

Solving Mixed Integer Linear Programs using Cutting Planes

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Mixed Integer Linear Program and Linear Program Relaxation

MILP (standard form)

$$\min_x c^T x$$

$$\text{s.t. } x \in F$$

$$:= \{x \mid Ax \leq b, x \in \mathbb{Z}_{\geq 0}^{n_1} \times \mathbb{R}_{\geq 0}^{n-n_1}\}$$

$$c, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

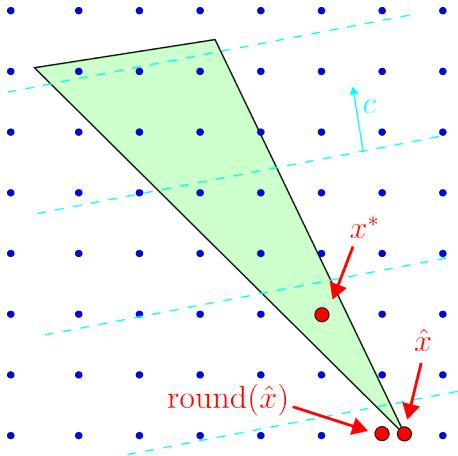
LP Relaxation

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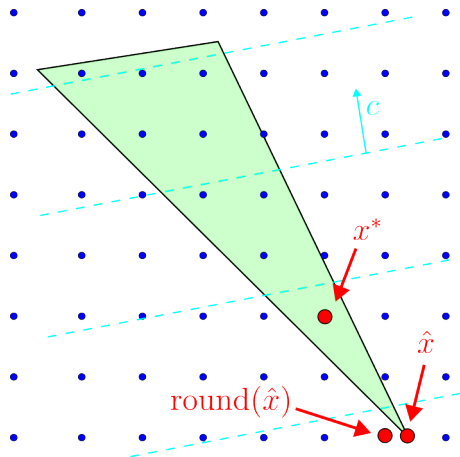
$$:= \{x \mid Ax \leq b, x \in \mathbb{R}_{\geq 0}^n\}$$

Observations



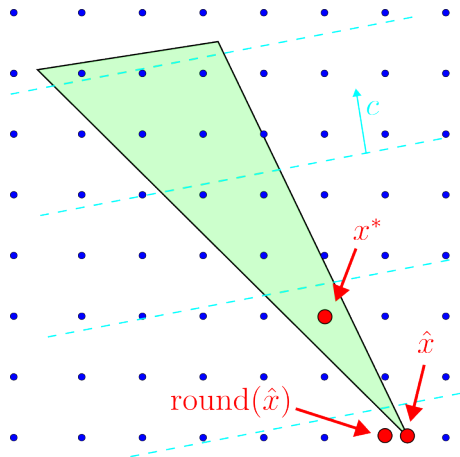
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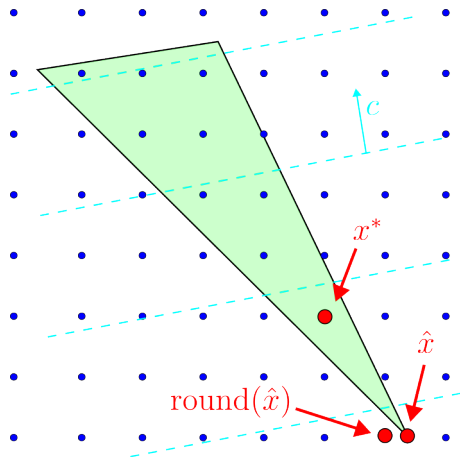
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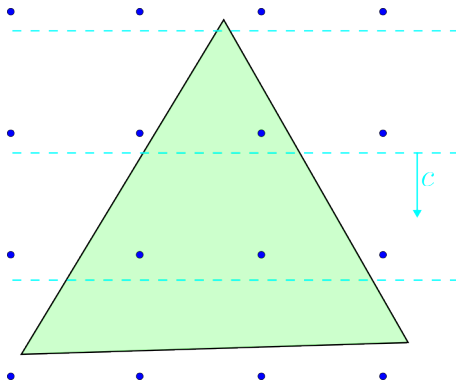
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- The relaxed solution gives a lower bound:
 $c^\top \hat{x} \leq c^\top x^*$.
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- If \hat{x} is already feasible, then $c^\top \hat{x} = c^\top x^*$.

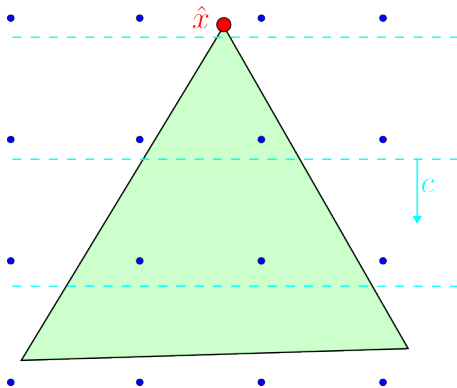
Cutting Planes

Solve the problem relaxation. If integer constraints are violated, add additional inequalities to the problem.



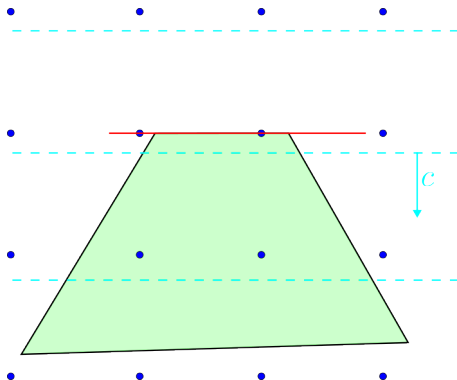
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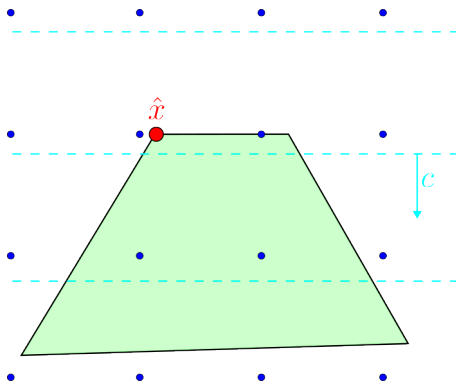
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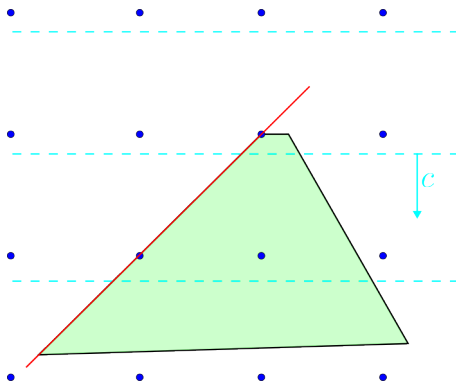
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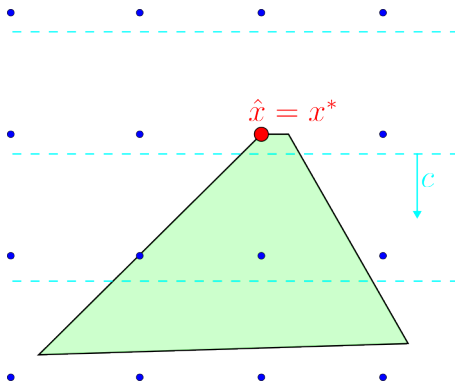
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 - For example: $x \leq 2$ is a valid inequality for $\{x \in \mathbb{Z}_{\geq 0} \mid x \leq 2.718\}$
- A cutting plane (or cut) w.r.t. $\hat{x} \in P \setminus F$ is any valid inequality $a^\top x \leq r$ for F such that:

$$a^\top \hat{x} > r$$

Cutting Planes Algorithm

- 1: $LP \leftarrow$ Relaxation of the MILP
- 2: **repeat**
- 3: $\hat{x} \leftarrow$ Optimal solution of the LP
- 4: **if** $(\hat{x}_1, \dots, \hat{x}_{n_1}) \notin \mathbb{Z}^{n_1}$ **then**
- 5: Add a cut w.r.t. \hat{x} to the LP
- 6: **until** $(\hat{x}_1, \dots, \hat{x}_{n_1}) \in \mathbb{Z}^{n_1}$
- 7: **return** \hat{x}

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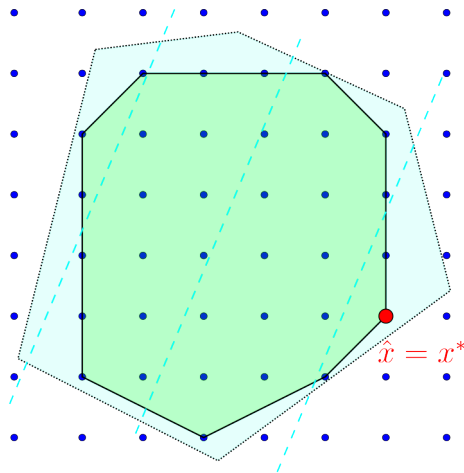
Cutting Strategy

Question: How to generate “good” and useful cuts?

- Good: Cut away as much as possible (while staying feasible)
- Useful: Cut away the optimal solution of the relaxation

Convex Hull

- The relaxed solution \hat{x} in $\text{conv}(F)$ also solves the MILP.
- But computing the convex hull is infeasible (exponential).



Integer Part and Fractional Part

Any real number $a \in \mathbb{R}$ can be expressed as

$$a = \lfloor a \rfloor + f_a$$

for some unique $\lfloor a \rfloor \in \mathbb{Z}$ and $f_a \in [0, 1)$.

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 $\lceil a \rceil = \min\{z \in \mathbb{Z} \mid z \geq a\}$
- $a \in \mathbb{Z}$ and $a \leq b \Rightarrow a \leq \lfloor b \rfloor$
- $a \in \mathbb{Z}$ and $a \geq b \Rightarrow a \geq \lceil b \rceil$

Chvátal–Gomory Inequality for Integer Linear Programs

Let $\sum_{j=1}^n a_{ij}x_j \leq b_i$ for an Integer Linear Program ($x \in \mathbb{Z}_{\geq 0}^n$). Then the following inequalities are valid for any $\alpha \geq 0$:

- | | | |
|---|--|----------------------|
| 1 | $\sum_{j=1}^n \alpha a_{ij}x_j \leq \alpha b_i$ | $\alpha \geq 0$ |
| 2 | $\sum_{j=1}^n \lfloor \alpha a_{ij} \rfloor x_j \leq \alpha b_i$ | $x_j \geq 0$ |
| 3 | $\sum_{j=1}^n \lfloor \alpha a_{ij} \rfloor x_j \leq \lfloor \alpha b_i \rfloor$ | $x_j \in \mathbb{Z}$ |

Example

Illustration

Chvátal–Gomory Inequality

If not all variables are integer, these inequalities are not valid. Show that it does not work for mixed integer problem

Basic Mixed Integer Rounding Inequalities I

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x + y \leq b\} \quad (1)$$

and

$$x \geq \lceil b \rceil \text{ is a valid inequality for } \{-x + y \leq -b\} \quad (2)$$

Basic Mixed Integer Rounding Inequalities I

Hello World

Basic Mixed Integer Rounding Inequalities II

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x - \frac{1}{f_b - 1} \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x - y \leq b\} \quad (1)$$

and

$$x + \frac{1}{f_b} \geq \lceil b \rceil \text{ is a valid inequality for } \{-x - y \leq -b\} \quad (2)$$

Basic Mixed Integer Rounding Inequalities II

General Mixed Integer Rounding Inequality

Let $F_{MIR} = \{(x, y) \in \mathbb{Z}_{\geq 0}^2 \times \mathbb{R}_{\geq 0} \mid a_1x_1 + a_2x_2 - y \leq b\}$ where $a \in \mathbb{R}^2$, $b \in \mathbb{R} \setminus \mathbb{Z}$ and assume that $f_1 \leq f_b \leq f_2$. Then the inequality

$$\lfloor a_1 \rfloor x_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f_b}{1 - f_b} \right) x_2 - \frac{1}{1 - f_b} y \leq \lfloor b \rfloor$$

is valid for F_{MIR} .

General Mixed Integer Rounding Inequality

Simplex Algorithm

Simplex finds $\hat{x} \in P \times \mathbb{R}_{\geq 0}^{N-n}$ and creates the optimal simplex tableau:

i -th row in the simplex tableau

$$x_{B_i} + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i$$

- x_1, \dots, x_{n_1} : Integer problem variables
- x_{n_1+1}, \dots, x_n : Real problem variables
- x_{n+1}, \dots, x_N : (Real) slack variables
- $B = \{B_1, \dots, B_m\}$: Basic variables
- $NB = \{1, \dots, N\} \setminus B$: Nonbasic variables ($\hat{x}_j = 0$ for $j \in NB$)

Gomory Mixed Integer Cut

Let $N_1 = NB \cap \{1, \dots, n_1\}$, $N_2 = NB \cap \{n_1 + 1, \dots, x_N\}$. Consider the i -th row in the optimal simplex tableau

$$x_{B_i} + \sum_{j \in N_1} \bar{a}_{ij} x_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i$$

and assume $B_i \leq n_1$ but $\hat{x}_{B_i} = \bar{b}_i \notin \mathbb{Z}$. Then the Gomory Mixed Integer Cut

$$x_{B_i} + \sum_{\substack{j \in N_1 \\ f_{ij} \leq f_i}} \lfloor \bar{a}_{ij} \rfloor x_j + \sum_{\substack{j \in N_1 \\ f_{ij} > f_i}} \left(\lfloor \bar{a}_{ij} \rfloor + \frac{f_{ij} - f_i}{1 - f_i} \right) x_j + \sum_{\substack{j \in N_2 \\ \bar{a}_{ij} < 0}} \left(\frac{\bar{a}_{ij}}{1 - f_i} \right) x_j \leq \lfloor \bar{b}_i \rfloor$$

is a valid inequality for F that is not satisfied by \hat{x} .

Gomory Mixed Integer Cut

Cutting Planes Algorithm

- Let a MILP be given with feasible region $F = \{x \in \mathbb{Z}_{\geq 0}^{n_1} \times \mathbb{R}_{\geq 0}^{n-n_1} \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- The relaxation is the LP obtained by removing the integer constraints, so its feasible region is the polyhedron $P = \{x \in \mathbb{R}_{\geq 0}^n \mid Ax \leq b\}$.
- Repeat the following two steps until $\hat{x} \in F$:
 - 1 Solve the LP using the Simplex Algorithm and obtain $\hat{x} \in P$
 - 2 TODO

Project Demonstration

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- Only adding an arbitrary, single cutting plane is very inefficient if the problem dimension is large.
 - Heuristic to evaluate the efficiency of a cutting plane (e.g. euclidean distance to \hat{x}).
 - Add multiple cutting planes in each iteration.
- Other cutting plane strategies exist.

Branch & Bound

- Like Cutting Planes, we solve the problem relaxation and add constraints until an optimal solution has been found.
- Divide & Conquer

Branching

Cut off the non-integer neighborhood of $x_i^* \notin \mathbb{Z} \Rightarrow$ Two new relaxation problems

Bounding

- Initially, we only know that $-\infty < c^T x_{MILP}^* < \infty$ (not very helpful).
- Improve bounds until Upper Bound – Lower Bound $< \epsilon$:
 - Lower Bound: For any LP-Relaxation, we have $c^T x_{LP}^* \leq c^T x_{MILP}^*$
 - Upper Bound: By definition, $c^T x_{MILP}^* \leq c^T x$ for any feasible $x \in F_{MILP}$

Branch & Cut

- Hello World
- Cutting Planes + Branch & Bound = Branch & Cut

Questions?

References