EE5137 Stochastic Processes: Problem Set 1
Assigned: 14/01/22, Due: 21/01/22

There are six non-optional problems in this problem set.

1. Exercise 1.1 (Gallager's book)
2. Exercise 1.2 (Gallager's book)
3. Exercise 1.3 (Gallager's book)
4. We say that $A_1, A_2, A_3, \ldots$ is an *increasing* sequence of events if $A_i \subset A_{i+1}$ for all $i \geq 1$. Let

$$A = \lim_{i\to\infty} A_i = \bigcup_{i=1}^{\infty} A_i.$$

   (a) Show carefully using the axioms of probability we stated in class that

$$\Pr\{A\} = \lim_{i\to\infty} \Pr\{A_i\}$$

   (b) Formulate an analogous result for a *decreasing* sequence of events.

5. In Section 1.2.1 of Gallager's book, we saw that given a sample space Ω a σ*-algebra* $\mathfrak{F}$ of Ω is a collection of subsets of Ω that satisfies (i) $\Omega \in \mathfrak{F}$; (ii) For any sequence of sets $A_1, A_2, \ldots \in \mathfrak{F}$, $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{F}$; and (iii) For every $A \in \mathfrak{F}$, $\Omega \setminus A \in \mathfrak{F}$. The elements of $\mathfrak{F}$ are called *events* in probability theory and $\mathfrak{F}$-*measurable sets* in measure theory.

   (a) Show that if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are σ-algebras so is $\mathfrak{F}_1 \cap \mathfrak{F}_2$;
   (b) Is it true that if $\{\mathfrak{F}_\alpha\}_{\alpha\in\mathcal{I}}$ is a family of σ-algebras, so is $\bigcap_{\alpha\in\mathcal{I}} \mathfrak{F}_\alpha$?
   (c) Consider parts (a) and (b) for unions.

6. (Strengthened Union Bound) Let $A_1, \ldots, A_n$ be arbitrary events. Prove that

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} \leq \min_{1\leq k\leq n} \left(\sum_{i=1}^{n} \Pr\{A_i\} - \sum_{i=1\,:\, i\neq k}^{n} \Pr\{A_i \cap A_k\} \right).$$

   Hint: For any two sets C and D, $C = (C \cap D) \cup (C \cap D^c)$.

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7. (Optional) Often, by using the union bound or its variants (such as Question 6 or Gallager's ρ-trick[1]), it is easy to upper bound probabilities. Lower bounding probabilities is often harder, but very useful. Let $A_1, \ldots, A_n$ be arbitrary events. Prove that

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} \geq \sum_{i=1}^{n} \frac{\Pr\{A_i\}^2}{\sum_{j=1}^{n} \Pr\{A_i \cap A_j\}}.$$

[1]This says that $\Pr\{\cup_{i=1}^{n} A_i\} \leq \left(\sum_{i=1}^{n} \Pr\{A_i\}\right)^{\rho}$ for any $0 \leq \rho \leq 1$. Prove this.

This bound is called de Caen's lower bound. Obviously from the form of the inequality, you've to use the Cauchy–Schwarz inequality somewhere.

8. (Optional) This is another lower bound on the union of n events $A_1, \ldots, A_n$. Prove that

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} \geq \frac{\sum_{i,j} \Pr\{A_i\} \Pr\{A_j\}}{\sum_{i,j} \Pr\{A_i \cap A_j\}}.$$

This bound is called the Chung–Erdős inequality. Obviously from the form of the inequality, you've to use the Cauchy–Schwarz inequality somewhere.