EE5137 2021/22 (Sem 2): Quiz 1 (Total 25 points)

Name:______________________________

Matriculation Number:_______________

Score:______________________________

You have 1.0 hours for this quiz. There are SIX (6) printed pages. You're allowed 1 sheet of handwritten notes. Please provide *careful explanations* for all your solutions.

1. Let X and Y be two independent Bernoulli (i.e., $\{0,1\}$-valued) random variables with

$$\Pr(X=1)=\Pr(Y=1)=1/2.$$

   (a) (2 points) Are the random variables $X+Y\in\{0,1,2\}$ and $|X-Y|\in\{0,1\}$ independent? Explain carefully.

(b) (3 points) We say that two random variables A and B are *uncorrelated* if

$$\mathbb{E}[AB] = \mathbb{E}[A]\mathbb{E}[B].$$

Are the random variables $X + Y$ and $|X - Y|$ uncorrelated? Explain carefully.

2. In machine learning and statistics, sub-Gaussian random variables play very important roles. We say that a zero-mean random variable X is *sub-Gaussian with variance proxy* σ^2, written as $X \sim \mathrm{subG}(\sigma^2)$, if its moment generating function $g_X(r)$ satisfies

$$g_X(r) = \mathbb{E}[e^{rX}] \leq \exp\left(\frac{r^2\sigma^2}{2}\right) \qquad \forall r \in \mathbb{R}.$$

(a) (2 points) If $X_i \sim \mathrm{subG}(\sigma_i^2)$ and the X_i's are zero-mean and independent, then what is the (smallest) variance proxy of $\sum_{i=1}^n X_i$?

(b) (3 points) If $X \sim \mathrm{subG}(\sigma^2)$ with zero mean, show that for any $t \geq 0$,

$$\Pr(X \geq t) \leq \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

(c) (5 points) Let $X \sim \text{subG}(\sigma^2)$ with zero mean. Fix an integer $k \geq 1$. Use part (b) to find the best functions $f(k, \sigma^2)$ and $g(k)$ (i.e., those resulting in the tightest bound) such that

$$\mathbb{E}\left[|X|^k\right] \leq f(k, \sigma^2)\, \Gamma\big(g(k)\big) \quad \text{where} \quad \Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, \mathrm{d}x.$$

3. Let Y be a uniform random variable on $[0, 1]$. Given Y, we then toss a coin with bias Y repeatedly (i.e., the probability of seeing Head equals Y). The outcomes of the coin tosses are denoted by $X_1, X_2, \ldots \in \{H, T\}$.

   (a) (5 points) Suppose that among the first 2 coin tosses, one is Head and one is Tail. Find the conditional cumulative distribution function of Y, i.e., find

   $$F_{Y|\{X_1,X_2\}=\{H,T\}}(y) := \Pr\left(Y \leq y \mid \{X_1, X_2\} = \{H, T\}\right) \qquad \forall\, y \in [0, 1].$$

   Hint: Figuring out $\Pr(\{X_1, X_2\} = \{H, T\})$ *first would get you some marks. Think of using iterated expectations.*

(b) (5 points) Suppose that among the first n coin tosses, we observe k Heads. What is the probability that the $(n+1)$-st coin toss shows Head? More precisely, compute

$$\Pr\left(X_{n+1} = H \mid k \text{ Heads among } X_1, \ldots, X_n\right).$$

Hint: You can assume the following without proof. For $n \in \mathbb{N}$ *and* $k \in \{0, 1, \ldots, n\}$,

$$\int_0^1 y^k (1-y)^{n-k} \, \mathrm{d}y = \frac{k!(n-k)!}{(n+1)!}.$$