Session E: Interpolation

E.1 Introduction

In this exercise we see how the concept of finite differences can help us with the problem of interpolation. The problem can be summarized as follows: suppose we have evaluated the function $f(m)$ at certain mesh points $f(0), f(1), f(2), \ldots, f(n)$ and we wish to determine a value at an intermediate (non-integer) value of m. Consider the effect of the forward shift operator, E, which we met in the last session.

$$\begin{aligned} f(1) &= E\{f(0)\} \\ f(2) &= E\{f(1)\} = E^2\{f(0)\} \\ \ldots \\ f(n) &= E^n\{f(0)\} \end{aligned}$$

Raising E to an integer power, n, takes us to the n^{th} measurement away from our starting point. As you might expect, to obtain an intermediate point we need to evaluate $f(m) = E^m\{f(0)\}$, where m takes a non-integer value. What does E^m mean? We can approximate it by noting that the E operator can be expressed in terms of the forward difference operator, $E = 1 + \Delta$ (see Session D). Hence, $f(m) = (1+\Delta)^m\{f(0)\}$.

We can expand $(1+\Delta)^m$ in terms of a power series such that:

$$f(m) = \left(1 + \frac{m}{1!}\Delta + \frac{m(m-1)}{2!}\Delta^2 + \frac{m(m-1)(m-2)}{3!}\Delta^3 + \ldots\right)\{f(0)\}$$

This is a standard result for a binomial series and is known as Newton's interpolation formula. If you are unsure of its origin, look up Maclaurin or Taylor series in any standard mathematics textbook. It is important to realise that by using this expansion the accuracy we can expect to obtain will depend on the number of terms we keep.

E.2 First order approximation

The simplest approach is to keep only the first expansion term:

$$\begin{aligned} f(m) &= \left(1 + \frac{m}{1!}\Delta\right)\{f(0)\} \\ \therefore \quad f(m) &= (1 + m\Delta)\{f(0)\} \\ \therefore \quad f(m) &= f(0) + m\left[f(1) - f(0)\right] \end{aligned}$$

Collecting the common terms, we can write this as:

$$f(m) = (1-m)f(0) + mf(1) \tag{E.1}$$

which we recognize is nothing more than linear interpolation between the two mesh points $f(m=0)$ and $f(m=1)$. If we wish to interpolate linearly between a pair of data points at higher values of x, then, using the notation of Session D, the origin of our mesh is no longer at $x = a_0$, but at $x = a_0 + mh$, where h is the interval between our measured values. As long as we are clear about our use of notation (i.e. whether we are using $f(x)$ or $f(m)$) we can still use the above formula. We translate between the two notations using $f(m) \equiv f(x = a_0 + mh)$. Note that there is no reason why the interval, h, should be unity, or even an integer. However, as long as m is seen to be a *fraction* of the distance between measured points at $m = 0$ and $m = 1$, equation E.1 can be used directly. For example, if the mesh points $m = 0, 1, 2, \ldots$ are at $x = 7.5, 8.25, 9.0, \ldots$ then $a_0 = 7.5$ and $h = 0.75$. Thus, interpolating linearly between the first two mesh points equation E.1 gives:

$$f(m) = (1 - m)f(m = 0) + mf(m = 1)$$

which is equivalent to:

$$f(m) = (1 - m)f(x = 7.5) + mf(x = 8.25)$$

The point I wish to make here is that the origin of the mesh ($m = 0$) can be chosen, as a matter of convenience, to coincide with any measured data point. If we want to evaluate the function at a value of $x = 7.575$ (the interpolation point), then $m = step/h = 0.075/0.75 = 0.1$.

In general, for Δx lying between $x = a_0$ and $x = a_0 + h$, equation E.1 becomes:

$$f(x + \Delta x) = \left[1 - \frac{\Delta x}{h}\right] f(x = a_0) + \frac{\Delta x}{h} f(x = a_0 + h)$$

Exercise E.1

1. Produce a spreadsheet with a table of values for $f(x) = 1 + x^3$ for $x = 0, 1, 2, 3, 4, 5$.
2. Calculate linear interpolation values for $x = 0.0, 0.1, 0.2, 0.3, \ldots 0.9, 1.0$.
3. Calculate the algebraic values at the same points and hence the error associated with using linear interpolation.
4. Repeat the process for linear interpolation between $x = 4$ and $x = 5$ in steps of 0.1.
5. Produce a further table with $f(x) = 1 + x^3$ for $x = 0, 2.5, 5$. Now interpolate linearly between the first two points in this table at $x = 0, 0.25, 0.5, 0.75, \ldots 2.5$.
   Use an Excel chart to compare the results with those for part 1 of this exercise.

E.3 Second order approximation

If we keep two terms from Newton's interpolation formula, we find:

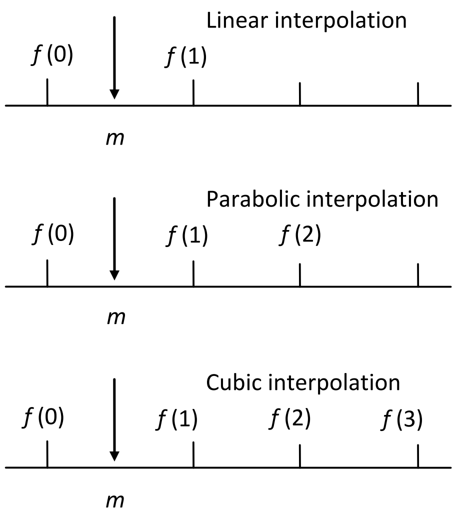
$$\begin{aligned}
& f(m) \approx \left(1 + \frac{m}{1!}\Delta + \frac{m(m-1)}{2!}\Delta^2\right)\{f(0)\} \\
\therefore\quad & f(m) = f(0) + m\,[f(1) - f(0)] + \frac{m(m-1)}{2}\Delta\{f(1) - f(0)\} \\
\therefore\quad & f(m) = f(0) + m\,[f(1) - f(0)] + \frac{m(m-1)}{2}\,[f(2) - f(1) - f(1) + f(0)] \\
\therefore\quad & f(m) = f(0)\left[1 - m + \frac{m(m-1)}{2}\right] + f(1)\,[m - m(m-1)] + f(2)\left[\frac{m(m-1)}{2}\right]
\end{aligned}$$

This has the effect of fitting a parabola through three points, $f(0), f(1), f(2)$ and we can use the resulting expression to calculate interpolated values between the measured points.

Exercise E.2

1. Reuse the spreadsheet from Exercise E.1 with its table of values for $f(x) = 1 + x^3$ for $x = 0, 1, 2, 3, 4, 5$.
2. Now calculate the parabolic interpolation values for $x = 0.0, 0.1, 0.2, \ldots 0.9, 1.0$.
3. Again calculate the algebraic values at the same points and hence the error associated with parabolic interpolation. Compare this with the error from the linear interpolation.
4. Again repeat the process for parabolic interpolation between $x = 4$ and $x = 5$ in steps of 0.1. Note that you will need to extend your basic table of $f(x) = 1 + x^3$.

By increasing the number of terms in our expansion we can increase our accuracy. But note that by using the forward difference operator Δ in this fashion we are including only *forward* values of $f(x)$ in our interpolation. This is illustrated in the sketch below:



Remember that in our notation we have defined $f(0)$ as the known value of the function nearest and below the intermediate point m at which we wish to interpolate.

That is, $f(0) = f(m = 0) = f(x = a_0)$.

Exercise E.3

1. Repeat Exercise E.2 but extend the formulae to 3$^{\text{rd}}$ and 4$^{\text{th}}$ order differences.
2. What happens to the error associated with the differences as we go to 3$^{\text{rd}}$ and 4$^{\text{th}}$ order differences? If you are not sure why it happens, construct a table of successively higher order forward differences $(\Delta, \Delta^2, \Delta^3, \Delta^4, \ldots)$ for both $1 + x^3$ and say, $1 + x^5$.

E.4 Using other difference operators

There is no particular reason why we should use the forward difference operator. We could just as well use the backward difference operator using the relationship derived in Session D:

$$E = (1 - \nabla)^{-1}$$

in this case, we would find:

$$\begin{aligned} f(m) &= (1-\nabla)^{-m}\{f(0)\} \\ f(m) &= \left(1 + \frac{m}{1!}\nabla + \frac{m(m+1)}{2!}\nabla^2 + \frac{m(m+1)(m+2)}{3!}\nabla^3 + \dots\right)\{f(0)\} \end{aligned}$$

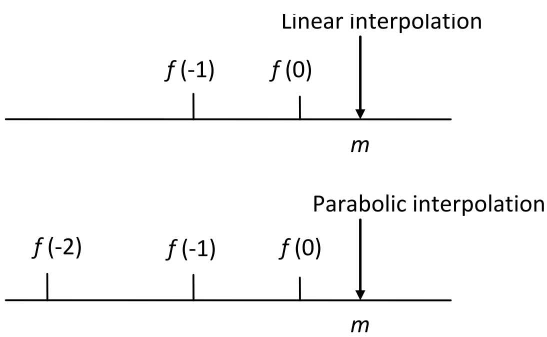
The corresponding expression for linear interpolation now becomes:

$$\begin{aligned} f(m) &\approx f(0) + m\nabla f(0) \\ &= f(0) + m\left[f(0) - f(-1)\right] \end{aligned}$$

and for parabolic interpolation we obtain:

$$\begin{aligned} f(m) &\approx \left(1 + m\nabla + \frac{m(m+1)}{2}\nabla^2\right)\{f(0)\} \\ &= f(0) + m\left[f(0) - f(-1)\right] + \frac{m(m+1)}{2}\nabla\{f(0) - f(-1)\} \\ &= f(0) + m\left[f(0) - f(-1)\right] + \frac{m(m+1)}{2}\left[f(0) - f(-1) - f(-1) + f(-2)\right] \\ &= f(0)\left[1 + m + \frac{m(m+1)}{2}\right] - f(-1)\left[m + m(m+1)\right] + f(-2)\left[\frac{m(m+1)}{2}\right] \end{aligned}$$

We are now working with the points $f(0), f(-1), f(-2)$ as shown in the sketch below:



The choice of whether to use forward or backward difference is usually dictated by the problem at hand.

Exercise E.4

1. Create a spreadsheet table of $\sin(x)$, $\cos(x)$ and $\tan(x)$ for $x = 0, 10, 20, \dots, 180$ degrees. Be careful to check whether the spreadsheet functions use degrees or radians.

2. Use linear and parabolic forms of the forward and backward difference interpolation formulae to interpolate between a) 20 and 30° and also between b) 80 and 90° and c) 90 and 100°.

3. Why are there problems with some of the $\tan(x)$ interpolations?

We can combine the approaches of forward and backward differences by using the central difference operator, δ (and, as it happens, the averaging operator, μ).

$$E = \left(1 + \frac{\delta^2}{2} + \delta\mu\right)$$

To convince yourself of this, write the δ and μ operators in terms of the forward difference operator (Session D). You should find that:

$$\begin{aligned} \delta &= E^{1/2} - E^{-1/2} \\ 2\mu &= E^{1/2} + E^{-1/2} \end{aligned}$$

Hence $\delta^2/2$ is given by:

$$\frac{\delta^2}{2} = \frac{1}{2}(E^{1/2} - E^{-1/2})^2 = \frac{1}{2}(E - 2 + E^{-1})$$

and the $\delta\mu$ term by:

$$\begin{aligned} \delta\mu &= \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2}) \\ &= \frac{1}{2}(E - E^{-1}) \end{aligned}$$

To interpolate we require E^m, which evidently is given by:

$$E^m = \left(1 + \frac{\delta^2}{2} + \delta\mu\right)^m$$

Expanding this expression we obtain (without proof) Stirling's interpolation formula:

$$\begin{aligned} f(m) = &\left(1 + \frac{m^2}{2!}\delta^2 + \frac{m^2(m^2-1)}{4!}\delta^4 + \dots\right)\{f(0)\} \\ &+ \left(\frac{m}{1!}\delta + \frac{m(m^2-1)}{3!}\delta^3 + \dots\right)\mu\{f(0)\} \end{aligned}$$

In its simplest form:

$$\begin{aligned} f(m) &\approx f(0) + m\delta\mu\{f(0)\} \\ &= f(0) + \frac{m}{2}(E - E^{-1})\{f(0)\} \qquad \text{(from the expansion of } \delta\mu \text{ above)} \\ &= f(0) + \frac{m}{2}\left[f(1) - f(-1)\right] \end{aligned}$$

Exercise E.5

1. Reuse the spreadsheet table of $\sin(x)$, $\cos(x)$ and $\tan(x)$ for $x = 0, 10, 20, \ldots 180$ degrees. Use the simple form (first order) expansion of the central difference formula to interpolate between a) 20 and 30° and also between b) 80 and 90° and c) 90 and 100°.
2. Repeat to second order by including the $(m^2\delta^2/2!)\{f(0)\}$ term. Is the interpolation of the $\tan(x)$ function working better around 90°?

Summary

In numerical analysis we often use power series to represent a given function or to approximate the action of a particular operator. The important point is that the accuracy of this approximation will depend on the number of terms that you keep in the expansion. What you should take away from this exercise is that if the data represents a set of discrete measurements of an underlying linear function, than a linear approximation is sufficient to give you an exact solution. If the underlying physical function is quadratic, then a quadratic approximation will give you an *exact* solution. Adding additional terms will not improve the accuracy further. As is often the case in real-life situations, you need to know something about the physical system that you are trying to model in order that you can make an informed decision about the number of terms you need to keep to obtain the required accuracy. Beyond these considerations, you also have a choice of operators at your disposal to determine how you implement your solution. This choice is often determined by factors such as simplicity and efficiency, as well as accuracy.

References

Here are some references that you may find useful:

Introduction to numerical analysis, F. B. Hildebrand.
Introduction to numerical analysis, J. Stoer and R. Bulirsch.
Introduction to numerical analysis, G. P. Weeg and Georgia B. Reed

The library catalogue has quite a few titles in stack QA297 with *Numerical Analysis* in the title. If you find a particularly good one, please let me know!