Take-home Final
Phy 426, 2017
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DUE: Thu 11 Apr, 2017, 1000 Exam to be completed independently. Open book, open notes are fine. Show all work, define any constants you need that I don't provide, check your units, etc. Except as noted, the density of the fluid is ρ, gravity is g, the kinematic viscosity ν, and the fluid can be assumed Bousinesque and incompressible.

Please try to make it readable. I will deduct up to 10% for illegible chicken scratches, so please take the time to recopy your work.

The value of each question is indicated in square brackets, the total is out of 75.

Question 1. Viscous flow around a rotating cylinder

Consider an infinite viscous fluid with a solid cylinder of radius a spinning with rotation rate Ω:

1. [7] Show that the velocity of the fluid is given $u_{theta} = \Omega a^2/r$, where r is the distance from the axis of rotation of the cylinder.
2. [10] Show that the rate of work done by the cylinder on the flow is equal to the viscous dissipation in the flow.
3. [8] *Sketch* what happens to the velocity in time field if the cylinder suddenly stops rotating. Indicate an appropriate relationship between the spatial and temporal scales in your sketch.

Question 2. Lossy standing waves

Consider the quasi-steady response in a rectangular basin H deep, W wide, with a vertical wall at one end to forcing at a tidal frequency ω.

1. [4] A distance L from the far end, the pressure is measured with a gauge to vary as $p = p_o \cos \omega t$. Assuming no energy losses in the basin, what is the water height as a function of x, and t?
2. [4] What would the functional dependence of $u(x, t)$ be?
3. [10] Suppose we measure u at $x = L$ from the far end, and it is given my $u = u_o \sin(\omega t + \phi)$. what is the average rate of energy loss inside the embayment in terms of p_o and u_o?
4. [7] If $\phi << 1$, $p_o << \rho g H$, and $L << \frac{2\pi\sqrt{gH}}{\omega}$, approximate the above just in terms of p_o and ϕ.

Question 3. Flow around a Headland

Consider a headland protruding into a rectangular channel of free-stream depth H. The headland is approximated by a circle with radius R. Assume that R is much less than the width of the channel, and that the free-stream speed far from the headland is U along the channel. The flow separates from the headland at the tip.

1. [4] What is the fastest speed along the headland?
2. [4] Assume that the pressure in the separation bubble behind the tip is the same as at the tip. What is the total drag on the flow exerted by the headland?
3. [4] Assuming that the separation vortex is a Rankine vortex with a radius similar to the headland (i.e. approximately R) and outer speed given by the speed of the flow past the headland, what does the surface of the water look like?
4. [4] Suppose the mean flow suddenly turns off. Where and how fast will this eddy start to move?
5. [5] Suppose the eddy travels around, with its vorticity core largely intact, and that it slowly shoals into water h deep, where $h < H$. Describe the surface height as a function of distance from the centre of the eddy at this point.

Answer Key for Exam $\boxed{\text{A}}$

Question 1. Viscous flow around a rotating cylinder

Consider an infinite viscous fluid with a solid cylinder of radius a spinning with rotation rate Ω:

1. [7] Show that the velocity of the fluid is given $u_{theta} = \Omega a^2/r$, where r is the distance from the axis of rotation of the cylinder.

   Answer: There is no preferred direction to the motion, so all $\partial/\partial\theta$ terms must be zero in the Navier-Stokes equations. Similarly there can't be any radial velocity so $u_r = 0$ everywhere in the flow. So only the θ direction equation of motion needs to be solved:

   $$\nu\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}u_\theta - \frac{u_\theta}{r^2}\right) = 0 \tag{1}$$

   This can be rewritten as

   $$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) = 0 \tag{2}$$

   which can be integrated to get $u_\theta = Ar + \frac{B}{r}$, and the boundary conditions give $u_\theta = \Omega a^2/r$.

2. [10] Show that the rate of work done by the cylinder on the flow is equal to the viscous dissipation in the flow.

   Answer: The work done is

   $$W = \int_0^{2\pi} a u_\theta(a)\tau_{r\theta}(a)d\theta \tag{3}$$

   The viscous stress perpendicular to the θ-direction flow is $\rho\nu r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = -2\rho\nu\Omega a^2/r^2 = -2\rho\nu\Omega$ so

   $$W = 4\pi\rho\nu a^2\Omega^2 \tag{4}$$

   The dissipation rate in a fluid is given by $= 2\rho\nu e_{ij}e_{ij}$ (See chapter 4). The strain rates are given in Appendix B in cylindrical co-ordinates, and the only terms that do not involve $u_r = 0$ and $\partial/\partial\theta$ are $e_{r\theta} = e_{\theta r} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = \Omega a^2/r^2$. So this means that $\epsilon = 2\rho\nu(e_{r\theta}^2 + e_{\theta r}^2) = 4\rho\nu\Omega^2 a^4/r^4$. Integrating over the whole fluid:

   $$\begin{aligned} D &= \int_a^\infty \int_0^{2\pi} \epsilon r d\theta dr \\ &= 8\pi\rho\nu\Omega^2 a^4 \int_a^\infty r^{-3} dr \\ &= 4\pi\rho\nu a^2\Omega^2 \end{aligned}$$

3. [8] *Sketch* what happens to the velocity in time field if the cylinder suddenly stops rotating. Indicate an appropriate relationship between the spatial and temporal scales in your sketch.

Question 2. Lossy standing waves

Consider the quasi-steady response in a rectangular basin H deep, W wide, with a vertical wall at one end to forcing at a tidal frequency ω.

1. [4] A distance L from the far end, the pressure is measured with a gauge to vary as $p = p_o \cos \omega t$. Assuming no energy losses in the basin, what is the water height as a function of x, and t?

   Answer: This is just a standing wave so we have

$$
\begin{aligned}
p &= (A/2)\left(\cos\left(kx-\omega t\right)+\cos\left(kx+\omega t\right)\right) && (5)\\
&= A\cos(kx)\cos(\omega t) && (6)\\
&= p_o\frac{\cos(kx)}{\cos(kL)}\cos(\omega t) && (7)
\end{aligned}
$$

   and the waves are long, so the surface height is given by:

$$
\eta(x,t) = \frac{p_0}{\rho g}\frac{\cos(kx)}{\cos(kL)}\cos(\omega t) \qquad (8)
$$

2. [4] What would the functional dependence of $u(x,t)$ be?

   Answer: For long waves

$$
\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \qquad (9)
$$

   so,

$$
\begin{aligned}
u(x,t) &= \frac{p_0}{2\rho}\frac{k}{\omega}\left(-\cos\left(kx-\omega t\right)+\cos\left(kx+\omega t\right)\right) && (10)\\
&= \frac{p_0}{\rho}\frac{k}{\omega}\frac{\sin(kx)}{\cos(kL)}\sin(\omega t) && (11)
\end{aligned}
$$

   using the same angle identities we used for the standing wave above. Note that u and eta are out of phase and that $u(x=0)=0$.

3. [10] Suppose we measure u at $x=L$ from the far end, and it is given my $u = u_o \sin\left(\omega t + \phi\right)$. what is the average rate of energy loss inside the embayment in terms of p_o and u_o?

   Answer: The instantaneous energy flux into the inlet is given by

$$
\begin{aligned}
F &= \int_o^H up\,\mathrm{d}z && (12)\\
&= Hup && (13)
\end{aligned}
$$

   and then to get the mean, we average over the time period $T = 2\pi/\omega$:

$$
\begin{aligned}
\langle F\rangle &= \frac{H}{T}\int_o^T up\mathrm{d}t && (14)\\
&= \frac{H}{T}p_o u_o\int_o^T \sin\left(\omega t+\phi\right)\cos(\omega t)\mathrm{d}t && (15)\\
&= \frac{H}{T}p_o u_o\int_o^T \left(\sin\omega t\cos\omega t\cos\phi + \cos^2\omega t\cos\phi\right)\mathrm{d}t && (16)
\end{aligned}
$$

The first integrand vanishes, and the second is easy to evaluate:

$$< F >= \frac{H p_o u_o}{2} \cos\phi \tag{17}$$

The rate of energy loss is simply this net flux into the bay. Note that if $\phi = 0$, there is not energy loss, and we are back to a standing wave.

4. [7] If $\phi << 1$, $p_o << \rho g H$, and $L << \frac{2\pi\sqrt{gH}}{\omega}$, approximate the above just in terms of p_o and ϕ.

Answer: In this limit, the small phase change means that the bay largely rises and falls in qudrature. Also, the seasurface slope is small, so the flux of water into the bay is equal to the seasurface height change:

$$Hu = L\frac{\partial \eta}{\partial t} \tag{18}$$

so, given our variables:

$$u_o = (L/H)\omega\frac{p_o}{\rho g} \tag{19}$$

so, we have

$$< F >= \frac{p_o^2}{2\rho g} \cos\phi \tag{20}$$

Question 3. Flow around a Headland

Consider a headland protruding into a rectangular channel of free-stream depth H. The headland is approximated by a circle with radius R. Assume that R is much less than the width of the channel, and that the free-stream speed far from the headland is U along the channel. The flow separates from the headland at the tip.

1. [4] What is the fastest speed along the headland?

Answer: The flow in the channel is just the same as flow around a sphere, if the channel is wide enough. Therefore, assuming the flow is irrotational:

$$u_\theta = -2U\sin\theta \tag{21}$$

which has a maximum at $\theta = \pi/2$, and $max(u) = 2U$ in the downstream direction.

2. [4] Assume that the pressure in the separation bubble behind the tip is the same as at the tip. What is the total drag on the flow exerted by the headland?

Answer: The drag on the headland is simply the rpessure force exerted in the x-direction:

$$D = R\int_0^{\pi} p(\theta)\cos\theta \mathrm{d}\theta. \tag{22}$$

The pressure on the surface is given by $p(\theta) = \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)$ for $\pi/2 < \theta < \pi$, and $-3/2\rho U^2$, for $0 < \theta < \pi/2$. For the back half:

$$D_2 = -3/2\rho R U^2. \tag{23}$$

For the front half

$$\begin{aligned} D_1 &= R\int_{\pi/2}^{\pi} \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right)\cos\theta \mathrm{d}\theta. &(24)\\ &= R\frac{1}{2}\rho U^2(-1+\frac{4}{3}). &(25)\\ &= \frac{1}{6}\rho R U^2 &(26)\end{aligned}$$

So, the total drag is $-\frac{4}{3}\rho R U^2$.

3. [4] Assuming that the separation vortex is a Rankine vortex with a radius similar to the headland (i.e. approximately R) and outer speed given by the speed of the flow past the headland, what does the surface of the water look like?

Answer: The azimuthal velocity in a Rankine vortex is given by $u_\theta = 2Ur/R$. The water surface provides the pressure gradient that keeps the water moving in a circle:

$$\begin{aligned} -g\frac{\partial\eta}{\partial r} &= -\frac{u_\theta^2}{r} &(27)\\ &= -\frac{4U^2}{R^2}r &(28)\end{aligned}$$

so, $\eta(r) = \frac{2U^2}{g}(\frac{r^2}{R^2} - 1)$ for $r < R$, assuming $\eta = 0$ at $r = R$. Outside the vortex, the surface can also be determined, but it will likely be swamped by other flow factors.

4. [4] Suppose the mean flow suddenly turns off. Where and how fast will this eddy start to move?

Answer: The method of images tells us that the eddy will be self-advected. Because the eddy is in solid body rotation, the actual interaction will be complicated, but lets approximate by thinking about where the center of the eddy will move. In that case, if the eddy is R from the wall, it is $2R$ from its image eddy. The velocity outside the core is just $u_\theta(r) = 2UR/r$, so the self-advection is at speed U towards the headland. Note that this is consistent with the eddy being held in place by the oncoming flow.

5. [5] Suppose the eddy travels around, with its vorticity core largely intact, and that it slowly shoals into water h deep, where $h < H$. Describe the surface height as a function of distance from the centre of the eddy at this point.

Answer: Conservation of potential vorticity tells us that