Example: Inverse Kinematics — 5R1P manipulator

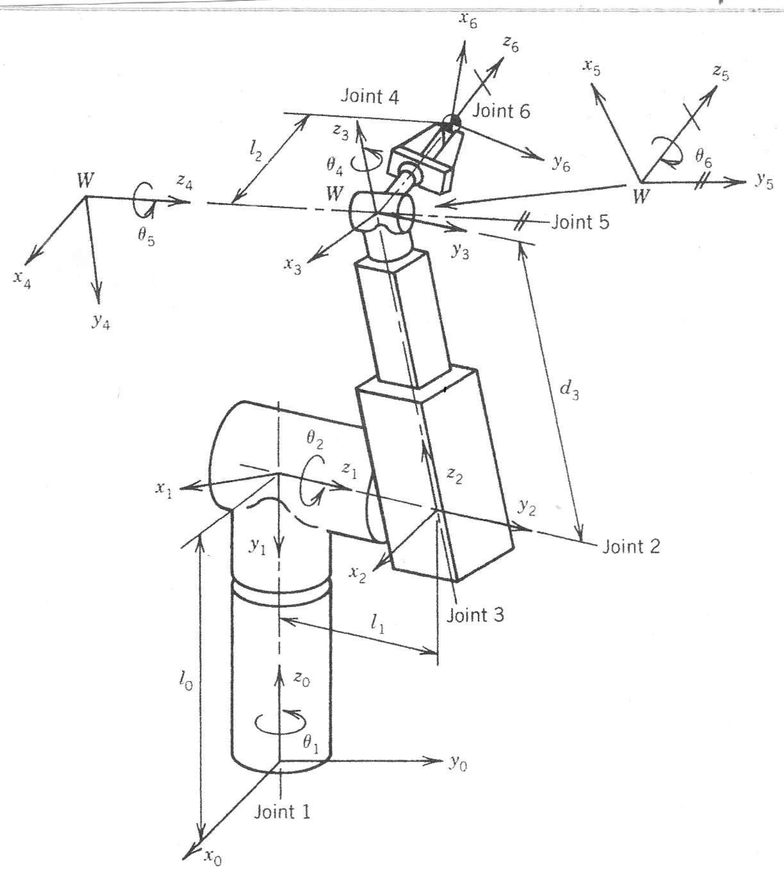

Figure 2-14: 5-R-1-P manipulator.

5-R-1-P (5 revolute joints and 1 prismatic joint) has 6 DOFs.

A manipulator arm must have at least 6 DOFs to locate its end effector at an arbitrary point w/ an arbitrary orientation in space.

Redundant manipulator $\equiv$ manipulator arm w/ more than 6 DOFs ($\exists$ an infinite number of solutions to the kinematic equation).

Kinematic equation: $T = A_1^0 A_2^1 A_3^2 A_4^3 A_5^4 A_6^5$ ——— ①

Postmultiplying both sides by $(A_6^5)^{-1}$:

$$\underbrace{T (A_6^5)^{-1}}_{\text{a function of } \theta_6} = \underbrace{A_1^0 A_2^1 A_3^2 A_4^3 A_5^4}_{\text{involves all other joint displacements}} \quad \text{——— ②}$$

Further premultiplying both sides by $(A_1^0)^{-1}$:

$$(A_1^0)^{-1} T (A_6^5)^{-1} = A_2^1 A_3^2 A_4^3 A_5^4 \quad \text{——— ③}$$

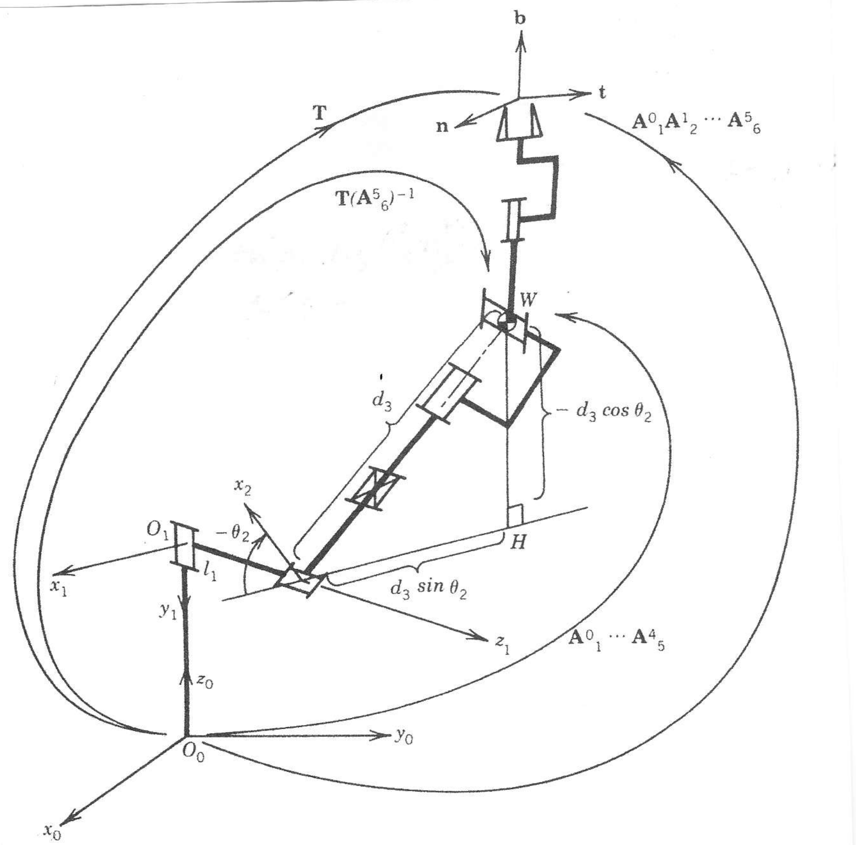

Figure 2-15 : Skeleton structure of the 5-R-1-P manipulator.

Both sides of eqn (2) represent the position and orientation of the frame attached to link 5 with reference to the base frame through two different paths reaching the same frame.

Point W is the origin of coordinate frame 5

== 4th column of the 4×4 matrix in eqn (2).

== 4th column of the 4×4 matrix in eqn (3):

$$x_w^1 = \begin{pmatrix} d_3 s_2 \\ -d_3 c_2 \\ l_1 \end{pmatrix} \quad (4)$$

(as observed from Figure 2-15).

The desired end-effector position and orientation, $T = \begin{bmatrix} n_x & t_x & b_x & p_x \\ n_y & t_y & b_y & p_y \\ n_z & t_z & b_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

substituting into LHS of eqn (3), we obtain

$$x_w^1 = \begin{bmatrix} p_x^* c_1 + p_y^* s_1 \\ -p_z^* + l_0 \\ -p_x^* s_1 + p_y^* c_1 \end{bmatrix} \quad (5)$$

where p_x^*, p_y^*, p_z^* represent the coordinates of point W, and

$p_x^* = p_x - l_2 b_x$; $p_y^* = p_y - l_2 b_y$; $p_z^* = p_z - l_2 b_z$.

Equating (4) and (5),

$$d_3 s_2 = p_x^* c_1 + p_y^* s_1 \quad \text{——— (6)}$$

$$-d_3 c_2 = -p_z^* + l_0 \quad \text{——— (7)}$$

$$l_2 = -p_x^* s_1 + p_y^* c_1 \quad \text{——— (8)}$$

To solve last eqn, let $t = \tan\left(\frac{\theta_1}{2}\right)$

$$c_1 = \cos\theta_1 = \frac{1-t^2}{1+t^2} \quad \text{and} \quad s_1 = \sin\theta_1 = \frac{2t}{1+t^2} \quad \text{——— (9)}$$

Substitute (9) into (8),

$$(l_1 + p_y^*)t^2 + 2p_x^* t + l_1 - p_y^* = 0.$$

Solving the above eqn for t,

$$\theta_1 = 2\tan^{-1}\left[\frac{-p_x^* \pm \sqrt{p_x^{*2} + p_y^{*2} - l_1^2}}{l_1 + p_y^*}\right] \quad \text{——— (10)}$$

Note the quantity under square root must be +ve, otherwise there is no solution $\Rightarrow$ the end-effector position is out of reach.
Eqn (10) can have two solutions due to $\pm$, $\Rightarrow$ two configurations of shoulder joints

Dividing both sides of (6) by (7),

$$\theta_2 = \tan^{-1}\left[\frac{p_x^* c_1 + p_y^* s_1}{p_z^* - l_0}\right] \quad \text{——— (11)}$$

d_3 can be obtained by taking the sum of the squares of equs (6) and (7):

$$d_3 = \pm\sqrt{(p_x^* c_1 + p_y^* s_1)^2 + (p_z^* - l_0)^2} \quad \text{——— (12)}$$

Note d_3 is always +ve.

After the first 3 joint displacements are determined, we solve the kinematic equation for the last 3 joint displacements.

$$[A^0_1(\theta_1) A^1_2(\theta_2) A^2_3(d_3)]^{-1} T = A^3_4(\theta_4) A^4_5(\theta_5) A^5_6(\theta_6) \quad \text{(13)}$$

Both sides of eqn represent position and orientation of end-effector viewed from third frame.

θ_1, θ_2 and d_3 have been determined.

$$T' = [A^0_1 A^1_2 A^2_3]^{-1} T = \begin{bmatrix} n_x' & t_x' & b_x' & p_x' \\ n_y' & t_y' & b_y' & p_y' \\ n_z' & t_z' & b_z' & p_z' \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Premultiplying eqn (13) by $[A^3_4(\theta_4)]^{-1}$

$$(A^3_4)^{-1} T' = \begin{bmatrix} * & * & b_z' C_4 + b_y' S_4 & * \\ -n_z' & -t_z' & -b_z' & * \\ -n_x' S_4 + n_y' C_4 & -t_x' S_4 + t_y' C_4 & -b_x' S_4 + b_y' C_4 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^4_5 A^5_6 = \begin{bmatrix} C_5 C_6 & -C_5 S_6 & S_5 & * \\ S_5 C_6 & -S_5 S_6 & -C_5 & * \\ S_6 & C_6 & 0 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$*$: irrelevant in present calculation.

Compare [3, 3] elements :

$$-b_x' S_4 + b_y' C_4 = 0 \implies \theta_4 = \tan^{-1}\left(\frac{b_y'}{b_x'}\right) \quad \text{(14)}$$

From [1, 3] and [2, 3] elements,

$$\theta_5 = \tan^{-1}\left(\frac{b_x' C_4 + b_y' S_4}{b_z'}\right)$$

where C_4 and S_4 are evaluated by eqn (14)

From [3, 1] and [3, 2] elements, similarly,

$$\theta_6 = \tan^{-1}\left(\frac{-n_x' S_4 + n_y' C_4}{-t_x' S_4 + t_y' C_4}\right)$$

$\Rightarrow$ All 6 joint displacements are obtained.

Source : "Robot Analysis and Control" by Haruhiko Asada and Jean-Jacques E. Slotine (1986)