Reliability

Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered.

This definition breaks down into four basic parts:

-Probability,

-Adequate performance,

-Time,

-Operating conditions.

The ·first part, probability, provides the numerical input for the assessment of reliability and also the first index of system adequacy. In many instances it is the most significant index, but there are many more parameters calculated and used. These are discussed later.

The other three parts--adequate performance, time and operating conditions--are all engineering parameters, and probability theory is of no assistance in this part of the assessment. Often, only the engineer responsible for a particular system can satisfactorily supply information relating to these.

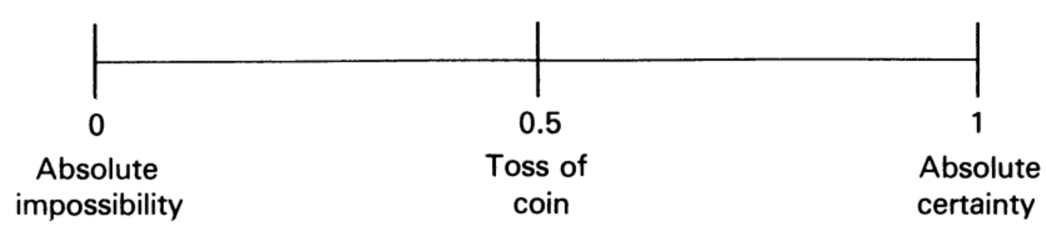
The 'time intended' may be continuous or very sporadic and the 'operating conditions' may be perfectly uniform or extremely variable, as in the propulsion phases associated with space rockets and in the take-off, cruising and landing of commercial flights.

Probability

The word 'probability' is used frequently in a loose sense implying that a certain event has a good chance of occurring

It is important to realize that it has a strict technical meaning and is a scientific 'measure of chance', i.e., it defines quantitatively the likelihood of an event or events. Mathematically it is a numerical index that can vary between zero which defines an absolute impossibility to unity which defines an absolute certainty.

For example, the probability that a man will live for ever is zero, and the probability that one day he will die is unity.



$$\left.\begin{aligned} P(\text{success}) &= \frac{\text{number of successes}}{\text{number of possible outcomes}} \\ P(\text{failure}) &= \frac{\text{number of failures}}{\text{number of possible outcomes}} \end{aligned}\right\}$$

Therefore if $s =$ number of ways success can occur and

$f =$ number of ways failure can occur:

$$P(\text{success}) = p = \frac{s}{s+f}$$

$$P(\text{failure}) = q = \frac{f}{s+f}$$

and $\quad p + q = 1$

Example 1. Consider a coin and the probability of getting a head or a tail in a single toss.

In this example s = f = 1.

Therefore the probability of getting a head or a tail in a single throw is 1/2.

Example 2.Consider a die and the probability of getting a 4 from a single throw.

If a 4 is called success, then s = 1 and f = 5 since there are five ways of not getting a 4.

Therefore the probability of getting a 4 is 1/6 and the probability of not getting a 4 is 5/6.

Example3 Consider 2 dice and the probability of getting a total of 9 spots in a single throw of both dice.

In this case, the successful outcomes are: (3+6),(4+5),(5+4) and (6+3)=4 ways=s

The failed outcomes are:

(1 + 1), (1 +2), (1 +3), (1 +4), (1 +5), (1 +6)

(2+ 1), (2+2), (2+3), (2+4), (2+5), (2+6)

(3+ 1), (3+2), (3+3), (3+4), (3+5)

(4+ 1), (4+2), (4+3), (4+4), (4+6)

(5 + 1), (5 +2), (5 + 3), (5 + 5), (5 +6)

(6+1),(6+2), (6+4),(6+5),(6+6)=32 ways=f

Therefore, the probability of getting a total of 9 spots in a single throw of two dice = 4/36 = 1/9

and similarly, the probability of not getting a total of 9 spots is 32/36 = 8/9.

Example 4.From 6 men and 5 women, how many committees of 6 members can be formed when each committee must contain at least 3 women? The condition that at least 3 women must be present on each committee is satisfied in three ways, when (a) there are 3 women and 3 men, or (b) there are 4 women and 2 men, or (c) there are 5 women and 1 man.

This example is not concerned with the order, only the total composition of the committees.

a) It can be selected in 5C3 x 6C3 ways
b) (b) can be selected in SC4 X 6C2 ways
c) (c) can be selected in sCs X 6C1 ways

Giving the total number of committees as (sC3 x 6C3) + (sC4 x 6C2) + (sCs x 6Cl) = 281
This number of possible committees can be compared with 462 committees when no conditions are imposed on the composition of each committee.

Rules for combining probabilities

Rule 1-Independent event.
Two events are said to be independent if the occurrence of one event does not affect the probability of occurrence of the other event.

Example 5. Throwing a die and tossing a coin are independent events since which face of the die is uppermost does not affect the outcome of tossing a coin.

Rule 2-Mutually exclusive event.
Two events are said to be mutually exclusive (or disjoint) if they cannot happen at the same time

Example 6 : When throwing a single die, the events 1 spot, 2 spots, 3 spots, 4 spots, 5 spots, 6 spots are all mutually exclusive because two or more cannot occur simultaneously.

Rule 3-Complementary events
Two outcomes of an event are said to be complementary if, when one outcome does not occur, the other must

Example 7. When tossing a coin, the outcomes head and tail are complementary

since: P(head) + P(tail) = 1 or P(head) = P(tail)

Rule 4-Conditional events

Conditional events are events which occur conditionally on the occurrence of another event or events.

Example : Consider two events A and B and also consider the probability of event A occurring under the condition that event B has occurred.

This is described mathematically as P(A I B) in which the vertical bar is interpreted as GIVEN and the complete probability expression is interpreted as the 'conditional probability of A occurring GIVEN that B has occurred'.

$$P(A \mid B) = \frac{\text{number of ways A and B can occur}}{\text{number of ways B can occur}}$$

$$P(A \cap B) = \frac{(A \cap B)}{S}$$

$$\text{similarly} \quad P(B) = \frac{B}{S}$$

$$\text{therefore} \quad P(A \mid B) = \frac{S \cdot P(A \cap B)}{S \cdot P(B)}$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$\text{similarly} \quad P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Rule 5-Simultaneous occurrence of events

The simultaneous occurrence of two events A and B is the occurrence of BOTH A AND B.

Example

In this rule there are two cases to consider, the first is when the two events are independent and the second is when they are dependent.

(a) Events are independent

If two events are independent, then the probability of occurrence of each event is not influenced by the probability of occurrence of the other.

In this case P(A / B) = P(A) and P(B / A) = P(B) ,

the probability that they both occur is: P(A n B) = P(A) . P(B)

Example 8 An engineer selects two components A and B. The probability that component A is good is 0.9 and the probability that component B is good is 0.95.

Therefore the probability of both components being good is:

P(A good n B good) = P(A good) . P(B good) = 0.9xO.95 =0.855

(b) **Events are dependent**

If two events are not independent, then the probability of occurrence of one event is influenced by the probability of occurrence of the other.

Thus: p(AnB)=p(B / A) ·P(A) =P(A/B) ·P(B) (2.11)

Example 9. One card is drawn from a standard pack of 52 playing cards. Let A be the event that it is a red card and B be the event that it is a court or face card. What is the probability that both A and B occur

P(A) = 26/52 Given that A has occurred,

the sample space for B is 26 states of which 6 states are those of a court or face card

Therefore P(B I A) = 6/26

P(A n B) = 6/26 x 26/52 = 6/52

Alternatively P(B) = 12/52

Given that B has occurred,

the sample space for A is 12 states of which 6 states are those of a red card,

Therefore P(A I B) = 6/12

and p(AnB) = 6/12 x 12/52 = 6/52

Rule 6-Occurrence of at least one of two events

The occurrence of at least one of two events A and B is the occurrence of A OR B OR BOTH. and is expressed as: (AUB), (A OR B) or (A+B)

(a) Events are independent but not mutually exclusive

There are two ways in which the appropriate equation can be deduced; using an analytical method and using the Venn diagram. Consider first the analytical technique.

P(A U B) = P(A OR B OR BOTH A AND B)

= 1-P(NOT A AND NOT B)

=1-p(AnB) =1-P(A) 'P(B)

= 1-(1-P(A»· (1-P(B)

= P(A) + P(B)- P(A)· P(B)

= P(A) + P(B) - P(A n B)

Example 10: An engineer selects two components A and B. The probability that component A is good is 0.9 and the probability that component B is good is 0.95.

The probability that component A or component B or both is good:

P(A good U B good) = P(A good) + P(B good) - P(A good) . P(B good)

= 0.9+0.95-0.9x 0.95 =0.995

(b) Events are independent and mutually exclusive

In the case of events A and B being mutually exclusive, then the probability of their simultaneous occurrence P(A) . P(B) must be zero by definition.

Therefore P(A U B) = P(A) + P(B)

Rule 7-Application of conditional probability

This principle can be extended to consider the occurrence of an event A which is dependent upon a number of mutually exclusive events B_i

p(AnB)=p(A IB)· P(B)

Using Equation

, the following set of equations can be deduced for each B_i

P(A n B1) = P(A I B1) • P(B1)

P(A n B2) =P(A I B2) • P(B2)

P(A n B;) =P(A I B;) . P(BJ)

P(A n Bn) = P(A I Bn) . P(Bn)

On combing

$$\sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)$$

Example 11

A certain item is manufactured at two plants. Plant 1 makes 70% of the requirement and plant 2 makes 30%. From plant 1, 90% meet a particular standard and from plant 2 only 80%.

Evaluate, (a) out of every 100 items purchased by a customer, how many will be up to standard and, (b) given that an item is standard, what is the probability that it was made in plant 2?

consider A as the event that the item is up to standard, Bl as the event that the item is made in plant 1 and B2 as the event that the item is made in plant 2.

Therefore, p(AIB1)=0.9,

p(AIB2)=0.8,

P(B1) =0.7,

P(B2) =0.3

P(A) =0.9xO.7+0.8xO.3

=0.63 + 0.24 = 0.87

and out of every 100 items purchased by the customer 100xO.87 = 87

b) The probability that the item comes from plant 2 given that it was standard, is p(B2 1 A).

in this case when i = 2).

From part (a), the probability that the component is standard AND comes from plant 2,

P(A n B2) = 0.24 and the probability that it is standard, P(A) = 0.87. Therefore P(B I A) = p(AnB2) 2 P(A) = 0.24/0.87 = 0.276

Example 11

The probability that a 30-year-old man will survive a fixed time period is 0.995. An insurance company offers him a $2000 insurance policy for this period for a premium of $20. What is the company's expected gain?

Gain = + $20.00 if man lives = -$1980.00

if man dies Probability that he lives = 0.995

Probability that he dies = 0.005 / 1.000

expected gain = (+20) xO.995 +(-1980) xO.005 =$10

This expected gain must be positive and greater than some minimum value for the company to make a profit.

Properties of the binomial distribution

From the previous section, the binomial distribution can be represented by the general expression: (p+q)"

For this expression to be applicable, four specific conditions are required.

These are: (a) there must be a fixed number of trials, i.e., n is known,

(b) each trial must result in either a success or a failure,

i.e., only two outcomes are possible and p + q = 1,

(c) all trials must have identical probabilities of success and therefore of failure, i.e., the values of p and q remain constant, and

(d) all trials must be independent (this property follows from (c) since the probability of success in trial i must be constant and not affected by the outcome of trials 1,2, ... , (i -1))

$$\begin{aligned} P_r &= \frac{n!}{r!\,(n-r)!}\,p^r q^{n-r} \\ &= {}_nC_r p^r q^{n-r} \\ &= {}_nC_r p^r (1-p)^{n-r} \end{aligned}$$

AND

$$(p+q)^n = \sum_{r=0}^{n} {}_nC_r p^r q^{n-r} = 1$$

Example 12

.A coin is tossed 5 times. Evaluate the probability of each possible outcome and draw the probability mass (density) function and the probability distribution function.

In this example

n = 5,

p = q = 1/2.

Number of heads	tails	*Individual probability* expression	value	*Cumulative probability*
0	5	${}_5C_0(1/2)^0(1/2)^5$	1/32	1/32
1	4	${}_5C_1(1/2)^1(1/2)^4$	5/32	6/32
2	3	${}_5C_2(1/2)^2(1/2)^3$	10/32	16/32
3	2	${}_5C_3(1/2)^3(1/2)^2$	10/32	26/32
4	1	${}_5C_4(1/2)^4(1/2)^1$	5/32	31/32
5	0	${}_5C_5(1/2)^5(1/2)^0$	1/32	32/32
			$\Sigma = 1$	

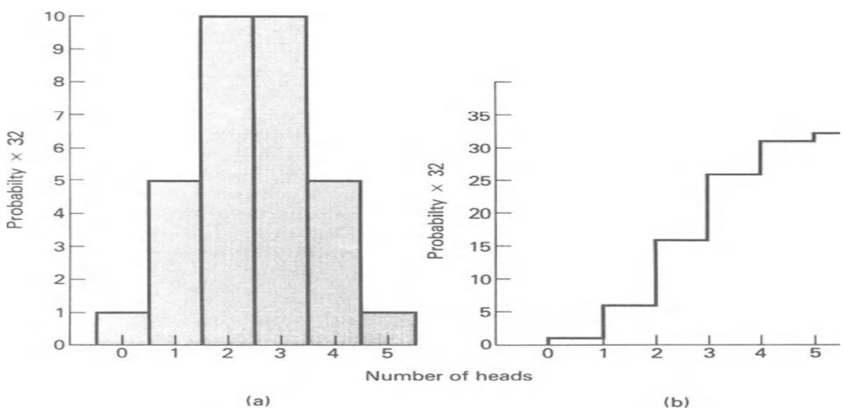


Fig. Results for Example (a) Probability density (mass) function. (b) Probability distribution function

Example 13.

Consider the case in which the probability of success in a single trial is 1/4 and four trials are to be made. Evaluate the individual and cumulative probabilities of success in this case and draw the two respective probability functions.

In this example

n = 4,

p = 1/4,

q = 3/4,

Number of successes	Number of failures	*Individual probability*	*Cumulative probability*
0	4	$(3/4)^4 = 81/256$	81/256
1	3	$4(1/4)\ (3/4)^3 = 108/256$	189/256
2	2	$6(1/4)^2(3/4)^2 = 54/256$	243/256
3	1	$4(1/4)^3(3/4) = 12/256$	255/256
4	0	$(1/4)^4 = 1/256$	256/256
		$\Sigma = 1$	

Expected value and standard deviation

Two of the most important parameters of a distribution are the expected, or mean value, and the standard deviation. The binomial distribution is a discrete random variable and therefore the expected value and standard deviation can be evaluated

First consider the expected value

$$E(X) = \sum_{x=0}^{n} x \cdot {}_nC_x p^x q^{n-x}$$
$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

As the contribution to this summation made by x = 0 is zero then

$$E(X) = \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} p^x q^{n-x}$$
$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!} p \cdot p^{x-1} q^{n-x}$$
$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x}$$

Letting $n-1=m$ and $x-1=y$, then:

$$E(X) = np \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y q^{m-y}$$

But

$\sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y q^{m-y}$ is the complete binomial expansion of $(p+q)^m$ and therefore is equal to unity.

Therefore $E(X) = np$

Therefore, for a binomial distribution, the expected number of successes is equal to the number of trials multiplied by the probability of success or conversely, the expected number of failures is equal to the number of trials multiplied by the probability of failure.

Now consider evaluation of the standard deviation

$$V(X) = E(X^2) - E^2(X)$$

and $$E(X^2) = \sum_{x=0}^{n} x^2 \cdot {}_nC_x p^x q^{n-x}$$
$$= \sum_{x=0}^{n} x(x-1) \cdot {}_nC_x p^x q^{n-x} + \sum_{x=0}^{n} x \cdot {}_nC_x p^x q^{n-x}$$
$$= \sum_{x=0}^{n} x(x-1) \cdot {}_nC_x p^x q^{n-x} + np$$

since the second summation is the expected value of $(p+q)^n$. Also, using the same logic as that used to deduce $E(X)$

$$E(X^2) = np + \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!\,(n-x)!} p^2 p^{x-2} q^{n-x}$$
$$= np + p^2 n(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x}$$
$$= np + p^2 n(n-1) \cdot 1$$
$$= n^2p^2 + np - np^2$$

and $$V(X) = (n^2p^2 + np - np^2) - (np)^2$$
$$= np(1-p)$$
$$= npq$$
and $$\sigma = \sqrt{npq}$$

Example 14 A product is claimed to be 90% free of defects. What is the expected value and standard deviation of the number of defects in a sample of 4?

In this example n = 4, p(defect) = 0.1, q(no defect) = 0.9.

E(defects) = 4 x 0.1 = 0.4

u(defects) = √4 x 0.1 x 0.9 = 0.6

The same result can be achieved in a more tedious way by the direct application

Defects	*Individual probability*	$E(X)$	$E(X^2)$
0	0.6561	—	—
1	0.2916	0.2916	0.2916
2	0.0486	0.0972	0.1944
3	0.0036	0.0108	0.0324
4	0.0001	0.0004	0.0016
	1.0000	0.4000	0.5200

therefore $E(\text{defects}) = 0.4$

$$\sigma(\text{defects}) = \sqrt{0.52 - 0.4^2} = 0.6$$

Poisson distribution

The Poisson distribution represents the probability of an isolated event occurring a specified number of times in a given interval of time or space when the rate of occurrence (hazard rate) in a continuum of time or space is fixed. The occurrence of events must be affected by chance alone. A particular characteristic feature of the Poisson distribution is that only the occurrence of an event is counted, its non-occurrence is not. This is one of the essential differences between the Poisson and binomial distributions because in the latter, both the occurrence and non-occurrence of an event must be counted.

Examples of this are

-Number of lightning strokes in a period

-Number of telephone calls in a period

-Number of faults on a system.

Derivation of the Poisson distribution

The Poisson distribution which has the hazard rate is constant, the hazard rate is generally termed the failure rate, a term which is more widely recognized by system engineers.

λ = average failure rate or average number of failures per unit time.

Let dt be a sufficiently small interval of time such that the probability of more than one failure occurring during this interval is negligible and can be neglected.

Therefore, λ dt = probability of failure in the interval dt, i.e., in the period (t, t+dt)

(a) Zero failures

Let $P_x(t)$ be the probability of failure occurring x times in the interval (0, t), then because:

Probability of zero failures in the interval (0, t + dt)

= probability of zero failures in the interval (0, t) x probability of zero failures in the interval (t, t+dt).

$P_o(t+dt) = P_o(t)(1- A\ dt)$

Assuming event independence

Assuming event independence

$$\frac{P_0(t+dt)-P_0(t)}{dt}=-\lambda P_0(t)$$

As $dt \to 0$, i.e., it becomes incrementally small

$$\frac{dP_0(t)}{dt}=-\lambda P_0(t)$$

which, by integrating, becomes

$$\ln P_0(t)=-\lambda t+C$$

At $t=0$, the component is known to be operating. Therefore at $t=0$, $P_0(0)=1$, $\ln P_0(t)=0$ and $C=0$, giving:

$$P_0(t)=e^{-\lambda t}$$

This is the first term of the Poisson distribution and gives the probability of zero failures occurring in a specified time period.

It shows that if

$$\lambda(t)=\lambda, \text{ a constant}$$

then, for zero failures

$$R(t)=e^{-\lambda t}$$

$$Q(t)=1-e^{-\lambda t}$$

$$f(t)=\frac{-dR(t)}{dt}=\lambda e^{-\lambda t}$$

(b) *Multiple failures*

If $P_x(t)$ is defined as the probability of failure occurring x times in the interval $(0, t)$, then

$$\begin{aligned}P_x(t+dt)=P_x(t)&[P(\text{zero failures in } t, t+dt)]\\ &+P_{x-1}(t)[P(\text{one failure in } t, t+dt)]\\ &+P_{x-2}(t)[P(\text{two failures in } t, t+dt)]+\dots\\ &+P_0(t)[P(x \text{ failures in } t, t+dt)]\end{aligned}$$

It is assumed however that the interval dt is sufficiently small that the probability of more than one failure in this interval is negligible.

$$\begin{aligned} P_x(t+\mathrm{d}t) &= P_x(t)[P(\text{zero failures in } t, t+\mathrm{d}t)] \\ &\quad + P_{x-1}(t)[P(\text{one failure in } t, t+\mathrm{d}t)] \\ &= P_x(t)(1-\lambda\,\mathrm{d}t)+P_{x-1}(t)(\lambda\,\mathrm{d}t) \\ &= P_x(t)-\lambda\,\mathrm{d}t[P_x(t)-P_{x-1}(t)] \end{aligned}$$

from which,

$$P_x(t)=\frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

Expected value

The expected value of a discrete distribution is given by

$$E(x)=\sum_{x=0}^{\infty} xP_x$$

where, for the Poisson distribution, x = number of failures and P_x is probability of x failures in the time period of interest.

$$\begin{aligned} E(x) &= \sum_{x=0}^{\infty} x\frac{(\lambda t)^x e^{-\lambda t}}{x!} \\ &= \sum_{x=1}^{\infty} x\frac{(\lambda t)^x e^{-\lambda t}}{x!} \qquad \text{since the term for } x=0 \text{ is zero} \\ &= \lambda t \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1} e^{-\lambda t}}{(x-1)!} \\ &= \lambda t \end{aligned}$$

since the summation of the probabilities for all x must be unity

Therefore the equation is also written as

$$P_x(t)=\frac{\mu^x e^{-\mu}}{x!}$$

if μ is defined as the expected value E(x).

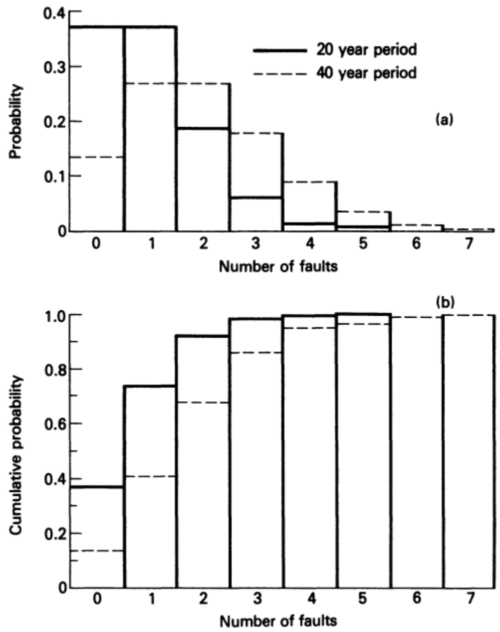
EXAMPLE 15

In a large system the average number of cable faults per year per 100 km of cable is 0.5. Consider a specified piece of cable 10 km long and evaluate the probabilities of 0, 1, 2, etc., faults occurring in (a) a 20 year period, and (b) a 40 year period. Assuming the average failure rate data to be valid for the 10 km cable and for the two periods being considered, the expected failure rate , is,

$$\lambda=\frac{0.5\times 10}{100}=0.05 \text{ f/yr.}$$

(a) for a 20 year period, E(x) = 0.05 x 20 = 1.0

and $P_x = \dfrac{1.0^x e^{-1.0}}{x!}$ for $x = 0, 1, 2, \ldots$



Relationship of Poisson and the binomial distribution

The probability of an event succeeding r times in n trials was given in

$$P_r = \frac{n!}{r!\,(n-r)!} p^r q^{n-r}$$

If $n \gg r$

$$\frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-r+1)$$

$$\simeq n^r$$

thus, $P_r = \dfrac{n^r}{r!} p^r q^{n-r}$

Also, if p is very small and r is small compared to n

$$q^{n-r} \simeq (1-p)^n$$

thus $P_r = \dfrac{(np)^r}{r!}(1-p)^n$

$$= \frac{(np)^r}{r!}\left[1 - np + \frac{n(n-1)}{2!}(-p)^2 + \ldots\right]$$

If n is large, $n(n-1) \simeq n^2$

thus $P_r = \dfrac{(np)^r}{r!}\left[1 - np + \dfrac{(np)^2}{2!} + \ldots\right]$

$$= \frac{(np)^r}{r!} e^{-np}$$

Therefore

np = ʎt and r = x

The standard deviation of the binomial distribution was

$$\sigma = \sqrt{npq}$$

If it is assumed that the value of p is very small, i.e., the condition which makes the Poisson and binomial distributions equivalent, then

$$\sqrt{q} \simeq 1 \text{ and } \sigma \simeq \sqrt{np}.$$

Example 16 The probability of success in a single trial is 0.1. Calculate the probability that in 10 trials there will be exactly two successes using (a) the binomial distribution, and (b) the Poisson distribution

(a) P(2) = 10C2 0.1^2 x 0.9^8 10! 2 8

$$= \frac{10!}{2!\,8!} 0.1^2 \times 0.9^8$$

= 0.1937

(b) np=10xO.l=1.0 1.02

therefore

$$P(2) = \frac{1.0^2}{2!} e^{-1.0}$$

$$= 0.1839$$

Repeat Example 16. when the number of trials is 20 and the probability of success in a single trial is 0.005.

(a) $$P(2) = \frac{20!}{2!\,18!} \times 0.005^2 \times 0.995^{18}$$

$$= 0.0043$$

(b) $$np = 20 \times 0.005 = 0.1$$

therefore, $$P(2) = \frac{0.1^2}{2!} e^{-0.1}$$

$$= 0.0045$$

The normal distribution

The normal probability distribution, sometimes referred to as the Gaussian distribution, is probably the most important and widely used distribution in the entire field of statistics and probability. Although having some important applications in reliability evaluation, it is of less significance in this field than many other distributions.

The probability density function of a normal distribution may be expressed in general terms as:

$$f(x) = \frac{1}{\beta\sqrt{2\pi}} \exp\left[-\frac{(x-\alpha)^2}{2\beta^2}\right]$$

If the mean value μ and standard deviation σ is evaluated for this expression, it can be shown that

$$\mu = \alpha \quad \text{and} \quad \sigma = \beta$$

Therefore, the expression for the probability density function of a normal distribution is always written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

The discussion concerning probability density functions indicates that for a continuous distribution

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

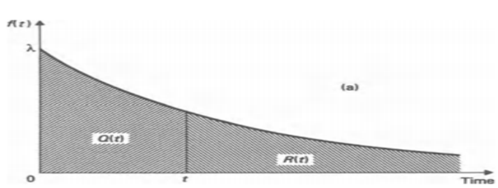
$$\text{Therefore} \quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \mathrm{d}x = 1$$

The exponential distribution

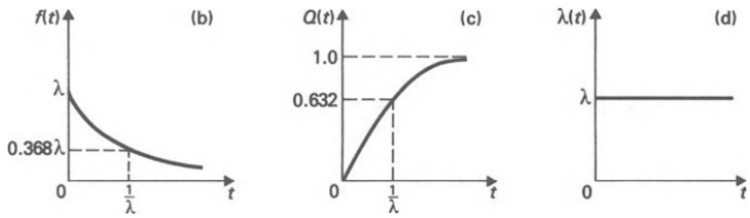
The exponential, or strictly the negative exponential, distribution is probably the most widely known and used distribution in reliability evaluation of systems.

The most important factor for it to be applicable is that the hazard rate should be constant, in which case it is defined as the failure rate ʎ. This is essentially the same requirement as that of the Poisson distribution and it can be argued that the negative exponential distribution is only a special case of the Poisson distribution.

$$f(t) = \frac{-\mathrm{d}R(t)}{\mathrm{d}t} = \lambda e^{-\lambda t}$$



Exponential reliability functions. (a) Areas showing Q(t) and R(t).



(c) Failure density function. (c) Cumulative failure distribution. (d) Hazard rate

$$Q(t) = \int_0^t \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda t}$$

$$R(t) = \int_t^\infty \lambda e^{-\lambda t}\, dt = e^{-\lambda t}$$

The failure density function f(t), cumulative failure distribution Q(t) and hazard rate $\lambda(t)$are shown in Figures, c and d, respectively.

Mean value and standard deviation

The expected value of a continuous random variable having a range (0, ∞) is given by

$$E(x) = \int_0^\infty x \cdot f(x)\, dx$$

In the case of the failure density function of the exponential distribution, this becomes

$$E(t) = \int_0^\infty t f(t)\, dt$$

$$= \int_0^\infty \lambda t \cdot e^{-\lambda t}\, dt$$

This can be integrated by parts: Letting

$$u = t \text{ and } \quad dv = \lambda e^{-\lambda t}\, dt$$

that is, $v = -e^{-\lambda t}$

then $\int u\, dv = uv - \int v\, du$

that is, $E(t) = [-te^{-\lambda t}]_0^\infty - \int_0^\infty -e^{-\lambda t}\,dt$

$$= [-te^{-\lambda t}]_0^\infty - \left[\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty$$

$$= 0 + \frac{1}{\lambda}$$

$$E(t) = 1/\lambda$$

Similarly, the standard deviation, σ, of the exponential distribution can be found from Equation

$$\sigma^2 = \int_0^\infty t^2 \lambda e^{-\lambda t}\,dt - E^2(t)$$

Integrating by parts as before gives

$$\sigma^2 = [-t^2 e^{-\lambda t}]_0^\infty - \int_0^\infty -2te^{-\lambda t}\,dt - E^2(t)$$

$$= 0 + \frac{2}{\lambda}\int_0^\infty \lambda te^{-\lambda t}\,dt - E^2(t)$$

$$= 0 + \frac{2}{\lambda}\cdot\frac{1}{\lambda} - \frac{1}{\lambda^2}$$

thus $\sigma = \frac{1}{\lambda}$

The Weibull distribution

The Weibull distribution, in common with a small number of other distributions such as the gamma and lognormal distributions, has one very important property; the distribution has no specific characteristic shape

The failure density function of the Weibull distribution is defined as

$$f(t) = \frac{\beta t^{\beta-1}}{\alpha^\beta} \exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]$$

where $t \geqslant 0$, $\beta > 0$ and $\alpha > 0$.

The survivor function is, from Equation

$$R(t) = \int_t^\infty f(t)\,dt$$

$$= \exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]$$

the cumulative failure distribution is

$$Q(t) = 1 - R(t)$$

$$= 1 - \exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]$$

and the hazard rate is, from Equation

$$\lambda(t) = \frac{f(t)}{R(t)}$$

$$= \frac{\beta t^{\beta-1}}{\alpha^\beta}$$

There are two particular cases that can be deduced from the Weibull distribution;

the first is when β= 1 and the second when β = 2.

a) For β = 1.

In this case, the above equations reduce to

$$f(t)=\frac{1}{\alpha}\exp\left[-\frac{t}{\alpha}\right]$$

$$\lambda(t)=\frac{1}{\alpha}$$

Equations are identical to those for the exponential distribution if a = 1/ ʎ, that is, the value of a represents the mean time to failure (MITF).

(b) For β = 2.

In this case, Equations 6.39 and 6.42 reduce to

$$f(t)=\frac{2t}{\alpha^2}\exp\left[-\frac{t^2}{\alpha^2}\right]$$

$$\lambda(t)=\frac{2t}{\alpha^2}$$

The expected value of the Weibull distribution is given by

$$E(t)=\int_0^\infty t\cdot\frac{\beta t^{\beta-1}}{\alpha^\beta}\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\mathrm{d}t$$

$$=\alpha\Gamma\left(\frac{1}{\beta}+1\right)$$

where Γ is the gamma function defined as

$$\Gamma(\gamma)=\int_0^\infty t^{\gamma-1}e^{-t}\,\mathrm{d}t$$

which, for integer values of γ, reduces to $\Gamma(\gamma)=(\gamma-1)!$

The standard deviation of the Weibull distribution is given by

$$\sigma^2=\int_0^\infty t^2\cdot\frac{\beta t^{\beta-1}}{\alpha^\beta}\exp\left[-\left(\frac{t}{\alpha}\right)^\beta\right]\mathrm{d}t-E^2(t)$$

$$=\alpha^2\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma^2\left(\frac{1}{\beta}+1\right)\right]$$

The conditional or *a posteriori* probability of failure $Q_c(t)$ can be found from Equation

$$Q_c(t) = \frac{\int_T^{T+t} f(t)\,dt}{\int_T^{\infty} f(t)\,dt}$$

$$= 1 - \exp\left[-\frac{(T+t)^\beta - T^\beta}{\alpha^\beta}\right]$$

which, for $\beta = 1$, gives the exponential case of

$$Q_c(t) = 1 - e^{-t/\alpha}$$