NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF ENGINEERING

EXAMINATION FOR

(Semester I: 2021/2022)

EE5103 / ME5403– COMPUTER CONTROL SYSTEMS

November 2021 - Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES:

1. Please write only your Student Number. Do not write your name.
2. This paper contains **FOUR** (4) questions and comprises **SEVEN** (7) printed pages.
3. Answer all **FOUR** (4) questions.
4. Students should write the answers for each question on a new page.
5. The **TOTAL** marks are 100.
6. This is an **Open BOOK** examination.

Q.1

A system is described by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

where α is a constant parameter, $x(k) = [x_1(k), x_2(k)]^T$ is the state vector, $y(k)$ is the output, $u(k)$ is the input, and $\omega(k)$ is the disturbance.

(a) Find the range of α such that the open loop system is stable.

(2 marks)

(b) Assuming that there is no disturbance and the state variables are accessible, design a deadbeat state feedback controller.

(6 marks)

(c) Assuming that there is no disturbance and only the output $y(k)$ is available, design a deadbeat observer to estimate the state variables, and use these estimates to design an output-feedback controller.

(6 marks)

(d) Assuming that the disturbance is an unknown constant, design a deadbeat observer to estimate both the state variables and the disturbance, and use these estimates to design an output-feedback controller such that the effect of the disturbance may be completely eliminated.

(6 marks)

(e) Assuming that there are time-delays in both the state variables and the input, the corresponding model of the system is given as

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & \alpha \end{bmatrix} x(k-1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k-3).$$

Derive the standard form of the state space model for this time-delayed system. What is the order of the system? Justify your answers.

(5 marks)

Q.2

A system is described by

$$y(k+1) = y(k-1) + 2u(k) + u(k-1) + v(k+1) + v(k)$$

where u(k) and y(k) are the input and output of the system, v(k) is an unknown disturbance. Assume that the sampling period h=1.

(a) Rewrite the system equations in the form of

$$A(q)y(k) = B(q)u(k) + C(q)v(k)$$

where A(q), B(q) and C(q) are polynomials in the forward-shift operator q. What is the open loop transfer function from the input u to output y? What is the open loop transfer function from the disturbance, v, to the output, y?

(4 marks)

(b) Assume that the disturbance v(k) is an unknown constant. Design a controller in the form of

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k)$$

such that the effect of the disturbance v(k) on the system output may be completely rejected and the closed loop transfer function from the command signal, $u_c(k)$, to the system output, y(k), follows the reference model, $\frac{1}{z^2}$. Let the order of the controller be as low as possible.

(15 marks)

(c) Assume that the disturbance v(k) is a ramp, described by

$$v(k) = ck, k \geq 0$$

where the slope constant parameter c is unknown. Is it still possible to design a controller to meet the same performance requirements as that in part (b)? Justify your answer.

(6 marks)

Q.3 Consider the process

$$
\begin{aligned}
x(k+1) &= Ax(k) + w(k) \\
y(k) &= Cx(k) + v(k) \\
A &= \begin{bmatrix} 0.8 & 0 \\ 0.8 & 1 \end{bmatrix} \\
C &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\
R_1 &= E[w(k)w(k)^T] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
R_2 &= E[v(k)^2] = 0.5
\end{aligned}
$$

which is also the model used by the Kalman filter

$$
\begin{aligned}
K_f(k) &= P(k|k-1)C^T(CP(k|k-1)C^T + R_2)^{-1} \\
K(k) &= (AP(k|k-1)C^T)(CP(k|k-1)C^T + R_2)^{-1} \\
\hat{x}(k|k) &= \hat{x}(k|k-1) + K_f(k)\big(y(k) - C\hat{x}(k|k-1)\big) \\
\hat{x}(k+1|k) &= A\hat{x}(k|k-1) + Bu(k) + K(k)\big(y(k) - C\hat{x}(k|k-1)\big) \\
P(k|k) &= P(k|k-1) - P(k|k-1)C^T(CP(k|k-1)C^T + R_2)^{-1}CP(k|k-1) \\
P(k+1|k) &= AP(k|k-1)A^T - K(k)(CP(k|k-1)C^T + R_2)K^T(k) + R_1
\end{aligned}
$$

where $w(k)$ and $v(k)$ are zero-mean independent Gaussian random variables. The output and state are given by $y(k)$ and $x(k)$ respectively and

the covariance matrix

$$P(2|1) = \begin{bmatrix} 2.28 & 1.68 \\ 1.68 & 2.58 \end{bmatrix}$$

the estimate

$$\hat{x}(2|1) = \begin{bmatrix} 0.8 \\ 1.8 \end{bmatrix}$$

and the measurement $y(2) = 2$.

(a) Find $K_f(2)$ and $K(2)$

(4 marks)

(b) Find $\hat{x}(2|2)$, E$[x(2) - \hat{x}(2|2)]$ and E$\{[x(2) - \hat{x}(2|2)]\,[x(2) - \hat{x}(2|2)]^T\}$

(6 marks)

(c) Find $\hat{x}(3|2)$ and E$\{[x(3) - \hat{x}(3|2)]\,[x(3) - \hat{x}(3|2)]^T\}$

(4 marks)

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Q.3 (continued)

(d) Consider the process

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

where the input, output and state are given by $u(k)$, $y(k)$ and $x(k)$ respectively and

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0 \\ 0.8 & 1 \end{bmatrix} \\ C &= \begin{bmatrix} 0 & 1 \end{bmatrix} \end{aligned}$$

The steady-state Kalman filter is used as an observer in a model predictive control system giving observer poles at $0.32 \pm j0.29$. Find the steady-state Kalman filter gains.

(6 marks)

Q.4 Consider the first-order process

$$x_p(k+1) = a_p x_p(k) + \left(1 - a_p\right)u(k) \tag{4.1}$$

$$y(k) = x_p(k) \tag{4.2}$$

where $a_p < 1$. The input and output are given by $u(k)$ and $y(k)$ respectively.

(a) Find the open-loop transfer function $\frac{Y(z)}{U(z)}$.

(2 marks)

(b) Augment the process in (4.1) and (4.2) with an integrator to give

$$\begin{aligned} x(k+1) &= Ax(k) + B\Delta u(k) \\ y(k) &= Cx(k) \end{aligned}$$

where $\Delta u(k) = u(k) - u(k-1)$. Find A, B, C and $x(k)$.

(4 marks)

(c) The augmented process in Part (b) is controlled by a model predictive controller

$$\begin{aligned} \Delta u(k) &= K_r r(k) - K_{mpc} x(k) \\ K_r &= (\Phi^T \Phi + \bar{R})^{-1} \Phi^T \bar{R}_s \\ K_{mpc} &= (\Phi^T \Phi + \bar{R})^{-1} \Phi^T F \end{aligned}$$

with parameters $r_w = 0$, $N_c = 1$, $N_p = n$, giving the closed-loop transfer function

$$\frac{Y(z)}{R(z)} = C\left[zI - (A - BK_{mpc})\right]^{-1} BK_r$$

where $R(z)$ is the set-point. Find $\bar{R}_s$, F, $\bar{R}$ and Φ in terms of a_p and n.

(4 marks)

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Q.4 (continued)

(d) Find $\lim_{n\to\infty} \frac{1}{n} \Phi^T \Phi$

(4 marks)

(e) Find $\lim_{n\to\infty} \frac{1}{n} \Phi^T \bar{R}_s$

(4 marks)

(f) Find $\lim_{n\to\infty} \frac{1}{n} \Phi^T F$.

Hint: $\lim_{n\to\infty} \frac{1}{n} \{(1-a_p)a_p + (1-a_p^2)(a_p + a_p^2) + \cdots + (1-a_p^n)(a_p + \cdots + a_p^n)\} = \frac{a_p}{1-a_p}$

(6 marks)

(g) When $n = \infty$, find K_r, K_{mpc}, and the relationship between the closed-loop transfer function $\frac{Y(z)}{R(z)}$ and the open-loop transfer function $\frac{Y(z)}{U(z)}$.

(6 marks)

Appendix A - **Table of Laplace Transform and Z Transform**

The following table contains some frequently used time functions x(t), and their Laplace transforms X(s) and Z transforms X(z).

Entry #	Laplace Domain	Time Domain	Z Domain (t=kT)
1	1	$\delta(t)$ unit impulse	1
2	$\frac{1}{s}$	$u(t)$ unit step	$\frac{z}{z-1}$
3	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
4	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$
5		$b^k \quad (b=e^{-aT})$	$\frac{z}{z-b}$
6	$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{\left(z-e^{-aT}\right)^2}$
7	$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$	$\frac{z\left(1-e^{-aT}\right)}{a(z-1)\left(z-e^{-aT}\right)}$
8	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$\frac{z\left(e^{-aT}-e^{-bT}\right)}{\left(z-e^{-aT}\right)\left(z-e^{-bT}\right)}$
9	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}-\frac{e^{-at}}{a(b-a)}-\frac{e^{-bt}}{b(a-b)}$	
10	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}\left(1-e^{-at}-ate^{-at}\right)$	
11	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
12	$\frac{b}{s^2+b^2}$	$\sin(bt)$	$\frac{z\sin(bT)}{z^2-2z\cos(bT)+1}$
13	$\frac{s}{s^2+b^2}$	$\cos(bt)$	$\frac{z(z-\cos(bT))}{z^2-2z\cos(bT)+1}$
14	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin(bt)$	$\frac{ze^{-aT}\sin(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$
15	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos(bt)$	$\frac{z^2-ze^{-aT}\cos(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$

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