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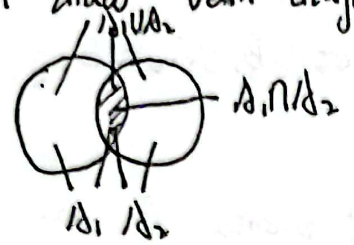
1. $Pr\{A_1 \cup A_2\} + Pr\{A_1 \cap A_2\} = Pr\{A_1\} + Pr\{A_2\}$.

. We can draw Venn diagrams.



~~Left~~ Double counted: $A_1 \cap A_2$

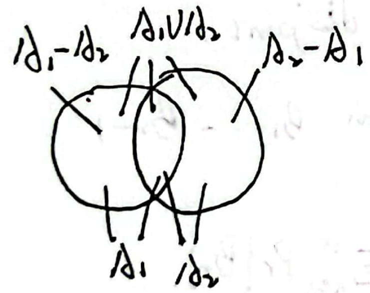
Counted onece: $A_1 \cup A_2 - A_1 \cap A_2 = A_1 \setminus A_2 + A_2 \setminus A_1$

2. (a). For two arbitrary events A_1 and A_2, show that:

① $A_1 \cup A_2 = A_1 \cup (A_2 - A_1)$

② A_1 and $A_2 - A_1$ are disjoint.

Pf: ①. We car draw the Venn diagram:



In this digram, we can see that $A_1 \cup A_2 = (A_1 - A_2) \cup (A_1 \cap A_2) \cup (A_2 - A_1)$

$= A_1 \cup (A_2 - A_1)$

$= (A_1 - A_2) \cup A_2$

②. Suppose that w is a sample point. w is in exactly one of:

A_1 or $A_2 - A_1$, but not both. (In the diagram), So, A_1 and $A_2 - A_1$ are disjoint.

2.(b)

Pf: $B_n = A_n - \bigcup_{m=1}^{n-1} A_m$, So $B_1 = A_1$, $B_2 = A_2 - A_1$, $B_3 = A_3 - A_1 \cup A_2$ ....

In Q.(a), we got $A_1 \cup A_2 = A_1 \cup (A_2 - A_1)$; A_1 and $A_2 - A_1$ are disjoint.

So, B_1 and B_2 are disjoint. $A_1 \cup A_2 = B_1 \cup B_2$. , Let $B_1 \cup B_2 = C_1$,

$\Rightarrow B_3 = A_3 - A_1 \cup A_2 = A_3 - B_1 \cup B_2 = A_3 - C_1$

Because $\{A_n; n \geq 1\}$ is an arbitrary sequence of events, A_3 and C_1 are also arbitrary. It's satisfy the Q.(a) condition, Let C_1 play the role of A_1, and $A_3 \rightarrow A_2$.

So, C_1 and $A_3 - C_1$ are disjoint, $A_3 \cup C_1 = C_1 \cup (A_3 - C_1) = C_1 \cup B_3$

$\Rightarrow A_1 \cup A_2 \cup A_3 = B_1 \cup B_2 \cup B_3$

Let $C_n = \bigcup_{i=1}^{n+1} A_i$, we can use induction to prove. $C_n = \bigcup_{i=1}^{n+1} B_i$. Assume $C_{n-1} = \bigcup_{i=1}^{n} B_i$

$C_n = \bigcup_{i=1}^{n+1} A_i = C_{n-1} \cup A_{n+1} = C_{n-1} \cup (A_{n+1} - C_{n-1}) = C_{n-1} \cup B_{n+1} = \bigcup_{i=1}^{n+1} B_i$

We can use Q.(a)'s conclution, let C_{n-1} play the role of A_1, B_{n+1} play the role of A_2, C_{n-1} and $B_{n+1} = A_{n+1} - C_{n-1}$ are disjoint.

Since $C_{n-1} = \bigcup_{i=1}^{n} B_i$, B_n is disjoint from $B_1 \cdots B_{n-1}$

2.(c). Show that $Pr\{\bigcup_{n=1}^{\infty} A_n\} = Pr\{\bigcup_{n=1}^{\infty} B_n\} = \sum_{n=1}^{\infty} Pr\{B_n\}$.

Pf: ① In Q2.(b), we know that. $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$

When $n \to \infty$, we can got: $\lim_{n\to\infty} \bigcup_{i=1}^{n} A_i = \lim_{n\to\infty} \bigcup_{i=1}^{n} B_i$

From axiom (iii), we know that. $\lim_{n\to\infty} \bigcup_{i=1}^{n} A_i = \bigcup_{n=1}^{\infty} A_n$

So, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \Rightarrow Pr\{\bigcup_{n=1}^{\infty} A_n\} = Pr\{\bigcup_{n=1}^{\infty} B_n\}$, first equation is true.

② The second equation is the axiom (iii) of probability. ($B_1, B_2 \ldots$ is disjoint)

$Pr\{\bigcup_{n=1}^{\infty} B_i\} = \sum_{n=1}^{\infty} Pr(B_i)$.

2.(d) Show that for each n. $Pr\{B_n\} \le Pr\{A_n\}$. Use this to show that
$Pr\{\bigcup_{n=1}^{\infty} A_n\} \le \sum_{n=1}^{\infty} Pr\{A_n\}$.

P.f: ① Because $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$, we can see that every events in B_n is A_n's events, B_n is a subset of A_n: $B_n \subseteq A_n$.

So. $Pr\{B_n\} \le Pr\{A_n\}$.

② In Q2.(c), we got that: $Pr\{\bigcup_{n=1}^{\infty} A_n\} = Pr\{\bigcup_{n=1}^{\infty} B_n\} = \sum_{n=1}^{\infty} Pr\{B_n\}$.

Because $Pr\{B_n\} \le Pr\{A_n\}$. $\sum_{n=1}^{\infty} Pr\{B_n\} \le \sum_{n=1}^{\infty} Pr\{A_n\}$.

Thus. $Pr\{\bigcup_{n=1}^{\infty} A_n\} = \sum_{n=1}^{\infty} Pr\{B_n\} \le \sum_{n=1}^{\infty} Pr\{A_n\}$.

2.(e) Show that $Pr\{\bigcup_{n=1}^{\infty} A_n\} = \lim_{n\to\infty} Pr\{\bigcup_{n=1}^{m} A_n\}$.

P.f: From Q.2(c), we know $Pr\{\bigcup_{n=1}^{\infty} A_n\} = \sum_{i=1}^{\infty} Pr\{B_n\} = \lim_{n\to\infty} \sum_{i=1}^{n} Pr(B_i)$ ①

Because B_n's are disjoint,

$$\sum_{i=1}^{n} Pr(B_i) = Pr\{\bigcup_{i=1}^{n} B_i\} = Pr\{\bigcup_{i=1}^{n} A_i\}. \quad ②$$

$\Rightarrow$ Combine ① and ②

$$Pr\{\bigcup_{n=1}^{\infty} A_n\} = \lim_{n\to\infty} Pr\{\bigcup_{i=1}^{n} A_i\}.$$

2.(f) Show that $Pr\{\bigcap_{n=1}^{\infty} A_n\} = \lim_{n\to\infty} Pr\{\bigcap_{n=1}^{m} A_n\}$.

P.f: Using De Morgan's equalities:

$$Pr\{\bigcap_{n=1}^{\infty} A_n\} = Pr\{[\bigcup_{n=1}^{\infty} A_n^c]^c\} = 1 - Pr\{\bigcup_{n=1}^{\infty} A_n^c\}$$
$$= 1 - \lim_{n\to\infty} Pr\{\bigcup_{i=1}^{n} A_n^c\}$$
$$= \lim_{n\to\infty} Pr\{\bigcap_{i=1}^{n} A_n\}.$$

3. Select 5 cards from a 52-card deck, what is the probability that all four aces are in the first five cards of the deck.

Solution: ~~The~~ If we choose random cards, there are C_{52}^{5} events.

If 4 aces are in the deck, there are $C_4^4 \cdot C_{48}^1$ events.

$$P_r = \frac{C_4^4 \cdot C_{48}^1}{C_{52}^5} = \frac{1 \cdot 48}{\frac{52!}{5! \cdot (52-5)!}} = \frac{1}{54145}$$

~~4. (a) Pf: Since F_1 and F_2 are σ-algebras, assume sample space is Ω_1 and Ω_2~~

~~Let~~

4. (a) ① Since F_1 and F_2 are σ-algebras on the sample space Ω, Ω is also a event of $F_1 \cap F_2$. (axiom i)

~~② Let $A \in F_1 \cap F_2$, so $A^c \in$~~

② Let $A_1, A_2, \ldots \in F_1 \cap F_2$, it means $\forall i \in \mathbb{N}$, $A_i \in F_1$, and $A_i \in F_2$

So we have that $\cup_{i=1}^{\infty} A_i \in F_1$ and $\cup_{i=1}^{\infty} A_i \in F_2$

$$\Rightarrow \cup_{i=1}^{\infty} A_i \in F_1 \cap F_2 \text{ (axiom ii)}$$

③ Let $A \in F_1 \cap F_2$, so $A^c \in F_1$, and $A^c \in F_2$

$$\Rightarrow A^c \in F_1 \cap F_2 \text{ (axiom iii)}$$

In conclution, $F_1 \cap F_2$ is a σ-algebra.

4.(b)

Axiom i: Because $\{F_\alpha\}_{\alpha\in I}$ is a family of σ-algebra on sample space Ω.

$\Omega \in \{F_\alpha\}_{\alpha\in I}$

$\Rightarrow \Omega \in \bigcap_{\alpha\in I} F_\alpha$

Axiom ii: Let $A_1, A_2 \ldots \in \bigcap_{\alpha\in I} F_\alpha$. $\forall i \in \mathbb{N}$. $A_i \in \{F_\alpha\}_{\alpha\in I}$.

Since $\{F_\alpha\}_{\alpha\in I}$ is a family of σ-algebra

$\bigcup_{n=1}^{\infty} A_n \in \{F_\alpha\}_{\alpha\in I}$.

So: $\bigcup_{n=1}^{\infty} A_n \in \bigcap_{\alpha\in I} F_\alpha$

Axiom iii: ~~Let~~ For every $A \in \bigcap_{\alpha\in I} F_\alpha$. $A \in \{F_\alpha\}_{\alpha\in I}$.

Since $\{F_\alpha\}_{\alpha\in I}$ is a family of σ-algebra.

$A^c \in \{F_\alpha\}_{\alpha\in I} \Rightarrow A^c \in \bigcap_{\alpha\in I} F_\alpha$.

So, $\bigcap_{\alpha\in I} F_\alpha$ is a σ-algebra

4.(c)

Axiom i: Because $\{F_\alpha\}_{\alpha\in I}$ is a family of σ-algebra on sample space Ω.

So $\bigcup_{\alpha\in I} F_\alpha$ is also on sample space Ω.

$\Rightarrow \Omega \in \bigcup_{\alpha\in I} F_\alpha$.

Axiom ii: Let $A_1, A_2 \ldots \in \bigcup_{\alpha\in I} F_\alpha$. $\forall i \in \mathbb{N}$, A_i, we can find at least one σ-algebra F_n which satisfy: $A_i \in F_n$.

Since F_n is a σ-algebra: all of it's events $A_i, A_{i+1} \ldots \in F_n$.

$\bigcup_{n=1}^{\infty} A_n \in F_n$

So: $\bigcup_{n=1}^{\infty} A_i \in \bigcup_{\alpha\in I} F_\alpha$

Axiom iii: For every $A \in \bigcup_{\alpha\in I} F_\alpha$, we can find at least one σ-algebra F_n from $\{F_\alpha\}_{\alpha\in I}$.

$A \in F_n$ and $A^c \in F_n$.

Because $F_n \in \{F_\alpha\}_{\alpha\in I}$

So: $F_n \in \bigcup_{\alpha\in I} F_\alpha$. $A \in \bigcup_{\alpha\in I} F_\alpha$, $A^c \in \bigcup_{\alpha\in I} F_\alpha$.

5. Because $A \cup B = A \cup (B - A)$, LHS can be expressed as:

$$LHS = Pr\left\{\bigcup_{i=1: i \neq k}^{n} A_i \cup A_k\right\} = Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i \cup \left(A_k - \bigcup_{i=1: i\neq k}^{n} A_i\right)\right\}$$

Since A and $B-A$ are disjoint.

$$LHS = Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i\right\} + Pr\left\{A_k - \bigcup_{i=1: i\neq k}^{n} A_i\right\}$$

$$= Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i\right\} + Pr(A_k) - Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i \cap A_k\right\}.$$

Compare with RHS, we just need to prove:

$$Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i\right\} - Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i \cap A_k\right\} \leq \sum_{i=1: i\neq k}^{n} Pr\{A_i\} - \sum_{i=1: i\neq k}^{n} Pr\{A_i \cap A_k\}$$

$$= \sum_{i=1: i\neq k}^{n} \left\{Pr(A_i) - Pr(A_i \cap A_k)\right\}.$$

$$= \sum_{i=1: i\neq k}^{n} Pr(A_i - A_k).$$

We can use induction to prove that:

Assume: $Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i\right\} - Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_k\right\} \leq \sum_{i=1: i\neq k}^{n-1} Pr(A_i - A_k).$ ①

Since $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$, $Pr(A \cap B) \geq 0$.

~~We~~ We have: $Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cup A_n\right\} = Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i\right\} + Pr(A_n) - Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_n\right\}.$ ②

$$Pr\left\{\left(\bigcup_{i=1: i\neq k}^{n-1} A_i \cup A_n\right) \cap A_k\right\} = Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_k\right\} + Pr(A_n \cap A_k) - Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_k \cap A_n\right\} ③$$

Combine ② and ③, we get:

$$Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i\right\} - Pr\left\{\bigcup_{i=1: i\neq k}^{n} A_i \cap A_k\right\} = \left[Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i\right\} - Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_k\right\}\right] + \left[Pr(A_n) - Pr(A_n \cap A_k)\right]$$
$$+ \left[Pr\left(\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_n \cap A_k\right) - Pr\left(\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_n\right)\right]$$

$$\leq \sum_{i=1: i\neq k}^{n} Pr(A_i - A_k) + \left[Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_n \cap A_k\right\} - Pr\left\{\bigcup_{i=1: i\neq k}^{n-1} A_i \cap A_n\right\}.\right]$$

Because $Pr(A \cap B \cap C) \leq Pr(A \cap B)$, So: $RHS \leq \sum_{i=1: i\neq k}^{n} Pr(A_i - A_k)$.

Therefore. $LHS \leq RHS$.