EE5137 Semester 1 2018/9: Quiz 1 (Total 24 points)

Name:______________________________

Matriculation Number:_______________

Score:______________________________

You have 1.0 hour for this quiz. There are FOUR (4) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. (8 points) [Distribution Functions] Which of the following functions is a cumulative distribution function (CDF)? For those which are, compute the probability density function (PDF). For those which are not, explain what fails.

   (a)

$$F_X(x) = \begin{cases} 1 - e^{-x^2} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

(b)

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{1}{3} & 0 < y \leq \frac{1}{2} \\ 1 & y > \frac{1}{2}. \end{cases}$$

2. (8 points) [Strengthened Union Bound]

   Let $A_1, \dots, A_n$ be arbitrary events. Prove that

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} \le \min_{1\le k\le n}\left(\sum_{i=1}^{n}\Pr\{A_i\} - \sum_{i=1\,:\,i\neq k}^{n}\Pr\{A_i\cap A_k\}\right).$$

   Hint: For any two sets C and D,

$$C = (C\cap D)\cup(C\cap D^c)$$

3. [Conditional Expectations] (8 points)

   In this problem, we will calculate the expectation of a geometric random variable using the formula for iterated expectations. Let N be a geometric random variable with parameter p, i.e., N is the number of coin flips until Head appears and $\Pr(\text{Heads}) = p$. In other words $p_N(n) = (1-p)^{n-1}p$ for $n = 1, 2, \ldots$. Define the random variable

$$Y = \begin{cases} 1 & \text{first flip is Heads} \\ 0 & \text{else} \end{cases}$$

   (i) Calculate $\mathbb{E}[N|Y = y]$ for $y = 1$.

   (ii) Calculate $\mathbb{E}[N|Y = y]$ for $y = 0$ in terms of $\mathbb{E}[N]$.

   (iii) Now use the law of iterated expectations to deduce $\mathbb{E}[N]$.