Controllability and Observability

Chong-Jin Ong

Department of Mechanical Engineering,
National University of Singapore

Outline

Introduction

- This chapter deals with two concepts that are not found in classical or frequency analysis.
- Controllability and Observability are two unique features of the state space analysis.
- These concepts were first introduced by E.G. Gilbert and R.F. Kalman in the 1960s.
- Explain why cancellation of unstable poles are undesirable even if perfect cancellation is possible.
- The concepts are first illustrated via several motivating examples.

Example

Example 1: Consider the system

$$
\begin{aligned}
\dot{x}_1 &= u \\
\dot{x}_2 &= u
\end{aligned}
$$

If $x_1(0) = x_2(0)$, then $x_1(t) = x_2(t)$ for all time and all control $u(t)$.

Example 2: Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 0 \\
y &= x_2
\end{aligned}
$$

Then $y(t) =$ constant for all choices of $x_1(0)$ and $x_2(0)$. Observing $y(t)$ does not tell us what x_1 is doing.

Motivating Example

A more interesting example is one given by the following diagram. It consists of two carts coupled by a spring. A force f is applied onto both carts via some means within the system.

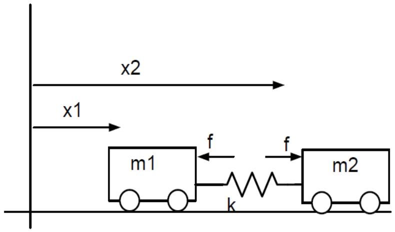

Figure: The Two cart example

The equations of motion of the system are

$$\dot{x}_1 = x_3, \qquad \dot{x}_2 = x_4$$

$$\dot{x}_3 = -\frac{k}{m_1}(x_1 - x_2) - \frac{f}{m_1}, \quad \dot{x}_4 = -\frac{k}{m_2}(x_2 - x_1) + \frac{f}{m_2}$$

Motivating Example

Rewritten in standard state space representation, the state equation is

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{m_1} \\ \frac{1}{m_2} \end{pmatrix} f$$

From law of physics, f can change the relative distance between the two carts $(x_2 - x_1)$ but it cannot change x_1 and x_2 **independently**.

Example

Example 4: Consider the system given below,

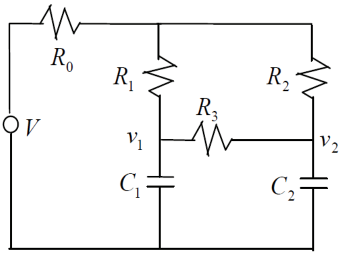

Figure: The unobservable system

The equations describing the system are;

$$\dot{v}_1 = -\frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2})v_1 + \frac{1}{C_1 R_3}v_2 + \frac{1}{C_1 R_1}V$$
$$\dot{v}_2 = \frac{1}{C_2 R_3}v_1 - \frac{1}{C_2}(\frac{1}{R_2} + \frac{1}{R_3})v_2 + \frac{1}{C_2 R_2}V$$

Example

Consider the voltage across R_3, $\bar{v} = v_1 - v_2$ then

$$\dot{\bar{v}} = -\left[\frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_3}) + \frac{1}{C_2 R_3}\right] + \left[\frac{1}{C_1 R_3} + \frac{1}{C_2}(\frac{1}{R_2} + \frac{1}{R_3})\right] + \frac{R_2 C_2 - R_1 C_1}{C_1 C_2 R_1 R_2} V$$

If the bridge is balanced, then $R_1 C_1 = R_2 C_2$, the coefficient of V vanishes and

$$\dot{\bar{v}} = -(\frac{R_1 + R_2 + R_3}{C_1 R_1 R_3})\bar{v}$$

which implies that $\bar{v}$ is not influenced by V and the voltage $\bar{v}$ decays from $\bar{v}(0)$ to zero, i.e., $\bar{v}$ is not controllable.

Controllability and Observability

- The above shows practical examples of uncontrollable and unobservable systems.
- This chapter studies analysis tools that checks for controllability and observability.
- Given (A, B, C, D), answer the questions:
  - Can we drive $x(t)$ wherever we desire using $u(t)$?
  - Can we track $x(t)$ by observing $y(t)$?
- We assume complete knowledge of (A, B, C, D) in answering these questions.

Controllability

- **Definition**: A LTI system (A, B, C, D) (or more precisely (A, B)) is controllable if there exists an input $u(t), 0 \leq t \leq t_1$ that drives the system from **any** initial state $x(0) = x_0$ to **any** other state $x(t_1) = x_1$ in a **finite** time t_1. Otherwise, (A, B) is said to be uncontrollable.
- Possible to define the above for time-varying systems (more complicated since reaching x_1 may depend on t_0). The above assumes $t_0 = 0$ WLOG.
- **Theorem 4.1** The n-dimensional LTI system with matrices (A, B) is controllable if and only if any of the following condition is satisfied:
  1. The Controllability Grammian
  $$W(0,t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau = \int_0^t e^{A(t-\tau)} B B^T e^{A^T (t-\tau)} d\tau$$
  is nonsingular for all $t \geq 0$.
  2. The $n \times nr$ matrix
  $$U = \left[B \ AB \ \cdots \ A^{n-1}B\right]$$
  is full row rank.
  3. The $n \times (n+r)$ matrix $[A - \lambda I \ B]$ has full row rank at every eigenvalue λ of A.

Controllability

Case 1: ($\Rightarrow$) Suppose $W(0,t)$ is non-singular for all $t > 0$ and $x(t_1) = x_1$. Choose the input as

$$u(t) = -B^T e^{A^T(t_1 - t)} W^{-1}(0, t_1)(e^{At_1}x_0 - x_1)$$

Using this input on the system, the state

$$\begin{aligned} x(t_1) &= e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \\ &= e^{At_1}x_0 - \int_0^{t_1} e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}W^{-1}(0,t_1)(e^{At_1}x_0 - x_1)d\tau \\ &= e^{At_1}x_0 - (e^{At_1}x_0 - x_1) = x_1 \end{aligned}$$

($\Leftarrow$) Suppose $W(0,t)$ is singular for some $t_1 > 0$, and that the system is controllable. We show that this leads to a contradiction. Since $W(0,t_1)$ is singular, there exists a non-zero α such that

$$\alpha^T W(0,t_1)\alpha = \int_0^{t_1} \alpha^T e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}\alpha = 0$$

$$\text{Or, } \int_0^{t_1} \|\alpha^T e^{A(t_1-\tau)}B\|_2^2 d\tau = 0$$

Since $\|\cdot\|$ is a non-negative function, this means that

$$\alpha^T e^{A(t_1-\tau)}B = 0 \text{ for all } t > \tau > 0 \tag{1}$$

Controllability

Since the system is controllable, this means that any state is reachable by an appropriate control in finite time. Suppose we want to reach the state $x(t_1) = \alpha + e^{At_1}x_0$ at time t_1. Then

$$x(t_1) = \alpha + e^{At_1}x_0 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

$$\text{or, } \alpha = \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

This also means that

$$\alpha^T\alpha = \int_0^{t_1} \alpha^T e^{A(t_1-\tau)}Bu(\tau)d\tau = 0$$

from (1) which then implies that $\alpha = 0$ which contradicts that $\alpha \neq 0$.

Controllability

Case (2) ($\Rightarrow$ and $\Leftarrow$): Various ways of showing this exist and this one given here is more intuitive but less rigorous. It shows a clear connection to Caley-Hamilton Principle. Recall that

$$x_1 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

$$e^{-At_1}x_1 - x_0 = \int_0^{t_1} e^{-At_1}e^{A(t_1-\tau)}Bu(\tau)d\tau = \int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau \tag{2}$$

and from CH Principle,

$$e^{-A\tau} = \sum_{k=0}^{n-1}\alpha_k(\tau)A^k$$

Using this into (2),

$$e^{-At_1}x_1 - x_0 = \sum_{k=0}^{n-1}A^kB\int_0^{t_1}\alpha_k(\tau)u(\tau)d\tau = \sum_{k=0}^{n-1}A^kB\beta_k$$

where $\beta_k = \int_0^{t_1}\alpha_k(\tau)u(\tau)d\tau$.

- The above equation can only be satisfied identically for any x_0, x_1 and t_1 if and only if $\begin{bmatrix} B & AB \cdots A^{n-1}B \end{bmatrix}$ is full row rank

Controllability

Case (3) ($\Rightarrow$) We want to show that

$$U \text{ is full rank } \Rightarrow \{[A - \lambda I \;\; B] \text{ has full row rank at every } \lambda \text{ of } A.\}$$

Suppose this is not true i.e., there exists a λ_1 and a non-zero q s.t.

$$q^T[A - \lambda_1 I \;\; B] = 0$$

This implies that

$$q^T A - q^T \lambda_1 = 0 \text{ and } q^T B = 0,$$

$$\begin{aligned} q^T[B \; AB \cdots A^{n-1}B] &= [q^T B \; q^T AB \cdots q^T A^{n-1}B] \\ &= [q^T B \quad \lambda_1 q^T B \;\; \cdots \;\; \lambda_1^{n-1} q^T B] = 0 \end{aligned}$$

Hence, U is not full row rank.

Controllability

Case (3) ($\Leftarrow$) We want to show that

$$\{[A - \lambda I \ B] \text{ has full row rank at every } \lambda \text{ of } A.\} \Rightarrow \{U \text{ is full rank }\}$$

Need an additional result that controllability is invariant under similarity transformation, a result to be shown later. Using that, and suppose rank $U = n - m$ for some integer $m \geq 1$, (A, B) can be expressed as

$$\bar{A} = PAP^{-1} = \begin{pmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{pmatrix} \ \bar{B} = PB = \begin{pmatrix} \bar{B}_c \\ 0 \end{pmatrix}$$

where $\bar{A}_{\bar{c}}$ is $m \times m$. Let λ_1 be an eigenvalue and q_1 be the left eigenvector of $\bar{A}_{\bar{c}}$ i.e., $q_1^T \bar{A}_{\bar{c}} = \lambda_1 \bar{A}_{\bar{c}}$ or $(q_1^T - \lambda_1 I)\bar{A}_{\bar{c}} = 0$. We form the n vector q as $q^T = [0 \ q_1^T]$ where 0 is a $(n - m)$ row vector. Then, we have

$$q^T[\bar{A} - \lambda_1 I \ \bar{B}] = \begin{pmatrix} 0 & q_1^T \end{pmatrix} \begin{pmatrix} \bar{A}_c - \lambda_1 I & \bar{A}_{12} & \bar{B}_c \\ 0 & \bar{A}_{\bar{c}} - \lambda_1 I & 0 \end{pmatrix} = 0$$

This implies that $[A - \lambda I \ B]$ is not full rank at eigenvalue λ_1. (q.e.d.)

Controllability

- The $n \times nr$ matrix $U := [B\ AB\ \cdots\ A^{n-1}B]$ is commonly known as the controllability matrix.
- In the case of single-input system, $r = 1$ and U is a square matrix with controllability ensured when U is non-singular.
- Controllability is a concept on the ability of a system to move from one point to another under the influence of $u(t)$.
- It does NOT take into account physical constraint.
- Knowing that a system is controllable does not mean that it can be controlled in practice.
- However, knowing that a system is not controllable means one should not try to move system state arbitrarily.

Example

Consider the system given below:

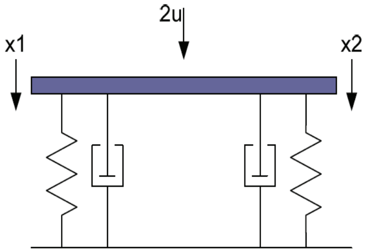

Figure: The Beam example

The equation of the system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} u$$

Note that the zero-input system is asymptotically stable.

Controllability

- Questions: Is there an input $u(t)$ such that, when applied, brings the platform system to equilibrium in 2 seconds starting from $x_1(0) = 10$ and $x_2(0) = -1$?
- Answer: First check that rank$[B\ AB]$=rank$\left(\begin{pmatrix} 0.5 & -0.25 \\ 1 & 1 \end{pmatrix}\right) = 2$.
- Hence, it is possible to bring the system to the origin in 2 second. The expression is

$$u(t) = -B^T e^{A^T(t_1-t)} W^{-1}(0,t_1)(e^{At_1}x_0 - x_1) \text{ with}$$

$$W(0,2) = \int_0^2 \begin{pmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \begin{pmatrix} 0.5 & 1 \end{pmatrix} \begin{pmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix} d\tau$$
$$= \begin{pmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{pmatrix}$$

- The corresponding $u(t) = -58.82e^{0.5t} + 27.96e^{t}$.

Example

Consider two cases: (1) no constraint on u; and (2) $-10 \le u(t) \le 10$,

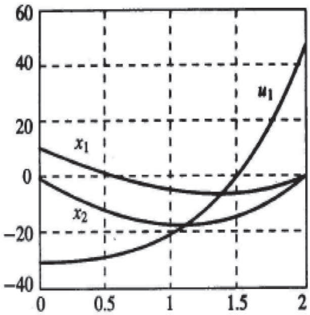

Figure: Plots of $x_1(t), x_2(t)$ and $u(t)$ for transfer from $x(0) = (10, -1)^T$ in 2 seconds.

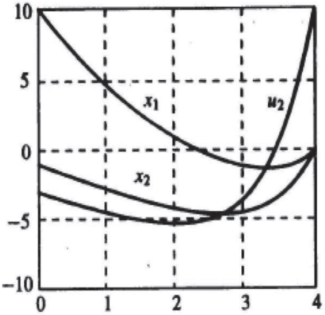

Figure: Plots of $x_1(t), x_2(t)$ and $u(t)$ for transfer from $x(0) = (10, -1)^T$ in 4 seconds.

Controllability under Similarity Transformation

- What happens to Controllability of a system when it undergoes a coordinate change?
- Consider the case of $x = T\bar{x}$, the new state equations are

$$\dot{\bar{x}} = T^{-1}AT\bar{x} + T^{-1}Bu$$
$$y = CTx + Du$$

The new controllability matrix is

$$\bar{U} = \left[T^{-1}B\,T^{-1}AB \cdots T^{-1}A^{n-1}B\right] = T^{-1}\left[B\,AB \cdots A^{n-1}B\right] = T^{-1}U$$

- Hence, rank of $\bar{U}$ = rank of U (why?) and Controllability is not affected by similarity transformation.

Reachable States and Controllability Grammian

- A closely related concept to Controllability is Reachability.
- Reachability has the same definition of Controllability except that $x(0) = 0$.
- A state x_ℓ is reachable if there exists a $u(t)$ such that

$$x_\ell = \int_0^\ell e^{A(\ell - t)} B u(t) dt$$

- The set of reachable state, $\mathcal{R}$, is the collection of all x_ℓ over all possible u.
- Note that $\mathcal{R}$ is a linear subspace of $\mathbb{R}^n$: for all $\alpha, \beta \in \mathbb{R}$,

$$x_a, x_b \in \mathcal{R} \Rightarrow \alpha x_a + \beta x_b \in \mathcal{R}$$

- Recall $W(0, \ell) = \int_0^\ell e^{A(\ell - \tau)} B B^T e^{A^T(\ell - \tau)} d\tau$. We claim that

$$\mathcal{R} = \text{Range space of } W(0, \ell)$$

Proof is omitted.

Controllability Grammian is time-independent

- We now show that rank W is independent of the terminal time ℓ. To do so, we note that
$$\operatorname{rank}(W(0,\ell)) = \operatorname{rank}\left[B\ AB \cdots A^{n-1}B\right] = \operatorname{rank}(U)$$
The special case when W is full rank is already established earlier.
- The proof is given in the next page. However, here is an intuitive reasoning.
- Since controllability requires that $W(0,\ell)$ to be non-singular for all $\ell > 0$ and the full rank condition of U is independent of ℓ. This provides plausible reason to the validity of the above.

Controllability Grammian is time-independent

Proof: (Optional) $(\Rightarrow)$

$$q^T W(0,\ell) = 0 \Rightarrow \int_0^{\ell} q^T e^{A(\ell-\tau)} B B^T e^{A^T(\ell-\tau)} q d\tau = 0$$

$$\Rightarrow q^T e^{A(\ell-\tau)} B = 0, \text{ for all } \tau \in [0,\ell]. \tag{3}$$

Let $\tau = \ell$ then the above implies $q^T B = 0$. Differentiate (3) once yields

$$(-1) q^T A e^{A(\ell-\tau)} B = 0$$

which implies, when $\tau = \ell$,

$$(-1) q^T AB = 0$$

Repeating the differentiation $n-1$ times yields

$$q^T \left[B \, (-1)AB \cdots (-1)^k A^k B, \cdots (-1)^{n-1} A^{n-1} B \right] = 0$$

which implies that $q^T \left[B \; AB \cdots A^{n-1} B \right] = 0$. This means $rank(W(0,\ell)) \leq rank(U)$.
$(\Leftarrow)$ Let q^T be in the left nullspace of U. Then

$$q^T \left[B \; AB \cdots A^{n-1} B \right] = 0 \Rightarrow q^T W(0,\ell) = 0 \text{ for any } \ell > 0$$

This implies $rank(U) \leq rank(W(0,\ell))$.

- Using $(\Rightarrow)$ and $(\Leftarrow)$, $rank(W(0,\ell)) = rank(U)$.
- Since U is independent of ℓ, so is $rank(U)$ and $rank(W)$.
- WLOG, denote $W = W(0,\infty)$.

Computation of Controllability Grammian

- The expression of $W = \int_0^\infty e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau$ is hard to compute.
- There is another expression that allows easy computation. Note that the W satisfies the Lyapunov Equation

$$AW + WA^T = -BB^T$$

such that the solution of the above yields W.

- This follows because (assuming A is stable)

$$AW + WA^T = \int_0^\infty \frac{d}{dt}(e^{At} B B^T e^{A^T t}) dt = -BB^T$$