

When attempting to find the solution to the following question:

”How can the average number of collisions necessary to thermalise a fission neutron (slow it down from 2 MeV to 1 eV) in deuterium be calculated? ”

When deriving the number of collision required  $N$  for a given  $\alpha = (\frac{A-1}{A+1})^2$  (where  $A$  is the atomic mass), several fellow students has stumbled across the following misconception:

## 1 Incorrect method

An intuitive method is to do the following:

$$\overline{E_f} = E_i \frac{(\alpha + 1)}{2} = \int P(E) E dE \quad (1)$$

$$\overline{\Delta u} = \ln \left( \frac{E_i}{\overline{E_f}} \right) \quad (2)$$

$$N = \frac{\ln \left( \frac{2MeV}{1eV} \right)}{\overline{\Delta u}} \quad (3)$$

Which corresponds to the procedure as follows:

1. Find the average energy lost  $\overline{E}$
2. Calculate the lethargy change it represents
3. Find the number of collisions (each with energy  $\overline{E}$  lost) required to slow it down.

In a single equation, this is represented as:

$$N = \frac{u(2MeV) - 2(1eV)}{\Delta u \left( \int EP(E)dE \right)} N = \frac{\ln(20000000)}{\ln\left(\frac{1+\alpha}{2}\right)} \quad (4)$$

Which gives the incorrect answer of 22.7 collisions for 2MeV neutrons moderated by Deuterium  $A = 2$

## 2 Correct method

However, the correct way to do so is

$$\overline{\Delta u} = \int (\Delta u) P(\Delta u) d(\Delta u) \quad (5)$$

$$= \int_{E_f=\alpha E_i}^{E_f=E_i} \Delta u(E_f) P(\Delta u(E_f)) \frac{d(\Delta u)}{dE_f} dE_f \quad (6)$$

where the conversion to  $\Delta u$  is carried out as follows:

$$\Delta u(E_f) = \ln \left( \frac{E_i}{E_f} \right) \quad (7)$$

and the probability distribution in  $u$  space can be converted back into probability distribution in  $E_f$  space in a straightforward manner:

$$P(E_f)dE_f = P(\Delta u(E_f))d(\Delta u) \quad (8)$$

$$P(\Delta u(E_f))\frac{d(\Delta u)}{dE_f} = P(E_f) \quad (9)$$

simplifying equation 6

$$\overline{\Delta u} = \int_{E_f=\alpha E_i}^{E_f=E_i} (\Delta u(E_f))P(E_f)dE_f \quad (10)$$

$$= \int_{E_f=\alpha E_i}^{E_f=E_i} \ln\left(\frac{E_i}{E_f}\right) P(E_f)dE_f \quad (11)$$

$$(12)$$

Which will be evaluated to

$$\overline{\Delta u} = 1 + \frac{\alpha}{1+\alpha} \ln(\alpha) \quad (13)$$

$$N = \frac{\ln\left(\frac{2\text{MeV}}{1\text{eV}}\right)}{\overline{\Delta u}} \quad (14)$$

Which gives 20.8 collisions for 2MeV neutrons moderated by Deuterium  $A = 2$

The numerical accuracy of this answer can be verified with the code

NeutronLethargy/MonteCarlo.py

in the repository

<https://github.com/OceanNuclear/NeutronLethargy>

### 3 Explanation

The correct method does the following:

\*Insert figure 1

Multiplying  $P(E)$  by  $\Delta u(E)$  gives a quantity with the dimension of "Lethargy per unit energy".

Integrating  $\Delta u(E)P(E)$  over  $E$  gives the mean lethargy change.

Alternatively, if one wishes to work with  $\Delta u$ ,

\*Insert Figure 2

In figure ??, multiplying  $P(\Delta u)$  by  $\Delta u$  gives a dimensionless quantity.

Integrating  $P(\Delta u)\Delta u$  over  $\Delta u$  gives the mean lethargy change.

Note that figure ?? is oriented in the opposite direction as figure ??, neutrons starts with a high initial energy  $E_i$  (RHS of the former graph)/ zero lethargy (RHS of the latter graph)  $\Delta u = 0$ , and scatter down to lower energy/ greater lethargy, with the minimum energy achievable  $= \alpha E_i$ / maximum lethargy gained  $= \ln(\alpha)$  in a single scatter

However, in section 1, the following method is used (See figure ??) :

\*Insert Figure 3

This will give an incorrect answer because the average  $E_f$  as determined by the rectangular area is different from the average as determined by the area under the curve  $\Delta u P(E)$  (product of the two curves drawn).

Due to the non-linearity of the  $\Delta u$  vs  $E$  a small 'excursion' to the left will contribute a much larger increment in lethargy gain than decrement in lethargy gain when there is a small 'excursion' to the right.

Note that if  $\alpha$  is very close to 1, then this non-linearity can be ignored, i.e. at very large atomic masses, e.g.  $^{238}\text{U}$ , the above two methods should give very similar results.

This means that a collision losing slightly more than  $\overline{E}$  requires multiple collision losing slightly less energy than  $\overline{E}$  to counter its effect. This problem is actually a confusion between arithmetic mean (section 1) and geometric mean (section 2) in disguise. The former method falsely assumes that an arithmetic mean can be done, finding the arithmetic average of the energy lost  $\Delta E = E_i - E_f$ , instead of the geometric average of  $\Delta E$  (which is identical to the arithmetic average of  $\Delta u$ ).

## 4 Acknowledgement

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