

Solving Mixed Integer Linear Programs with Cutting Planes

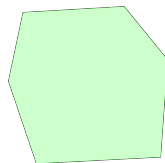
Tobias Kohler

May 16, 2024

Mixed Integer Linear Program

- Optimize linear objective function s.t. linear constraints and some integer constraints.
- Sometimes, decision variables are discrete: Distribution of patients/supplies/vehicles... or binary/logical variables.

Mixed Integer Linear Program and Linear Program Relaxation



MILP (standard form)

$$\min_x \quad c^T x$$

$$\text{s.t.} \quad x \in F$$

$$:= \{x \mid Ax \leq b, x \in \mathbb{Z}_{\geq 0}^{n_1} \times \mathbb{R}_{\geq 0}^{n-n_1}\}$$

$$c, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

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LP Relaxation

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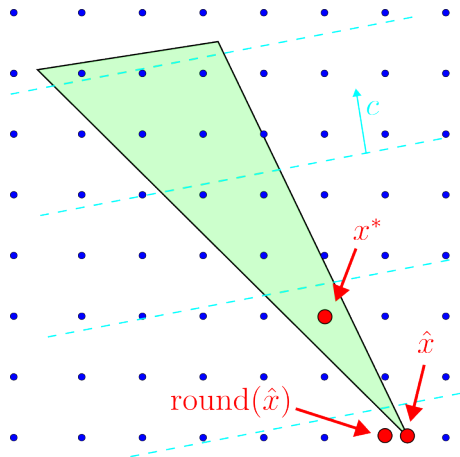
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Notation

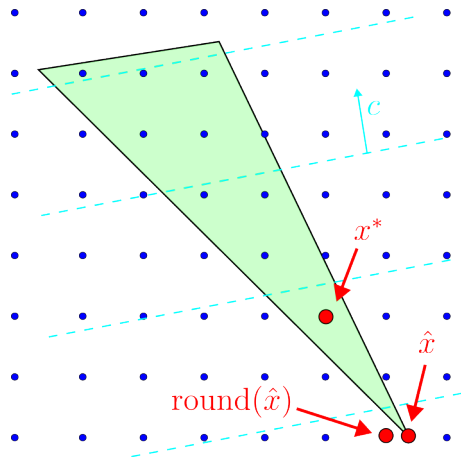
- x_1, \dots, x_{n_1} : Integer variables
- x_{n_1}, \dots, x_n : Real variables
- F, x^* : Feasible region and optimal solution of the MILP
- P, \hat{x} : Feasible region and optimal solution of the LP-relaxation
($F = P \cap \mathbb{Z}_{\geq 0}^{n_1} \times \mathbb{R}_{\geq 0}^{n-n_1}$)
- $n_1 = 0 \Rightarrow$ Linear Program (LP)
- $n_1 = n \Rightarrow$ Integer Linear Program (ILP)
- $x_i \in \{0, 1\} \Rightarrow$ (Mixed) Binary Program ((M)BP)

Observations



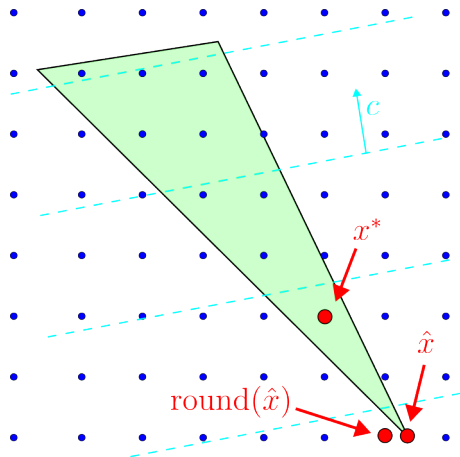
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- In general, $\text{round}(\hat{x}) \neq x^*$

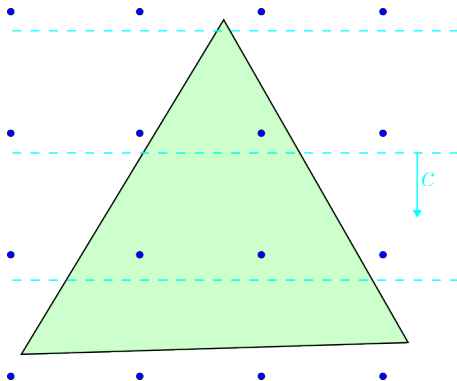
Observations



- \hat{x} can be found at a vertex of P (Simplex Algorithm).
- In general, $\text{round}(\hat{x}) \neq x^*$
- If \hat{x} is already feasible, then $c^\top \hat{x} = c^\top x^*$.

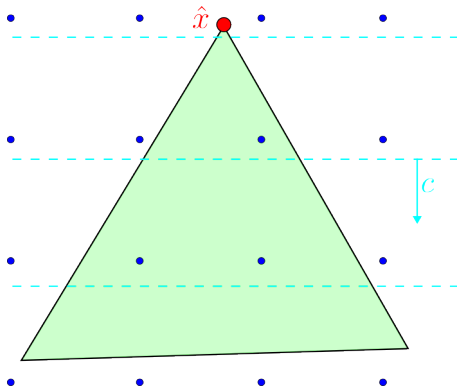
Cutting Planes

Solve the problem relaxation. If integer constraints are violated, add additional inequalities to the problem that cut off the relaxed solution.



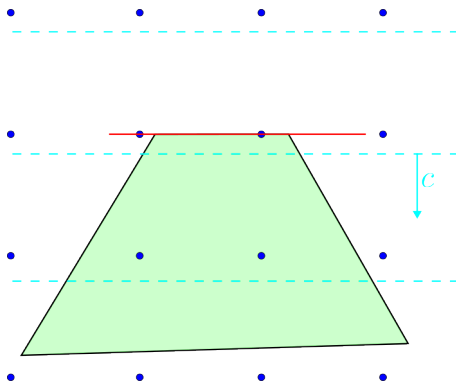
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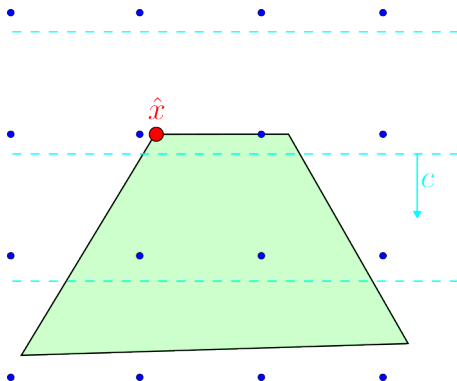
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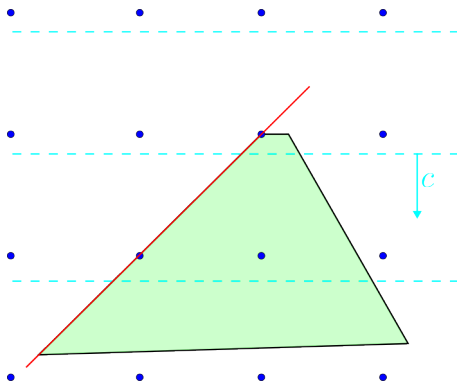
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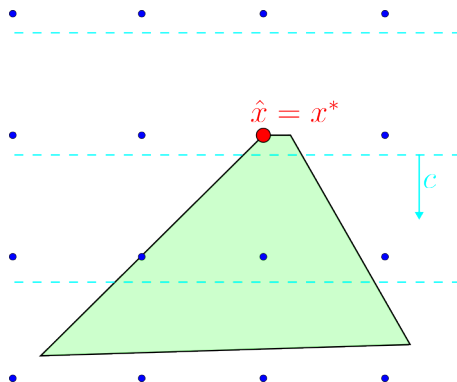
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Valid Inequalities and Cuts

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Valid Inequalities and Cuts

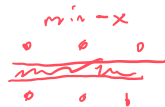
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 - For example: $x \leq 2$ is a valid inequality for $\{x \in \mathbb{Z}_{\geq 0} \mid x \leq 2.718\}$
- A cutting plane (or cut) w.r.t. $\hat{x} \in P \setminus F$ is any valid inequality $a^\top x \leq r$ for F such that:

$$a^\top \hat{x} > r$$

Cutting Planes Algorithm

$$F \subseteq P$$

$$P = \emptyset \Rightarrow F = \emptyset$$



- 1: $LP \leftarrow$ Relaxation of the MILP
- 2: **repeat**
- 3: $\hat{x} \leftarrow$ Optimal solution of the LP
- 4: If the LP is infeasible, return (null, INFEASIBLE)
- 5: If the LP is unbounded and no integer constraints are violated,
6: return (\hat{x} , UNBOUNDED)
- 7: **if** ($\hat{x}_1, \dots, \hat{x}_{n_1} \notin \mathbb{Z}^{n_1}$) **then**
- 8: Add a cut w.r.t. \hat{x} to the LP
- 9: **until** ($\hat{x}_1, \dots, \hat{x}_{n_1} \in \mathbb{Z}^{n_1}$)
- 10: **return** (\hat{x} , OPTIMAL)

$$\tilde{x}_i = \lfloor \hat{x}_i \rfloor$$

$$f_{x_i} = x_i - \lfloor \hat{x}_i \rfloor < \epsilon$$
$$> 1 - \epsilon$$

Cutting Strategy

Question: How to generate “good” and useful cuts?

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- Good: Cut away as much as possible (while staying feasible)

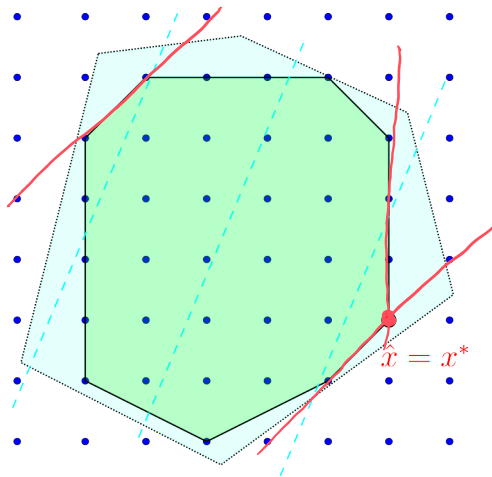
Cutting Strategy

Question: How to generate “good” and useful cuts?

- Good: Cut away as much as possible (while staying feasible)
- Useful: Cut away the optimal solution of the relaxation

Convex Hull

- The relaxed solution \hat{x} in $\text{conv}(F)$ also solves the MILP.
- But computing the convex hull is infeasible.
- Our goal is instead to approximate the convex hull in a neighborhood of x^* .



Chvátal–Gomory Inequality for Integer Linear Programs

Let $\sum_{j=1}^n a_{ij}x_j \leq b_i$ for an Integer Linear Program ($x \in \mathbb{Z}_{\geq 0}^n$). Then the following inequalities are valid for any $\alpha \geq 0$:

$$\boxed{1} \quad \sum_{j=1}^n \alpha a_{ij}x_j \leq \alpha b_i \qquad \alpha \geq 0$$

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- 1 $\sum_{j=1}^n \alpha a_{ij}x_j \leq \alpha b_i$ $\alpha \geq 0$
- 2 $\sum_{j=1}^n \lfloor \alpha a_{ij} \rfloor x_j \leq \alpha b_i$ $x_j \geq 0$

Chvátal–Gomory Inequality for Integer Linear Programs

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2 $\sum_{j=1}^n \lfloor \alpha a_{ij} \rfloor x_j \leq \alpha b_i$

3 $\sum_{j=1}^n \lfloor \alpha a_{ij} \rfloor x_j \leq \lfloor \alpha b_i \rfloor$

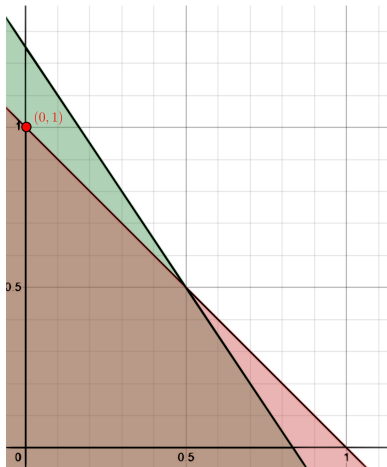
$$\begin{aligned} a &\leq b \\ \lfloor a \rfloor &\leq \lfloor b \rfloor \end{aligned}$$

$$\alpha \geq 0$$

$$x_j \geq 0$$

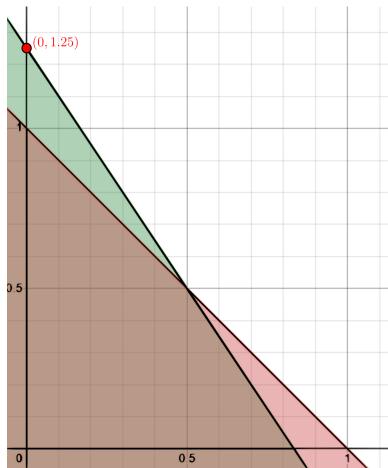
$$x_j \in \mathbb{Z}$$

ILP vs. MILP



- $\min_{x,y} -y$
s.t. $\frac{3}{2}x + y \leq \frac{5}{4}, (x, y) \in \mathbb{Z}_{\geq 0}^2$
- $(x^*, y^*) = (0, 1)$
- $\lfloor \frac{3}{2} \rfloor \cdot 0 + \lfloor 1 \rfloor \cdot 1 = 1 \leq \lfloor \frac{5}{4} \rfloor \checkmark$

ILP vs. MILP



- $\min_{x,y} -y$
s.t. $\frac{3}{2}x + y \leq \frac{5}{4}, (x, y) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$
- $(x^*, y^*) = (0, \frac{5}{4})$
- $\lfloor \frac{3}{2} \rfloor \cdot 0 + \lfloor 1 \rfloor \cdot \frac{5}{4} = \frac{5}{4} \not\leq \lfloor \frac{5}{4} \rfloor$ ✗

Basic Mixed Integer Rounding Inequalities I

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x + y \leq b\} \quad (1)$$

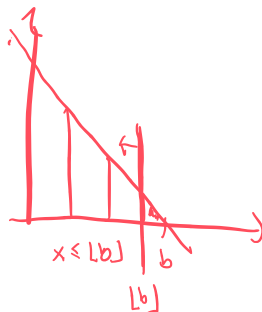
and

$$x \geq \lceil b \rceil \text{ is a valid inequality for } \{-x + y \leq -b\} \quad (2)$$

Basic Mixed Integer Rounding Inequalities I

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$$\begin{aligned} x &\leq x + y \leq b \\ \lfloor x \rfloor &\leq \lfloor b \rfloor \\ \parallel & \\ x & \end{aligned}$$



Basic Mixed Integer Rounding Inequalities II

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

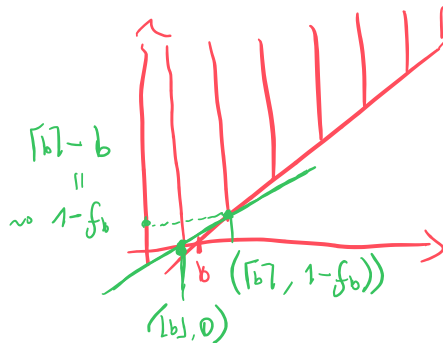
$$x - \frac{1}{1 - f_b} y \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x - y \leq b\} \quad (1)$$

and

$$x + \frac{1}{f_b} y \geq \lceil b \rceil \text{ is a valid inequality for } \{-x - y \leq -b\} \quad (2)$$

Basic Mixed Integer Rounding Inequalities II

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then $x - \frac{1}{1-f_b}y \leq \lfloor b \rfloor$ is a valid inequality for $\{x - y \leq b\}$



General Mixed Integer Rounding Inequality

Let $F_{MIR} = \{(x, y) \in \mathbb{Z}_{\geq 0}^2 \times \mathbb{R}_{\geq 0} \mid a_1x_1 + a_2x_2 - y \leq b\}$ where $a \in \mathbb{R}^2$, $b \in \mathbb{R} \setminus \mathbb{Z}$ and assume that $f_1 \leq f_b < f_2$. Then the inequality

$$\lfloor a_1 \rfloor x_1 + \underbrace{\left(\lfloor a_2 \rfloor + \frac{f_2 - f_b}{1 - f_b} \right)}_{\varepsilon(0,1)} x_2 - \frac{1}{1 - f_b} y \leq \lfloor b \rfloor$$

is valid for F_{MIR} .

Simplex Algorithm

Simplex finds $\hat{x} \in P \times \mathbb{R}_{\geq 0}^{N-n}$ and creates the optimal simplex tableau:

i -th row in the simplex tableau

$$x_{B_i} + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i$$

- x_1, \dots, x_{n_1} : Integral decision variables
- x_{n_1+1}, \dots, x_n : Real decision variables
- x_{n+1}, \dots, x_N : (Real) slack variables
- $B = \{B_1, \dots, B_m\}$: Basic variables
- $NB = \{1, \dots, N\} \setminus B$: Nonbasic variables ($\hat{x}_j = 0$ for $j \in NB$)

Gomory Mixed Integer Cut

Let $N_1 = NB \cap \{1, \dots, n_1\}$, $N_2 = NB \cap \{n_1 + 1, \dots, N\}$. Consider the i -th row in the optimal simplex tableau

$$x_{B_i} + \sum_{j \in N_1} \bar{a}_{ij} x_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i$$

and assume $B_i \leq n_1$ but $\hat{x}_{B_i} = \bar{b}_i \notin \mathbb{Z}$. Then the Gomory Mixed Integer Cut

$$x_{B_i} + \sum_{\substack{j \in N_1 \\ f_{ij} \leq f_i}} \lfloor \bar{a}_{ij} \rfloor x_j + \sum_{\substack{j \in N_1 \\ f_{ij} > f_i}} \left(\lfloor \bar{a}_{ij} \rfloor + \frac{f_{ij} - f_i}{1 - f_i} \right) x_j + \sum_{\substack{j \in N_2 \\ \bar{a}_{ij} < 0}} \left(\frac{\bar{a}_{ij}}{1 - f_i} \right) x_j \leq \lfloor \bar{b}_i \rfloor$$

is a valid inequality for F that is not satisfied by \hat{x} .

Project Demonstration

- Simplex Solver
- Mixed Integer Gomory Cut
- 2D Visualisation

Cutting Planes Selection

- Only adding an arbitrary, single cutting plane is very inefficient if the problem dimension is large.

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 - Evaluate the efficiency of a cutting plane based on some heuristics (for example euclidean distance to \hat{x}).

Cutting Planes Selection

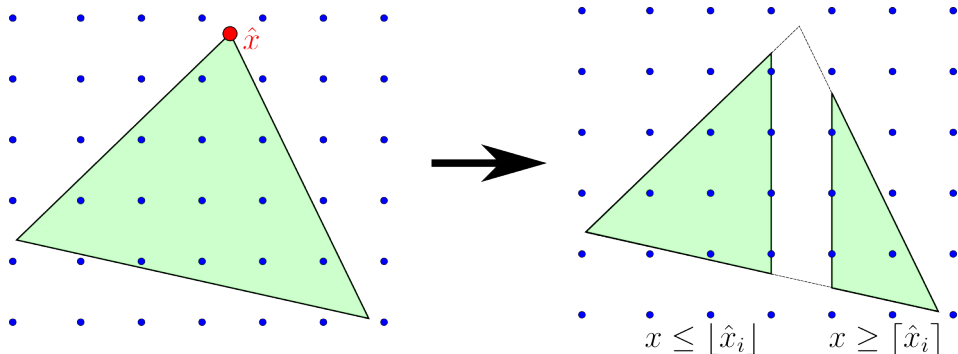
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 - Add multiple cutting planes in each iteration.
- For binary programs: Knapsack Covers or GUB (generalized upper bound) covers.

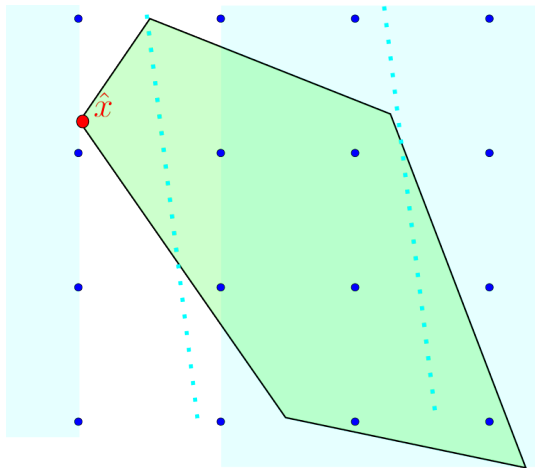
The Branch in Branch & Bound

Divide & Conquer: Cut off the non-integer neighborhood of $\hat{x}_i \notin \mathbb{Z}$ and obtain two new relaxation problems, one with $x_i \leq \lfloor \hat{x}_i \rfloor$ and one with $x_i \geq \lceil \hat{x}_i \rceil$. The resulting data structure is a binary tree of problems.

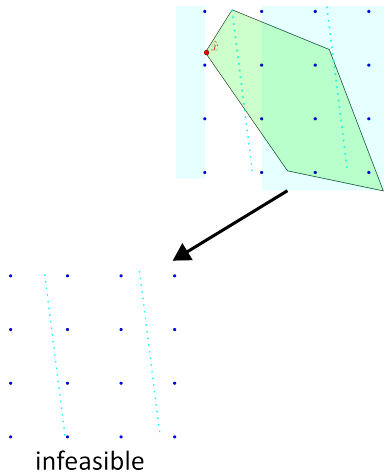


The Bound in Branch & Bound

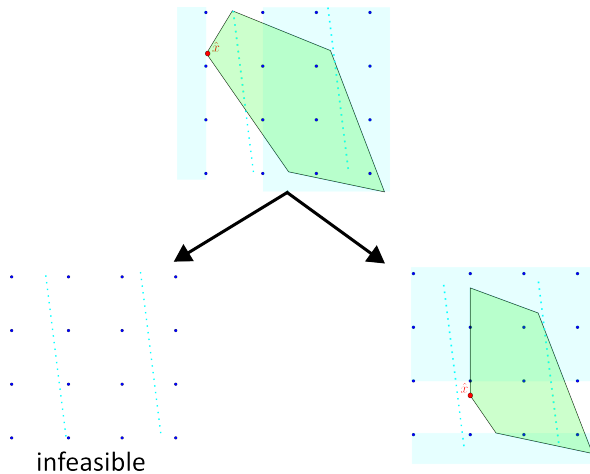
- Initially, we only know that $-\infty < c^T x^* < \infty$ (not very helpful).
- Improve bounds:
 - Lower Bound: For any LP-Relaxation, we have $c^T \hat{x} \leq c^T x^*$
 - Upper Bound: By definition, $c^T x^* \leq c^T x$ for any feasible $x \in F$
- For an optimal solution \hat{x} of a subproblem:
 - if $c^T \hat{x} \geq$ upper bound, prune tree (stop branching).
 - if \hat{x} is feasible, update upper bound and prune.
 - if \hat{x} is infeasible, update lower bound and branch.
 - stop if tree is completely pruned or upper bound — lower bound $< \epsilon$.



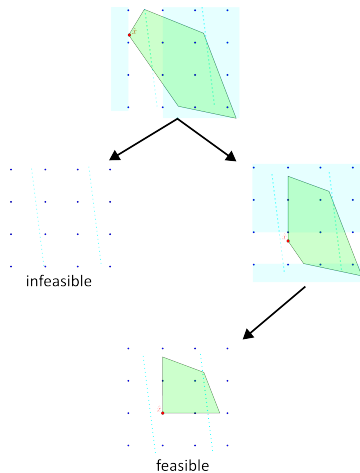
Example



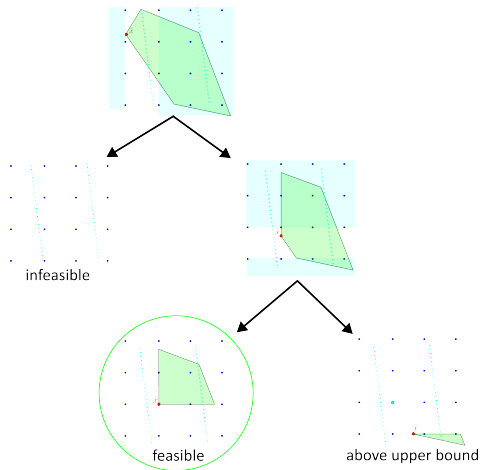
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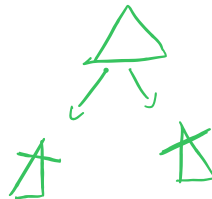


Subproblem and Branching Variable Selection Strategies

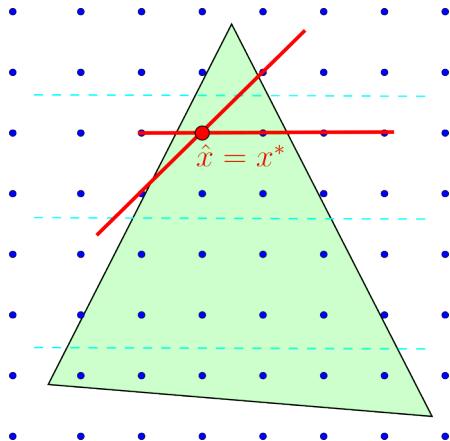
- Subproblem selection:
 - Depth-First: Descend quickly to obtain a good upper bound (obtain a feasible solution fast).
 - Best-First: Pick the active node with the current lower bound (obtain a good lower bound).
- Branching variable selection:
 - Most fractional: $i = \arg \max_{1 \leq i \leq n_1} \min(f_i, 1 - f_i)$ (f_i close to $\frac{1}{2}$).
 - Multiple variables at once.

Cutting Planes + Branch & Bound = Branch & Cut

- Cutting Planes or Branch & Bound on their own are inefficient in practice.
- Combine the two to Branch & Cut. This works like Branch & Bound but with additionally adding cuts before branching.
- Used in practice.



Questions?



References



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