Demystifying neutron lethargy
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2019-03-18

When attempting to find the solution to the following question:

" How can the average number of collisions necessary to thermalise a fission neutron (slow it down from 2 MeV to 1 eV) in deuterium be calculated? "

When deriving the number of collision required N for a given $\alpha = (\frac{A-1}{A+1})^2$ (where A is the atomic mass), several fellow students have stumbled across the following misconception:

1 Incorrect method

An intutive method is to do the following:

$$\overline{E_f} = \int P(E) E dE = E_i \frac{(\alpha+1)}{2} \tag{1}$$

$$\overline{\Delta u} = ln\left(\frac{E_i}{\overline{E_f}}\right) \tag{2}$$

$$N = \frac{ln\left(\frac{2MeV}{1eV}\right)}{\overline{\Delta u}} \tag{3}$$

Which corresponds to the procedure as follows:

1. Find the average energy lost $\overline{E}$
2. Calculate the lethargy change it represents
3. Find the number of collisions (each with energy $\overline{E}$ lost) required to slow it down.

In a single equation, this is represented as:

$$N = \frac{u(2MeV) - 2(1eV)}{\Delta u\left(\int EP(E)dE\right)} N = \frac{ln\,(2000000)}{ln\left(\frac{1+\alpha}{2}\right)} \tag{4}$$

Which gives the incorrect answer of 22.7 collisions for 2MeV neutrons moderated by Deuterium $A = 2$

2 Correct method

However, the correct way to do so is

$$\overline{\Delta u} = \int (\Delta u) P\big(\Delta u\big) d(\Delta u) \tag{5}$$

$$= \int_{E_f=\alpha E_i}^{E_f=E_i} \Delta u(E_f) P\big(\Delta u(E_f)\big) \frac{d(\Delta u)}{dE_f} dE_f \tag{6}$$

where the conversion to Δu is carried out as follows:

$$\Delta u(E_f) = ln\left(\frac{E_i}{E_f}\right) \tag{7}$$

and the probability distribution in u space can be converted back into probability distribution in E_f space in a straightforward manner:

$$P(E_f)dE_f = P(\Delta u(E_f))d(\Delta u) \tag{8}$$

$$P(\Delta u(E_f))\frac{d(\Delta u)}{dE_f} = P(E_f) \tag{9}$$

simplifying equation 6

$$\overline{\Delta u} = \int_{E_f=\alpha E_i}^{E_f=E_i} \big(\Delta u(E_f)\big)P\big(E_f\big)dE_f \tag{10}$$

$$= \int_{E_f=\alpha E_i}^{E_f=E_i} ln\left(\frac{E_i}{E_f}\right)P\big(E_f\big)dE_f \tag{11}$$

$$\tag{12}$$

Which will be evaluated to

$$\overline{\Delta u} = 1 + \frac{\alpha}{1+\alpha}ln(\alpha) \tag{13}$$

$$N = \frac{ln\left(\frac{2MeV}{1eV}\right)}{\overline{\Delta u}} \tag{14}$$

Which gives 20.8 collisions for 2MeV neutrons moderated by Deuterium $A = 2$

The numerical accuracy of this answer can be verified with the code

- MonteCarlo.py

in the repository

- https://github.com/OceanNuclear/NeutronLethargy

3 Explanation

The correct method does the following (Figure 1):

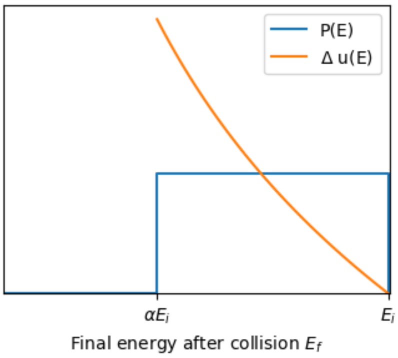

Figure 1: Multiplying $P(E_f)$ by $\Delta u(E_f)$ gives a quantity with the dimension of " Lethargy per unit energy" .

Integrating $\Delta u(E_f)P(E_f)$ over E_f gives the mean lethargy change $\overline{\Delta u}$.

Alternatively, if one wishes to visualize it in terms of Δu,

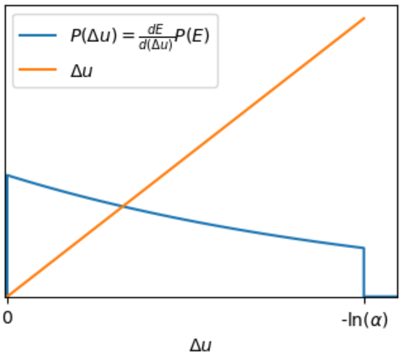

Figure 2: Multiplying $P(\Delta u)$ by Δu gives a dimensionless quantity.
The probability density function as a function of Δu is given by equation 6.

Integrating $\Delta u P(\Delta u)$ over Δu gives the mean lethargy change.

Note that figure 1 is oriented in the opposite direction as figure 2, neutrons starts with a high initial energy E_i (RHS of figure 1)/ zero lethargy (RHS of figure 2) $\Delta u = 0$, and scatter down to lower energy/ greater lethargy, with the minimum energy achievable $= \alpha E_i$/ maximum lethargy gained $= ln(\alpha)$ in a single scatter.

However, in section 1, the lethargy of the average average energy (green dotted line) is used instead.

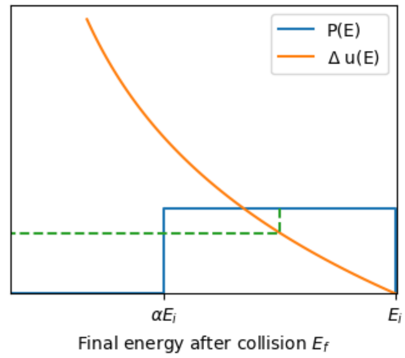

Figure 3: The mid point of the rectangular probability density function is used, and the (wrong) mean lethargy change $\overline{\Delta u}$ is inferred from that.

This will give an incorrect answer because the average E_f as determined by the rectangular area is different from the average as determined by the area under the curve $\Delta u P(\Delta u)$ (product of the two solid curves drawn).

Due to the non-linearity of the $\Delta u(E)$ curve in figure 3 a small 'excursion' to the left will contribute a much larger increment in lethargy gain than decrement in lethargy gain when there is a small 'excursion' to the right.

This means that a collision losing slightly more than $\overline{E}$ requires multiple collision losing slightly less energy than $\overline{E}$ to counter its effect. This problem is actually a confusion between arithematic mean (section 1) and geometric mean (section 2) in disguise. The former method falsely assumes that an arithmatic mean can be used, finding the arithmatic average of the energy lost $\Delta E = E_i - E_f$, instead of the geometric average of ΔE (which is identical to the arithmetic average of Δu).

Note that if α is very close to 1, then this non-linearity can be ignored, i.e. at very large atomic masses, e.g. ^{238}U, the above two methods should give very similar results.

4 Physical intuition

Consider 4 hypothetical collisional events:

4.1 No variation in factor of energy loss

If all collisional events gives away *exactly* the same amount of energy as in the mid-point of the distrbution (green line in figure 3), then the lethargy lost calculated by both methods in section 1 and 2 will be the same.

- Factor of reduction in energy after each collisional event =

  `0.6, 0.6, 0.6, 0.6`

- Total energy lost = $0.6^4 = 0.1296$
- Using section 1 's method : Energy lost = exp(4*ln(0.6)) = 0.1296
- Using section 2 's method: Energy lost = exp(ln(0.6)+ln(0.6)+ln(0.6)+ln(0.6)) = 0.1296

4.2 Accounting for variation in factor of energy loss

However, using another set of hypothetical energy loss factors, with the same arithmatic average (0.6) as the set before, but with a non-zero standard deviation:

- Factor of reduction in energy after each collisional event =

  `0.5, 0.7, 0.5, 0.7`

- Total energy lost = $0.6^4 = 0.1225$
- Using section 1 's method : Energy lost = exp(4*ln(0.6)) = 0.1296
- Using section 2 's method: Energy lost = exp(ln(0.5)+ln(0.7)+ln(0.5)+ln(0.7)) = 0.1225

This demonstrates how the numbers on the left hand side of the green line in figure 3 will skew the total energy lost towards a smaller value.

5 Acknowledgement

Many thanks to Sam Dyson for his insight and discussion on this topic, as well as providing the inspiration for the last section.