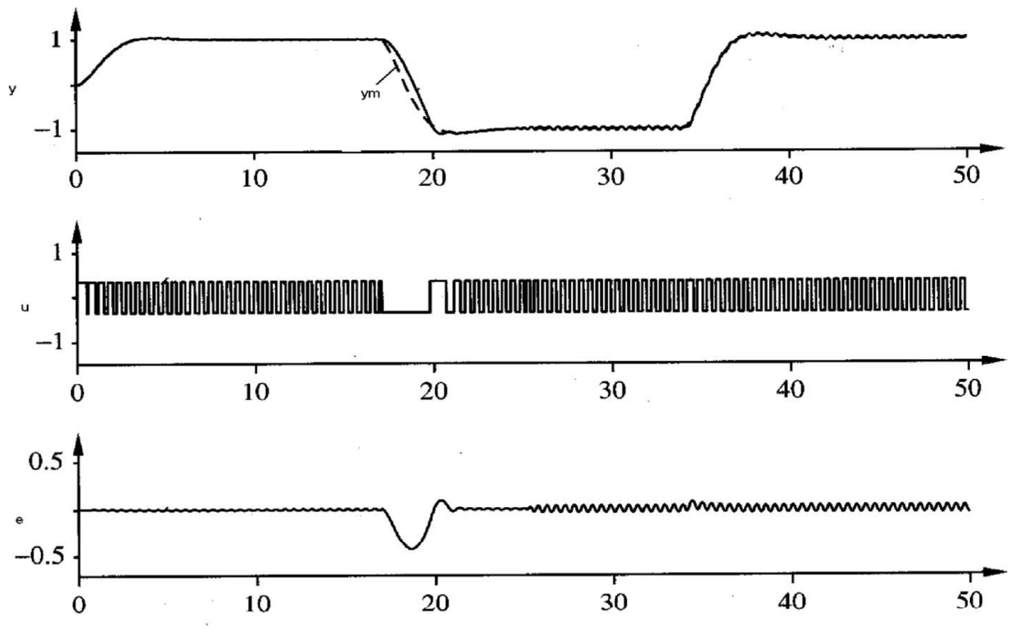
or $\omega_u' = 10$ rad/s. To get this oscillating frequency, a lead network $G_f(s)$ is added such that

$$\arg G_f(i\omega_u') + \arg G_p(i\omega_u') = -\pi$$

The fig. shows a simulation of the system in (Example A basic SOAS) with the compensation network

$$G_f(s) = 1.2\frac{s+5}{s+15}$$

The gain is increase by a factor of 5 at $t = 25$. It is seen that the lead network decreases the amplitude of the oscillation.



1
y
ym
−1
0 10 20 30 40 50
1
u
−1
0 10 20 30 40 50
0.5
e
−0.5
0 10 20 30 40 50

To determine ω_u'

$$
\begin{aligned}
d|G_p(j\omega_u')| &= e_0 = 0.01 \\
|G_p(j\omega_u')| &= \frac{0.01}{0.35} = 0.0286 \\
G_p(j\omega_u') &= \frac{K\alpha}{j\omega(j\omega+1)(j\omega+\alpha)} \\
&= \frac{K\alpha}{(-\omega^2(1+\alpha)+j(\omega\alpha-\omega^3)} \\
|G_p(j\omega_u')| &= \frac{K\alpha}{\sqrt{\omega_u'^4(1+\alpha)^2+\omega_u'^2(\alpha-\omega_u'^2)^2}}
\end{aligned}
$$

gives $\omega_u' = 10$.

To determine $G_f(s)$

$$
\arg G_f(j\omega_u') + \arg G_p(\omega_u') = -\pi
$$

$$\arg K_f \frac{s+T_1}{s+T_2} - \frac{\pi}{2} - \arctan \omega_u' - \arctan \frac{\omega_u'}{\alpha} = -\pi$$
$$\arctan \frac{\omega_u'}{T_1} - \arctan \frac{\omega_u'}{T_2} - \frac{\pi}{2} - \arctan \omega_u' - \arctan \frac{\omega_u'}{\alpha} = -\pi$$

Choose $T_1 = 5$; $T_2 = 15$ to satisfy the equation. Choose $K_f = 1.2$ so that $|K_f \frac{j\omega_u' + T_1}{j\omega_u' + T_2}| \approx 1$.

GAIN-SCHEDULING

Gain scheduling is an attempt to apply the well developed linear control methodology to the control of nonlinear systems. The idea of gain scheduling is to select a number of operating points which cover the range of the system operation. Then, at each of these points, the designer makes a linear time-invariant approximation to the plant dynamics and designs a linear controller for each linearized plant. Between operating points, the parameters of the compensators are then interpolated.

Consider the cascaded spherical tanks below.

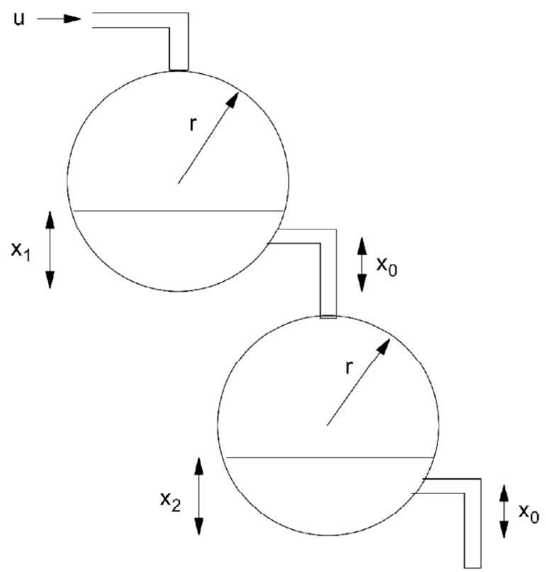


Figure 1: Spherical Tank

The rate of change in volume is given by

$$\begin{aligned} v &= \frac{1}{3}\pi x^2(3r - x) \\ \frac{dv}{dt} &= \pi(2rx - x^2)\dot{x} \end{aligned}$$

This must be equal to the rate of nett inflow. Therefore

$$\begin{aligned} u - K\sqrt{x_1 - x_0} &= \pi(2rx_1 - x_1^2)\dot{x}_1 \\ K\sqrt{x_1 - x_0} - K\sqrt{x_2 - x_0} &= \pi(2rx_2 - x_2^2)\dot{x}_2 \end{aligned}$$

Rearrange and name the two equations as g and h.

$$\begin{aligned} \dot{x}_1 &= \frac{u - K\sqrt{x_1 - x_0}}{\pi(2rx_1 - x_1^2)} = g \\ \dot{x}_2 &= \frac{K\sqrt{x_1 - x_0} - K\sqrt{x_2 - x_0}}{\pi(2rx_2 - x_2^2)} = h \end{aligned}$$

From Taylor expansion:

$$\dot{x}_1 = g|_{\bar{x}_1,\bar{x}_2,\bar{u}} + \left.\frac{dg}{dx_1}\right|_{\bar{x}_1,\bar{x}_2,\bar{u}} \Delta x_1 + \left.\frac{dg}{dx_2}\right|_{\bar{x}_1,\bar{x}_2,\bar{u}} \Delta x_2 + \left.\frac{dg}{du}\right|_{\bar{x}_1,\bar{x}_2,\bar{u}} \Delta u$$

$$\dot{x}_2 = h|_{\bar{x}_1,\bar{x}_2,\bar{u}} + \left.\frac{dh}{dx_1}\right|_{\bar{x}_1,\bar{x}_2,\bar{u}} \Delta x_1 + \left.\frac{dh}{dx_2}\right|_{\bar{x}_1,\bar{x}_2,\bar{u}} \Delta x_2 + \left.\frac{dh}{du}\right|_{\bar{x}_1,\bar{x}_2,\bar{u}} \Delta u$$

At steady state

$$g|_{\bar{x}_1,\bar{x}_2,\bar{u}} = h|_{\bar{x}_1,\bar{x}_2,\bar{u}} = 0$$

$$g|_{\bar{x}_1,\bar{x}_2,\bar{u}} = \frac{\bar{u} - K\sqrt{\bar{x}_1 - x_0}}{\pi(2r\bar{x}_1 - \bar{x}_1^2)} = 0 \Rightarrow \bar{u} = K\sqrt{\bar{x}_1 - x_0}$$

$$h|_{\bar{x}_1,\bar{x}_2,\bar{u}} = \frac{K(\sqrt{\bar{x}_1 - x_0} - \sqrt{\bar{x}_2 - x_0})}{\pi(2r\bar{x}_2 - \bar{x}_2^2)} = 0 \Rightarrow \bar{x}_1 = \bar{x}_2 = (\frac{\bar{u}}{K})^2 + x_0$$

The linearized model in state space form

$$\begin{bmatrix} \dot{\Delta} x_1 \\ \dot{\Delta} x_2 \end{bmatrix} = \begin{bmatrix} \frac{dg}{dx_1} & \frac{dg}{dx_2} \\ \frac{dh}{dx_1} & \frac{dh}{dx_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{dg}{du} \\ \frac{dh}{du} \end{bmatrix} \Delta u = A\Delta x + B\Delta u$$

$$\Delta y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = C\Delta x$$

where

$$\dot{\Delta} x_1 = \dot{x}_1 - g|_{\bar{x}_1, \bar{x}_2, \bar{u}}$$

$$\dot{\Delta} x_2 = \dot{x}_2 - h|_{\bar{x}_1, \bar{x}_2, \bar{u}}$$

$$\frac{dg}{du} = \frac{1}{\pi(2r\bar{x}_1 - \bar{x}_1^2)}$$

$$\frac{dg}{dx_1} = -\frac{dh}{dx_1}$$

$$\begin{aligned}
\frac{dg}{dx_2} &= 0\\
\frac{dh}{du} &= 0\\
\frac{dh}{dx_1} &= \frac{K(\bar{x}_1 - \bar{x}_0)^{-1/2}}{2\pi(2r\bar{x}_2 - \bar{x}_2^2)}\\
\frac{dh}{dx_2} &= -\frac{dh}{dx_1}
\end{aligned}$$

The linearized transfer function model

$$\frac{\Delta Y(s)}{\Delta U(s)} = C(sI - A)^{-1}B = \frac{\frac{2}{K}\sqrt{\bar{x}_2 - x_0}}{(\frac{2}{K}\pi\bar{x}_2(2r - \bar{x}_2)\sqrt{\bar{x}_2 - x_0}s + 1)^2}$$

The DC gain $= \frac{2}{K}\sqrt{\bar{x}_2 - x_0}$

The Time constant $= \frac{2}{K}\pi\bar{x}_2(2r - \bar{x}_2)\sqrt{\bar{x}_1 - x_0}$

The PID controller can be designed using pole-placement as follows. Consider the second-order model process

$$G_p(s) = \frac{K_p}{(1+sT_1)(1+sT_2)}$$

The PID controller can be written as

$$G_c(s) = \frac{K_c(1+sT_i+s^2T_iT_d)}{sT_i}$$

The characteristic equation of the closed-loop system becomes

$$s^3 + s^2\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{K_pK_cT_d}{T_1T_2}\right) + s\left(\frac{1}{T_1T_2}+\frac{K_pK_c}{T_1T_2}\right) + \frac{K_pK_c}{T_iT_1T_2} = 0$$

A suitable closed-loop characteristic equation is a third-order system is

$$(s+\alpha\omega)(s^2+2\zeta\omega s+\omega^2) = 0$$

Comparing coefficients in the last 2 equations gives

$$K_c = \frac{T_1T_2\omega^2(1+2\zeta\alpha)-1}{K_p}$$

$$T_i = \frac{T_1T_2\omega^2(1+2\zeta\alpha)-1}{T_1T_2\alpha\omega^3}$$

$$T_d = \frac{T_1T_2\omega(\alpha+2\zeta)-T_1-T_2}{\omega^2T_1T_2(1+2\zeta\alpha)-1}$$

where α, ζ and ω are user-specified.

TUTORIAL ON LEAST-SQUARES ESTIMATION

Solve simultaneous equations for the unknowns when there are more equations than unknowns. For example 3 equations 2 unknowns θ_1 and θ_2:

$$\begin{aligned}\phi_{11}\theta_1 + \phi_{12}\theta_2 &= y_1\\ \phi_{21}\theta_1 + \phi_{22}\theta_2 &= y_2\\ \phi_{31}\theta_1 + \phi_{32}\theta_2 &= y_3\end{aligned}$$

Organised in matrices give

$$\Phi\theta = Y$$

where

$$\Phi = \begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\\ \phi_{31} & \phi_{32}\end{bmatrix}; \quad \theta = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}; \quad Y = \begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$$

Choose the θ_1 and θ_2 that minimizes the error in the 3 equations. There many ways to define such an error, but the most convenient is the sum of squares:

$$
\begin{aligned}
V &= (y_1 - \phi_{11}\theta_1 - \phi_{12}\theta_2)^2 \\
&\quad +(y_2 - \phi_{21}\theta_1 - \phi_{22}\theta_2)^2 \\
&\quad +(y_3 - \phi_{31}\theta_1 - \phi_{32}\theta_2)^2 \\
&= (Y - \Phi\theta)^T(Y - \Phi\theta)
\end{aligned}
$$

If there is an exact solution to $\Phi\theta = Y$, the minimum error is $V = 0$ else the minimum is given by

$$
\begin{aligned}
\frac{dV}{d\theta_1} &= -(\phi_{11}y_1 + \phi_{21}y_2 + \phi_{31}y_3) \\
&\quad +(\phi_{11}^2 + \phi_{21}^2 + \phi_{31}^2)\theta_1 + (\phi_{11}\phi_{12} + \phi_{21}\phi_{22} + \phi_{31}\phi_{32})\theta_2 = 0 \\
\frac{dV}{d\theta_2} &= -(\phi_{12}y_1 + \phi_{22}y_2 + \phi_{32}y_3) \\
&\quad +(\phi_{11}\phi_{12} + \phi_{21}\phi_{22} + \phi_{31}\phi_{32})\theta_1 + (\phi_{12}^2 + \phi_{22}^2 + \phi_{32}^2)\theta_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV}{d\theta} &= -\Phi^T Y + \Phi^T \Phi\theta = 0 \\
\theta &= (\Phi^T\Phi)^{-1}\Phi^T Y
\end{aligned}
$$

if $\Phi^T\Phi$ is invertible.

REAL TIME PARAMETER ESTIMATION

INTRODUCTION

Parameter Identification and Adaptive Control:

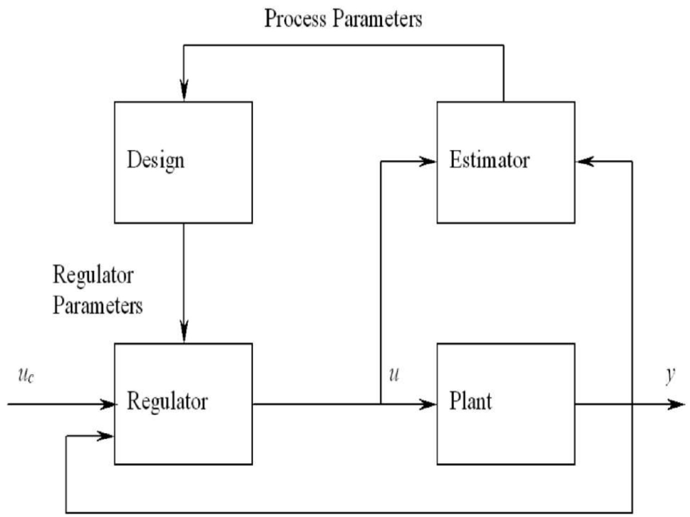


Figure 1. Block Diagram of a Self-tuning Regulator

Self-Tuning Regulator (STR) is based on the idea of separating the estimation of unknown parameters from the design of the controller.

PARAMETERIZATION

This course focuses on systems which are linear-in-the-parameters (LIPs). It depends on our ability to rearrange the model so that the predicted output is described as a linear function of a parameter vector θ.

Linear in the Parameters

From the system identification's point of view, it is highly desirable to have a model that gives an error that is linear in the parameters.

Suppose there exists some vector of measured variables, $\phi(t)$, such that the model output $y(t)$ can be expressed as

$$y = \varphi^T \theta \tag{1}$$

The parameterization (1) is LIP because y depends linearly on θ.

In some cases, it is straightforward to express the model in this form.

A Simple Example

Consider a sine wave

$$y = A \sin(t + \alpha) \tag{2}$$

is linear in A but is nonlinear in α. It does not immediately fit into the form (1). However, a linear representation can be achieved by re-expressing the model (2) in the following form:

$$y = \theta_1 \sin(t) + \theta_2 \cos(t) = \varphi^T \theta \tag{3}$$

where $\varphi^T = [\sin(t) \quad \cos(t)]$.

Discrete time Linear Models

Discrete time shift operator linear models can be expressed by

$$A(q)y(t) = B(q)u(t) \tag{4}$$

where

$$\begin{aligned} A(q) &= q^n + a_{n-1}q^{n-1} + \ldots + a_0 \\ B(q) &= b_m q^m + b_{m-1}q^{m-1} + \ldots + b_0 \\ qx(t) &= x(t+1) \end{aligned}$$

By dividing both sides by q^n, equation (4) can be explicitly expressed as

$$y(t) = -a_{n-1}y(t-1) - \ldots - a_0 y(t-n)$$

$$+b_m u(t-n+m) + \ldots + b_0 u(t-n) \tag{5}$$
$$= \varphi^T \theta \tag{6}$$

which is LIPs, with

$$\varphi^T = \left[\, y(t-1)\ \ldots\ y(t-n)\ u(t-n+m)\ \ldots\ u(t-n) \,\right]$$
$$\theta^T = \left[\, -a_{n-1}\ \ldots\ a_0\ b_m\ \ldots\ b_0 \,\right]$$

Least Squares and Regression Models

The least-squares method is simple for a mathematical model that can be written in the form

$$y(i) = \varphi_1(i)\theta_1^0 + \varphi_2(i)\theta_2^0 + \ldots \varphi_n(i)\theta_n^0 \tag{7}$$

where y is the observed variable, θ_1^0, θ_2^0, …, θ_n^0 are parameters of the model to be determined, and φ_1, φ_2, …, φ_n are known functions that may depend on other known variables. The vectors

$$\begin{aligned} \varphi^T(i) &= [\varphi_1(i) \;\; \varphi_2(i) \;\; \ldots \;\; \varphi_n(i)] \\ \theta^0 &= \left[\theta_1^0 \;\; \theta_2^0 \;\; \ldots \;\; \theta_n^0\right]^T \end{aligned}$$

are defined. Discrete time with time index i, is assumed. Pairs of observations $\{(y(i),\ \varphi_n(i),\ i = 1,\ 2,\ \ldots,\ t\}$ are obtained from an experiment. The parameter θ, is chosen to minimize the least-squares loss function

$$V(\theta, t) = \frac{1}{2} \sum_{i=1}^{t} \left(y(i) - \varphi^T(i)\theta\right)^2 \tag{8}$$

Introduce the notations

$$\begin{aligned} Y(t) &= \left[\, y(1) \;\, y(2) \;\, \ldots \;\, y(t) \,\right]^T \\ E(t) &= \left[\, \varepsilon(1) \;\, \varepsilon(2) \;\, \ldots \;\, \varepsilon(t) \,\right]^T \end{aligned}$$

$$\Phi(t) = \begin{bmatrix} \varphi^T(1) \\ \vdots \\ \varphi^T(t) \end{bmatrix}$$

$$P = \left(\Phi^T(t)\Phi(t)\right)^{-1} = \left(\sum_{i=1}^{t} \varphi(i)\varphi^T(i)\right)^{-1} \tag{9}$$

where

$$\varepsilon(i) = y(i) - \hat{y}(i) = y(i) - \varphi^T(i)\theta$$

With these notations the loss function (8) can be written as

$$V(\theta, t) = \frac{1}{2}\sum_{i=1}^{t} \varepsilon^2(i) = \frac{1}{2}E^T E = \frac{1}{2}||E||^2$$

where

$$E = Y - \hat{Y} = Y - \Phi\theta \tag{10}$$

The solution to the least-squares problem is given by the following theorem.

Theorem: Least-squares estimation

The function of Eq. (8) is mimimal for parameters $\hat{\theta}$ such that

$$\Phi^T \Phi \hat{\theta} = \Phi^T Y \tag{11}$$

If the matrix $\Phi^T \Phi$ is nonsingular, the minimum is unique and given by

$$\hat{\theta} = (\Phi^T \Phi)^{-1} \Phi^T Y \tag{12}$$

Proof: The loss function of Eq. (8) can be written as

$$\begin{aligned} 2V(\theta, t) &= E^T E = (Y - \Phi\theta)^T (Y - \Phi\theta) \\ &= Y^T Y - Y^T \Phi\theta - \theta^T \Phi^T Y + \theta^T \Phi^T \Phi\theta \end{aligned} \tag{13}$$

Since the matrix $\Phi^T\Phi$ is always nonnegative definite, the function V has a minimum. The loss function is quadratic in θ. The minimum can be found in many ways. One way is to determine the gradient of Eq. (13) with respect to θ:

$$\frac{\partial V}{\partial \theta} = -\Phi^T Y + \Phi^T \Phi \theta = 0 \tag{14}$$

where the minimum $\partial V/\partial\theta = 0$ provides the normal equation

$$\Phi^T \Phi \hat{\theta} = \Phi^T Y \tag{15}$$

Notice that Eqs. (14) and (15) are necessary conditions for obtaining a minimum. If the positive semidefinite matrix $\Phi^T\Phi$ is assumed to be invertible, then we can also show sufficiency by completing the square:

$$2V(\theta, t) = Y^T Y - Y^T \Phi \theta - \theta^T \Phi^T Y + \theta^T \Phi^T \Phi \theta$$

$$
\begin{aligned}
&+Y^T\Phi\left(\Phi^T\Phi\right)^{-1}\Phi^TY - Y^T\Phi\left(\Phi^T\Phi\right)^{-1}\Phi^TY \\
= \; & Y^T\left(I - \Phi\left(\Phi^T\Phi\right)^{-1}\Phi^T\right)Y \\
&+\left(\theta - \left(\Phi^T\Phi\right)^{-1}\Phi^TY\right)^T\Phi^T\Phi\left(\theta - \left(\Phi^T\Phi\right)^{-1}\Phi^TY\right)
\end{aligned}
$$

the first term on the right-hand side is independent of θ. The second term is always positive. It may be concluded that $V(\theta, t)$ has a unique minimum given by

$$\theta = \hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^TY$$

and the theorem is proven.

Remark 1. The matrix $\left(\Phi^T\Phi\right)^{-1}\Phi^T$ is called the pseudo-inverse of Φ if the matrix $\Phi^T\Phi$ is nonsingular.

Remark 2. Eq. (12) can be written as

$$\begin{aligned}\hat{\theta}(t) &= \left(\sum_{i=1}^{t} \varphi(i)\varphi^T(i)\right)^{-1} \left(\sum_{i=1}^{t} \varphi(i)y(i)\right) \\ &= P(t)\left(\sum_{i=1}^{t} \varphi(i)y(i)\right) \qquad (16)\end{aligned}$$

Remark 3. The condition that the matrix $\Phi^T\Phi$ is invertible is called an excitation condition.

Remark 4. The least-squares criterion weights all errors $\varepsilon(t)$ equally.

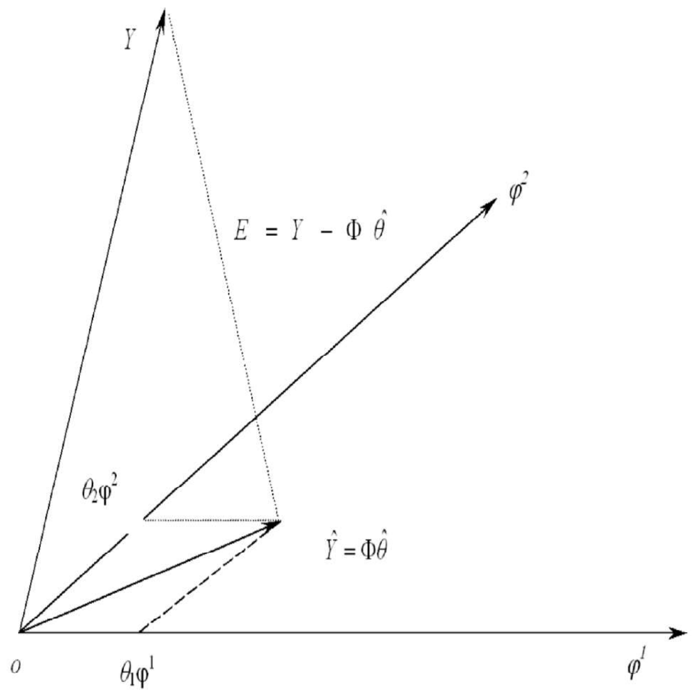
Different weighting of the errors can be accounted for by changing the loss function (8) to

$$V = \frac{1}{2}E^T W E$$

where W is a diagonal matrix with the weights in the diagonal. The least-squares estimate is then given by

$$\hat{\theta} = \left(\Phi^T W \Phi\right)^{-1} \Phi^T W Y$$

Geometric Interpretation



The least-squares problem can be interpreted as a geometric problem in R^t, where t is the number of observations. The figure shows the situation with 2 parameters and three observations. The vectors φ^1

and φ^2 spans a plane if they are linearly independent. The predicted output $\hat{Y}$ lies in the plane spanned by φ^1 and φ^2. The error $E = Y - \hat{Y}$ is smallest when E is orthogonal to this plane. In general, Eq. (10) can be written as

$$\begin{bmatrix} \varepsilon(1) \\ \varepsilon(2) \\ \vdots \\ \varepsilon(t) \end{bmatrix} = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(t) \end{bmatrix} - \begin{bmatrix} \varphi_1(1) \\ \varphi_1(1) \\ \vdots \\ \varphi_1(1) \end{bmatrix} \theta_1 - \cdots - \begin{bmatrix} \varphi_n(1) \\ \varphi_n(1) \\ \vdots \\ \varphi_n(1) \end{bmatrix} \theta_n$$

or

$$E = Y - \varphi^1\theta_1 - \varphi^2\theta_2 - \ldots - \varphi^n\theta_n$$

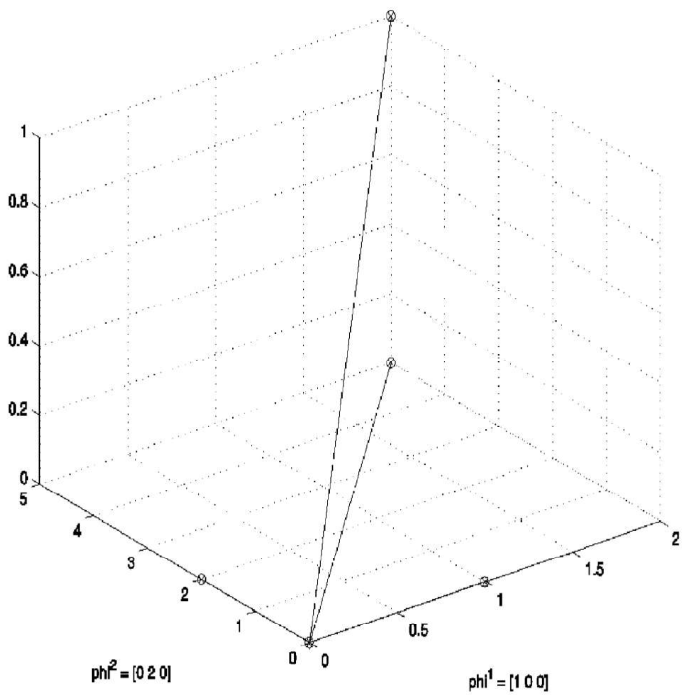
where φ^i are the columns of the matrix Φ. The least-squares problem is to find constant θ_1, …, θ_n such that the vector Y is approximated as well as possible by a linear combination of the vectors φ^1, φ^2, …,

φ^n. Let Y be the vector in the span of φ^1, φ^2, …, φ^n, which is the best approximation, and let $E = Y - \hat{Y}$. The vector E is smallest when it is orthogonal to all vectors φ^i. This gives

$$\left(\varphi^i\right)^T \left(Y - \varphi^1\theta_1 - \varphi^2\theta_2 - \ldots - \varphi^n\theta_n\right) = 0 \quad i = 1, \ldots, t$$

which is identical to the normal equation (11). The vector θ is unique if the vectors φ^1, φ^2, …, φ^n are linearly independent.

Geometric Interpretation Example



$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$\hat{\theta} = \left(\Phi^T \Phi\right)^{-1} \Phi^T Y = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix}$$

Statistical Interpretation

The least squares estimates $\hat{\theta}$ is a random variable whose properties can be analyzed. Two properties are important in this respect: bias and covariance. The term bias refers to the systematic error which can occur in the parameter estimate. The term covariance is related to the spread of estimates arising from the random errors.

Assume that the process is

$$y(i) = \varphi^T \theta^0 + e(i) \tag{17}$$

where θ^0 is the vector of true parameters and $\{e(i),\ i = 1, 2, \ldots\}$ is a sequence of independent random variables with zero mean. Equation (10) can be written as

$$Y = \Phi\theta^0 + E$$

Multiply by $\left(\Phi^T\Phi\right)^{-1}\Phi^T$ gives

$$\left(\Phi^T\Phi\right)^{-1}\Phi^T Y = \hat{\theta} = \theta^0 + \left(\Phi^T\Phi\right)^{-1}\Phi^T E \tag{18}$$

Provided that E is independent of Φ^T, which is equivalent to saying that $e(i)$ is independent of $\varphi(i)$, the mathematical expectation of $\hat{\theta}$ is equal to θ^0.

Theorem: Statistical properties of least-squares estimation

Consider the estimate in Eq. (12) and assume that data is generated from Eq. (17), where $\{e(i),\ i = 1, 2, \ldots\}$ is a sequence of independent random variables with zero mean and variance σ^2. Let $\mathcal{E}$ denote expectations and cov the covariance of a random variable. If $\Phi^T\Phi$ is nonsingular, then

$$\text{(i) } \mathcal{E}\left[\hat{\theta}(t)\right] = \theta^0$$

$$\text{(ii) } \operatorname{cov}\,\hat{\theta}(t) = \sigma^2\left(\Phi^T\Phi\right)^{-1}$$

Proof

Proofs of the above statements (i) and (ii) are straightforward according to the following calculations all of which rely on the assumption of E is independent of Φ^T

(i) From Eq. (18)

$$\begin{aligned}\hat{\theta} &= \theta^0 + \left(\Phi^T \Phi\right)^{-1} \Phi^T E \\ \mathcal{E}\left[\hat{\theta}(t)\right] &= \theta^0\end{aligned}$$

ii)

$$\begin{aligned}\text{cov }\hat{\theta}(t) &= \mathcal{E}\left[\left(\hat{\theta} - \theta^0\right)\left(\hat{\theta} - \theta^0\right)^T\right] \\ &= \mathcal{E}\left[\left(\Phi^T \Phi\right)^{-1} \Phi^T E E^T \Phi \left(\Phi^T \Phi\right)^{-1}\right] \\ &= \left(\Phi^T \Phi\right)^{-1} \Phi^T \mathcal{E}\left[E E^T\right] \Phi \left(\Phi^T \Phi\right)^{-1} \\ &= \sigma^2 \left(\Phi^T \Phi\right)^{-1}\end{aligned}$$

Recursive Computations

In adaptive controllers, the observations are obtained sequentially in real time. It is desirable to make the computations recursive in order to save computation time. The computations can be arranged in such a way that the results obtained at time $t-1$ can be used to get the estimates at time t.

From Eq. (9)

$$\begin{aligned} P^{-1}(t) &= \Phi^T(t)\Phi(t) = \sum_{i=1}^{t} \varphi(i)\varphi^T(i) && (19) \\ &= \sum_{i=1}^{t-1} \varphi(i)\varphi^T(i) + \varphi(t)\varphi^T(t) \\ &= P^{-1}(t-1) + \varphi(t)\varphi^T(t) && (20) \end{aligned}$$

The least-squares estimate $\hat{\theta}(t)$ is given by Eq. (16):

$$\hat{\theta}(t) = P(t)\left(\sum_{i=1}^{t} \varphi(i)y(i)\right) = P(t)\left(\sum_{i=1}^{t-1} \varphi(i)y(i) + \varphi(t)y(t)\right) \quad (21)$$

It follows from Eq (16)

$$\begin{aligned} \sum_{i=1}^{t-1} \varphi(i)y(i) &= P^{-1}(t-1)\hat{\theta}(t-1) \\ &= P^{-1}(t)\hat{\theta}(t-1) - \varphi(t)\varphi^T(t)\hat{\theta}(t-1) \end{aligned} \quad (22)$$

using Eq (20).

From Eqs. (21) and (22), the estimate at time t can now be written as

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) - P(t)\varphi(t)\varphi^T(t)\hat{\theta}(t-1) + P(t)\varphi(t)y(t) \\ &= \hat{\theta}(t-1) + P(t)\varphi(t)\left(y(t) - \varphi^T(t)\hat{\theta}(t-1)\right) \\ &= \hat{\theta}(t-1) + K(t)\epsilon(t) \end{aligned}$$

where

$$\begin{aligned} K(t) &= P(t)\varphi(t) \\ \varepsilon(t) &= y(t) - \varphi^T \hat{\theta}(t-1) \end{aligned}$$

To proceed, it is necessary to derive a recursive equation for $P(t)$ rather than for $P(t)^{-1}$ as in Equation (20). The following lemma is useful.

Lemma - Matrix inversion lemma

Let A, C and $C^{-1} + DA^{-1}B$ be nonsingular square matrices. Then,

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1} DA^{-1}$$

Proof: Multiply by $A + BCD$ and verify the identity.

Apply the Lemma to $P(t)$ and using Equation (20), we get

$$P(t) = \left(\Phi^T(t)\Phi(t)\right)^{-1} = \left(\Phi^T(t-1)\Phi(t-1) + \varphi(t)\varphi^T(t)\right)^{-1}$$

$$
\begin{aligned}
&= \left(P(t-1)^{-1} + \varphi(t)\varphi^T(t)\right)^{-1} \\
&= P(t-1) - P(t-1)\varphi(t)\left(I + \varphi^T(t)P(t-1)\varphi(t)\right)^{-1}\varphi^T(t)P(t-1)
\end{aligned}
$$

giving

$$
K(t) = P(t)\varphi(t) = P(t-1)\varphi(t)\left(I + \varphi^T(t)P(t-1)\varphi(t)\right)^{-1}
$$

Theorem: Recursive least-squares estimation (RLS)

Assume that the matrix $\Phi(t)$ has full rank, that is $\Phi^T(t)\Phi(t)$ is non-singular, for all $t > t_0$. Given $\hat{\theta}(t_0)$ and $P(t_0) = \left(\Phi^T(t_0)\Phi(t_0)\right)^{-1}$, the least-squares estimate $\hat{\theta}(t)$ then satisfies the recursive equations

$$
\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\left(y(t) - \varphi^T(t)\hat{\theta}(t-1)\right) && (23) \\
K(t) &= P(t)\varphi(t) = P(t-1)\varphi(t)\left(I + \varphi^T(t)P(t-1)\varphi(t)\right)^{-1} && (24) \\
P(t) &= P(t-1) - P(t-1)\varphi(t)\left(I + \varphi^T(t)P(t-1)\varphi(t)\right)^{-1}\varphi^T(t)P(t-1) \\
&= \left(I - K(t)\varphi^T(t)\right)P(t-1) && (25)
\end{aligned}
$$

Remark: Equation (23) has strong intuitive appeal. The estimate $\hat{\theta}(t)$ is obtained by adding a correction to the previous estimate $\hat{\theta}(t-1)$. The correction is proportional to $y(t) - \varphi^T(t)\hat{\theta}(t-1)$, where the last term can be interpreted as the value of y at time t predicted by the model of Eq. (7). The correction term is thus proportional to the difference between the measured value of $y(t)$ and the prediction of $y(t)$ based on the previous parameter estimate. The components of the vector $K(t)$ are weighting factors that tell how the correction and the previous estimate should be combined.

Notice that in Eq (19) the matrix $P(t)$ is defined only when the matrix $\Phi^T(t)\Phi(t)$ is nonsingular. Since

$$\Phi^T(t)\Phi(t) = \sum_{i=1}^{t} \varphi(i)\varphi^T(i)$$

it follows that $\Phi^T\Phi$ is always singular if $t < n$. To obtain an initial condition for P, it is thus necessary to choose $t = t_0$ such that $\Phi^T(t_0)\Phi(t_0)$ is nonsingular. The initial condition is then

$$P(t_0) = \left(\Phi^T(t_0)\Phi(t_0)\right)^{-1}$$

From Eq. (12):

$$\hat{\theta}(t_0) = P(t_0)\Phi^T(t_0)Y(t_0)$$

The recursive equation can then be used for $t > t_0$. It is, however, often convenient to use the recursive equations in all steps. If the recursive equations are started with the initial condition

$$P(0) = P_0$$

where P_0 is positive definite, then

$$P(t) = \left(P_0^{-1} + \Phi^T(t)\Phi(t)\right)^{-1}$$

Notice that $P(t)$ can be made arbitrarily close to $\left(\Phi^T(t)\Phi(t)\right)^{-1}$ by choosing P_0 sufficiently large.

Time-Varying Parameters

In the least-squares model (7) the parameters θ_i^0 are assumed to be constant. Two cases can be covered by simple extensions of the least-squares method. In one such case parameters are assumed to change abruptly but infrequently; in the other case the parameters are changing continuously but slowly. The case of abrupt parameter changes can be covered by resetting. The matrix P in the least-squares algorithm (Recursive least-squares estimation theorem) is then periodically reset to αI, where α is a large number. This implies that the gain $K(t)$ in the estimator becomes large and the estimate can be updated with a larger step. One approach for the case of slowly time-varying parameters is to replace the least-squares criterion of Eq. (8) with

$$V(\theta, t) = \frac{1}{2} \sum_{i=1}^{t} \lambda^{t-i} \left(y(i) - \varphi_k^T(i)\theta \right)^2$$

where λ is a parameter such that $0 < \lambda < 1$. The parameter λ is called the forgetting factor. The most recent data is given unit weight, but data that is n time units old is weighted by λ^n. the method is called exponential forgetting.

Theorem: Recursive least squares with exponential forgetting

Assume that the matrix $\Phi(t)$ has full rank for $t > t_0$. The parameter

θ, which minimizes Eq. (26), is given recursively by

$$\begin{aligned} \hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\left(y(t) - \varphi^T(t)\hat{\theta}(t-1)\right) \\ K(t) &= P(t)\varphi(t) = P(t-1)\varphi(t)\left(\lambda I + \varphi^T(t)P(t-1)\varphi(t)\right)^{-1} \\ P(t) &= \left(I - K(t)\varphi^T(t)\right)P(t-1)/\lambda \end{aligned} \tag{26}$$

A disadvantage of exponential forgetting is that data is discounted even if $K(t) = 0$. This condition gives $\hat{\theta}(t) = \hat{\theta}(t-1)$ and implies that $y(t)$ does not contain any new information about the parameter θ. In this case it follows from Eqs (26) that the matrix P increases exponentially with rate λ. Several ways to avoid this will be discussed later.

ESTIMATING PARAMETERS IN DYNAMICAL SYSTEMS

Finite-Impulse Response (FIR) Models

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + \ldots + b_n u(t-n)$$

or

$$y(t) = \varphi^T(t-1)\theta$$

where

$$\begin{aligned} \theta^T &= [b_1 \quad \ldots \quad b_n] \\ \varphi^T(t-1) &= [u(t-1) \quad \ldots \quad u(t-n)] \end{aligned}$$

The estimator is then given by the RLS Theorem.

Transfer Function Models

$$A(q)y(t) = B(q)u(t) \tag{27}$$

where

$$A(q) = q^n + aq^{n-1} + \ldots + a_n$$

$$B(q) = b_1 q^{m-1} + b_2 q^{m-2} + \ldots + b_m$$

Equation (27) can be written as

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) =$$
$$b_1 u(t+m-n-1) + \ldots + b_m u(t-n)$$

Introduce the parameter vector

$$\theta^T = [a_1 \ \ \ldots \ \ a_n \ \ b_1 \ \ \ldots \ \ b_m]$$

and the regression vector

$$\varphi^T(t-1) = [-y(t-1) \ \ \ldots \ \ -y(t-n) \ \ u(t+m-n-1) \ \ \ldots \ \ u(t-n)]$$

Therefore

$$y(t) = \varphi^T(t-1)\theta$$

The parameter estimates can be obtained by applying the least-squares estimation theorem. The matrix Φ is given by

$$\Phi = \begin{bmatrix} \varphi^t(n) \\ \vdots \\ \varphi^T(t-1) \end{bmatrix}$$

The method described will work when when the disturbances can be described as white noise added to the right-hand side of Eq. (27):

$$A(q)y(t) = B(q)u(t) + e(t)$$

Nonlinear Models

The essential restriction is that the models be linear in the parameters so that they can be written as linear regression models.

Example: Nonlinear system

$$y(t) + ay(t-1) = b_1 u(t-1) + b_2 \sin(u(t-1))$$

Let

$$\theta = [a \;\; b_1 \;\; b_2]^T$$

and

$$\varphi^T(t) = [-y(t) \;\; u(t) \;\; \sin(u(t))]$$

The model can be written as

$$y(t) = \varphi^T(t-1)\theta$$

the model is linear in the parameters, and the least-squares method can be used to estimate θ.

ADAPTIVE CONTROL TUTORIAL

1) Consider the unstable system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = Ax + Bu$$

which has the transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s-1)}$$

(a) Design a variable-structure controller. Choose the switching line as

$$\sigma = x_1 + x_2$$

and the amplitude of the "sign" function as 0.5.

(b) Draw the states as a function of time. The initial conditions are

$x_1(0) = 1.5$ and $x_2(0) = 0$.

(c) Draw the phase plane trajectory.

2) The model of a first-order system is given by

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) + d(t) \\ y(t) &= x(t) \end{aligned}$$

where b is a positive constant, $a^- \leq a \leq a^+$, $|x| \leq x^+$ are within known bounds, $u(t)$ is the control signal and disturbance, $d(t)$, is bounded as $|d(t)| \leq d^+$.

Define the sliding variable as

$$\sigma = -x(t),$$

the sliding controller as

$$u(t) = M\text{sign}(\sigma)$$

and the Lyapunov function as

$$V = \frac{\sigma^2}{2b}$$

The goal is to design a controller such that $x(t) = 0$ is an asymptotically stable solution.

(a) Determine $y(t)$ analytically, given $a^- = a^+ = a = 0.5$, $b = 1$, $d^+ = d = 0$, $M = 2$ and $x(0) = 1$.

(b) If $M > g^+$ then $\dot{V} < 0$. Determine g^+ in terms of a^+, x^+ and d^+.

3) Consider the system

$$y(t) = b_1 u(t-1) + b_2 u(t-2)$$

a) Given

t	0	1	2	3	4
$u(t)$	1	1	1	1	
$y(t)$			2	2	2

Find least-square estimates for $\hat{b}_1$ and $\hat{b}_2$.

b) Given

t	0	1	2	3	4
$u(t)$	0	1	1	1	
$y(t)$			1	2	2

Find least-square estimates for $\hat{b}_1$ and $\hat{b}_2$.

4) Consider the system

$$y(t) = ay(t-1) + e(t) + ce(t-1)$$

where $\{e(t)\}$ is a zero mean white noise sequence of variance σ. It is estimated as

$$y(t) = \hat{a}y(t-1) + e(t)$$

using least squares. Determine the asymptotic values of the estimate $\hat{a}$.

5) In an adaptive controller the process parameters are estimated according to the model

$$y(t) + a_1 y(t-1) = b_0 u(t-1) + b_1 u(t-2) + e(t)$$

The controller is given as

$$u(t) = u(t-1) - y(t)$$

The reference value is thus zero. Consider the case in which the controller parameters are constant. a) Show that the parameters a_1, a_2, b_0, b_1 cannot be uniquely determined. b) Show that with the controller

$$u(t) = u(t-1) + u(t-2) - y(t)$$

all process parameters can be estimated uniquely.

SOLUTION FOR ADAPTIVE CONTROL TUTORIAL

1(a) From lecture notes

$$\begin{aligned} u(t) &= -\frac{p^T f}{p^T g} - \frac{k}{p^T g}\text{sign}(\sigma(x)) \\ &= -2x_1 - 0.5\text{sign}(\sigma(x)) \end{aligned}$$

where

$$\begin{aligned} k &= 0.5 \\ p^T &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\ f &= \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \\ g &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

1(b) Substituting for $u(t)$ gives

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}(-2x_1 - 0.5\mathrm{sign}(\sigma))\\ &= \begin{bmatrix}-x_1 - 0.5\mathrm{sign}(\sigma)\\ x_1\end{bmatrix}\end{aligned}$$

At

$$\begin{aligned} t &= 0\\ x_1(0) &= 1.5;\\ x_2(0) &= 0 => \sigma > 0\end{aligned}$$

$$\begin{aligned} sX_1(s) - x_1(0) &= -X_1(s) - \frac{0.5}{s}\\ x_1(t) &= 2e^{-t} - 0.5\end{aligned}$$

$$\begin{aligned}
x_2(t) &= \int x_1 dt = -1.5e^{-t} - 0.5(t + e^{-t}) + c \\
x_2(0) &= 0 => c = 2 \\
x_2(t) &= -2e^{-t} - 0.5t + 2
\end{aligned}$$

When we reach the sliding line

$$\sigma = x_1(t) + x_2(t) = 0$$

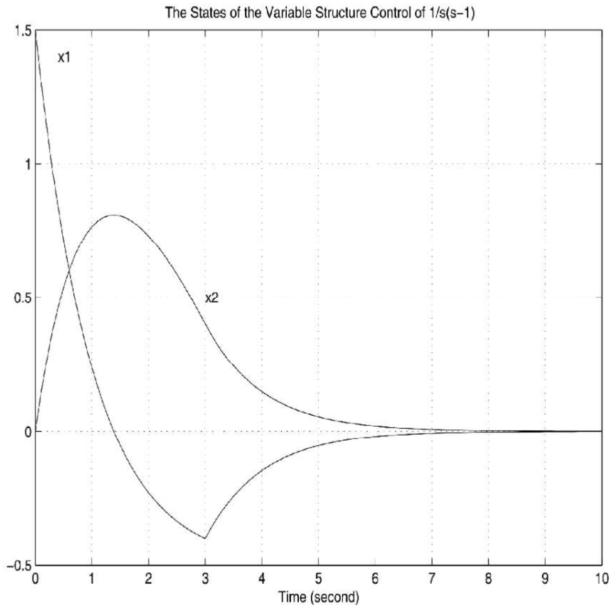
giving $t = 3$.

For $t \geq 3$, we are on the sliding line

$$\begin{bmatrix} \dot{x}_1(t+3) \\ \dot{x}_2(t+3) \end{bmatrix} = \begin{bmatrix} -x_1(t+3) \\ x_1(t+3) \end{bmatrix}$$

$$\begin{aligned}
se^{3s}X_1(s) - x_1(3) &= -e^{3s}X_1(s) \\
X_1(s) &= \frac{x_1(3)e^{-3s}}{s+1} = \frac{(2e^{-3} - 0.5)e^{-3s}}{s+1} = \frac{-0.4e^{-3s}}{s+1}
\end{aligned}$$

$$
\begin{aligned}
x_1(t) &= -0.4e^{-(t-3)} \\
x_2(t) &= \int x_1(t)dt = 0.4e^{-(t-3)} + c \\
x_2(3) &= -2e^{-3} - 0.5(3) + 2 = 0.4 => c = 0 \\
x_2(t) &= 0.4e^{-(t-3)}
\end{aligned}
$$



1(c) Phase trajectory

For $0 \leq t < 3$

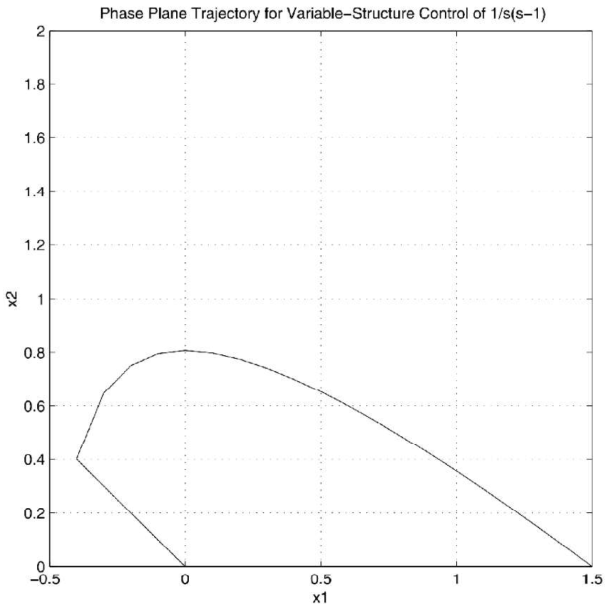
$$
\begin{aligned}
x_1(t) &= 2e^{-t} - 0.5 \\
t &= \ln\frac{2}{x_1 + 0.5}
\end{aligned}
$$

Substitute into

$$
\begin{aligned}
x_2(t) &= -2e^{-t} - 0.5t + 2 \\
&= 1.5 - x_1 - 0.5\ln\frac{2}{x_1 + 0.5}
\end{aligned}
$$

For $t > 3$, we are on the sliding line

$$x_1 + x_2 = 0$$



2(a)

$$
\begin{aligned}
\dot{x} &= 0.5x + u \quad 0 \leq t \leq t_\sigma \\
sX(s) - x(0) &= 0.5X(s) + U(s) \\
X(s) &= \frac{x(0)}{s-0.5} + \frac{1}{s-0.5}U(s) \\
\sigma &= -x
\end{aligned}
$$

$$
\begin{aligned}
u &= -M = -2 \\
X(s) &= \frac{1}{s-0.5} - \frac{2}{s(s-0.5)} \\
x(t) &= e^{0.5t} + 4(1-e^{0.5t}) \\
y(t) &= x(t) = 4 - 3e^{0.5t} \\
\sigma &= -x = 0 \\
4 - 3e^{-0.5t_\sigma} &= 0 \\
3e^{0.5t_\sigma} &= 4 \\
0.5t_\sigma &= \ln\frac{4}{3} \\
t_\sigma &= 0.58 \\
y(t) &= 0 \text{ for } t > t_\sigma
\end{aligned}
$$

2(b)

$$
\dot{x} = ax + bu + d
$$

$$
\begin{aligned}
y &= x \\
\sigma &= -x \\
u &= M\text{sign}(\sigma) \\
V &= \frac{\sigma^2}{2b} \\
\dot{V} &= \frac{\sigma}{b}\dot{\sigma} = \frac{\sigma}{b}(-\dot{x}) = \frac{\sigma}{b}(-ax - bu - d) \\
&= \sigma\left(\frac{-ax - d}{b} - u\right) \\
&= \sigma\left(\frac{-ax - d}{b} - M\text{sign}(\sigma)\right)
\end{aligned}
$$

$$
\text{For } \dot{V} < 0 \Rightarrow M > g^+ = \frac{a^+x^+ + d^+}{b}
$$

3a)

$$
\begin{aligned}
\Phi &= \begin{bmatrix} u(1) & u(0) \\ u(2) & u(1) \\ u(3) & u(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \\
\Phi^T\Phi &= \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \\
V &= \left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right)^T \left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\
&= 3\left(2 - b_1 - b_2\right)^2
\end{aligned}
$$

According to Eq. (2.5)

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Substitute into V gives $V = 0$ the minimum. The estimates $\hat{b}_1$ and $\hat{b}_2$ are not unique. Note that the dimension of

$$\Phi^T \Phi = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

depend upon the number of unknown parameters, not the number of data samples. Here it is singular and hence a unique least-squares solution cannot be obtained because $\Phi^T \Phi$ is noninvertible. This corresponds to a set of simultaneous equations where the number of equations is less than the number of unknowns.

3b)

$$
\begin{aligned}
\Phi &= \begin{bmatrix} u(1) & u(0) \\ u(2) & u(1) \\ u(3) & u(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \\
\Phi^T \Phi &= \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \\
V &= \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right)^T \left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) \\
&= (1 - b_1)^2 + 2\,(2 - b_1 - b_2)^2
\end{aligned}
$$

According to Eq. (2.5)

$$\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Substitute into V gives $V = 0$ the minimum. The estimates $\hat{b}_1$ and $\hat{b}_2$ are unique. Note that

$$\Phi^T \Phi = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

is invertible.

4) The model can be written as

$$Y = \Phi\hat{\theta}$$

where

$$\begin{aligned}
Y(t) &= \begin{bmatrix} y(1) \; y(2) \; \dots \; y(t) \end{bmatrix} \\
\Phi(t) &= \begin{bmatrix} -y(0) \; -y(1) \; \dots \; -y(t-1) \end{bmatrix}^T \\
\hat{\theta}(t) &= \hat{a}(t)
\end{aligned}$$

The least-square estimate is given by

$$\begin{aligned}
\hat{\theta} &= \left(\Phi^T\Phi\right)^{-1}\Phi^T Y \\
\hat{a}(t) &= \frac{\Sigma_{i=1}^{t} y(i)y(i-1)}{\Sigma_{i=1}^{t} y^2(i-1)}
\end{aligned}$$

Let

$$y(t) = ay(t-1) + \xi(t) \tag{1}$$

where

$$\xi(t) = e(t) + ce(t-1) \tag{2}$$

Therefore

$$\hat{a}(t) = a + \frac{\Sigma_{i=1}^{t}\xi(i)y(i-1)}{\Sigma_{i=1}^{t}y^2(i-1)}$$

$$\hat{a}(t) = a + \frac{\mathcal{E}\left[\xi(i)y(i-1)\right]}{\mathcal{E}\left[y^2(i-1)\right]}$$

for large t, assuming ergodicity. From Eq. (2)

$$\mathcal{E}\left[\xi(t)\xi(t)\right] = \left(1+c^2\right)\sigma^2$$

$$\mathcal{E}\left[\xi(t)\xi(t-1)\right] = c\sigma^2$$

$$\mathcal{E}\left[\xi(t)\xi(t-k)\right] = 0 \quad k = 2, 3, 4, \ldots$$

Using Eqs. (1) and (2) and defining $y(0) = 0$ gives

$y(1) = \xi(1);\ \ y(2) = \xi(2) + a\xi(1);\ \ y(3) = \xi(3) + a\xi(2) + a^2\xi(1);$

$$y(t-1) = \xi(t-1) + a\xi(t-2) + a^2\xi(t-3) + \ldots$$

This leads to

$$\mathcal{E}\left[\xi(i)y(i-1)\right] = \mathcal{E}\left[\xi(i)\xi(i-1)\right] = c\sigma^2$$

$$\mathcal{E}\left[y^2(t-1)\right] =$$

$$\left(1 + a^2 + a^4 + \ldots\right)\mathcal{E}\left[\xi^2(t-1)\right] + 2a\left(1 + a^2 + a^4 + \ldots\right)\mathcal{E}\left[\xi(t-1)\xi(t-2)\right]$$

$$= \frac{1+c^2}{1-a^2}\sigma^2 + \frac{2ac}{1-a^2}\sigma^2$$

Finally

$$\hat{a} = a + \frac{c(1-a^2)}{1+c^2+2ac}$$

This shows that a bias is present unless c is zero.

5a) Consider 3 samples.

$$\Phi = \begin{bmatrix} -y(t-1) & u(t-1) & u(t-2) \\ -y(t) & u(t) & u(t-1) \\ -y(t+1) & u(t+1) & u(t) \end{bmatrix}$$

$$\begin{aligned} \det\Phi &= -y(t-1)[u^2(t) - u(t-1)u(t+1)] \\ &\quad +y(t)[u(t)u(t-1) - u(t+1)u(t-2)] \\ &\quad -y(t+1)[u^2(t-1) - u(t)u(t-2)] \end{aligned}$$

Substitue $y(t) = -u(t) + u(t-1)$ gives $\det\Phi = 0$. Hence Φ is singular. The estimates $\hat{\theta}$ are not unique.

3b) Substitute $y(t) = -u(t) + u(t-1) + u(t-2)$ gives

$$\begin{aligned} \det\Phi &= -u(t-3)u^2(t) + u(t-3)u(t-1)u(t+1) \\ &\quad +u^2(t-2)u(t) - u(t-2)u(t+1)u(t-2) \end{aligned}$$

$$+u^2(t-1)u(t-2) - u(t-1)u^2(t-1)$$

If $\det \Phi \neq 0$ then Φ is nonsingular and the estimates $\hat{\theta}$ are unique.

ADAPTIVE CONTROL TUTORIAL 1

Consider the plant

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1-s}{(1+s)^2}$$

under relay feedback control

$$u = \begin{cases} 1 & \text{if } r - y > 0 \\ -1 & \text{if } r - y \leq 0 \end{cases}$$

where the reference $r = 0$.

1. Use the describing function method to find the limit cycle approximately. Hence find the ultimate gain K_u and ultimate period T_u.

2. Draw the phase plane trajectory of the system. Take the initial states as 0.
3. Use the result of (b) to find K_u and T_u.
4. Compute T_u accurately.
5. Replace the relay with a gain K and determine K_u and T_u.
6. Find the describing function of the relay with amplitude M and hysteresis h.

SOLUTION FOR ADAPTIVE CONTROL TUTORIAL 1

(1) The limit cycle is obtained when

$$\begin{aligned}
G(j\omega) &= -\frac{1}{N(A,\omega)} \\
N(A) &= \frac{4}{\pi A} \\
G(j\omega) &= \frac{1-j\omega}{(1+j\omega)^2} = -\frac{\pi A}{4} \\
4 - j4\omega &= -\pi A(1-\omega^2+2j\omega)
\end{aligned}$$

Compare real and imaginary parts give

$$\begin{aligned}
A &= \frac{2}{\pi} \\
\omega &= \sqrt{3}
\end{aligned}$$

Therefore limit cycle is approximately a sine wave of

$$\frac{2}{\pi}\sin\sqrt{3}t$$

$$\begin{aligned} K_u &= \frac{4d}{a\pi} = 2 \\ T_u &= \frac{2\pi}{\sqrt{3}} \end{aligned}$$

(2)

$$\frac{Y(s)}{U(s)} = \frac{1-s}{(1+s)^2}$$

Let

$$Q(s) = \frac{Y(s)}{1-s} = \frac{u(s)}{(1+s)^2}$$

Therefore

$$\begin{aligned}(s^2+2s+1)Q(s) &= u(s)\\ \ddot{q}+2\dot{q}+q &= u\end{aligned}$$

Define

$$\begin{aligned}x_1 &= = q\\ x_2 &= = \dot{q}\end{aligned}$$

From

$$\begin{aligned}Y(s) &= -sQ(s)+Q(s)\\ y &= -\dot{q}+q\end{aligned}$$

gives the state space description of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now with the relay

$$u = \begin{cases} 1 & \text{for } y = x_1 - x_2 < 0 \\ -1 & \text{for } y = x_1 - x_2 \geq 0 \end{cases}$$

Substitute for u gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ f(y) \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ -x_1 - 2x_2 - f(y) \end{bmatrix}$$

$$
\begin{aligned}
sX_1(s) - x_{10} &= X_2(s) \\
sX_2(s) - x_{20} &= -X_1(s) - 2X_2(s) - \frac{f(y)}{s}
\end{aligned}
$$

where

$$
f(y) = \begin{cases} 1 & \text{for } y > 0 \\ -1 & \text{for } y \le 0 \end{cases}
$$

Solving the simultaneous equations give

$$
\begin{aligned}
X_1(s) &= -\frac{x_{20} + x_{10} + f(y)}{s(s+1)^2} + \frac{x_{20}}{s(s+1)} + \frac{x_{10}}{s} \\
x_1(t) &= -(x_{20} + x_{10} + f(y))(1 - e^{-t} - te^{-t}) + x_{20}(1 - e^{-t}) + x_{10} \\
X_2(s) &= -\frac{x_{20} + x_{10} + f(y)}{(s+1)^2} + \frac{x_{20}}{s+1} \\
x_2(t) &= -(x_{20} + x_{10} + f(y))te^{-t} + x_{20}e^{-t}
\end{aligned}
$$

Condition for switching of the relay is $x_1 = x_2$ giving

$$e^{-t}(2t(x_{20} + x_{10} + f(y)) - x_{20} + x_{10} + f(y)) = f(y)$$

Assume that at $t = 0$, $y < 0$, $x_{10} = x_{20} = 0$.

$$\begin{aligned} x_1(t) &= (1 - e^{-t} - te^{-t}) \\ x_2(t) &= te^{-t} \end{aligned}$$

Relay switches at

$$e^{-t}(-2t - 1) = -1t = 1.3$$

solving by trial and error.

<u>t_1 interval; $y > 0$</u>

where $t = t_1 + 1.3$

At $t_1 = 0$;

$$x_{10} = x_{20} = 1.3e^{-1.3} = 0.36.$$

For $t_1 > 0$

$$\begin{aligned} x_1(t_1) &= -1 + 1.36e^{-t_1} + 1.72t_1e^{-t_1} \\ x_2(t_1) &= 0.36e^{-t_1} - 1.72t_1e^{-t_1} \end{aligned}$$

Relay switches at

$$\begin{aligned} e^{-t_1}(3.44t_1 + 1) &= 1 \\ t_1 &= 2.1 \end{aligned}$$

solving by trial and error.

$\underline{t_2 \text{ interval; } y < 0}$

where $t = t_2 + 2.1 + 1.3$

At $t_2 = 0$;

$$x_{10} = x_{20} = -1.72(2.1)e^{-2.1} + 0.36e^{-2.1} = -0.4$$

For $t_2 > 0$

$$\begin{aligned} x_1(t_2) &= 1 - 1.4e^{-t_2} - 1.8t_2e^{-t_2} \\ x_2(t_2) &= -0.4e^{-t_2} + 1.8t_2e^{-t_2} \end{aligned}$$

Relay switches at

$$\begin{aligned} e^{-t_2}(-3.6t_2 - 1) &= -1 \\ t_2 &= 2.2 \end{aligned}$$

solving by trial and error.

t_3 interval; $y > 0$

where $t = t_3 + 2.2 + 2.1 + 1.3$

At $t_3 = 0$;

$$x_{10} = x_{20} = 1.8(2.2)e^{-2.2} - 0.4e^{-2.2} = 0.4$$

For $t_3 > 0$

$$\begin{aligned} x_1(t_3) &= -1 + 1.4e^{-t_3} + 1.8t_3e^{-t_3} \\ x_2(t_3) &= 0.4e^{-t_3} - 1.8t_3e^{-t_3} \end{aligned}$$

Relay switches at

$$e^{-t_3}(3.6t_3 + 1) = 1$$
$$t_3 = 2.2$$

solving by trial and error.

<u>t_4 interval; $y < 0$</u>
where $t = t_4 + 2.2 + 2.2 + 2.1 + 1.3$

At $t_4 = 0$;

$$x_{10} = x_{20} = 0.4e^{-2.2} - 1.8(2.2)e^{-2.2} = -0.4$$

which is of the same initial condition at $t_2 = 0$. Therefore the solution for the t_4 interval is the same as the t_2 interval — limit cycle is reached.

(3)

$T_u = 2 \times 2.2 = 4.4$. The amplitude of oscillation at steady-state can be obtained from interval t_3.

$$
\begin{aligned}
y &= x_1 - x_2 = -1 + e^{-t_3} + 3.6t_3e^{-t_3} \\
\frac{dy}{dt_3} &= e^{-t_3}(2.6 - 3.6t_3) = 0 \\
t_3 &= 0.7 \\
y_{max} &= -1 + e^{-0.7} + 3.6(0.7)e^{-0.7} = 0.75 \\
K_u &= N(A) = \frac{4}{\pi A} = \frac{4}{\pi 0.75} = 1.7
\end{aligned}
$$

(4)

$$
\begin{aligned}
G_p(s) &= \frac{1}{(s+1)^2} - \frac{s}{(s+1)^2} \\
H_h(z) &= \frac{(1-e^{-h}(1+h))z + e^{-h}(e^{-h}+h-1)}{z^2 - 2e^{-h}z + e^{-2h}} - \frac{(z-1)he^{-h}}{(z-e^{-h})^2} \\
H_h(-1) &= 0
\end{aligned}
$$

$$
\begin{aligned}
h &= 2.178 \\
T_u &= 4.356
\end{aligned}
$$

(5)

$$
\begin{aligned}
K_u G(j\omega_u) &= -1 \\
\frac{K_u(1-j\omega_u)}{(1+j\omega)^2} &= -1 \\
-K_u + jK_u\omega_u &= 1-\omega_u^2+2j\omega_u
\end{aligned}
$$

Compare real and Imaginary parts give

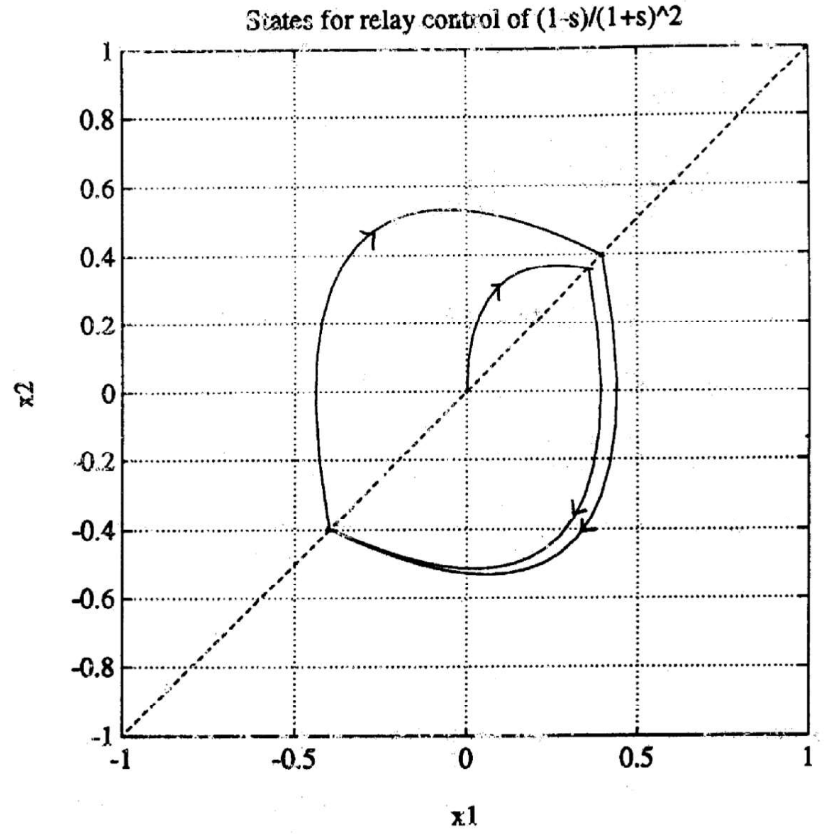
$$
\begin{aligned}
K_u &= 2 \\
\omega_u &= \sqrt{3} \\
T_u &= \frac{2\pi}{\sqrt{3}} = 3.627
\end{aligned}
$$

(6)

$$A\sin(\omega t_1) = h$$
$$\omega t_1 = \sin^{-1}(\frac{h}{A})$$
$$\cos(\omega t_1) = \frac{\sqrt{A^2 - h^2}}{A}$$

$$\begin{aligned} a_1 &= \frac{2}{\pi}\int_{\omega t_1}^{\pi+\omega t_1} M\cos(\omega t)d\omega t \\ &= \frac{2M}{\pi}(\sin(\omega t))|_{\omega t_1}^{\pi+\omega t_1} \\ &= -\frac{4Mh}{\pi A} \\ b_1 &= \frac{2}{\pi}\int_{\omega t_1}^{\pi+\omega t_1} M\sin(\omega t)d\omega t \end{aligned}$$

$$
\begin{aligned}
&= \frac{2M}{\pi}(-\cos(\omega t))|_{\omega t_1}^{\pi+\omega t_1} \\
&= \frac{4M}{\pi A}\sqrt{A^2 - h^2} \\
N(A) &= \frac{1}{A}(b_1 + ja_1) = \frac{4M}{\pi A}(\frac{\sqrt{A^2 - h^2}}{A} - j\frac{h}{A})
\end{aligned}
$$


States for relay control of (1-s)/(1+s)^2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
x1
x2

Tutorial Question on Self-Oscillating Adaptive System

Q.3 Consider Figure Q3. The process and relay are given as

$$G_p(s) = \frac{1}{s}e^{-s}$$

$$u_2 = \begin{cases} 0.1 & \text{if } u_1 > 0 \\ -0.1 & \text{if } u_1 < 0 \end{cases}$$

(a) What is the amplitude and frequency of the oscillation in y if

$$G_c(s) = 1 \quad ?$$

(b) Design a self-oscillating adaptive system to give a gain margin of 2 and oscillation amplitude of 0.04 in y.

(c) For the system you designed in part (b), what are the amplitude and frequency of the oscillation in y if the process is changed to

$$G_p(s) = \frac{3}{s}e^{-0.5s} \quad ?$$

(d) What is the gain margin of the system in part (c) ?

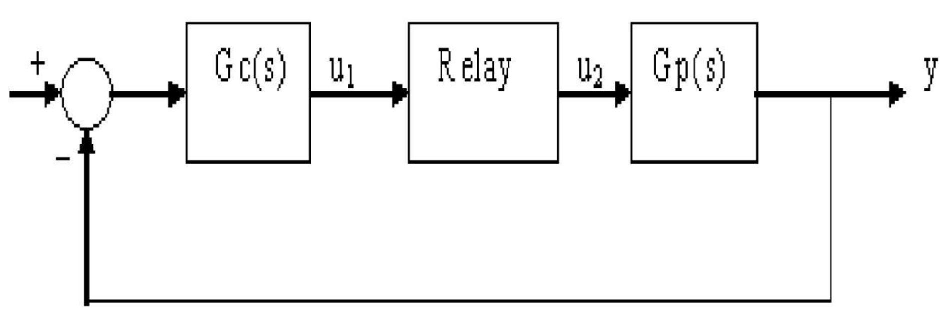


Figure Q3

<u>Tutorial Solution on Self-Oscillating Adaptive System</u>

Q3a

$$\begin{aligned}\angle G_p(j\omega) &= -\frac{\pi}{2} - \omega = -\pi \\ \omega &= \frac{\pi}{2} \\ d\left|G_j(j\frac{\pi}{2})\right| &= e \\ 0.1\left|\frac{1}{j\frac{\pi}{2}}\right| &= \frac{0.2}{\pi}\end{aligned}$$

Q3b
Calculate the frequency of oscillation to satisfy the amplitude of oscillation

$$\begin{aligned}0.1\left|\frac{1}{j\omega}e^{-j\omega L}\right| &= \frac{0.1}{\omega} = 0.04 \\ \omega &= 2.5\end{aligned}$$

Choose a lead-compensator

$$G_c(s) = K_c\frac{s+T_1}{s+T_2}$$

to satisfy the phase

$$\arctan\frac{\omega}{T_1} - \arctan\frac{\omega}{T_2} - \frac{\pi}{2} - \omega = -\pi$$

Choose

$$\begin{aligned}T_1 &= 1 \\ T_2 &= 10\end{aligned}$$

Calculate K_c such that

$$\begin{aligned}K_c\frac{\sqrt{\omega^2+T_1^2}}{\sqrt{\omega^2+T_2^2}} &= K_c\frac{\sqrt{2.5^2+1^2}}{\sqrt{2.5^2+10^2}} = 1 \\ K_c &= 3.8\end{aligned}$$

Q3c
When the plant change

$$\begin{aligned}\arctan\frac{\omega}{1} - \arctan\frac{\omega}{10} - \frac{\pi}{2} - 0.5\omega &= -\pi \\ \omega &= 5 \\ 0.1\left|\frac{3}{j\omega}e^{-j0.5\omega}\right| &= \frac{0.3}{5} = 0.06\end{aligned}$$

Q3d
The gain margin is 2.

EE5104 Adaptive Control Project

Consider the system

$$\begin{aligned}\dot{x}_1 &= ax_1 + bu + d \\ \dot{x}_2 &= x_1\end{aligned}$$

The switching surface is defined by

$$\sigma = c_1 x_1 + c_2 x_2$$

where a, b, c_1 and c_2 are known a priori, and d is a bounded disturbance with $|d| \le d_{\max}$.

(1) Design a variable structure controller such that $x = 0$ is an asymptotically stable solution.

(2) Simulate the dynamic behaviour of the system assuming that $a = 2$, $b = 1$, $c_1 = c_2 = 1$, $d = 0.9\sin(628t)$, $d_{\max} = 0.9$ and $\mu = 0.5$. Consider both sign() and sat() function. For the sat() function, let $\varepsilon = 0.01$. Give the phase portrait, plots of states and control signal versus time and discuss the results.

Adaptive Control Project EE6104

Iterative feedback tuning (IFT) is a flexible methodology for tuning controllers of arbitrary structure. The key feature is that closed-loop experimental data is used to directly compute a change of the controller parameters such that some performance objective is improved. Since no modeling step is required, the method is relatively simple to use. Thorough presentations of the method are provided in Hjlmarsson (2002). There is also a Special Issue on IFT in Control Engineering Practice (2003). IFT is also applied to relay auto-tuning (Ho et al., 2003). Given the simplicity of the scheme, it became clear that this new scheme had wide-ranging potential, from the optimal tuning of simple PID controllers to the systematic design of controllers of increasing complexity that have to meet some pre-specified specifications. In particular, the IFT method is appealing to process control engineers because under this scheme, the controller parameters can be successively improved without ever opening the loop. In addition, the idea of improving the performance of an already operating controller, on the basis of closed loop data, corresponds to a natural way of thinking.

In the area of semiconductor, the trend is towards larger wafer size and the linewidth going below 100nm. One of the challenges is to control the resist thickness and uniformity to a tight tolerance in order to minimize the thin-film interference effect on the linewidth. In Lee et. al (2002) the Generalized Predictive Controller (GPC) was used to improve resist thickness control and uniformity through the softbake process. There is however, no adaptation and a fixed controller may not be suitable for different batches of photoresist and wafers.

In this project, replace the GPC controller with the PI controller and apply IFT to tune the PI parameters for the above semiconductor manufacturing problem (Lee et. al, 2002). Show through simulation that the PI controller can adapt to new batches of photoresist and wafers. There is no need to consider constraint on the control signal.

Reference

Hjalmarsson, H. (2002). Iterative feedback tuning – an overview. International Journal on Adaptive Control and Signal Processing, Volume 16, pp. 373-395.

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Ho W.K., Y Hong, A. Hansson, H. Hajlmarsson and J.W. Deng (2003). Relay Auto-Tuning of PID controllers using Iterative Feedback Tuning, Automatica Volume 39, pp. 149-175.

Lee L.L., C.D. Schaper and W.K. Ho (2002). Real-Time Predictive Control of Photoresist Film Thickness Uniformity, IEEE Trans. On Semiconductor Manufacturing, Volume 15, No 1, pp. 51-59.

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