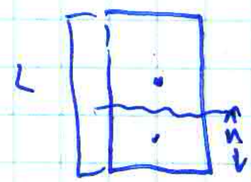
Assignment #1

Q1

Block:



$$h\rho_0 = L\rho \quad \Rightarrow \quad h = L\frac{\rho_0}{\rho}$$

Tilting? I found it easier to rotate reference frame:

Now gravity points

$$\underset{\sim}{F_1} = \rho g L d\left(-\theta, -1\right)$$

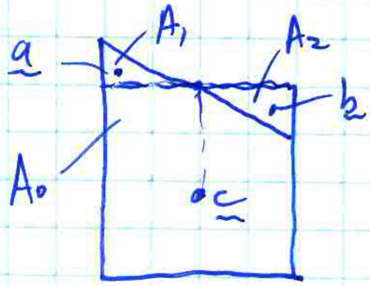
$$\underset{\sim}{F_2} = \rho g L d\left(\theta, 1\right)$$

Now to get torques you need to know where. Choosing origin where the block was tilted

$$\underset{\sim}{r_1} = \left(0, \frac{L}{2} - h\right)$$

$$\tau_1 = \underset{\sim}{r_1} \times \underset{\sim}{F_1} = \rho g L d\,(+\theta)\left(\frac{L}{2} - h\right)$$

$\underset{\sim}{r_2}$ requires a bit of geometry to find c.o.m of displaced water



$$\underset{\sim}{r_2} = \frac{\underset{\sim}{a} A_1 - \underset{\sim}{b} A_2 + \underset{\sim}{c} A_0}{A_0}$$

$$\underset{\sim}{a} = \left(-\frac{2}{3}\frac{d}{2},\ \frac{\theta}{3}\frac{d}{2} \right)$$

$$\underset{\sim}{b} = \left(\frac{2}{3}\frac{d}{2},\ -\frac{\theta}{3}\frac{d}{2} \right)$$

$$\underset{\sim}{c} = \left(0,\ -\frac{h}{2} \right)$$

$$A_0 = dh$$

$$A_1 = A_2 = \frac{1}{2}\left(\frac{d}{2}\right)^2 \theta$$

$$\underset{\sim}{r_2} = \left(\frac{-\frac{2}{3} d \frac{d^2}{8}\theta,\ -\frac{dh^2}{2} + \cancel{\theta^2 \text{ terms}}}{dh} \right)$$

$$= \left(-\frac{2}{3}\frac{d^2}{h}\frac{1}{8}\theta,\ -\frac{h}{2} \right)$$

so $$\underset{-}{\tau_2} = \left(-\frac{2}{3}\frac{d^2}{h}\frac{\theta}{8} + \frac{h}{2}\theta \right) \rho g L d$$

$$\tau_1 + \tau_2 = \rho g L d \theta \left(\frac{1}{12}\frac{d^2}{h} + \frac{h}{2} + \frac{L}{2} - h \right)$$

$$= \frac{\rho g L d \theta}{h} \left(\frac{1}{12} d^2 - \frac{h^2}{2} + \frac{hL}{2} \right)$$

Stable if < 0

So $\tau_1 + \tau_2 < 0$ if stable

$$\Rightarrow \quad -\frac{d^2}{h}\frac{1}{12} + \cancel{\frac{h}{2}} + \frac{L}{2} - \frac{h}{2} < 0$$

$$-\frac{d^2}{h} + 6(L-h) < 0$$

$$\frac{d^2}{h} > 6(L-h)$$

$$\boxed{d^2 > 6L^2 \frac{\rho_0}{\rho}\left(1 - \frac{\rho_0}{\rho}\right)}$$

10

1.2 Suppose block unstable

ie $\frac{d^2}{L^2} < 6\, r(1-r)$ $\quad r \equiv \frac{\rho_0}{\rho}$

is it stable if you flip it

ie is $\frac{L^2}{d^2} > 6\, r(1-r)$?

well, we know

$$\frac{L^2}{d^2} > \frac{1}{6\; r(1-r)}$$

which is stable if

$$> 6\, r(1-r)$$

$\therefore$ stable if

$$\frac{1}{6\, r(1-r)} > 6\, r(1-r)$$

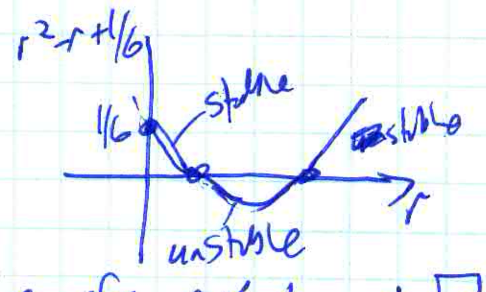
$$\Rightarrow r^2 - r + \frac{1}{6} > 0$$

so this has zeros at

$$r = \frac{1}{2} \mp \frac{1}{2}\sqrt{\frac{1}{3}}$$

also at $r=0$ $= \frac{1}{6}$

$\therefore$ convex up

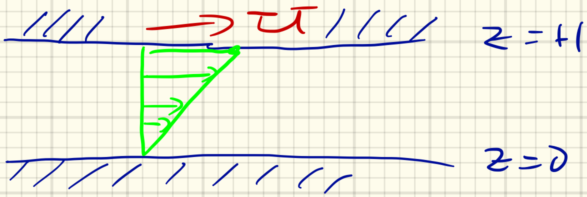


stable if $r < \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{3}}$ or $r > \frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{3}}$

unstable again if r between these

(5)

Ass 1 Q2



$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} \qquad u(z,t)$$

B.C. $u(z, t=0) = \emptyset$

$u(z=0) = \emptyset$

$u(z=H) = U$

1) steady state?

$$\nu \frac{\partial^2 u_0}{\partial z^2} = 0 \quad \Rightarrow \quad u_0 = Az + b$$

or $\boxed{u_0 = U \frac{z}{H}}$

2) $w = u - u_0 \quad \Rightarrow \quad \frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial z^2}$

$w(z, t=\emptyset) = -U \frac{z}{H}$

$w(z, t \to \infty) = \emptyset$

$w(z=0) = \emptyset$

$w(z=H) = \emptyset$

3) $w = T(t)\, Z(z)$

$$T' Z = \nu T Z''$$

$$\Rightarrow \frac{T'}{\nu T} = \frac{Z''}{Z} = c_j^2$$

c_j^2 is some constant to be determined

so $\frac{Z''}{Z} = c_j^2$ $\qquad Z(0) = Z(H) = 0$

$\Rightarrow$

$$Z = \sum_{j=0}^{\infty} A_j \sin(k_j z) \qquad ; \quad k_j = \frac{j\pi}{H}$$

First : $\boxed{-k_j^2 = c_j^2}$ which we need below

Second $\quad z = -\frac{\tau z}{H} = \sum_{j=0}^{\infty} A_j \sin(k_j z)$

$$\frac{H}{2} A_j = -\frac{\tau}{H}\int_0^H z \sin\left(\frac{j\pi z}{H}\right) dz$$

$y = \frac{j\pi z}{H}$; $dy = \frac{j\pi dz}{H}$

$$A_j = -\frac{2\tau}{H^2}\frac{H^2}{j^2\pi^2}\int_0^{j\pi} y \sin y \, dy$$

$$= \frac{2\tau}{j^2\pi^2}\left(y\cos y + \sin y\right)\Big|_0^{j\pi}$$

$$= \frac{2\tau}{j^2\pi^2}\left(j\pi \cos j\pi\right) = \frac{2\tau}{j\pi}(-1)^j$$

now for time dependence

$$T' = \nu c^2 T$$

$$= -\nu \frac{j^2 \pi^2}{H^2} T$$

$$\boxed{T = e^{\omega_j t}}$$

$$\boxed{\omega_j = -\nu \frac{j^2 \pi^2}{H^2}}$$

< 0 ∴ decays w/ time.

so

$$\boxed{\frac{u}{U} = \frac{z}{H} + \frac{2}{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j}{j} e^{\omega_j t} \sin\left(\frac{j \pi z}{H}\right)}$$

Full soln

$$\omega_j = -\nu \frac{j^2 \pi^2}{H^2}$$

$j^2 \uparrow_j |\omega_j| \uparrow$ so decays faster in time

/10

✓

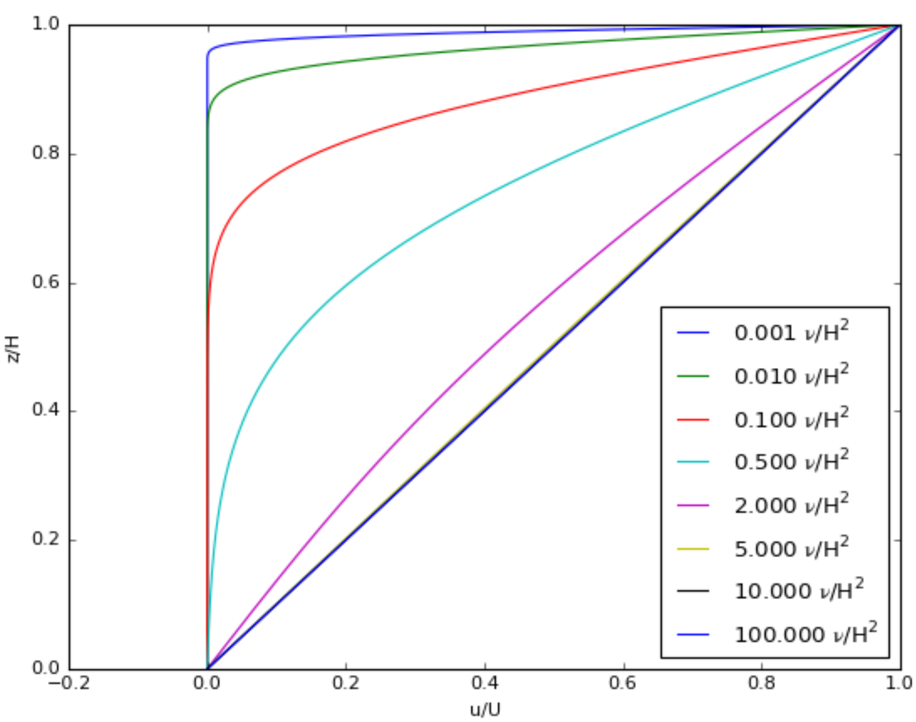
In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib notebook
```

```
/Users/jklymak/anaconda2/lib/python2.7/site-packages/matplotlib/__init__.py:878:
UserWarning: axes.color_cycle is deprecated and replaced with axes.prop_cycle; p
lease use the latter.
  warnings.warn(self.msg_depr % (key, alt_key))
```

In [2]:
```
z = np.linspace(0.,1.,1000)
```

In [19]:
```
fig,ax = plt.subplots()
j = np.arange(1,10000)
A = (2./j/np.pi)*(-1.)**(j)
nu = 0.1

for t in  [0.0001/nu,0.001/nu,0.01/nu,0.05/nu, 0.2/nu, 0.5/nu,1./nu,10./nu]:
    # this is the steady-state solution...
    Z = 0.+z;
    # these are the anomalies per vertical mode
    for jj in j:
        om = -nu*(jj*np.pi)**2
        Z += A[jj-1]*np.exp(om*t)*np.sin(jj*np.pi*z)
    ax.plot(Z,z,label=r'%1.3f $\nu/H^2$'%t)
ax.legend(loc=4)
ax.set_xlabel('u/U')
ax.set_ylabel('z/H')
```

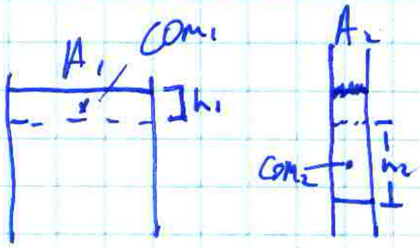


Out[19]: <matplotlib.text.Text at 0x110341e50>

In [ ]:

Q3 Hydraulic Jack

Work done to move water $= \Delta PE$



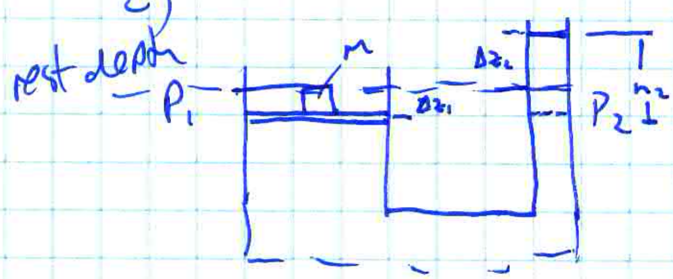
The water ~~th~~ on right is moved to A_1 pipe

mass $m = \rho A_1 h_1$

com moves by $\frac{h_2}{2} + \frac{h_1}{2} = \frac{h_1}{2}\left(\frac{A_1}{A_2}+1\right)$

so $\Delta PE = \frac{\rho g A_1 h_1^2}{2}\left(\frac{A_1}{A_2}+1\right)$ = work done to move water.

2)



$P_1 = \frac{mg}{A_1}$

$P_2 = P_1 = \rho g h_2$

$h_2 = \frac{m}{\rho A_1}$

That's how much higher level in small pipe is now than plunger in big pipe.

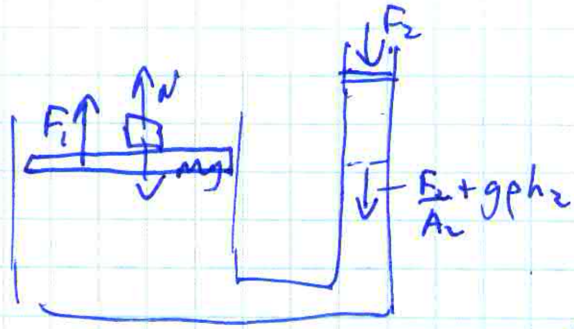
want $\Delta z_1 A_1 = \Delta z_2 A_2$

$= (h_2 - \Delta z_1) A_2$

$\Delta z_1 = \frac{h_2 A_2}{A_1 + A_2} = \frac{m A_2}{\rho A_1 (A_1 + A_2)}$

$\Delta z_2 = \frac{m}{\rho} \frac{1}{(A_1 + A_2)} > \Delta z_1$ ✓

Q3 3)



$$P_2 = P_1$$

~~$F_2/A_2 = F_1/A_1$~~

~~$F_1 = F_2 \frac{A_2}{A_1} \ldots F_2$~~

$$\frac{F_2}{A_2} = \frac{F_1}{A_1}$$

4) Suppose

$W_1 = F_1 u_1$

$W_2 = F_2 u_2$

if pipe 2 moves at u_2

then pipe 1 moves $u_1 = \frac{u_2 A_2}{A_1}$

and $W_1 = \frac{A_1}{A_2} F_2 u_2 \frac{A_2}{A_1} = W_2$

and ~~water~~ work done on water at pipe 2 is same as work done at pipe 1

/8

Q4

$$\underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = \underset{\sim}{\omega}\times\underset{\sim}{u} + \nabla\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{u}}{2}\right)$$

suffices to show for x-component

$$\underset{\sim}{u}\cdot\nabla u = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\underset{\sim}{\omega}\times\underset{\sim}{u} = \begin{matrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ u & v & w \end{matrix} \qquad i\text{th component} = \omega_y w - \omega_z v$$

$$\omega_y = ? \quad \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ u & v & w \end{vmatrix} \qquad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

6

$$\therefore\ \underset{\sim}{\omega}\times\underset{\sim}{u}\Big|_i = w\frac{\partial u}{\partial z} - \cancel{w\frac{\partial w}{\partial x}} - \cancel{v\frac{\partial v}{\partial x}} + v\frac{\partial u}{\partial y}$$

$$\nabla\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{u}}{2}\right)\Bigg|_i = u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial v}{\partial x}} + \cancel{w\frac{\partial w}{\partial x}}$$

adding: $\underset{\sim}{\omega}\times\underset{\sim}{u}\Big|_i + \nabla\left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{u}}{2}\right)$

$$= w\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$= \underset{\sim}{u}\cdot\nabla u.$$

Q4 index notation..

$$\underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = u_j \frac{\partial u_i}{\partial x_j}$$

$$\underset{\sim}{\omega} \times \underset{\sim}{u} = \varepsilon_{jki} \, \omega_j u_k$$

$$= \varepsilon_{jki} \, \varepsilon_{lmj} \frac{\partial u_l}{\partial x_m} u_k$$

$$= + \varepsilon_{ikj} \, \varepsilon_{jlm} \frac{\partial u_l}{\partial x_m} u_k$$

$$= \left(+ \delta_{il} \delta_{km} - \delta_{im} \delta_{kl} \right) \frac{\partial u_l}{\partial x_m} u_k$$

$$= + \frac{\partial u_i}{\partial x_k} u_k - \cancel{\frac{\partial u_k}{\partial x_i}} u_k$$

$$\nabla \left(\frac{u \cdot u}{2} \right) = \frac{1}{2} \frac{\partial}{\partial x_i} (u_j \cdot u_j)$$

$$= u_j \cancel{\frac{\partial u_j}{\partial x_i}}$$

∴ proven.

/34