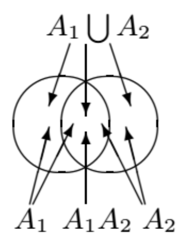
EE5137 Stochastic Processes: Problem Set 1
Assigned: 14/01/22, Due: 21/01/22

There are five non-optional problems in this problem set.

1. Exercise 1.1 (Gallager's book) Let A_1 and A_2 be arbitrary events and show that $\Pr\{A_1 \cup A_2\} + \Pr\{A_1 A_2\} = \Pr\{A_1\} + \Pr\{A_2\}$. Explain which parts of the sample space are being double-counted on both sides of this equation and which parts are being counted one.
   Solution: As in the the figure below, $A_1 \cap A_2$ is part of $A_1 \cup A_2$ and is thus being double counted on the left side of the equation. It is also being double counted on the right (and is in fact the meaning of $A_1 \cap A_2$ as those sample points that are both in A_1 and in A_2).

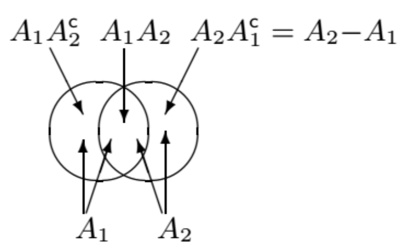


2. Exercise 1.2 (Gallager's book) This exercise derives the probability of an arbitrary (non-disjoint) union of events, derives the union bound, and derives some useful limit expressions.

   (a) For 2 arbitrary events A_1 and A_2, show that

$$A_1 \cup A_2 = A_1 \cup (A_2 - A_1), \tag{1}$$

   where $A_2 \setminus A_1 = A_2 \cap A_1^c$. Show that A_1 and $A_2 \setminus A_1$ are disjoint. Hint: This is what Venn diagrams were invented for.
   Solution: Note that each sample point ω is in A_1 or A_1^c, but not both. Thus each ω is in exactly one of $A_1, A_1^c \cap A_2$ or $A_1^c \cap A_2^c$. In the first two cases, ω is both sides of $(A.1)$ and in the last case it is in neither. Thus the both sides of $(A.1)$ are identical. Also, as pointed out above, A_1 and $A_2 - A_1$ are disjoint. These results are intuitively obvious from the Venn diagram.



(b) For any $n \geq 2$ and arbitrary events $A_1, A_2, \ldots, A_n$, define $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$. Show that $B_1, B_2, \ldots$ are disjoint events and show that for each $n \geq 2$, $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$. Hint: Use induction.

Solution: Let $B_1 = A_1$. From (a) B_1 and B_2 are disjoint and (from (A.1)), $A_1 \cup A_2 = B_1 \cup B_2$. Let $C_n = \bigcup_{i=1}^{n} A_i$. We use induction to prove that $C_n = \bigcup_{i=1}^{n} B_i$ and that the B_n are disjoint. We have seen that $C_2 = B_1 \bigcup B_2$, which forms the basis for the induction. We assume that $C_{n-1} = \bigcup_{n=1}^{i-1} B_i$ and prove that $C_n = \bigcup_{i=1}^{n} B_i$.

$$\begin{aligned} C_n &= C_{n-1} \cup A_n = C_{n-1} \cup (A_n \cap C_{n-1}^c) \\ &= C_{n-1} \cup B_n = \bigcup_{i=1}^{n} B_i. \end{aligned}$$

In the second equality, we used (1), letting C_{n-1} play the role of A_1 and A_n play the role of A_2. From this same application of (1), we also see that C_{n-1} and $B_n = A_n \setminus C_{n-1}$ are disjoint. Since $C_{n-1} = \bigcup_{i=1}^{n-1} B_i$, this also shows that B_n is disjoint from $B_1, B_2, \ldots, B_{n-1}$.

(c) Show that

$$\Pr\left\{ \bigcup_{n=1}^{\infty} A_n \right\} = \Pr\left\{ \bigcup_{n=1}^{\infty} B_n \right\} = \sum_{n=1}^{\infty} \Pr\{B_n\}.$$

Solution: If $\omega \in \bigcup_{n=1}^{\infty} A_n$, then it is in A_n for some $n \geq 1$. Thus $\omega \in \bigcup_{i=1}^{n} B_i$, and thus $\omega \in \bigcup_{n=1}^{\infty} B_n$. The same argument works the other way, so $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. This establishes the first equality above, and the second is the third axiom of probability.

(d) Show that for each n, $\Pr\{B_n\} \leq \Pr\{A_n\}$. Use this to show that

$$\Pr\left\{ \bigcup_{n=1}^{\infty} A_n \right\} \leq \sum_{n=1}^{\infty} \Pr\{A_n\}.$$

Solution: Since $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$, we see that $\omega \in B_n$ implies that $\omega \in A_n$, i.e., that $B_n \subset A_n$. From (1.5) in the textbook, this implies that $\Pr\{B_n\} \leq \Pr\{A_n\}$ for each n. Thus

$$\Pr\left\{ \bigcup_{n=1}^{\infty} A_n \right\} = \sum_{n=1}^{\infty} \Pr\{B_n\} \leq \sum_{n=1}^{\infty} \Pr\{A_n\}.$$

(e) Show that $\Pr\{\bigcup_{n=1}^{\infty} A_n\} = \lim_{n\to\infty} \Pr\{\bigcup_{i=1}^{n} A_i\}$. Hint: Combine (c) and (b). Note that this says that the probability of a limit is equal to the limit of the probabilities. This might well appear to be obvious without a proof, but you will see situations later where similar appearing interchanges cannot be made.

Solution: From (c),

$$\Pr\left\{ \bigcup_{n=1}^{\infty} A_n \right\} = \sum_{n=1}^{\infty} \Pr\{B_n\} = \lim_{k\to\infty} \sum_{n=1}^{k} \Pr\{B_n\}.$$

From (b), however,

$$\sum_{n=1}^{k} \Pr\{B_n\} = \Pr\left\{ \bigcup_{n=1}^{k} B_n \right\} = \Pr\left\{ \bigcup_{n=1}^{k} A_n \right\}.$$

Combining the first equation with the limit in k of the second yields the desired result.

(f) Show that $\Pr\left\{\bigcap_{n=1}^{\infty} A_n\right\} = \lim_{n\to\infty} \Pr\left\{\bigcap_{i=1}^{n} A_i\right\}$. Hint: Remember De Morgan's inequalities.
Solution: Using De Morgan's equalities,

$$\Pr\Big\{\bigcap_{n=1}^{\infty} A_n\Big\} = 1 - \Pr\Big\{\bigcup_{n=1}^{\infty} A_n^c\Big\} = 1 - \lim_{k\to\infty} \Pr\Big\{\bigcup_{n=1}^{k} A_n^c\Big\}$$
$$= \lim_{k\to\infty} \Pr\Big\{\bigcap_{n=1}^{k} A_n\Big\}.$$

3. Exercise 1.3 (Gallager's book) Find the probability that a five card poker hand, chosen randomly from a 52 card deck, contains 4 aces. That is, if all 52! arrangements of a deck of cards are equally likely, what is the probability that all 4 aces are in the first 5 cards of the deck.
Solution: The ace of spades can be in any of the first 5 positions, the ace of hearts in any of the remaining positions out of the first 5, and so forth for the other two aces. The remaining 48 cards can be in any of the remaining 48 positions. Thus there are (5.4.3.2)48! permutations of the 52 cards for which the first 5 cards contains 4 aces. Thus,

$$\Pr\{4 \text{ aces}\} = \frac{5!48!}{52!} = \frac{5.4.3.2}{52.51.50.49} = 1.847 \times 10^{-5}.$$

4. In Section 1.2.1 of Gallager's book, we saw that given a sample space Ω a σ-*algebra* $\mathfrak{F}$ of Ω is a collection of subsets of Ω that satisfies (i) $\Omega \in \mathfrak{F}$; (ii) For any sequence of sets $A_1, A_2, \ldots \in \mathfrak{F}$, $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{F}$; and (iii) For every $A \in \mathfrak{F}$, $\Omega \setminus A \in \mathfrak{F}$. The elements of $\mathfrak{F}$ are called *events* in probability theory and $\mathfrak{F}$-*measurable sets* in measure theory.

(a) Show that if $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are σ-algebras so is $\mathfrak{F}_1 \cap \mathfrak{F}_2$;

(b) Is it true that if $\{\mathfrak{F}_\alpha\}_{\alpha\in\mathcal{I}}$ is a family of σ-algebras, so is $\bigcap_{\alpha\in\mathcal{I}} \mathfrak{F}_\alpha$?

(c) Consider parts (a) and (b) for unions.

Solution:

(a) Let $\mathfrak{F}_i, i = 1, 2$ be σ-algebras in Ω. Consider $\mathfrak{F} = \mathfrak{F}_1 \cap \mathfrak{F}_2$. Since $\Omega \in \mathfrak{F}_i, i = 1, 2$, hence, $\Omega \in \mathfrak{F}$. Next, for any $A \in \mathfrak{F}$, $A \in \mathfrak{F}_i$ for each $i = 1, 2$. Since each of the $\mathfrak{F}_i$ is a σ-algebra, so $A^c = \Omega\setminus A \in \mathfrak{F}_i$ for each $i = 1, 2$. Hence, $A^c = \mathfrak{F}$. Finally, let $\{A_j\}_{j=1}^{\infty} \subset \mathfrak{F}$ be a sequence in $\mathfrak{F}$. Then each of the A_j's belongs to both $\mathfrak{F}_i$ for $i = 1, 2$. Since each $\mathfrak{F}_i$ is a σ-algebra, so $\cup_{j=1}^{\infty} A_j \in \mathfrak{F}_i$ for each $i = 1, 2$. Hence, by the definition of intersection, $\cup_{j=1}^{\infty} A_j \in \mathfrak{F}$ as desired. So $\mathfrak{F}$ is a σ-algebra as all three axioms hold.

(b) Yes, the same proof goes through.

(c) If $\mathfrak{F}_i, i = 1, 2$ are σ-algebras, $\mathfrak{F}_1 \cup \mathfrak{F}_2$ need not be a σ-algebra. Consider $\Omega = \{a, b, c\}$ and $\mathfrak{F}_1 = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$ and $\mathfrak{F}_2 = \{\emptyset, \{b\}, \{a, c\}, \Omega\}$. Then $\mathfrak{F}_1 \cup \mathfrak{F}_2 = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \Omega\}$ is not a σ-algebra because $\{a\} \cup \{b\} = \{a, b\}$ is not in $\mathfrak{F}_1 \cup \mathfrak{F}_2$ and σ-algebras are closed under countable union (we proved this in class).

5. (Strengthened Union Bound) Let $A_1, \ldots, A_n$ be arbitrary events. Prove that

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} \le \min_{1\le k\le n} \left(\sum_{i=1}^{n} \Pr\{A_i\} - \sum_{i=1\,:\,i\ne k}^{n} \Pr\{A_i \cap A_k\}\right).$$

Hint: For any two sets C *and* D,

$$C = (C \cap D) \cup (C \cap D^c)$$

Solution: Using the hint, we have

$$\bigcup_{i=1}^{n} A_i = \left[\left(\bigcup_{i=1}^{n} A_i\right) \cap A_k\right] \cup \left[\left(\bigcup_{i=1}^{n} A_i\right) \cap A_k^c\right].$$

for any $1 \le k \le n$. But this is equivalent to

$$\bigcup_{i=1}^{n} A_i = A_k \cup \left[\bigcup_{i=1}^{n} (A_i \cap A_k^c)\right].$$

Taking probabilities,

$$\begin{aligned}
\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} &= \Pr\left\{A_k \cup \left[\bigcup_{i=1}^{n} (A_i \cap A_k^c)\right]\right\} \\
&\le \Pr\{A_k\} + \sum_{i=1}^{n} \Pr\{A_i \cap A_k^c\} \\
&= \Pr\{A_k\} + \sum_{i=1, i\neq k}^{n} \Pr\{A_i \cap A_k^c\}, \qquad \text{(because } A_k \cap A_k^c = \emptyset) \\
&= \Pr\{A_k\} + \sum_{i=1, i\neq k}^{n} [\Pr\{A_i\} - \Pr(A_i \cap A_k\}] \\
&= \sum_{i=1}^{n} \Pr\{A_i\} - \sum_{i=1, i\neq k}^{n} \Pr(A_i \cap A_k\}
\end{aligned}$$

Since the bound holds for all $1 \le k \le n$, we can minimize the right-hand-side to yield

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} \le \min_{1\le k\le n} \left(\sum_{i=1}^{n} \Pr\{A_i\} - \sum_{i=1\,:\, i\neq k}^{n} \Pr\{A_i \cap A_k\}\right).$$

as desired.

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6. (Optional) Often, by using the union bound or its variants (such as Question 6 or Gallager's ρ-trick[1]), it is easy to upper bound probabilities. Lower bounding probabilities is often harder, but very useful. Let $A_1, \ldots, A_n$ be arbitrary events. Prove that

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} \ge \sum_{i=1}^{n} \frac{\Pr\{A_i\}^2}{\sum_{j=1}^{n} \Pr\{A_i \cap A_j\}}.$$

This bound is called de Caen's lower bound. Obviously from the form of the inequality, you've to use the Cauchy–Schwarz inequality somewhere.

Solution: First assume that the sample space Ω is finite. For each $\omega \in \Omega$, let $\deg(\omega)$ be the number of A_i's that that contain ω, i.e., $\deg(\omega) = |\{i \in \{1, 2, \ldots, n\} : \omega \in A_i\}|$. Put $p(\omega) = \Pr\{\{\omega\}\}$. Then we have

$$\Pr\left\{\bigcup_{i=1}^{n} A_i\right\} = \sum_{i=1}^{n} \sum_{\omega \in A_i} \frac{p(\omega)}{\deg(\omega)}. \tag{2}$$

[1]This says that $\Pr\{\cup_{i=1}^{n} A_i\} \le \left(\sum_{i=1}^{n} \Pr\{A_i\}\right)^{\rho}$ for any $0 \le \rho \le 1$. Prove this.

This holds because the probability of the union the A_i's is exactly $\sum_{\omega\in\cup_{i=1}^n A_i} p(\omega)$ and we sum over the indices i, then if ω appears in exactly one of the A_i's the contribution is $p(\omega)$; if ω appears in exactly k of the A_i's, its contribution is $p(\omega)/k$ and so on. Next, consider an application of the Cauchy–Schwarz inequality as follows:

$$\left(\sum_{\omega\in A}\frac{p(\omega)}{\deg(\omega)}\right)\left(\sum_{\omega\in A}p(\omega)\deg(\omega)\right)\geq\left(\sum_{\omega\in A}\left[\frac{p(\omega)}{\deg(\omega)}\right]^{1/2}[p(\omega)\deg(\omega)]^{1/2}\right)^2=\left(\sum_{\omega\in A}p(\omega)\right)^2. \quad (3)$$

Combining (2) and (3) yields

$$\Pr\left\{\bigcup_{i=1}^n A_i\right\}\geq\sum_{i=1}^n\frac{(\sum_{\omega\in A_i}p(\omega))^2}{\sum_{\omega\in A_i}p(\omega)\deg(\omega)}.$$

Finally, notice by the definition of probability that $\sum_{\omega\in A_i}p(\omega)=\Pr\{A_i\}$ and it is easy to check that $\sum_{\omega\in A_i}p(\omega)\deg(\omega)=\sum_{j=1}^n\Pr\{A_i\cap A_j\}$. Thus,

$$\Pr\left\{\bigcup_{i=1}^n A_i\right\}\geq\sum_{i=1}^n\frac{\Pr\{A_i\}^2}{\sum_{j=1}^n\Pr\{A_i\cap A_j\}},$$

as desired.

7. (Optional) This is another lower bound on the union of n events $A_1,\ldots,A_n$. Prove that

$$\Pr\left\{\bigcup_{i=1}^n A_i\right\}\geq\frac{\sum_{i,j}\Pr\{A_i\}\Pr\{A_j\}}{\sum_{i,j}\Pr\{A_i\cap A_j\}}.$$

This bound is called the Chung–Erdős inequality. Obviously from the form of the inequality, you've to use the Cauchy–Schwarz inequality somewhere.

Solution: Let $A=\cup_{i=1}^n A_i$ and put $S_n=\sum_{i=1}^n\mathbb{1}\{A_i\}$. Then the event $A=\{S_n>0\}$. Now consider

$$\sum_{i=1}^n\Pr\{A_i\}=\sum_{i=1}^n\mathbb{E}\big[\mathbb{1}\{A_i\}\big]=\mathbb{E}\left[\sum_{i=1}^n\mathbb{1}\{A_i\}\right]=\mathbb{E}\big[S_n\big]=\mathbb{E}\big[S_n\mathbb{1}\{S_n>0\}\big]. \quad (4)$$

The only "difficult" equality to justify is the last one. Let $Y=S_n\mathbb{1}\{S_n>0\}$. Then $\Pr\{Y=0\}=\Pr\{S_n=0\}$ and $\Pr\{S_n=a\}=\Pr\{Y=a\}$ for $a=1,\ldots,n$. Hence, $\mathbb{E}[S_n]=\mathbb{E}[Y]$. By applying the Cauchy–Schwarz inequality $(\mathbb{E}[XY])^2\leq\mathbb{E}[X^2]\mathbb{E}[Y^2])$ to (4), we have

$$\left(\sum_{i=1}^n\Pr\{A_i\}\right)^2\leq\mathbb{E}\big[S_n^2\big]\mathbb{E}\big[\mathbb{1}\{S_n>0\}^2\big]=\mathbb{E}\big[S_n^2\big]\mathbb{E}\big[\mathbb{1}\{S_n>0\}\big]=\mathbb{E}\big[S_n^2\big]\Pr\{S_n>0\}=\mathbb{E}\big[S_n^2\big]\Pr\{A\}.$$

Now notice that the LHS is $\sum_{i,j}\Pr\{A_i\}\Pr\{A_j\}$ and

$$\mathbb{E}\big[S_n^2\big]=\mathbb{E}\left[\Big(\sum_{i=1}^n\mathbb{1}\{A_i\}\Big)^2\right]=\sum_{i,j}\mathbb{E}\big[\mathbb{1}\{A_i\cap A_j\}\big]=\sum_{i,j}\Pr\{A_i\cap A_j\},$$

which completes the proof.