UNIT - III

Reliability Functions

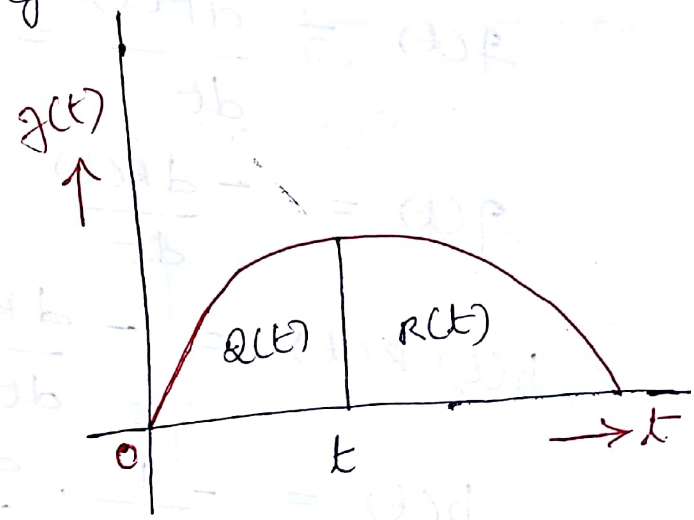
Exponential Distribution :-

The exponential or strictly the negative exponential distribution is probably the most widely known and used distribution in reliability evaluation of systems. The most important factor for it to be applicable is that the hazard rate should be constant i.e the failure rate (λ). In practice the negative exponential distribution has a much wider degree of significance than just that of first failure and is extensively used in the analysis of repairable systems in which the component cycle between operating or up states and failure or down states.

Time Dependent Probabilities :-

Let $f(t)$ be the probability density function of failure and $F(t)$ be the probability distribution function upto time t.

$R(t)$ — Reliability of the component as a function of time



$$R(t) + Q(t) = 1$$

$$R(t) + F(t) = 1$$

$$\therefore F(t) = \int_0^t f(t)\,dt$$

$$R(t) = 1 - \int_0^t f(t)\,dt$$

$$= \int_t^\infty f(t)\,dt$$

Hazard Rate Function [h(t)]:

It is defined as the probability of the component will fail in the next interval survived until time t and t to t+Δt

$$h(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[P(\text{failure in } (t, t+\Delta t)) / \text{working at time } t \right]$$

$$h(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{P(t \le T \le t+\Delta t)}{T > t} \right]$$

$$h(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\frac{P(t \le T \le t+\Delta t) \cap P(T > t)}{P(T > t)} \right]$$

$$h(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot \frac{f(t)\,\Delta t}{R(t)}.$$

$$\therefore h(t) = \frac{f(t)}{R(t)}$$

$$f(t) = h(t)\,R(t)$$

we know

$$F(t) = \int_0^t f(t)\,dt$$

$$f(t) = \frac{dF(t)}{dt} = \frac{d}{dt}\left[1 - R(t)\right]$$

$$f(t) = \frac{-dR(t)}{dt}$$

$$h(t)\,R(t) = -\frac{dR(t)}{dt}$$

$$h(t) = -\frac{1}{R(t)} \cdot \frac{dR(t)}{dt}$$

$$-\int_0^t h(t)\,dt = \int \frac{dR(t)}{R(t)}$$

$$R(t) = e^{-\int_0^t h(t)\,dt}$$

If $h(t)$ is constant $= \lambda$ ie constant hazard rate function or failure rate

$$R(t) = e^{-\int_0^t \lambda\,dt} = e^{-\lambda t}$$

$$\therefore \boxed{R(t) = e^{-\lambda t}}$$

The probability of a component surviving for a time t if the hazard rate is constant ie

$$R(t) = e^{-\lambda t}$$

The failure density function is

$$f(t) = \frac{-dR(t)}{dt} = \lambda e^{-\lambda t}$$

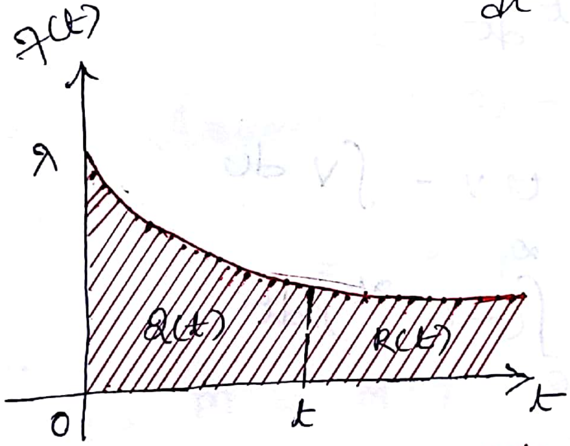
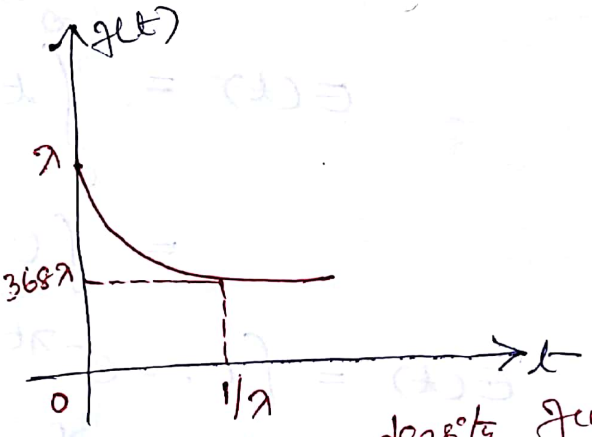
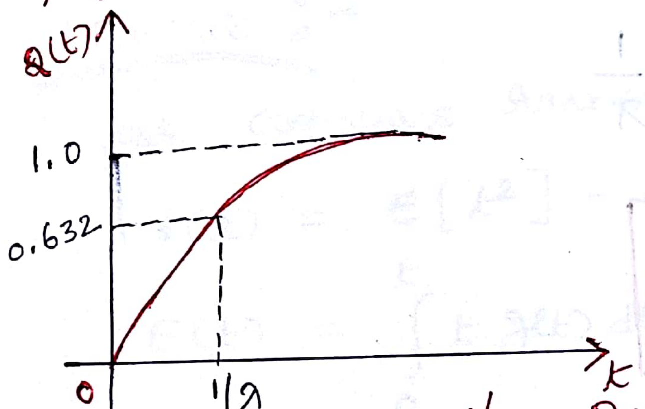


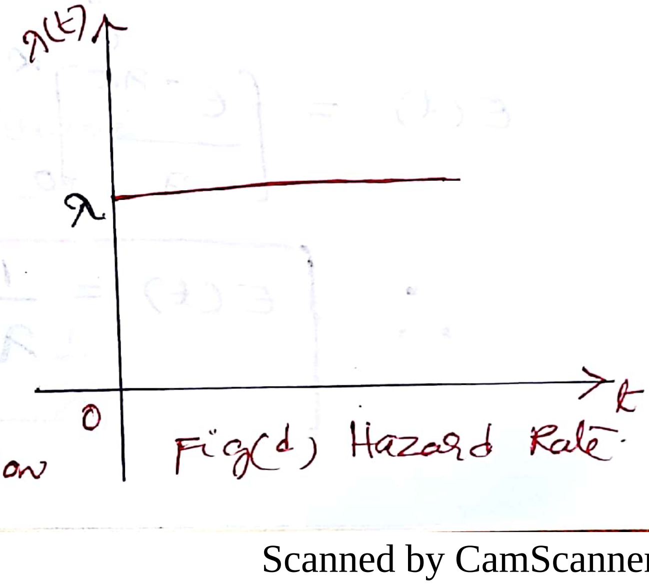
Fig (a) shows Q(t) & R(t)



Fig(b) Failure density function



Fig(c) Cumulative Failure Distribution



Fig(d) Hazard Rate

$$R(t) = \int_0^t \lambda e^{-\lambda t} dt = e^{-\lambda t}$$

$$Q(t) = \int_0^t \lambda e^{-\lambda t} dt = 1 - e^{-\lambda t}$$

The failure density function $f(t)$, Cummulative failure distribution $Q(t)$ and hazard rate $\lambda(t)$ are shown in fig (b), (c) and (d) respectively.

Expected Value :-

$$E(t) = \int_0^\infty t\, f(t)\, dt$$

we know $R(t) = e^{-\lambda t}$

$$f(t) = -\frac{d}{dt} R(t)$$

$$= -\frac{d}{dt} (e^{-\lambda t}) = \lambda e^{-\lambda t}$$

$$E(t) = \int_0^\infty t\, \lambda e^{-\lambda t} dt$$

$$= \int u\, dv = uv - \int v\, du$$

$$E(t) = [t . -e^{-\lambda t}]_0^\infty - \int_0^\infty (-e^{-\lambda t})\, dt$$

$$= 0 + \int_0^\infty e^{-\lambda t} dt$$

$$E(t) = \left[\frac{e^{-\lambda t}}{-\lambda}\right]_0^\infty = \frac{1}{\lambda}$$

$$\therefore \boxed{E(t) = \frac{1}{\lambda}}$$

The expected value of an exponentially distributed reliability function is equal to the reciprocal of the constant hazard rate function.

MTTF (m) mean time to failure $= E(t) = \frac{1}{\lambda}$

Relationship Between MTTF & R(t) :-

$$m = MTTF = E(t) = \int_0^t t\, g(t)\, dt$$

$$= \int_0^t t \left(-\frac{dR(t)}{dt}\right) dt$$

$$= \int_0^\infty -t\, dR(t)$$

$$\int u\, dv = uv - \int v\, du$$

$$= \left[-t\, R(t)\right]_0^\infty - \int_0^\infty -R(t)\, dt$$

Assume $R(\infty) \rightarrow 0$

$R(0) \rightarrow 1$

$$= 0 + \int_0^\infty R(t)\, dt$$

$$\therefore m = MTTF = \int_0^\infty R(t)\, dt$$

Varience :-

For continuous random Variable

$$V(t) = E[t^2] - [E(t)]^2$$

$$E(t) = \int_0^t t\, g(t)\, dt = \frac{1}{\lambda}$$

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$$E(t^2) = \int_0^\infty t^2 g(t)\,dt$$

$$E(t^2) = \int_0^\infty t^2 \lambda e^{-\lambda t}\,dt$$

Let $u = t^2$

$dv = -d\,[e^{-\lambda t}]$

$$\int u\,dv = uv - \int v\,du$$

$$E(t^2) = \left[t^2(-e^{-\lambda t})\right]_0^\infty - \int_0^\infty -(e^{-\lambda t})\,2t\,dt$$

$$= 0 + \int_0^\infty 2t\,e^{-\lambda t}\,dt$$

$$E(t^2) = \frac{2}{\lambda}\int_0^\infty \lambda t\,e^{-\lambda t}\,dt$$

$$E(t^2) = \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$V(t) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Standard Deviation $\sigma = \sqrt{V(t)} = \sqrt{1/\lambda^2} = 1/$

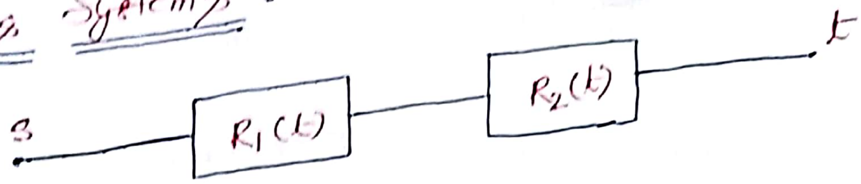
$$\therefore \boxed{\sigma = \frac{1}{\lambda}}$$

The standard deviation of an exponential distributio reliability function is also equal to the reciproca of the failure rate.

Applications of Exponential Distribution to Network Reliability Evaluation :-

1. Series Systems
2. Parallel Systems

1. Series Systems :-

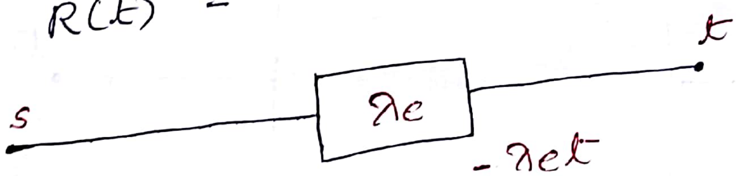


$R_1(t) = e^{-\lambda_1 t}$

$R_2(t) = e^{-\lambda_2 t}$

λ_1, λ_2 are the component failure rate are constant hazard rate function

$$R(t) = R_1(t) \cdot R_2(t)$$
$$= e^{-(\lambda_1 + \lambda_2)t}$$
$$R(t) = e^{-\lambda_e t}$$



$$R(t) = e^{-\lambda_e t}$$

where λ_e = Equivalent failure rate

$$MTTF = m = \frac{1}{\lambda_e} = \frac{1}{\lambda_1 + \lambda_2}$$

For 'n' Components Connected in Series

$R_1(t) = e^{-\lambda_1 t}$

$R_2(t) = e^{-\lambda_2 t}$

$\vdots$

$R_n(t) = e^{-\lambda_n t}$

$$R(t) = e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdots\cdots e^{-\lambda_n t}$$

$$= e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t}$$

$$= e^{-\lambda_e t}$$

where $\lambda_e = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

$$= \sum_{i=1}^{n} \lambda_i$$

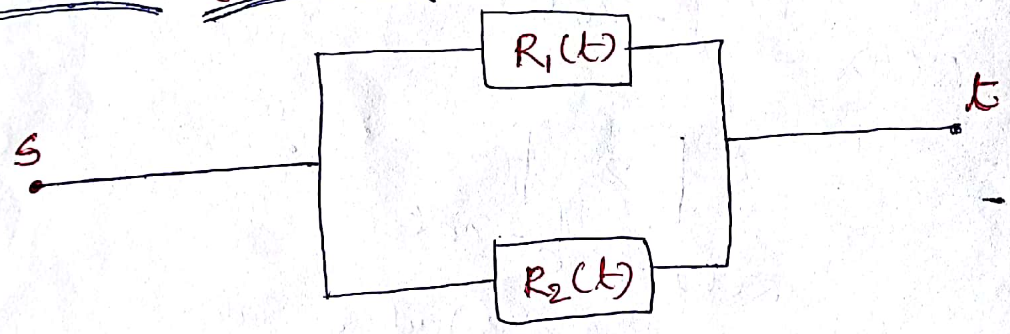
$$R(t) = R_1(t) R_2(t) \cdots\cdots R_n(t)$$

$$= \prod_{i=1}^{n} R_i(t)$$

$$MTTF = m = \frac{1}{\lambda_e} = \frac{1}{\sum_{i=1}^{n} \lambda_i}$$

Redundancy $\rightarrow$ No longer needed (or) useful

2. Parallel Systems :-



$$R_1(t) = e^{-\lambda_1 t}$$

$$R_2(t) = e^{-\lambda_2 t}$$

$$Q(t) = Q_1(t) Q_2(t)$$

$$= [1 - R_1(t)][1 - R_2(t)]$$

$$Q(t) = 1 - R_1(t) + R_2(t) + R_1(t) R_2(t)$$

$$R(t) = 1 - Q(t)$$

$$= R_1(t) + R_2(t) - R_1(t) R_2(t)$$

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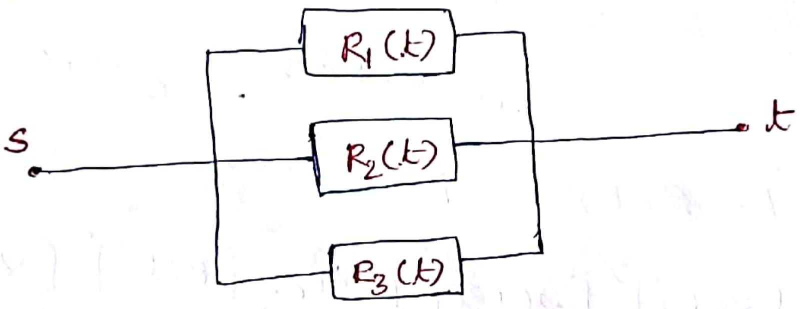
$$R(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-\lambda_1 t} \cdot e^{-\lambda_2 t}$$

$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

$$m = MTTF = \int_0^\infty R(t)\,dt = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$$

Consider there are 3 Components



Applying union rule

$$R(t) = R_1(t) + R_2(t) + R_3(t) - R_1(t)R_2(t) - R_2(t)R_3(t) - R_3(t)R_1(t) + R_1(t)R_2(t)R_3(t)$$

$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-\lambda_1 t} e^{-\lambda_2 t} - e^{-\lambda_2 t} e^{-\lambda_3 t} - e^{-\lambda_3 t} e^{-\lambda_1 t} + e^{-\lambda_1 t} e^{-\lambda_2 t} e^{-\lambda_3 t}$$

$$R(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} + e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2)t} - e^{-(\lambda_2+\lambda_3)t} - e^{-(\lambda_3+\lambda_1)t} + e^{-(\lambda_1+\lambda_2+\lambda_3)}$$

$$m = MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1+\lambda_2} - \frac{1}{\lambda_2+\lambda_3} - \frac{1}{\lambda_3+\lambda_1} + \frac{1}{\lambda_1+\lambda_2+\lambda_3}$$

$$= \sum_{i=1}^{n} \frac{1}{\lambda_i} - \sum_{\substack{i=1\\ j=1\\ i\neq j}}^{3} \frac{1}{(\lambda_i+\lambda_j)} + \frac{1}{\sum_{i=1}^{3} \lambda_i}$$

PROBLEMS :-

1. Consider a system comprising of four identical units having a failure rate of 0.1 f/yr. Evaluate the system surviving for five years if atleast 2 units must operate successfully.

$t = 5$ years

Sol.

$$R(t) = e^{-\lambda t} = e^{-0.1t}$$

$$= e^{-0.1\times 5} = e^{-0.5}$$

$$Q(t) = 1 - R(t) = 1 - e^{-0.5}$$

$$R_S(t) = {}^4C_2 [R(t)]^2 [Q(t)]^2 + {}^4C_3 [R(t)]^3 [Q(t)]^1 + {}^4C_4 [R(t)]^4 [Q(t)]^0$$

$$R_S(t) = 0.8282$$

2. An electronic circuit consists of 6 transistors each having a failure rate of 10^{-6} f/hr, 4 diodes each having a failure rate of 0.5×10^{-6} f/hr, 3 capacitors 0.2×10^{-6} f/hr, 10 resistors - 5×10^{-6} f/hr and 2 switches - 2×10^{-6} f/hr.

1. Evaluate the equivalent failure rate of the system and the probability of system surviving 1000 hours.
2. If two such circuits are connected in parallel find the probability of surviving for 1000 hrs; if only one of the circuits is required for system success.

Sol. 1.

6T — 10^{-6} f/hr $\rightarrow 6\lambda_T$

4D — 0.5×10^{-6} f/hr $\rightarrow 4\lambda_D$

3C — 0.2×10^{-6} f/hr $\rightarrow 3\lambda_C$

$10R - 5\times 10^{-6}$ f/hr $\rightarrow 10\lambda_R$

$2S - 2\times 10^{-6}$ f/hr $\rightarrow 2\lambda_S$

$$\lambda_e = 6\lambda_T + 4\lambda_D + 3\lambda_C + 10\lambda_R + 2\lambda_S$$

$$= 6\times 10^{-6} + 4\times 0.5\times 10^{-6} + 3\times 0.2\times 10^{-6} + 10\times 5\times 10^{-6} + 2\times 2\times 10^{-6}$$

$$\lambda_e = 6.26\times 10^{-5} \text{ f/hr.}$$

$$R(S) = e^{-\lambda_e t}$$

$$= e^{-6.26\times 10^{-5}\times 10^{3}}$$

$$= \underline{0.9393}$$

2. Let λ_{e_1} & λ_{e_2} be the failure rates of the circuit connected in parallel.

$$\lambda_{e_1} = \lambda_{e_2} = \lambda_e = 6.26\times 10^{-5} \text{ f/hr}$$

$$R(S) = e^{-\lambda_{e_1} t} + e^{-\lambda_{e_2} t} - e^{-(\lambda_{e_1}+\lambda_{e_2})t}$$

$$= e^{-(6.26\times 10^{-5}\times 10^{3})} - e^{-6\times 26\times 10^{-5}\times 10^{3}} - e^{-(6.26\times 10^{-5}+6.26\times 10^{-5})10^{3}}$$

$$= \underline{0.9963}$$

3. It is observed that the failure pattern of an electronic system follows an exponential distribution with mean time to failure of 1000 hrs. What is the probability that the system failure occurs within 750 hours?

Sol

$$MTTF = 1000 \text{ hrs} = \frac{1}{\lambda}$$

Probability of failure

$$Q[1000] = 1 - e^{-\lambda t}$$

$$= 1 - e^{-750/1000}$$

$$= 0.528$$

4. It is found that the random variations with respect to time in the output voltage of a particular system are exponentially distributed with a mean value of 100V. What is the probability that the o/p voltage will be found at any time to lie in the range 90 - 110V ?

Sol MTTF with a mean value of 100V.

Probability Density Function

$$f_d(t) = \lambda e^{-\lambda t}$$

$$MTTF = \frac{1}{\lambda} = 100 \Rightarrow \lambda = \frac{1}{100}$$

The probability that the Voltage is within a value (V) [which corresponds to the probability that the failure occurs within time t] i.e,

$$F(t) = 1 - e^{-\lambda t} \quad ie \quad F(V) = 1 - e^{-\lambda V}.$$

The probability that the voltage lies b/w V_1 and V_2 is

$$F(V_2) - F(V_1) = [1 - e^{-\lambda V_2}] - [1 - e^{-\lambda V_1}]$$

$$= e^{-\lambda V_1} - e^{-\lambda V_2}$$

with $V_1 = 90V$, $V_2 = 110V$, & $\lambda = 1/100$

$$F(110) - F(90) = e^{-90\lambda} - e^{-100\lambda}$$

$$= e^{-90/100} - e^{-100/100}$$

$$= 0.074$$

5. A Control system of three seperate sub-systems all components of which have reliabilities that are exponentially distributed. The subsystems are
(a) a single component having a failure rates of 1×10^{-6} f/hr,
(b) Two identical components having a failure rate of 8×10^{-6} f/hr for which one component must operate for success and
(c) Three components having failure rates of 5×10^{-6} f/hr, 2×10^{-6} f/hr & 10×10^{-6} f/hr and for which two components must operate for success.

If all subsystems must be successful for satisfactory system operation. Evaluate the probability of surviving for a period of 5000 hrs.

Sol.

(a) $R(a) = e^{-\lambda t}$

$= e^{-[1 \times 10^{-6} \times 5000]}$

$= 0.9950$

(b) $R(b) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}$

$= 2e^{-(8 \times 10^{-6} \times 5000)} - e^{-(2 \times 8 \times 10^{-6} \times 5000)}$

$= 0.9985$

(c) $R(c) = R_1(t) R_2(t) R_3(t) + R_1(t) R_2(t) Q_3(t) + R_1(t) Q_2(t) R_3(t) + Q_1(t) R_2(t) R_3(t)$

$R(c) = e^{-(\lambda_1 + \lambda_2 + \lambda_3)t} + e^{-(\lambda_1 + \lambda_2)t}[1 - e^{-\lambda_3 t}] + e^{-(\lambda_1 + \lambda_3)t}[1 - e^{-\lambda_2 t}] + e^{-(\lambda_2 + \lambda_3)t}[1 - e^{-\lambda_1 t}]$

$= 0.9981$

Probability of Surviving 5000 hrs

$= R(a)\, R(b)\, R(c)$

$= 0.9916$

Measures of Reliability :-

1. MTTF [Mean Time to Failure] :-

If all the specimens donot fail at the same time, they have different times to failure. If t_1 is the time to failure for the first and t_2 is the time for the second specimen and t_N is the time to failure for N^{th} specimen.

The MTTF for N specimens will be

$$MTTF = \left[\frac{t_1 + t_2 + \cdots + t_N}{N}\right]$$

$$= \frac{1}{N} \sum_{i=1}^{N} t_i$$

In terms of expected value

$$m = -\int_0^\infty t\, R(t)\, dt$$

Integration by parts.

$$m = \left[-t\, R(t)\right]_0^\infty + \int_0^\infty R(t)\, dt$$

$$\therefore m = \int_0^\infty R(t)\, dt = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}$$

For Series System :-

$$m = \int_0^\infty R_s(t)\, dt$$

$$m = \int_0^\infty \exp\left(-\sum_{i=1}^{N} \lambda_i t\right) dt$$

$$m = \frac{1}{\sum_{i=1}^{N} \lambda_i} = \frac{1}{\lambda_1 + \lambda_2 + \dots + \lambda_N}.$$

For parallel system :-

$$m = \int_0^\infty R_p(t)\, dt$$

$$= \int_0^\infty \exp(-\lambda_1 t) + \exp(-\lambda_2 t) - \exp(-\lambda_1 + \lambda_2) t \; dt$$

$$m = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}.$$

2. MTBF [Mean Time Between Failures] :-

It is the average value of time intervals between successive failures of equipment.

$$MTBF = \frac{\text{Equipment Operating Time}}{\text{Number of observed failures}}$$

For series system :-

$$MTBF = \frac{1}{\lambda_1 + \lambda_2 + \dots + \lambda_n} = \frac{1}{n\lambda}$$

For parallel system :-

$$MTBF = m = \frac{1}{\lambda} + \frac{1}{2\lambda} + \frac{1}{3\lambda} + \dots + \frac{1}{n\lambda}$$

where λ = failure rate

n = Number of Components in parallel

3. MTTR [Mean Time to Repair]:-

It is the mean time required for repair of an equipment. It doesnot include waiting time.

$$MTTR = r = \frac{1}{\mu}$$

$$MTBF = MTTR + MTTF$$

$$T = m + r$$

$$F = \frac{1}{T} = \frac{1}{m+r}$$

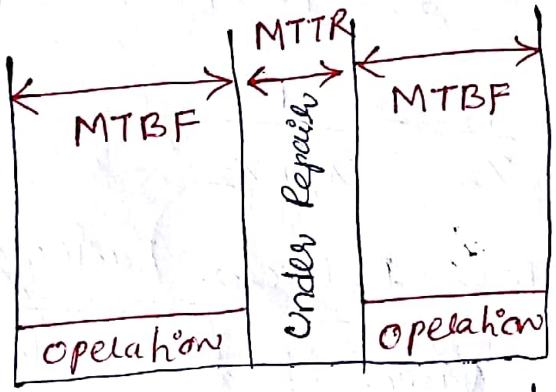
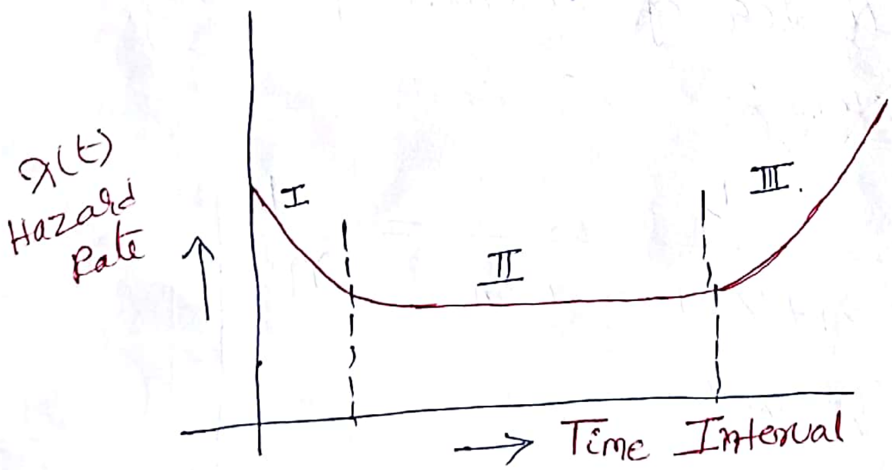


Fig: Represents Measures of Reliability.

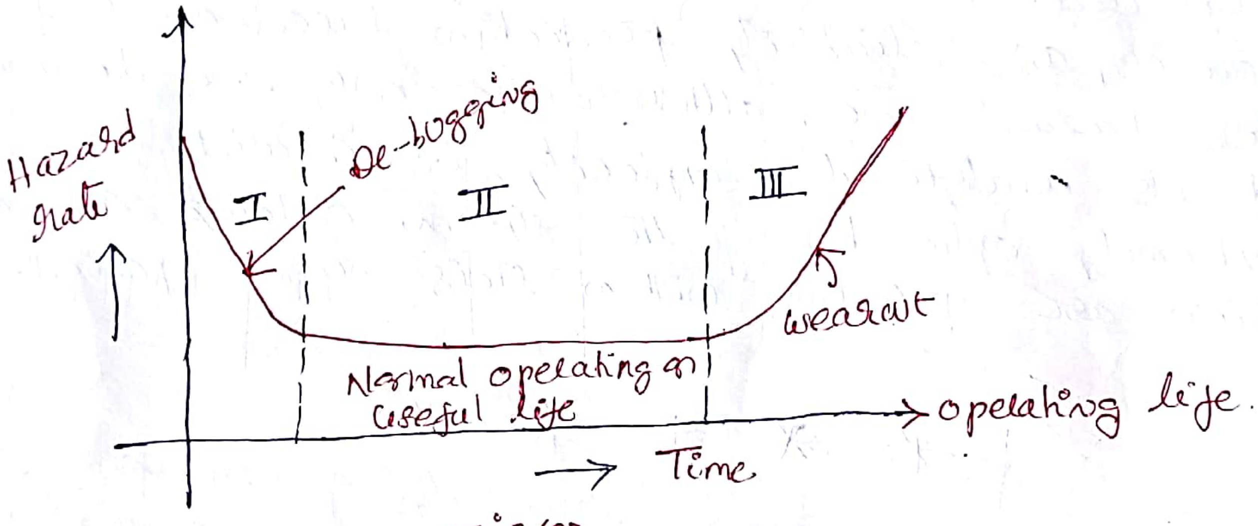
Bath-Tub Curve :-



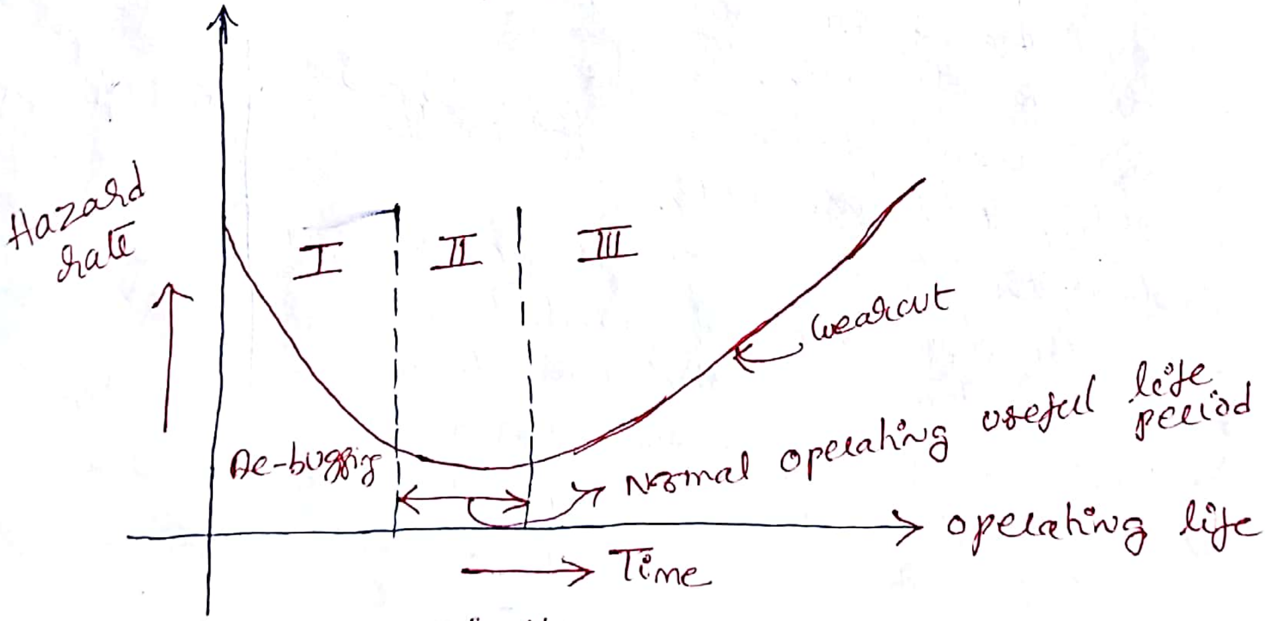
The shape is often referred to as a bath-tub curve for self evident reasons and can generally be divided into 3 distinct regions. Region I is known by various names, such as the infant (baby) mortality or de-bugging phase, and could be due to (defect) manufacturing errors or improper design. In this region the hazard rate decreases as a function of time or age.

Region II is the known as the useful life period or normal operating phase and is characterized by a constant hazard rate. In this region failures occur by chance and this is the only region in which the exponential distribution is valid.

Region III represents the wearout or fatigue (make very tired) phase and is characterized by a rapidly increasing hazard rate with time.



Fig(a)



Fig(b)

The fig(a) and (b) are the typical electronic components and mechanical equipment respectively. In these two examples which generally cover the two extreme cases, electronic components are usually associated with a very brief useful life.

Many components and systems, including power systems components and mechanical devices, can be made to remain within their useful life period for the bulk of their economically feasible life by constant and careful preventive maintenance. In this way, the components of the systems are not allowed to enter an advanced wearout state before they are replaced. This is an extremely important assumption however, as reliability prediction based on useful life hazard rates, although frequently done because of its analytical simplicity, is invalid and extremely optimistic if the system contains components which are operating within their wearout period.

* * * * *

Pbs

1. A Component has a reliability of 0.9 for a mission time of 50hrs. what is the reliability for a mission time of 100hrs.
2. Define hazard function and derive the relation b/w reliability $R(t)$ and $h(t)$.
3. Explain the terms MTTF, MTTR & MTBF.
4. Derive the expressions for Reliability of series & parallel systems in terms of exponential distribution.
5. Evaluate the prob of system survival for a time period of 1000hrs, if at least three out of the five units must be success. If the failure rate of each unit is 0.2×10^{-3} /hrs.