Cutting Plane Methods

Tobias Kohler

May 16, 2024

Integer Rounding

The generation of cuts is based on some simple observations. If $x \in \mathbb{Z}$ and $x \leq b$, then $x \leq \lfloor b \rfloor$

Chvátal–Gomory Inequality for ILPs

Consider the inequality $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ for an Integer Linear Program ($x \in \mathbb{Z}^n_{\geq 0}$).
Then the following inequalities are valid for any $\alpha \geq 0$:

1. $\sum_{j=1}^{n} \alpha a_{ij}x_j \leq \alpha b_i$ $\qquad \alpha \geq 0$
2. $\sum_{j=1}^{n} \lfloor \alpha a_{ij} \rfloor x_j \leq \alpha b_i$ $\qquad x_j \geq 0$
3. $\sum_{j=1}^{n} \lfloor \alpha a_{ij} \rfloor x_j \leq \lfloor \alpha b_i \rfloor$ $\qquad x_j \in \mathbb{Z}$

Chvátal–Gomory Inequality for ILPs

Illustration

Basic Mixed Integer Rounding Inequalities I

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then the following hold:

$$x + y \leq b \Rightarrow x \leq \lfloor b \rfloor \quad (1)$$

$$-x + y \leq -b \Rightarrow x \geq \lceil b \rceil \quad (2)$$

Basic Mixed Integer Rounding Inequalities II

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then the following hold:

$$x - y \leq b \Rightarrow x - \frac{1}{f_b - 1} \leq \lfloor b \rfloor \tag{3}$$

$$-x - y \leq -b \Rightarrow x + \frac{1}{f_b} \geq \lceil b \rceil \tag{4}$$

where $f_b = b - \lfloor b \rfloor \in (0, 1)$ is the fractional part of b.

General Mixed Integer Rounding Inequality

Let $F_{MIR} = \{(x, y) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{R}_{\geq 0} \mid a_1x_1 + a_2x2 - y \leq b\}$ and $f_i = a_i - \lfloor a_i \rfloor$ for $i = 1, 2$ where $a \in \mathbb{R}^2$, $b \in \mathbb{R} \setminus \mathbb{Z}$ and suppose that $f_1 \leq f_b \leq f_2$. Then the inequality

$$\lfloor a_1 \rfloor x_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f_b}{1 - f_b} \right) x_2 - \frac{1}{1 - f_b} y \leq \lfloor b \rfloor$$

is valid for F_{MIR}.

Simplex Algorithm

Recall that the Simplex Algorithm transforms a LP with a feasible polyhedron P_{LP} to slack-form and produces an optimal solution $x^* \in P_{LP} \times \mathbb{R}^{N-n}_{\geq 0}$ and equalities of the form:

The $i-$th row in the simplex tableau

$$x_{B_i} + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i$$

where $x_{n+1}, \ldots, x_N \geq 0$ are slack variables, $B = \{B_1, \ldots, B_m\}$ are the indices of the basic variables and $NB = \{1, \ldots, N\} \setminus B$ are the indices of the nonbasic variables ($x_j^* = 0$ for $j \in NB$).

Cutting Planes Algorithm

- Let a MILP be given with feasible region $F_{MILP} = \{x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0} \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- The relaxation is the LP obtained by removing the integer constraints, so its feasible region is the polyhedron $P_{LP} = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$.
- Repeat the following two steps until $x^* \in F_{MILP}$:
  1. Solve the LP using the Simplex Algorithm and obtain $x^* \in P_{LP}$
  2. If $x^* \notin F_{MILP}$, add an inequality that is valid for F_{MILP} and not satisfied by x^*.

Gomory Mixed Integer Cut

Let $N_1 = NB \cap \{1, \ldots, n_1\}$, $N_2 = NB \cap \{n_1 + 1, \ldots, x_N\}$. Consider the $i-$th row in the optimal simplex tableau

$$x_{B_i} + \sum_{j \in N_1} \bar{a}_{ij} x_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i$$

and assume $B_i \in N_1$ but $x^*_{B_i} = \bar{b}_i \notin \mathbb{Z}$. Then the <u>Gomory Mixed Integer Cut</u>

$$\boxed{\sum_{\substack{j \in N_1 \\ f_{ij} \leq f_i}} \lfloor \bar{a}_{ij} \rfloor x_j + \sum_{\substack{j \in N_1 \\ f_{ij} > f_i}} \left(\lfloor \bar{a}_{ij} \rfloor + \frac{f_{ij} - f_i}{1 - f_i} \right) x_j + \sum_{\substack{j \in N_2 \\ \bar{a}_{ij} < 0}} \left(\frac{\bar{a}_{ij}}{1 - f_i} \right) x_j \leq \lfloor \bar{b}_i \rfloor}$$

is a valid inequality for F_{MILP} that is not satisfied by x^*.

References

Temporary page!

LaTeX was unable to guess the total number of pages correctly. As there was some unprocessed data that should have been added to the final page this ext
page has been added to receive it.
If you rerun the document (without altering it) this surplus page will go away, because LaTeX now knows how many pages to expect for this document.