1 Proof of the Independent Increments Property of the Merged Process

Let $N_\mathrm{A} = \{N_\mathrm{A}(t) : t > 0\}$ and $N_\mathrm{B} = \{N_\mathrm{B}(t) : t > 0\}$ be two independent Poisson counting processes with rates λ_A and λ_B respectively. We know $N(t) := N_\mathrm{A}(t) + N_\mathrm{B}(t)$ is a Poisson random variable with mean $(\lambda_\mathrm{A} + \lambda_\mathrm{B})t$. Let's show that $N = \{N(t) : t > 0\}$ has the independent increments property.

Let $t_1 < t_2 < t_3$ be three distinct times.

Claim 1. *$\tilde{N}(t_1, t_2) = \tilde{N}_\mathrm{A}(t_1, t_2) + \tilde{N}_\mathrm{B}(t_1, t_2)$ is independent of $\tilde{N}(t_2, t_3) = \tilde{N}_\mathrm{A}(t_2, t_3) + \tilde{N}_\mathrm{B}(t_2, t_3)$.*

This proves that the merged process has the independent increments property.

Proof. Consider $n, m \in \mathbb{N} \cup \{0\}$. We have

$$
\begin{aligned}
&\Pr\left(\tilde{N}(t_1,t_2) = n, \tilde{N}(t_2,t_3) = m\right) \\
&= \Pr\left(\tilde{N}_\mathrm{A}(t_1,t_2) + \tilde{N}_\mathrm{B}(t_1,t_2) = n, \tilde{N}_\mathrm{A}(t_2,t_3) + \tilde{N}_\mathrm{B}(t_2,t_3) = m\right) && (1)\\
&= \sum_{n_\mathrm{B}, m_\mathrm{B}} \Pr\left(\tilde{N}_\mathrm{A}(t_1,t_2) + \tilde{N}_\mathrm{B}(t_1,t_2) = n, \tilde{N}_\mathrm{A}(t_2,t_3) + \tilde{N}_\mathrm{B}(t_2,t_3) = m \mid \tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}, \tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right) \\
&\qquad \times \Pr\left(\tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}, \tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right) && (2)\\
&= \sum_{n_\mathrm{B}, m_\mathrm{B}} \Pr\left(\tilde{N}_\mathrm{A}(t_1,t_2) = n - n_\mathrm{B}, \tilde{N}_\mathrm{A}(t_2,t_3) = m - m_\mathrm{B} \mid \tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}, \tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right) \\
&\qquad \times \Pr\left(\tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}, \tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right) && (3)\\
&= \sum_{n_\mathrm{B}, m_\mathrm{B}} \Pr\left(\tilde{N}_\mathrm{A}(t_1,t_2) = n - n_\mathrm{B}\right) \Pr\left(\tilde{N}_\mathrm{A}(t_2,t_3) = m - m_\mathrm{B}\right) \Pr\left(\tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}\right) \Pr\left(\tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right) && (4)\\
&= \left[\sum_{n_\mathrm{B}} \Pr\left(\tilde{N}_\mathrm{A}(t_1,t_2) = n - n_\mathrm{B}\right) \Pr\left(\tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}\right)\right] \left[\sum_{m_\mathrm{B}} \Pr\left(\tilde{N}_\mathrm{A}(t_2,t_3) = m - m_\mathrm{B}\right) \Pr\left(\tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right)\right] && (5)\\
&= \left[\sum_{n_\mathrm{B}} \Pr\left(\tilde{N}_\mathrm{A}(t_1,t_2) = n - n_\mathrm{B} \mid \tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}\right) \Pr\left(\tilde{N}_\mathrm{B}(t_1,t_2) = n_\mathrm{B}\right)\right] \\
&\qquad \times \left[\sum_{m_\mathrm{B}} \Pr\left(\tilde{N}_\mathrm{A}(t_2,t_3) = m - m_\mathrm{B} \mid \tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right) \Pr\left(\tilde{N}_\mathrm{B}(t_2,t_3) = m_\mathrm{B}\right)\right] && (6)\\
&= \Pr\left(\tilde{N}_\mathrm{A}(t_1,t_2) + \tilde{N}_\mathrm{B}(t_1,t_2) = n\right) \Pr\left(\tilde{N}_\mathrm{A}(t_2,t_3) + \tilde{N}_\mathrm{B}(t_2,t_3) = m\right) && (7)\\
&= \Pr\left(\tilde{N}(t_1,t_2) = n\right) \Pr\left(\tilde{N}(t_2,t_3) = m\right) && (8)
\end{aligned}
$$

where

- Equality (1) follows from the fact that $\tilde{N}(t,s) = \tilde{N}_\mathrm{A}(t,s) + \tilde{N}_\mathrm{B}(t,s)$;
- Equality (2) follows from the law of total probability where $n_\mathrm{B} = 0, 1, \ldots, n$ and $m_\mathrm{B} = 0, 1, \ldots, m$;
- Equality (3) follows by substituting $\tilde{N}_\mathrm{B}(t_1,t_2)$ and $\tilde{N}_\mathrm{B}(t_2,t_3)$ by n_B and m_B respectively;
- Equality (4) follows from the independence of the two constituent Poisson processes N_A and N_B so we can drop the conditioning and the fact that N_A and N_B have the independent increments property so we can split the probability of the joint event $\{\tilde{N}_\mathrm{A}(t_1,t_2) = n - n_\mathrm{B}, \tilde{N}_\mathrm{A}(t_2,t_3) = m - m_\mathrm{B}\}$ into the product of the probabilities of the constituent events;

- Equality (5) splits the double sum into two parts;
- Equality (6) follows from the independence of the two constituent Poisson processes N_{A} and N_{B} so we can insert the events $\{\tilde{N}_{\mathrm{B}}(t_1, t_2) = n_{\mathrm{B}}\}$ and $\{\tilde{N}_{\mathrm{B}}(t_2, t_3) = m_{\mathrm{B}}\}$;
- Equality (7) follows from two applications of the law of total probability;
- Equality (8) follows from the definition of the merged process N.

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