Session H: Finite Difference Solutions of Differential Equations, Part 1

H.1 Introduction

In this session we use the method of finite differences to help us solve differential equations. There are two broad classes of differential equations: (a) those that can be solved in a *closed form* using an incremental procedure, and (b) those that require a matrix inversion technique. We are primarily concerned with differential equations that fall into the first category.

Supposing we wish to solve the following *homogeneous* equation. (Reminder: homogeneous means that all terms are proportional to derivatives of y, or y itself. There is no term that is a function of x alone, or a constant.)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

By *solve* we mean that we wish to find the function, $y(x)$, that satisfies the above equation for all x. We shall find that our finite difference solution will be an approximation of the function $y(x)$ evaluated at discrete points, $x = a_0, a_0 + h, a_0 + 2h, \ldots, a_0 + mh$, starting from some initial value. Our accuracy will be determined by the step size, h. If we require better accuracy we can decrease the step size, but this comes at the cost of an increase in computation time. This may be particularly severe in 3-dimensional problems, but need not concern us too much here.

H.2 Order of the difference equation

To solve for the function $y(x)$ we must first express the derivatives in finite difference terms. We are free to choose the particular finite difference form that suits us best. Suppose we write the above equation (perhaps a little perversely) in the following form:

$$\frac{1}{h^2}\Delta^2 y(x) - \frac{1}{h}\nabla y(x) = 0$$

Then, about a specific point $x = a_0 + jh$, we would obtain:

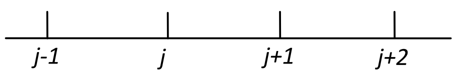
$$y_{j+2} - 2y_{j+1} + (1-h)y_j + hy_{j-1} = 0$$

It is sometimes convenient to work with a scale whereby the step size, h, is equal to 1. (We are free to do this. It amounts to a change of variable, $x \to x'$, where x' is simply x multiplied by a constant.) In this case we would have:

$$y_{j+2} - 2y_{j+1} + y_{j-1} = 0 \tag{H.1}$$

This means that we are required to evaluate our function at points $j+2, j+1$ and $j-1$, but not at j itself. Of course, if we chose to continue with a general value of h, the y_j term would not cancel.

In this example we find that we are working across 3 mesh intervals, as shown schematically in the figure below. Equation H.1 is therefore known as a *3rd order* difference equation.



Note that the order of the difference equation is a function of the finite difference approximation that we chose and is **not** connected with the order of the differential equation we are trying to solve, which in this example is only of order 2.

Exercise H.1

1. Verify that equation H.1 is obtained with the specified difference relationships.
2. Determine the equivalent form if the double differential were to be replaced with a central difference representation up to order δ^2.
3. What order difference equation would this represent?

H.3 First order differential equations

The two types of first order differential equations that you are most likely to encounter are a) those which are separable and b) linear equations. In the following we shall consider separable equations as for this class of equations it is rather straightforward to obtain the analytic solution, which we can compare with the finite difference solution. Consider a simple first order differential equation, which happens to be inhomogeneous:

$$\frac{dy}{dx} = f(x) \tag{H.2}$$

Since the function, f, is independent of y, equation H.2 can be solved analytically by integration:

$$y(x) - y(0) = \int_{x'=0}^{x'=x} f(x')dx'$$

Thus, you can see that to obtain a solution that is valid for all x an initial value of y must be specified as part of the problem, namely $y(x=0)$. Now we will attempt to find the solution using a forward difference representation of the differential. At some arbitrary point $x_j = a_0 + jh$:

$$\left.\frac{dy}{dx}\right|_{x_j} \approx \frac{(y_{j+1} - y_j)}{h} \equiv f_j$$

Here, f_j is just the first order difference approximation of the derivatives (see Session F). We can easily show that the function at any given point is given by:

$$\begin{aligned}
y_1 &= y_0 + hf_0 \\
y_2 &= y_1 + hf_1 = y_0 + h(f_1 + f_0) \\
&\vdots \\
y_n &= y_0 + h\sum_{k=0}^{n-1} f_k
\end{aligned}$$

The final expression is in fact the solution we require, since it gives us the function at any mesh point. Notice that, once again, the solution requires an initial value of the function y. The term y_0 actually represents the solution to the homogeneous equation $\frac{dy}{dx} = 0$, and the term $h\sum_{k=0}^{n-1} f_k$ the particular solution.

H.3.1 General first order differential equations

Extending our previous example, consider the general homogeneous equation below. Notice that the coefficient of the y term is not constant, but is also a function of x.

$$\frac{dy}{dx} - a(x)y(x) = 0$$

This equation is also separable. In this case the analytic solution is:

$$\begin{aligned}
\frac{dy}{dx} - a(x)y(x) &= 0 \\
\int_{y_0}^{y(x)} \frac{dy'}{y'} &= \int_0^x a(x')dx' \\
\ln\frac{y(x)}{y_0} &= \int_0^x a(x')dx' \\
y(x) &= y_0 \exp\left(\int_0^x a(x')dx'\right)
\end{aligned}$$

Let's compare this with the numerical solution. We can solve this equation using finite differences by writing:

$$\frac{(y_{j+1} - y_j)}{h} = a_j y_j$$

Once again, assuming that we know the solution at $j = 0$, that is the value y_0, we obtain:

$$\begin{aligned}
y_1 &= y_0 + ha_0y_0 = y_0(1 + ha_0) \\
y_2 &= y_1 + ha_1y_1 = y_0(1 + ha_0)(1 + ha_1) \\
&\vdots \\
y_n &= y_0 \prod_{k=0}^{n-1}(1 + ha_k)
\end{aligned}$$

We can make this look more like the analytic solution by rewriting the solution as:

$$y_n = y_0 \exp\left(\ln\left[\prod_{k=0}^{n-1}(1 + ha_k)\right]\right)$$

Clearly, the combination of taking logs and the exponent is a unity operation, just like multiplying and dividing by a constant. We can write this modified solution in a slightly different form by recalling that $\ln(a.b) = \ln(a) + \ln(b)$. That is, the natural log of a product is equal to a sum of natural logs. This changes the product into a sum, such that:

$$y_n = y_0 \exp\left(\sum_{k=0}^{n-1} \ln(1 + ha_k)\right)$$

which is now directly comparable to the analytic solution.

H.4 General solutions to homogeneous equations

Now that we have shown how our numerical solution is equivalent to the analytic solution in a couple of simple cases we move on to consider a slightly more challenging problem, which we will expand upon in the next session.

Consider the second order linear differential equation below:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$$

At the j^{th} point on the mesh, this can be written in forward difference form as:

$$\frac{1}{h^2}\Delta^2 y_j - 2\frac{1}{h}\Delta y_j - y_j = 0$$

or

$$y_{j+2} - 2(1+h)y_{j+1} + (1+2h-h^2)y_j = 0$$

Again, if we are able to work with $h = 1$, we obtain the following relation between neighbouring evaluations of the function, y:

$$y_{j+2} - 4y_{j+1} + 2y_j = 0 \tag{H.3}$$

From this we can see that, in general, we can write the homogeneous equation in finite difference form as:

$$\sum_{i=0}^{n} a_i y_{j+n-i} = 0 \tag{H.4}$$

where the number of terms is determined by the order of the difference equation, n. In our specific example, $n = 2$ and the coefficients are given by $a_0 = 1$, $a_1 = -2(1+h)$ and $a_2 = (1+2h-h^2)$, or, if $h = 1$, $a_0 = 1$, $a_1 = -4$ and $a_2 = 2$.

The above equation will have a solution of the form $y_j = \beta^j$ or $y_j = e^{mj}$. You can verify that these are valid solutions by substituting them into the original equation. Substituting $y_j = \beta^j$ into the finite difference form of the differential equation (equation H.4 – we will look at exponential solutions next time) we obtain:

$$\sum_{i=0}^{n} a_i \beta^{j+n-i} = 0$$

We only get a non-trivial ($y_j = \beta^j \neq 0$) solution if:

$$\sum_{i=0}^{n} a_i \beta^{n-i} = 0 \tag{H.5}$$

that is, β is the solution to an n^{th} order polynomial equation, H.5. There will be n of these solutions. Let us call them β_k with $k = 1$ to n. The full solution will then be some linear combination of these solutions:

$$y_j = \sum_{k=1}^{n} b_k \beta_k^j$$

We need to consider the possibility that one of the roots of the polynomial equation H.5 is repeated. If one root, say β_1, is repeated s times, then the full solution becomes:

$$y_j = \beta_1^j \sum_{k=1}^{s+1} b_k j^{k-1} + \sum_{k=s+2}^{n} b_k \beta_k^j$$

Note that in solving the polynomial equation H.5 you will determine the functional form of β. The coefficients b_k can only be found by applying certain boundary conditions, which must be specified in the problem you are trying to solve.

Exercise H.2

1. From equation H.3, show that the solution is of the form:

$$y_j = b_1(2+\sqrt{2})^j + b_2(2-\sqrt{2})^j$$

2. For the difference equation:

$$y_{j+2} - 6y_{j+1} + 11y_j - 6y_{j-1} = 0$$

   show that the solution is of the form:

$$y_j = b_1 + b_2 2^j + b_3 3^j$$

3. For the difference equation:

$$y_{j+2} - 4y_{j+1} + 4y_j = 0$$

   show that the solution is of the form:

$$y_j = b_1 2^j + b_2 2^j j$$

Summary

In this short exercise we have seen how we can choose a particular representation of the differential operators using finite differences to solve simple differential equations. In the case of first order equations, we found that the finite difference representation at a given point allowed us to iterate to a general solution that was valid for all x. In the case of second (or higher) order equations, the finite difference representation yields, for a particular choice of the solution $y_j = \beta^j$, a polynomial which can be solved to find the general solution. The general solution is a linear combination of the roots of the polynomial.