EE5137 Stochastic Processes: Problem Set 2
Assigned: 21/01/22, Due: 28/01/22

There are six (6) non-optional problems in this problem set. You have one week to do this problem set. My advice is to get started soon.

1. Exercise 1.14 (Gallager's book)

2. Exercise 1.19(a) (Gallager's book)

3. Exercise 1.20 (Gallager's book)

4. Exercise 1.22 (Gallager's book)

   Note that there's a typo in the book. $p_Y(m) = \mu^n \exp(-\mu)/n!$ *should be* $p_Y(n) = \mu^n \exp(-\mu)/n!$

5. Suppose there are n different types of coupons, and each day we acquire a single coupon uniformly at random from the n types. The coupon collector problem asks: "How many days before we collect *at least one* of each type?"

   Let's formulate this precisely. We will count the time before seeing each new coupon type. Let X_i be the random variable that denotes the number of days to see a new type of coupon after seeing the i-th new type of coupon. The quantity
   $$c_n = \mathbb{E}\left[\sum_{i=0}^{n-1} X_i\right]$$
   gives us the total number of days before we see all n types on average. Show that $c_n \approx n \ln n$ when n is large. Make this precise.

   You can assume that the harmonic number $H_n = \sum_{i=1}^{n} 1/i$ *behaves as* $H_n = \ln n + \gamma + o(1)$ *where* $\gamma = 0.5772156649\ldots$ *is the Euler–Mascheroni constant.*

6. We toss a biased coin n times. The probability of heads, denoted by y, is the value of a random variable Y with a given mean μ and variance σ^2. Let X_i be a Bernoulli random variable that models the outcome of the i-th toss (i.e., $X_i = 1$ if the i-th toss is a head). In other words, for each $1 \le i \le n$,
   $$X_i = \begin{cases} 1 & \text{w.p. } Y \\ 0 & \text{w.p. } 1 - Y \end{cases},$$
   where $Y \in [0, 1]$ is a random variable with
   $$\mathbb{E}[Y] = \mu, \quad \text{and} \quad \mathrm{Var}(Y) = \sigma^2.$$
   We assume that $X_1, X_2, \ldots, X_n$ are conditionally independent given the event $\{Y = y\}$ for each $y \in [0, 1]$. let
   $$S_n = X_1 + X_2 + \ldots + X_n$$
   be the total number of heads in the n tosses.

(a) (5 points) Use the law of iterated expectations to find $\mathbb{E}[X_i]$ and $\mathbb{E}[S_n]$.

(b) (3 points) Using the fact that $X_i^2 = X_i$, show that $\mathrm{Var}(X_i) = \mu - \mu^2$.

(c) (5 points) Using the law of iterated expectations, find

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j], \quad \text{for} \quad i \neq j.$$

Are X_i and X_j independent?

(d) (7 points) By writing $\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$, show that

$$\mathrm{Var}(S_n) = \mathbb{E}[\mathrm{Var}(S_n|Y)] + \mathrm{Var}(\mathbb{E}[S_n|Y]), \tag{1}$$

where $\mathrm{Var}(S_n|Y)$ is the random variable that takes on the value $\mathrm{Var}(S_n|Y = y)$ with probability $\Pr(Y = y)$.

(e) (5 points) Calculate the variance of S_n by using the formula (1) in part (d) above.

This was an exam question in 2017.

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7. (Optional) Exercise 1.6 (Gallager's book)

8. (Optional) Exercise 1.16 (Gallager's book)

9. (Optional) A *round robin* tournament of n contestants is one in which each of the $\binom{n}{2}$ pairs of contestants plays each other exactly once, with the outcome of any play being that one of the contestants wins and the other loses. Suppose the players are initially numbered $1, 2, \ldots, n$. The permutation $i_1, i_2, \ldots, i_n$ is called a *Hamiltonian permutation* if i_1 beats i_2, i_2 beats i_3, …, and i_{n-1} beats i_n. Show that there is an outcome of the round robin for which the number of Hamiltonian permutations is at least $n!/2^{n-1}$.

   Hint: Use the probabilistic method.

10. (Optional) [Knockout Football]

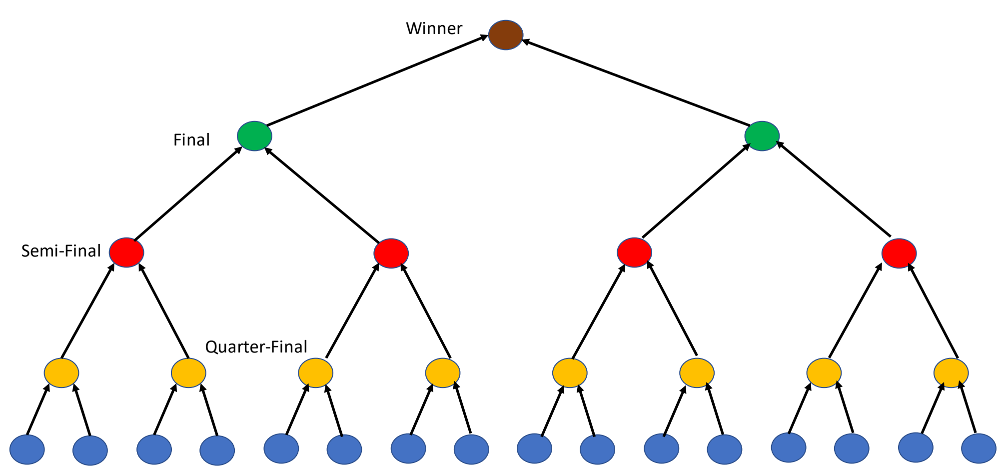

Figure 1: Figure for 16 teams

In the knockout phase of a football tournament, there are 32 teams *of equal skill* that compete in an elimination tournament. This proceeds in a number of rounds in which teams compete in pairs; any losing team retires from the tournament. See Fig. 1 for an illustration with 16 teams. What is the probability that two given teams will compete against each other? Generalize your answer to 2^k teams.

The following argument is wrong but the answer is right. There has to be 31 *games to knock out all but the ultimate winner. There are* $\binom{32}{2}$ *possible pairs, so that the probability of a given pair being selected for a particular match is* $1/\binom{32}{2} = 1/(16 \cdot 31)$. *Since the selection of the teams in the different matches is mutually exclusive, the probability of a given pair being selected is* 31 *times this, which is* $1/16$. *Why is this wrong and what's the correct way of doing it?*

This problem is taken from Problem 297 of *Five Hundred Mathematical Challenges* (Mathematical Association of America, 1996).