EE5137 2019/20 (Sem 2): Quiz 2 (Total 30 points)

Name:______________________________

Matriculation Number:_______________

Score:______________________________

You have 1.0 hour for this quiz. There are SEVEN (7) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. (a) (5 points) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda > 0$. Let T_1 be an exponential random variable with probability density function

$$f_{T_1}(t) = \begin{cases} \nu \exp(-\nu t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_1)$, the number of Poisson arrivals of the first process in the interval $[0, T_1]$?

(b) (5 points) Let $\{N(t) : t > 0\}$ be as in part (a). Now, let T_2 be an Erlang random variable of order 2 with probability density function

$$f_{T_2}(t) = \begin{cases} \nu^2 t \exp(-\nu t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_2)$, the number of Poisson arrivals of the first process in the interval $[0, T_2]$?

Hint: Drawing a figure might be helpful.

2. Buses arrive at a certain bus stop according to a Poisson process with rate 2 per hour. Passengers arrive according to an independent Poisson process with rate 10 per hour. The instant a bus arrives, all passengers at the stop at that instant board the bus and the bus departs.

   You may use the following fact without proof: An exponential random variable with rate λ, i.e., density
$$f_X(x) = \lambda \exp(-\lambda x) \mathbb{1}_{x \geq 0}$$
   has mean $1/\lambda$ and variance $1/\lambda^2$.

   (a) (2 points) Assume that there are currently no passengers at the bus stop. What is the probability that the next bus will pick up no passengers? Explain.

   (b) (2 points) If you arrive at the stop at noon, what is the expected amount of time you will have to wait until the next arrival of any type (bus or passenger)? Explain.

(c) (2 points) At 2:00pm, there are 2 passengers waiting for the bus. Given this information, what is the expected arrival time of the next bus after 2:00pm? Explain.

(d) (2 points) Show that

$$\mathrm{Var}(Y) = \mathbb{E}[\mathrm{Var}(Y|T)] + \mathrm{Var}(\mathbb{E}[Y|T])$$

where $\mathrm{Var}(Y|T) := \mathbb{E}[Y^2|T] - (\mathbb{E}[Y|T])^2$.

(e) (4 points) Assume that there are currently no passengers at the bus stop. Let Y be the number of people at the stop when the next bus arrives. Find $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$, showing all your work. For $\mathrm{Var}(Y)$, use part (d).

3. There are three types of MRT trains—Red, Blue and Green. MRT trains arrive at Kent Ridge NUS station according to a Poisson process with an arrival rate of λ trains per minute. Trains arriving after 5.00pm always arrive in the following order: Red always comes first; followed by Blue; then by Green. An EE5137 student Alice *must* take the Green train. In the following problems, δ is a very small constant. You may use the fact that an Erlang distribution of order k with rate λ is

$$f_{S_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}, \qquad t \geq 0$$

and that for any continuous function g,

$$\int_t^{t+\delta} g(s)\,\mathrm{d}s \approx g(t)\delta.$$

(a) (2 points) On week 1, Alice arrives at Kent Ridge at exactly 5:00pm. What is the probability that she will wait between t and $t+\delta$ minutes for her green MRT train?

(b) (3 points) However, Alice's lecturer has a poor grasp of time, ending his lecture late on week 2. Therefore, Alice reaches the Kent Ridge station at 5.05pm. What is the probability that her green MRT train arrives between τ and $\tau+\delta$ minutes <u>after</u> Alice? Note that τ may be negative.

(c) (3 points) Fortunately, Alice's friend Bob has been Kent Ridge since 5:00pm. He told Alice that she hasn't missed her Green bus; the Red train came at 5:03pm, but the Blue and Green trains have not arrived yet. Assuming that Bob is reliable, what is the probability that Alice will wait between τ and $\tau + \delta$ minutes for her bus?