Liu Ukihao

A0232935A

1.(a) Find the joint PMF of $N(t), N(t+s)$ for $s>0$.

$$P(N(t)=m, N(t+s)=n) = P(N(t)=m, N(s)=n-m)$$

$$= P(N(t)=m)\cdot P(N(s)=n-m)$$

$$= \frac{(\lambda t)^m \cdot e^{-\lambda t}}{m!} \cdot \frac{(\lambda s)^{n-m}\, e^{-\lambda s}}{(n-m)!}$$

$$= \binom{n}{m} e^{-\lambda(t+s)} \cdot \frac{(\lambda t)^m (\lambda s)^{n-m}}{n!}$$

(b) $N(t+s) = N(t) + \tilde{N}(t, t+s)$

$$E[N(t)\cdot N(t+s)] = E[N^2(t)] + E[N(t)\cdot \tilde{N}(t, t+s)]$$

$$= E[N^2(t)] + E[N(t)]\cdot E[N(s)]$$

$$= (\lambda t)^2 + \lambda t + \lambda t \cdot \lambda s.$$

(c) $E[\tilde{N}(t_1,t_3)\cdot \tilde{N}(t_2,t_4)] = E[\{\tilde{N}(t_1,t_2)+\tilde{N}(t_2,t_3)\}\cdot\{\tilde{N}(t_2,t_3)+\tilde{N}(t_3,t_4)\}]$

$$= E[\tilde{N}(t_1,t_2)\cdot \tilde{N}(t_2,t_4)] + E[\tilde{N}^2(t_2,t_3)] + E[\tilde{N}(t_2,t_3)\cdot \tilde{N}(t_3,t_4)]$$

$$= \lambda^2(t_2-t_1)(t_4-t_2) + \lambda^2(t_3-t_2)^2 + \lambda(t_3-t_2) + \lambda^2(t_3-t_2)(t_4-t_3)$$

$$= \lambda^2(t_3-t_1)\cdot(t_4-t_2) + \lambda(t_3-t_2)$$

2. (a) Find the joint probability density of $S_1, S_2, \ldots S_{n-1}$ conditional on $S_n = t$.

$$f_{S_1,S_2,\ldots S_{n-1}|S_n}(S_1,S_2\ldots S_{n-1}|S_n) = \frac{f_{S_1,S_2,\ldots S_n}(S_1,S_2,\ldots S_n)}{f_{S_n}(S_n)}$$

$$= \frac{\lambda^n e^{-\lambda t}}{\frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}}$$

$$= \frac{(n-1)!}{t^{n-1}}$$

(b) $P_r\{X_1 > \tau | S_n = t\} = P_r\{X_1 > \tau \mid N(t) = n\}$.

for all ~~$\ast$~~ $S_i > \tau$, $1 \le i \le n$. Because $S_1 \ldots S_{n-1}$ are IID and uniformly distributed

from $(\tau, t]$.

So: $P_r\{X_1 > \tau | S_n = t\} = (\frac{t-\tau}{t})^n$.

(c) $P_r\{X_i > \tau | S_n = t\}$. Because $X_1 \ldots X_n$ are IID.

$\Rightarrow P_r\{X_i > \tau | S_n = t\} = P_r\{X_1 > \tau | S_n = t\} = (\frac{t-\tau}{t})^n$.

(d) To find $f_{S_i|N(t)}(S_i|n)$, look at n uniformly distributed rv's in $(0,t]$. The probability that one of these lies in the interval $(S_i, S_i+dt]$ is $\frac{n\,dt}{t}$. Out of the remain $n-1$, the probability that $i-1$ lies in the interval $(0, S_i]$ is given by the binomial distribution with probability of success $\frac{S_i}{t}$.

$$f_{S_i|N(t)}(S_i|t)\,dt = \binom{n-1}{i-1}\left(\frac{S_i}{t}\right)^{i-1} \cdot \left(\frac{t-S_i}{t}\right)^{n-i} \cdot \frac{n\,dt}{t}$$

$$= \frac{(n-1)!\, S_i^{i-1} (t-S_i)^{n-i}}{(i-1)!\,(n-i)! \cdot t^{n-1}} \cdot \frac{n\,dt}{t}$$

$$\Rightarrow f_{S_i|N(t)}(S_i|t) = \frac{n!\, S_i^{i-1} (t-S_i)^{n-i}}{(i-1)! \cdot (n-i)! \cdot t^n}$$

3.(a) For a Poisson process of rate λ, find $Pr\{N(t)=n|S_1=\tau\}$ for $t>\tau$ and $n\geq 1$

$S_1=\tau$, the number of arrivals $N(t)$ in $(0,t]$ is 1 plus the number in $(\tau,t]$.

$$Pr\{N(t)=n|S_1=\tau\} = Pr\{N(\tau,t)=n-1\} = \frac{[\lambda(t-\tau)]^{n-1}e^{-\lambda(t-\tau)}}{(n-1)!}$$

(b) Using this, find $f_{S_1}(\tau|N(t)=n)$.

$$f_{S_1|N(t)}(\tau|n) = \frac{f_{S_1,N(t)}(\tau,n)}{f_{N(t)}(n)} = \frac{f_{S_1,N(t)}(\tau,n)}{f_{S_1}(\tau)}\cdot\frac{f_{S_1}(\tau)}{f_{N(t)}(n)}$$

$$= f_{N(t)|S_1}(n|\tau)\cdot\frac{f_{S_1}(\tau)}{f_{N(t)}(n)}$$

$$= \frac{[\lambda(t-\tau)]^{n-1}\cdot e^{-\lambda(t-\tau)}}{(n-1)!}\cdot\frac{\lambda\cdot\tau^0\cdot e^{-\lambda\tau}}{1}\cdot\frac{n!}{e^{-\lambda t}\cdot\lambda^n\cdot t^n}$$

$$= \frac{n(t-\tau)^{n-1}}{t^n}$$

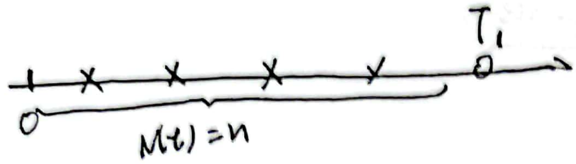
(c) Eq 2.41: $Pr\{S_1>\tau|N(t)=n\} = [\frac{t-\tau}{t}]^n$. The derivative of this with respect to τ is.

$$-f_{S_1|N(t)}(\tau|t) = -\frac{n(t-\tau)^{n-1}}{t^n}$$

4.

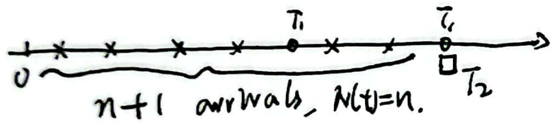
(a)

$N(T_1)$ means there are n arrivals before T_1 arrive.



$$N(T_1) = \left(\frac{\lambda}{\lambda+\nu}\right)^n \cdot \frac{\nu}{\lambda+\nu}$$

(b) Becaue T_2 is an erlang of order 2. $N(T_2)$ means, there are n arrivals before T_2 arrive, or n arrivals before ~~1st~~ 2nd arrive of T_1.



$$N(T_2) = \binom{n+1}{n}\left(\frac{\lambda}{\lambda+\nu}\right)^n \cdot \frac{\nu}{\lambda+\nu} \cdot \frac{\nu}{\lambda+\nu}$$

$$= (n+1) \cdot \left(\frac{\lambda}{\lambda+\nu}\right)^n \cdot \left(\frac{\nu}{\lambda+\nu}\right)^2$$

5.

$$X(t) = \sum_{i=1}^{N_t} e^{-\theta(t-S_i)}$$

$$E[X(t)|N_t=n] = E\left(\sum_{i=1}^{n} e^{-\theta(t-U_i)}\right)$$

Where $U_1, U_2 \dots U_n$ are IID and distributed uniformly over $[0,t]$.

Therefore,

$$E[X(t)|N(t)=n] = n \cdot E[e^{-\theta(t-U_i)}] = n \cdot \frac{1-e^{-\theta t}}{\theta t} = N(t) \cdot \frac{1-e^{-\theta t}}{\theta t}$$

leading to

$$E[X(t)] = \frac{\lambda(1-e^{-\theta t})}{\theta}$$

6.

(a) Find the mean and variance of $M(t)$.

$$E[M(t)] = E[N(t)] - E[\lambda t] = \lambda t - \lambda t = 0.$$

$$\begin{aligned}
Var[M(t)] &= E[M^2(t)] - (E[M(t)])^2 \\
&= E[N^2(t) + (\lambda t)^2 - 2\lambda t \cdot N(t)] - 0 \\
&= E[N^2(t)] + E[(\lambda t)^2] - E[2\lambda t \cdot N(t)] \\
&= (\lambda t)^2 + \lambda t + (\lambda t)^2 - 2\lambda t \cdot \lambda t \\
&= \lambda t.
\end{aligned}$$

6.

(b) $M(t) = N(t) - \lambda t$

Because $N(t)$ is IIP, SIP.

So $M(t)$ are IIP, & SIP

(c)