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Abstract

This document explores the intuition for exponentiation of complex numbers

1 Introduction

To most physics students and graduates, two of the three basic arithmetic operations of complex numbers $z \in \mathbb{C}$ has been made intuitive:

operation	representation	intuition
addition	$z_1 + z_2$	translation of the z_1 plane by the vector z_2
multiplication	$(z_1)(z_2)$	dilation of the complex plane by a factor $\|z_2\|$ and rotation by a factor $arg(\|z_2\|)$.

However, the third commonly used arithmetic operation, exponentiation ${z_1}^{z_2}$, is not made obvious/intuitive as part of their physics training.

This text aims to provide intuition for that.

2 Set up and calculation

In the previous two operations we have visualized z_1 as a member on the infinitely extending complex plane, while z_2 describes an operation on this plane[1]. We will use the same approach in the following set up:

Let $z_1 = e^x e^{iy}$. And for convenience, let's denote the $z_2 = a + ib$.

Then an operation $z_1^{z_2}$ will give

$$z_3 = z_1^{z_2} = e^{ax} e^{iay} \cdot e^{ibx} e^{-by} \tag{1}$$

$$= e^{ax-by} e^{iay+bx} \tag{2}$$

$$= e^{(ax-by)+i(ay+bx)} \tag{3}$$

3 Trick

The trick here is then to recognize the matrix-algebra nature of equation 3.

It the real part of the exponent looks like the dot product $(a, -b) \cdot (x, y)$, while the imaginary part of the exponent looks like the dot product $(b, a) \cdot (x, y)$.

[1]This asymmetric nature of z_1 and z_2 makes the communtativity of complex multiplication unintuitive. But that's a story for another time

Thus we can re-write the the real and imaginary part of the resulting z_3 into a 2D-vector, given by the transformation

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

4 Intuition

Therefore the intuition of raising z_1 to the power of z_2 can be broken down into four steps

1. Decompose z_1 into $x = ln(|z|), y = arg(z)$. This will turn the polar coordinates lines on the argand diagram 1 into a cartesian coordinate with $-\infty < x < \infty$,$-\pi \leq y \leq \pi$ (figure 2). The point $(-\infty, -\pi < y \geq \pi)$ in figure 2 corresponds to the origin in figure 1.
2. Contruct a matrix: $\underline{\underline{\mathbf{M}}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ by decomposing z_2 into $z_2 = a + ib$.
3. Apply this matrix transformation $\underline{\underline{\mathbf{M}}}$ (equation 4) on figure 2.
4. Now exponentiate the transformed figure 2 (inverse transformation of what we've done in step 1).

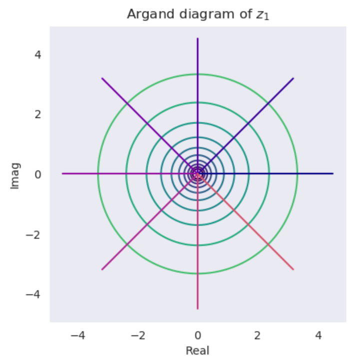

Figure 1: Input complex plane with polar coordinate lines overlaid. Exponentiating figure 2 will return this figure. The origin in figure 2 corresponds to the point $(1, 0)$ in this figure.

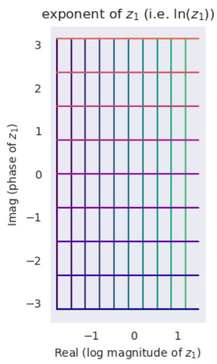

Figure 2: The output complex plane with cartesian coordinates overlaid. Each point on the cartesian coordinate lines corresponds to a point on the polar coordinate lines in figure 1. This can be obtained by taking the logarithm of figure 1 to give $x = ln(|z|), y = arg(z)$.

The lines in the figures have corresponding colours: the deep purple line gets mapped to the deep purple line after transformation, etc.

5 Correspondence to our expectation in special cases

When $a = n, b = 0$, $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ becomes a simple scaling matrix with no skewing component, agreeing with our expectation that ${z_1}^n$ simply multiplies itself by n times.

If both the real and imaginary parts are zero, then the transformation in step 3 will squash everything down to the origin; and step four will transform the origin back to the number $1+0i$, confirming our expectation that $z_1^0 = 1$.

6 Example: i^i

For the case of raising any complex number z_1 to the power of i,

$$z_2 = 0 + 1i \tag{5}$$

Therefore, using the checklist laid out in section 4:

1. Step 1 is as described in section 4. At this point $z_1 = i = (0, 1)$ will be transformed to $ln(z_1) = 0 + arg(i)i = (0, arg(i)) = (0, \frac{\pi}{2})$
2. Step 2 will give us a rotation matrix $\underline{\underline{\mathbf{M}}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which is a rotation matrix that rotates everything anti-clockwise by 90°.
3. Applying this on z_1 will give us a rotated figure 2, rotated to the left by 90°. If we consider only the point $z_1 = i$, it will now be mapped from $(0, \frac{\pi}{2})$ (step 1) to the point $(-\frac{\pi}{2}, 0)$
4. Exponentiating this result will give us the solution z_3. In this case we will exponentiate $-\frac{\pi}{2} + 0i$ into $e^{-\frac{\pi}{2}} \approx 0.20788$

7 Interesting results

In fact when z_2 is purely imaginary, the unit circle of z_1 (whose exponent has real part=0) will be mapped to the real number line in the output z_3. This is because when $a = 0$, the matrix transformation becomes a 90-degrees rotation plus a scaling operation, swapping the real and the imaginary part of the exponent.

8 Multiple solutions

Of course, the point (x, y) and the point $(x, y + 2n\pi)$ on figure 2 both maps to the same point on figure 1. But these two points may be transformed differently by step 3. Therefore one get multiple answers of z_3 depending on which represenation of z_1 they have chosen for themselves on figure 2 to represent z_2.

An example of this is that i^i actually has mutliple solutions,

$$i^i = e^{-\frac{\pi}{2}} e^{2n\pi} = 0.20788(e^{2n\pi}) \qquad \forall n \in \mathbb{Z} \tag{6}$$