EE5103/ME5403 Computer Control Systems: Homework #1 Solution

Semester 1 Y2021/2022

Q1 Solution

a) Applying Laplace transform we can get:

$$L(sI(s)-i(0))+RI(s)+\frac{1}{Cs}I(s)=E_i(s) \tag{1.1}$$

$$\frac{1}{C}\frac{1}{s}I(s)=E_o(s) \tag{1.2}$$

Assuming the initial condition is zero, then we have

$$\frac{1}{Cs}\frac{E_i(s)}{Ls+R+\frac{1}{Cs}}=E_o(s) \tag{1.3}$$

Then the transfer function is as follows:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{E_o(s)}{E_i(s)}=\frac{1}{LCs^2+RCs+1}=\frac{1}{s^2+2s+1} \tag{1.4}$$

b) Since $u=e_i, y=e_{o,}x_2=\dot{e}_o, x_1=e_o$

$$\frac{1}{C}\int idt=e_o \Rightarrow x_2=\dot{e}_o=\frac{1}{C}i \Rightarrow \frac{di}{dt}=C\dot{x}_2 \tag{1.5}$$

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=e_i \Rightarrow LC\dot{x}_2+RCx_2+x_1=u \tag{1.6}$$

Thus, the above equations yield

$$\dot{x}_1=\dot{e}_o=x_2 \tag{1.7}$$

$$\dot{x}_2=-\frac{1}{LC}x_1-\frac{R}{L}x_2+\frac{1}{LC}u \tag{1.8}$$

$$y=e_o=x_1 \tag{1.9}$$

The state space model is

$$\begin{gathered}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{LC}\end{bmatrix}u=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u\\ y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{gathered} \tag{1.10}$$

c) Denote the above state space model as

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma u(k) \\
y(k) &= Cx(k)
\end{aligned} \tag{1.11}
$$

Then we have

$$
\Phi = e^{Ah} = e^{A} \tag{1.12}
$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ is the state transition matrix in (1.10). To find e^{At}, we can use the Laplace transform. Let $f(t) = e^{At}$, and its Laplace transform is

$$
F(s) = \mathcal{L}[f(t)] = (sI - A)^{-1} = \frac{1}{s^2 + 2s + 1} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} \tag{1.13}
$$

Now it is straightforward to derive the inverse Laplace transform

$$
f(t) = \mathcal{L}^{-1}[F(s)] = \begin{bmatrix} e^{-t}(1+t) & te^{-t} \\ -te^{-t} & e^{-t}(1-t) \end{bmatrix} \tag{1.14}
$$

Let $t = h = 1$, we have

$$
\Phi = e^{A} = f(1) = \begin{bmatrix} 0.7358 & 0.3679 \\ -0.3679 & 0 \end{bmatrix} \tag{1.15}
$$

To compute the input matrix Γ:

$$
\begin{aligned}
\Gamma &= \int_0^h e^{A\tau} d\tau\, B = \begin{bmatrix} \int_0^h e^{-\tau}(1+\tau) d\tau & \int_0^h \tau e^{-\tau} d\tau \\ -\int_0^h \tau e^{-\tau} d\tau & \int_0^h e^{-\tau}(1-\tau) d\tau \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \int_0^1 \tau e^{-\tau} d\tau \\ \int_0^1 e^{-\tau}(1-\tau) d\tau \end{bmatrix} = \begin{bmatrix} 1 - 2e^{-1} \\ e^{-1} \end{bmatrix} = \begin{bmatrix} 0.2642 \\ 0.3679 \end{bmatrix}
\end{aligned} \tag{1.16}
$$

Thus, the state space model for sampled discrete-time system is

$$
\begin{aligned}
x(k+1) &= \begin{bmatrix} 0.7358 & 0.3679 \\ -0.3679 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0.2642 \\ 0.3679 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)
\end{aligned} \tag{1.17}
$$

d) Applying Z transform on discrete state space model (1.11) yields

$$
\begin{aligned}
zX(z) - zx(0) &= \Phi X(z) + \Gamma U(z) \\
Y(z) &= CX(z)
\end{aligned} \tag{1.18}
$$

Then we can get

$$Y(z) = CX(z) = C(zI-\Phi)^{-1}\Gamma U(z) + zC(zI-\Phi)^{-1}x(0) \tag{1.19}$$

Assuming zero initial conditions we can get the z transfer function as

$$\text{H(z)}=\frac{Y(z)}{U(z)} = C(zI-\Phi)^{-1}\Gamma = \frac{0.2642\text{ z} + 0.1354}{z^2 - 0.7352z + 0.1354} \tag{1.20}$$

Therefore, we can get the input-output model as

$$\begin{aligned} &z^2Y(z) - 0.7352zY(z) + 0.1354Y(z) = 0.2642zU(z) + 0.1354U(z) \\ &\Rightarrow y(k+1) = 0.7352y(k) - 0.1354y(k-1) + 0.2642u(k) + 0.1354u(k-1), k \geq 1 \end{aligned} \tag{1.21}$$

e) We know that

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.22}$$

$$U(z) = Z[u(k)] = \frac{z}{z-1} \tag{1.23}$$

Substituting $x(0), \text{U(z)}$ into equation (1.19) yields

$$\begin{aligned} Y(z) &= C(zI-\Phi)^{-1}zX(0) + C(zI-\Phi)^{-1}\Gamma U(z) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-0.7358 & 0.3096 \\ -0.3679 & z \end{bmatrix}^{-1} (z\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2642 \\ 0.3679 \end{bmatrix}\frac{z}{z-1}) \\ &= \frac{z}{z-1} \end{aligned} \tag{1.24}$$

Then, by checking the z transform table, the output sequence is

$$y(k) = 1 \qquad (k \geq 0). \tag{1.25}$$

Q2 Solution

a) Poles: $s^2(s+2) = 0 \Rightarrow s_1 = 0, s_2 = 0, s_3 = -2$.

Since there are two identical poles s_1 and s_2 with a real part of 0, the system is unstable.

For the inverse system, its pole will be the zero of this original system, i.e., $s_0 = 1$. Therefore, the inverse system is unstable.

(Note: for a system $G(s) = \frac{N(s)}{D(s)}$, by *inverse* we mean the system $G'(s) = \frac{D(s)}{N(s)}$.)

b) Mapping the above poles onto z-plane, we have poles for the discrete sampled system,

$$
\begin{aligned}
z_1 &= e^{s_1 h} = 1 \Rightarrow |z_1| = 1 \\
z_2 &= e^{s_2 h} = 1 \Rightarrow |z_2| = 1 \\
z_3 &= e^{s_3 h} = e^{-2h} \Rightarrow |z_3| = e^{-2h} < e^0 = 1 (h > 0)
\end{aligned}
\tag{2.1}
$$

Since there are multiple poles with magnitude 1, the sampled system is still unstable.

c) Since there is no direct mapping for zeros between s-domain and z-domain, we have to derive the z-transfer function first.

$$
G(s) = \frac{s-1}{s^2(s+2)} = \frac{0.75}{s} - \frac{0.5}{s^2} - \frac{0.75}{s+2}
\tag{2.2}
$$

From table 2.1, we can find the sampled system is

$$
\begin{aligned}
G(z) &= 0.75 \times \frac{h}{z-1} - 0.5 \times \frac{h^2(z+1)}{2(z-1)^2} - 0.325 \times \frac{1-e^{-2h}}{z-e^{-2h}} \\
&= \frac{[3e^{-2h} - 2h^2 + 6h - 3]z^2 + [e^{-2h}(2h^2 - 6h - 6) - (2h^2 + 6h - 6)]z + e^{-2h}(2h^2 + 6h + 3) - 3}{8(z-1)^2(z-e^{-2h})}
\end{aligned}
\tag{2.3}
$$

Now we need to find the zeros of the samples system (2.3), that is, the poles of the inverse sampled system, to check the stability of the inverse sampled system. There are two cases to be considered depending on whether the coefficient of the 2nd-order term is zero.

- If $3e^{-2h} - 2h^2 + 6h - 3$ is zero, there will only be one zero for (2.3). To find such h, we can resort to the MATLAB function *fzero* and get the solution to be $h = 2.3735$. Inserting h into (2.3), we can get the single zero as $z = -0.1405$. Since this zero lies inside the unit circle, this special sampling period $h = 2.3735$ can give a stable inverse.

- If $3e^{-2h} - 2h^2 + 6h - 3$ is not zero, then two zeros can be found for (2.3). The MATLAB function *roots* can be used to get the roots for this 2nd-order polynomial. We vary h in range $[0.01, 5]$ and plot the two zeros shown in Figure 1. As can be seen, the magnitude of z_1 is always bigger than 1. There exists $h \to 0$, $z_1 \to 1$ and $h \to \infty$, $z_1 \to -1$. Therefore, z_1 is always a zero outside the unit circle and the inverse system is unstable.

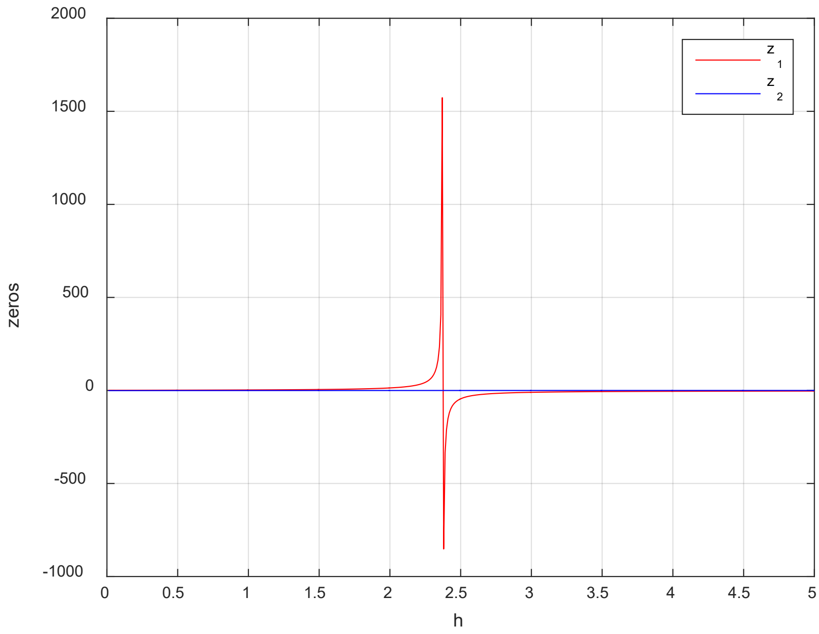

Figure 1 Zeros of the sampled system of case 2.

For your reference, the MATLAB code of this question is given below.

```
%% case 1
fun = @(h) 3*exp(-2*h) - 2*h*h+6*h-3;
x = fzero(fun, 3);
%% case 2
N = 500;
h = linspace(0.01, 5, N);
P = [3*exp(-2*h) - 2*h.*h+6*h-3;
    exp(-2*h).*(2*h.*h-6*h-6) - (2*h.*h+6*h-6);
    exp(-2*h).*(2*h.*h+6*h+3) - 3];
% each column of P is a group of three coefficients
Z = NaN(2, N);
for ii = 1:N
    Z(:,ii) = roots(P(:,ii));
end
% plot: each column of Z is the two zeros
figure;
plot(h, Z(1,:), 'r', h, Z(2,:), 'b');
legend('z_1', 'z_2');
xlabel('h'); ylabel('zeros');
grid on;
```

To sum up the above two cases, we can see that it is possible to get a stable inverse by choosing $h = 2.3735$. For all other sampling time h's, the inverse system is unstable.

As can be seen, sampling will not change the stability of the system due to the pole mapping relation between continuous-time and discrete-time domain. However, there is no simple relation *between the zeros. Furthermore, even the original system has no zeros, after sampling, it may have zeros.*

Q3 Solution

a) The state transition matrix, input matrix and output matrix are

$$\Phi = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \Gamma = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.1}$$

The characteristic equation and the poles are

$$\det(\lambda I - \Phi) = \lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = -2 \tag{3.2}$$

Since the poles are NOT in the unit circle of the z-plane, the system is unstable.

Controllability matrix is

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \tag{3.3}$$

Since W_c is nonsingular, the system is controllable.

The observability matrix is

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{3.4}$$

Since W_o is nonsingular, the system is observable.

b) The z transfer function is

$$\begin{aligned} H(z) &= C(zI - \Phi)^{-1}\Gamma \\ &= \frac{z+1}{(z-3)(z+2)} \end{aligned} \tag{3.5}$$

Since $H(z) = \dfrac{Y(z)}{U(z)}$, we can get

$$z^2 Y(z) - zY(z) - 6Y(z) = zU(z) + U(z) \tag{3.6}$$

Applying the equivalence of z to shift operator in time domain yields

$$y(k+2) - y(k+1) - 6y(k) = u(k+1) + u(k) \tag{3.7}$$

That is,

$$y(k+1) = y(k) + 6y(k-1) + u(k) + u(k-1). \tag{3.7}$$

c) Applying z transform to the controller signal gives

$$U(z) = Z[\mathrm{u(k)}] = K(U_c(z) - Y(z)) \tag{3.8}$$

Since we have known $Y(z) = H(z)U(z)$, then it follows that

$$Y(z) = H(z)K(U_c(z) - Y(z)) \tag{3.9}$$

Thus, it can be derived that

$$\frac{Y(z)}{U_c(z)} = \frac{KH(z)}{1+KH(z)} = \frac{K(z+1)}{z^2 + (K-1)z + K - 6} \tag{3.10}$$

Of course, another simpler way to derive above transfer function is to use the formula for transfer function of the closed loop, where KH(z) is both feedforward TF and open loop TF in that formula.

(Note: for this question, you can also work it out by first writing the input-output relation from $u_c(k)$ to $y(k)$ and then applying Z transform.)

d) The characteristic polynomial is $z^2 + (K-1)z + K - 6$. Then Jury's test can be listed in a table as follows.

ID	Operation	Result		
(1)	Get coefficients	1	K-1	K-6
(2)	Reverse	K-6	K-1	1
(3)	(1)-(2)*(K-6)	(K-5)(7-K)	(K-1)(7-K)	
(4)	Reverse	(K-1)(7-K)	(K-5)(7-K)	
(5)	(3) - (4)* $\frac{(K-1)(7-K)}{(K-5)(7-K)}$	$(K-5)(7-K) - \frac{(K-1)^2(7-K)^2}{(K-5)(7-K)}$		

According to Jury's test, if the system is stable, the first element of all the **odd** rows should be positive (in blue), i.e.,

$$\begin{cases} (K-5)(7-K) > 0 \\ (K-5)(7-K) - \dfrac{(K-1)^2(7-K)^2}{(K-5)(7-K)} > 0 \end{cases} \tag{3.11}$$

From the first inequality in (3.12), we can get $5 < K < 7$.

Then, on the basis of $5 < K < 7$, for the second inequality in (3.12), we can get $K < 3$.

Therefore, the final answer is

$$\begin{cases} 5 < K < 7 \\ K < 3 \end{cases} \Rightarrow \varnothing \tag{3.12}$$

Therefore, there does not exist such K to stabilize the closed-loop system.

e) Since the closed-loop system cannot be stabilized, the steady-state error will be infinity for a unit step. This example shows that you need to first determine the stability of the system before applying the formula for the step response and frequency response. Those formulas are only applicable to stable systems.

Q4 Solution

a) Applying z transform by assuming zero initial conditions can yield

$$zY(z)=3z^{-1}Y(z)-2z^{-2}Y(z)+U(z)-2z^{-1}U(z)+z^{-2}U(z) \tag{4.1}$$

Then, the transfer function is derived to be

$$H(z)=\frac{Y(z)}{U(z)}=\frac{(z-1)^2}{(z-1)^2(z+2)} \tag{4.2}$$

Poles: $(z-1)^2(z+2)=0 \Rightarrow z_1=1, z_2=1, z_3=-2$

Since z_3 is outside the unit circle, the system is unstable.

Zeros: $(z-1)^2=0 \Rightarrow z_1=z_2=1$.

The multiple zeros with magnitude equal 1. Thus, the inverse system is also unstable.

b) Observable but uncontrollable?

$$H(z)=\frac{z^2-2z+1}{z^3-3z+2} \tag{4.3}$$

whose observable canonical form is

$$\Phi=\begin{bmatrix}0 & 1 & 0\\ 3 & 0 & 1\\ -2 & 0 & 0\end{bmatrix},\quad \Gamma=\begin{bmatrix}1\\ -2\\ 1\end{bmatrix},\quad C=\begin{bmatrix}1 & 0 & 0\end{bmatrix} \tag{4.4}$$

The corresponding observability and controllability matrix of (4.4) are

$$W_o=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\qquad W_c=\begin{bmatrix}1 & -2 & 4\\ -2 & 4 & -8\\ 1 & -2 & 4\end{bmatrix} \tag{4.5}$$

and their ranks are $rank(W_o)=3$ and $rank(W_c)=1$ respectively. However, is (4.4) a true realization of $H(z)$ in (4.2)? To verify it, we write down the difference equation of (4.4) as

$$\begin{aligned}x_1(k+1)&=x_2(k)+u(k)\\ x_2(k+1)&=3x_1(k)+x_3(k)-2u(k)\\ x_3(k+1)&=-2x_1(k)+u(k)\end{aligned} \tag{4.6}$$

Then, for the input-output model we have

$$\begin{aligned}
&y(k) \\
&= x_1(k) \\
&= x_2(k-1) + u(k-1) \\
&= 3x_1(k-2) + x_3(k-2) - 2u(k-2) + u(k-1) \\
&= 3x_1(k-2) - 2x_1(k-3) + u(k-3) - 2u(k-2) + u(k-1) \\
&= 3y(k-2) - 2y(k-3) + u(k-1) - 2u(k-2) + u(k-3)
\end{aligned} \tag{4.7}$$

So,

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z^2 - 2z + 1}{z^3 - 3z + 2} = \frac{(z-1)^2}{(z+2)(z-1)^2} \tag{4.8}$$

As can be seen, equation (4.8) is exactly the input-output model of the original system $\mathrm{H(z)}$ in (4.2). That is to say, this is indeed a realization of the given original system.

Therefore, it is possible to realize the system such that it is observable but not controllable.

(*Note: for this question, if you just give the final conclusion, you get 1 mark; if you present justifications for your conclusion with some example or theoretical proof, you get the full marks*.)

c) Controllable but unobservable?

Similarly, the controllable canonical form of $\mathrm{H(z)}$ in (4.2) is

$$\Phi = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \tag{4.9}$$

and the associated observability and controllability matrix are

$$W_c = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad W_o = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 4 & -8 & 4 \end{bmatrix} \tag{4.10}$$

whose ranks are $rank(W_c) = 3$ and $rank(W_o) = 1$ respectively. The input-output model of the state space model (4.9) is

$$\begin{aligned}
x_1(k+1) &= 3x_2(k) - 2x_3(k) + u(k) \\
x_2(k+1) &= x_1(k) \\
x_3(k+1) &= x_2(k) \\
y(k) &= x_1(k) - 2x_2(k) + x_3(k)
\end{aligned} \tag{4.11}$$

which further leads to

$$\begin{aligned}
&y(k) \\
&= x_1(k) - 2x_2(k) + x_3(k) \\
&= 3x_2(k-1) - 2x_3(k-1) + u(k-1) - 2x_1(k-1) + x_2(k-1) \\
&= -2x_1(k-1) + 4x_2(k-1) - 2x_3(k-1) + u(k-1) \\
&= -2y(k-1) + u(k-1)
\end{aligned} \tag{4.12}$$

So, it is obvious that the transfer function for this input-output model is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{z+2} \tag{4.13}$$

As can be seen, equation (4.13) is different from the given transfer function H(z) in (4.2). That is to say, this realization is not valid. This is the non-minimal realization of $\frac{1}{z+2}$.

Therefore, it is impossible to realize the system such that it is controllable but not observable.

d) Both controllable and observable?

It is not possible since there are common poles and zeros in the transfer function as

$$H(z) = \frac{z^2 - 2z + 1}{z^3 - 3z + 2} = \frac{(z-1)^2}{(z+2)(z-1)^2} \tag{4.14}$$

You can also draw this conclusion by the result from part c). If the system is controllable, it must be equivalent to the controllable canonical form. But we already showed in part c) that the controllable canonical form is not the correct realization.