EE5103/ME5403: Computer Control Systems

Xiang Cheng

Associate Professor

Department of Electrical & Computer Engineering
The National University of Singapore

Phone: 65166210 Office: Block E4-08-07
Email: elexc@nus.edu.sg

Lecturers

- Part I: Dr. Xiang Cheng
- Part II: Dr. Ho Weng Khuen

Teaching Assistants

Mr. Yue Xianghu (email: xianghu.yue@u.nus.edu)

Mr. Yang Xin (email: e0674612@u.nus.edu)

Mr. Wang Jiadong (email: jiadong.wang@u.nus.edu)

Course Outline

1. **Background Overview (Lecture One)**

2. **Discrete System Analysis (Lectures Two and Three)**

3. **Controller Design (Lectures Four, Five and Six)**

4. **Kalman Filter (Lectures Seven, Eight and Nine)**

5. **Model Predictive Control (Lectures Ten, Eleven and Twelve)**

References: K.J.Astrom and B Wittenmark, ``Computer Controlled Systems, Theory and Design", Prentice-Hall, NJ, 1997.

Liuping Wang 'Model Predictive Control System Design and Implementation Using MATLAB', Springer-Verlag, 2009

J.M. Maciejowski, 'Predictive Control with Constraints', Prentice Hall, 2002

Assessment:

- Continuous Assessment (CA): 40%
- 20%: Three homework assignments for part I.
- 20%: Assignments for part II
- Final Exam: 60%, 27 Nov, 2021
- The format of the exam is to be determined in due course.

Simulation Tools: MATLAB with SIMULINK toolbox

All NUS students are allowed to install a free copy of student version of MATLAB to their computers.

What do I expect from you?

1. Be prepared. Roughly go through the lecture notes before the class.

2. I am going to spoon-feed you **with lots of questions !**

These questions are designed to arouse your interest and to help you to figure out most of the stuff by **your own thinking**!

<u>The best way to learn anything is to discover it by yourself!</u>

You will HAVE FUN by actively thinking and discussing these questions.

It will be a WASTE of TIME if you just want to know the answers without thinking!

3. Do the homework assignments by yourself.

You can discuss the questions with your classmates.

But DO NOT copy and paste!

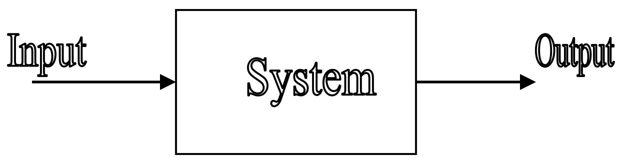
Systems and Control Background Overview (Warm-up Class)

- **<u>What is the Control Theory all about?</u>**

- **How to take care of systems.**

- **<u>What is a system?</u>**

- **A group of interacting components subject to various inputs and producing various outputs.**



NUS National University of Singapore

- Who can give me one example of a system?
- What is the largest system you can imagine?

- The whole universe !

- What is the input to the whole universe?

So a system may have no input!

- What is the smallest system you can imagine?

An electron or even a quark!

- Is a single electron (or a quark, string) a system?

Yes or No.

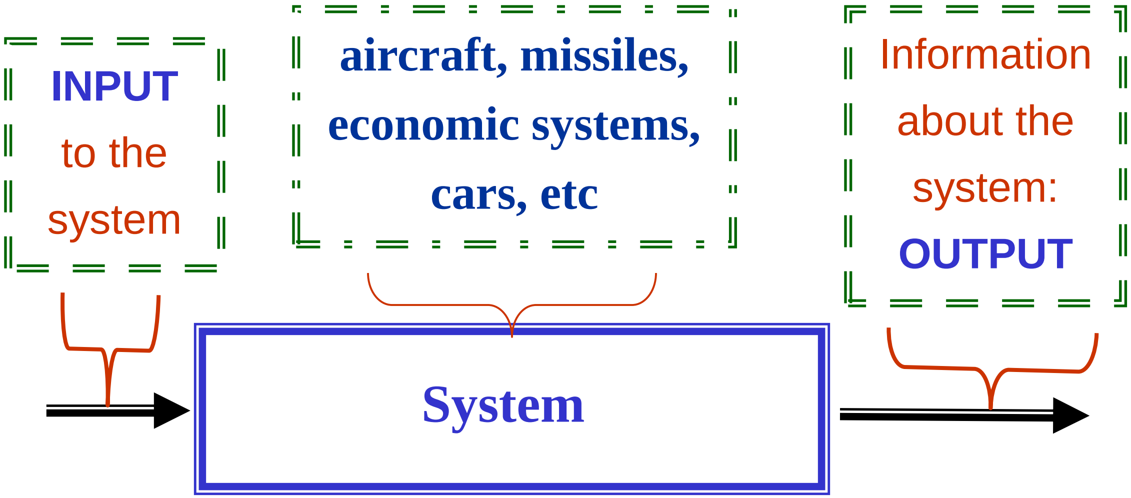
So a system does not necessarily have internal parts!

- Is human mind (not brain) a system?

- This is certainly debatable!

- Now you can see the difficulty in defining the "system" precisely!
- There is no universally accepted definition!
- But don't worry, when you deal with any specific systems in the real world, you don't need a precise definition. We will happily leave it to the philosophers.

What is a system from control engineering point of view?



<u>What is Output?</u> The attributes which you are interested, and can be observed.

<u>What is Input?</u> The variables which affect the outputs of the system.

<u>Does noise affect the output of the system?</u> Of course!

<u>Do we consider noise as input?</u> This is debatable!

Normally, we call those variables which are directly “under our control” as inputs.

Therefore, noise is usually not considered as input.

Sometimes, we call those variables which are out of our hands as **<u>disturbances.</u>**

- **How to describe or study a system?**
- Usually we need to build mathematical models.
- A **mathematical model** is a description of a system using mathematical language.
- **Is this the only way?**
- This is not the only way! But the most efficient way in many cases.

How many mathematical models can you build for a system, say, Airbus A380?

- It depends upon who you are and what you are looking at!
- Different models serve different purposes.

When we build mathematical models, some might work well,

some might not.

Therefore, does it make sense to say that some model (the one working well) is true while others may be wrong?

For instance, can we say that Newton's Second Law is "true"?

- **Mathematic model is always an approximation of the reality.**

Everybody believed that Newton's law is true until Einstein showed Relativity.

- If Newton's law is not "**TRUE**", every model can be "**WRONG**"!

You need this mentality for doing research!

Now let's turn our attention to the behavior of systems in our daily life.

- **Many systems are not operating in a completely desired fashion.**

- **<u>Who can give me one example of a poor system?</u>**

- Economical systems: inflation and depression; recession.

- Human systems: Cancer, heart diseases, Covid-19

- Industrial systems: un-productivity and non-profitability

- Political systems: Wars.

- Ecological systems: pests and drought; green house effect.

- Geological systems: earthquakes; floods

  - **<u>That's why we need to study this course: how to control a system!</u>**

  - **<u>Control</u>**: Change the behavior of the system, for better or worse!

•**How to improve the behavior of a system?What is the simple but expensive way?**

1. Build a new system; examples?

2. Replace or change the subsystems; examples?

•There is another possibility.

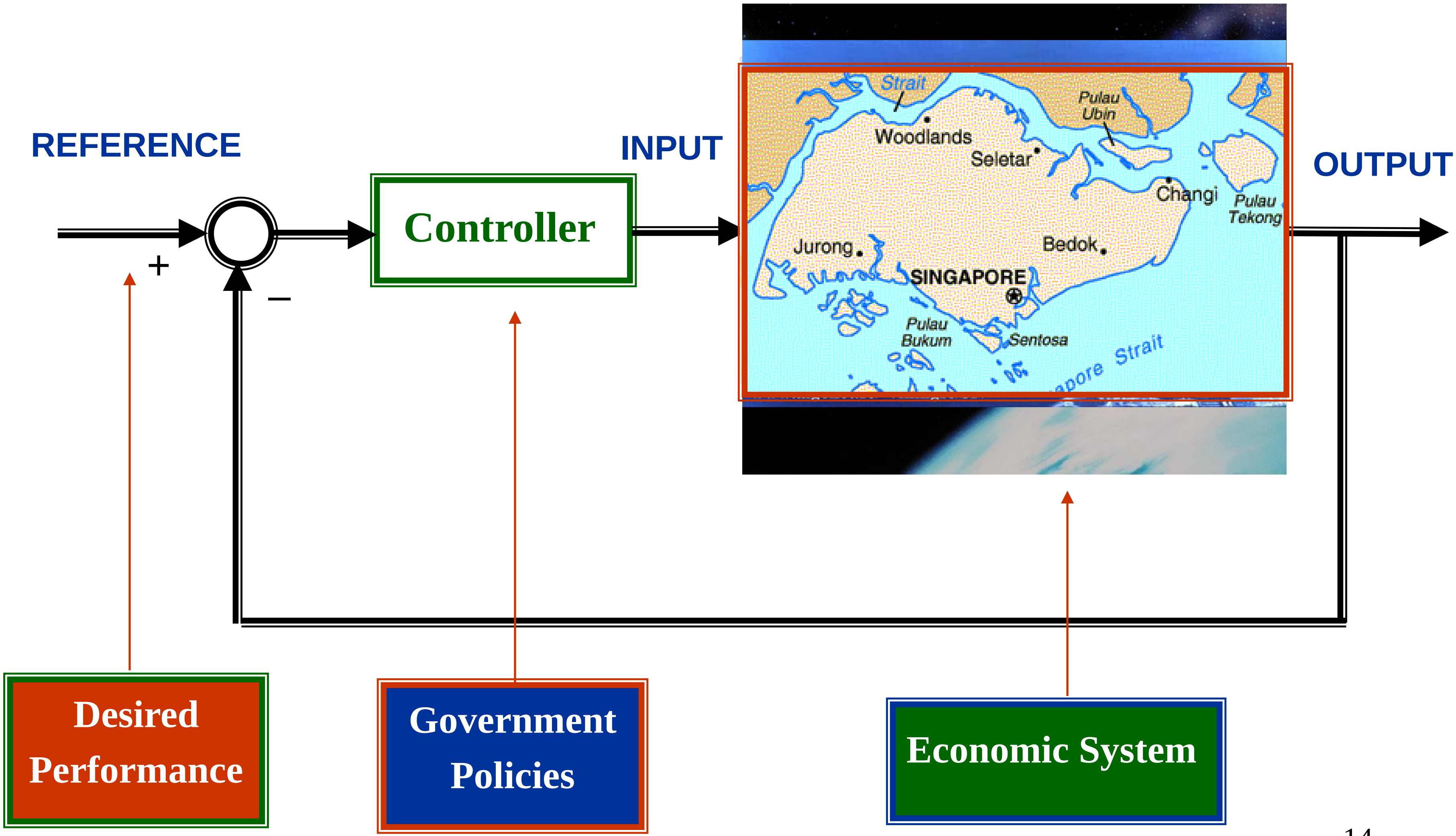
One day, you had a fever, and went to see a doctor, and the doctor gave you some medicine, and you recovered the next day after taking the medicine.

3. Change the inputs!

•**Control theory mostly concerns about the third way.**

•**Control problem: How to apply proper inputs such that the overall system meets certain performance requirements.**

Some Control Systems Examples:



Can you give me one example of control system around you now?

How to do it? How do we achieve such goals? Is there any fundamental principle to guide us to design control systems?

- Let's take lessons from our own daily life experience:

- **Does anybody know how to drive a car?**

- **How to steer your car within the fast lane in the express highway?**

- **How many steps are involved?**

- **Three interconnected steps!**

- Observe the heading---Compare with the lane lines ---Adjust the wheel

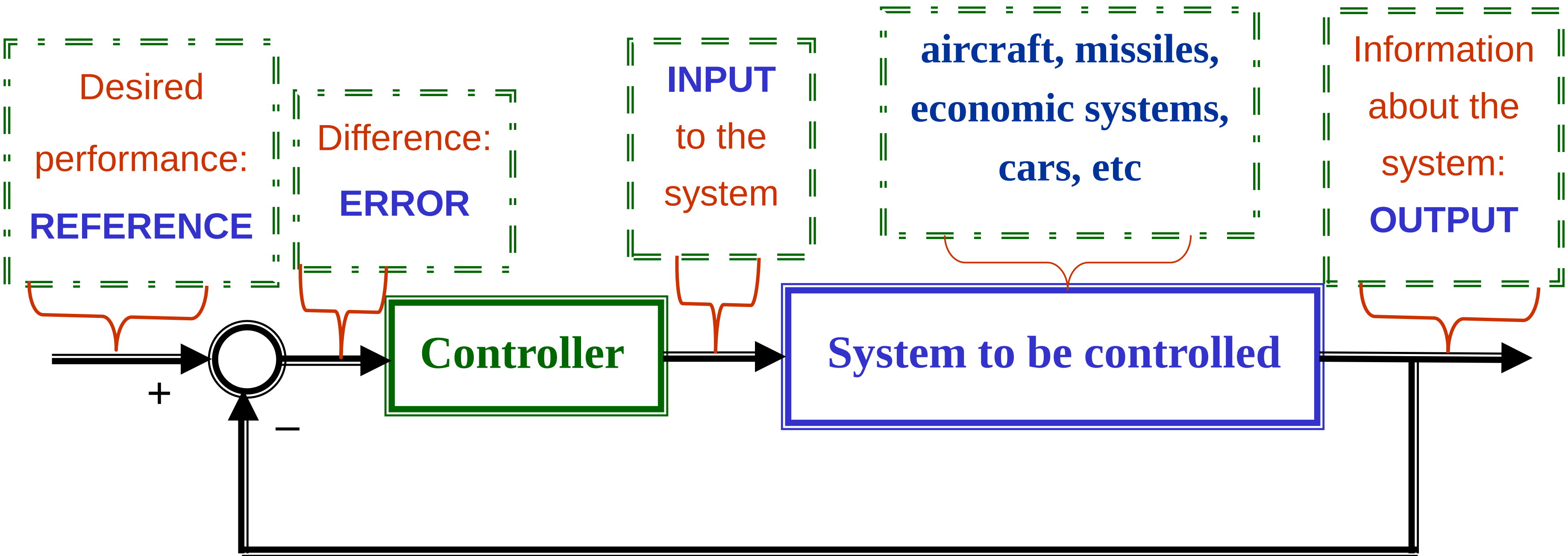
- **Trio: Measure —Compare –-Correct**

- **And we have a name for this principle. What do we call it?**

- Fundamental Concept of Control: **Feedback**

How to apply this idea of feedback to build control system is the central problem of the control theory.

What is a feedback control system?



Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

Key Issues: 1) How to describe the system to be controlled? (Modelling)

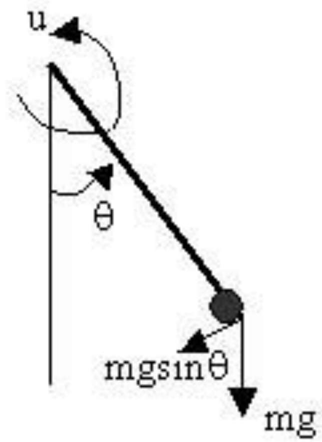
2) How to design the controller? (Control)

But why can feedback help?

Example: Robotic Manipulator

There are many links in the real world robots. Let's consider the simplest robotic manipulator here.

A **single-link** rotational joint using a motor placed at the pivot.



What is the output?

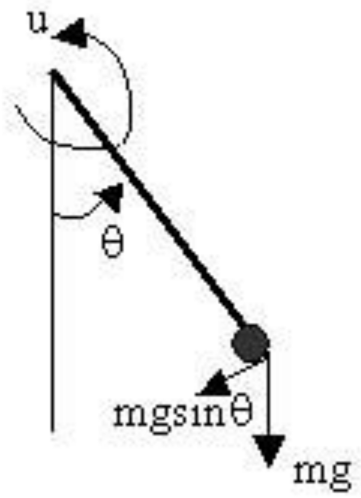
The angle of the arm.

What is the input?

- The torque produced by the motor.

After the system is set up, what is the next step for control system design?

Once the system is chosen,
the first step is building a mathematical model if possible.



We assume the friction is negligible, and all of the mass is concentrated at the end.

- **Which Physics law can we use to build the model?**

From Newton's second law, the change rate of the angular momentum is equal to the torque, we have

$$ml^2\ddot{\theta}(t) = u(t) - mgl\sin\theta(t)$$

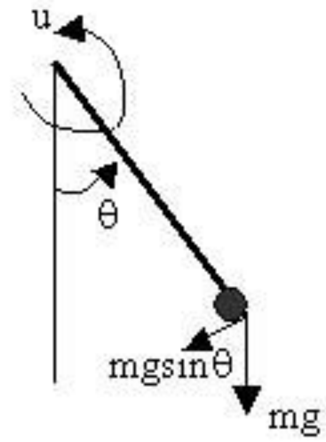
To avoid having to keep track of constants, let

$$ml^2 = mgl = 1$$

$$\ddot{\theta}(t) + \sin\theta(t) = u(t)$$

Let's first try to find out the equilibrium positions, where

$$\ddot{\theta}=0, \dot{\theta}=0, u=0$$



<u>Without looking at the mathematical equations, can you tell what the equilibrium positions are from physics?</u>

$\theta = 0, \dot{\theta} = 0$ and $\theta = \pi, \dot{\theta} = 0$

<u>How to find out the equilibrium points from the mathematical model?</u>

$$\ddot{\theta}(t)+\sin\theta(t)=u(t)$$

Apply the definition of equilibrium point: $\ddot{\theta}=0, \dot{\theta}=0, u=0$

$$\sin\theta(t)=0$$

<u>How many solutions can we find?</u>

$\theta = 0, \dot{\theta} = 0$ ----Stable position

$\theta = \pi, \dot{\theta} = 0$ --- Unstable position

- **<u>These two positions are fundamentally different. What is the difference?</u>**

The fundamental concept of stability

•**Stable**: small deviation from the equilibrium position will remain small.

•**Asymptotic Stable**: small deviation from the equilibrium position will be completely eliminated (decrease to zero) in the end.

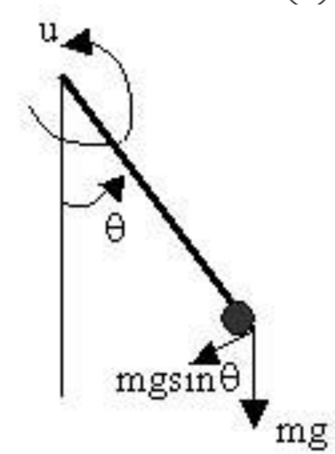
• **Unstable**: small deviation from the equilibrium position will result in huge deviation in the long run.

The first requirement for any control system is that the overall system has to be stable!

Why is stability the primary concern for any control system?

Disturbances (something out of our hands) are unavoidable in reality!

If a system is unstable, then any small deviations caused by those disturbances would spoil the whole system!



Control objective: apply torque u(t) to stabilize the robotic arm (inverted pendulum) such that the pendulum will maintain the upright position.

$$\theta = \pi, \dot{\theta} = 0$$

Stabilization: In many of the control problems in industry, the objective is to make the system stable by proper design of the controller, such that any small deviation from the desired point (set-point or reference value) will be corrected. This will also be the central topic of the course.

- A Stabilization Example

$$\ddot{\theta}(t) + \sin\theta(t) = u(t)$$

Since it is just a second-order differential equation, why don't we try to solve it directly?

Is there any analytical (close form) solution to this equation?

No.

Do we really need to get the solution first before design the controller?

No. The objective of control system is to change the system behavior.

We do not care whether we can solve the equations or not!

This is not a math course!

Instead we should ask the right question:

Can we change this complex system into a simple system which meets all the requirements?

$$\ddot{\theta}(t) + \sin\theta(t) = u(t)$$

Let's figure out how to change the behavior of the system step by step

Can we choose u(t) properly such that the nonlinear term $\sin\theta(t)$ disappears?

Remember that you can choose any control input u(t)

In other words, the control input u(t) can be any function of θ

- **What is the simplest choice of u(t) to get rid of $\sin\theta(t)$?**

$$u(t) = \sin\theta(t)$$

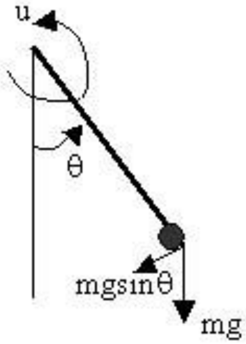
Then the closed loop system becomes

$$\ddot{\theta}(t) = 0$$

Is this system, i.e. double integrator, stable?

No! It will rotate forever if the initial speed is not zero!

$$\ddot{\theta}(t)+\sin\theta(t)=u(t)$$



So we have to add something more to make the system stable.

$$u(t)=\sin\theta(t)-1.4\dot{\theta}(t)-\theta(t)$$

the closed loop: $$\ddot{\theta}(t)+1.4\dot{\theta}(t)+\theta(t)=0$$

Is this what we want? What is the final value of θ ?

•ZERO!

How to make the angle to be $180^o(\pi)$ instead of zero?

We need to add one more term to the control input.

$$u(t)=u_{fb}+u_{ff}=\sin\theta(t)-1.4\dot{\theta}(t)-\theta(t)+u_{ff}$$

$$\ddot{\theta}(t)+1.4\dot{\theta}(t)+\theta(t)=u_{ff}(t)$$

•**What should we choose here?**

$$u(t)=\sin\theta(t)-1.4\dot{\theta}(t)-\theta(t)+\pi$$

•**Can we make the robotic arm stay at any other position, say, θ^*?**

•**Just let** $$u(t)=\sin\theta(t)-1.4\dot{\theta}(t)-\theta(t)+\theta^*$$

Let's take a close look at what happens before and after feedback control.

Without any feedback control, the system is described by

$$\ddot{\theta}(t) + \sin\theta(t) = u(t)$$

which is a complex nonlinear system, whose analytical solutions are not even known.

With a feedback controller:

$$u(t) = \sin\theta(t) - 1.4\dot{\theta}(t) - \theta(t) + u_{ff}(t)$$

The overall system (or closed loop) becomes

$$\ddot{\theta}(t) + 1.4\dot{\theta}(t) + \theta(t) = u_{ff}(t)$$

which is a standard second order system, whose properties are well known!

<u>**The magic of feedback: it can easily change one system into another one!**</u>

Therefore, the objective of the control system design is to use feedback to change the dynamic property, i.e. the behavior of the system, to meet specific performance requirement.

Let's take a closer look at the feedback controller,

$$u(t) = \sin\theta(t) - 1.4\dot{\theta}(t) - \theta(t) + u_{ff}(t)$$

In theory, we can design any controller we like. In reality, can we implement this?

How many sensors do you need to make it work?

We need sensors to measure the angle, $\theta(t)$ and the angular speed, $\dot{\theta}(t)$

There is no free lunch. This is the price we have to pay for the feedback controller: we need sensors to measure the outputs of the system.

Another issue: is this controller a linear controller or nonlinear controller?

This controller is nonlinear. Sometimes, it is not easy to implement nonlinear controller especially for analog controller.

Let's assume that we can only implement linear controller, can we solve this problem?

Break

(A good example of control system)

- ***10 Amazing Robots***

Can we approximate nonlinear functions by linear ones? ----- linearization

What is the mathematical theory behind linearization?

- You already learned that in Calculus! Call you recall it?

  - **Taylor Series:**

$$f(x) = f(0) + f'(0)x + O(x^2)$$

  - **Ignore the higher order terms:**

$$f(x) \approx f(0) + f'(0)x$$

- **What is the condition such that the approximation is a good one?**

  - x is sufficiently small!

Linearization

•If only **small** deviations are of interest, we can linearize the system around the equilibrium point.

For small $y = \theta - \pi$

$$\sin\theta = \sin(y+\pi) = -\sin y \approx -y$$
$$\ddot{\theta} = \ddot{y}$$

we replace the nonlinear equation $\ddot{\theta}(t) + \sin\theta(t) = u(t)$

$$\ddot{y}(t) - y(t) = u(t)$$

But does it always work?

The good news is that we have **Linearization principle:**

Design based on linearizations works locally for the original nonlinear system.

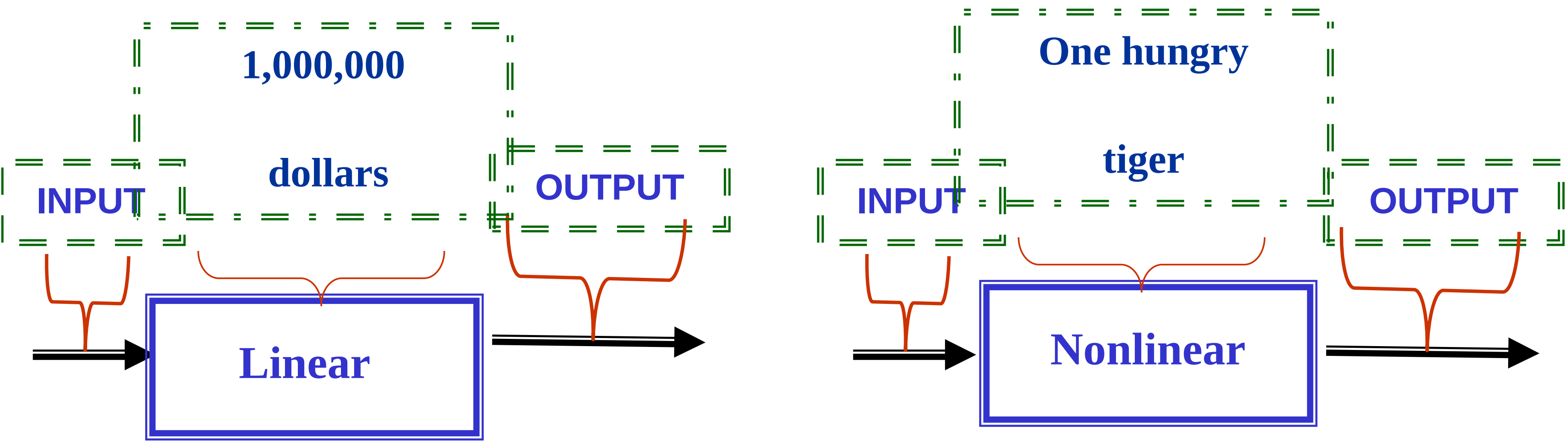
The bad news is that the design may not work well if the deviations from the operating points are large.

•**Why is linear system much easier to analyze than nonlinear system?**

•**What's the fundamental difference between linear and nonlinear systems?**

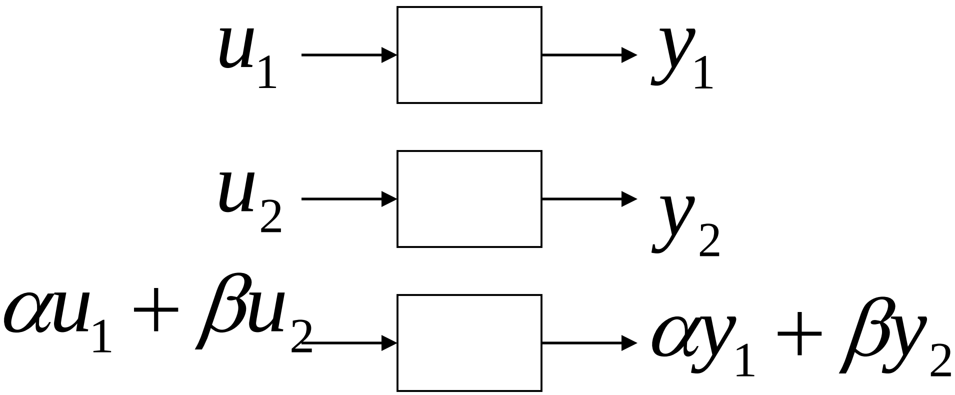
•**How to determine whether a system is linear or nonlinear in real world?**

Consider following million-dollar problem



Do you want to risk your life to win the million dollars by opening the right box? and how?

Fundamental property of linear system----Superposition Principle



Why is superposition principle so important?

Suppose you try to figure out the response of the system to a complicated input u. And you have no clue on how to do that.

But, if we can let input u be resolved into a set of **basis functions**

$$u = a_1\varphi_1 + a_2\varphi_2 + \ldots.$$

For each basis function, the output can be easily computed as $A(\varphi_1), A(\varphi_2)\ldots.$

If the system is linear, the output for input u is simply

$$A(u) = a_1 A(\varphi_1) + a_2 A(\varphi_2) + \ldots.$$

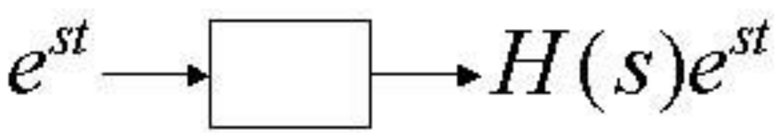
We are going to apply this **superposition principle** to determine the response y(t) subject to the input u(t) for our single-link robot:

$$\ddot{y}(t) - y(t) = u(t)$$

The trick of superposition principle is to choose proper component functions whose response can be easily obtained.

- **For what kind of input, the output response can be easily computed?**

Let's try the exponential function e^{st}



Assume $y(t) = H(s)e^{st}$. Plug it into the pendulum equation, we have

$$s^2 H(s)e^{st} - H(s)e^{st} = e^{st}$$

$$H(s) = \frac{1}{s^2 - 1}$$

Hence $H(s)e^{st}$ is indeed the response to e^{st}

Can we decompose u(t) by the exponentials e^{st} ?

$$u(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}\,ds$$

•Integration==summation

What is F(s)?

$$F(s) = \int_0^\infty u(t)e^{-st}\,dt$$

We have a famous name for F(s), what is it?

F(s) is the Laplace transform of u(t)

- **Why is Laplace transform so useful for analyzing linear system?**
- It is due to superposition principle.

Is Laplace transform also useful for nonlinear system?

Not really. Superposition principle does not hold for nonlinear systems!

The most important property of Laplace transform

$$L(y(t)) = Y(s)$$

$$L(y'(t)) = sY(s) - y(0)$$

$$L[y^{(n)}(t)] = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \ldots - y^{(n-1)}(0)$$

If the initial conditions are zero, we simply replace the differential operator d/dt with s:

$$L[y^{(n)}(t)] = s^n Y(s)$$

$$\frac{d}{dt} \Leftrightarrow s$$

Now we are ready to apply the Laplace transform to our example.

$$\ddot{y}(t) - y(t) = u(t)$$

- Assume zero initial conditions, we have

$$s^2 Y(s) - Y(s) = U(s)$$

Can you solve this equation to get Y(s)?

This is a simple **algebraic equation**! Everyone should be able to solve for Y(s)

$$Y(s) = \frac{1}{s^2 - 1} U(s) = H(s) U(s)$$

How to get y(t)? How to do inverse Laplace transform?

$$y(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} Y(s) e^{st} ds$$

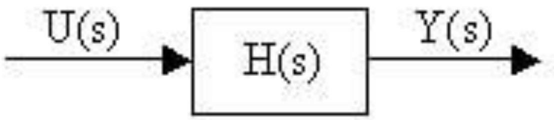
- **Did you ever use this formula to find out time-domain signal y(t)?** No.

What is the EASY way?

Partial Fraction Expansion: $Y(s) = \frac{1}{s^2 - 1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$

Use the Transform Table: $y(t) = \frac{1}{2}(e^t - e^{-t})$

Transfer function: H(s) -----Transfer the input into output



<u>Since Y(s)=H(s)U(s), does the same input always produce the same output?</u>

The famous counter-example

Y(s)=H(s)U(s) only if the initial conditions are all zero.

If initial conditions are not zero, there are extra terms in the solution!.

$$\ddot{y}(t) - y(t) = u(t)$$

$$s^2 Y(s) - sy(0) - \dot{y}(0) - Y(s) = U(s)$$

$$Y(s) = \frac{1}{s^2 - 1}(sy(0) + \dot{y}(0)) + H(s)U(s)$$

The response of the system is affected by both the input and the initial conditions.

The transfer function can describe the response due to input only!

Transfer function:

A representation of the system which is equivalent to differential equation.

$$\frac{d}{dt} \Leftrightarrow s$$

Given a transfer function, how to write down the differential equation?

$$\frac{Y(s)}{U(s)} = \frac{s+2}{s^2-1} \qquad s^2Y(s) - Y(s) = sU(s) + 2U(s)$$

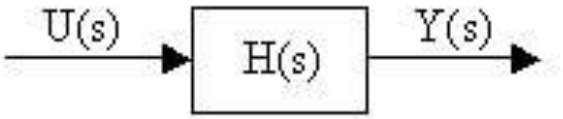
$$\frac{d}{dt} \Leftrightarrow s \qquad \frac{d^2}{dt^2}y(t) - y(t) = \frac{d}{dt}u(t) + 2u(t)$$

Given a differential equation, how to write down the transfer function?

$$\dddot{y}(t) + y(t) = \ddot{u}(t) + 2\dot{u}(t) \qquad \frac{d^3}{dt^3}y(t) + y(t) = \frac{d^2}{dt^2}u(t) + 2\frac{d}{dt}u(t)$$

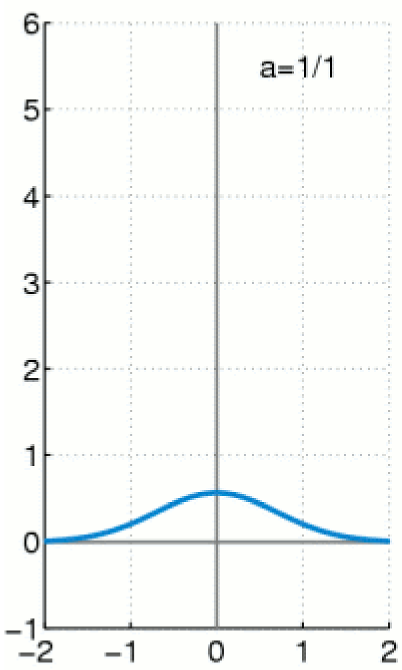
$$\frac{d}{dt} \Leftrightarrow s \qquad s^3Y(s) + Y(s) = s^2U(s) + 2sU(s) \qquad Y(s) = \frac{s^2+2s}{s^3+1}U(s)$$

Transfer function: H(s) -----Transfer the input into output



Let's consider the response of the system when the input is an impulse.

<u>What is an impulse signal?</u>



Abstraction of signal which is concentrated

during a very short period, and zero elsewhere.

$$u(t) = \delta(t)$$

$$U(s) = 1, Y(s) = H(s)U(s) = H(s)$$

Transfer function is the impulse response in s-domain!

Can we tell whether a system is stable or not by simply looking at the transfer function?

For instance, is the inverted pendulum stable or unstable by checking

$$H(s) = \frac{1}{s^2 - 1}$$

$$H(s) = \frac{Q(s)}{P(s)}$$

What are the poles? P(s)=0

What are the zeros? Q(s)=0

What decide the stability of the system, the poles or zeros?

- The poles. **But WHY?**

The relation between poles and stability:

Partial Fraction expansion of Transfer function (impulse response) H(s)

$$H(s) = \frac{1}{s^2 - 1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

If λ is the pole, then $\frac{1}{s-\lambda}$ must be one component in H(s)

Therefore, in the time domain,

$e^{\lambda t}$ is one component of the impulse response

$$\lambda = \sigma + j\omega$$

$$e^{\lambda t} = e^{\sigma t} e^{j\omega t}$$

σ ---Real part of pole---stability

ω ---Imaginary part of pole---oscillation

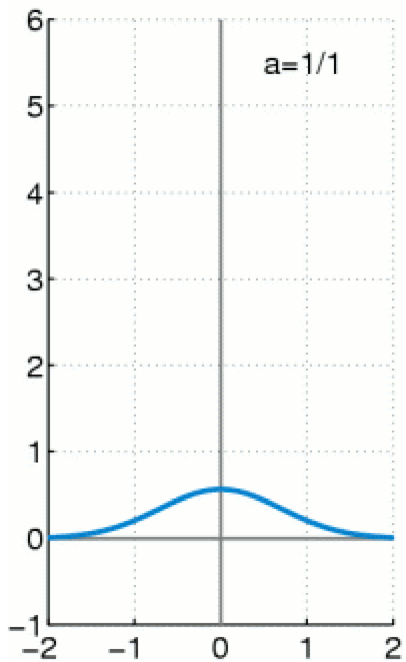
<u>The poles decide whether the impulse response blows up or not.</u>

How is impulse response related to the stability of the system?

•The stability concerns the behavior of the system when there is a perturbation (deviation) from the equilibrium point (origin).

What is the impulse signal?

Abstraction of signal which is concentrated during a very short period, and zero elsewhere.



What would happen if the impulse signal is applied to the system in equilibrium position?

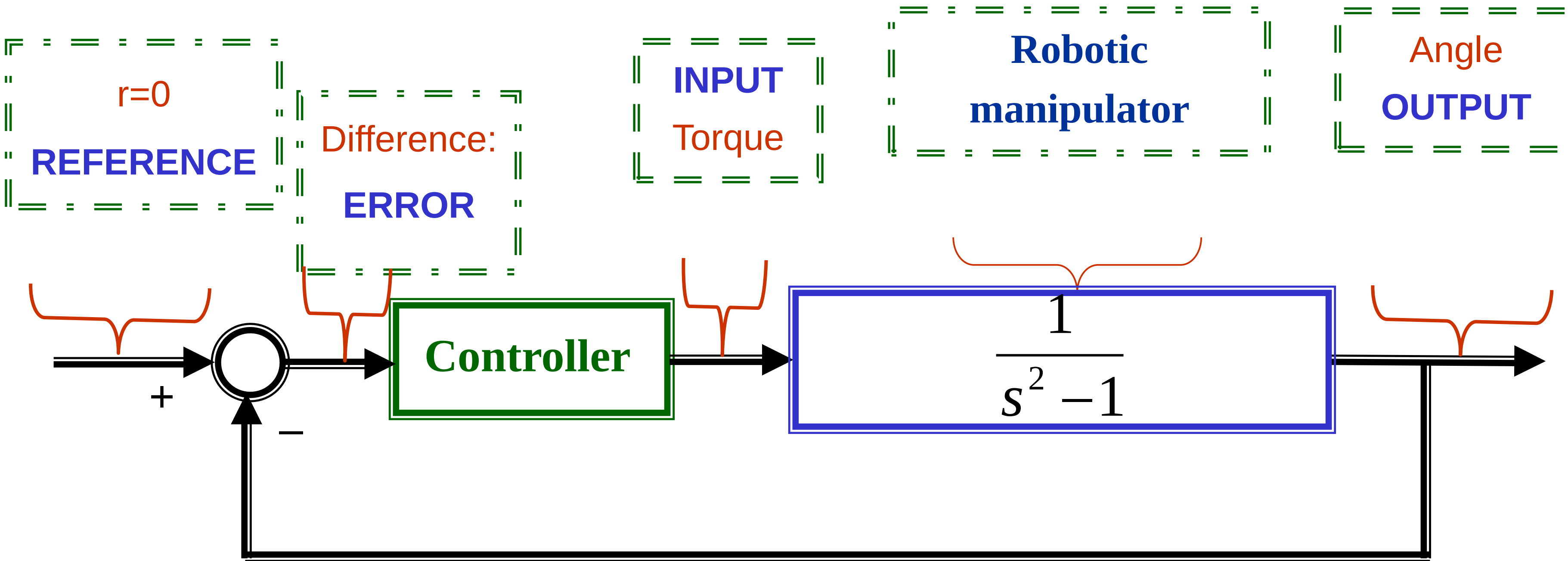
•Example

•The system will be pushed away from its equilibrium position.

•Impulse response describes the behavior of the system where there is deviation from the equilibrium point.

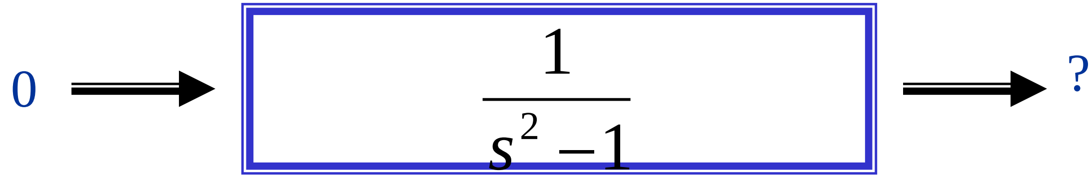
Therefore, impulse response is closely related to stability!

Control of the robotic manipulator



Objective: To bring the system OUTPUT to zero for any small nonzero initial conditions.

<u>What would happen without any control action?</u>



The system is unstable, and the robotic arm cannot stand by itself .

Open loop control –control without feedback

$$y(0)=1, y(0)'=-2.$$

$$Y(s)=\frac{1}{s^2-1}(sy(0)+y'(0)+U(s))=\frac{1}{s^2-1}(sy(0)+y'(0))+H(s)U(s)$$

$$=\frac{1}{s^2-1}(s-2)+\frac{1}{s^2-1}U(s)$$

<u>How to choose U(s) such that y(t) goes to zero?</u>

There are many solutions. Let’s choose one

$$U(s)=\frac{3}{s+2} \qquad u(t)=3e^{-2t}$$

$$Y(s)=\frac{1}{s^2-1}(s-2)+\frac{1}{s^2-1}\frac{3}{s+2}=\frac{1}{s+2} \qquad y(t)=e^{-2t}$$

<u>But does this controller really work in reality?</u>

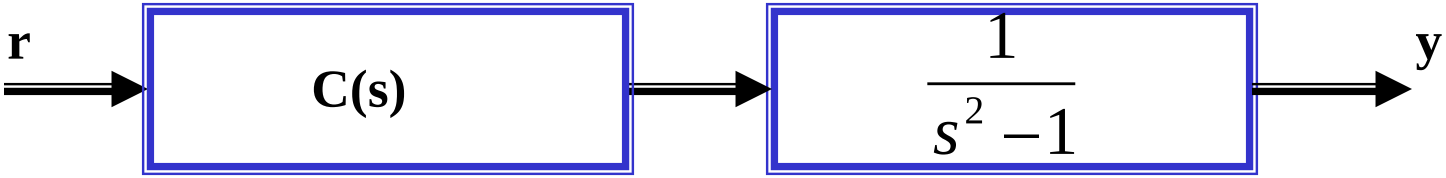
<u>If there is little noise in your measurement (inevitable!)</u>

$$y(0)'=-2+\varepsilon$$

$$Y(s)=\frac{\varepsilon}{s^2-1}+\frac{1}{s+2}$$

<u>Would the output still converge to zero?</u> **<u>No.</u>**

How about other types of open-loop control?



The transfer function from r to y is $$C(s)\frac{1}{s^2-1}$$

Is it possible to choose the controller c(s) such that the whole system is stable?

How about C(s)=(s-1)?

Then $$C(s)\frac{1}{s^2-1}=\frac{(s-1)}{(s+1)(s-1)}=\frac{1}{s+1}$$

Can we cancel out the poles and zeros in the transfer function?

Can we cancel out the poles and zeros in the transfer function?

The answer can be found out in the time-domain!

Let's write down the differential equations and compare them.

$$\frac{1}{s+1} \Longrightarrow \dot{y} + y = u$$

$$\frac{s-1}{s^2-1} \Longrightarrow \ddot{y} - y = \dot{u} - u$$

Consider the case when there are no inputs, u=0.

$$\dot{y} + y = 0 \Longrightarrow sY(s) - y(0) + Y(s) = 0 \Longrightarrow Y(s) = \frac{y(0)}{s+1}$$

$$\ddot{y} - y = 0 \Longrightarrow s^2Y(s) - sy(0) - \dot{y}(0) - Y(s) = 0 \Longrightarrow Y(s) = \frac{sy(0) + \dot{y}(0)}{s^2-1}$$

Are these two solutions the same or different?

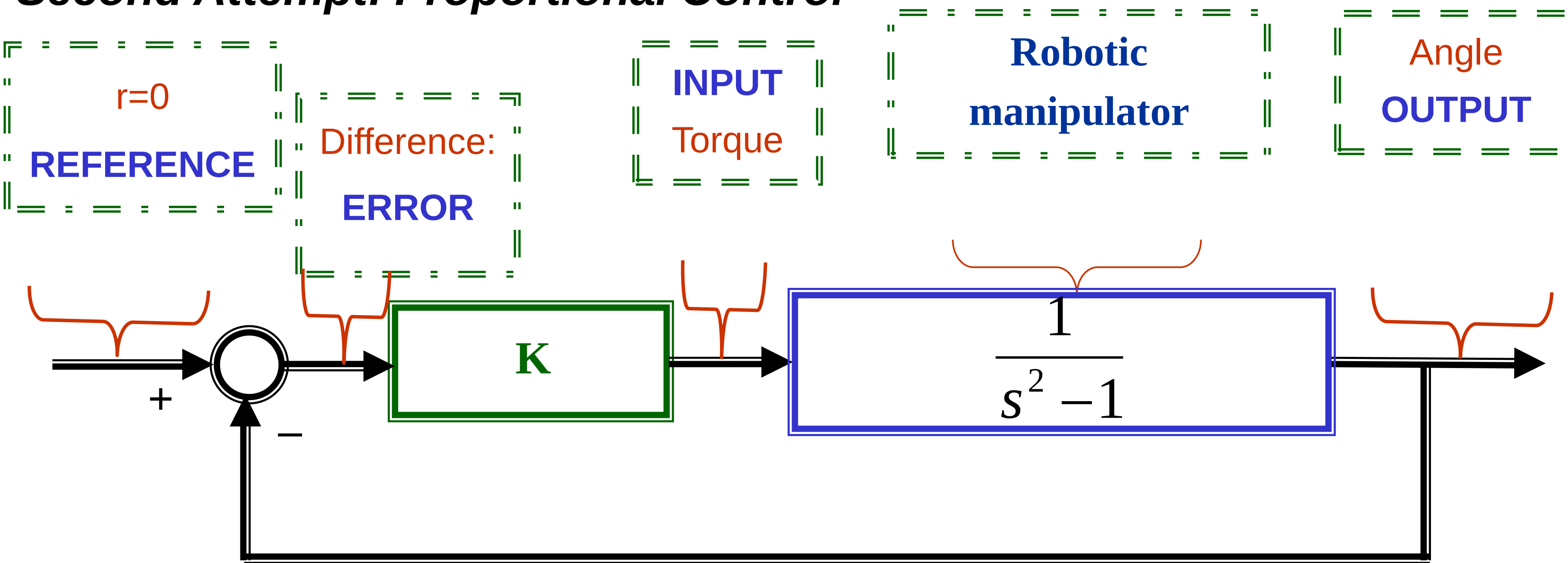
The first one goes to zero, the second one blows up!

•Be careful about pole-zero cancellation!

The unstable poles cannot be changed by the open loop control!

What is the simplest feedback controller you can imagine?

Second Attempt: Proportional Control



What is the closed loop transfer function?

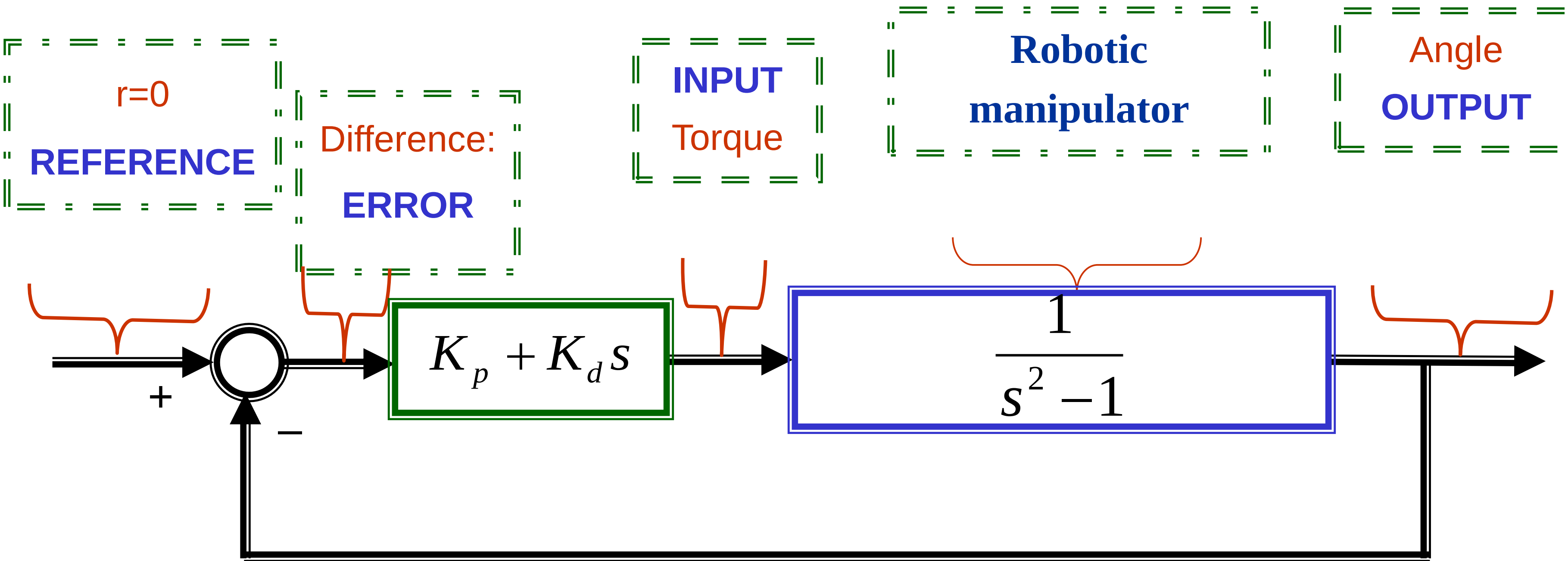
How to write down transfer functions for basic block diagrams?

$$\frac{Y(s)}{R(s)} = \frac{K\frac{1}{s^2-1}}{1+K\frac{1}{s^2-1}} = \frac{K}{s^2-1+K} \quad \Longrightarrow \quad s = \pm\sqrt{1-K}$$

Does this controller meet our goal: make y(t) go to zero?

No. The system is at most marginally stable, not asymptotically stable.

Third Attempt: Proportional and Derivative Control



What is the closed loop transfer function?

$$\frac{Y(s)}{R(s)} = \frac{(K_p + K_d s)\frac{1}{s^2-1}}{1+(K_p + K_d s)\frac{1}{s^2-1}} = \frac{K_p + K_d s}{s^2 + K_d s + K_p - 1} \Longrightarrow \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

We can choose the control parameters to match any specifications of the damping ratio and natural frequencies, or the step responses.

$$K_d = 2\zeta\omega_n, \quad K_p - 1 = \omega_n^2$$

State-space approach

•The concept of state was introduced by Poincare at the end of 19th century.

•It was only till the late 1950s that it was introduced into the control theory.

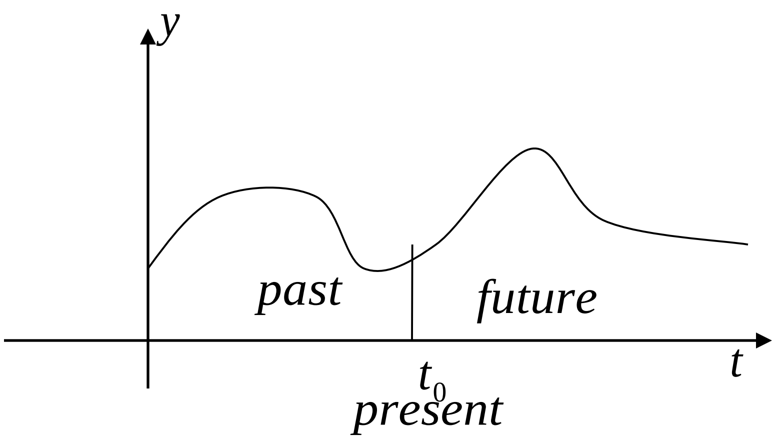
•Modern control theory---state space approach

•Classical control theory--- the transfer function approach.

Poincare(1854-1912)

What is the state of a system?

State at present: the information needed to predict the future assuming the current and all the future inputs are known.



How to find out the state variables for a system?

What information is really needed for predicting the future assuming we have the complete information about the past and the present of the system combining with the current and future inputs?

NUS National University of Singapore

•Example 1: Inverted Pendulum (robotic manipulator) problem:

$$\ddot{y}(t) - y(t) = u(t)$$

At time t=0, what do we need to know in order to determine y(t) in the future? Do we need to know all the values of the output in the past?

Let's find out the solution first by Laplace Transform :

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = U(s)$$

$$Y(s) = \frac{1}{s^2 - 1}(sy(0) + y'(0)) + \frac{U(s)}{s^2 - 1}$$

What do we need to know at time t=0 in order to predict y(t) in the future?

The state at t=0 is

x(0)=[y(0),y'(0)]

What is the Order of the system? The dimension of the state vector.

In the inverted pendulum case, x(t)=[y(t),y'(t)]
it is second order system.

How about x(t)=[y(t)+y'(t), y(t)-y'(t)]?

Surely it is equivalent to [y(t),y'(t)].

How many state vectors can you identify?

There are infinite number of possible state vectors! They are related by state transformations.

Example Two: Perfect delay problem:

$$y(t) = y(t-1)$$

<u>What's the state of the system?</u>

<u>How to predict the future value of y(t) for all t>0?</u>

What is y(0.1)? y(0.1)=y(0.1-1)=y(-0.9)

What is y(0.2)? y(0.2)=y(0.2-1)=y(-0.8)

What is y(0.5)? y(0.5)=y(0.5-1)=y(-0.5)

What is y(10)? y(10)=y(10-1)=y(10-2)=…=y(1)=y(0)

<u>How many values of y in the past and the present do you need ?</u>

Infinite!

$$x(0) = y_{[-1,0]}$$

<u>What is the order of the system (dimension of the state) ?</u>

Infinity!

Once the state vector is chosen, how to derive the state space representation?

Example: robotic manipulator (inverted pendulum).

$$U(s) \longrightarrow \boxed{\frac{1}{s^2-1}} \longrightarrow Y(s)$$

$$\ddot{y}(t) - y(t) = u(t)$$

Given the current state x(t)=[y(t),y'(t)], how to predict its immediate future?

We need to compute the change rate (derivative) of the state!

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{y}(t) = y(t) + u(t) = x_1(t) + u(t)$$

$$y(t) = x_1(t)$$

Matrix Form

$$\dot{x} = Ax + bu$$

$$y = cx$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{x} = Ax + bu$$

$$y = cx$$

At present time t, given the state x(t) and the input signals, **<u>how to predict the future?</u>**

We can do it step by step.

Let's try to predict the state at next instant, t+Δ, where Δ is arbitrarily small.

$$x(t+\Delta) \approx x(t) + \dot{x}(\mathrm{t})\,\Delta$$

Do we know $\dot{x}(\mathrm{t})$? $\dot{x}(\mathrm{t})=\mathrm{Ax(t)+bu(t)}$

$$x(t+\Delta) \approx x(t) + (Ax(t) + u(t))\Delta$$

Once we have predicted $x(t+\Delta)$, we can try to predict $x(t+2\Delta)$

$$x(t+2\Delta) \approx x(t+\Delta) + \dot{x}(\mathrm{t}+\Delta)\,\Delta$$

$$x(t+2\Delta) \approx x(t+\Delta) + (Ax(t+\Delta) + u(t+\Delta))\Delta$$

So you can see it is pretty easy to make predictions once the state x(t) is given, plus all the input signals are known in advance!

$$\dot{x} = Ax + bu$$

$$y = cx$$

Since this is a linear differential equation, we can even obtain an analytical solution.

If x is a scalar (not a vector), using Laplace Transform

$$sX(s) - x(0) = AX(s) + bU(s)$$

$$X(s) = \frac{1}{s-A}x(0) + \frac{bU(s)}{s-A} \quad \Longrightarrow \quad x(t) = e^{At}x(0) + e^{At}bu(t)$$

WRONG

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

<u>**Now if x is a vector, can we still write the solution in such a form?**</u> YES!

$$e^{At} = I + At + \ldots + A^k \frac{t^k}{k!} + \ldots$$

•Taylor Series!

$$\frac{de^{At}}{dt} = Ae^{At}$$

How to calculate e^{At} ? Can you directly use the definition?

$$e^{At} = I + At + \ldots + A^k \frac{t^k}{k!} + \ldots$$

Let's solve the equation again, now assuming x is a vector.

$$\dot{x} = Ax + bu \quad \Longrightarrow \quad sX(s) - x(0) = AX(s) + bU(s)$$

$$sX(s) - AX(s) = x(0) + bU(s) \quad \Longrightarrow \quad X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}bU(s)$$

Let's compare it to the solution in time domain,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

What is the Laplace transform of e^{At} ?

$$e^{At} \quad \Longrightarrow \quad (sI - A)^{-1}$$

Similar to $\frac{1}{s - A}$ in the scalar case.

What is output y in the s-domain?

$$Y(s) = cX(s) = c(sI - A)^{-1}x(0) + c(sI - A)^{-1}bU(s)$$

If initial conditions x(0)=0,

$$Y(s) = c(sI - A)^{-1}bU(s)$$

What is the transfer function H(s)?

$$Y(s) = H(s)U(s) = c(sI - A)^{-1}bU(s)$$

$$H(s) = c(sI - A)^{-1}b$$

If the state space model {A,b,c} is given, the transfer function can be easily obtained.

What is the denominator in $H(s) = c(sI - A)^{-1}b = \dfrac{Q(s)}{P(s)}$**?**

$$(sI - A)^{-1} = \frac{\begin{bmatrix} * & * \\ * & * \end{bmatrix}}{\det\{(sI - A)\}} \quad \Longrightarrow \quad P(s) = \det\{(sI - A)\}$$

How to find out the poles of the system?

$$\mathrm{P(s)} = \det\{(sI - A)\} = 0$$

- **Do you know the other meaning for the roots of this equation in Linear Algebra?**

Eigenvectors and Eigenvalues of Matrix A

The eigenvector is the vector that does not change direction after the transformation.

$$Av = sv$$

- But the magnitude changes by a factor, s, which we call eigenvalue.

How to calculate the eigenvalues of A?

$$(A - sI)v = 0$$

What is the condition for existence of non-zero eigenvectors?

$$\det\{(A - sI)\} = 0$$

$$\Longrightarrow \quad \det\{(sI - A)\} = 0$$

$\det\{(sI - A)\}$ is called the characteristic polynomial (C.P.) of matrix A.

The eigenvalues of A $\Longrightarrow$ **The poles of the system**

Historical Background

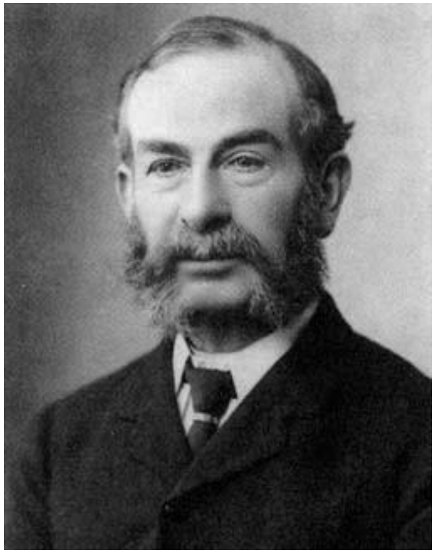
Who started this whole area of control theory?

•The whole area began with a dynamics analysis of the governor in the steam engine by the physicist James Clerk Maxwell in 1868 .

Maxwell(1831–1879)

•Maxwell's classmate Edward John Routh generalized the results of Maxwell for the general class of linear systems.

•Independently, Adolf Hurwitz analyzed system stability using differential equations in 1877. This result is called the **Routh-Hurwitz theorem.**



Routh (1831-1907)

Hurwitz (1859-1919)

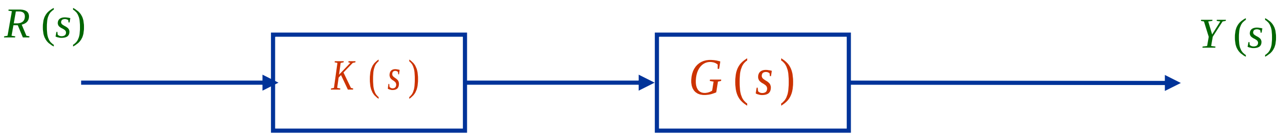
•**What is the oldest feedback control system in human history?**

•Water Clocks in Egypt, India and China around 6000 years ago.

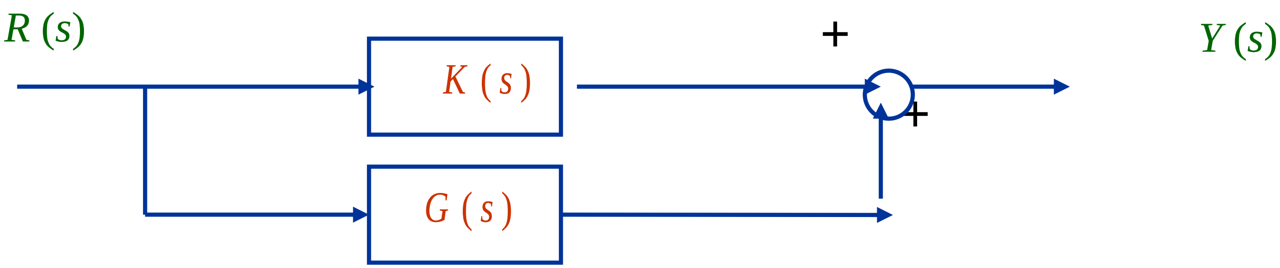
Q & A…

THANK YOU !

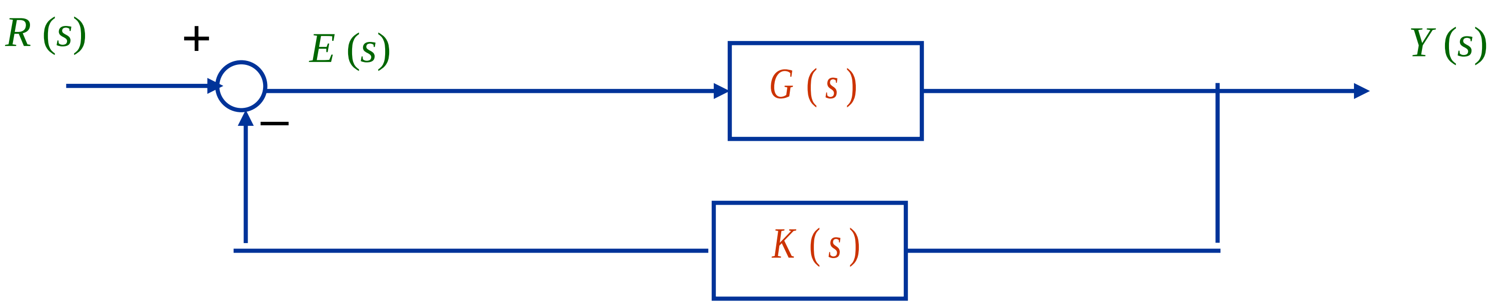
Transfer functions Y/R for basic block diagrams



$K(s)G(s)$



$K(s)+G(s)$



$$Y(s) = G(s)E(s) = G(s)\left[R(s) - K(s)Y(s)\right] \quad \Rightarrow \quad \left[1 + G(s)K(s)\right]Y(s) = G(s)R(s)$$

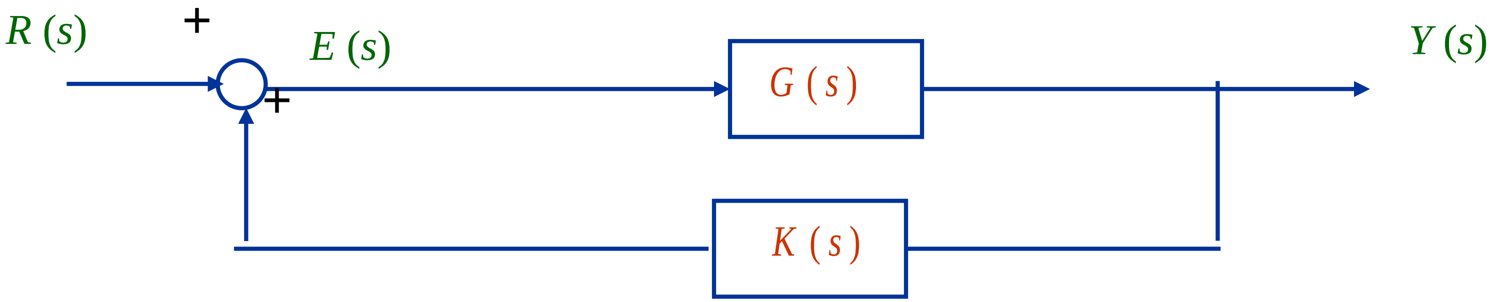
$$\Rightarrow \boxed{H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)K(s)}}$$

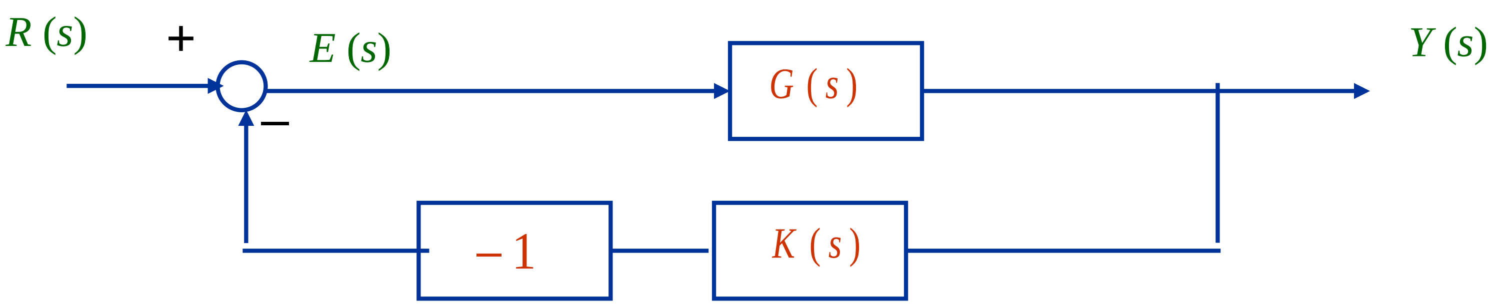
Closed-loop transfer function from *R* to *Y*.

G(s)--- Feedforward TF

G(s)K(s)----open loop TF

Transfer functions Y/R for other block diagrams





What is the Feedforward TF ?

$G(s)$

What is the open loop TF?

$G(s)K(s)(-1)$

$$\Rightarrow \quad \boxed{H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)K(s)}}$$

Closed-loop transfer function from R to Y.

return

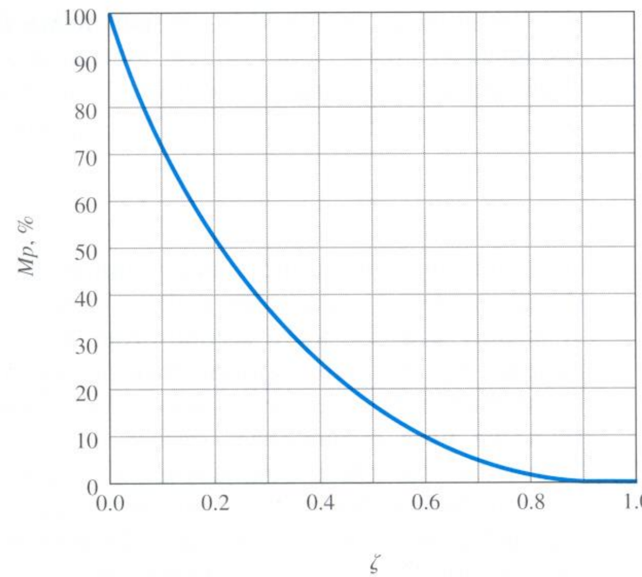
Control System Design with Time-domain Specifications

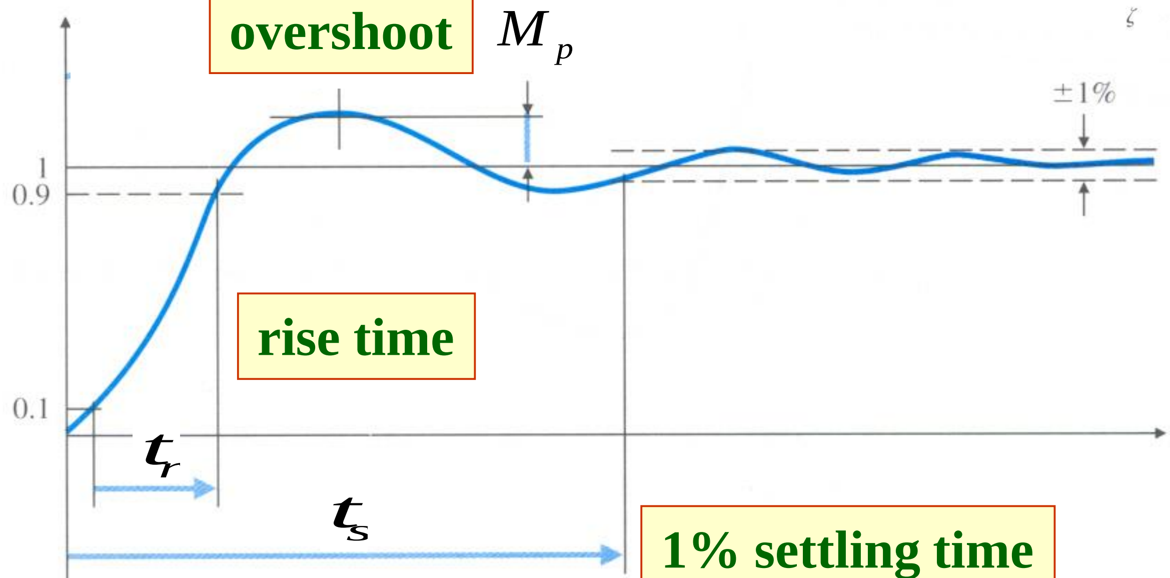
$R(s) = 1/s$

$r = 1$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$Y(s)$





$$t_r \cong \frac{1.8}{\omega_n}$$

$$t_s \cong \frac{4.6}{\zeta\omega_n}$$

return