Session C′

1. We have seen that we can calculate the error on a function $z(p_{1,}p_2)$ of two variables $p_1(x_1,x_2),\ p_2(x_1,x_2)$ defined in terms of the primary variables $x_1,\ x_2$ whose error matrix we know, following the three steps

   a. Differentiate z with respect to the variables $p_1,\ p_2$ in terms of which it is defined.
   b. Transform the $p_{1,}\ p_2$ dependence to $x_1,\ x_2$ dependence using an appropriate transformation matrix
   c. Square the result and take expectation values for the errors, which should now be expressed in terms of $\overline{\delta z}$ and the errors on the primary variables $\overline{\delta x_1^2}, \overline{\delta x_2^2}$.

   Use this method to determine the error on the function $z = \dfrac{p_2}{p_1} = \dfrac{x^2}{x}$, and show that if the correlation between p_1 and p_2 is properly taken into account, the "common sense" result that $\overline{\delta z^2} = \sigma^2$ is recovered.

2. The least-squares method can be used to determine estimates for a set of variables by finding the best fit to a set of data. The error on the fit variables can be estimated at the point where the least squares function is minimised. For the two-variable case we have

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2}\frac{\partial^2 S}{\partial p_1^2} & \frac{1}{2}\frac{\partial^2 S}{\partial p_1 \partial p_2} \\ \frac{1}{2}\frac{\partial^2 S}{\partial p_1 \partial p_2} & \frac{1}{2}\frac{\partial^2 S}{\partial p_2^2} \end{pmatrix}$$

   Show that for a fit to a straight line, $y = p_0 + p_1 x$ this leads to

$$\mathbf{A} = \begin{pmatrix} [\omega x^2] & [\omega x^1] \\ [\omega x^1] & [\omega x^0] \end{pmatrix}$$

   Where $\mathbf{A}^{-1}$ is the covariance matrix $\begin{pmatrix} \sigma_{p_2}^2 & \mathrm{cov}(p_1, p_2) \\ \mathrm{cov}(p_1, p_2) & \sigma_{p_1}^2 \end{pmatrix}$.
   Consider the compound pendulum problem presented in last week's problem sheed (problem C4). Data are provided in table 1 (*different from those in*

problem C4) for h^2 (x-axis) and t^2h (y-axis). Let the best values for the intercept and gradient of the line be p_0 and p_1 respectively. From this we are able to obtain an estimate for the acceleration due to gravity as

$$g = \frac{4\pi^2}{p_1}$$

and the radius of gyration, k as

$$k = \left(\frac{p_0}{p_1}\right)^{\frac{1}{2}} .$$

Determine estimates based on the data in table 1 for g and k, and calculate the corresponding errors. Note that for the radius of gyration k the correlation between the fit variables must be taken into account.

Table 1. Measurements from a compound pendulum experiment. σ is the error on ht^2.

h^2	ht^2	σ
0.170982	1.04030	0.014
0.177662	1.04090	0.01
0.126025	0.83300	0.008
0.127092	0.85090	0.004
0.093025	0.69930	0.01
0.091506	0.69630	0.011
0.060270	0.58850	0.009
0.057600	0.55700	0.015
0.042230	0.51650	0.009
0.023562	0.41960	0.025
0.013110	0.38850	0.01
0.041820	0.51410	0.007
0.022500	0.41340	0.02
0.013572	0.38900	0.007