Solving Mixed Integer Linear Programs with Cutting Planes

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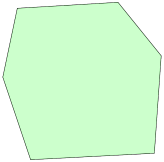
Mixed Integer Linear Program and Linear Program Relaxation



MILP (standard form)

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & x \in F \\ & := \{x \mid Ax \leq b,\ x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0}\} \end{aligned}$$

LP Relaxation

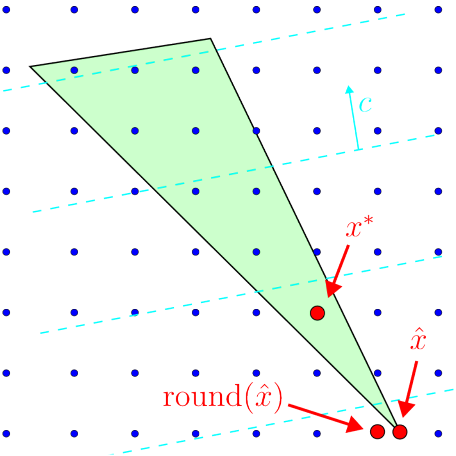
$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & x \in P \\ & := \{x \mid Ax \leq b,\ x \in \mathbb{R}^{n}_{\geq 0}\} \end{aligned}$$

$c,\ x \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$

Notation

- $x_1, \ldots, x_{n_1}$: Integer variables
- $x_{n_1}, \ldots, x_n$: Real variables
- F, x^* : Feasible region and optimal solution of the MILP
- P, $\hat{x}$: Feasible region and optimal solution of the LP-relaxation
  $(F = P \cap \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0})$
- $n_1 = 0 \Rightarrow$ Linear Program (LP)
- $n_1 = n \Rightarrow$ Integer Linear Program (ILP)
- $x_i \in \{0, 1\} \Rightarrow$ (Mixed) Binary Program ((M)BP)

Observations



- $\hat{x}$ can be found at a vertex of P (Simplex Algorithm).

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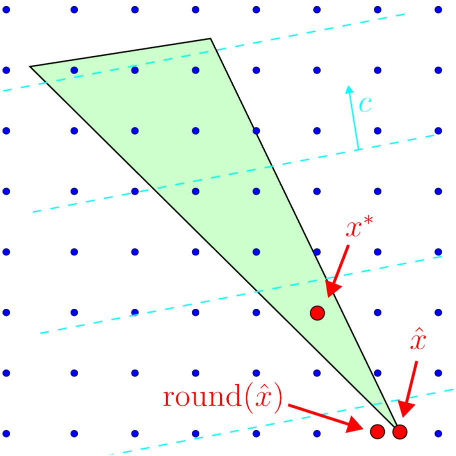

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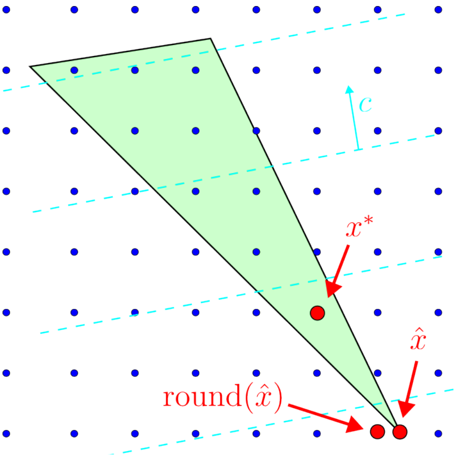
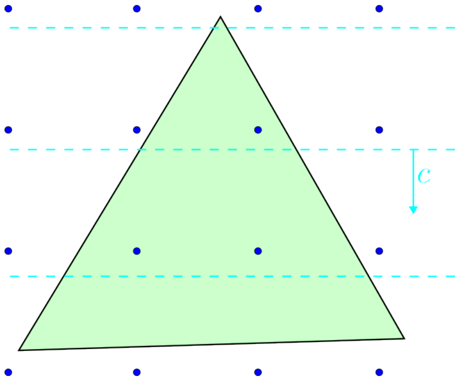


- $\hat{x}$ can be found at a vertex of P (Simplex Algorithm).
- In general, round($\hat{x}$) $\neq x^*$
- If $\hat{x}$ is already feasible, then $c^\top \hat{x} = c^\top x^*$.

Cutting Planes

Solve the problem relaxation. If integer constraints are violated, add additional inequalities to the problem that cut off the relaxed solution.



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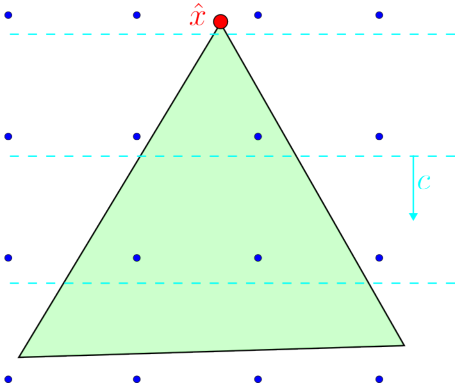

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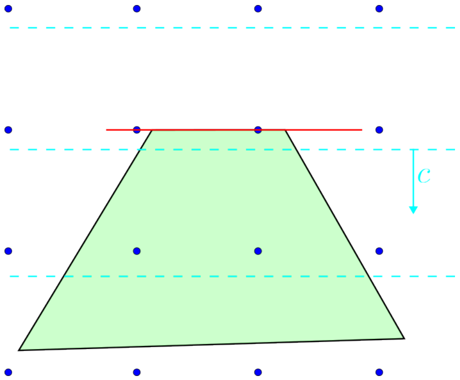

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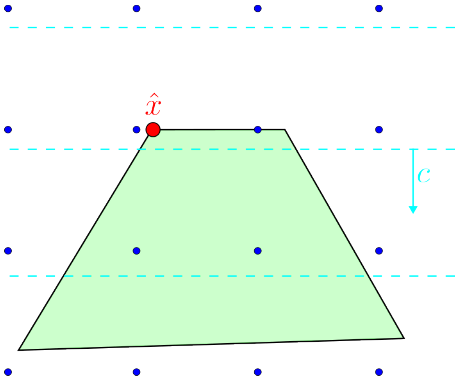

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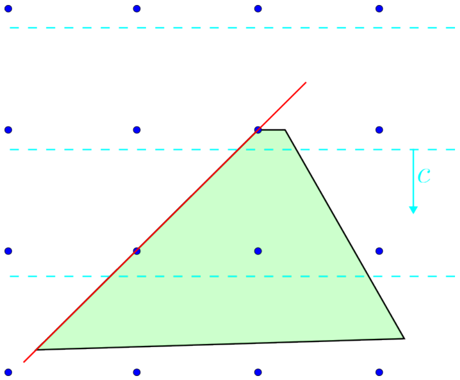

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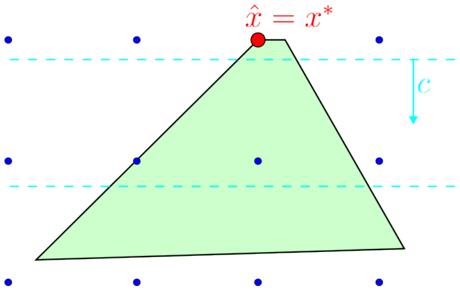

Valid Inequalities and Cuts

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  - For example: $x \leq 2$ is a valid inequality for $\{x \in \mathbb{Z}_{\geq 0} \mid x \leq 2.718\}$
- A cutting plane (or cut) w.r.t. $\hat{x} \in P \setminus F$ is any valid inequality $a^\top x \leq r$ for F such that:

$$a^\top \hat{x} > r$$

Cutting Planes Algorithm

LP $\leftarrow$ Relaxation of the MILP
repeat
    $\hat{x} \leftarrow$ Optimal solution of the LP
    if $(\hat{x}_1, \dots, \hat{x}_{n_1}) \notin \mathbb{Z}^{n_1}$ **then**
        Add a cut w.r.t. $\hat{x}$ to the LP
until $(\hat{x}_1, \dots, \hat{x}_{n_1}) \in \mathbb{Z}^{n_1}$
return $\hat{x}$

Cutting Strategy

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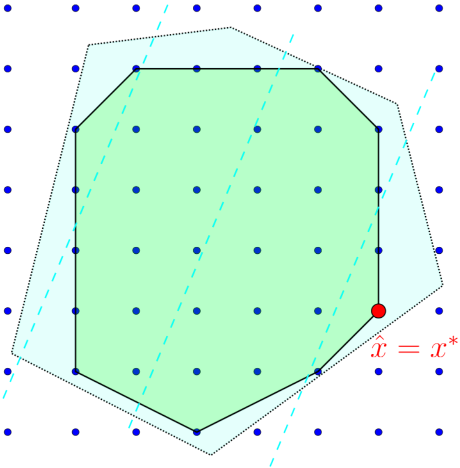
- Good: Cut away as much as possible (while staying feasible)
- Useful: Cut away the optimal solution of the relaxation

Convex Hull

- The relaxed solution $\hat{x}$ in conv(F) also solves the MILP.
- But computing the convex hull is infeasible.
- Our goal is instead to approximate the convex hull in a neighborhood of x^*.



Integer Part and Fractional Part

Any real number $a \in \mathbb{R}$ can be expressed as

$$a = \lfloor a \rfloor + f_a$$

for some unique $\lfloor a \rfloor \in \mathbb{Z}$ and $f_a \in [0, 1)$.

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- $a \in \mathbb{Z}$ and $a \leq b \Rightarrow a \leq \lfloor b \rfloor$
- $a \in \mathbb{Z}$ and $a \geq b \Rightarrow a \geq \lceil b \rceil$

Chvátal–Gomory Inequality for Integer Linear Programs

Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ for an Integer Linear Program ($x \in \mathbb{Z}^n_{\geq 0}$). Then the following inequalities are valid for any $\alpha \geq 0$:

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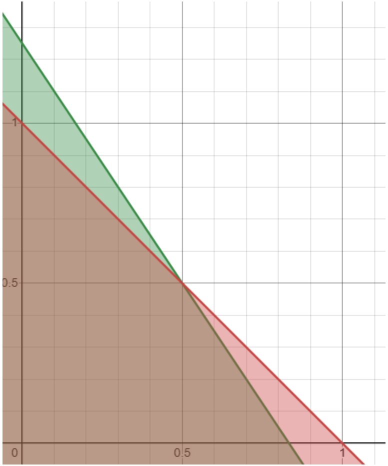
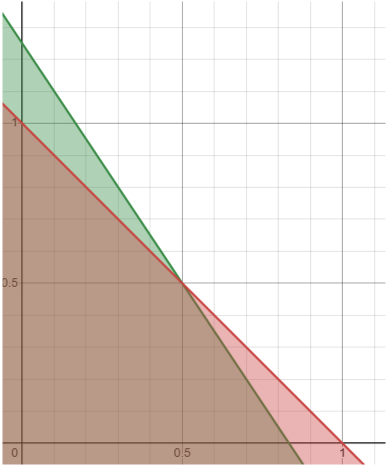
3. $\sum_{j=1}^{n} \lfloor \alpha a_{ij} \rfloor x_j \leq \lfloor \alpha b_i \rfloor$ $\qquad x_j \in \mathbb{Z}$

ILP vs. MILP



- $\min_{x,y} -y$
  s.t. $\frac{3}{2}x + y \leq \frac{5}{4}$, $(x, y) \in \mathbb{Z}^2_{\geq 0}$
- $(x^*, y^*) = (0, 1)$
- $\lfloor \frac{3}{2} \rfloor \cdot 0 + \lfloor 1 \rfloor \cdot 1 = 1 \leq \lfloor \frac{5}{4} \rfloor$ ✓

ILP vs. MILP



- $\min_{x,y} -y$
  s.t. $\frac{3}{2}x + y \leq \frac{5}{4}$, $(x, y) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$
- $(x^*, y^*) = (0, \frac{5}{4})$
- $\lfloor \frac{3}{2} \rfloor \cdot 0 + \lfloor 1 \rfloor \cdot \frac{5}{4} = \frac{5}{4} > \lfloor \frac{5}{4} \rfloor$ ✗

Basic Mixed Integer Rounding Inequalities I

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x + y \leq b\} \tag{1}$$

and

$$x \geq \lceil b \rceil \text{ is a valid inequality for } \{-x + y \leq -b\} \tag{2}$$

Basic Mixed Integer Rounding Inequalities I

Basic Mixed Integer Rounding Inequalities II

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x - \frac{1}{f_b - 1} \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x - y \leq b\} \tag{1}$$

and

$$x + \frac{1}{f_b} \geq \lceil b \rceil \text{ is a valid inequality for } \{-x - y \leq -b\} \tag{2}$$

Basic Mixed Integer Rounding Inequalities II

General Mixed Integer Rounding Inequality

Let $F_{MIR} = \{(x, y) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{R}_{\geq 0} \mid a_1 x_1 + a_2 x2 - y \leq b\}$ where $a \in \mathbb{R}^2$, $b \in \mathbb{R} \setminus \mathbb{Z}$ and assume that $f_1 \leq f_b \leq f_2$. Then the inequality

$$\lfloor a_1 \rfloor x_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f_b}{1 - f_b} \right) x_2 - \frac{1}{1 - f_b} y \leq \lfloor b \rfloor$$

is valid for F_{MIR}.

General Mixed Integer Rounding Inequality

Simplex Algorithm

Simplex finds $\hat{x} \in P \times \mathbb{R}^{N-n}_{\geq 0}$ and creates the optimal simplex tableau:

$i-$th row in the simplex tableau

$$x_{B_i} + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i$$

- $x_1, \ldots, x_{n_1}$: Integer problem variables
- $x_{n_1+1}, \ldots, x_n$: Real problem variables
- $x_{n+1}, \ldots, x_N$: (Real) slack variables

- $B = \{B_1, \ldots, B_m\}$: Basic variables
- $NB = \{1, \ldots, N\} \setminus B$: Nonbasic variables ($\hat{x}_j = 0$ for $j \in NB$)

Gomory Mixed Integer Cut

Let $N_1 = NB \cap \{1, \ldots, n_1\}$, $N_2 = NB \cap \{n_1 + 1, \ldots, x_N\}$. Consider the $i-$th row in the optimal simplex tableau

$$x_{B_i} + \sum_{j \in N_1} \bar{a}_{ij} x_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i$$

and assume $B_i \leq n_1$ but $\hat{x}_{B_i} = \bar{b}_i \notin \mathbb{Z}$. Then the Gomory Mixed Integer Cut

$$\boxed{x_{B_i} + \sum_{\substack{j \in N_1 \\ f_{ij} \leq f_i}} \lfloor \bar{a}_{ij} \rfloor x_j + \sum_{\substack{j \in N_1 \\ f_{ij} > f_i}} \left(\lfloor \bar{a}_{ij} \rfloor + \frac{f_{ij} - f_i}{1 - f_i} \right) x_j + \sum_{\substack{j \in N_2 \\ \bar{a}_{ij} < 0}} \left(\frac{\bar{a}_{ij}}{1 - f_i} \right) x_j \leq \lfloor \bar{b}_i \rfloor}$$

is a valid inequality for F that is not satisfied by $\hat{x}$.

Gomory Mixed Integer Cut

Cutting Planes Algorithm

- Let a MILP be given with feasible region $F = \{x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0} \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- The relaxation is the LP obtained by removing the integer constraints, so its feasible region is the polyhedron $P = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$.
- Repeat the following two steps until $\hat{x} \in F$:
  1. Solve the LP using the Simplex Algorithm and obtain $\hat{x} \in P$
  2. If $\hat{x} \notin F$, pick $i \in \{1, ..., n_1\}$ s.t. $\hat{x}_i \notin \mathbb{Z}$ and add the corresponding CMI-Cut to the LP.

Project Demonstration

C++ simplex solver and mixed integer Gomory cut. Python visualiser for 2D problems.

Selecting Cutting Planes

- Only adding an arbitrary, single cutting plane is very inefficient if the problem dimension is large.

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  - Heuristic to evaluate the efficiency of a cutting plane (e.g. euclidean distance to $\hat{x}$).
  - Add multiple cutting planes in each iteration.
- Other cutting plane strategies exist.

Complexity

Branch & Bound

- Like Cutting Planes, we solve the problem relaxation and add constraints until an optimal solution has been found.
- Divide & Conquer

Branching

Cut off the non-integer neighborhood of $x_i^* \notin \mathbb{Z} \Rightarrow$ Two new relaxation problems

Bounding

- Initially, we only know that $-\infty < c^\top x^*_{MILP} < \infty$ (not very helpful).
- Improve bounds until Upper Bound $-$ Lower Bound $< \epsilon$:
  - Lower Bound: For any LP-Relaxation, we have $c^\top x^*_{LP} \leq c^\top x^*_{MILP}$
  - Upper Bound: By definition, $c^\top x^*_{MILP} \leq c^\top x$ for any feasible $x \in F_{MILP}$

Branch & Cut

- Hello World
- Cutting Planes + Branch & Bound = Branch & Cut

Questions?

References