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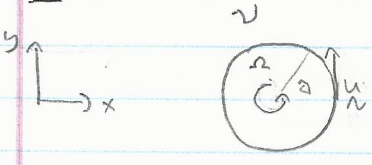
PHYS 426

Take Home Final

April 10, 2017

1.



1) cylinder rotates as solid body: $u_r = 0$
for $0 \le r \le a$ $\quad u_\theta = r\Omega\,\hat{\theta}$ $\quad u_\theta(a) = a\Omega$

no slip boundary: at $r = a$ $\quad \underset{\sim}{u}_{fluid} = \underset{\sim}{u}_{cyl} = a\Omega\hat{\theta}$
$r = \infty$ $\quad \underset{\sim}{u} = 0$

Vorticity $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{u} \equiv \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = 2\Omega\hat{k} = \omega_0\hat{k}$

$$\frac{D}{Dt}(\underset{\sim}{\omega}) = (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} + \nu\nabla^2(\underset{\sim}{\omega})$$

in steady state $\quad 0 = (\cancelto{0}{(\nabla\times\underset{\sim}{u})\cdot\nabla})\underset{\sim}{u} + \nu\nabla^2(\nabla\times\underset{\sim}{u})$
in polar coordinates:

$$0 = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)$$

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} = 0$$ can rewrite as ODE:

$$\frac{d^2 u_\theta}{dr^2} + \frac{d}{dr}\left(\frac{u_\theta}{r}\right) = 0 \Rightarrow \frac{du_\theta}{dr} + \frac{u_\theta}{r} = C_1$$

$$\frac{d(r\,u_\theta)}{dr} = C_1 r$$

$$r\,u_\theta = \frac{1}{2}C_1 r^2 + C_2$$

$$u_\theta = \frac{1}{2}C_1 r + \frac{C_2}{r}$$

$u_\theta(\infty) = 0 \Rightarrow C_1 = 0$
$u_\theta(a) = a\Omega = \frac{C_2}{a} \Rightarrow C_2 = a^2\Omega$

$$\boxed{u_\theta = \frac{a^2\Omega}{r}}$$

✓ 2) Bernoulli: $\frac{D}{Dt}\left[\frac{1}{2}|u|^2 + gz + \frac{P}{\rho}\right] = u \cdot F + \frac{1}{\rho}\cancel{\frac{\partial P}{\partial t}}^{0}$

in steady state $u \cdot F = 0$

$$u \cdot F = \underbrace{\nabla\cdot(u \cdot \tau)}_{\text{work done by cyl on fluid}} - \underbrace{\nu(\nabla u)^2}_{\text{viscous dissipation}} = 0$$

$$\int \nabla\cdot(u\cdot\tau) = \oint_{cyl} u_\theta \tau_\theta \cdot dA \qquad \tau_\theta = \nu\cancel{\frac{\partial u_\theta}{\partial \theta}}^{0} + \nu\frac{\partial u_\theta}{\partial r} + \nu\cancel{\frac{\partial u_\theta}{\partial z}}^{0}$$

$$= \nu\int_{cyl} u_\theta \frac{\partial u_\theta}{\partial r}\, dA = \nu\int_0^{2\pi}\left(\frac{a^2\Omega}{r}\right)\left(-\frac{a^2\Omega}{r^2}\right)d\theta$$

$$= -2\pi\nu\frac{a^4\Omega^2}{r^3} \text{ work} \qquad \nabla u_\theta = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\cancel{\frac{\partial u_\theta}{\partial \theta}}^{0} + \cancel{\frac{\partial u_\theta}{\partial z}}^{0}$$

$$\int_{vol}\nu\left(\frac{\partial u_\theta}{\partial r}\right)^2 dV = \nu\int_0^{2\pi}\int_a^\infty\left(-\frac{a^2\Omega}{r^2}\right)^2 dr\, d\theta$$

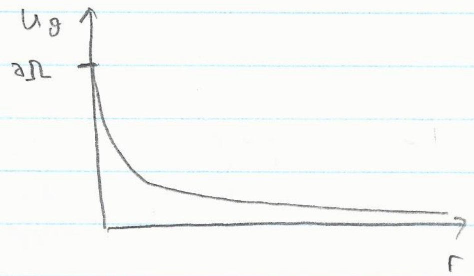
Same!

$$= 2\pi\nu\int_a^\infty\left(\frac{a^4\Omega^2}{r^4}\right)dr = 2\pi\nu\left[-\frac{a^4\Omega^2}{3r^3}\right|_a^\infty$$

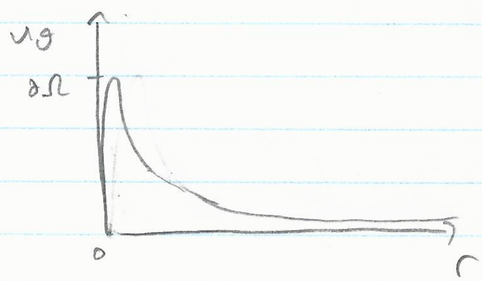
$$= -2\pi\nu\frac{a^4\Omega^2}{r^3} \text{ dissipation}$$

hence rate of work equals viscous dissipation

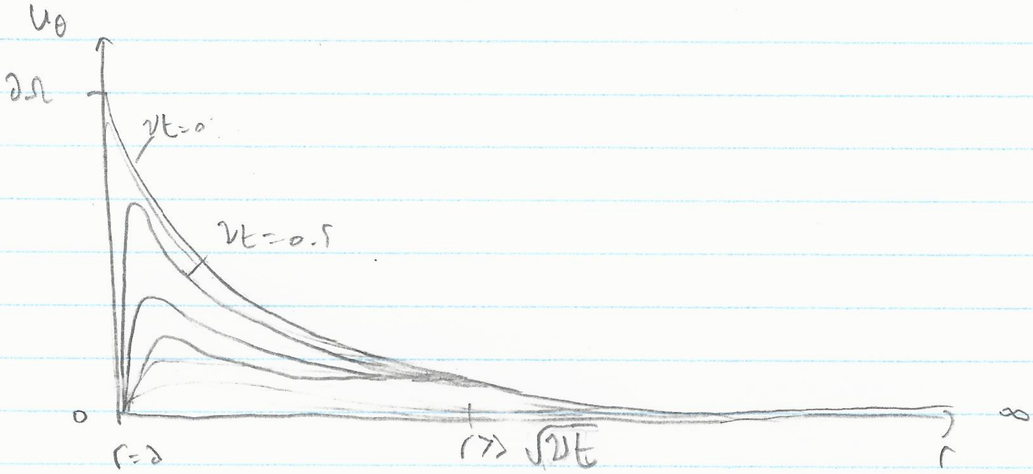
3) when cylinder rotating



when cylinder stops $t=0$



Sketch of temporal evolution of Velocity field



$\sqrt{\nu t}$ units [m]

$u_\theta = \dfrac{\Gamma}{2\pi r}$ - circulation $\qquad \Gamma = 2\pi \Omega a^2$

① $\dfrac{\partial u_\theta}{\partial t} = \nu \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}(r u_\theta)\right]$ $\qquad u_\theta(r,0) = \dfrac{\Gamma}{2\pi r}$

$u_\theta(a,t) = 0$

$u_\theta(r\to\infty, t) = \dfrac{\Gamma}{2\pi r}$ →

letting non-dimensional velocity

$$u' \equiv \frac{u_\theta}{\Gamma/2\pi r} = f(r,t,\nu) = F(\eta)$$

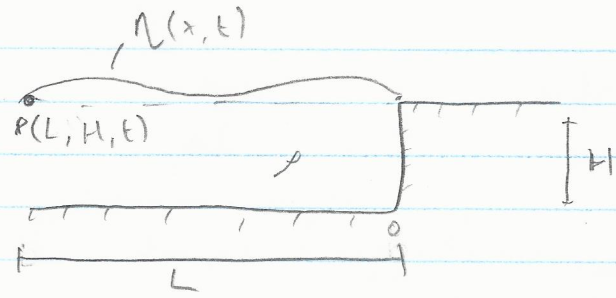
non dimensional grouping $\dfrac{r}{\sqrt{\nu t}}$

$$\eta = \frac{r^2}{4\nu t} \quad \text{sub into (1)}$$

$$F'' + F' = 0 \qquad \begin{matrix} F(\infty) = 1 \\ F(0) = 0 \end{matrix} \Rightarrow F = 1 - e^{-\eta}$$

$$u_\theta = \frac{\Gamma}{2\pi r}\left[1 - e^{-r^2/4\nu t}\right]$$ form of sketches

2.



$P(L,t) = P_0 \cos(\omega t)$

g: gravitational constant

1) hydrostatic approximation: $\frac{1}{\rho_0}\frac{\partial P}{\partial x} = g\frac{\partial \eta}{\partial x} \Rightarrow \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$

assuming there is no fluid above the basin $(P \geq (H+\eta) = 0)$

$P(L,H,t) = \rho g \eta$ $\qquad \omega t = \frac{n\pi}{2} \Rightarrow P = 0$

$(n = 1, 3, 5 \ldots)$

$P_0 \cos(\omega t) = \rho g \eta$

let $\eta_0 = \frac{P_0}{\rho g} \Rightarrow \eta(L,t) = \eta_0 \cos(\omega t)$

$\eta(x,t) = Ae^{i(kx-\omega t)} + Be^{i(-kx-\omega t)}$ $\qquad \frac{\omega}{k} = \sqrt{gH}$

no energy loss $|A| = |B|$

$\eta(L,t) = \eta_0 \cos(\omega t) = Re(\eta_0 e^{-i\omega t})$

$\Rightarrow A + B = \eta_0$

$u(0,t) = 0 \Rightarrow \frac{\partial \eta(0,t)}{\partial x} = 0 \Rightarrow ikAe^{i(\ -\omega t)} - ikBe^{i(\ -\omega t)} = 0$

$A = B \Rightarrow B = \frac{\eta_0 e^{ikL}}{e^{ikL} + e^{-ikL}} = \frac{\eta_0 e^{ikL}}{2\cos kL}$

$A = \frac{\eta_0}{2}\frac{e^{-ikL}}{\cos kL}$

$$\boxed{\eta = \eta_0 \frac{\cos[kx]}{\cos[kL]} \cos[\omega t]}$$

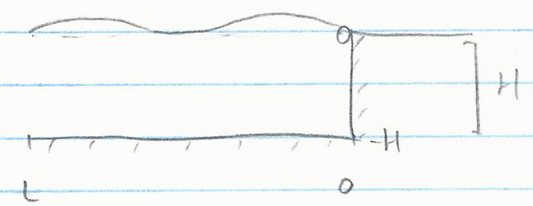
2) from hydrostatic: $\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}$

$$\frac{\partial u}{\partial t} = -gk\eta_0 \frac{\sin(k(x))}{\cos(kL)} \cos\omega t$$

$$\boxed{u(x,t) = -\frac{gk}{\omega}\eta_0 \frac{\sin(kx)}{\cos(kL)} \sin\omega t}$$

$u(x,t)$ functionally dependent on $g, \omega, \rho, P_0, k, L$ & t (with H written above)

3)



$$u(L,t) = u_0 \sin(\omega t + \phi)$$

No E loss: $u_{No}(L,t) = -\frac{gk}{\omega}\left(\frac{P_0}{\rho g}\right)\frac{\sin(kL)}{\cos(kL)} \sin\omega t = \frac{u k P_0}{\omega \rho} \tan(kL) \sin(\omega t)$

E loss: $u(L,t) = u_0 \sin(\omega t + \phi)$

$$\langle KE_{no} \rangle = \int_{-H}^{\eta} \frac{1}{2} u_{No}^2 \, dz = (\overset{0}{\cancel{\eta}} + H)\frac{1}{2}u^2$$

$$KE_{no} = \frac{H}{2}\frac{k^2 P_0^2}{\omega^2 \rho^2} \tan^2(kL)\sin^2(\omega t)$$

$$\langle KE_{no} \rangle_? = \frac{H}{2}\frac{k^2 P_0^2}{\omega^2 \rho^2}\left(\frac{1}{2}\right)$$

$$APE_{no} = \frac{1}{2}\rho g \eta^2 = \frac{1}{2}\rho g \left(\frac{P_0}{\rho g}\right)^2 \cos^2(\omega t)$$

$$\langle APE_{no} \rangle = \frac{P_0^2}{4\rho g}$$

$$\langle E_{No} \rangle_? = \frac{H}{4}\frac{k^2 P_0^2}{\omega^2 \rho^2} + \frac{P_0^2}{4\rho g}$$

$$KE = \int_{-H}^{\eta} \frac{1}{2} u^2 dz = \frac{H}{2} u^2 = \frac{H}{2} u_0^2 \sin^2(\omega t + \phi)$$

$$\langle KE \rangle_\lambda = \frac{H u_0^2}{4} = \langle APE_\lambda \rangle$$

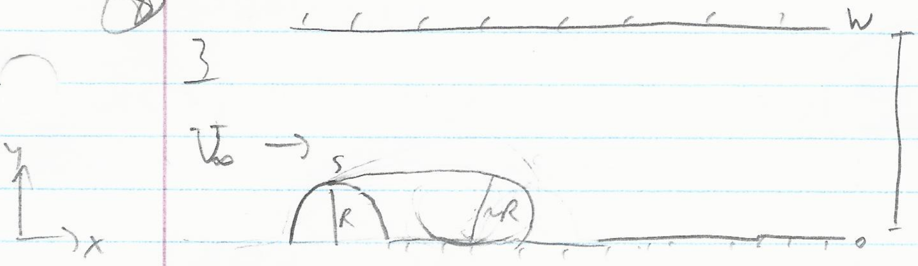
$$\left\langle \frac{dE}{dt} \right\rangle_\lambda = \langle E_{N_0} \rangle_\lambda - \langle E \rangle_\lambda$$

$$= \frac{P_0^2}{4} \underbrace{\left[\frac{H k^2}{\omega^2 \rho^2} + \frac{1}{\rho g} \right]}_{A} - \frac{H u_0^2}{2}$$

$$= \frac{P_0^2 - H u_0^2}{4} \left[A \right]$$

☆ 4) $\phi \ll 1$ $\quad p_0 \ll \rho g H$ $\quad L \ll \frac{2\pi \sqrt{gH}}{\omega}$

3



S = separation point

$u_r = 0$ } at surface $\Rightarrow$ $\psi(R,\theta) = 0$
$u_\theta = 0$ } $\frac{\partial \psi}{\partial r}(R,\theta) = 0$

1) outside of boundary layer (assuming viscous flow)

at $x = 2R$ to W
$\psi = 0$

$\psi = U_\infty \left(r - \frac{R^2}{r^2}\right)\sin\theta \Big|_{\theta=0}^{\pi}$

$\phi = U_\infty \left(r - \frac{R^2}{r^2}\right)\cos\theta$

$u_\theta = -U_\infty\left(1 + \frac{R^2}{r^2}\right)\sin\theta = -2U_\infty \sin\theta$ at $r = R$

$\underline{|u_\theta|_{max} = 2U_\infty}$

2) $P_{bubble} = P_s$ $P_\infty + \frac{1}{2}\rho U_\infty^2 = P_s + \frac{1}{2}\rho u_{\theta_s}^2$

$P_s = P_\infty - \frac{1}{2}\rho U_\infty^2 (1 - 4\sin^2\theta)$

$P = \text{const}$ $\theta \geq \pi/2$

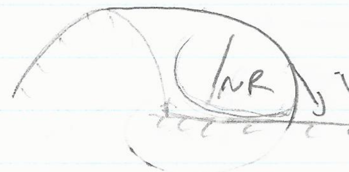
$D = -\int_0^{\pi/2} R\cos\theta \, P_s \, d\theta$ $\boxed{D = -\frac{1}{2}\rho U_\infty^2 R}$

defining drag & pressure coefficients $C_D \equiv \frac{D}{\rho U_\infty^2 R}$

$C_p \equiv \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2}$

$C_D = -\frac{1}{2}\int_0^{\pi/2} C_p \cos\theta \, d\theta$

3)



$$u_\theta = \begin{cases} \Gamma r/(2\pi R^2) & r \le R \\ \Gamma/(2\pi r) & r > R \end{cases}$$

Surface of water in vortex ($r \sim R$)
rough and turbulent
outside vortex smoother, with
u_θ flow along surface of headland
& channel edge (looks like an eddy)

4) At the moment the flow
turns off, the vortex will
travel back towards the headland
with initial speed $-U_\infty$ because
of negative C_p where
vortex is.

5) $h < H \rightarrow u_\theta$ decreases $\frac{\omega_0}{H} = \frac{\omega_f}{h}$

Surface height will
be parabolic with $\eta = h - \frac{R^2}{r^2}$