EE5137 : Stochastic Processes (Spring 2022)
Some Additional Notes on Poisson Processes

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February 16, 2022

In this document, we provide some supplementary material to Lecture 6 on 18 February 2022. Sections 1, 3 and 4 are solutions to some exercises that I will do in class. You need to know everything here.

1 Single-Server Queues

Queueing theorists use a standard notation of characters separated by slashes to describe common types of queueing systems. The first character describes the arrival process to the queue. M stands for memoryless and means a Poisson arrival process; D stands for deterministic and means that the interarrival interval is fixed and non-random; G stands for general interarrival distribution. We assume that the interarrival intervals are IID (thus making the arrival process a renewal process), but many authors use GI to explicitly indicate IID interarrivals. The second character describes the service process. The same letters are used, with M indicating exponentially distributed service times. The third character gives the number of servers. It is assumed, when this notation is used, that the service times are IID, independent of the arrival epochs, and independent of which server is used.

Consider the single-server M/M/1 queue as shown in Fig. 1. The queueing system follows a Poisson process with rate λ. This single-server queue serves customers with a service time distribution $F(y) = 1-\exp(-\mu y)$, i.e., the service process is also a Poisson process but with a rate μ. Both the queuing system and the service times are memoryless processes. One can think of λ packets entering the queue per second. One can also think of the server serving μ packets per second. The probability that the k^{th} packet arrives before the j^{th} departure is $\Pr(S_{1k} < S_{2j})$ where S_{1k} is the k^{th} arrival epoch of the first Poisson process.

We will cover Markov chains later, but just to give you a flavor of what is to come, consider the Markov chain in Fig. 2. The state being k means that there are k packets (or customers) in the system. Let $P(i,j) = \Pr(\text{transition from state } i \text{ to } j)$. As $\delta \to 0$, we get

$$P(0,0) = 1-\lambda\delta, \quad P(j,j+1) = \lambda\delta, \quad P(j,j-1) = \mu\delta, \quad P(j,j) = 1-\lambda\delta-\mu\delta. \tag{1}$$

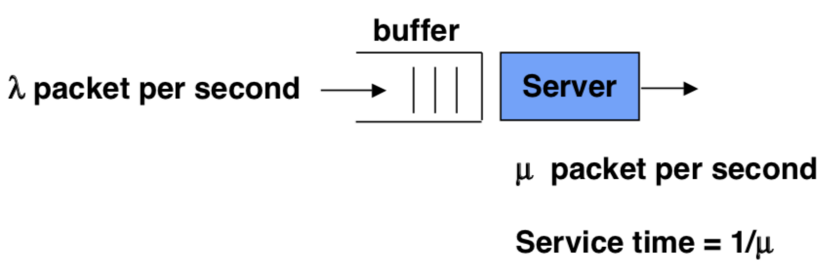

Figure 1: A single-server queue

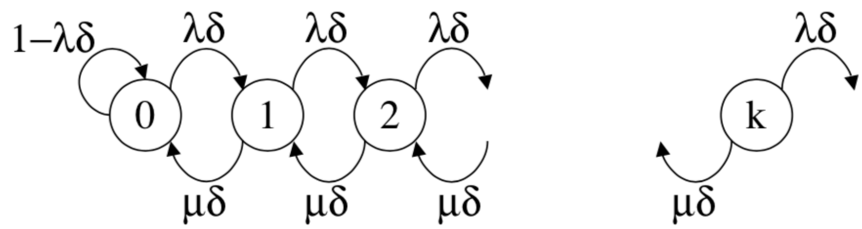

Figure 2: Markov chain for a single-server queue

This is an example of a so-called "birth-death" process with countably infinitely many states. Note that $\lambda\delta$ and $\mu\delta$ are the flow rates between states. Let $P(k)$ be the probability of being in state k and similarly for S_{2j}. One can show that at equilibrium,

$$\lambda P(k) = \mu P(k+1). \tag{2}$$

This means that

$$P(k) = \left(\frac{\lambda}{\mu}\right)^k P(0) =: \rho^k P(0) \tag{3}$$

Since $\sum_k P(k) = 1$, we obtain that $P(0) = 1 - \rho$ assuming that $\rho < 1$, which occurs if and only if $\lambda < \mu$. This means that $P(k) = \rho^k(1-\rho)$. The average queue size is

$$N = \sum_{k=0}^{\infty} kP(k) = \sum_{k=0}^{\infty} k\rho^k(1-\rho) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}. \tag{4}$$

Note that $\lambda < \mu$ means that the the arrival of the packets to the buffer is smaller than the service rate. Intuitively speaking in the long run, all the packets will be served before they "accumulate" in the buffer.

2 Merging of Poisson Processes

People with letters to mail arrive at the post office according to a Poisson process with rate λ_1, while people with packages to mail arrive at the post office according to an independent Poisson process with rate λ_2. The merged process is a Poisson process with rate $\lambda_1 + \lambda_2$.

Focus on a small interval of length δ. Then

$$\Pr(\text{0 arrivals in the merged process}) \approx (1-\lambda_1\delta)(1-\lambda_2\delta) \approx 1 - (\lambda_1+\lambda_2)\delta \tag{5}$$

$$\Pr(\text{1 arrival in the merged process}) \approx \lambda_1\delta(1-\lambda_2\delta) + \lambda_2\delta(1-\lambda_1\delta) \approx (\lambda_1+\lambda_2)\delta \tag{6}$$

Note that we ignored terms of order δ^2 since this is much smaller than δ when δ is small.

Given an arrival has just been recorded, what is the probability that it is an arrival of a person with a *letter* to mail? Again focus on a small interval of length δ around the current time. We seek the probability

$$\Pr(\text{1 arrival of a person with a letter} \,|\, \text{1 arrival}) \tag{7}$$

Using the definition of conditional probabilities, and ignoring the small probability of more than one arrival, this is

$$\frac{\Pr(\text{1 arrival of a person with a letter})}{\Pr(\text{1 arrival})} \approx \frac{\lambda_1\delta}{(\lambda_1+\lambda_2)\delta} = \frac{\lambda_1}{\lambda_1+\lambda_2}. \tag{8}$$

Generalizing this calculation, we let L_k be the event that the k-th arrival corresponds to an arrival of a person with a letter to mail, and we have

$$\Pr(L_k) = \frac{\lambda_1}{\lambda_1+\lambda_2}. \tag{9}$$

Furthermore since distinct arrivals occur at different times, and since, for Poisson processes, events at different times are independent, it follows that the events $L_1, L_2, \ldots$ are mutually independent.

3 Number of Poisson Arrivals during an Exponentially Distributed Interval

Consider a Poisson process with parameter λ and an independent random variable T which is exponential with parameter ν. Find the PMF of the number of Poisson arrivals during the interval $[0, T]$.

Solution: Let us view T as the first arrival in an independent Poisson process with parameter ν and merge this process with the original Poisson process. Each arrival in the merged process comes from the original process with probability $\frac{\lambda}{\lambda+\nu}$, independent of all other arrivals (see Section 2 of this document). If we regard each arrival from the merged process as a trial, and an arrival from the new process as a success, we note that the number K of trials/arrivals until the first success has a geometric PMF, of the form

$$P_K(k) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^{k-1}, \quad k = 1, 2, 3 \ldots \tag{10}$$

Now the number L of arrivals from the original Poisson process until the first "success" is equal to $K - 1$ and its PMF is

$$P_L(l) = P_K(l+1) = \left(\frac{\nu}{\lambda+\nu}\right)\left(\frac{\lambda}{\lambda+\nu}\right)^{l}, \quad l = 0, 1, 2, \ldots \tag{11}$$

4 Counting Processing with a Random Rate (Gallager Problem 2.18)

Consider a counting process in which the rate is a rv Λ with probability density

$$f_\Lambda(\lambda) = \text{Exp}(\alpha) = \alpha \exp(-\alpha\lambda)\mathbf{1}\{\lambda \geq 0\}. \tag{12}$$

Conditional on a given sample value λ for the rate, the counting process is a Poisson process of rate λ (i.e., nature first chooses a sample value λ and then generates a sample path of a Poisson process of that rate λ).

(a) What is $\Pr(N(t) = n|\Lambda = \lambda)$, where $N(t)$ is the number of arrivals in the interval $(0, t]$ for some given $t > 0$?

Solution: Conditional on $\Lambda = \lambda$, $N(t)$ is a usual Poisson counting process with rate λ so

$$\Pr(N(t) = n|\Lambda = \lambda) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \ldots \tag{13}$$

(b) Show that $\Pr(N(t) = n)$, the unconditional PMF for $N(t)$, is given by

$$\Pr(N(t) = n) = \frac{\alpha t^n}{(t+\alpha)^{n+1}}, \quad n = 0, 1, 2, \ldots \tag{14}$$

Solution: The straightforward approach is to average the conditional distribution over λ, i.e.,

$$\Pr(N(t) = n) = \int_0^\infty \frac{(\lambda t)^n e^{\lambda t}}{n!} \alpha e^{-\alpha\lambda}\, \mathrm{d}\lambda \tag{15}$$

$$= \frac{\alpha t^n}{(t+\alpha)^n} \int_0^\infty \frac{[\lambda(t+\alpha)]^n e^{-\lambda(t+\alpha)}}{n!}\, \mathrm{d}\lambda \tag{16}$$

$$= \frac{\alpha t^n}{(t+\alpha)^{n+1}} \int_0^\infty \frac{x^n e^{-x}}{n!}\, \mathrm{d}x = \frac{\alpha t^n}{(t+\alpha)^{n+1}} \tag{17}$$

where we changed the variable of integration from λ to $x = \lambda(t+\alpha)$ and then recognized the integral as the integral of an Erlang density of order $n+1$ with unit rate.

The solution can also be written as pq^n where $p = \frac{\alpha}{\alpha+t}$ and $q = \frac{t}{\alpha+t}$. This suggests a different interpretation for this result. $\Pr(N(t) = n)$ for a Poisson process (PP) of rate λ is a function of λt and n only. The rv $N(t)$ for a PP of rate λ thus has the same distribution as $N(\lambda t)$ for a PP of unit rate and thus $N(t)$ for a PP of variable rate Λ has the same distribution as $N(\Lambda t)$ for a PP of rate 1.

Since Λt is an exponential rv of parameter α/t, we see that $N(\Lambda t)$ is the number of arrivals of a PP of unit rate before the first arrival from an independent PP of rate α/t. This unit rate PP and rate α/t PP are independent and the combined process has rate $1 + \alpha/t$. The event $\{N(\Lambda t) = n\}$ then has the probability of n arrivals from the unit rate PP followed by one arrival from the α/t rate process, thus yielding the probability $q^n p$ (a geometric distribution).

(c) Find $f_\Lambda(\lambda|N(t) = n)$, the density of Λ conditioned on $N(t) = n$.

Solution: Using Bayes' law with the answers in (a) and (b), we get

$$f_{\Lambda|N(t)}(\lambda|n) = \frac{\lambda^n e^{-\lambda(\alpha+t)}(\alpha+t)^{n+1}}{n!}, \quad \lambda \geq 0. \tag{18}$$

This is an Erlang PDF of order $n+1$ and can be interpreted (after a little work) in the same way as (b).

(d) Find $\mathbb{E}[\Lambda|N(t) = n]$ and interpret your result for very small t with $n = 0$ and for very large t with n large.

Solution: Since Λ conditional on $N(t) = n$ is Erlang, it is the sum of $n+1$ IID rv's, each of mean $1/(t+\alpha)$. Thus

$$\mathbb{E}[\Lambda|N(t) = n] = \frac{n+1}{t+\alpha}. \tag{19}$$

For $N(t) = 0$ and $t \ll \alpha$, this is close to $1/\alpha$, which is the mean of Λ. This is not surprising since it has little effect on the distribution of Λ. For n large and $t \gg \alpha$, $\mathbb{E}[\Lambda|N(t) = n] \approx n/t$.

(e) Find $\mathbb{E}[\Lambda|N(t) = n, S_1, S_2, \ldots, S_n]$ and $\mathbb{E}[\Lambda|N(t) = n, N(\tau) = m]$ for some $\tau < t$.

Solution: From Theorem 2.5.1, $S_1, \ldots, S_n$ are uniformly distributed, subject to $0 < S_1 < \ldots < S_n < t$, given $N(t) = n$ and $\Lambda = \lambda$. Thus, conditional on $N(t) = n$, Λ is statistically independent of $S_1, \ldots, S_n$, i.e.,

$$\mathbb{E}[\Lambda|N(t) = n, S_1, S_2, \ldots, S_n] = \mathbb{E}[\Lambda|N(t) = n] = \frac{n+1}{t+\alpha}. \tag{20}$$

Conditional on $N(t) = n$, $N(\tau)$ is determined by $S_1, \ldots, S_n$ for $\tau < t$ and thus

$$\mathbb{E}[\Lambda|N(t) = n, N(\tau) = m] = \mathbb{E}[\Lambda|N(t) = n] = \frac{n+1}{t+\alpha}. \tag{21}$$

This corresponds to one's intuition; given the number of arrivals in $(0, t]$, it makes no difference where the individual arrivals occur.