EE5137 Stochastic Processes: Problem Set 8
Assigned: 11/03/22, Due: 18/03/22

There are six (6) non-optional problems in this problem set.

1. Exercise 4.10 (Gallager's book)

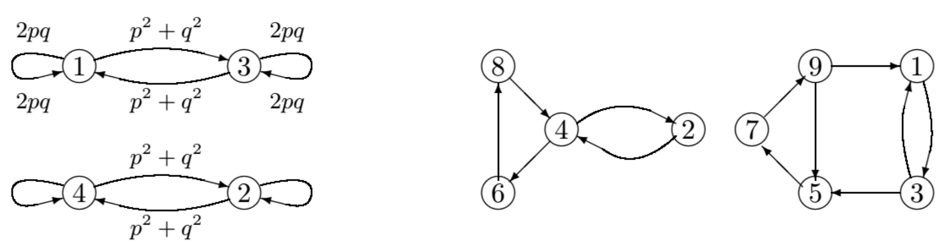
   (a) Find the steady-state probabilities for each of the Markov chains in Figure 4.2. Assume that all clockwise probabilities in the first graph are the same, say p, and assume that $P_{4,5} = P_{4,1}$ in the second graph.

   Solution: Solution: These probabilities can be found in a straightforward but tedious fashion by solving (4.8). Note that $\pi = \pi[P]$ is a set of M linear equations of which only $M-1$ are linearly independent and $\sum_i \pi_i = 1$ provides the needed extra equation. The solutions are $\pi_i = 1/4$ for each state in the first graph and $\pi_i = 1/10$ for all but state 4 in the second graph; $\pi_4 = 1/5$.

   One learns more by trying to find π by inspection. For the first graph, the π are clearly equal by symmetry. For the second graph, states 1 and 5 are immediately accessible only from state 4 and are thus equally likely and each has half the probability of 4. The probabilities on the states of each loop should be the same, leading to the answer above. It would be prudent to check this answer by (4.8), but that is certainly easier than solving (4.8).

   (b) Find the matrices $[P^2]$ for the same chains. Draw the graphs for the Markov chains represented by $[P^2]$, i.e., the graph of two step transitions for the original chains. Find the steady-state probabilities for these two-step chains. Explain why your steady-state probabilities are not unique.

   Solution: Let $q = 1 - p$ in the first graph. In the second graph, all transitions out of states 3, 4, and 9 have probability 1/2. All other transitions have probability 1.



   One steady-state probability for the first chain is $\pi_1 = \pi_3 = 1/2$ and the other is $\pi_2 = \pi_4 = 1/2$. These are the steady-state probabilities for the two recurrent classes of $[P^2]$. The second chain also has two recurrent classes. The steady-state probabilities for the first are $\pi_2 = \pi_6 = \pi_8 = 0.2$ and $\pi_4 = 0.4$. Those for the second are $\pi_1 = \pi_3 = \pi_5 = \pi_7 = \pi_9 = 0.2$.

   (c) Find $\lim_{n\to\infty}[P^{2n}]$ for each of the chains.

   Solution: The limit for each chain is block diagonal with one block being the even numbers and the other the odd numbers. Within a block, the rows are the same. For the first chain, the blocks are $(1,3)$ and $(2,4)$. We have $\lim_{n\to\infty} P_{ij}^{2n} = 1/2$ for i, j both odd or both even; it is 0 otherwise.

For the second chain, within the even block, $\lim_{n\to\infty} P_{ij}^n = 0.2$ for $j \neq 4$ and 0.4 for $j = 4$. For the odd block, $\lim_{n\to\infty} P_{ij}^n = 0.2$ for all odd i, j.

2. Consider a Markov chain with given transition probabilities and with a single recurrent class that is aperiodic. Assume that for $n \geq 500$, the n-step transition probabilities are very close to the steady-state probabilities.

   (a) Find an approximate formula for $\Pr(X_{1000} = j, X_{1001} = k, X_{2000} = k | X_0 = i)$.
   Solution: Let p_{ij} be the transition probabilities and let π_i be the steady-state probabilities. We then have

   $$P(X_{1000} = j, X_{1001} = k, X_{2000} = l | X_0 = i) = p_{ij}^{1000} p_{jk} p_{kl}^{999} \approx \pi_j p_{jk} \pi_l.$$

   Note that $p_{ij}^n = \Pr(X_n = j \mid X_0 = i)$ are the n-step transition probabilities.

   (b) Find an approximate formula for $\Pr(X_{1000} = i | X_{1001} = j)$
   Solution: Using Bayes' rule, we have

   $$\mathbb{P}(X_{1000} = i | X_{1001} = j) = \frac{P(X_{1000} = i, X_{1001} = j)}{P(X_{1001} = j)} = \frac{\pi_i p_{ij}}{\pi_j}.$$

3. A coin having probability $p, 0 < p < 1$, of landing heads is tossed continually. We are interested in the length of consecutive heads in the tosses. Define $X_n = k$ if the coin comes up heads in all of the most recent k tosses (from the $(n-k+1)$-st up to the nth), but tails in the $(n-k)$-th toss. On the contrary, if the coin comes up tails in the n-th toss, then let $X_n = 0$. For example, for the outcome of 15 tosses HHHHTTHHTHTTHHH, the value of X_n's are

   $$(X_1, \ldots, X_{15}) = (1, 2, 3, 4, 0, 0, 1, 2, 0, 1, 0, 0, 1, 2, 3)$$

   Observe that $\{X_n : n \geq 0\}$ is a Markov chain with infinite state-space $\mathcal{S} = \{0, 1, 2, \ldots\}$ and

   $$X_n = \begin{cases} X_{n-1} + 1 & \text{with probability } p \text{ (if the coin comes up heads in the } n\text{th toss)} \\ 0 & \text{with probability } 1-p \text{ (if the coin comes up tails in the } n\text{th toss)} \end{cases}$$

   (a) Find the limiting distribution of this chain, i.e., find

   $$\pi_i = \lim_{n\to\infty} \Pr(X_n = i), \quad i = 0, 1, 2, \ldots$$

   Solution: Any state $k > 0$ communicates with state 0 because we have a positive probability for the following cycle:

   $$0 \to 1 \to 2 \to \ldots \to k \to 0.$$

   The transition probabilities for this Markov chain are

   $$P_{i,i+1} = p, \quad P_{i,0} = 1 - p, \quad \forall i = 0, 1, 2, \ldots$$

   Other P_{ij}'s are 0.
   The limiting distribution is simply the solution to the balanced equation:

   $$\pi_i = \sum_j \pi_j P_{ji} = p\pi_{i-1} = p^2 \pi_{i-2} = \ldots = p^i \pi_0.$$

   To make sure that $\sum_{i=0}^{\infty} \pi_i = 1$,

   $$1 = \sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} p^i = \frac{\pi_0}{1-p}.$$

   we must have $\pi_0 = 1 - p$. The limiting distribution is thus

   $$\pi_i = \lim_{n\to\infty} \Pr(X_n = i) = (1-p)p^i, \qquad i = 0, 1, 2, \ldots.$$

(b) Let T_k be the first time that k consecutive heads have appeared. In other words, $T_k = m$ if and only if at m-th toss the Markov chain $\{X_n : n \geq 0\}$ reaches state k for the first time. Explain why that $\mathbb{E}[T_k]$'s satisfy the recursive equation

$$p\mathbb{E}[T_k] = \mathbb{E}[T_{k-1}] + 1, \quad k = 2, 3, 4, \ldots$$

and also show that $\mathbb{E}[T_1] = 1/p$.

Solution: To reach state k from state 0, the process must first reach state $k-1$. It takes T_{k-1} steps to reach state $k-1$. If the in the next toss the coin comes up heads, then state k is reached and so we have $T_k = 1 + T_{k-1}$. However, if the next toss comes up tails, then the process goes back to state 0 and one need another T_k' steps to reach state k. Here T_k' has the same distribution as T_k. Moreover, T_k' is independent of the past, and hence is independent of T_{k-1}. To sum up,

$$T_k = \begin{cases} T_{k-1} + 1 & \text{w.p. } p \\ T_{k-1} + 1 + T_k' & \text{w.p. } 1-p \end{cases}$$

So

$$\begin{aligned} \mathbb{E}[T_k|T_{k-1}] &= T_{k-1} + 1 + (1-p)\mathbb{E}[T_k'|T_{k-1}] \\ &\stackrel{(a)}{=} T_{k-1} + 1 + (1-p)\mathbb{E}[T_k'] \\ &\stackrel{(b)}{=} T_{k-1} + 1 + (1-p)\mathbb{E}[T_k] \end{aligned}$$

where (a) arises from the fact that T_k' is independent of T_{k-1} and (b) is because T_k' has the same distribution as T_k. Hence, Hence,

$$\mathbb{E}[T_k] = \mathbb{E}[\mathbb{E}[T_k|T_{k-1}]] = \mathbb{E}[T_{k-1} + 1 + (1-p)\mathbb{E}[T_k]] = \mathbb{E}[T_{k-1}] + 1 + (1-p)\mathbb{E}[T_k],$$

which is the same as

$$p\mathbb{E}[T_k] = \mathbb{E}[T_{k-1}] + 1.$$

For T_1, observe that T_1 has a geometric distribution since $T_1 = n$ if and only if the coin comes up tails in the first $n-1$ tosses and heads in the nth toss. So

$$\Pr(T_1 = n) = (1-p)^{n-1}p, \quad n = 1, 2, \ldots$$

One can then show that $\mathbb{E}[T_1] = 1/p$.

(c) Solve the recursive equation in part (b) to find $\mathbb{E}[T_k]$.

Solution: We have that

$$\begin{aligned} \mathbb{E}[T_1] &= \frac{1}{p} \\ \mathbb{E}[T_2] &= \frac{\mathbb{E}[T_1] + 1}{p} = \frac{1}{p} + \frac{1}{p^2} \\ \mathbb{E}[T_3] &= \frac{\mathbb{E}[T_2] + 1}{p} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \end{aligned}$$

and so on and thus we see that

$$\mathbb{E}[T_k] = \sum_{j=1}^{k} \frac{1}{p^j}.$$

4. We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ϵ where $0 < \epsilon < 1$, or we select a ball at random so that each ball

has probability $(1-\epsilon)/n$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the state-state distribution of the number of white balls?

Solution: Let $i = 0, 1, 2, \ldots, n$ be the states, with state i indicating that there are exactly i white balls. The nonzero transition probabilities are

$$p_{00} = \varepsilon; \quad p_{01} = 1-\varepsilon; \quad p_{nn} = \varepsilon; \quad p_{n,n-1} = 1-\varepsilon;$$
$$p_{i,i-1} = (1-\varepsilon)\frac{i}{n}; \quad p_{i,i+1} = (1-\varepsilon)\frac{n-i}{n}, \quad i = 1, 2, \ldots, n-1.$$

The chain has a single recurrent class, which is aperiodic. In addition, it is a birth-death process. The local balance equations take the form

$$\pi_i(1-\varepsilon)\frac{n-i}{n} = \pi_{i+1}(1-\varepsilon)\frac{i+1}{n} \quad i = 0, 1, 2, \ldots, n-1,$$

which leads to

$$\pi_i = \frac{n(n-1)\cdots(n-i+1)}{1.2.\cdots i}\pi_0 = \frac{n!}{i!(n-i)!}\pi_0 = \binom{n}{i}\pi_0.$$

We recognize that this has the form of a binomial distribution, so that for the probabilities to add to 1, we must have $\pi_0 = 1/2^n$. Therefore, the steady-state probabilities are given by

$$\pi_j = \binom{n}{j}\left(\frac{1}{2}\right)^n, \quad j = 0, 1, 2, \ldots, n.$$

5. Each of two urns contains m balls. Out of the total of $2m$ balls, m are white and m are black. A ball is simultaneously selected from each urn and moved to the other urn, and the process is indefinitely repeated. What is the steady-state distribution of the number of white balls in each urn?

   Solution: Let $j = 1, 2, \ldots, m$ be the states, with state j corresponding to the first urn containing j white balls. The nonzero transition probabilities are

   $$p_{j,j-1} = \left(\frac{j}{m}\right)^2, \quad p_{j,j+1} = \left(\frac{m-j}{m}\right)^2, \quad p_{jj} = \frac{2j(m-j)}{m^2}.$$

   The chain has a single recurrent class that is aperiodic. This chain is a birth-death process and the steady-state probabilities can be found by solving the local balance equations:

   $$\pi_j\left(\frac{m-j}{m}\right)^2 = \pi_{j+1}\left(\frac{j+1}{m}\right)^2, \quad j = 0, 1, 2, \ldots, m-1.$$

   The solution is of the form

   $$\pi_j = \pi_0\left(\frac{m(m-1)\cdots(m-j+1)}{1.2.\cdots j}\right)^2 = \pi_0\left(\frac{m!}{j!(m-j)!}\right)^2 = \pi\binom{m}{j}^2.$$

   We recognize this as having the form of the hypergeometric distribution (Problem 61 of Chapter 1, with $n = 2m$ and $k = m$), which implies that $\pi_0 = \binom{2m}{m}$, and

   $$\pi_j = \frac{\binom{m}{j}^2}{\binom{2m}{m}}, \quad j = 1, 2, \ldots, m.$$

6. An auto insurance company classifies its customers in three categories: bad, satisfactory and preferred. No one moves from bad to preferred or from preferred to bad in one year. 40% of the customers in the bad category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become bad; 20% of those in the preferred category are downgraded to satisfactory.

   Solution: The transition matrix is

$$[P] = \begin{bmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

   We will find the limiting fraction of drivers in each of these categories from the components of the stationary distribution vector π, which satisfies the following equation:

$$\pi = \pi[P].$$

   This is equivalent to the following system of equations:

$$\begin{aligned} \pi_1 &= 0.6\pi_1 + 0.1\pi_2 \\ \pi_2 &= 0.4\pi_1 + 0.6\pi_2 + 0.2\pi_3 \\ \pi_3 &= 0.3\pi_2 + 0.8\pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

   This has the following solution

$$\pi = \frac{1}{11}(1, 4, 6)$$

   Thus, the limiting fraction of drivers in the bad category is $1/11$, in the satisfactory category $4/11$ and in the preferred category $6/11$.

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7. (Optional) Exercise 4.5 (Gallager's book)

   (a) (**Proof of Theorem 4.2.11)**) Show that an ergodic Markov chain with $M > 1$ states must contain a cycle with $\tau < M$ states. Hint: Use ergodicity to show that the smallest cycle cannot contain M states.

   Solution: The states in any cycle (not counting the initial state) are distinct and thus the number of steps in a cycle is at most M. A recurrent chain must contain cycles, since for each pair of states $l \neq j$, there is a walk from l to j and then back to l; if any state i other than l is repeated in this walk, the first i and all subsequent states before the second i can be eliminated. This can be done repeatedly until a cycle remains.

   Finally, suppose a cycle contains M states. If there is any transition $P_{ij} > 0$ for which (i, j) is not a transition on that cycle, then that transition can be added to the cycle and all the transitions between i and j on the existing cycle can be omitted, thus creating a cycle of fewer than M steps. If there are no nonzero transitions other than those in a cycle with M steps, then the Markov chain is periodic with period M and thus not ergodic.

   (b) Let l be a fixed state on a fixed cycle of length $\tau < M$. Let $\mathcal{T}(m)$ be the set of states accessible from l in m steps. Show that for each $m \geq 1$, $\mathcal{T}(m) \subseteq \mathcal{T}(m + \tau)$. Hint: For any given state $j \in \mathcal{T}(m)$, show how to construct a walk of $m + \tau$ steps from l to j from the assumed walk of m steps.

   Solution: Let j be any state in $\mathcal{T}(m)$. Then there is an m-step walk from l to j. There is also a cycle of τ steps from state l to l. Concatenate this cycle (as a walk) with the above m step walk from l to j, yielding a walk of $\tau + m$ steps from l to j. Thus $j \in \mathcal{T}(m + \tau)$ and it follows that $\mathcal{T}(m) \subseteq \mathcal{T}(m + \tau)$.

(c) Define $\mathcal{T}(0)$ to be singleton set $\{l\}$ and show that

$$\mathcal{T}(0) \subseteq \mathcal{T}(\tau) \subseteq \mathcal{T}(2\tau) \subseteq \cdots \subseteq \mathcal{T}(n\tau) \subseteq \cdots .$$

Solution: Since $\mathcal{T}(0) = \{l\}$ and $l \in \mathcal{T}(\tau)$, we see that $\mathcal{T}(0) \subseteq \mathcal{T}(\tau)$. Next, for each $n \geq 1$, use (b), with $m = n\tau$, to see that $\mathcal{T}(n\tau) \subseteq \mathcal{T}(n\tau + \tau)$. Thus each subset inequality above is satisfied.

(d) Show that if one of the inclusions above is satisfied with equality, then all subsequent inclusions are satisfied with equality. Show from this that at most the first $M-1$ inclusions can be satisfied with strict inequality and that $\mathcal{T}(n\tau) = \mathcal{T}((M-1)\tau)$ for all $n \geq M-1$.

Solution: We first show that if $\mathcal{T}((k+1)\tau) = \mathcal{T}(k\tau)$ for some k, then $\mathcal{T}(n\tau) = \mathcal{T}(k\tau)$ for all $n > k$. Note that $\mathcal{T}((k+1)\tau)$ is the set of states reached in τ steps from $\mathcal{T}(k\tau)$. Similarly $\mathcal{T}((k+2)\tau)$ is the set of states reached in τ steps from $\mathcal{T}((k+1)\tau)$. Thus if $\mathcal{T}((k+1)\tau) = \mathcal{T}(k\tau)$ then also $\mathcal{T}((k+2)\tau) = \mathcal{T}((k+1)\tau)$. Using induction, $\mathcal{T}(n\tau) = \mathcal{T}(k\tau)$ for all $n \geq k$. Now if k is the smallest integer for which $\mathcal{T}((k+1)\tau) = \mathcal{T}(k\tau)$, then the size of $\mathcal{T}(n\tau)$ must increase for each $n < k$. Since $|\mathcal{T}(0)| = 1$, we see that $|T(n\tau)| \geq n+1$ for $n \leq k$. Since M is the total number of states, we see that $k \leq M-1$. Thus $\mathcal{T}(n\tau) = \mathcal{T}((M-1)\tau)$ for all $n \geq M-1$.

(e) Show that all states are included in $\mathcal{T}((M-1)\tau)$.

Solution: For any t such that $P_{ll}^t > 0$, we can repeat the argument in part (b), replacing τ by t to see that for any $m \geq 1$, $\mathcal{T}(m) \subset \mathcal{T}(m+t)$. Thus we have

$$\mathcal{T}((M-1)\tau) \subseteq \mathcal{T}((M-1)\tau + t) \subseteq \cdots \subseteq \mathcal{T}((M-1)\tau + t\tau) = T((M-1)\tau),$$

where (d) was used in the final equality. This shows that all the inclusions above are satisfied with equality and thus that $\mathcal{T}((M-1)\tau) = \mathcal{T}((M-1)\tau + kt)$, for all $k \leq \tau$. Using $t$in place of τ in the argument in (d), this can be extended to

$$\mathcal{T}((M-1)\tau) = \mathcal{T}((M-1)\tau + kt), \quad \text{for all} \quad k \geq 1.$$

Since the chain is ergodic, we can choose t so that both $P_{ll}^t > 0$ and $gcd(t, \tau) = 1$. From elementary number theory, integers $k \geq 1$ and $j \geq 1$ can then be chosen so that $kt = jr + 1$. Thus

$$\mathcal{T}((M-1)\tau) = \mathcal{T}((M-1)\tau + kt) = \mathcal{T}((M-1+j)\tau + 1) = \mathcal{T}((M-1)\tau + 1) \tag{1}$$

As in (d), $\mathcal{T}((M-1)\tau + 2)$, is the set of states reachable in one step from $\mathcal{T}((M-1)\tau + 1$. From (1),this is the set of states reachable from $\mathcal{T}((M-1)\tau)$ in 1 step, i.e.,

$$\mathcal{T}((M-1)\tau + 2) = T((M-1)\tau + 1) = \mathcal{T}((M-1)).$$

Extending this,

$$\mathcal{T}((M-1)\tau) = \mathcal{T}((M-1)\tau + m), \quad \text{for all} \quad m \geq 1.$$

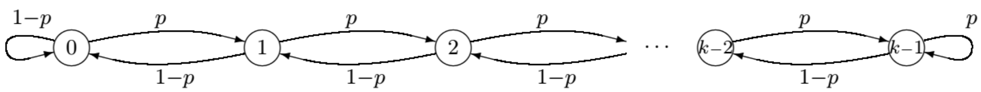
This means that $\mathcal{T}((M-1)\tau)$ contains all states that can ever occur from time $((M-1)\tau$ on, and thus must contain all states since the chain is recurrent.

(f) Show that $P_{ij}^{(M-1)^2+1} > 0$ for all i, j.

Solution: We have shown that all states are accessible from state l at all times $\tau(M-1)$ or later, and since $\tau \leq M-1$, all are accessible at all times $n \geq (M-1)^2$. The same applies to any state on a cycle of length at most $M-1$. It is possible (as in Figure 4.4), for some states to be only on a cycle of length M. Any such state can reach the cycle in the proof in at most $M - \tau$ steps. Using this path to reach a state on the cycle and following this by paths of length $\tau(M-1)$, all states can reach all other states at all times greater than or equal to

$$\tau(M-1) + M - \tau \leq (M-1)^2 + 1.$$

The above derivation assumed $M > 1$. The case $M = 1$ is obvious, so the theorem is proven.



8. (Optional) Exercise 4.9 (Gallager's book)

(a) Find the steady-state probabilities $\pi_0, \cdots, \pi_{k-1}$ for the Markov chain below. Express your answer in terms of the ratio $\rho = p/q$ where $q = 1 - p$. Pay particular attention to the special case $\rho = 1$.
Solution: The steady-state equations, using the abbreviation $q = 1 - p$ are

$$\begin{aligned} \pi_0 &= q\pi_0 + q\pi_1, \\ \pi_j &= p\pi_{j-1} + q\pi_{j+1}; \quad \text{for} \quad 1 \le j \le k-2, \\ \pi_{k-1} &= p\pi_{k-2} + p\pi_{k-1}. \end{aligned}$$

Simplifying the first equation, we get $p\pi_0 = q\pi_1$. Substituting $q\pi_1$ for $p\pi_0$ in the second equation, we get $\pi_1 = q\pi_1 + q\pi_2$. Simplifying, we get $p\pi_1 = q\pi_2$.
We can then use induction. Using the inductive hypothesis $p\pi_{j-1} = q\pi_j$ (which has been verified for $j = 1, 2$) on $\pi_j = p\pi_{j-1} + q\pi_{j+1}$, we get

$$p\pi_j = q\pi_{j+1} \quad \text{for} \quad 1 \le j \le k-2.$$

Combining these equations, $\pi_j = \rho\pi_{j-1}$ for $1 \le j \le k-1$, so $\pi_j = \rho^j \pi_0$ for $1 \le j \le k-1$. Normalizing by the fact that the steady-state probabilities sum to 1, and taking $\rho \neq 1$,

$$\pi_0 \left(\sum_{j=0}^{k-1} \rho^j \right) = 1 \quad \text{so} \quad \pi_0 = \frac{1-\rho}{1-\rho^k}, \quad \text{and} \quad \pi_j = \rho^j \frac{1-\rho}{1-\rho^k}. \tag{2}$$

For $\rho = 1, \rho^j = 1$ and $\pi_j = 1/k$ for $0 \le j \le k-1$. Note that we did not use the final steady-state equation. The reason is that $\pi = \pi[P]$ specifies π only within a scale factor, so the k equations can not be linearly independent and the kth equation can not provide anything new.
Note also that the general result here is most simply stated as $p\pi_{j-1} = q\pi_j$. This says that the steady-state probability of a transition from $j-1$ to j is the same as that from j to $j-1$. This is intuitively obvious since in any sample path, the total number of transitions from $j-1$ to j is within 1 of the transitions in the opposite direction. This important intuitive notion will become precise after studying the strong law of numbers.

(b) Sketch $\pi_0, \cdots, \pi_{k-1}$. Give one sketch for $\rho = 1/2$, one for $\rho = 1$, and one for $\rho = 2$.
Solution: We see from (2) that for $\rho \neq 1$, π_j is geometrically distributed for $1 \le j \le k-1$. For $\rho > 1$, it is geometrically increasing in j and for $\rho < 1$, it is geometrically decreasing. For $\rho = 1$, it is constant. If you can't visualize this without a sketch, you should draw the sketch yourself.

(c) Find the limit of π_0 as k approaches ∞; give separate answers for $\rho < 1, \rho = 1$, and $\rho > 1$. Find limiting values of π_{k-1} for the same cases.
Solution:

$$\lim_{k\to\infty} \pi_0 = \begin{cases} \lim_{k\to\infty} \frac{1-\rho}{1-\rho^k} = 1 - \rho, & \text{for} \quad \rho < 1, \\ \lim_{k\to\infty} \frac{1}{k} = 0, & \text{for} \quad \rho = 1, . \\ \lim_{k\to\infty} \frac{\rho-1}{\rho^k - 1} = 0, & \text{for} \quad \rho > 1 \end{cases}$$

Note that the approach to 0 for $\rho = 1$ is harmonic in k and that for $\rho > 1$ is geometric in k. For state $k-1$, the analogous result is

$$\lim_{k\to\infty} \pi_0 = \begin{cases} \lim_{k\to\infty} \rho^{k-1} \frac{1-\rho}{1-\rho^k} = 0, & \text{for} \quad \rho < 1, \\ \lim_{k\to\infty} \frac{1}{k} = 0, & \text{for} \quad \rho = 0, \\ \lim_{k\to\infty} \rho^{k-1} \frac{\rho-1}{\rho^k-1} = 1 - \frac{1}{\rho}, & \text{for} \quad \rho > 1 \end{cases}.$$

This shows us that for $\rho > 1$, the probability is still geometric in the states, but the large probabilities are clustered at the highest states. We will see in the next chapter that with a countable number of states, all states have probability zero when $\rho \geq 1$, and this shows us how that limit is approached with increasing k.