EE5137 Stochastic Processes: Problem Set 6
Assigned: 18/02/22, Due: 04/03/22

There are six (6) non-optional problems in this problem set.

1. Exercise 2.10 (Gallager's book)

   Solution:

   (a) Note that $N(t+s)$ is the number of arrivals in $(0,t]$ plus the number in $(t,t+s)$. In order to find the joint distribution of $N(t)$ and $N(t+s)$, it makes sense to express $N(t+s)$ as $N(t)+\tilde{N}(t,t+s)$ and to use the independent increment property to see that $\tilde{N}(t,t+s)$ is independent of $N(t)$. Thus for $m > n$,

$$p_{N(t)N(t+s)}(n,m) = \Pr\{N(t)=n\}\Pr\{\tilde{N}(t,t+s)=m-n\} \tag{1}$$
$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!} \times \frac{(\lambda s)^{m-n}e^{-\lambda s}}{(m-n)!}, \tag{2}$$

   where we have used the stationary increment property to see that $\tilde{N}(t,t+s)$ has the same distribution as $N(s)$. This solution can be rearranged in various ways, of which the most interesting is

$$p_{N(t)N(t+s)}(n,m) = \frac{(\lambda(t+s))^m e^{-\lambda(s+t)}}{m!} \times \binom{m}{n}\left(\frac{t}{t+s}\right)^n \left(\frac{s}{t+s}\right)^{m-n}, \tag{3}$$

   where the first term is $p_{N(t+s)}(m)$ (the probability of m arrivals in $(0,t+s]$) and the second, conditional on the first, is the binomial probability that n of those m arrivals occur in $(0,t)$.

   (b) Again expressing $N(t+s) = N(t) + \tilde{N}(t,t+s)$,

$$\mathbb{E}[N(t).N(t+s)] = \mathbb{E}[N^2(t)] + \mathbb{E}[N(t)\tilde{N}(t,t+s)] \tag{4}$$
$$= \mathbb{E}[N^2(t)] + \mathbb{E}[N(t)]\mathbb{E}[N(s)] \tag{5}$$
$$= \lambda t + \lambda^2 t^2 + \lambda t \lambda s. \tag{6}$$

   In the final step, we have used the fact (from Table 1.2 or a simple calculation) that the mean of a Poisson rv with PMF $(\lambda t)^n \exp(-\lambda t)/n!$ is λt and the variance is also λt (thus the second moment is $\lambda t + (\lambda t)^2$). This mean and variance was also derived in Excercise 2.2 and can also be calculated by looking at the limit of shrinking Bernoulli processes.

   (c) This is straightforward generalization of what was done in (b). We break up $\tilde{N}(t_1,t_3)$ as $\tilde{N}(t_1,t_2)+$

$\tilde{N}(t_2, t_3)$ and break up $\tilde{N}(t_2, t_4)$ as $\tilde{N}(t_2, t_3) + \tilde{N}(t_3, t_4)$. The interval $(t_2, t_3]$ is shared. Thus,

$$
\begin{aligned}
&\mathbb{E}[\tilde{N}(t_1, t_3)\tilde{N}(t_2, t_4)] \\
&= \mathbb{E}[[\tilde{N}(t_1, t_2) + \tilde{N}(t_2, t_3)][\tilde{N}(t_2, t_3) + \tilde{N}(t_3, t_4)]] && (7)\\
&= \mathbb{E}[\tilde{N}(t_1, t_2)\tilde{N}(t_2, t_3)] + \mathbb{E}[\tilde{N}(t_1, t_2)\tilde{N}(t_3, t_4)] + \mathbb{E}[\tilde{N}(t_2, t_3)^2] + \mathbb{E}[\tilde{N}(t_2, t_3)\tilde{N}(t_3, t_4)]] && (8)\\
&= \mathbb{E}[\tilde{N}(t_1, t_2)(\tilde{N}(t_2, t_3) + \tilde{N}(t_3, t_4))] + \mathbb{E}[\tilde{N}(t_2, t_3)^2] + \mathbb{E}[\tilde{N}(t_2, t_3)\tilde{N}(t_3, t_4)] && (9)\\
&= \mathbb{E}[\tilde{N}(t_1, t_2)\tilde{N}(t_2, t_4)] + \mathbb{E}[\tilde{N}^2(t_2, t_3)] + \mathbb{E}[\tilde{N}(t_2, t_3)\tilde{N}(t_3, t_4)] && (10)\\
&= \lambda^2(t_2 - t_1)(t_4 - t_2) + \lambda^2(t_3 - t_2)^2 + \lambda(t_3 - t_2) + \lambda^2(t_3 - t_2)(t_4 - t_3) && (11)\\
&= \lambda^2(t_3 - t_1)(t_4 - t_2) + \lambda(t_3 - t_2). && (12)
\end{aligned}
$$

2. Exercise 2.16 (Gallager's book)

(a) For a Poisson counting process of rate λ, find the joint probability density of $S_1, S_2, \cdots, S_{n-1}$ conditions on $S_n = t$.
Solution: The joint density of $S_1, S_2, \cdots, S_n$ is given in equation (2.15) in the book as

$$f_{S_1,\cdots,S_n}(s_1, \cdots, s_n) = \lambda^n \exp(-\lambda s_n).$$

The marginal density of S_n is the Erlang density. The conditional density of the ratio of these, i.e.,

$$f_{S_1,\cdots,S_{n-1}|S_n}(s_1, \cdots, s_{n-1}|s_n) = \frac{\lambda^n \exp(-\lambda s_n)}{\lambda^n s_n^{n-1} e^{-\lambda s_n}/(n-1)!} = \frac{(n-1)!}{s_n^{n-1}}. \tag{13}$$

(b) Find $\Pr\{X_1 > \tau | S_n = t\}$.
Solution: We first use Bayes' law to find the density, $f_{X_1|S_n}(\tau|t)$.

$$
\begin{aligned}
f_{X_1|S_n}(\tau|t) &= \frac{f_{X_1}(\tau) f_{S_n|X_1}(t|\tau)}{f_{S_n}(t)} = \frac{\lambda e^{-\lambda\tau}\lambda^{n-1}(t-\tau)^{n-2}e^{-\lambda(t-\tau)}/(n-2)!}{\lambda^n t^{n-1}e^{-\lambda t}/(n-1)!} && (14)\\
&= \frac{(t-\tau)^{n-2}(n-1)}{t^{n-1}}, \quad \text{for} \quad \tau < t. && (15)
\end{aligned}
$$

Integrating this with respect to τ, we get

$$\Pr\{X_1 > \tau | S_n = t\} = \left[\frac{t-\tau}{t}\right]^{n-1}. \tag{16}$$

(c) Find $\Pr\{X_i > \tau | S_n = t\}$ for $1 \le i \le n$.
Solution: The condition here is $X_1 + X_2 + \cdots + X_n = t$. Since $X_1, X_2, \cdots, X_n$ are IID without the condition is symmetric in $X_1, X_2, \cdots, X_n$, we see that $X_1, X_2, \cdots, X_n$ are identically distributed conditional on $S_n = t$. Thus, from (b),

$$\Pr\{X_i > \tau | S_n = t\} = \left[\frac{t-\tau}{t}\right]^{n-1}, \quad 1 \le i \le n. \tag{17}$$

(d) Find the density $f_{S_i|S_n}(s_i|t)$ for $1 \le i \le n-1$.
Solution: We can use Bayes' law in the same way as (b), getting

$$f_{S_i|S_n}(s_i|t) = \frac{s_i^{i-1}(t-s_i)^{n-i-1}(n-1)!}{t^{n-1}(i-1)!(n-i-1)!}. \tag{18}$$

(e) Give an explanation for the striking similarity between the condition $N(t) = n-1$ and the condition $S_n = t$.

Solution: The solutions to (c) and (d) are the same as (2.45) and (2.46) respectively for $N(t) = n-1$. The condition $N(t) = n-1$ and the condition $S_n = t$ both imply that the number of arrivals in $(0, t)$ is $n-1$. In addition, $N(t) = n-1$ implies that the first arrival after $(0, t)$ is strictly after t, whereas $S_n = t$ implies that the first arrival after $(0, t)$ is at t. Because of the independent increment property, this additional implication does not affect the distribution of $S_1, S_2, \cdots, S_{n-1}$. See the solution to Exercise 2.13(b) for a further discussion of this point.

The important fact here is that the equivalence of the arrival distributions in $(0, t)$, given these slightly different conditions, is a valuable aid in problem solving, since either approach can be used.

3. Exercise 2.17 (Gallager's book)

(a) For a Poisson process of rate λ, find $\Pr\{N(t) = n | S_1 = \tau\}$ for $t > \tau$ and $n \geq 1$.

Solution: Given that $S_1 = \tau$, the number, $N(t)$, of arrivals in $(0, t]$ is 1 plus the number in $(\tau, t]$. This later number $\tilde{N}(\tau, t)$ is Poisson with mean $\lambda(t-\tau)$. Thus,

$$\Pr\{N(t) = n | S_1 = \tau\} = \Pr\{\tilde{N}(\tau, t) = n-1\} = \frac{[\lambda(t-\tau)]^{n-1} e^{-\lambda(t-\tau)}}{(n-1)!}. \tag{19}$$

(b) Using this, find $f_{S_1|N(t)}(\tau|n)$.

Solution: Using Bayes' law ,

$$f_{S_1|N(t)}(\tau|n) = \frac{n(t-\tau)^{n-1}}{t^n}. \tag{20}$$

(c) Check your answer against (2.41).

Solution: Eq. (2.41) is $\Pr\{S_1 > \tau | N(t) = n\} = [(t-\tau)/t]^n$. The derivative of this with respect to τ is $-f_{S_1|N(t)}(\tau|t)$, which clearly checks with (b).

4. (a) Let $\{N(t) : t > 0\}$ be a Poisson counting process with rate $\lambda > 0$. Let T_1 be an exponential random variable independent of $\{N(t) : t > 0\}$ with probability density function

$$f_{T_1}(t) = \begin{cases} \nu \exp(-\nu t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_1)$, the number of Poisson arrivals of the first process in the interval $[0, T_1]$?

Solution: Let the Poisson processes with rates λ and ν be called PP of the first and second types respectively. Given an arrival, it is of the first type with probability $p = \frac{\lambda}{\lambda+\nu}$. Thus $N(T_1)$ represents the number of arrivals of the first type before the first arrival of the second type. This is a geometric random variable starting from 0, i.e.,

$$\Pr(N(T_1) = n) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^n, \quad n = 0, 1, 2, \ldots$$

(b) Let $\{N(t) : t > 0\}$ be as in part (a). Now, let T_2 be an Erlang random variable of order 2 independent of $\{N(t) : t > 0\}$ with probability density function

$$f_{T_2}(t) = \begin{cases} \nu^2 t \exp(-\nu t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for some $\nu > 0$. What is the distribution (probability mass function) of $N(T_2)$, the number of Poisson arrivals of the first process in the interval $[0, T_2]$?

Hint: Drawing a figure might be helpful.

Solution: Let the Poisson processes with rates λ and ν be called PP of the first and second types respectively. Given an arrival, it is of the first type with probability $p = \frac{\lambda}{\lambda+\nu}$. Thus $N(T_2)$ represents the number of arrivals of the first type before the second arrival of the second type. This is a negative binomial random variable starting from 0, i.e.,

$$\begin{aligned}\Pr(N(T_2) = n) &= \binom{n+2-1}{1}\left(\frac{\nu}{\lambda+\nu}\right)^2\left(\frac{\lambda}{\lambda+\nu}\right)^n \\ &= (n+1)\left(\frac{\nu}{\lambda+\nu}\right)^2\left(\frac{\lambda}{\lambda+\nu}\right)^n, \quad n = 0, 1, 2, \ldots\end{aligned}$$

5. Each arrival in a homogeneous Poisson process with rate λ causes a *shock*. Its effect s time units later equals $e^{-\theta s}$. Denote by $X(t)$ the total effect of all the shocks from the interval $[0, t]$ at time t. Compute the expectation $\mathbb{E}[X(t)]$.

Solutions: We can write

$$X(t) = \sum_{i=1}^{N(t)} e^{-\theta(t-S_i)},$$

where $S_1, S_2, \ldots$ denote the arrival times. First, we compute the conditional expectation where we condition on the event $\{N(t) = n\}$:

$$\mathbb{E}[X(t) \mid N(t) = n] = \mathbb{E}\left[\sum_{i=1}^{N(t)} e^{-\theta(t-S_i)} \,\middle|\, N(t) = n\right].$$

Now conditioned on $\{N(t) = n\}$, the joint distribution of the random variables $S_1, \ldots, S_n$ is equal to that of the order statistics of $U_1, \ldots, U_n$, i.i.d. random variables distributed on $[0, t]$. In other words

$$\begin{aligned}S_1 &= U_{(1)} = \min\{U_1, \ldots, U_n\} \\ S_2 &= U_{(2)} = \min\{\{U_1, \ldots, U_n\} \setminus \{S_1\}\} \\ S_3 &= U_{(3)} = \min\{\{U_1, \ldots, U_n\} \setminus \{S_1, S_2\}\} \\ &\vdots \\ S_n &= U_{(n)} = \max\{U_1, \ldots, U_n\}.\end{aligned}$$

Thus, the conditional expectation writes

$$\mathbb{E}[X(t) \mid N(t) = n] = \mathbb{E}\left[\sum_{i=1}^{n} e^{-\theta(t-U_{(i)})}\right]$$

But since addition is commutative,

$$\mathbb{E}[X(t) \mid N(t) = n] = \mathbb{E}\left[\sum_{i=1}^{n} e^{-\theta(t-U_i)}\right]$$

Thus,

$$\mathbb{E}[X(t) \mid N(t) = n] = n\mathbb{E}[e^{-\theta(t-U_1)}] = n\frac{1-e^{-\theta t}}{\theta t},$$

or altneraitvely,

$$\mathbb{E}[X(t) \mid N(t)] = N(t)\frac{1-e^{-\theta t}}{\theta t}.$$

By iterated expectations,

$$\mathbb{E}[X(t)] = \mathbb{E}[\mathbb{E}[X(t) \mid N(t)]] = \lambda\frac{1-e^{-\theta t}}{\theta}.$$

6. Let $\{N(t) : t > 0\}$ be the Poisson counting process with rate λ. The *compensated Poisson process* is defined as

$$M(t) = N(t) - \lambda t$$

Let $\mathcal{F}_t := \{M(\tau) : 0 < \tau \leq t\}$ be the process up to and including time t. A *continuous-time martingale* $\{X(t) : t > 0\}$ is a stochastic process satisfying

$$\mathbb{E}[|X(t)|] < \infty \quad \text{and} \quad \mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s) \quad \text{a.s.} \quad \forall\, t > s > 0.$$

(a) Find the mean and variance of $M(t)$.

Solution: Note that $\mathbb{E}[N(t)] = \mathrm{Var}(N(t)) = \lambda t$. We thus have

$$\mathbb{E}[M(t)] = \mathbb{E}[N(t) - \lambda t] = \lambda t - \lambda t = 0,$$

and

$$\mathrm{Var}(M(t)) = \mathrm{Var}(N(t) - \lambda t) = \mathrm{Var}(N(t)) = \lambda t.$$

(b) Does $\{M(\tau) : 0 < \tau \leq t\}$ have the (a) stationary increments property and (b) independent increments property?

Solution: Recall that the SIP means that $M(t' - t) \stackrel{\mathrm{d}}{=} M(t') - M(t)$ for $0 < t < t'$. Note that $M(t' - t) = N(t' - t) - \lambda(t' - t)$ and $M(t') - M(t) = N(t') - N(t) - \lambda(t' - t)$. Since $N(t' - t) \stackrel{\mathrm{d}}{=} N(t') - N(t)$, the SIP holds.

The IIP means that $M(t) - M(s) = N(t) - N(s) - \lambda(t - s)$ is independent of $M(s) = N(s) - \lambda s$. Since $N(t) - N(s)$ is independent of $N(s)$, the IIP carries over to the process $M(t)$.

(c) Show that $\{M(\tau) : 0 < \tau \leq t\}$ is a continuous-time martingale.

Solution: First, we need to show that $\mathbb{E}[|M(t)|] < \infty$. For this we note that $\mathbb{E}[|N(t) - \lambda t|] \leq \mathbb{E}[|N(t)|] + \lambda t$. Since $N(t)$ is a non-negative rv, $\mathbb{E}[|N(t)|] = \mathbb{E}[N(t)] = \lambda t$. This means that $\mathbb{E}[|N(t) - \lambda t|] < 2\lambda t < \infty$.

Next, we check that $\mathbb{E}[M(t) \mid \mathcal{F}_s] = M(s)$ almost surely for $t > s > 0$. Consider,

$$\begin{aligned}
\mathbb{E}[M(t) - M(s) \mid \mathcal{F}_s] &= \mathbb{E}[N(t) - N(s) - \lambda(t - s) \mid \mathcal{F}_s] \\
&= \mathbb{E}[N(t) - N(s) \mid \mathcal{F}_s] - \lambda(t - s) \\
&= \mathbb{E}[N(t) - N(s)] - \lambda(t - s) \\
&= \lambda(t - s) - \lambda(t - s) = 0
\end{aligned}$$

where the third equality follows from the fact that $N(t) - N(s) \perp\!\!\!\perp \mathcal{F}_s$.

(d) Let $\tilde{M}(t, t + \delta) = M(t + \delta) - M(t)$. Show using the SIP, IIP, and the incremental property of the Poisson process (Eqn. (2.19) of Gallager) that

$$\mathbb{E}[\tilde{M}(t, t + \delta)^2 \mid \mathcal{F}_t] = \lambda\delta + o(\delta).$$

Solution: Consider,

$$\begin{aligned}
\mathbb{E}[\tilde{M}(t, t + \delta)^2 \mid \mathcal{F}_t] &= \mathbb{E}[M(t + \delta)^2 - 2M(t + \delta)M(t) + M(t)^2 \mid \mathcal{F}_t] \\
&= \mathbb{E}[(N(t + \delta) - \lambda(t + \delta))^2 - 2(N(t + \delta) - \lambda(t + \delta))(N(t) - \lambda t) \\
&\qquad + (N(t) - \lambda t)^2 \mid \mathcal{F}_t] \\
&= \mathbb{E}[(N(t + \delta) - N(t))^2 \mid \mathcal{F}_t] + 2\lambda t \mathbb{E}[N(t + \delta) - N(t) \mid \mathcal{F}_t] \\
&\qquad - 2\lambda(t + \delta)\mathbb{E}[(N(t + \delta) - N(t)) \mid \mathcal{F}_t] + \lambda^2\delta^2 \\
&= \lambda\delta + (\lambda\delta)^2 + 2\lambda^2 t\delta - 2\lambda^2(t + \delta)\delta + \lambda^2\delta^2 \\
&= \lambda\delta
\end{aligned}$$

as desired. In fact the $o(\delta)$ term is 0!

7. (Optional) Continuing from Problem 5(c), we have the following interesting converse result due to Shinzo Watanabe[1]. A *counting process* $\{N(t) : t > 0\}$ is a continuous-time stochastic process with $N(0) = 0$ and N is constant except for jumps of $+1$. Show that if $\{N(t) : t > 0\}$ is a counting process and $\{M(t) = N(t) - \lambda t : t > 0\}$ is a (continuous-time) martingale, then $\{N(t) : t > 0\}$ is a Poisson process of rate λ. This is yet another characterization of a Poisson process.

   Hint: It can be shown using Itô's formula and the fact that $M(t)$ is a martingale that

$$X(t) = \exp\big(uN(t) - (e^u - 1)\lambda t\big)$$

   is a martingale. Use this and transforms (moment generating functions) that $N(t)$ is a Poisson process.

   Solution: Look at the proof of Theorem 4 here.

   `https://almostsuremath.com/2010/06/24/poisson-processes/`

8. (Optional) Exercise 2.13 (Gallager's book)

   (a) Show that the arrival epochs of a Poisson process satisfy

$$f_{\mathbf{S}^{(n)}|S_{n+1}}(\mathbf{s}^{(n)}|s_{n+1}) = n!/s_{n+1}^n. \tag{21}$$

   Hint: This is easy if you use only the results of Section 2.2.2.
   Solution: Note that $\mathbf{S}^{(n)}$ is shorthand for $S_1, S_2, \dots, S_n$. Using Bayes's law,

$$f_{\mathbf{S}^{(n)}|S_{n+1}}(\mathbf{s}^{(n)}|s_{n+1}) = \frac{f_{\mathbf{S}^{(n)}}(\mathbf{s}^{(n)}) f_{S_{n+1}|\mathbf{S}^{(n)}}(s_{n+1}|\mathbf{s}^{(n)})}{f_{S_{n+1}}(s_{n+1})}. \tag{22}$$

   From (2.15), $f_{\mathbf{S}^{(n)}}(\mathbf{s}^{(n)}) = \lambda^n e^{-\lambda s_n}$. Also

$$f_{S_{n+1}|\mathbf{S}^{(n)}}(s_{n+1}|\mathbf{s}^{(n)}) = \lambda e^{-\lambda(s_{n+1}-s_n)}.$$

   Finally, $f_{S_{n+1}}(s_{n+1})$ is Erlang. Combining these terms,

$$f_{\mathbf{S}^{(n)}|S_{n+1}}(\mathbf{s}^{(n)}|s_{n+1}) = n!/s_{n+1}^n. \tag{23}$$

   (b) Contrast this with the result of Theorem 2.5.1.

   Solution: (a) says that $S_1, S_2, \dots, S_n$ are uniformly distributed (subject to the ordering constraint) between 0 and t for any sample value t for S_{n+1}. Theorem 2.5.1 says that they are uniformly distributed between 0 and t given $N(t) = n$. The conditions $\{S_{n+1} = t\}$ and $\{N(t) = n\}$ each imply that there are n arrivals in $(0, t)$. The conditions $\{S_{n+1} = t\}$ in addition specifies that arrival $n + 1$ is at epoch t, whereas $\{N(t) = n\}$ specifies that arrival $n + 1$ is in (t, ∞). From the independent increment property, the arrival epochs in $(0, t)$ are independent of those in $[t, \infty)$ and thus the conditional joint distribution of $\{\mathbf{S}^{(n)}\}$ is the same for each conditioning event.

   One might ask whether this equivalence of conditional distributions provides a rigorous way of answering (a). The answer is yes if the above argument is spelled out in more detail. It is simpler, however, to use the approach in (a). The equivalence approach is more insightful, on the other hand, so it is worthwhile to understand both approaches.

[1] S. Watanabe, "On discontinuous additive functionals and Lévy measures of Markov processes". Japanese J. Maths. 34, 1964.