Solving Mixed Integer Linear Programs using Cutting Planes

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Mixed Integer Linear Program

Mixed Integer Linear Program (MILP) and Relaxation

MILP (standard form)

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & x \in F_{MILP} \\ & :=\{x \mid Ax \leq b,\, x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0}\} \end{aligned}$$

$c, x \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$

LP Relaxation

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & x \in P_{LP} \\ & :=\{x \mid Ax \leq b,\, x \in \mathbb{R}^{n}_{\geq 0}\} \end{aligned}$$

Example

$$\begin{aligned} \min_{x,y} \quad & -y \\ \text{s.t.} \quad & 3x + 2y \leq 6 \\ & -3x + y \leq 0 \\ & (x,\ y) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0} \end{aligned}$$

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- If $x^*_{LP} \in F_{MILP}$, then $c^\top x^*_{LP} = c^\top x^*_{MILP}$.
- x^*_{LP} can be found at a vertex of P_{LP} (Simplex Algorithm).

Cutting Planes

Cutting Planes

Illustration

Valid Inequalities and Cuts

- An inequality $a^\top x \leq r$ is valid for a set F_{MILP} if $a^\top x \leq r$ is satisfied for all $x \in F_{MILP}$.
- For $x^*_{LP} \in P_{LP} \setminus F_{MILP}$ we define a cutting plane (or cut) w.r.t. x^*_{LP} as any valid inequality $a^\top x \leq r$ for F_{MILP} such that:

$$a^\top x^*_{LP} > r$$

Cutting Planes Algorithm

```
LP ← Relaxation of the MILP
repeat
    x* ← Optimal solution of the LP
    if (x*_1, ..., x*_{n_1}) ∉ ℤ^{n_1} then
        Add a cut w.r.t. x* to the LP
until (x*_1, ..., x*_{n_1}) ∈ ℤ^{n_1}
return x*
```

Cutting Strategy

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- Good: Cut away as much as possible (while staying feasible)
- Useful: Cut away the optimal solution of the relaxation

Convex Hull

Create convex hull - equivalent to relaxation. But too expensive (exponential!)

Integer Part and Fractional Part

Any real number $a \in \mathbb{R}$ can be expressed as

$$a = \lfloor a \rfloor + f_a$$

for some unique $\lfloor a \rfloor \in \mathbb{Z}$ and $f_a \in [0, 1)$.

- $\lfloor a \rfloor = \max\{z \in \mathbb{Z} \mid z \leq a\}$ is the <u>integer part</u> of a.
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- $\lfloor -a \rfloor = -\lceil a \rceil$ where $\lceil a \rceil = \min\{z \in \mathbb{Z} \mid z \geq a\}$
- $a \in \mathbb{Z}$ and $a \leq b \Rightarrow a \leq \lfloor b \rfloor$
- $a \in \mathbb{Z}$ and $a \geq b \Rightarrow a \geq \lceil b \rceil$

Chvátal–Gomory Inequality for Integer Linear Programs

Let $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for an Integer Linear Program ($x \in \mathbb{Z}^n_{\geq 0}$). Then the following inequalities are valid for any $\alpha \geq 0$:

1. $\sum_{j=1}^{n} \alpha a_{ij} x_j \leq \alpha b_i$ $\qquad \alpha \geq 0$
2. $\sum_{j=1}^{n} \lfloor \alpha a_{ij} \rfloor x_j \leq \alpha b_i$ $\qquad x_j \geq 0$
3. $\sum_{j=1}^{n} \lfloor \alpha a_{ij} \rfloor x_j \leq \lfloor \alpha b_i \rfloor$ $\qquad x_j \in \mathbb{Z}$

Example

Illustration

Chvátal–Gomory Inequality

If not all variables are integer, these inequalities are not valid. Show that it does not work for mixed integer problem

Basic Mixed Integer Rounding Inequalities I

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x + y \leq b\} \tag{1}$$

and

$$x \geq \lceil b \rceil \text{ is a valid inequality for } \{-x + y \leq -b\} \tag{2}$$

Basic Mixed Integer Rounding Inequalities I

Hello World

Basic Mixed Integer Rounding Inequalities II

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x - \frac{1}{f_b - 1} \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x - y \leq b\} \tag{1}$$

and

$$x + \frac{1}{f_b} \geq \lceil b \rceil \text{ is a valid inequality for } \{-x - y \leq -b\} \tag{2}$$

Basic Mixed Integer Rounding Inequalities II

General Mixed Integer Rounding Inequality

Let $F_{MIR} = \{(x, y) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{R}_{\geq 0} \mid a_1x_1 + a_2x2 - y \leq b\}$ where $a \in \mathbb{R}^2$, $b \in \mathbb{R} \setminus \mathbb{Z}$ and assume that $f_1 \leq f_b \leq f_2$. Then the inequality

$$\lfloor a_1 \rfloor x_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f_b}{1 - f_b} \right) x_2 - \frac{1}{1 - f_b} y \leq \lfloor b \rfloor$$

is valid for F_{MIR}.

General Mixed Integer Rounding Inequality

Simplex Algorithm

Simplex finds $x^* \in P_{LP} \times \mathbb{R}_{\geq 0}^{N-n}$ and creates the optimal simplex tableau:

$i-$th row in the simplex tableau

$$x_{B_i} + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i$$

- $x_1, \ldots, x_{n_1}$: Integer problem variables
- $x_{n_1+1}, \ldots, x_n$: Real problem variables
- $x_{n+1}, \ldots, x_N$: (Real) slack variables

- $B = \{B_1, \ldots, B_m\}$: Basic variables
- $NB = \{1, \ldots, N\} \setminus B$: Nonbasic variables ($x_j^* = 0$ for $j \in NB$)

Gomory Mixed Integer Cut

Let $N_1 = NB \cap \{1, ..., n_1\}$, $N_2 = NB \cap \{n_1 + 1, ..., x_N\}$. Consider the $i-$th row in the optimal simplex tableau

$$x_{B_i} + \sum_{j \in N_1} \bar{a}_{ij} x_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i$$

and assume $B_i \leq n_1$ but $x^*_{B_i} = \bar{b}_i \notin \mathbb{Z}$. Then the Gomory Mixed Integer Cut

$$\boxed{x_{B_i} + \sum_{\substack{j \in N_1 \\ f_{ij} \leq f_i}} \lfloor \bar{a}_{ij} \rfloor x_j + \sum_{\substack{j \in N_1 \\ f_{ij} > f_i}} \left(\lfloor \bar{a}_{ij} \rfloor + \frac{f_{ij} - f_i}{1 - f_i} \right) x_j + \sum_{\substack{j \in N_2 \\ \bar{a}_{ij} < 0}} \left(\frac{\bar{a}_{ij}}{1 - f_i} \right) x_j \leq \lfloor \bar{b}_i \rfloor}$$

is a valid inequality for F_{MILP} that is not satisfied by x^*.

Gomory Mixed Integer Cut

Cutting Planes Algorithm

- Let a MILP be given with feasible region $F_{MILP} = \{x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0} \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- The relaxation is the LP obtained by removing the integer constraints, so its feasible region is the polyhedron $P_{LP} = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$.
- Repeat the following two steps until $x^* \in F_{MILP}$:
  1. Solve the LP using the Simplex Algorithm and obtain $x^* \in P_{LP}$
  2. TODO

Project Demonstration

Selecting Cutting Planes

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Selecting Cutting Planes

- Only adding an arbitrary, single cutting plane is very inefficient if the problem dimension is large.
  - Heuristic to evaluate the efficiency of a cutting plane (e.g. euclidean distance to x^*).
  - Add multiple cutting planes in each iteration.
- Other cutting plane strategies exist.

Branch & Bound

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- Similar to Cutting Planes - Solve Problem Relaxation, add constraints until solution is found
- Divide & Conquer

Branch & Cut

Branch & Cut

- Hello World
- Cutting Planes + Branch & Bound = Branch & Cut

References