Solving Mixed Integer Linear Programs with Cutting Planes

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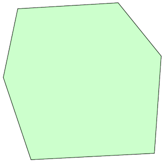
Mixed Integer Linear Program

- Optimize linear objective function s.t. linear constraints and some integer constraints.
- Sometimes, decision variables are discrete: Distribution of patients/supplies/vehicles... or binary/logical variables.

Mixed Integer Linear Program and Linear Program Relaxation



MILP (standard form)

$$\min_x \quad c^\top x$$

$$x = (\underbrace{x_1, \dots x_{n_1}}_{\in \mathbb{Z}}, \underbrace{x_{n_1+1} \dots x_n}_{\mathbb{R}})$$

$$\text{s.t.} \quad x \in F$$

$$:= \{x \mid Ax \leq b,\ x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0}\}$$

$c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

LP Relaxation

$$\min_x \quad c^\top x$$

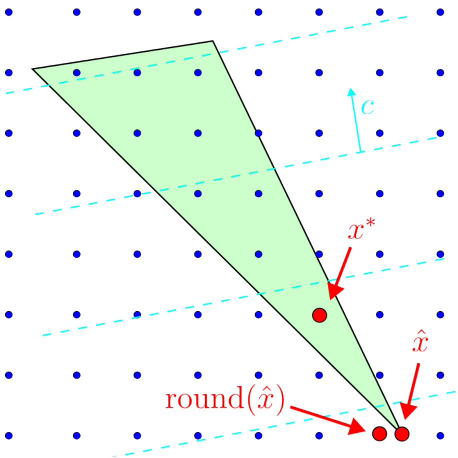
$$\text{s.t.} \quad x \in P$$

$$:= \{x \mid Ax \leq b,\ x \in \mathbb{R}^n_{\geq 0}\}$$

Notation

- $x_1, \ldots, x_{n_1}$: Integer variables
- $x_{n_1}, \ldots, x_n$: Real variables
- F, x^* : Feasible region and optimal solution of the MILP
- P, $\hat{x}$: Feasible region and optimal solution of the LP-relaxation
  $(F = P \cap \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0})$
- $n_1 = 0 \Rightarrow$ Linear Program (LP)
- $n_1 = n \Rightarrow$ Integer Linear Program (ILP)
- $x_i \in \{0, 1\} \Rightarrow$ (Mixed) Binary Program ((M)BP)

Observations



- $\hat{x}$ can be found at a vertex of P (Simplex Algorithm).

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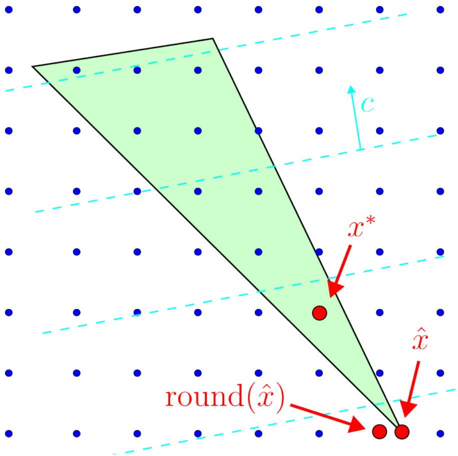

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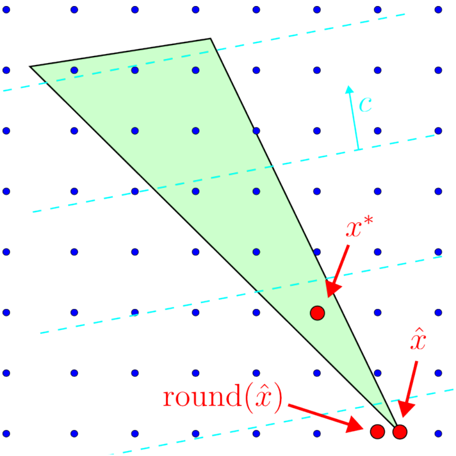
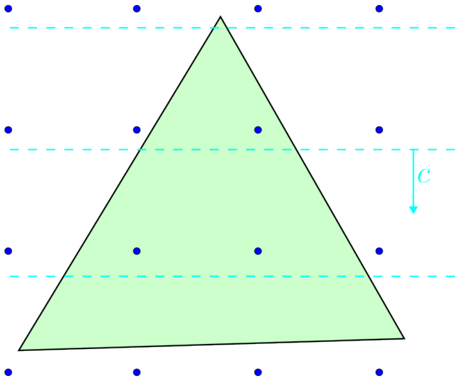


- $\hat{x}$ can be found at a vertex of P (Simplex Algorithm).
- In general, round($\hat{x}$) $\neq x^*$
- If $\hat{x}$ is already feasible, then $c^\top \hat{x} = c^\top x^*$.

Cutting Planes

Solve the problem relaxation. If integer constraints are violated, add additional inequalities to the problem that cut off the relaxed solution.



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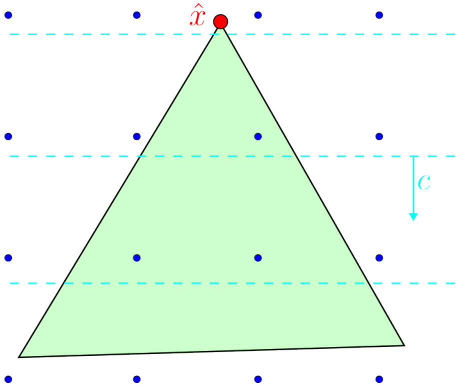

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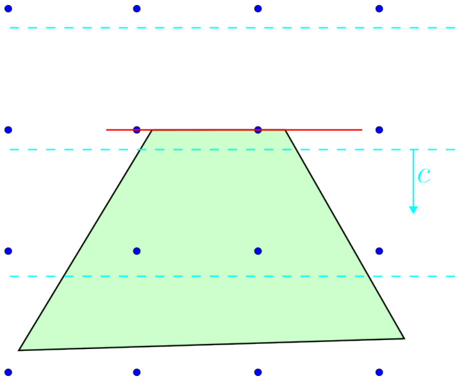

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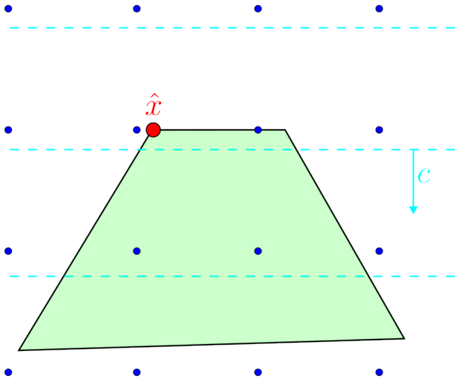

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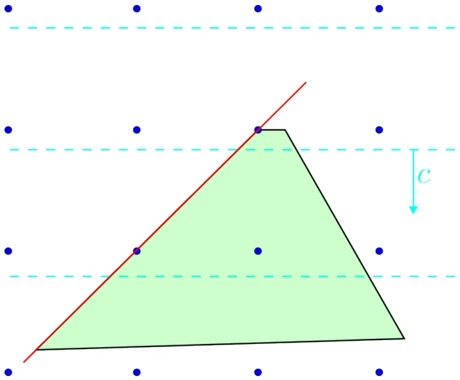

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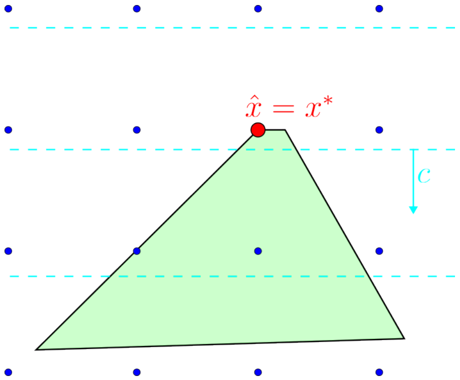

Valid Inequalities and Cuts

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  - For example: $x \leq 2$ is a valid inequality for $\{x \in \mathbb{Z}_{\geq 0} \mid x \leq 2.718\}$
- A cutting plane (or cut) w.r.t. $\hat{x} \in P \setminus F$ is any valid inequality $a^\top x \leq r$ for F such that:

$$a^\top \hat{x} > r$$

Cutting Planes Algorithm

LP $\leftarrow$ Relaxation of the MILP
repeat
    $\hat{x} \leftarrow$ Optimal solution of the LP
    if $(\hat{x}_1, \ldots, \hat{x}_{n_1}) \notin \mathbb{Z}^{n_1}$ **then**
        Add a cut w.r.t. $\hat{x}$ to the LP
until $(\hat{x}_1, \ldots, \hat{x}_{n_1}) \in \mathbb{Z}^{n_1}$
return $\hat{x}$

Cutting Strategy

Question: How to generate "good" and useful cuts?

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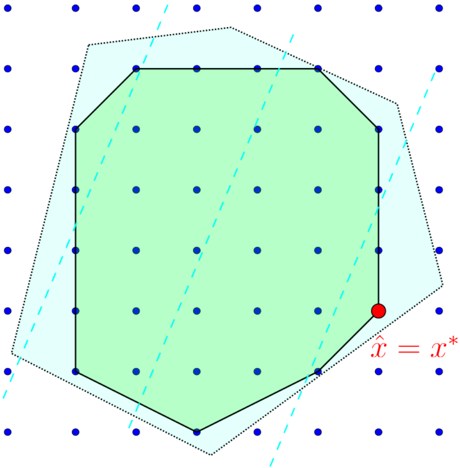
- Good: Cut away as much as possible (while staying feasible)
- Useful: Cut away the optimal solution of the relaxation

Convex Hull

- The relaxed solution $\hat{x}$ in $\text{conv}(F)$ also solves the MILP.
- But computing the convex hull is infeasible.
- Our goal is instead to approximate the convex hull in a neighborhood of x^*.



Integer Part and Fractional Part

Any real number $a \in \mathbb{R}$ can be expressed as

$$a = \lfloor a \rfloor + f_a$$

for some unique $\lfloor a \rfloor \in \mathbb{Z}$ and $f_a \in [0, 1)$.

- $\lfloor a \rfloor = \max\{z \in \mathbb{Z} \mid z \leq a\}$ is the <u>integer part</u> of a.
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- $a \in \mathbb{Z}$ and $a \leq b \Rightarrow a \leq \lfloor b \rfloor$

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- $a \in \mathbb{Z}$ and $a \leq b \Rightarrow a \leq \lfloor b \rfloor$
- $a \in \mathbb{Z}$ and $a \geq b \Rightarrow a \geq \lceil b \rceil$

Chvátal–Gomory Inequality for Integer Linear Programs

Let $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ for an Integer Linear Program ($x \in \mathbb{Z}^n_{\geq 0}$). Then the following inequalities are valid for any $\alpha \geq 0$:

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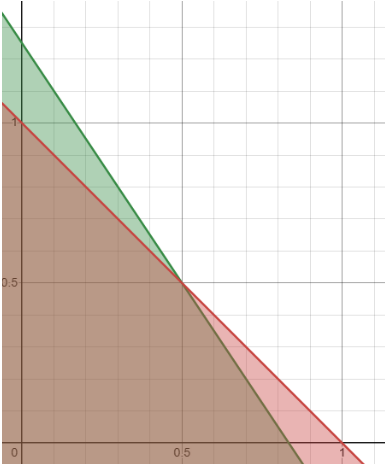
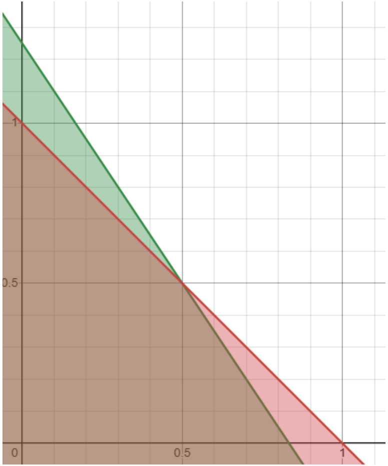
3. $\sum_{j=1}^{n} \lfloor \alpha a_{ij} \rfloor x_j \leq \lfloor \alpha b_i \rfloor$ $\qquad x_j \in \mathbb{Z}$

ILP vs. MILP



- $\min_{x,y} -y$
  s.t. $\frac{3}{2}x + y \leq \frac{5}{4}$, $(x, y) \in \mathbb{Z}^2_{\geq 0}$
- $(x^*, y^*) = (0, 1)$
- $\lfloor \frac{3}{2} \rfloor \cdot 0 + \lfloor 1 \rfloor \cdot 1 = 1 \leq \lfloor \frac{5}{4} \rfloor$ ✓

ILP vs. MILP



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- $(x^*, y^*) = (0, \frac{5}{4})$
- $\lfloor \frac{3}{2} \rfloor \cdot 0 + \lfloor 1 \rfloor \cdot \frac{5}{4} = \frac{5}{4} > \lfloor \frac{5}{4} \rfloor$ ✗

Basic Mixed Integer Rounding Inequalities I

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x + y \leq b\} \tag{1}$$

and

$$x \geq \lceil b \rceil \text{ is a valid inequality for } \{-x + y \leq -b\} \tag{2}$$

Basic Mixed Integer Rounding Inequalities I

Basic Mixed Integer Rounding Inequalities II

Let $x \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{R}_{\geq 0}$, $b \in \mathbb{R}_{>0} \setminus \mathbb{Z}$. Then

$$x - \frac{1}{f_b - 1} \leq \lfloor b \rfloor \text{ is a valid inequality for } \{x - y \leq b\} \tag{1}$$

and

$$x + \frac{1}{f_b} \geq \lceil b \rceil \text{ is a valid inequality for } \{-x - y \leq -b\} \tag{2}$$

Basic Mixed Integer Rounding Inequalities II

General Mixed Integer Rounding Inequality

Let $F_{MIR} = \{(x, y) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{R}_{\geq 0} \mid a_1 x_1 + a_2 x2 - y \leq b\}$ where $a \in \mathbb{R}^2$, $b \in \mathbb{R} \setminus \mathbb{Z}$ and assume that $f_1 \leq f_b \leq f_2$. Then the inequality

$$\lfloor a_1 \rfloor x_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f_b}{1 - f_b} \right) x_2 - \frac{1}{1 - f_b} y \leq \lfloor b \rfloor$$

is valid for F_{MIR}.

General Mixed Integer Rounding Inequality

Simplex Algorithm

Simplex finds $\hat{x} \in P \times \mathbb{R}^{N-n}_{\geq 0}$ and creates the optimal simplex tableau:

$i-$th row in the simplex tableau

$$x_{B_i} + \sum_{j \in NB} \bar{a}_{ij} x_j = \bar{b}_i$$

- $x_1, \ldots, x_{n_1}$: Integral decision variables
- $x_{n_1+1}, \ldots, x_n$: Real decision variables
- $x_{n+1}, \ldots, x_N$: (Real) slack variables

- $B = \{B_1, \ldots, B_m\}$: Basic variables
- $NB = \{1, \ldots, N\} \setminus B$: Nonbasic variables ($\hat{x}_j = 0$ for $j \in NB$)

Gomory Mixed Integer Cut

Let $N_1 = NB \cap \{1, \ldots, n_1\}$, $N_2 = NB \cap \{n_1 + 1, \ldots, N\}$. Consider the $i-$th row in the optimal simplex tableau

$$x_{B_i} + \sum_{j \in N_1} \bar{a}_{ij} x_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i$$

and assume $B_i \leq n_1$ but $\hat{x}_{B_i} = \bar{b}_i \notin \mathbb{Z}$. Then the Gomory Mixed Integer Cut

$$\boxed{x_{B_i} + \sum_{\substack{j \in N_1 \\ f_{ij} \leq f_i}} \lfloor \bar{a}_{ij} \rfloor x_j + \sum_{\substack{j \in N_1 \\ f_{ij} > f_i}} \left(\lfloor \bar{a}_{ij} \rfloor + \frac{f_{ij} - f_i}{1 - f_i} \right) x_j + \sum_{\substack{j \in N_2 \\ \bar{a}_{ij} < 0}} \left(\frac{\bar{a}_{ij}}{1 - f_i} \right) x_j \leq \lfloor \bar{b}_i \rfloor}$$

is a valid inequality for F that is not satisfied by $\hat{x}$.

Gomory Mixed Integer Cut

Cutting Planes Algorithm

- Let a MILP be given with feasible region $F = \{x \in \mathbb{Z}^{n_1}_{\geq 0} \times \mathbb{R}^{n-n_1}_{\geq 0} \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.
- The relaxation is the LP obtained by removing the integer constraints, so its feasible region is the polyhedron $P = \{x \in \mathbb{R}^n_{\geq 0} \mid Ax \leq b\}$.
- Repeat the following two steps until $\hat{x} \in F$:
  1. Solve the LP using the Simplex Algorithm and obtain $\hat{x} \in P$
  2. If the problem is infeasible ($P = \emptyset$), return INEASIBLE. If the problem is unbounded and no integer constraints are violated, return UNBOUNDED
  3. If $\hat{x} \notin F$, pick $i \in \{1, ..., n_1\}$ s.t. $\hat{x}_i \notin \mathbb{Z}$ and add the corresponding GMI-Cut to the LP.

Project Demonstration

- Simplex Solver
- Mixed Integer Gomory Cut
- 2D Visualisation

Cutting Planes Selection

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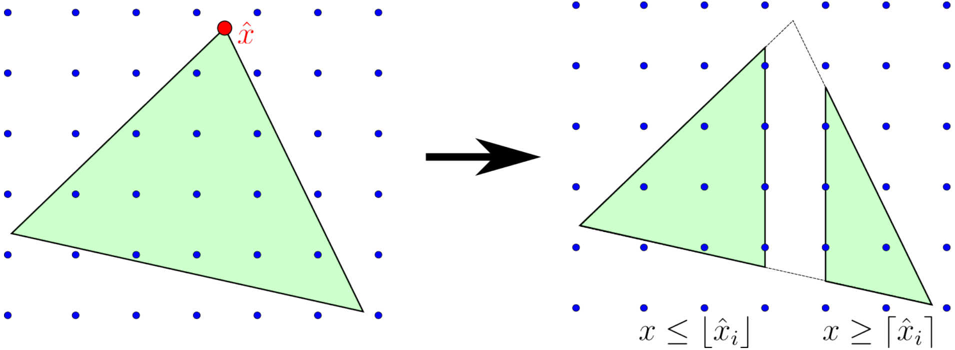
- Other cutting plane strategies exist:
  - Knapsack Covers or GUB (generalized upper bound) covers for binary programs.

Branch & Bound

- Like Cutting Planes, we solve the problem relaxation and add constraints until an optimal solution has been found.
- Divide & Conquer approach

Branching

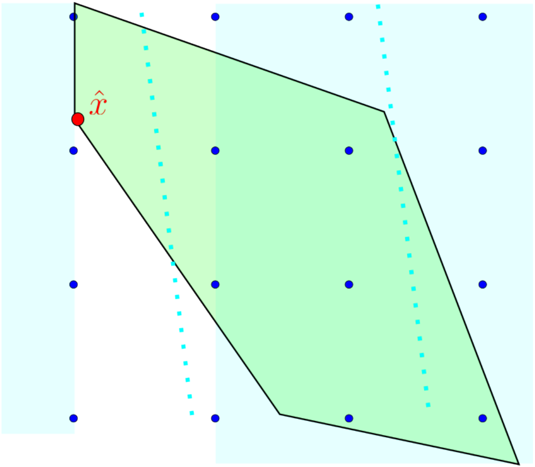
Cut off the non-integer neighborhood of $\hat{x}_i \notin \mathbb{Z}$ and obtain two new relaxation problems, one with $x_i \leq \lfloor \hat{x}_i \rfloor$ and one with $x_i \geq \lceil \hat{x}_i \rceil$. The resulting data structure is a binary tree of problems.



Bounding

- Initially, we only know that $-\infty < c^\top x^* < \infty$ (not very helpful).
- Improve bounds:
  - Lower Bound: For any LP-Relaxation, we have $c^\top \hat{x} \leq c^\top x^*$
  - Upper Bound: By definition, $c^\top x^* \leq c^\top x$ for any feasible $x \in F$
- For an optimal solution $\hat{x}$ of a subproblem:
  - if $c^\top \hat{x} \geq$ upper bound, prune tree (stop branching).
  - if $\hat{x}$ is feasible, update upper bound and prune.
  - if $\hat{x}$ is infeasible, update lower bound and branch.
  - stop if tree is completely pruned or upper bound $-$ lower bound $< \epsilon$.

Example



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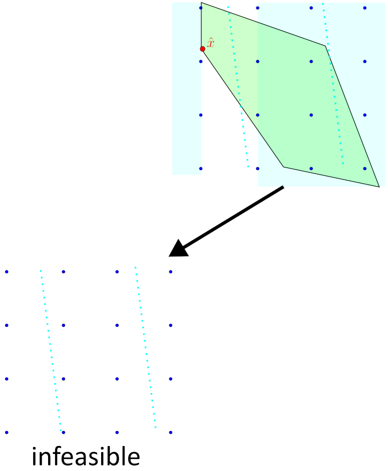

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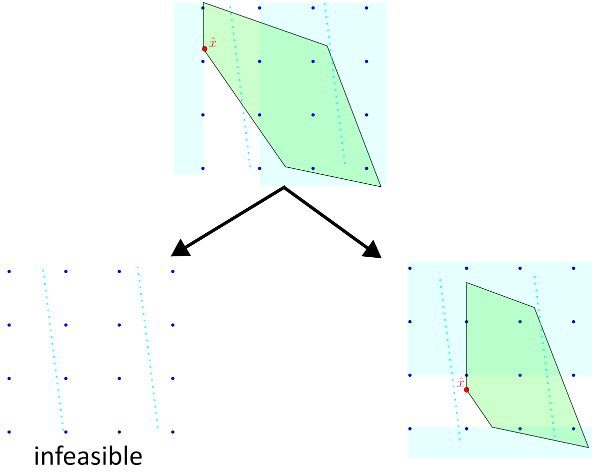

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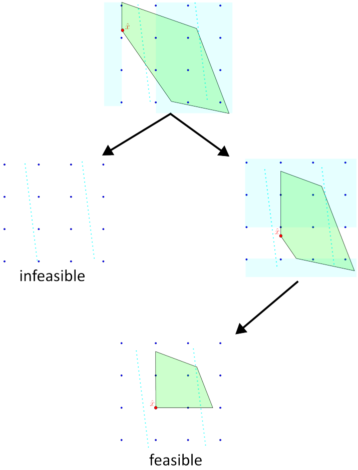

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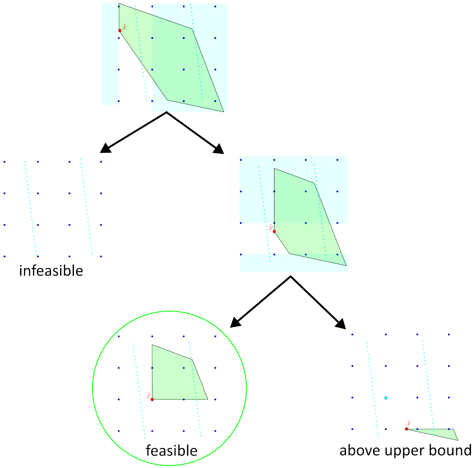

Subproblem and Branching Variable Selection Strategies

- Subproblem selection:
  - Depth-First: Descend quickly to obtain a good upper bound (obtain a feasible solution fast).
  - Best-First: Pick the active node with the current lower bound (obtain a good lower bound).
- Branching variable selection:
  - Most fractional: $i = \arg\max_{1 \leq i \leq n_1} \min(f_i, 1 - f_i)$ (f_i close to $\frac{1}{2}$).
  - Multiple variables at once.

Cutting Planes + Branch & Bound = Branch & Cut

- Cutting Planes or Branch & Bound on their own are inefficient in practice.
- Combine the two to Branch & Cut. This works like Branch & Bound but with additionally adding cuts before branching.
- Used most often in practice.

Questions?

References

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