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Abstract

This document explores the intuition for exponentiation of complex numbers

1 Introduction

To most physics students and graduates, two of the three basic arithmetic operations of complex numbers $z \in \mathbb{C}$ has been made intuitive:

operation	representation	intuition
addition	$z_1 + z_2$	translation of the z_1 plane by the vector z_2
multiplication	$(z_1)(z_2)$	dilation of the complex plane by a factor $\|z_2\|$ and rotation by a factor $arg(\|z_2\|)$.

However, the third commonly used arithmetic operation, exponentiation ${z_1}^{z_2}$, is not made obvious/intuitive as part of their physics training.

2 Set up and calculation

In the previous two operations we have visualized z_1 as a member on the infinitely extending complex plane, while z_2 describes an operation on this plane[1]. We will use the same approach in the following set up:

Let $z_1 = e^x e^{iy}$. And for convenience, let's denote the $z_2 = a + ib$.

Then an operation $z_1^{z_2}$ will give

$$
\begin{aligned}
z_3 = z_1^{z_2} &= e^{ax}e^{iay} \cdot e^{ibx}e^{-by} && (1)\\
&= e^{ax-by}e^{iay+bx} && (2)\\
&= e^{(ax-by)+i(ay+bx)} && (3)
\end{aligned}
$$

3 Trick

The trick here is then to recognize the matrix-algebra nature of equation 3.

It the real part of the exponent looks like the dot product $(a, -b) \cdot (x, y)$, while the imaginary part of the exponent looks like the dot product $(b, a) \cdot (x, y)$.

Thus we can re-write the the real and imaginary part of the resulting z_3 into a 2D-vector, given by the transformation

$$
\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}
$$

[1]This asymmetric nature of z_1 and z_2 makes the communtativity of complex multiplication unintuitive. But that's a story for another time

4 Intuition

Therefore the intuition of raising z_1 to the power of z_2 can be broken down into four steps

1. decompose z_1 into $x = ln(|z|), y = arg(z)$. This will turn the polar coordinates lines on the argand diagram 1 into a cartesian coordinate with $-\infty < x < \infty$,$-\pi \leq y \leq \pi$ (figure 2). The point ($-\infty$, whatever) in figure 2 corresponds to the origin in figure 1.
2. contruct a matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ by decomposing z_2 into $z_2 = a + ib$.
3. apply the matrix transformation (equation 4)
4. now exponentiate (inverse transformation of what we've done in step 1).

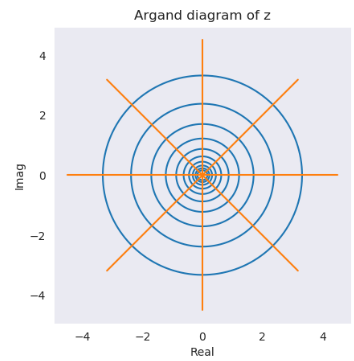

Figure 1: Argand diagram with polar coordinate lines overlaid. Exponentiating figure 2 will give this diagram. The origin in figure 2 corresponds to the point $(1, 0)$ in figure 1.

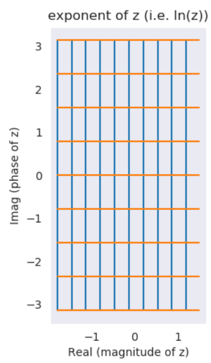

Figure 2: Cartesian coordinates of the exponent of a complex number z. This can be obtained by taking the logarithm of figure 1 to give $x = ln(|z|), y = arg(z)$.

5 Correspondence to our expectation in special cases

When $a = n, b = 0$, $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ becomes a simple scaling matrix with no skewing component, agreeing with our expectation that ${z_1}^n$ simply multiplies itself by n times.

If both the real and imaginary parts are zero, then the transformation in step 3 will squash everything down to the origin; and step four will transform the origin back to the number $1 + 0i$, confirming our expectation that $z_1^0 = 1$.

6 Interesting results

When z_2 is purely imaginary, the unit circle of z_1 (whose exponent has real part=0) will be mapped to the real number line in the output z_3. This is because when $a = 0$, the matrix transformation becomes a 90-degrees rotation plus a scaling operation, swapping the real and the imaginary part of the exponent.

7 Multiple results

Of course, the point (x, y) and the point $(x, y + 2n\pi)$ on figure 2 both maps to the same point on figure 1. But these two points may be transformed differently by step 3.

Therefore one get multiple results of z_3 depending on which represenation of z_1 s/he has chosen for themselves on figure 2 to represent z_2.