EE5137 Stochastic Processes: Problem Set 9
Assigned: 18/03/22, Due: 25/03/22

There are six (6) non-optional problems in this problem set.

1. Exercise 4.12 (Gallager's book)

   Hint for Part (b): Choose j to maximize $|\pi_j^{(k)}|$ for the given k.

2. Exercise 4.16 (Gallager's book)

3. Exercise 4.17 (Gallager's book)

4. Consider the Markov chain whose transition probability matrix is

$$[P] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

   The first row/column corresponds to state 0 and the last row/column corresponds to state 5 and so on.

   (a) (4 marks) Classify the states $\{0, 1, 2, 3, 4, 5\}$ into classes.

   (b) (4 marks) Identify the recurrent and transient states.

   (c) (4 marks) Compute the period of each recurrent class.

   (d) (3 marks) Identify the ergodic states.

   (e) (5 marks) If the chain starts from state 1, find the steady state probabilities in each of the states $(\pi_0, \pi_1, \ldots, \pi_5)$.

   (f) (5 marks) Assuming again we start from state 1, evaluate

$$\lim_{n\to\infty} -\frac{1}{n} \log |[P^n]_{11} - \pi_1|,$$

   where $[M]_{ij}$ is the (i, j) element of the matrix $[M]$.

5. A fly moves along a straight line in unit increments. At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3 and stays in place with probability 0.4, independent of past movements. Two spiders are lurking at positions 1 and M; if a fly lands in positions 1 or M, it is captured by the spider and the process terminates. Let $j \in \{1, 2, \ldots, M\}$ be the position of the fly. The Markov chain $[P]$ is thus given by

$$p_{11} = 1 \quad p_{mm} = 1, \quad p_{ij} = \begin{cases} 0.3 & \text{if } |j-i| = 1 \\ 0.4 & \text{if } j = i \end{cases} \qquad \text{for} \quad i = 2, 3, \ldots, M-1.$$

Write down a system of linear equations to deduce the expected number of steps the fly takes given it starts from state j before being captured by one of the two spiders. This is denoted as v_j. For $M = 4$, solve your equations to find v_2 and v_3.

6. Consider the Markov chain

$$[P] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \\ 3/5 & 0 & 0 & 2/5 \end{bmatrix}$$

The steady-state probabilities are known to be

$$\pi_1 = \frac{6}{31} \quad \pi_2 = \frac{9}{31} \quad \pi_3 = \frac{6}{31} \quad \pi_4 = \frac{10}{31}.$$

Assume that the process is in state 1 before the first transmission.

(a) What's the probability that the process will be in state 1 just after the sixth transition?

(b) Determine the expected value and variance of the number of transitions up to and including the next transition during which the process returns to state 1.

(c) What is (approximately) the probability that the state of the system resulting from transition 1000 is neither the same as the state resulting from transition 999 nor the same as the state resulting from transition 1001?

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7. (Optional) Exercise 4.11 (Gallager's book)

8. (Optional) Exercise 4.18 (Gallager's book)

9. (Optional) Consider an m-state Markov chain where each state is either transient or absorbing (i.e., a state i is absorbing if $P_{ii} = 1$). Fix an absorbing state s. Show that the probabilities a_i of eventually reaching state s starting from a state i are the unique solutions to the equations

$$\begin{aligned} a_s &= 1 \\ a_i &= 0 \qquad &&\text{for all absorbing } i \neq s \\ a_i &= \sum_{j=1}^{m} P_{ij} a_j \qquad &&\text{for all transient } i \end{aligned}$$