UNIT - IV

MARKOV MODELING

Discrete Markov Chains :-

The markov approach can be applied to the random behaviour of systems that vary discretely or continuously with respect to time and space. This discrete or continuous random variation is known as a stochastic process.

The markov approach is applicable to those systems whose behaviour can be described by a probability distribution i.e. characterized by a constant hazard rate i.e. poisson's and exponential distribution, since only if the hazard rate is constant does the probability of making a transition between two states remain constant at all points of time. If this probability is a function of time or the number of discrete steps, then the process is non-stationary and designated as non-markovian.

In the general case of markov models, both time and space may either be discrete or continuous. In the particular case of system reliability evaluation, space is normally represented only as a discrete function since this represents the discrete and identifiable states in which the system and its components can reside, whereas time may either be discrete or continuous. The discrete case, generally known as Markov chains and the continuous case, generally known as Markov Process.

General Modelling Concept :-

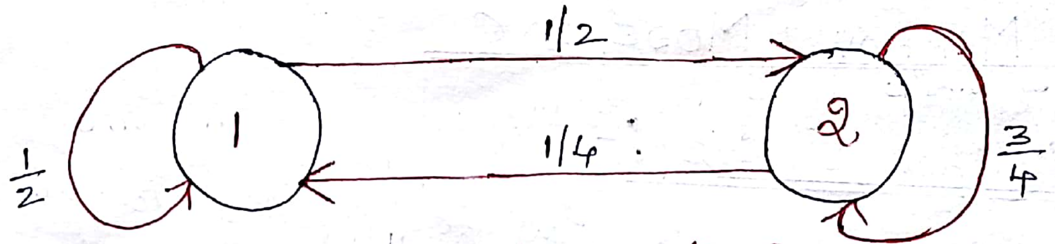

Fig: A two-state system

This is a discrete Markov chain, since the system is stationary and the movement between states occur in discrete steps.
Consider the first time interval and assume that the system is initially in state 1. The system can remain in state 1 with a probability of 1/2 (or) it can move into state 2 with a probability of 1/2.

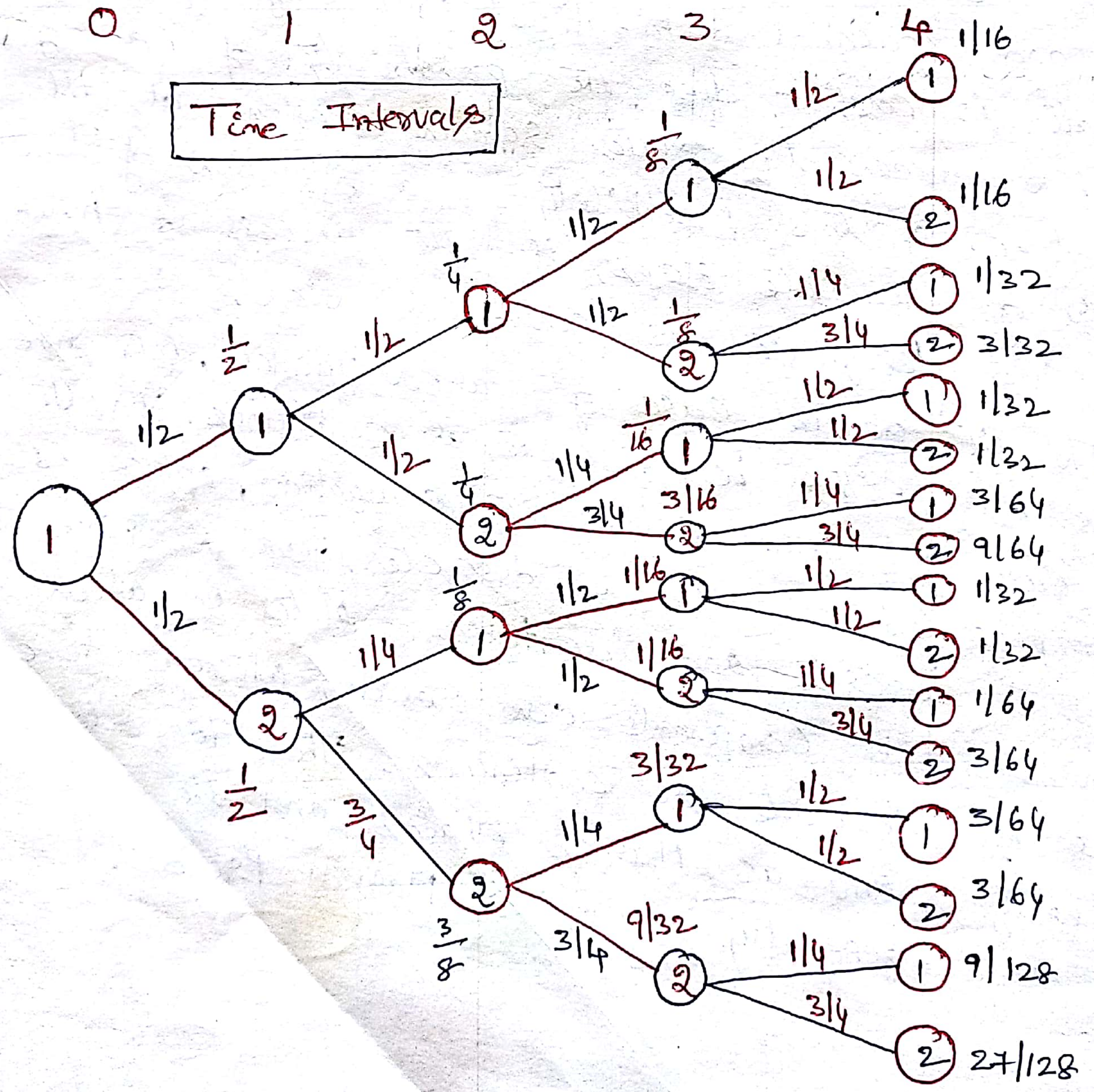

Fig: Tree diagram of the two-state system.

Time Interval	State Probabilities	
Initially in state 1	state 1	state 2
1st time interval	$\frac{1}{2} = 0.5$	$\frac{1}{2} = 0.5$
2nd time interval	$\frac{1}{4} + \frac{1}{8} = \frac{3}{8} = 0.375$	$\frac{1}{4} + \frac{3}{8} = \frac{5}{8} = 0.625$
3rd time interval	$\frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{3}{32} = \frac{11}{32} = 0.344$	$\frac{1}{8} + \frac{3}{16} + \frac{1}{16} + \frac{9}{32} = \frac{21}{32} = 0.656$
4th time interval	$\frac{1}{16} + \frac{1}{32} + \frac{1}{32} + \frac{3}{64} + \frac{1}{32} + \frac{1}{64} + \frac{3}{64} + \frac{9}{128} = \frac{43}{128} = 0.336$	$\frac{1}{16} + \frac{3}{32} + \frac{1}{32} + \frac{9}{64} + \frac{1}{32} + \frac{3}{64} + \frac{3}{64} + \frac{27}{128} = \frac{85}{128} = 0.664$

The tree diagram method is useful technique for illustrating the concept of Markov chains but it is totally impractical of large systems and a large number of times.

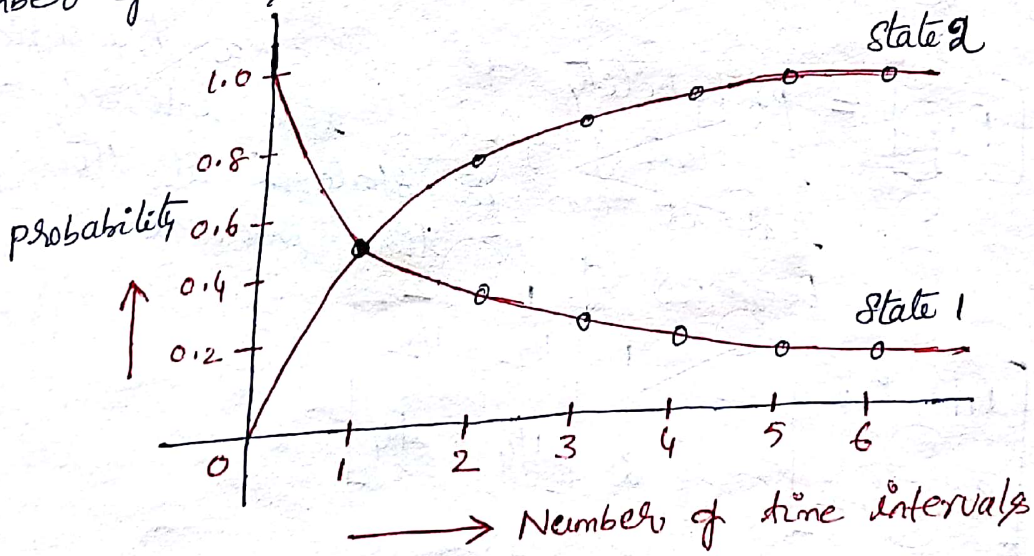

Fig: System Transient Behaviour

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Concept of STPM [Stochastic Transitional Probability Matrix]:

An n-state system, the general form of the matrix, which must be always be square as

$$P = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{matrix} & \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\ P_{31} & P_{32} & P_{33} & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

From state

$\longrightarrow$ to state

This matrix is known as the stochastic transitional probability matrix for the system, since it represents the transitional probabilities of the stochastic process. It should be noted that the summation of the probabilities in each row of the matrix must be unity.

$P = [P_{ij}]$ = probability of making a transition to state j after a time interval given that it was in state i at the beginning of the time interval.

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{array}{c|c} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

$$P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$= \begin{array}{c|c} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix} \end{array}$$

$$P^3 = P^2 \times P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \frac{11}{32} & \frac{21}{32} \\ 2 & \frac{21}{64} & \frac{43}{64} \end{array}$$

Recursive Relation :-

Consider initially $P[0] = [1 \quad 0]$ is known as initial probability vector.

$$P[1] = P(0)P$$

$$= [1 \quad 0] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P[2] = P(1)P$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

Similarly $P[3] = P(2)P$

$\vdots$

$P[N] = P(n-1)P$ Called Recursive Relation

$P[1] = P[0]P$

$P[2] = P[1]P$

$= P[0]P^2$

$P[3] = P[0]P^3$

$$\therefore \boxed{P(n) = P(0)P^n}$$

The repeated multiplication of STPM will converged to a finite value and this finite value is known as limiting state probabilities of the states.

Evaluation of limiting state Probabilities :-

Let P_1, P_2 be the LSP of the state 1 and 2 respectively.

Let α be the LSP vector.

$$\alpha = [P_1 \quad P_2]$$

The solution technique

$$\alpha P = \alpha$$

Consider one component repairable model with 2 states

$$[P_1 \quad P_2] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = [P_1 \quad P_2]$$

$$P_1 P_{11} + P_2 P_{21} = P_1$$

$$[1 - P_{11}] P_1 - P_2 P_{21} = 0 \rightarrow (1)$$

$$P_1 P_{12} + P_2 P_{22} = P_2$$

$$P_1 P_{12} + [P_{22} - 1] P_2 = 0 \rightarrow (2)$$

We know $P_{11} + P_{12} = 1$

$P_{21} + P_{22} = 1$

$$P_{12} P_1 - P_2 P_{21} = 0 \rightarrow (3)$$

$$-P_1 P_{12} + P_{21} P_2 = 0 \rightarrow (4)$$

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out of eq ③ & ④ only one of them is linearly independent but in order to solve for the LSP P_1 and P_2 another equation is required and is given by

$$P_1 + P_2 = 1 \longrightarrow ⑤$$

Writing in matrix form for eq ③ & ⑤

$$\begin{bmatrix} P_{12} & -P_{21} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using Cramer's rule

$$P_1 = \frac{\begin{bmatrix} 0 & -1/4 \\ 1 & 1 \end{bmatrix}}{\frac{3}{4}} = \frac{1}{3}$$

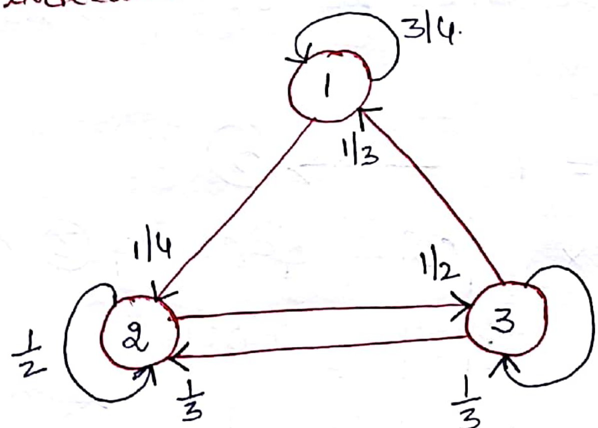
$$P_2 = \frac{\begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 1 \end{bmatrix}}{\frac{3}{4}} = \frac{2}{3}$$

Question

1. Distinguish b/w Markov chain & Markov process with the help of suitable example by writing STPM of each.
2. Write short notes on STPM.
3. Define Markov chains & explain the concept of stochastic transitional probability matrix and evaluation of limiting state probabilities.

Example :-

① Consider a 3 state system shown in fig. with transitional probability indicated. Evaluate the LSPs of each state.



Sol: The steps for solving the problem to be followed are

1. write STPM of various states.
2. check whether sum of each row elements are equal to 1.
3. To evaluate LSPs of state 1 to n

$$\alpha P = \alpha$$

4. Take any $n-1$ equations from the set of n equations.
5. Take n^{th} equation as $P_1 + P_2 + \ldots + P_n = 1$
6. Solve for P_1 to P_n using cramer's rule.

$$\text{STPM} = P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

$$\alpha = [P_1 \quad P_2 \quad P_3]$$

The solution technique is

$$\alpha P = \alpha$$

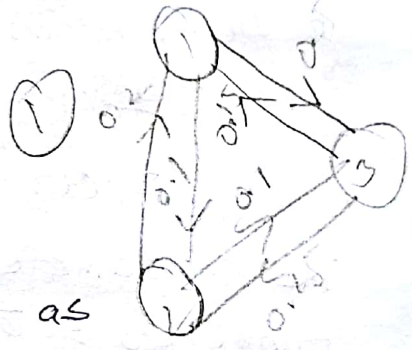
$$[P_1 \quad P_2 \quad P_3]\begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = [P_1 \quad P_2 \quad P_3]$$

$$P_1 \frac{3}{4} + P_3 \frac{1}{3} = P_1$$

$$-\frac{1}{4} P_1 + \frac{P_3}{3} = 0 \rightarrow ①$$

$P_2/2 + P_3/3 = P_3$

$$\frac{P_2}{2} - \frac{2}{3}P_3 = 0 \rightarrow (2)$$

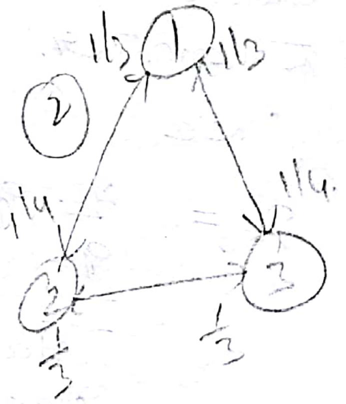


~~out of req (1) x~~

$$P_1 + P_2 + P_3 = 1 \rightarrow (3)$$

Writing eq (1), (2) and (3) in matrix form as

$$\begin{bmatrix} -1/4 & 0 & 1/3 \\ 0 & 1/2 & -2/3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



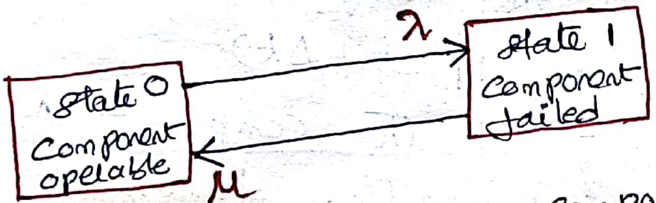
Solving these equations we get

$$P_1 = 4/11, \quad P_2 = 4/11, \quad P_3 = 3/11.$$

Continuous Markov Process :-

Transition Rate Concept :-

Consider the case of a single repairable component for which the failure rate and repair rate are characterized by the exponential distribution. ie they are constant



Let $P_0(t)$ = probability that the component is operable at time t

$P_1(t)$ = probability that the component is failed at time t

λ = failure rate

μ = repair rate

The failure density function for a component with a constant hazard rate of λ is given by

$$f(t) = \lambda e^{-\lambda t}$$

The density functions for the operating and failed states of the system shown in fig are

$$f_0(t) = \lambda e^{-\lambda t}$$
$$f_1(t) = \mu e^{-\mu t}$$

The parameters λ and μ are referred to as state transition rates since they represent the rate at which the system transits from one state of the system to another.

where

$$\lambda = \frac{\text{No. of failures of a component in a given period of time}}{\text{Total period of the time the component was operating.}}$$

$$\mu = \frac{\text{No. of repairs of a component in a given period of time}}{\text{Total period of the time the component is repaired}}$$

Evaluating Time dependent probabilities :-

The probability of the component functioning during the time $(t+\Delta t)$ is $P_1(t+\Delta t)$

= The probability of the component functioning at time t and not failing during t to $t+\Delta t$ + probability of component failed at time t and probability of component goes into operable state during Δt.

Let $P_1(t)$ be the probability of the component being functioning in operable state at time 't'

$P_2(t)$ be the probability of the component being in failed state at time 't'

$$P_1(t+\Delta t) = P_1(t)(1-\lambda\Delta t) + P_2(t)\mu\Delta t$$

$$P_1(t+\Delta t) - P_1(t) = -P_1(t)\lambda\Delta t + P_2(t)\mu\Delta t \quad \text{(or)}$$

$$\lim_{\Delta t \to 0} \frac{P_1(t+\Delta t) - P_1(t)}{\Delta t} = -\lambda P_1(t) + \mu P_2(t)$$

$$P_1'(t) = -\lambda P_1(t) + \mu P_2(t) \longrightarrow (1)$$

Let the probability of the component failing during the time $t+\Delta t$ is defined as $P_2(t+\Delta t)$

= The probability of the component functioning at time 't' and failed during t to $t+\Delta t$ + The probability of the component failed at time 't' and not transiting to other state

$$P_2(t+\Delta t) = P_1(t)\lambda\Delta t + P_2(t)(1-\mu\Delta t)$$

$$P_2(t+\Delta t) - P_2(t) = P_1(t)\lambda\Delta t - P_2(t)\mu\Delta t$$

$$\lim_{\Delta t \to 0} \frac{P_2(t+\Delta t) - P_2(t)}{\Delta t} = \lambda P_1(t) - \mu P_2(t)$$

$$P_2'(t) = \lambda P_1(t) - \mu P_2(t) \longrightarrow (2)$$

1. Evaluation of the expression for $P_1(t)$, $P_2(t)$ (time dependent prob. of each state) using LT approach as

$$\left.\begin{aligned} t\to\infty \quad P_1(t) &= P_1 \\ t\to\infty \quad P_2(t) &= P_2 \end{aligned}\right\} \text{ are LSP of the states 1 and 2 respectively.}$$

2. To evaluate LSP using
   (a) Differential Equation Approach
   (b) STPM Approach ($\alpha P = \alpha$)

From eq. (1) taking LT

$$sP_1(s) - P_1(0) = -\lambda P_1(s) + \mu P_2(s)$$

$$(s+\lambda)P_1(s) - \mu P_2(s) = P_1(0) \rightarrow (3)$$

From eq (2) taking LT

$$sP_2(s) - P_2(0) = \lambda P_1(s) - \mu P_2(s)$$

$$P_2(s)[s+\mu] - \lambda P_1(s) = P_2(0) \longrightarrow (4)$$

writing eqs. (3) and (4) in matrix form

$$\begin{bmatrix} s+\lambda & -\mu \\ -\lambda & s+\mu \end{bmatrix} \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix}$$

Applying Cramer's rule.

$$P_1(s) = \frac{\begin{bmatrix} P_1(0) & -\mu \\ P_2(0) & s+\mu \end{bmatrix}}{\Delta} = \frac{(s+\mu)P_1(0) + \mu P_2(0)}{\Delta}$$

where $\Delta = (s+\lambda)(s+\mu) - \lambda\mu$

$$= s^2 + s\mu + s\lambda + \lambda\mu - \lambda\mu$$

$$(5) \leftarrow = s^2 + (\mu+\lambda)s$$

$$= s[s + (\lambda+\mu)]$$

$$P_1(s) = \frac{sP_1(0) + \mu[P_1(0) + P_2(0)]}{s[s+(\lambda+\mu)]}$$

Assuming $P_1(0) = 1$ and $P_2(0) = 0$

$$P_1(s) = \underbrace{\frac{s}{s+(\lambda+\mu)}}_{(1)} + \underbrace{\frac{\mu}{s[s+(\lambda+\mu)]}}_{(2)}$$

Using Heavy side partial fraction expansion theorem

$$\frac{\mu}{s[s+(\lambda+\mu)]} = \frac{A}{s} + \frac{B}{s+(\lambda+\mu)}.$$

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$$A = \frac{\mu}{s+(\lambda+\mu)}\Big|_{s=0} = \frac{\mu}{\lambda+\mu}$$

$$B = \mu/s\Big|_{s=-(\lambda+\mu)} = \frac{\mu}{-(\lambda+\mu)}$$

Using inverse laplace transformation

$$= \frac{\mu}{(\lambda+\mu)} - \frac{\mu}{(\lambda+\mu)} e^{-(\lambda+\mu)t}$$

$$P_1(t) = \cancel{P_1(0)}\, e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}\left[1 - e^{-(\lambda+\mu)t}\right]$$

$$= e^{-(\lambda+\mu)t}\left[1 - \mu/\lambda+\mu\right] + \frac{\mu}{\lambda+\mu}$$

$$P_1(t) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} \longrightarrow (5)$$

$$P_2(s) = \frac{\begin{bmatrix} s+\lambda & P_1(0) \\ -\lambda & P_2(0) \end{bmatrix}}{\Delta} = \frac{(s+\lambda)P_2(0) + \lambda P_1(0)}{s[s+\lambda+\mu]}$$

$$= \frac{s\,P_2(0) + \lambda[P_1(0) + P_2(0)]}{s[s+(\lambda+\mu)]}$$

$$= \frac{P_2(0)}{s+(\lambda+\mu)} + \frac{\lambda[P_1(0)+P_2(0)]}{s[s+(\lambda+\mu)]}$$

Taking inverse LT.

$$P_2(t) = P_2(0)\, e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

Assuming $P_1(0) = 1$ and $P_2(0) = 0$

$$\therefore P_2(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \longrightarrow (6)$$

Let LSP of states ① and ② be P_1 and P_2

$$\lim_{t \to \infty} P_1(t) = P_1 \quad \& \quad \lim_{t \to \infty} P_2(t) = P_2$$

Substituting the above in eq ⑤ and ⑥, we get

$$P_1 = \frac{\mu}{\lambda + \mu}, \quad P_2 = \frac{\lambda}{\lambda + \mu}.$$

2. Differential Equation Approach :-

We know

$$P_1'(t) = -\lambda P_1(t) + \mu P_2(t)$$

$$P_2'(t) = \lambda P_1(t) - \mu P_2(t)$$

As $t \to \infty$ $P_1'(t) = 0$ & $P_2'(t) = 0$

$$0 = -\lambda P_1(t) + \mu P_2(t) \to ⑦$$

$$0 = \lambda P_1(t) - \mu P_2(t) \to ⑧$$

The above equations are not linearly independent

$$\therefore P_1 + P_2 = 1 \to ⑨$$

$$\therefore -\lambda P_1(t) + \mu P_2(t) = 0$$

$$P_1 + P_2 = 1$$

$$\begin{bmatrix} -\lambda & \mu \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P_1(t) = \frac{\begin{bmatrix} 0 & \mu \\ 1 & 1 \end{bmatrix}}{-(\lambda + \mu)} = \frac{\mu}{\lambda + \mu}$$

$$P_2(t) = \frac{\begin{bmatrix} -\lambda & 0 \\ 1 & 1 \end{bmatrix}}{-(\lambda + \mu)} = \frac{\lambda}{\lambda + \mu}.$$

3. Stochastic Transitional Probability Matrix [STPM]

To evaluate limiting state probabilities using STPM approach

$$\alpha = [P_1 \quad P_2]$$

$$P = \begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$$

$$\alpha P = \alpha$$

$$[P_1 \quad P_2]\begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix} = [P_1 \quad P_2]$$

$$P_1[1-\lambda\Delta t] + \mu\Delta t\, P_2 = P_1$$

$$-\lambda\Delta t\, P_1 + \mu\Delta t\, P_2 = 0$$

$$-\lambda P_1 + \mu P_2 = 0 \rightarrow (1)$$

$$\lambda\Delta t\, P_1 + P_2(1-\mu\Delta t) = P_2$$

$$\lambda\Delta t\, P_1 - \mu\Delta t\, P_2 = 0$$

$$\lambda P_1 - \mu P_2 = 0 \rightarrow (2)$$

The above equations are not linearly independent

$$\therefore P_1 + P_2 = 1 \rightarrow (3)$$

Solving eqns. (1) and (3), we get

$$\begin{bmatrix} -\lambda & \mu \\ 1 & 1 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P_1 = \frac{\begin{bmatrix} 0 & \mu \\ 1 & 1 \end{bmatrix}}{-(\lambda+\mu)} = \frac{\mu}{\lambda+\mu} \qquad P_2 = \frac{\begin{bmatrix} -\lambda & 0 \\ 1 & 1 \end{bmatrix}}{-(\lambda+\mu)} = \frac{\lambda}{\lambda+\mu}$$

Two - Component Repairable System :-

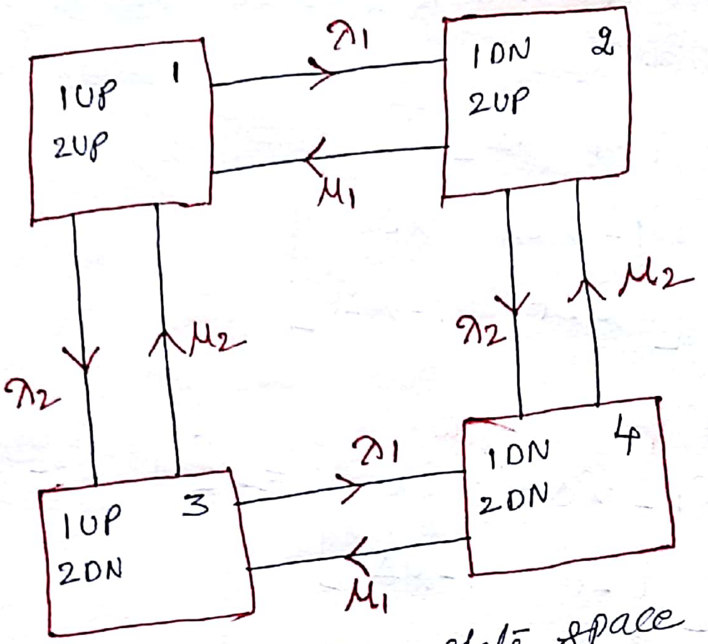

Fig. (1) : Complete state space diagram

Evaluation of LSP's :-

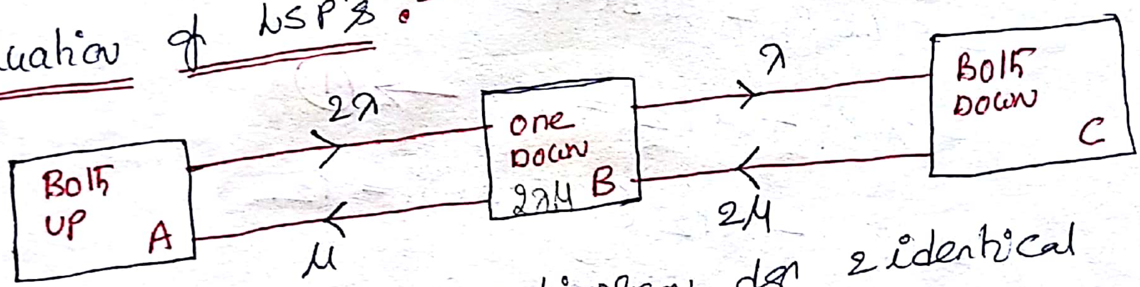

Fig. (2) : State space diagram for 2 identical Components

Minimum No. of states for n-identical Components is (n+1) states.

2^n maximum number of states for non-identical 'n' components.

$$\lambda_1 = \lambda_2 = \lambda \quad \text{[Assume Identical Components]}$$

$$\mu_1 = \mu_2 = \mu$$

To evaluate LSPs of states ABC :-

$$STPM = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 1-2\lambda & 2\lambda & 0 \\ \mu & 1-(\lambda+\mu) & \lambda \\ 0 & 2\mu & 1-2\mu \end{bmatrix} \end{array}$$

The solution is $\alpha P = \alpha$

where α is LSP vector

$$\alpha = [P_A \quad P_B \quad P_C]$$

$$[P_A \quad P_B \quad P_C] \begin{bmatrix} 1-2\lambda & 2\lambda & 0 \\ \mu & 1-(\lambda+\mu) & \lambda \\ 0 & 2\mu & 1-2\mu \end{bmatrix} = [P_A \quad P_B \quad P_C]$$

$$(1-2\lambda) P_A + P_B \mu = P_A$$

$$-2\lambda P_A + P_B \mu = 0 \longrightarrow (1)$$

$$2\lambda P_A + [1-(\lambda+\mu)] P_B + 2\mu P_C = P_B$$

$$2\lambda P_A - (\lambda+\mu) P_B + 2\mu P_C = 0 \longrightarrow (2)$$

$$\lambda P_B + (1-2\mu) P_C = P_C$$

$$\lambda P_B - 2\mu P_C = 0 \longrightarrow (3)$$

Writing eq (1), (2) and (3) in matrix form

$$\begin{bmatrix} -2\lambda & \mu & 0 \\ 2\lambda & -(\lambda+\mu) & 2\mu \\ 0 & \lambda & -2\mu \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

out of which only two of them linearly independent.

The linear equation is

$$P_A + P_B + P_C = 1.$$

$$\begin{bmatrix} -2\lambda & \mu & 0 \\ 0 & \lambda & -2\mu \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_A \\ P_B \\ P_C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving using Cramer's rule

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$$P_A = \frac{1}{\Delta}\begin{bmatrix} 0 & \mu & 0 \\ 0 & \lambda & -2\mu \\ 1 & 1 & 1 \end{bmatrix} = -2\mu^2$$

$$\Delta = -2\lambda(\lambda+2\mu) - \mu(2\mu)$$
$$= -2(\lambda+\mu)^2$$

$$P_A = \frac{\mu^2}{(\lambda+\mu)^2}$$

$$P_B = \frac{1}{\Delta}\begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & 0 & -2\mu \\ 1 & 1 & 1 \end{bmatrix} = 4\lambda\mu$$

$$P_B = \frac{-4\lambda\mu}{-2(\lambda+\mu)^2} = \frac{2\lambda\mu}{(\lambda+\mu)^2}$$

$$P_C = \frac{1}{\Delta}\begin{bmatrix} -2\lambda & \mu & 0 \\ 0 & \lambda & 0 \\ 1 & 1 & 1 \end{bmatrix} = -2\lambda^2$$

$$P_C = \frac{\lambda^2}{(\lambda+\mu)^2}$$

(2) Now, consider fig (1) i.e, for 2-component repairable model with non-identical components.

Let P_1, P_2, P_3, P_4 be the LSPs states 1 to 4

$$P_1 = \frac{\mu_1}{(\lambda_1+\mu_1)} \cdot \frac{\mu_2}{(\lambda_2+\mu_2)}$$

$$P_2 = \frac{\lambda_1}{(\lambda_1+\mu_1)} \cdot \frac{\mu_2}{(\lambda_2+\mu_2)}$$

$$P_3 = \frac{\lambda_2}{(\lambda_1+\mu_1)} \cdot \frac{\mu_1}{(\lambda_2+\mu_2)}$$

$$P_4 = \frac{\lambda_1}{(\lambda_1+\mu_1)} \cdot \frac{\lambda_2}{(\lambda_2+\mu_2)}$$

① 2- Component identical model.

Series system :-

$$R_S = P_A$$
$$Q_S = P_B + P_C = 1 - P_A$$

parallel system :-

$R_P = P_A + P_B$ — fully redundant system.

$Q_P = P_C \rightarrow$ unavailability of the system.

② 2- Component unidentical model :-

Series system :-

$$R_S = P_1$$
$$Q_S = 1 - P_1$$

parallel system :-

$$R_P = P_1 + P_2 + P_3$$
$$Q_P = P_4$$

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3- Component Repairable Model :-

Let P_1 to P_8 be the limiting state probabilities of states 1 to 8.

10, 11, 28.

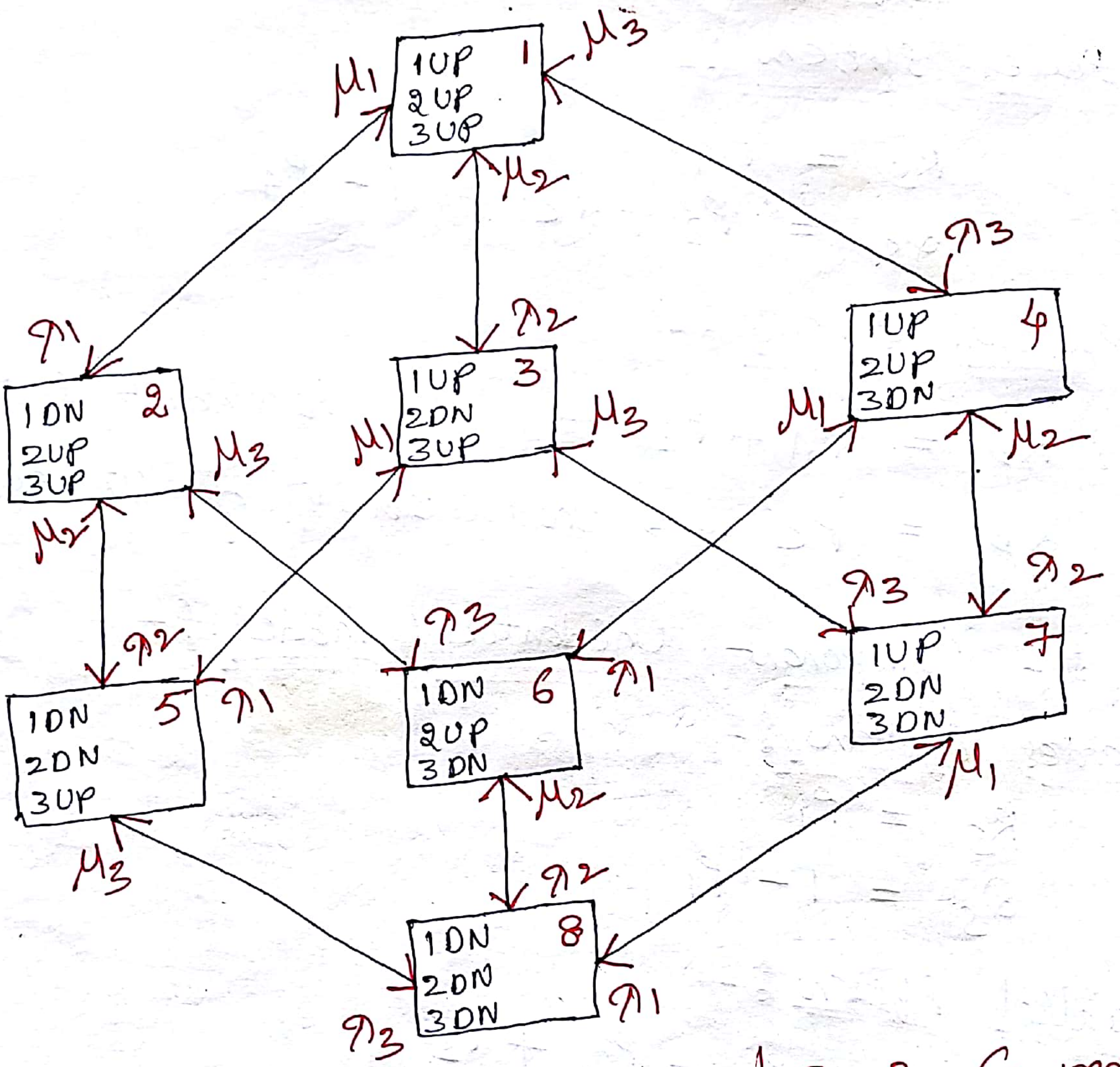


Fig. : State space diagram for 3 - Component repairable model.

Series System :-

$$R_S = P_1$$

$$Q_S = 1 - P_1$$

$$= P_2 + P_3 + P_4 + P_5 + P_6 + P_7 + P_8$$

Parallel System :-

$$Q_P = P_8$$

$$R_P = P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = 1 - P_8$$

Partially Redundant System :-

Atleast 2 out of 3 must function

$$R_S = P_1 + P_2 + P_3 + P_4$$

$$Q_S = P_5 + P_6 + P_7 + P_8.$$

1(a)

1. Explain the methods of calculating the steady state probabilities for a single component with repair by two state Markov process.

2. 2(b) Develop the expression for time dependent probability of a one component repairable model in terms of failure and repair rates of the component.

3. Develop the state space model of two component repairable system and hence obtain the expressions for limiting probabilities of the states if the states have identical transitional rates and the components have identical capacities.

2(a)

4. How the reliability evaluation is done for two-component repairable models with an example.

2(b).

5. For the markov process of a two state system determine the availability of each state as a function of time

1(c)

UNIT - V

3(a) 1. Explain how cumulative probability and cumulative frequency is evaluated for merged states.

2. (i) Write short notes on frequency and duration concept.

3. (ii) Explain one, two component separable models reliability evaluation using the concept of frequency of encountering states.

4. Define cumulative probability and explain the cumulative frequency of failure evaluation in generation of system reliability analysis. 3(b)

5. Develop the expression for evaluation of cumulative probabilities and cumulative frequencies of an n-component separable model. 3(c)

V-IV

6. The STPM of a discrete state system is given below.

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

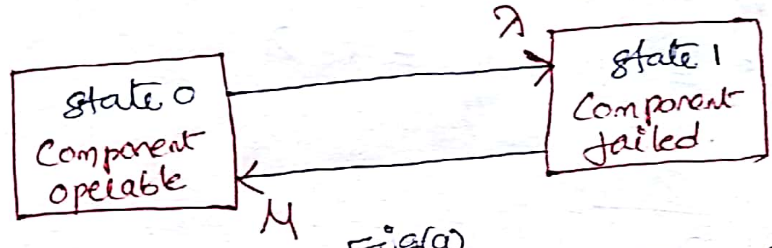
$P_1 = 0.3214$
$P_2 = 0.25$
$P_3 = 0.4285$

(1) Draw state space diagram
(2) Compute the LSPs of each state

FREQUENCY & DURATION TECHNIQUES

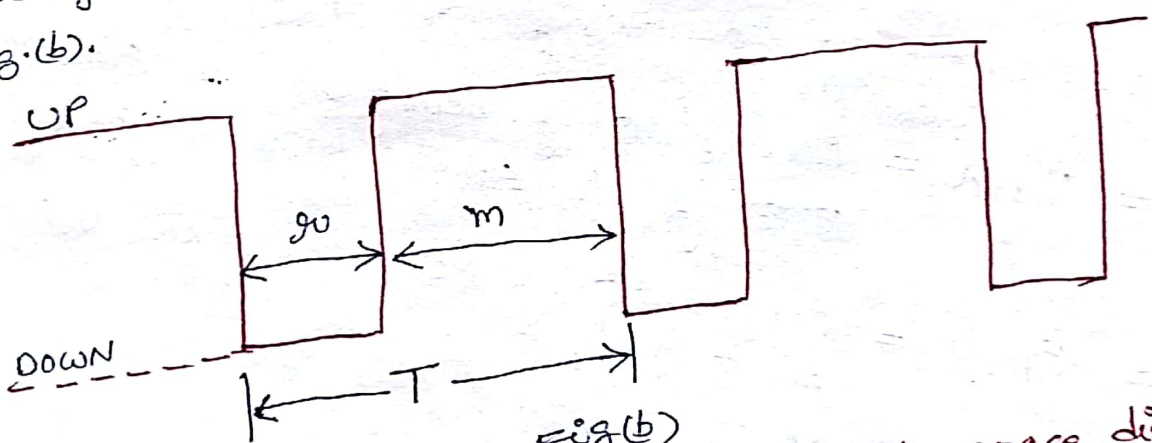
Frequency & Duration Concept :-

The basic concept associated with the frequency and duration technique are best in terms of the single repairable component used to describe the continuous Markov Process. The state space diagram of this system is shown in fig(a).



Fig(a)

The two system states and their associated transitions can be shown chronologically on a time graph. The mean values of up and down times can be used to give the average performance of this two state system shown in fig.(b).



Fig(b)

Fig.: Single Component System (a) State space diagram (b) Mean time/state diagram

Where λ = failure rate of the component
μ = repair rate of the component
m = mean operating time of the component
r = mean repair time of the component.

The probability of residing in the operable state (availability) and the probability of residing in the failed state (unavailability) given as

$$P_0 = \frac{\mu}{\lambda+\mu}$$

$$= \frac{\frac{1}{r}}{\frac{1}{m}+\frac{1}{r}} = \frac{m}{m+r} = \frac{m}{T} = mf$$

$$\therefore P_0 = mf$$

$$f = P_0 \times \frac{1}{m} = P_0 \lambda$$

$$P_1 = \frac{\lambda}{\lambda+\mu}$$

$$= \frac{1/m}{\frac{1}{m}+\frac{1}{r}} = \frac{r}{m+r} = \frac{r}{T} = rf.$$

$$\therefore P_1 = rf \Rightarrow f = P_1 \times \frac{1}{r} = P_1 \mu.$$

$\therefore$ Frequency of encountering the up state

$= P_0 \cdot \lambda =$ Probability of being in the state $\times$ Rate of departure from the state

$= P_1 \mu =$ Probability of NOT being in the state $\times$ Rate of entry into the state.

2. Two-Component Repairable Model :-

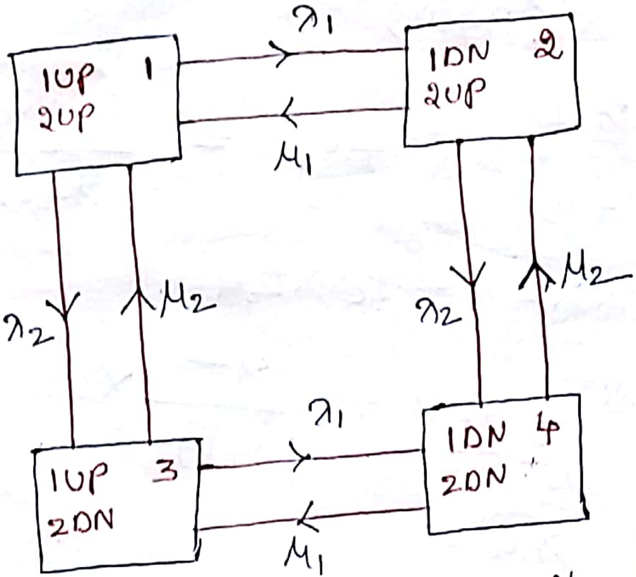
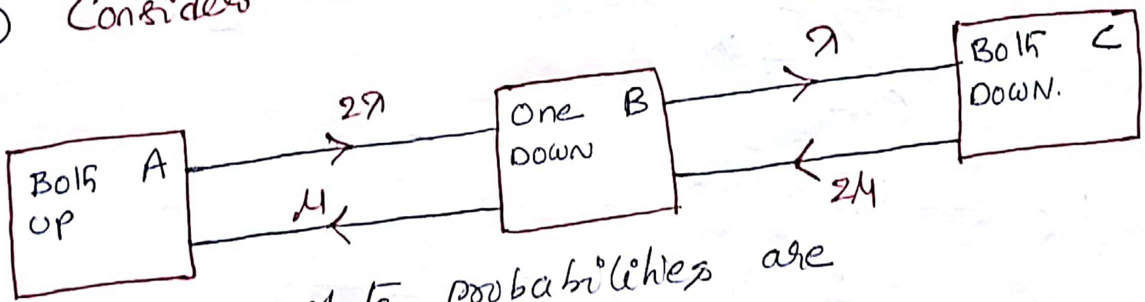


Fig.: Two Component Repairable Model. State space diagram

(a) Consider Identical Components.



The limiting state probabilities are

$$P_A = \mu^2 / (\lambda+\mu)^2$$

$$P_B = 2\lambda\mu / (\lambda+\mu)^2$$

$$P_C = \lambda^2 / (\lambda+\mu)^2$$

Let f_A, f_B and f_C be the frequency of encountering states A, B and C respectively.

$$f_A = P_A \times \text{rate of departure}$$

$$= \frac{\mu^2}{(\lambda+\mu)^2} \cdot 2\lambda \quad \text{(or)}$$

$$f_A = P_B \times \text{rate of entry} = P_B \times \mu$$

$$= \frac{2\lambda\mu}{(\lambda+\mu)^2} \times \mu = \frac{2\lambda\mu^2}{(\lambda+\mu)^2}$$

$$f_B = P_B \times \text{rate of departure}$$
$$= P_B \times (\lambda+\mu)$$
$$= \frac{2\lambda\mu}{(\lambda+\mu)^2} \times (\lambda+\mu) = \frac{2\lambda\mu}{(\lambda+\mu)}$$

$$f_B = P_A \times 2\lambda + P_C \times 2\mu$$
$$= \frac{\mu^2}{(\lambda+\mu)^2} \times 2\lambda + \frac{\lambda^2}{(\lambda+\mu)^2} \times 2\mu$$
$$= \frac{2\lambda\mu}{(\lambda+\mu)}$$

$$f_C = P_C \times \text{rate of departure}$$
$$= \frac{\lambda^2}{(\lambda+\mu)^2} \times 2\mu$$

$$P_B \times \text{rate of entry} = P_B \times \lambda$$

$$f_C = P_B \times \text{rate of entry}$$
$$= \frac{2\lambda\mu}{(\lambda+\mu)^2} \times \lambda = \frac{2\lambda^2\mu}{(\lambda+\mu)^2}$$

Mean cycle time

$$T_A = \frac{1}{f_A} = \frac{(\lambda+\mu)^2}{2\lambda\mu^2}, \quad T_B = \frac{1}{f_B} = \frac{(\lambda+\mu)}{2\lambda\mu}$$

$$T_C = \frac{1}{f_C} = \frac{(\lambda+\mu)^2}{2\lambda^2\mu}$$

State No.	Rate of Departure	Rate of Entry
A	2λ	μ
B	$\lambda+\mu$	$2(\lambda+\mu)$
C	2μ	λ

Let f_1, f_2, f_3 and f_4 be the frequency of encountering states 1 to 4 respectively.

The LSPs of the states 1 to 4 i.e., P_1 to P_4 can be expressed as

$$P_1 = \frac{\mu_1 \mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$P_2 = \frac{\lambda_1 \mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$P_3 = \frac{\mu_1 \lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$P_4 = \frac{\lambda_1 \lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}.$$

$$f_1 = P_1 \times \text{rate of departure}$$

$$= \frac{\mu_1 \mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \times (\lambda_1+\lambda_2)$$

$$f_1 = P_2 \mu_1 + P_3 \mu_2$$

$$= \frac{\lambda_1 \mu_1 \mu_2}{(\lambda_1+\mu_1)} + \frac{\lambda_2 \mu_1 \mu_2}{(\lambda_2+\mu_2)}.$$

Similarly. $f_2 = P_2 \times \text{rate of departure}$

$$= P_2 [\lambda_2+\mu_1]$$

$$= \frac{\lambda_1 \mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \times (\lambda_2+\mu_1)$$

$$f_2 = P_1 \lambda_1 + P_4 \mu_2$$

$f_3 = P_3 \times$ rate of departure

$= P_3 \times (\lambda_1 + \mu_2)$

$$= \frac{\mu_1 \lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \times (\lambda_1+\mu_2)$$

$f_3 = P_1 \lambda_2 + P_4 \mu_1$

$f_4 = P_4 \times [\mu_1 + \mu_2] = P_4 \times$ rate of departure

$$= \frac{\lambda_1 \lambda_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \times [\mu_1+\mu_2]$$

$f_4 = P_2 \lambda_2 + P_3 \lambda_1$

Mean Duration

$m_1 = \frac{1}{\lambda_1+\lambda_2}$ $\quad m_2 = \frac{1}{\mu_1+\lambda_2}$ $\quad m_3 = \frac{1}{\lambda_1+\mu_2}$

$m_4 = \frac{1}{\mu_1+\mu_2}$

State	Rate of Departure	Rate of Entry
1	$\lambda_1+\lambda_2$	$\mu_1+\mu_2$
2	$\lambda_2+\mu_1$	$\lambda_1+\mu_2$
3	$\lambda_1+\mu_2$	$\lambda_2+\mu_1$
4	$\mu_1+\mu_2$	$\lambda_1+\lambda_2$

Evaluation of Cumulative Probability, Cumulative Frequency & Equivalent Transition Rates :-

1. First obtain the individual state probabilities of each of state $P_i \; \forall \; i = 1$ to 2^n.
   where n is number of generating units. and obtain the capacity outage probability table.

2. Let f_i = frequency of encountering a state 'i' $\forall \; i = 1$ to 2^n
   = $P_i \times$ rate of transition

3. Let the identical capacity states be 'K' in number of the available states (K^th combined states).
   In each of the K^th combined state, the capacity of K^th combined state will be identical.
   Let 'r' such capacity states are present for K^th combined state.

$$C_1 = C_2 = \text{------} = C_r \text{ for } K^{th} \text{ combined state.}$$

The combined probability of state 'K' is

$$P_K = \sum_{i=1}^{r} P_i$$

The frequency of the combined state K is

$$f_K = \sum_{i=1}^{r} f_i$$

$$= \sum_{i=1}^{r} P_i \times \text{rate of transition}$$

4. Equivalent Transitional Rates :- Equivalent failure rate :-

$$\lambda_{K-L} = \frac{\sum_{i=1}^{r} P_i \lambda_{ij}}{P_K}$$

$$\lambda_{K-L} = \frac{\sum_{i=1}^{r} P_i \sum_{j \in L} \lambda_{ij}}{P_K = \sum_{i=1}^{r} P_i}$$

where λ_{K-L} is K^{th} combined state to its lower state (L).

Equivalent Repair Rate :-

$$\mu_{K-H} = \frac{\sum_{i=1}^{r} P_i \sum_{j \in H} \mu_{ij}}{P_K = \sum_{i=1}^{r} P_i}$$

where μ_{K-H} is K^{th} combined state to its higher margin states (H)

5. Cumulative Probabilities :-

Let there be 'N' combined states in the combined state model [lies b/w $(n+1)$ to 2^n]' where n is the number of components.

Let $P_A, P_B \ldots\ldots P_N$ be the cumulative probabilities of the combined states A, B to N respectively.

Where A is highest available capacity state

N is lowest available capacity state

P_{N-1} = Cumulative probability upto $(N-1)^{th}$ state starting from N^{th} state

$\therefore P_{N-1} = P_N + P_K$

6. Cumulative Frequency :-

Higher order　　Lower order

$$F_{N-1} = F_N + P_K [\mu_{K-H} - \lambda_{K-L}]$$

$$= F_N + P_K [(n+K) - (n-K)]$$

32. The prob density function

$$f(x) = 2000/x^3 \quad , x > 100$$
$$= 0 \quad \text{else where}$$

Expected life $E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{100}^{\infty} x\, f(x)\, dx$$

$$= \int_{100}^{\infty} x \cdot \frac{2000}{x^3}\, dx$$

$$= \int_{100}^{\infty} \frac{2000}{x^2}\, dx$$

$$= 2000 \left[\frac{x^{-2+1}}{-2+1} \right]_{100}^{\infty} = -2000 \left[0 - \frac{1}{100} \right]$$

Expected life $= 20$ years

1. A generating station has 3 generators, two rated for 30 MW and the third for 40 MW. The failure and repair rate of each unit are 0.5 f/year and 9.5 r/year respectively. Obtain the state space diagram and mark various transition rates. Hence evaluate the cumulative probability and cumulative frequency of various merged states.

Sol

1	2	3	Total
30 MW	30 MW	40 MW	100

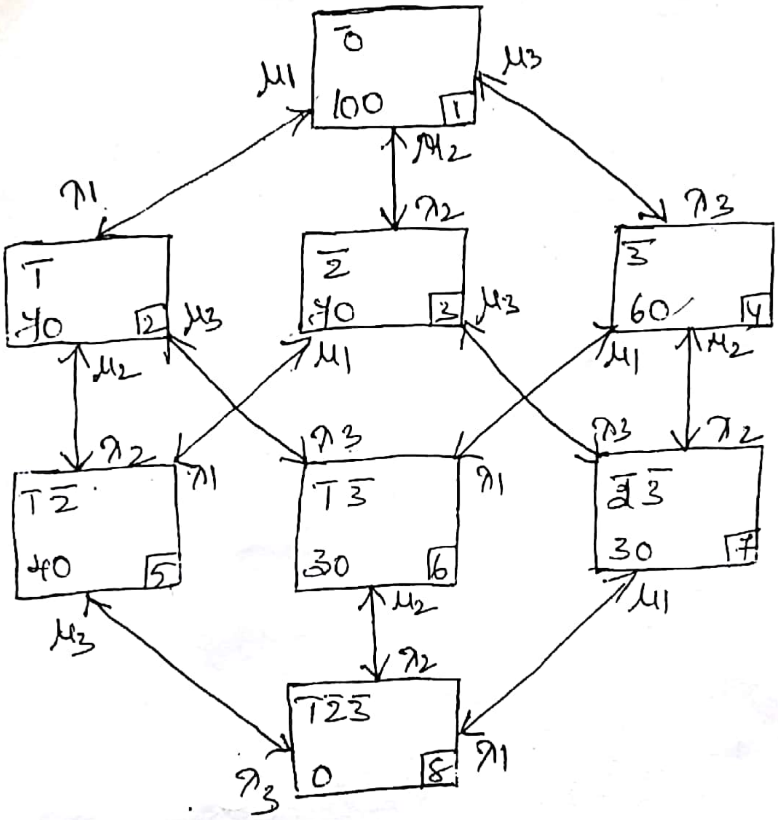
$\lambda_1 = \lambda_2 = \lambda_3 = 0.5$ f/yr

$\mu_1 = \mu_2 = \mu_3 = 9.5$ r/yr

$A = \mu/\lambda+\mu = 0.95 \quad U = 1 - A = 0.05$

f_6

f_8



The probabilities are

$$P_1 = A^3 = (0.95)^3 = 0.857375$$

$$P_2 = P_3 = P_4 = A^2U = (0.95)^2 \times 0.05 = 0.045125$$

$P_2+P_3+P_4 = A^2 \dots$ $= 3A^2$

$$P_5 = P_6 = P_7 = U^2A = (0.05)^2 \times 0.95 = 0.002375$$

$$P_8 = U^3 = (0.05)^3 = 0.000125$$

Frequency of encountering to each state

$f_1 = P_1 \times$ rate of transition
$= P_1[\lambda_1 + \lambda_2 + \lambda_3] = 1.2860625$

$f_2 = P_2[\lambda_2 + \lambda_3 + \lambda_4]$
$= 0.438125$

$f_3 = P_3[\lambda_1 + \lambda_3 + \mu_2] = 0.438125$

$f_4 = P_4[\lambda_1 + \lambda_2 + \lambda_3] = 0.438125$

$f_5 = P_5[\lambda_3 + \mu_1 + \mu_2] = 0.0463125$

Equi

λ

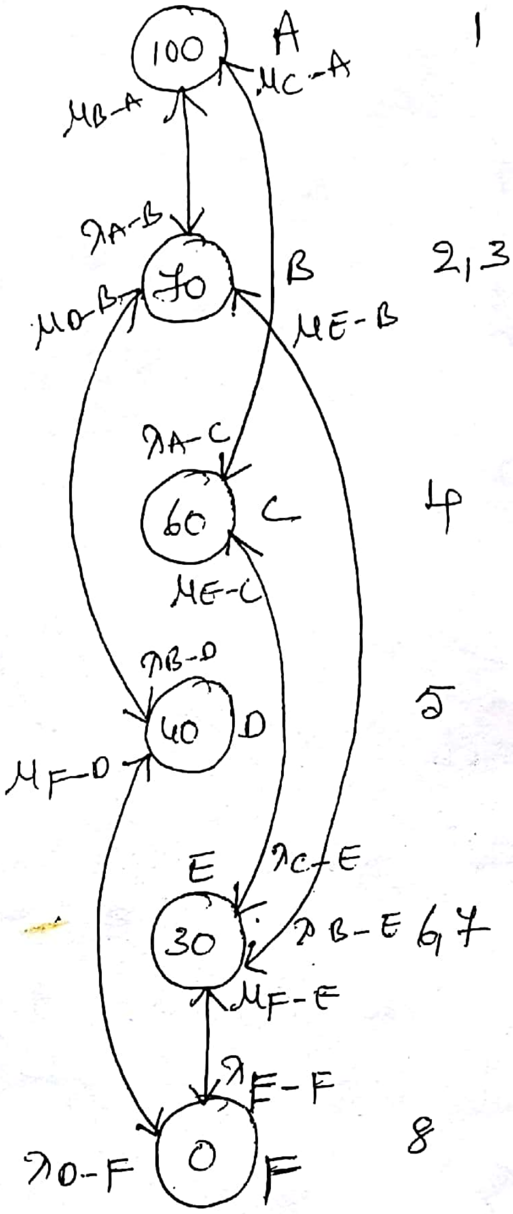
λ

λ_B

$2\lambda\,3$

$f_6 = P_6[\lambda_2 + \mu_1 + \mu_3] = 0.0463125 = f_7$

$f_8 = P_8[\mu_1 + \mu_2 + \mu_3] = 0.0035625$



Equivalent transition rates :-

$$\lambda_{A-B} = \frac{P_1[\lambda_1 + \lambda_2]}{P_1} = 1.0$$

1 - 2,3

$$\lambda_{A-C} = \frac{P_1[\lambda_3]}{P_1} = 0.5$$

1 - 4

$$\lambda^{2,3-5}_{B-D} = \frac{P_2[\lambda_2] + P_3[\lambda_1]}{\cancel{P_2 + P_2}\ P_2 + P_3} = 0.5$$

Individual states	Combined states	Cap out	Cap in	probability P_K	Probability λ_F
ϕ					
1	A $P_A = P_1$	0	100	P_1 0.857375	0.286025 $\rightarrow \lambda_1$
2, 3	B $P_B = P_2 + P_3$	30	70	P_2+P_3 0.07025	0.747625 $\rightarrow \lambda_2+\lambda_3$
4	C $P_C = P_4$	40	60	P_4 0.045125	0.6438125 $\rightarrow \lambda_4$
5	D $P_D = P_5$	60	40	P_5 0.002375	0.0063125 $\rightarrow \lambda_5$
6, 7	E $P_E = P_6 + P_7$	70	30	P_6+P_7 0.00675	0.092625 $\rightarrow \lambda_6+\lambda_7$
8	F $P_F = P_8$	100	0	P_8 0.000125	0.0035625 $\rightarrow \lambda_8$

$$\lambda_{\substack{C-E \\ 4 - 6,7}} = \frac{P_4 [\lambda_1 + \lambda_2]}{P_4} = 1.0$$

$$\lambda_{\substack{D-F \\ 5-8}} = \frac{P_5 [\lambda_3]}{P_5} = 0.5$$

$$\lambda_{\substack{B-E \\ 2,3 - 6,7}} = \frac{P_2 [\lambda_3 + 0] + P_3 [\lambda_3]}{P_2 + P_3}$$

$$= 0.5$$

$$\lambda_{\substack{E-F \\ 6,7 - 8}} = \frac{P_6 (\lambda_2) + P_7 (\lambda_1)}{P_6 + P_7}$$

$$= 0.5.$$

Equivalent Repair Rates :

$$\mu_{\substack{F-E \\ 8 - 6,7}} = \frac{P_8 [\mu_2 + \mu_1]}{P_8} = 19$$

$$\mu_{\substack{F-D \\ 8-5}} = \frac{P_8 [\mu_3]}{P_8} = 9.5$$

$$\mu_{\substack{E-C \\ 6,7 - 4}} = \frac{P_6 [\mu_1] + P_7 [\mu_2]}{P_6 + P_7}$$

$$= 9.5$$

$$\mu_{\substack{C-A \\ 4-1}} = \frac{P_4 [\mu_3]}{P_4} = 9.5$$

$$\mu_{\substack{E-B \\ 6,7 - 2,3}} = \frac{P_6 [\mu_3] + P_7 [\mu_3]}{P_6 + P_7} = 9.5$$

$$\mu_{B-A} = \frac{P_2[\mu_1] + P_3[\mu_2]}{P_2 + P_3} = 9.5$$
2,3 - 1

$$\mu_{D-B} = \frac{P_5[\mu_2 + \mu_1]}{P_5} = 19$$
5 - 2,3

P_B

F_B

P_A

F_A

Cumulative frequencies and probabilities :

$P_{N-1} = P_K + P_N$

$F_{N-1} = F_N + P_K[\mu_{K-H} - \lambda_{K-L}]$

I pb

$P_F = 0.000125$

$F_F = 0.0035625$

$P_E = P_F + P_E = 0.004875$

1.

$F_E = F_F + P_E[\mu_{E-C} + \mu_{E-B} - \lambda_{E-F}]$

$= 0.0914375$

$P_D = P_E + P_D = 0.00725$

$F_D = F_E + P_D[\mu_{D-B} - \lambda_{D-F}]$

The

$= 0.135375$

$P_C = P_D + P_C = 0.052375$

2. !

$F_C = F_D + P_C[\mu_{C-A} - \lambda_{C-E}]$

$= 0.5189375$

The

$$P_B = P_C + P_B = 0.162625$$

$$\therefore F_B = F_C + P_B \left[\mu_{B-A} - (\lambda_{B-D} + \lambda_{B-E}) \right].$$

$$= 1.2860625$$

$$P_A = P_B + P_A = 1.0$$

$$F_A = F_B + P_A \left[\cancel{\mu_A}^{1.0}_{0} - [\lambda_{A-B} + \lambda_{A-C}] \right]$$

$$= \underline{\underline{0}}$$