Tutorial 9 Decoupling Control

9.1 Consider the 2-input/2-output system described by

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -3 & -3 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} x.$$

Find the state-feedback decoupler.

Solution:
It is readily checked that

$$c_1^T A^0 B = C_1^T B = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \text{or} \quad \sigma_1 = 1;$$
$$c_2^T B = \begin{pmatrix} 0 & 0 \end{pmatrix},$$
$$c_2^T AB = \begin{pmatrix} 4 & 3 \end{pmatrix}, \quad \text{or} \quad \sigma_2 = 2,$$

so that

$$B^* = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix}.$$

Since B^* is nonsingular, the system can be decoupled by state feedback $u = -Kx + Fv$.
It is easily checked that

$$B^{*-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -4 & 1 \end{bmatrix} = F,$$

Use the constructive procedure of Theorem 9.1, one finds that

$$K = B^{*-1}C^* = \begin{bmatrix} -1 & 0 & 0 \\ 1.67 & 1.33 & 3 \end{bmatrix}.$$

The resulting closed loop transfer function is

$$H(s) = \begin{bmatrix} \dfrac{1}{s} & 0 \\ 0 & \dfrac{1}{s^2} \end{bmatrix}.$$

Now that the system is decoupled further stabilization is easily done on the single loop basis.

9.2 The linearized equations governing the equatorial motion of a satellite in a circular orbit is

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u,$$

$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x.$$

where the outputs and inputs are radial and tangential

$$y = \begin{bmatrix} r \\ \theta \end{bmatrix}, \qquad u = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}.$$

These are coupled by the orbital dynamics, as indicated by the transfer function matrix

$$G(s) = \begin{bmatrix} \dfrac{1}{s^2 + \omega^2} & \dfrac{2\omega}{s\left(s^2 + \omega^2\right)} \\ \dfrac{-2\omega}{s\left(s^2 + \omega^2\right)} & \dfrac{s^2 - 3\omega^2}{s^2\left(s^2 + \omega^2\right)} \end{bmatrix}.$$

Find a state-feedback decoupler.

Solution:
To determine whether this system can be decoupled, we compute

$$c_1^T B = \begin{bmatrix} 0 & 0 \end{bmatrix},$$
$$c_1^T AB = \begin{bmatrix} 1 & 0 \end{bmatrix}, \text{ or } \sigma_1 = 2.$$

$$c_2^T B = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$c_2^T AB = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ or } \sigma_2 = 2.$$

We then have

$$B^* = \begin{bmatrix} c_1^T AB \\ c_2^T AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

is nonsingular. Thus decoupling can be achieved with

$$\begin{aligned} u(t) &= -Kx(t) + Fw(t) \\ &= -B^{*-1}C^* x(t) + B^{*-1} w(t) \\ &= -\begin{bmatrix} 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -2\omega & 0 & 0 \end{bmatrix} x(t) + w(t) \end{aligned},$$

which results in the close-loop system

$$\dot{x} = (A - BK)x + BFw = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} w,$$

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$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x.$$

The transfer function matrix may easily be shown to be

$$H(s) = \begin{bmatrix} \dfrac{1}{s^2} & 0 \\ 0 & \dfrac{1}{s^2} \end{bmatrix}.$$

We design pole placement state feedback control for this decoupled process as follows

$$w = K_f x + \overline{w} = \begin{bmatrix} k_1 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \end{bmatrix} x + \begin{bmatrix} \overline{w}_r \\ \overline{w}_\theta \end{bmatrix},$$

resulting in the closed-loop system

$$\dot{x} = \left(A - BK + BK_f\right)x + B\overline{w} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ k_1 & k_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & k_3 & k_4 \end{bmatrix}x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\overline{w},$$

$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}x.$$

If we choose the feedback gains $k_1 = k_3 = -1$ and $k_2 = k_4 = -2$, each of the two independent subsystems will be stable with a double pole at $s = -1$. The transfer function matrix may easily be shown to be

$$H(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & 0 \\ \\ 0 & \dfrac{1}{(s+1)^2} \end{bmatrix}.$$

This closed-loop system is not only stable (perturbations from the nominal trajectory will always decay to zero), but adjustments to r and θ be made independently via the external inputs $\overline{w}_r$ and $\overline{w}_\theta$.

9.3 Design the controller in a unity output feedback configuration such that it decouples and stabilizes the plant

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+1} \\ \\ \dfrac{1}{s+2} & \dfrac{2}{s+2} \end{bmatrix}.$$

Solution:

Write

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \quad N_r = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \\ 0 & \dfrac{1}{s+2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Choose the decoupler

$$K_d = N_r^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

So that the decoupled plant, GK_d, is

$$GK_d = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix}.$$

It can be easily verified that the closed loop is stable in the unity feedback configuration. So a simple gain matrix can decouple and stabilize the system at the same time.